Transforming Fuzzy Description Logic $\mathcal{ALC}_{\mathcal{FL}}$ into Classical Description Logic $\mathcal{ALCH}$

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Abstract. In this paper, we present a satisfiability preserving transformation of the fuzzy Description Logic $\mathcal{ALC}_{\mathcal{FL}}$ into the classical Description Logic $\mathcal{ALCH}$. We can use the already existing DL systems to do the reasoning of $\mathcal{ALC}_{\mathcal{FL}}$ by applying the result of this paper. This work is inspired by Straccia, who has transformed the fuzzy Description Logic $\mathfrak{f}\mathcal{ALCH}$ into the classical Description Logic $\mathcal{ALCH}$.

1 Introduction

The Semantic Web is a vision for the future of the Web in which information is given explicit meaning, making it easier for machines to automatically process and integrate information available on the Web. While as a basic component of the Semantic Web, an ontology is a collection of information and is a document or file that formally defines the relations among terms. OWL[1] is a *Web Ontology Language* and is intended to provide a language that can be used to describe the classes and relations between them that are inherent in Web documents and applications. The OWL language provides three increasingly expressive sublanguages: OWL Lite, OWL DL, OWL Full. OWL DL is so named due to its correspondence with description logics. OWL DL was designed to support the existing Description Logic business segment and has desirable computational properties for reasoning systems. According to the corresponding relation between axioms of OWL ontology and terms of Description Logic, we can represent the knowledge base contained in the ontology in syntax of DLs.

Description Logics (DLs) [1] have been studied and applied successfully in a lot of fields. The concepts in classical DLs are usually interpreted as crisp sets, i.e., an individual either belongs to the set or not. In the real world, the answers to some questions are often not only yes or no, rather we may say that an individual is an instance of a concept only to some certain degree. We often say linguistic terms such as "Very", "More or Less" etc. to distinguish, e.g. between a young person and a very young person. In 1970s, the theory of approximate reasoning based on the notions of linguistic variable and fuzzy logic was introduced and developed by Zadeh [21–23]. Adverbs as "Very", "More or Less" and "Possibly"

[1] Please visit `http://www.w3.org/TR/owl-guide/` for more details.

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are called hedges in fuzzy DLs. Some approaches to handling uncertainty and vagueness in DL for the Semantic Web are described in [12].

A well known feature of DLs is the emphasis on reasoning as a central service. Some reasoning procedures for fuzzy DLs have been proposed in [18]. A transformation of $\mathfrak{f}\mathcal{ALCH}$ into $\mathcal{ALCH}$ has been presented by Straccia [19]. T

In this paper we consider the fuzzy linguistic description logic $\mathcal{ALC}_{\mathcal{FL}}$ [9] which is an instance of the description logic framework $\mathcal{L} - \mathcal{ALC}$ with the certainty lattice characterized by a hedge algebra (HA) and allows the modification by hedges. Because the certainty lattice is characterized by a HA, the modification by hedges becomes more natural than that in $\mathcal{ALC}_{\mathcal{FH}}$ [10] and $\mathcal{ALC}_{\mathcal{FLH}}$ [16] which extend fuzzy $\mathcal{ALC}$ by allowing the modification by hedges of HAs. We will present a satisfiability preserving transformation of $\mathcal{ALC}_{\mathcal{FL}}$ into $\mathcal{ALCH}$ which makes the reuse of the technical results of classical Dls for $\mathcal{ALC}_{\mathcal{FL}}$ feasible.

The remaining part of this paper is organized in the following way. First we state some preliminaries on $\mathcal{ALCH}$, hedge algebra and $\mathcal{ALC}_{\mathcal{FL}}$. Then we present the transformation of $\mathcal{ALC}_{\mathcal{FL}}$ into $\mathcal{ALCH}$. Finally we discuss the main result of the paper and identify some possibilities for further work.

2 Preliminaries

2.1 $\mathcal{ALCH}$

We consider the language $\mathcal{ALCH}$ (Attributive Language with Complement and role Hierarchy). In abstract notation, we use the letters A and B for concept names, the letter R for role names, and the letters C and D for concept terms.

Definition 1. *Let N_R and N_C be disjoint sets of role names and concept names. Let $A \in N_C$ and $R \in N_R$.* Concept terms *in $\mathcal{ALCH}$ are formed according to the following syntax rule:*

$$A|\top|\bot|C \sqcap D|C \sqcup D|\neg C|\forall R.C|\exists R.C$$

The semantics of concept terms are defined formally by interpretations.

Definition 2. *An* interpretation $\mathcal{I}$ *is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a nonempty set (*interpretation domain*) and $\cdot^{\mathcal{I}}$ is an* interpretation function *which assigns to each concept name A a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and to each role name R a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The interpretation of complex concept terms is extended by the following inductive definitions:*

$$\begin{aligned}
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \emptyset \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \forall d'.(d, d') \notin R^{\mathcal{I}} \text{ or } d' \in C^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \exists d'.(d, d') \in R^{\mathcal{I}} \text{ and } d' \in C^{\mathcal{I}}\}
\end{aligned}$$

A concept term C is *satisfiable* iff there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$, denoted by $\mathcal{I} \models C$. Two concept terms C and D are *equivalent* (denoted by $C \equiv D$) iff $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretation $\mathcal{I}$.

We have seen how we can form complex descriptions of concepts to describe classes of objects. Now, we introduce *terminological axioms*, which make statements about how concept terms and roles are related to each other respectively.

In the most general case, *terminological axiom* have the form $C \sqsubseteq D$ or $R \sqsubseteq S$, where C, D are concept terms, R, S are role names. This kind of terminological axioms are also called *inclusions*. A set of axioms of the form $R \sqsubseteq S$ is called *role hierarchy*. An interpretation $\mathcal{I}$ *satisfies* an inclusion $C \sqsubseteq D$ ($R \sqsubseteq S$) iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ ($R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$), denoted by $\mathcal{I} \models C \sqsubseteq D$ ($\mathcal{I} \models R \sqsubseteq S$).

A *terminology*, i.e., *TBox*, is a finite set of terminological axioms. An interpretation $\mathcal{I}$ *satisfies* (is a *model* of) a terminology $\mathcal{T}$ iff $\mathcal{I}$ *satisfies* each element in $\mathcal{T}$, denoted by $\mathcal{I} \models \mathcal{T}$.

Assertions define how individuals relate with each other and how individuals relate with concept terms. Let N_I be a set of individual names which is disjoint to N_R and N_C. An *assertion* α is an expression of the form $a : C$ or $(a, b) : R$, where $a, b \in N_I$, $R \in N_R$ and $C \in N_C$. A finite set of *assertions* is called *ABox*. An interpretation $\mathcal{I}$ *satisfies* a concept assertion $a : C$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$, denoted by $\mathcal{I} \models a : C$. $\mathcal{I}$ *satisfies* a role assertion $(a, b) : R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$, denoted by $\mathcal{I} \models (a, b) : R$. An interpretation $\mathcal{I}$ *satisfies* (is a *model* of) an ABox $\mathcal{A}$ iff $\mathcal{I}$ *satisfies* each assertion in $\mathcal{A}$, denoted by $\mathcal{I} \models \mathcal{A}$.

A *knowledge base* is of the form $\langle \mathcal{T}, \mathcal{A} \rangle$ where $\mathcal{T}$ is a TBox and $\mathcal{A}$ is an ABox. An interpretation $\mathcal{I}$ *satisfies* (is a *model* of, denoted by $\mathcal{I} \models \mathcal{K}$) a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ iff $\mathcal{I}$ *satisfies* both $\mathcal{T}$ and $\mathcal{A}$. We say that a knowledge base $\mathcal{K}$ *entails* an assertion α, denoted $\mathcal{K} \models \alpha$ iff each model of $\mathcal{K}$ satisfies α. Furthermore, let $\mathcal{T}$ be a TBox and let C, D be two concept terms. We say that D *subsumes* C with respect to $\mathcal{T}$ (denoted by $C \sqsubseteq_{\mathcal{T}} D$) iff for each model of $\mathcal{T}$, $\mathcal{I} \models C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

The problem of determining whether $\mathcal{K} \models \alpha$ is called *entailment problem*; the problem of determining whether $C \sqsubseteq_{\mathcal{T}} D$ is called *subsumption problem*; and the problem of determining whether $\mathcal{K}$ is satisfiable is called *satisfiability problem*. Entailment problem and subsumption problem can be reduced to satisfiability problem.

2.2 Linear Symmetric Hedge Algebra

In this section, we introduce linear symmetric Hedge Algebras (HAs). For general HAs, please refer to [13–15].

Let us consider a linguistic variable *TRUTH* with the domain $dom(TRUTH) = \{True, False, VeryTrue, VeryFalse, MoreTrue, MoreFalse, PossiblyTrue, \ldots\}$. This domain is an infinite partially ordered set, with a natural ordering $a < b$ meaning that b describes a larger degree of truth if we consider $True > False$. This set is generated from the basic elements (*generators*) $G = \{True, False\}$ by using *hedges*, i.e., unary operations from a finite set $H = \{Very, Possibly, More\}$. The $dom(TRUTH)$ which is a set of linguistic values can be represented as $X = \{\delta c \mid c \in G, \delta \in H^*\}$ where H^* is the Kleene star of H, From the algebraic point of view, the truth domain can be described as an abstract algebra $AX = (X, G, H, >)$.

To define relations between hedges, we introduce some notations first. We define that $H(x) = \{\sigma x \mid \sigma \in H^*\}$ for all $x \in X$. Let I be the identity hedge, i.e., $\forall x \in X. Ix = x$. The identity I is the least element. Each element of H is an *ordering operation*, i.e., $\forall h \in H$, $\forall x \in X$, either $hx > x$ or $hx < x$.

Definition 3. *[14] Let $h, k \in H$ be two hedges, for all $x \in X$ we define:*

- *h, k are* converse *if $hx < x$ iff $kx > x$;*
- *h, k are* compatible *if $hx < x$ iff $kx < x$;*
- *h modifies terms stronger or equal than k, denoted by $h \geq k$ if $hx \leq kx \leq x$ or $hx \geq kx \geq x$;*
- *$h > k$ if $h \geq k$ and $h \neq k$;*
- *h is* positive wrt k *if $hkx < kx < x$ or $hkx > kx > x$;*
- *h is* negative wrt k *if $kx < hkx < x$ or $kx > hkx > x$.*

$\mathcal{ALC}_{\mathcal{FL}}$ only considers symmetric HAs, i.e., there are exactly two generators as in the example $G = \{\mathit{True}, \mathit{False}\}$. Let $G = \{c^+, c^-\}$ where $c^+ > c^-$. c^+ and c^- are called *positive* and *negative generators* respectively. Because there are only two generators, the relations presented in Definition 3 divides the set H into two subsets $H^+ = \{h \in H \mid hc^+ > c^+\}$ and $H^- = \{h \in H \mid hc^+ < c^+\}$, i.e., every operation in H^+ is converse w.r.t. any operation in H^- and vice-versa, and the operations in the same subset are compatible with each other.

