Limit Property of a Multi-Choice Value and the Fuzzy Value

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1 Introduction

In ordinary voting games, each player's vote is yes or no. Some researchers have worked on modifications in order to deal with more than two options, and these modifications are divided into two major categories. One is multi-alternative games, defined by Bolger (1986, 1993), which enables players to choose among more than two independent alternatives; e.g., voting in an election in which more than two candidates are running. The other is multi-choice games, defined by Hsiao and Raghavan (1993) and Hsiao (1995), which enable players to choose among more than two participation levels, e.g., voting a yes/no or casting a blank vote. The latter means that the voter is not sufficiently in favor to vote yes, but not sufficiently against to vote no. Both modifications can be discussed not only in the class of voting games but also in broader class of cooperative games. In what follows we will focus on the relationship of multi-choice games and fuzzy games.

In multi-choice games, three values have been proposed as a generalization of the well-known Shapley value (Shapley 1953): those of Hsiao and Raghavan (1993) and Hsiao (1995), of Derks and Peters (1993), and of van den Nouweland et al. (1995). Although the value of Hsiao and Raghavan is derived from a set of axioms, the weight of each participation level must be defined exogenously. To calculate the value of Derks and Peters, we do not need weights, but the value depends on the step number of participation levels, which should be an ordinal number. The value of van den Nouweland et al. is advantageous, because it does not depend on exogenous numbers, and more so, it is capable of an interpretation using permutation like the Shapley value.

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Before the concept of multiple-choice gained importance, Aubin (1981, 1993) defined fuzzy games, where each player chooses a participation level in the interval $[0,\ 1]$. This definition seems closely related to the idea of the multi-choice games. The fuzzy value of Aubin is known as a generalized Shapley value in the fuzzy games.

In this chapter, we discuss two values: the multi-choice value of van den Nouweland et al. and the fuzzy value of Aubin. Our purpose is to show that the multi-choice value is consistent with the fuzzy value when the number of participation levels is sufficiently large. Section 2 describes the basic notation of multi-choice games and fuzzy games, and introduces the multi-choice value of Nouweland et al., and the fuzzy value of Aubin. Section 3 presents a tool, called a piecewise multilinear function, to extend a multi-choice game to a fuzzy game in a natural way. Section 4 contains the main results of this study, which show that the multi-choice value converges to the fuzzy value as the number of participation levels increases. A numerical example illustrates the result. Section 5 concludes.

2 Preliminaries

2.1 Multi-Choice Games

Let us review the multi-choice game defined by van den Nouweland et al. (1995). Let $N = \{1, 2, \ldots, n\}$ be the set of players van den Nouweland (1993) and Each player $i \in N$ chooses a participation level $s_i \in M_i = \{0, 1, \ldots, m_i\}$, where the number of levels $m_i \geq 1$ is an integer. That is, player i has $m_i + 1$ alternatives from which to choose a particular level of participation intensity. The n-tuple

$$s = (s_1, \cdots, s_n) \in \prod\nolimits_{i \in N} M_i$$

is called multi-choice coalition. Function

$$v : \prod\nolimits_{i \in N} M_i \to R \mathrm{with} v(0, \ldots, 0) = 0$$

is called the characteristic function; $v(s)$ is the value that N can gain as a group when the coalition is s. The triple $(N, (M_i)_{i \in N}, v)$ is called a multi-choice game. The set of all multi-choice games with player set N is denoted by MC^N. Note that, if $M_i = \{0, 1\}$ for all $i \in N$, the games in $MC^N$are equivalent to the usual cooperative games.

If N and $(M_i)_{i \in N}$ are well defined, we simply call v a multi-choice game. We assume that every multi-choice game v is monotonic nondecreasing with respect to s. In the following discussion, we also assume that each player has the same set of participation levels so that $M_i = M = \{0, 1, \ldots, m\}$ for all $i \in N$; and the set of all coalitions M^N.

The analogue of unanimity games for multi-choice games is minimal-effort games u_t defined by

$$u_t(s) = \begin{cases} 1, & \text{if } s_i \geq t_i \text{ for all } i \\ 0, & \text{otherwise} \end{cases}$$

where $s, t \in M^N$. We call t_i player i's required level of u_t. This means that the group gets 1 if every player $i \in N$ chooses a level higher than or equal to $t_i \in M$. Every multi-choice game v is described as a linear combination of u_t. This is an extended version of the well-known theorem that every vector is expressed as a linear combination of mutually orthogonal unit vectors.