Definition 4. *[9] An abstract algebra $AX = (X, G, H, >)$, where $H \neq \emptyset, G = \{c^+, c^-\}$ and $X = \{\sigma c \mid c \in G, \sigma \in H^*\}$ is called a* linear symmetric hedge algebra *if it satisfies the properties* (A1)-(A5).

(A1) Every hedge in H^+ is a converse operation of all operations in H^-.
(A2) Each hedge operation is either positive or negative w.r.t. the others, including itself.
(A3) The sets $H^+ \cup \{I\}$ and $H^- \cup \{I\}$ are linearly ordered with the I.
(A4) If $h \neq k$ and $hx < kx$ then $h'hx < k'kx$, for all $h, k, h', k' \in H$ and $x \in X$.
(A5) If $u \notin H(v)$ and $u \leq v$ ($u \geq v$) then $u \leq hv$ ($u \geq hv$), for any hedge h and $u, v \in X$.

Let $AX = (X, G, H, >)$ be a linear symmetric hedge algebra and $c \in G$. We define that, $\bar{c} = c^+$ if $c = c^-$ and $\bar{c} = c^-$ if $c = c^+$. Let $x \in X$ and $x = \sigma c$, where $\sigma \in H^*$. The *contradictory element* to x is $y = \sigma\bar{c}$ written $y = -x$.

[14] gave us the following proposition to compare elements in X.

Proposition 5. *Let $AX = (X, G, H, >)$ be a linear symmetric HA, $x = h_n \cdots h_1 u$ and $y = k_m \cdots k_1 u$ are two elements of X where $u \in X$. Then there exists an index $j \leq \min\{n, m\} + 1$ such that $h_i = k_i$ for all $i < j$, and*

(i) *$x < y$ iff $h_j x_j < k_j x_j$, where $x_j = h_{j-1} \cdots h_1 u$;*
(ii) *$x = y$ iff $n = m = j$ and $h_j x_j = k_j x_j$.*

In order to define the semantics of the hedge modification, we only consider monotonic HAs defined in [9] which also extended the order relation on $H^+ \cup \{I\}$ and $H^- \cup \{I\}$ to one on $H \cup \{I\}$. We will use "hedge algebra" instead of "linear symmetric hedge algebra" in the rest of this paper.

2.3 Inverse Mapping of Hedges

Fuzzy description logics represent the assessment "It is true that Tom is very old" by

$$(VeryOld)^{\mathcal{I}}(Tom)^{\mathcal{I}} = True. \tag{1}$$

In a fuzzy linguistic logic [21–23], the assessment "It is true that Tom is very old" and the assessment "It is very true that Tom is old" are equivalent, which means

$$(Old)^{\mathcal{I}}(Tom)^{\mathcal{I}} = VeryTrue, \tag{2}$$

and (1) has the same meaning. (In other word, a fuzzy interpretation $\mathcal{I}$ (Definition 8) satisfies an assertion $Tom : VeryOld \geq True$ if and only if $\mathcal{I}$ satisfies the assertion $Tom : Old \geq VeryTrue$.) This signifies that the modifier can be moved from concept term to truth value and vice versa. For any $h \in H$ and for any $\sigma \in H^*$, the rules of moving hedges [13] are as follows,

$$RT1 : (hC)^{\mathcal{I}}(d) = \sigma c \rightarrow (C)^{\mathcal{I}}(d) = \sigma hc$$
$$RT2 : (C)^{\mathcal{I}}(d) = \sigma hc \rightarrow (hC)^{\mathcal{I}}(d) = \sigma c.$$

where C is a concept term and $d \in \Delta^{\mathcal{I}}$.

Definition 6. *[9] Consider a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$ and a $h \in H$. A mapping $h^- : X \rightarrow X$ is called an* inverse mapping *of h iff it satisfies the following two properties,*

1. $h^-(\sigma hc) = \sigma c$.
2. $\sigma_1 c_1 > \sigma_2 c_2 \Leftrightarrow h^-(\sigma_1 c_1) > h^-(\sigma_2 c_2)$.

where $c, c_1, c_2 \in G$, $h \in H$ and $\sigma_1, \sigma_2 \in H^*$.

2.4 $\mathcal{ALC}_{\mathcal{FL}}$

$\mathcal{ALC}_{\mathcal{FL}}$ is a Description Logic in which the truth domain of interpretations is represented by a hedge algebra. The syntax of $\mathcal{ALC}_{\mathcal{FL}}$ is similar to that of $\mathcal{ALCH}$ except that $\mathcal{ALC}_{\mathcal{FL}}$ allows concept modifiers and does not include role hierarchy.

Definition 7. *Let H be a set of hedges. Let A be a concept name and R a role, complex concept terms denoted by C, D in $\mathcal{ALC}_{\mathcal{FL}}$ are formed according to the following syntax rule:*

$$A|\top|\bot|C \sqcap D|C \sqcup D|\neg C|\delta C|\forall R.C|\exists R.C$$

where $\delta \in H^*$.

In [15], HAs are extended by adding two artificial hedges inf and sup defined as $\inf(x) = \text{infimum}(H(x))$, $\sup(x) = \text{supremum}(H(x))$. If $H = \emptyset$, $H(c^+)$ and $H(c^-)$ are infinite, according to [15] $\inf(c^+) = \sup(c^-)$. Let $W = \inf(\mathit{True}) = \sup(\mathit{False})$ and let $\sup(\mathit{True})$ and $\inf(\mathit{False})$ be the greatest and the least elements of X respectively.

The semantics is based on the notion of interpretations.

Definition 8. *Let AX be a monotonic HA such that $AX = (X, \{\mathit{True}, \mathit{False}\}, H, >)$. A* fuzzy interpretation ($\mathfrak{f}$-interpretation) *$\mathcal{I}$ for $\mathcal{ALC}_{\mathcal{FL}}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a nonempty set and $\cdot^{\mathcal{I}}$ is an* interpretation function *mapping:*

- *individuals to elements in $\Delta^{\mathcal{I}}$;*
- *a concept C into a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to X$;*
- *a role R into a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to X$.*

For all $d \in \Delta^{\mathcal{I}}$ the interpretation function satisfies the following equations

$$\begin{aligned}
\top^{\mathcal{I}}(d) &= \sup(\mathit{True}),\\
\bot^{\mathcal{I}}(d) &= \inf(\mathit{False}),\\
(\neg C)^{\mathcal{I}}(d) &= -C^{\mathcal{I}}(d),\\
(C \sqcap D)^{\mathcal{I}}(d) &= \min(C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)),\\
(C \sqcup D)^{\mathcal{I}}(d) &= \max(C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)),\\
(\delta C)^{\mathcal{I}}(d) &= \delta^{-}(C^{\mathcal{I}}(d)),\\
(\forall R.C)^{\mathcal{I}}(d) &= \inf_{d' \in \Delta^{\mathcal{I}}}\{\max(-R^{\mathcal{I}}(d,d'), C^{\mathcal{I}}(d'))\},\\
(\exists R.C)^{\mathcal{I}}(d) &= \sup_{d' \in \Delta^{\mathcal{I}}}\{\min(R^{\mathcal{I}}(d,d'), C^{\mathcal{I}}(d'))\},
\end{aligned}$$

where $-x$ is the contradictory element of x, and δ^- is the inverse of the hedge chain δ.

Definition 9. *A* fuzzy assertion ($\mathfrak{f}$assertion) *is an expression of the form $\langle \alpha \bowtie \sigma c\rangle$ where α is of the form $a : C$ or $(a, b) : R$, $\bowtie \in \{\geq, >, \leq, <\}$ and $\sigma c \in X$.*

Formally, an $\mathfrak{f}$-interpretation $\mathcal{I}$ *satisfies* a fuzzy assertion $\langle a : C \geq \sigma c\rangle$ (respectively $\langle (a,b) : R \geq \sigma c\rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \sigma c$ (respectively $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq \sigma c$). An $\mathfrak{f}$-interpretation $\mathcal{I}$ *satisfies* a fuzzy assertion $\langle a : C \leq \sigma c\rangle$ (respectively $\langle (a,b) : R \leq \sigma c\rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \leq \sigma c$ (respectively $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \leq \sigma c$). Similarly for $>$ and $<$.

Concerning terminological axioms, an $\mathcal{ALC}_{\mathcal{FL}}$ terminology axiom is of the form $C \sqsubseteq D$, where C and D are $\mathcal{ALC}_{\mathcal{FL}}$ concept terms. From a semantics point of view, a $\mathfrak{f}$-interpretation $\mathcal{I}$ *satisfies* a fuzzy concept inclusion $C \sqsubseteq D$ iff $\forall d \in \Delta^{\mathcal{I}}.C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$. Two concept terms C, D are said to be *equivalent*, denoted by $C \equiv D$ iff $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all $\mathfrak{f}$-interpretations $\mathcal{I}$. Some properties concerning the hedge modification are showed in the following proposition [9].

Proposition 10. *We have the following semantical equivalence:*

$$\begin{aligned}
\delta(C \sqcap D) &\equiv \delta(C) \sqcap \delta(D)\\
\delta(C \sqcup D) &\equiv \delta(C) \sqcup \delta(D)\\
\delta_1(\delta_2 C) &\equiv (\delta_1\delta_2)C.
\end{aligned}$$

A *fuzzy knowledge base* (f*KB)* is $\langle \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{T}$ and $\mathcal{A}$ are finite sets of terminological axioms and fassertions respectively.

Example 11. *A fKB* $\mathfrak{f}\mathcal{K} = \langle \{A \sqsubseteq \forall R.\neg B\}, \{a : \forall R.C \geq \mathit{VeryTrue}\} \rangle$.

An f-interpretation $\mathcal{I}$ *satisfies* (is a *model* of) a TBox $\mathcal{T}$ iff $\mathcal{I}$ satisfies each element in $\mathcal{T}$. $\mathcal{I}$ *satisfies* (is a *model* of) an ABox $\mathcal{A}$ iff $\mathcal{I}$ satisfies each element in $\mathcal{A}$. $\mathcal{I}$ *satisfies* (is a *model* of) a fKB $\mathfrak{f}\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ iff $\mathcal{I}$ satisfies both $\mathcal{A}$ and $\mathcal{T}$. Given a fKB $\mathfrak{f}\mathcal{K}$ and a fassertion $\mathfrak{f}\alpha$. We say that $\mathfrak{f}\mathcal{K}$ *entails* $\mathfrak{f}\alpha$ (denoted $\mathfrak{f}\mathcal{K} \models \mathfrak{f}\alpha$) iff each model of $\mathfrak{f}\mathcal{K}$ satisfies $\mathfrak{f}\alpha$.

3 Transforming $\mathcal{ALC}_{\mathcal{FL}}$ into $\mathcal{ALCH}$

We will introduce a satisfiability preserving transformation from $\mathcal{ALC}_{\mathcal{FL}}$ into $\mathcal{ALCH}$ in this section. First, we illustrate the basic idea which is similar to the one in [19] which is the first efforts in this direction. There is also other more efficient representation in [3].

Consider a monotonic HA $AX = (X, \{\mathit{True}, \mathit{False}\}, H, >)$. In the following, we assume that $c \in \{c^+, c^-\}$ where $c^+ = \mathit{True}, c^- = \mathit{False}$, $\sigma \in H^*$, $\sigma c \in X$ and $\bowtie \in \{\geq, >, \leq, <\}$. Assume we have an $\mathcal{ALC}_{\mathcal{FL}}$ knowledge base, $\mathfrak{f}\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{A} = \{\mathfrak{f}\alpha_1, \mathfrak{f}\alpha_2, \mathfrak{f}\alpha_3, \mathfrak{f}\alpha_4\}$ and $\mathfrak{f}\alpha_1 = \langle a : A \geq \mathit{True} \rangle$, $\mathfrak{f}\alpha_2 = \langle b : A \geq \mathit{VeryTrue} \rangle$, $\mathfrak{f}\alpha_3 = \langle a : B \leq \mathit{False} \rangle$, and $\mathfrak{f}\alpha_4 = \langle b : B \leq \mathit{VeryFalse} \rangle$ where A, B are concept names. We introduce four new concept names: $A_{\geq \mathit{True}}$, $A_{\geq \mathit{VeryTrue}}$, $B_{\leq \mathit{False}}$ and $B_{\leq \mathit{VeryFalse}}$. The concept name $A_{\geq \mathit{True}}$ represents the set of individuals that are instances of A with degree greater and equal to *True*. The concept name $B_{\leq \mathit{VeryFalse}}$ represents the set of individuals that are instances of B with degree less and equal to *VeryFalse*. We can map the fuzzy assertions into classical assertions:

$\langle a : A \geq \mathit{True} \rangle \rightarrow \langle a : A_{\geq \mathit{True}} \rangle$,
$\langle b : A \geq \mathit{VeryTrue} \rangle \rightarrow \langle b : A_{\geq \mathit{VeryTrue}} \rangle$,
$\langle a : B \leq \mathit{False} \rangle \rightarrow \langle a : B_{\leq \mathit{False}} \rangle$,
$\langle b : B \leq \mathit{VeryFalse} \rangle \rightarrow \langle b : B_{\leq \mathit{VeryFalse}} \rangle$.

We also need to consider the relationships among the newly introduced concept names. Because $\mathit{VeryTrue} > \mathit{True}$, it is easy to get if a truth value $\sigma c \geq \mathit{VeryTrue}$ then $\sigma c \geq \mathit{True}$. Thus, we obtain a new inclusion $A_{\geq \mathit{VeryTrue}} \sqsubseteq A_{\geq \mathit{True}}$. Similarly for B, because $\mathit{VeryFalse} < \mathit{False}$, a truth value $\sigma c \leq \mathit{VeryFalse}$ implies $\sigma c \leq \mathit{False}$ too. Then the inclusion $B_{\leq \mathit{VeryFalse}} \sqsubseteq B_{\leq \mathit{False}}$ is obtained.

Now, let us proceed with the mappings. Let $\mathfrak{f}\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be an $\mathcal{ALC}_{\mathcal{FL}}$ knowledge base. We are going to transform $\mathfrak{f}\mathcal{K}$ into an $\mathcal{ALCH}$ knowledge base $\mathcal{K}$. We assume $\sigma c \in [\inf(\mathit{False}), \sup(\mathit{True})]$ and $\bowtie \in \{\geq, >, \leq, <\}$.

3.1 The Transformation of ABox

In order to transform $\mathcal{A}$, we define two mappings θ and ρ to map all the assertions in $\mathcal{A}$ into classical assertions. Notice that we do not allow assertions of the forms

$(a, b) : R < \sigma c$ and $(a, b) : R \leq \sigma c$ although they are legal forms of assertions in $\mathcal{ALC}_{\mathcal{FL}}$ because they related to 'negated role' which is not part of classical $\mathcal{ALCH}$.

We use the mapping ρ to encode the basic idea we present at the beginning of this section. The mapping ρ combines the $\mathcal{ALC}_{\mathcal{FL}}$ concept term, the $\bowtie$ and the fuzzy value σc together into one $\mathcal{ALCH}$ concept term.

Let A be a concept name, C, D be concept terms and R be a role name. For roles we have simply

$$\rho(R, \bowtie \sigma c) = R_{\bowtie \sigma c}.$$

For concept terms, the mapping ρ is inductively defined on the structures of concept terms:
For $\top$,

$$\rho(\top, \bowtie \sigma c) = \begin{cases} \top \text{ if } \bowtie \sigma c = \geq \sigma c \\ \top \text{ if } \bowtie \sigma c = > \sigma c, \sigma c < \sup(c^+) \\ \bot \text{ if } \bowtie \sigma c = > \sup(c^+) \\ \top \text{ if } \bowtie \sigma c = \leq \sup(c^+) \\ \bot \text{ if } \bowtie \sigma c = \leq \sigma c, \sigma c < \sup(c^+) \\ \bot \text{ if } \bowtie \sigma c = < \sigma c. \end{cases}$$

For $\bot$,

$$\rho(\bot, \bowtie \sigma c) = \begin{cases} \top \text{ if } \bowtie \sigma c = \geq \inf(c^-) \\ \bot \text{ if } \bowtie \sigma c = \geq \sigma c, \sigma c > \inf(c^-) \\ \bot \text{ if } \bowtie \sigma c = > \sigma c \\ \top \text{ if } \bowtie \sigma c = \leq \sigma c \\ \top \text{ if } \bowtie \sigma c = < \sigma c, \sigma c > \inf(c^-) \\ \bot \text{ if } \bowtie \sigma c = < \inf(c^-). \end{cases}$$

For concept name A,

$$\rho(A, \bowtie \sigma c) = A_{\bowtie \sigma c}.$$

For concept conjunction $C \sqcap D$,

$$\rho(C \sqcap D, \bowtie \sigma c) = \begin{cases} \rho(C, \bowtie \sigma c) \sqcap \rho(D, \bowtie \sigma c) \text{ if } \bowtie \in \{\geq, >\} \\ \rho(C, \bowtie \sigma c) \sqcup \rho(D, \bowtie \sigma c) \text{ if } \bowtie \in \{\leq, <\}. \end{cases}$$

For concept disjunction $C \sqcup D$,

$$\rho(C \sqcup D, \bowtie \sigma c) = \begin{cases} \rho(C, \bowtie \sigma c) \sqcup \rho(D, \bowtie \sigma c) \text{ if } \bowtie \in \{\geq, >\} \\ \rho(C, \bowtie \sigma c) \sqcap \rho(D, \bowtie \sigma c) \text{ if } \bowtie \in \{\leq, <\}. \end{cases}$$

For concept negation $\neg C$,

$$\rho(\neg C, \bowtie \sigma c) = \rho(C, \neg \bowtie \sigma \bar{c}),$$

where $\neg \geq \, = \, \leq, \neg > \, = \, <, \neg \leq \, = \, \geq, \neg < \, = \, >$.

For modifier concept δC,

$$\rho(\delta C, \bowtie \sigma c) = \rho(C, \bowtie \sigma\delta c).$$

For existential quantification $\exists R.C$,

$$\rho(\exists R.C, \bowtie \sigma c) = \begin{cases} \exists\rho(R, \bowtie \sigma c).\rho(C, \bowtie \sigma c) \text{ if } \bowtie \in \{\geq, >\} \\ \forall\rho(R, - \bowtie \sigma c).\rho(C, \bowtie \sigma c) \text{ if } \bowtie \in \{\leq, <\}, \end{cases}$$

where $- \leq = >$ and $- < = \geq$.

For universal quantification $\forall R.C$,

$$\rho(\forall R.C, \bowtie \sigma c) = \begin{cases} \forall\rho(R, + \bowtie \sigma\bar{c}).\rho(C, \bowtie \sigma c) \text{ if } \bowtie \in \{\geq, >\} \\ \exists\rho(R, \neg \bowtie \sigma\bar{c}).\rho(C, \bowtie \sigma c) \text{ if } \bowtie \in \{\leq, <\}, \end{cases}$$

where $+ \geq = >$ and $+ > = \geq$.

θ maps fuzzy assertions into classical assertions using ρ. Let $\mathfrak{f}\alpha$ be a $\mathfrak{f}$assertion in $\mathcal{A}$, we define it as follows.

$$\theta(\mathfrak{f}\alpha) = \begin{cases} a : \rho(C, \bowtie \sigma c) \text{ if } \mathfrak{f}\alpha = \langle a : C \bowtie \sigma c\rangle \\ (a, b) : \rho(R, \bowtie \sigma c) \text{ if } \mathfrak{f}\alpha = \langle (a, b) : R \bowtie \sigma c\rangle. \end{cases}$$

Example 12. *Let* $\mathfrak{f}\alpha = \langle a : Very(A \sqcap B) \leq LessFalse\rangle$*, then*

$$\begin{aligned} \theta(\mathfrak{f}\alpha) &= a : \rho(Very(A \sqcap B), \leq LessFalse) \\ &= a : \rho((A \sqcap B), \leq LessVeryFalse) \\ &= a : \rho(A, \leq LessVeryFalse) \sqcup \rho(B, \leq LessVeryFalse) \\ &= a : A_{\leq LessVeryFalse} \sqcup B_{\leq LessVeryFalse}. \end{aligned}$$

We extend θ to a set of $\mathfrak{f}$assertions $\mathcal{A}$ point-wise,

$$\theta(\mathcal{A}) = \{\theta(\mathfrak{f}\alpha) \mid \mathfrak{f}\alpha \in \mathcal{A}\}.$$

According to the rules above, we can see that $|\theta(\mathcal{A})|$ is linearly bounded by $|\mathcal{A}|$.

4 The Transformation of TBox

The new TBox is a union of two terminologies. One is the newly introduced TBox (denoted by $\mathcal{T}(N^{\mathfrak{f}\mathcal{K}})$ which is the terminology relating to the newly introduced concept names and role names. The other one is $\kappa(\mathfrak{f}\mathcal{K}, \mathcal{T})$ which is reduced by a mapping κ from the TBox of an $\mathcal{ALC}_{\mathcal{FL}}$ knowledge base.

4.1 The Newly Introduced TBox

Many new concept names and new role names are introduced when we transform an ABox. We need a set of terminological axioms to define the relationships among those new names.

We need to collect all the linguist terms σc that might be the subscript of a concept name or a role name. It means that not only the set of linguistic terms that appears in the original ABox but also the set of new linguist terms which are produced by applying the ρ for modifier concepts should be included. Let A be a concept name, R be a role name.

$$X^{\mathfrak{f}\mathcal{K}} = \{\sigma c \mid \langle \alpha \bowtie \sigma c \rangle \in \mathcal{A}\} \cup \{\sigma\delta c \mid \rho(\delta C, \bowtie \sigma c) = \rho(C, \bowtie \sigma\delta c)\}.$$

such that δC occurs in $\mathfrak{f}\mathcal{K}$.

We define a sorted set of linguistic terms,

$$\begin{aligned} N^{\mathfrak{f}\mathcal{K}} &= \{\inf(\mathit{False}), W, \sup(\mathit{True})\} \cup X^{\mathfrak{f}\mathcal{K}} \cup \{\sigma\bar{c} \mid \sigma c \in X^{\mathfrak{f}\mathcal{K}}\} \\ &= \{n_1, \ldots, n_{|N^{\mathfrak{f}\mathcal{K}}|}\} \end{aligned}$$

where $n_i < n_{i+1}$ for $1 \leq i \leq |N^{\mathfrak{f}\mathcal{K}}| - 1$ and $n_1 = \inf(\mathit{False}), n_{|N^{\mathfrak{f}\mathcal{K}}|} = \sup(\mathit{True})$.

Example 13. *Consider Example 11, the sorted set is,*

$$N^{\mathfrak{f}\mathcal{K}} = \{\inf(\mathit{False}), \mathit{VeryFalse}, W, \mathit{VeryTrue}, \sup(\mathit{True})\}.$$

Let $\mathcal{T}(N^{\mathfrak{f}\mathcal{K}})$ be the set of terminological axioms relating to the newly introduced concept names and role names.

Definition 14. *Let $\mathcal{A}^{\mathfrak{f}\mathcal{K}}$ and $\mathcal{R}^{\mathfrak{f}\mathcal{K}}$ be the sets of concept names and role names occurring in $\mathfrak{f}\mathcal{K}$ respectively. For each $A \in \mathcal{A}^{\mathfrak{f}\mathcal{K}}$, for each $R \in \mathcal{R}^{\mathfrak{f}\mathcal{K}}$, for each $1 \leq i \leq |N^{\mathfrak{f}K}| - 1$ and for each $2 \leq j \leq |N^{\mathfrak{f}K}| - 1$, $\mathcal{T}(N^{\mathfrak{f}\mathcal{K}})$ contains*

$$\begin{aligned} &A_{\geq n_{i+1}} \sqsubseteq A_{>n_i},\ A_{>n_j} \sqsubseteq A_{\geq n_j}, \\ &R_{\geq n_{i+1}} \sqsubseteq R_{>n_i},\ R_{>n_j} \sqsubseteq R_{\geq n_j}. \end{aligned}$$

where $n \in N^{\mathfrak{f}\mathcal{K}}$.

$n_{i+1} > n_i$ because $N^{\mathfrak{f}\mathcal{K}}$ is a sorted set. Then if an individual is an instance of a concept name with degree $\geq n_{i+1}$ then the degree is also $> n_i$. The first terminological axiom shows that if an individual is an instance of $A_{\geq n_{i+1}}$ then it is an instance of $A_{>n_i}$ as well. Similarly, if an individual is an instance of a concept name with degree $> n_i$ then the degree is also $\geq n_i$. The second terminological axiom shows that if an individual is an instance of $A_{>n_i}$ then it is also an instance of $A_{\geq n_i}$.

$\mathcal{T}(N^{\mathfrak{f}\mathcal{K}})$ contains $2|\mathcal{A}^{\mathfrak{f}\mathcal{K}}|(|\mathcal{N}^{\mathfrak{f}\mathcal{K}}| - 1)$ plus $2|\mathcal{R}^{\mathfrak{f}\mathcal{K}}|(|\mathcal{N}^{\mathfrak{f}\mathcal{K}}| - 1)$ terminological axioms.

Example 15. *Consider the $\mathcal{ALC}_{\mathcal{FL}}$ knowledge base in Example 11, the following is an excerpt of the $\mathcal{T}(N^{\mathfrak{f}\mathcal{K}})$,*

$$\begin{aligned} \mathcal{T}(N^{\mathfrak{f}\mathcal{K}}) = \{ &A_{\geq \sup(\mathit{True})} \sqsubseteq A_{> \mathit{VeryTrue}},\ A_{\geq \mathit{VeryTrue}} \sqsubseteq A_{>W}, \\ &A_{\geq W} \sqsubseteq A_{> \mathit{VeryFalse}},\ A_{\geq \mathit{VeryFalse}} \sqsubseteq A_{>\inf(\mathit{False})}\} \\ \cup \{ &A_{> \mathit{VeryTrue}} \sqsubseteq A_{\geq \mathit{VeryTrue}}, A_{>W} \sqsubseteq A_{\geq W}, \\ &A_{> \mathit{VeryFalse}} \sqsubseteq A_{\geq \mathit{VeryFalse}}\} \\ \cup \{ &\ldots, R_{\geq \sup(\mathit{True})} \sqsubseteq R_{> \mathit{VeryTrue}}, \ldots\}. \end{aligned}$$

4.2 The Mapping κ

κ maps the fuzzy TBox into the classical TBox.

Definition 16. *Let C, D be two concept terms and $C \sqsubseteq D \in \mathcal{T}$. For all $n \in N^{\mathfrak{f}\mathcal{K}}$*

$$\begin{aligned} \kappa(\mathfrak{f}\mathcal{K}, C \sqsubseteq D) = & \bigcup_{n \in N^{\mathfrak{f}\mathcal{K}}, \bowtie \in \{\geq, >\}} \{\rho(C, \bowtie n) \sqsubseteq \rho(D, \bowtie n)\} \\ & \bigcup_{n \in N^{\mathfrak{f}\mathcal{K}}, \bowtie \in \{\leq, <\}} \{\rho(D, \bowtie n) \sqsubseteq \rho(C, \bowtie n)\} \end{aligned} \tag{3}$$

We extend κ to a terminology $\mathcal{T}$ point-wise. For all $\tau \in \mathcal{T}$

$$\kappa(\mathfrak{f}\mathcal{K}, \mathcal{T}) = \cup_{\tau \in \mathcal{T}} \kappa(\mathfrak{f}\mathcal{K}, \tau).$$

κ reduces a terminological axiom in $\mathcal{ALC}_{\mathcal{FL}}$ into a set of $\mathcal{ALCH}$ terminology axioms.

4.3 The Satisfiability Preserving Theorem

Now we can define the *reduction* of $\mathfrak{f}\mathcal{K}$ into an $\mathcal{ALCH}$ knowledge base, denoted $\mathcal{K}(\mathfrak{f}\mathcal{K})$,

$$\mathcal{K}(\mathfrak{f}\mathcal{K}) = \langle \mathcal{T}(N^{\mathfrak{f}\mathcal{K}}) \cup \kappa(\mathfrak{f}\mathcal{K}, \mathcal{T}), \theta(\mathcal{A}) \rangle.$$

The transformation can be done in polynomial time. The soundness and completeness of the algorithm is guaranteed by the following satisfiability preserving theorem.

Theorem 17. *Let $\mathfrak{f}\mathcal{K}$ be an $\mathcal{ALC}_{\mathcal{FL}}$ knowledge base. Then $\mathfrak{f}\mathcal{K}$ is satisfiable iff the $\mathcal{ALCH}$ knowledge base $\mathcal{K}(\mathfrak{f}\mathcal{K})$ is satisfiable.*

Proof. Let $\mathfrak{f}\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be an $\mathcal{ALC}_{\mathcal{FL}}$ knowledge base , $\mathcal{K}(\mathfrak{f}\mathcal{K}) = \langle \mathcal{T}', \mathcal{A}' \rangle$ be the transformed $\mathcal{ALCH}$ knowledge base, where $\mathcal{T}' = \mathcal{T}(N^{\mathfrak{f}\mathcal{K}}) \cup \kappa(\mathfrak{f}\mathcal{K}, \mathcal{T})$ and $\mathcal{A}' = \theta(\mathcal{A})$. We define that $\rhd \in \{\geq, >\}$ and $\lhd \in \{\leq, <\}$.

Our goal is to prove that there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathfrak{f}\mathcal{K}$ if and only if there exists an interpretation $\mathcal{I}'$ such that $\mathcal{I}' \models \mathcal{K}(\mathfrak{f}\mathcal{K})$, where $\mathcal{I}$ is a fuzzy interpretation and $\mathcal{I}'$ is an $\mathcal{ALCH}$ interpretation.

$\Rightarrow$.) Assume $\mathcal{I}$ is an interpretation such that $\mathcal{I} \models \mathfrak{f}\mathcal{K}$. So $\mathcal{I} \models \mathcal{A}$ and $\mathcal{I} \models \mathcal{T}$. We construct an $\mathcal{ALCH}$ interpretation $\mathcal{I}'$:

- $\Delta^{\mathcal{I}'} := \Delta^{\mathcal{I}}$,
- $a^{\mathcal{I}'} := a^{\mathcal{I}}$ for all individual a,
- $A^{\mathcal{I}'}_{\bowtie\sigma c} := \{d \in \Delta^{\mathcal{I}'} \mid A^{\mathcal{I}}(d) \bowtie \sigma c\}$, for all concept name $A_{\bowtie\sigma c}$,
- $R^{\mathcal{I}'}_{\bowtie\sigma c} := \{(d, d') \in \Delta^{\mathcal{I}'} \times \Delta^{\mathcal{I}'} \mid R^{\mathcal{I}}(d, d') \bowtie \sigma c\}$, for all role name $R_{\bowtie\sigma c}$.

In order to show $\mathcal{I}' \models \mathcal{K}(\mathfrak{f}\mathcal{K})$, we have to show that $\mathcal{I}' \models \theta(\mathcal{A})$ and $\mathcal{I}' \models \mathcal{T}(N^{\mathfrak{f}\mathcal{K}}) \cup \kappa(\mathfrak{f}\mathcal{K}, \mathcal{T})$. Then it is sufficient to prove that:

1. for each $\alpha \bowtie \sigma c \in \mathcal{A}$, $\mathcal{I}' \models \theta(\alpha \bowtie \sigma c)$, and
2. $\mathcal{I}' \models \mathcal{T}(N^{\mathfrak{f}\mathcal{K}})$ and for each $C \sqsubseteq D \in \mathcal{T}$, $\mathcal{I}' \models \kappa(\mathfrak{f}\mathcal{K}, C \sqsubseteq D)$.

First, we need the following Lemma.

Lemma 18. *Let C be a concept term in $\mathcal{ALC}_{\mathcal{FL}}$. $C \neq \top$ and $C \neq \bot$. It follows that $(\rho(C, \bowtie \sigma c))^{\mathcal{I}'} = \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \bowtie \sigma c\}$.*

Proof. We use proof by induction.
Basic step:
Let R be a role name. Then
$(\rho(R, \bowtie \sigma c))^{\mathcal{I}'} = R^{\mathcal{I}'}_{\bowtie \sigma c} = \{(d, d') \in \Delta^{\mathcal{I}'} \times \Delta^{\mathcal{I}'} \mid R^{\mathcal{I}}(d, d') \bowtie \sigma c\}$.
Let A be a concept name. Then
$(\rho(A, \bowtie \sigma c))^{\mathcal{I}'} = A^{\mathcal{I}'}_{\bowtie \sigma c} = \{d \in \Delta^{\mathcal{I}'} \mid A^{\mathcal{I}}(d) \bowtie \sigma c\}$.
Inductive step:
Let C, D be concept terms. Assume
$(\rho(C, \bowtie \sigma c))^{\mathcal{I}'} = \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \bowtie \sigma c\}$ and
$(\rho(D, \bowtie \sigma c))^{\mathcal{I}'} = \{d \in \Delta^{\mathcal{I}'} \mid D^{\mathcal{I}}(d) \bowtie \sigma c\}$.
we prove inductively on the structures of concept terms.

Case $\neg C$.

$$\begin{aligned}(\rho(\neg C, \bowtie \sigma c))^{\mathcal{I}'} &= (\rho(C, \neg \bowtie \sigma \bar{c}))^{\mathcal{I}'} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \neg \bowtie \sigma \bar{c}\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid (\neg C)^{\mathcal{I}}(d) \bowtie \sigma c\}.\end{aligned}$$

Case δC.

$$\begin{aligned}(\rho(\delta C, \bowtie \sigma c))^{\mathcal{I}'} &= (\rho(C, \bowtie \sigma \delta c))^{\mathcal{I}'} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \bowtie \sigma \delta c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid (\delta C)^{\mathcal{I}}(d) \bowtie \sigma c\}.\end{aligned}$$

Case $C \sqcap D$.

$$\begin{aligned}(\rho(C \sqcap D, \rhd \sigma c))^{\mathcal{I}'} &= (\rho(C, \rhd \sigma c) \sqcap \rho(D, \rhd \sigma c))^{\mathcal{I}'} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \rhd \sigma c\} \cap \{d \in \Delta^{\mathcal{I}'} \mid D^{\mathcal{I}}(d) \rhd \sigma c\}. \\ &= \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \rhd \sigma c \wedge D^{\mathcal{I}}(d) \rhd \sigma c\}. \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \min(C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)) \rhd \sigma c\}. \\ &= \{d \in \Delta^{\mathcal{I}'} \mid (C \sqcap D)^{\mathcal{I}}(d) \rhd \sigma c\}.\end{aligned}$$

$$\begin{aligned}(\rho(C \sqcap D, \lhd \sigma c))^{\mathcal{I}'} &= (\rho(C, \lhd \sigma c) \sqcup \rho(D, \lhd \sigma c))^{\mathcal{I}'} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \lhd \sigma c\} \cup \{d \in \Delta^{\mathcal{I}'} \mid D^{\mathcal{I}}(d) \lhd \sigma c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \lhd \sigma c \vee D^{\mathcal{I}}(d) \lhd \sigma c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \min(C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)) \lhd \sigma c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid (C \sqcap D)^{\mathcal{I}}(d) \lhd \sigma c\}.\end{aligned}$$

Case $C \sqcup D$.

$$\begin{aligned}(\rho(C \sqcup D, \rhd \sigma c))^{\mathcal{I}'} &= (\rho(C, \rhd \sigma c) \sqcup \rho(D, \rhd \sigma c))^{\mathcal{I}'} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \rhd \sigma c\} \cup \{d \in \Delta^{\mathcal{I}'} \mid D^{\mathcal{I}}(d) \rhd \sigma c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \rhd \sigma c \vee D^{\mathcal{I}}(d) \rhd \sigma c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \max(C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)) \rhd \sigma c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid (C \sqcup D)^{\mathcal{I}}(d) \rhd \sigma c\}.\end{aligned}$$

$$\begin{aligned}(\rho(C \sqcup D, \lhd \sigma c))^{\mathcal{I}'} &= (\rho(C, \lhd \sigma c) \sqcap \rho(D, \lhd \sigma c))^{\mathcal{I}'} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \lhd \sigma c\ \} \cap \{d \in \Delta^{\mathcal{I}'} \mid D^{\mathcal{I}}(d) \lhd \sigma c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid C^{\mathcal{I}}(d) \lhd \sigma c \wedge D^{\mathcal{I}}(d) \lhd \sigma c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \max(C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)) \lhd \sigma c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid (C \sqcup D)^{\mathcal{I}}(d) \lhd \sigma c\}.\end{aligned}$$

Case $\forall R.C$**.**

$$\begin{aligned}(\rho(\forall R.C, \rhd \sigma c))^{\mathcal{I}'} &= (\forall \rho(R, + \rhd \sigma \bar{c}).\rho(C, \rhd \sigma c))^{\mathcal{I}'} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \forall d' \in \Delta^{\mathcal{I}'}.(d, d') \notin R^{\mathcal{I}'}_{+\rhd \sigma \bar{c}} \vee C^{\mathcal{I}}(d') \rhd \sigma c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \forall d' \in \Delta^{\mathcal{I}'}.(d, d') \in R^{\mathcal{I}'}_{\neg\rhd \sigma \bar{c}} \vee C^{\mathcal{I}}(d') \rhd \sigma c\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \textstyle\bigwedge_{d' \in \Delta^{\mathcal{I}'}} (R^{\mathcal{I}}(d, d') \neg \rhd \sigma \bar{c} \vee C^{\mathcal{I}}(d') \rhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \textstyle\bigwedge_{d' \in \Delta^{\mathcal{I}'}} (-R^{\mathcal{I}}(d, d') \rhd \sigma c \vee C^{\mathcal{I}}(d') \rhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \textstyle\bigwedge_{d' \in \Delta^{\mathcal{I}'}} (\max(-R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')) \rhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \inf_{d' \in \Delta^{\mathcal{I}'}} (\max(-R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\} \rhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid (\forall R.C)^{\mathcal{I}}(d) \rhd \sigma c)\}.\end{aligned}$$

$$\begin{aligned}(\rho(\forall R.C, \lhd \sigma c))^{\mathcal{I}'} &= (\exists \rho(R, \neg \lhd \sigma \bar{c}).\rho(C, \lhd \sigma c))^{\mathcal{I}'} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \exists d' \in \Delta^{\mathcal{I}'}.(d, d') \in R^{\mathcal{I}'}_{\neg\lhd \sigma \bar{c}} \wedge C^{\mathcal{I}}(d') \lhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \textstyle\bigvee_{d' \in \Delta^{\mathcal{I}'}} (R^{\mathcal{I}}(d, d') \neg \lhd \sigma \bar{c} \wedge C^{\mathcal{I}}(d') \lhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \textstyle\bigvee_{d' \in \Delta^{\mathcal{I}'}} (-R^{\mathcal{I}}(d, d') \lhd \sigma c \wedge C^{\mathcal{I}}(d') \lhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \textstyle\bigvee_{d' \in \Delta^{\mathcal{I}'}} (\max(-R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')) \lhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \inf_{d' \in \Delta^{\mathcal{I}'}} (\max(-R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\} \lhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid (\forall R.C)^{\mathcal{I}}(d) \lhd \sigma c)\}.\end{aligned}$$

Case $\exists R.C$**.**

$$\begin{aligned}(\rho(\exists R.C, \rhd \sigma c))^{\mathcal{I}'} &= (\exists \rho(R, \rhd \sigma c).\rho(C, \rhd \sigma c))^{\mathcal{I}'} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \exists d' \in \Delta^{\mathcal{I}'}.(d, d') \in R^{\mathcal{I}'}_{\rhd \sigma c} \wedge C^{\mathcal{I}}(d') \rhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \textstyle\bigvee_{d' \in \Delta^{\mathcal{I}'}} (R^{\mathcal{I}'}(d, d') \rhd \sigma c \wedge C^{\mathcal{I}}(d') \rhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \textstyle\bigvee_{d' \in \Delta^{\mathcal{I}'}} (\min(R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')) \rhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \sup_{d' \in \Delta^{\mathcal{I}'}} \{\min(R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\} \rhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid (\exists R.C)^{\mathcal{I}}(d) \rhd \sigma c)\}.\end{aligned}$$

$$\begin{aligned}(\rho(\exists R.C, \lhd \sigma c))^{\mathcal{I}'} &= (\forall \rho(R, - \lhd \sigma c).\rho(C, \lhd \sigma c))^{\mathcal{I}'} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \forall d' \in \Delta^{\mathcal{I}'}.(d, d') \notin R^{\mathcal{I}'}_{-\lhd \sigma c} \vee C^{\mathcal{I}}(d') \lhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \forall d' \in \Delta^{\mathcal{I}'}.(d, d') \in R^{\mathcal{I}'}_{\lhd \sigma c} \vee C^{\mathcal{I}}(d') \lhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \forall d' \in \Delta^{\mathcal{I}'}.(R^{\mathcal{I}}(d, d') \lhd \sigma c \vee C^{\mathcal{I}}(d') \lhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \textstyle\bigwedge_{d' \in \Delta^{\mathcal{I}'}} (\min(R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')) \lhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid \sup_{d' \in \Delta^{\mathcal{I}'}} (\min(R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\} \lhd \sigma c)\} \\ &= \{d \in \Delta^{\mathcal{I}'} \mid (\exists R.C)^{\mathcal{I}}(d) \lhd \sigma c)\}.\end{aligned}$$

In the following, we use $C_{\bowtie \sigma c}$ to represent $\rho(C, \bowtie \sigma c)$.

(1) Now we prove that $\mathcal{I}' \models \theta(\mathcal{A})$. Let $\alpha \bowtie \sigma c \in \mathcal{A}$. Then $\mathcal{I} \models \alpha \bowtie \sigma c$ because $\mathcal{I} \models \mathcal{A}$.

If σ is a role assertion of the form $(a,b):R$, then

$$\begin{aligned}\mathcal{I} \models (a,b):R \bowtie \sigma c &\Rightarrow R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \bowtie \sigma c \\ &\Rightarrow (a^{\mathcal{I}'}, b^{\mathcal{I}'}) \in R^{\mathcal{I}'}_{\bowtie\sigma c} \\ &\Rightarrow \mathcal{I}' \models (a,b):R_{\bowtie\sigma c}.\end{aligned}$$

For concept assertions, we inductively prove on the structure of concept term:

Case $\top$. For all interpretation $\mathcal{I}$ and for all $d \in \Delta^{\mathcal{I}}$, $\top^{\mathcal{I}}(d) = \sup(True)$, so $a:\top \geq \sigma c, a:\top > \sigma c$ if $\sigma c < \sup(True)$ and $a:\top \leq \sup(True)$ are valid, $a:\top$ is valid too. While $a:\top > \sup(True), a:\top \leq \sigma c$ if $\sigma c < \sup(True)$ and $a:\top < \sigma c$ are unsatisfiable, $a:\bot$ is unsatisfiable as well.

Case $\bot$. For all interpretation $\mathcal{I}$ and for all $d \in \Delta^{\mathcal{I}}$, $\bot^{\mathcal{I}}(d) = \inf(False)$, so $a:\bot \geq \inf(False), a:\bot < \sigma c$ if $\sigma c > \inf(False)$ and $a:\bot \leq \sigma c$ are valid, so is $a:\top$. While $a:\bot < \inf(False), a:\bot \geq \sigma c$ if $\sigma c > \inf(False)$ and $a:\bot > \sigma c$ are unsatisfiable. $a:\bot$ is also unsatisfiable.

Case concept name A. $\mathcal{I} \models a:A \bowtie \sigma c \Rightarrow A^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \sigma c \Rightarrow a^{\mathcal{I}'} \in A^{\mathcal{I}'}_{\bowtie\sigma c} \Rightarrow \mathcal{I}' \models a:A_{\bowtie\sigma c}$.

Case concept negation $\neg C$.

$$\begin{aligned}\mathcal{I} \models a:\neg C \bowtie \sigma c &\Rightarrow (\neg C)^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \sigma c \\ &\Rightarrow -C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \sigma c \\ &\Rightarrow C^{\mathcal{I}}(a^{\mathcal{I}}) \neg\bowtie \sigma\bar{c} \\ &\Rightarrow a^{\mathcal{I}} \in C^{\mathcal{I}'}_{\neg\bowtie\sigma\bar{c}} \\ &\Rightarrow a^{\mathcal{I}'} \in C^{\mathcal{I}'}_{\neg\bowtie\sigma\bar{c}} \\ &\Rightarrow \mathcal{I}' \models a:C_{\neg\bowtie\sigma\bar{c}}.\end{aligned}$$

Case modifier concept δC.

$$\begin{aligned}\mathcal{I} \models a:\delta C \bowtie \sigma c &\Rightarrow \mathcal{I} \models a:C \bowtie \sigma\delta c \\ &\Rightarrow C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \sigma\delta c \\ &\Rightarrow a^{\mathcal{I}} \in C^{\mathcal{I}'}_{\bowtie\sigma\delta c} \\ &\Rightarrow a^{\mathcal{I}'} \in C^{\mathcal{I}'}_{\bowtie\sigma\delta c} \\ &\Rightarrow \mathcal{I}' \models a:C_{\bowtie\sigma\delta c}.\end{aligned}$$

Case concept conjunction $C \sqcap D$.

$$\begin{aligned}\mathcal{I} \models a:C \sqcap D \rhd \sigma c &\Rightarrow (C \sqcap D)^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \sigma c \\ &\Rightarrow \min(C^{\mathcal{I}}(a^{\mathcal{I}}), D^{\mathcal{I}}(a^{\mathcal{I}})) \rhd \sigma c \\ &\Rightarrow (C^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \sigma c) \wedge (D^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \sigma c) \\ &\Rightarrow a^{\mathcal{I}} \in C^{\mathcal{I}'}_{\rhd\sigma c} \wedge a^{\mathcal{I}} \in D^{\mathcal{I}'}_{\rhd\sigma c} \\ &\Rightarrow a^{\mathcal{I}'} \in C^{\mathcal{I}'}_{\rhd\sigma c} \wedge a^{\mathcal{I}'} \in D^{\mathcal{I}'}_{\rhd\sigma c} \\ &\Rightarrow a^{\mathcal{I}'} \in C^{\mathcal{I}'}_{\rhd\sigma c} \cap D^{\mathcal{I}'}_{\rhd\sigma c} \\ &\Rightarrow \mathcal{I}' \models a:C_{\rhd\sigma c} \sqcap D_{\rhd\sigma c}.\end{aligned}$$

$$\begin{aligned}\mathcal{I} \models a : C \sqcap D \lhd \sigma c &\Rightarrow (C \sqcap D)^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \sigma c\\ &\Rightarrow \min(C^{\mathcal{I}}(a^{\mathcal{I}}), D^{\mathcal{I}}(a^{\mathcal{I}})) \lhd \sigma c\\ &\Rightarrow (C^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \sigma c) \vee (D^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \sigma c)\\ &\Rightarrow a^{\mathcal{I}} \in C^{\mathcal{I}'}_{\lhd \sigma c} \vee a^{\mathcal{I}} \in D^{\mathcal{I}'}_{\lhd \sigma c}\\ &\Rightarrow a^{\mathcal{I}'} \in C^{\mathcal{I}'}_{\lhd \sigma c} \vee a^{\mathcal{I}'} \in D^{\mathcal{I}'}_{\lhd \sigma c}\\ &\Rightarrow a^{\mathcal{I}'} \in C^{\mathcal{I}'}_{\lhd \sigma c} \cup D^{\mathcal{I}'}_{\lhd \sigma c}\\ &\Rightarrow \mathcal{I}' \models a : C_{\lhd \sigma c} \sqcup D_{\lhd \sigma c}.\end{aligned}$$

Case concept disjunction $C \sqcup D$.

$$\begin{aligned}\mathcal{I} \models a : C \sqcup D \rhd \sigma c &\Rightarrow (C \sqcup D)^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \sigma c\\ &\Rightarrow \max(C^{\mathcal{I}}(a^{\mathcal{I}}), D^{\mathcal{I}}(a^{\mathcal{I}})) \rhd \sigma c\\ &\Rightarrow (C^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \sigma c) \vee (D^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \sigma c)\\ &\Rightarrow a^{\mathcal{I}} \in C^{\mathcal{I}'}_{\rhd \sigma c} \vee a^{\mathcal{I}} \in D^{\mathcal{I}'}_{\rhd \sigma c}\\ &\Rightarrow a^{\mathcal{I}'} \in C^{\mathcal{I}'}_{\rhd \sigma c} \vee a^{\mathcal{I}'} \in D^{\mathcal{I}'}_{\rhd \sigma c}\\ &\Rightarrow a^{\mathcal{I}'} \in C^{\mathcal{I}'}_{\rhd \sigma c} \cup D^{\mathcal{I}'}_{\rhd \sigma c}\\ &\Rightarrow \mathcal{I}' \models a : C_{\rhd \sigma c} \sqcup D_{\rhd \sigma c}.\end{aligned}$$

$$\begin{aligned}\mathcal{I} \models a : C \sqcup D \lhd \sigma c &\Rightarrow (C \sqcup D)^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \sigma c\\ &\Rightarrow \max(C^{\mathcal{I}}(a^{\mathcal{I}}), D^{\mathcal{I}}(a^{\mathcal{I}})) \lhd \sigma c\\ &\Rightarrow (C^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \sigma c) \wedge (D^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \sigma c)\\ &\Rightarrow a^{\mathcal{I}} \in C^{\mathcal{I}'}_{\lhd \sigma c} \wedge a^{\mathcal{I}} \in D^{\mathcal{I}'}_{\lhd \sigma c}\\ &\Rightarrow a^{\mathcal{I}'} \in C^{\mathcal{I}'}_{\lhd \sigma c} \wedge a^{\mathcal{I}'} \in D^{\mathcal{I}'}_{\lhd \sigma c}\\ &\Rightarrow a^{\mathcal{I}'} \in C^{\mathcal{I}'}_{\lhd \sigma c} \cap D^{\mathcal{I}'}_{\lhd \sigma c}\\ &\Rightarrow \mathcal{I}' \models a : C_{\lhd \sigma c} \sqcap D_{\lhd \sigma c}.\end{aligned}$$

Case universal quantification $\forall R.C$.

$$\begin{aligned}&\mathcal{I} \models a : \forall R.C \rhd \sigma c\\ &\quad\Rightarrow (\forall R.C)^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \sigma c\\ &\quad\Rightarrow \inf_{d' \in \Delta^{\mathcal{I}}}\{\max(-R^{\mathcal{I}}(a^{\mathcal{I}}, d'), C^{\mathcal{I}}(d'))\} \rhd \sigma c\\ &\quad\Rightarrow \textstyle\bigwedge_{d' \in \Delta^{\mathcal{I}}}(\max(-R^{\mathcal{I}}(a^{\mathcal{I}}, d'), C^{\mathcal{I}}(d')) \rhd \sigma c)\\ &\quad\Rightarrow \textstyle\bigwedge_{d' \in \Delta^{\mathcal{I}}}((-R^{\mathcal{I}}(a^{\mathcal{I}}, d') \rhd \sigma c) \vee (C^{\mathcal{I}}(d') \rhd \sigma c))\\ &\quad\Rightarrow \textstyle\bigwedge_{d' \in \Delta^{\mathcal{I}}}((R^{\mathcal{I}}(a^{\mathcal{I}}, d') \neg \rhd \sigma \bar{c}) \vee (C^{\mathcal{I}}(d') \rhd \sigma c))\\ &\quad\Rightarrow \forall d' \in \Delta^{\mathcal{I}}.(((a^{\mathcal{I}}, d') \in R^{\mathcal{I}'}_{\neg \rhd \sigma \bar{c}}) \vee (d' \in C^{\mathcal{I}'}_{\rhd \sigma c}))\\ &\quad\Rightarrow \forall d' \in \Delta^{\mathcal{I}}.(((a^{\mathcal{I}}, d') \notin R^{\mathcal{I}'}_{+\rhd \sigma \bar{c}}) \vee (d' \in C^{\mathcal{I}'}_{\rhd \sigma c}))\\ &\quad\Rightarrow a^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid \forall d' \in \Delta^{\mathcal{I}} : (d, d') \notin R^{\mathcal{I}'}_{+\rhd \sigma \bar{c}} \vee d' \in C^{\mathcal{I}'}_{\rhd \sigma c}\}\\ &\quad\Rightarrow a^{\mathcal{I}'} = \{d \in \Delta^{\mathcal{I}'} \mid \forall d' \in \Delta^{\mathcal{I}'} : (d, d') \notin R^{\mathcal{I}'}_{+\rhd \sigma \bar{c}} \vee d' \in C^{\mathcal{I}'}_{\rhd \sigma c}\}\\ &\quad\Rightarrow a^{\mathcal{I}'} \in (\forall R_{+\rhd \sigma \bar{c}}.C_{\rhd \sigma c})^{\mathcal{I}'}\\ &\quad\Rightarrow \mathcal{I}' \models a : \forall R_{+\rhd \sigma \bar{c}}.C_{\rhd \sigma c}.\end{aligned}$$

$$
\begin{aligned}
&\mathcal{I} \models a : \forall R.C \lhd \sigma c \\
&\quad \Rightarrow (\forall R.C)^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \sigma c \\
&\quad \Rightarrow \inf_{d' \in \Delta^{\mathcal{I}}}\{\max(-R^{\mathcal{I}}(a^{\mathcal{I}}, d'), C^{\mathcal{I}}(d'))\} \lhd \sigma c \\
&\quad \Rightarrow \bigvee_{d' \in \Delta^{\mathcal{I}}}(\max(-R^{\mathcal{I}}(a^{\mathcal{I}}, d'), C^{\mathcal{I}}(d')) \lhd \sigma c) \\
&\quad \Rightarrow \bigvee_{d' \in \Delta^{\mathcal{I}}}((-R^{\mathcal{I}}(a^{\mathcal{I}}, d') \lhd \sigma c) \wedge (C^{\mathcal{I}}(d') \lhd \sigma c)) \\
&\quad \Rightarrow \bigvee_{d' \in \Delta^{\mathcal{I}}}((R^{\mathcal{I}}(a^{\mathcal{I}}, d') \neg \lhd \sigma \bar{c}) \wedge (C^{\mathcal{I}}(d') \lhd \sigma c)) \\
&\quad \Rightarrow \exists d' \in \Delta^{\mathcal{I}}.(((a^{\mathcal{I}}, d') \in R^{\mathcal{I}'}_{\neg \lhd \sigma \bar{c}}) \wedge (d' \in C^{\mathcal{I}'}_{\lhd \sigma c})) \\
&\quad \Rightarrow a^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid \exists d' \in \Delta^{\mathcal{I}} : (d, d') \in R^{\mathcal{I}'}_{\neg \lhd \sigma \bar{c}} \wedge d' \in C^{\mathcal{I}'}_{\lhd \sigma c}\} \\
&\quad \Rightarrow a^{\mathcal{I}'} = \{d \in \Delta^{\mathcal{I}'} \mid \exists d' \in \Delta^{\mathcal{I}'} : (d, d') \in R^{\mathcal{I}'}_{\neg \lhd \sigma \bar{c}} \wedge d' \in C^{\mathcal{I}'}_{\lhd \sigma c}\} \\
&\quad \Rightarrow a^{\mathcal{I}'} \in (\exists R_{\neg \lhd \sigma \bar{c}}.C_{\lhd \sigma c})^{\mathcal{I}'} \\
&\quad \Rightarrow \mathcal{I}' \models a : \exists R_{\neg \lhd \sigma \bar{c}}.C_{\lhd \sigma c}.
\end{aligned}
$$

Case existential quantification $\exists R.C$.

$$
\begin{aligned}
&\mathcal{I} \models a : \exists R.C \rhd \sigma c \\
&\quad \Rightarrow (\exists R.C)^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \sigma c \\
&\quad \Rightarrow \sup_{d' \in \Delta^{\mathcal{I}}}\{\min(R^{\mathcal{I}}(a^{\mathcal{I}}, d'), C^{\mathcal{I}}(d'))\} \rhd \sigma c \\
&\quad \Rightarrow \bigvee_{d' \in \Delta^{\mathcal{I}}}(\min(R^{\mathcal{I}}(a^{\mathcal{I}}, d'), C^{\mathcal{I}}(d')) \rhd \sigma c) \\
&\quad \Rightarrow \bigvee_{d' \in \Delta^{\mathcal{I}}}((R^{\mathcal{I}}(a^{\mathcal{I}}, d') \rhd \sigma c) \wedge (C^{\mathcal{I}}(d') \rhd \sigma c)) \\
&\quad \Rightarrow \exists d' \in \Delta^{\mathcal{I}}.(((a^{\mathcal{I}}, d') \in R^{\mathcal{I}'}_{\rhd \sigma c}) \wedge (d' \in C^{\mathcal{I}'}_{\rhd \sigma c})) \\
&\quad \Rightarrow a^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid \exists d' \in \Delta^{\mathcal{I}} : (d, d') \in R^{\mathcal{I}'}_{\rhd \sigma c} \wedge d' \in C^{\mathcal{I}'}_{\rhd \sigma c}\} \\
&\quad \Rightarrow a^{\mathcal{I}'} = \{d \in \Delta^{\mathcal{I}'} \mid \exists d' \in \Delta^{\mathcal{I}'} : (d, d') \in R^{\mathcal{I}'}_{\rhd \sigma c} \wedge d' \in C^{\mathcal{I}'}_{\rhd \sigma c}\} \\
&\quad \Rightarrow a^{\mathcal{I}'} \in (\exists R_{\rhd \sigma c}.C_{\rhd \sigma c})^{\mathcal{I}'} \\
&\quad \Rightarrow \mathcal{I}' \models a : \exists R_{\rhd \sigma c}.C_{\rhd \sigma c}.
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{I} \models a : \exists R.C \lhd \sigma c \\
&\quad \Rightarrow (\exists R.C)^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \sigma c \\
&\quad \Rightarrow \sup_{d' \in \Delta^{\mathcal{I}}}\{\min(R^{\mathcal{I}}(a^{\mathcal{I}}, d'), C^{\mathcal{I}}(d'))\} \lhd \sigma c \\
&\quad \Rightarrow \bigwedge_{d' \in \Delta^{\mathcal{I}}}(\min(R^{\mathcal{I}}(a^{\mathcal{I}}, d'), C^{\mathcal{I}}(d')) \lhd \sigma c) \\
&\quad \Rightarrow \bigwedge_{d' \in \Delta^{\mathcal{I}}}((R^{\mathcal{I}}(a^{\mathcal{I}}, d') \lhd \sigma c) \vee (C^{\mathcal{I}}(d') \lhd \sigma c)) \\
&\quad \Rightarrow \forall d' \in \Delta^{\mathcal{I}}.(((a^{\mathcal{I}}, d') \in R^{\mathcal{I}'}_{\lhd \sigma c}) \vee (d' \in C^{\mathcal{I}'}_{\lhd \sigma c})) \\
&\quad \Rightarrow \forall d' \in \Delta^{\mathcal{I}}.(((a^{\mathcal{I}}, d') \notin R^{\mathcal{I}'}_{-\lhd \sigma c}) \vee (d' \in C^{\mathcal{I}'}_{\rhd \sigma c})) \\
&\quad \Rightarrow a^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid \forall d' \in \Delta^{\mathcal{I}} : (d, d') \notin R^{\mathcal{I}'}_{-\lhd \sigma c} \vee d' \in C^{\mathcal{I}'}_{\lhd \sigma c}\} \\
&\quad \Rightarrow a^{\mathcal{I}'} = \{d \in \Delta^{\mathcal{I}'} \mid \forall d' \in \Delta^{\mathcal{I}'} : (d, d') \notin R^{\mathcal{I}'}_{-\lhd \sigma c} \vee d' \in C^{\mathcal{I}'}_{\lhd \sigma c}\} \\
&\quad \Rightarrow a^{\mathcal{I}'} \in (\forall R_{-\lhd \sigma c}.C_{\lhd \sigma c})^{\mathcal{I}'} \\
&\quad \Rightarrow \mathcal{I}' \models a : \forall R_{-\lhd \sigma c}.C_{\lhd \sigma c}.
\end{aligned}
$$

The proof shows that for each $\alpha \bowtie \sigma c \in \mathcal{A}$ if $\mathcal{I} \models \alpha \bowtie \sigma c$ then $\mathcal{I}' \models \theta(\alpha \bowtie \sigma c)$ which implies that $\mathcal{I} \models \mathcal{A} \Rightarrow \mathcal{I}' \models \theta(\mathcal{A})$.

(2) Now we prove that $\mathcal{I}' \models \mathcal{T}(N^{\mathfrak{f}\mathcal{K}}) \cup \kappa(\mathfrak{f}\mathcal{K}, \mathcal{T})$.

It is trivial that $\mathcal{I}' \models \mathcal{T}(N^{\mathfrak{f}\mathcal{K}})$ according to our basic idea.

Let $C \sqsubseteq D \in \mathcal{T}$, then for all $\sigma c \in \mathcal{N}^{\mathfrak{f}\mathcal{K}}$, $C_{\rhd \sigma c} \sqsubseteq D_{\rhd \sigma c} \in \kappa(\mathfrak{f}\mathcal{K}, C \sqsubseteq D)$ and $D_{\lhd \sigma c} \sqsubseteq C_{\lhd \sigma c} \in \kappa(\mathfrak{f}\mathcal{K}, C \sqsubseteq D)$.

$$\begin{aligned}\mathcal{I} \models C \sqsubseteq D &\Rightarrow \forall d \in \Delta^{\mathcal{I}}.C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)\\ &\Rightarrow \text{ if } C^{\mathcal{I}}(d) \rhd \sigma c \text{ then } D^{\mathcal{I}}(d) \rhd \sigma c\\ &\Rightarrow \text{ if } d \in C^{\mathcal{I}'}_{\rhd \sigma c} \text{ then } d \in D^{\mathcal{I}'}_{\rhd \sigma c}\\ &\Rightarrow C^{\mathcal{I}'}_{\rhd \sigma c} \subseteq D^{\mathcal{I}'}_{\rhd \sigma c}\\ &\Rightarrow \mathcal{I}' \models C_{\rhd \sigma c} \sqsubseteq D_{\rhd \sigma c}.\end{aligned}$$

$$\begin{aligned}\mathcal{I} \models C \sqsubseteq D &\Rightarrow \forall d \in \Delta^{\mathcal{I}}.C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)\\ &\Rightarrow \text{ if } D^{\mathcal{I}}(d) \lhd \sigma c \text{ then } C^{\mathcal{I}}(d) \lhd \sigma c\\ &\Rightarrow \text{ if } d \in D^{\mathcal{I}'}_{\lhd \sigma c} \text{ then } d \in C^{\mathcal{I}'}_{\lhd \sigma c}\\ &\Rightarrow D^{\mathcal{I}'}_{\lhd \sigma c} \subseteq C^{\mathcal{I}'}_{\lhd \sigma c}\\ &\Rightarrow \mathcal{I}' \models D_{\lhd \sigma c} \sqsubseteq C_{\lhd \sigma c}.\end{aligned}$$

So for each $C \sqsubseteq D \in \mathcal{T}$, if $\mathcal{I} \models C \sqsubseteq D$ then $\mathcal{I}' \models \{C_{\rhd \sigma c} \sqsubseteq D_{\rhd \sigma c}, D_{\lhd \sigma c} \sqsubseteq C_{\lhd \sigma c}\}$. It follows that $\mathcal{I}' \models \kappa(\mathfrak{f}\mathcal{K}, C \sqsubseteq D)$. So $\mathcal{I}' \models \mathcal{T}(N^{\mathfrak{f}\mathcal{K}}) \cup \kappa(\mathfrak{f}\mathcal{K}, C \sqsubseteq D)$.

$\Leftarrow$.) Let $\mathcal{I}'$ be a finite model of $\mathcal{K}(\mathfrak{f}\mathcal{K})$ whose domain $\Delta^{\mathcal{I}'}$ is finite. We build an $\mathcal{ALC}_{\mathcal{FL}}$ interpretation $\mathcal{I}$ such that

- $\Delta^{\mathcal{I}} := \Delta^{\mathcal{I}'}$,
- $a^{\mathcal{I}} := a^{\mathcal{I}'}$ for all individual a,
- $\forall d \in \Delta^{\mathcal{I}}.A^{\mathcal{I}}(d) := \sigma' c'$ for all concept name A, where
  Let $\sigma_1 c_1 = \sup\{\sigma c \mid d \in A^{\mathcal{I}'}_{\rhd \sigma c}\}$, $\sigma_2 c_2 = \inf\{\sigma c \mid d \in A^{\mathcal{I}'}_{\lhd \sigma c}\}$ and $\delta \in H^*$ such that for all $\delta' \in H^*$ and $\delta' \neq \delta$, $\delta' \sigma c > \delta \sigma c > \sigma c$.
  1. Since $\mathcal{K}(\mathfrak{f}\mathcal{K})$ is satisfiable, if $\sigma_1 c_1 = \sigma_2 c_2$ then $\sigma' c' = \sigma_1 c_1 = \sigma_2 c_2$,
  2. otherwise if $\sigma_1 c_1 < \sigma_2 c_2$, $\sigma' c' = \delta \sigma_1 c_1$.

  If $\forall \sigma c.d \notin A^{\mathcal{I}'}_{\bowtie \sigma c}$, $\sigma' c' = \inf(False)$.
- $\forall d, d' \in \Delta^{\mathcal{I}}.R^{\mathcal{I}}(d, d') := \sigma' c'$ for all role name R, where
  Let $\sigma_1 c_1 = \sup\{\sigma c \mid (d, d') \in R^{\mathcal{I}'}_{\rhd \sigma c}\}$, $\sigma_2 c_2 = \inf\{\sigma c \mid (d, d') \in R^{\mathcal{I}'}_{\lhd \sigma c}\}$ and $\delta \in H^*$ such that for all $\delta' \in H^*$ and $\delta' \neq \delta$, $\delta' \sigma c > \delta \sigma c > \sigma c$.
  1. Since $\mathcal{K}(\mathfrak{f}\mathcal{K})$ is satisfiable, if $\sigma_1 c_1 = \sigma_2 c_2$ then $\sigma' c' = \sigma_1 c_1 = \sigma_2 c_2$,
  2. otherwise if $\sigma_1 c_1 < \sigma_2 c_2$, $\sigma' c' = \delta \sigma_1 c_1$.

  If $\forall \sigma c.(d, d') \notin R^{\mathcal{I}'}_{\bowtie \sigma c}$, $\sigma' c' = \inf(False)$.

We have the following Lemma from our basic idea and the definition of the interpretation $\mathcal{I}$.

Lemma 19. *For all σc and for all $d, d' \in \Delta^{\mathcal{I}'}$, $d \in C^{\mathcal{I}'}_{\bowtie \sigma c} \Rightarrow C^{\mathcal{I}}(d) \bowtie \sigma c$ and $(d, d') \in R^{\mathcal{I}'}_{\bowtie \sigma c} \Rightarrow R^{\mathcal{I}}(d, d') \bowtie \sigma c$.*

Proof. Please refer to [20].

(1) For ABox, the proof is exactly the reverse processes of that of the $\Rightarrow$.) from which we can prove that if $\mathcal{I}' \models \theta(\mathcal{A}')$ then $\mathcal{I} \models \mathcal{A}$.

(2) For all $\sigma c \in \mathcal{N}^{\mathfrak{f}\mathcal{K}}$, $C_{\rhd \sigma c} \sqsubseteq D_{\rhd \sigma c} \in \kappa(\mathfrak{f}\mathcal{K}, \mathcal{T})$, then for all $d \in C^{\mathcal{I}'}_{\rhd \sigma c}$, $d \in D^{\mathcal{I}'}_{\rhd \sigma c}$. Therefore, if $C^{\mathcal{I}}(d) \geq \sigma c$ then $D^{\mathcal{I}}(d) \geq \sigma c$.

Assume $\mathcal{I}' \models \mathcal{T}'$ and $\mathcal{I} \not\models C \sqsubseteq D$ where $C \sqsubseteq D \in \mathcal{T}$. So there exists a $d' \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(d') > D^{\mathcal{I}}(d')$. Consider $C^{\mathcal{I}}(d') = \sigma' c'$. Of course $C^{\mathcal{I}}(d') \geq \sigma' c'$. Therefore, $D^{\mathcal{I}}(d') \geq \sigma' c'$. From the hypothesis it follows that $\sigma' c' = C^{\mathcal{I}}(d') > D^{\mathcal{I}}(d') \geq \sigma' c'$, which contradicts the hypothesis. So $\mathcal{I} \models \mathcal{T}$.

5 Discussion

In this paper, we have presented a satisfiability preserving transformation of $\mathcal{ALC}_{\mathcal{FL}}$ into $\mathcal{ALCH}$ which is with general TBox and role hierarchy. Since all other reasoning tasks such as entailment problem and subsumption problem can be reduced to satisfiability problem, this result allows for algorithms and complexity results that were found for $\mathcal{ALCH}$ to be applied to $\mathcal{ALC}_{\mathcal{FL}}$.

As for the complexity of the transformation, we know that,

1. $|\theta(\mathcal{A})|$ is linearly bounded by $|\mathcal{A}|$;
2. $|\mathcal{T}(N^{\mathfrak{f}\mathcal{K}})| = 2|\mathcal{A}^{\mathfrak{f}\mathcal{K}}|(|\mathcal{N}^{\mathfrak{f}\mathcal{K}}| - 1) + 2|\mathcal{R}^{\mathfrak{f}\mathcal{K}}|(|\mathcal{N}^{\mathfrak{f}\mathcal{K}}| - 1)$;
3. $\kappa(\mathfrak{f}\mathcal{K}, \mathcal{T})$ contains at most $4|\mathcal{T}||N^{\mathfrak{f}\mathcal{K}}|$.

Therefore, the resulted classical knowledge base (at most polynomial size) can be constructed in polynomial time.

The work of Straccia [19] transforms fuzzy $\mathcal{ALCH}$ into classical $\mathcal{ALCH}$. The truth domains of fuzzy $\mathcal{ALCH}$ is different from that of $\mathcal{ALC}_{\mathcal{FL}}$. $\mathcal{ALC}_{\mathcal{FL}}$ uses hedges as the fuzzy extension and the truth domain of interpretations is represented by a hedge algebra. Moreover, the hedges occur not only in the fuzzy values but also in concept terms. Thus there is one more rule for dealing with modifier concept terms in our current work.

Many approaches to transformation various fuzzy DLs into classical DLs have been proposed. Boillo et al. [3] proposed a reasoning preserving reduction for the fuzzy DL $\mathcal{SROIQ}$ under Gödel semantics to the crisp case. In the reduction, concept and role modifiers are allowed. While the truth domains of fuzzy DL $\mathcal{SROIQ}$ is not represented by a hedge algebra either. Bobillo and Straccia [5] have proposed a general framework for fuzzy DLs with a finite chain of degrees of truth N which can be seen as a finite totally ordered set of linguistic terms or labels. They also provided a a reasoning preserving reduction to the crisp case. Bobillo and Straccia [6] have shown that a fuzzy extension of SROIQ is decidable over a finite set of truth values by presenting a reasoning preserving procedure to obtain a non-fuzzy representation for the logic. This fuzzy extension of the logic SROIQ is the logic behind the language OWL 2. This reduction makes it possible to reuse current representation languages as well as currently available reasoners for ontologies.

There exist some reasoners for fuzzy DLs, e.g. *FiRE* [17], *GURDL* [7], *DeLorean* [2], *GERDS* [8], *YADLR* [11] and *fuzzyDL* [4]. Among them, *fuzzyDL* allows modifiers defined in terms of linear hedges and triangular functions and *DeLorean* supports triangularly-modified concept. So the approaches to transformation variety of fuzzy DLs into classical DLs make it possible to use the already existing resources for classical DL systems to do the reasoning of fuzzy DLs without adapting fuzzy DLs to some other fuzzy language.

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