For arbitrary v, dividend $\Delta_v(s)$ is given *recursively* by

$$\Delta_v(0, \ldots, 0) = 0, \text{ and}$$
$$\Delta_v(s) = v(s) - \sum_{r \leq s : r \neq s} \Delta_v(r).$$

This dividend corresponds to the coefficient of the linear combination, that is,

$$v(s) = \sum_{t \in M^N} \Delta_v(t) u_t(s).$$

Note that $\Delta_v(s) < +\infty$, because

$$\Delta_v(s) = v(s) + (n-1)v(s-I) - \sum_{i \in N} v(s - I + e^i),$$

where e^i is the unit n-vector whose ith component equals 1, and $I = (1, \ldots, 1)$. We also note that

$$\sum_{s \in M^N} \Delta_v(s) = v(m, \ldots, m).$$

2.2 Generalized Shapley Value

Van den Nouweland et al. (1995) proposed a generalized Shapley value. Let us define an order with m^n elements by bijection $\sigma : N \times (M - \{0\}) \to \{1, \ldots, m^n\}$. An order σ is said to be admissible if it satisfies $\sigma(i, j) < \sigma(i, j+1)$ for all [1] $i \in N$ and $j \in \{1, \ldots, m-1\}$; then there are

$$\frac{(mn)!}{(m!)^n}$$

admissible orders. The set of all admissible orders for a game v is denoted by $\Xi(v)$.

[1] For other solution concepts for multi-choice games such as a core, see van den Nouweland(1993), van den Nouweland et al.(1995), and Branzei et al.(2005).

Given an admissible order σ, let kth coalition be $s^{\sigma,k}$, where

$$s_i^{\sigma,k} = \max\{j \in M | \sigma(i,j) < k\} \cup \{0\}$$

for all $i \in N$, and the marginal contribution of i given j be

$$w_{i,j}^{\sigma} = v(s^{\sigma,\sigma(i,j)}) - v(s^{\sigma,\sigma(i,j)-1})$$

for all $i \in N$ and $j \in \{1,\ldots,m\}$.

Definition 1 (van den Nouweland et al. 1995) Let $v \in MC^N$. The multi-choice value $\varphi(v)$ is the expected marginal contribution of v over all admissible orders, i.e.,

$$\varphi_{i,j}(v) = \frac{(m!)^n}{(mn)!} \sum_{\sigma} w_{i,j}^{\sigma}$$

for all $i \in N$ and $j \in \{1,\cdots,m\}$.

This value is defined by an $m \times n$ matrix. In this chapter, adding a column of figures, let us define the total multi-choice value $\Phi(v) = (\Phi_1,\ldots,\Phi_n)$ as

$$\Phi_i(v) = \sum_{j=1}^{m} \varphi_{i,j}(v).$$

2.3 Fuzzy Games

Aubin (1981, 1993) discussed a fuzzy generalization of the ordinary cooperative games. Let $N = \{1,\ldots,n\}$ be the set of players. Each player $i \in N$ chooses among participation ($s_i = 1$), nonparticipation ($s_i = 0$), and fuzzy participation ($s_i \in (0,1)$). The n-tuple $s = (s_1,\ldots,s_n) \in [0,1]^N$ is called the fuzzy coalition. The fuzzy game is the pair (N, v^F), where $v^F : [0,1]^N \to R$ is continuously differentiable and satisfies $v^F(0,\ldots,0) = 0$. The set of all fuzzy games with player set N is denoted by FG^N.

Aubin (1981) developed the concept of generalized fuzzy games from an axiomatic approach, similar to Shapley (1953).

Definition 2 (Aubin 1981) Let $v^F \in FG^N$. The fuzzy value is defined by

$$\Theta(v^F) = \int_0^1 \nabla v^F(t,\ldots,t)dt,$$

where $\nabla v^F(\cdot)$ is the gradient of v^F.

In other words, the fuzzy value evaluates the gradient of v^F only on the main diagonal of n-cube $[0, 1]^N$.

3 Fuzzy Games with Piecewise Multilinear Functions

In this section, we present a way to derive a fuzzy game $v^F \in FG^N$ from a multi-choice game $v \in MC^N$. Without loss of generality, renumber the participation levels of a multi-choice game so that the highest level m equals $\frac{m}{m} = 1$, i.e., $M = \{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\}$. Then, the multi-choice game v is defined only on the grid points in n-cube $[0, 1]^N$. We fill the in-between space of this cube using multi linear functions.

Define $v^F(s) = v(s)$ if s is on a grid point; otherwise, define $v^F(s)$ using a piecewise multilinear function $z : M^N \to [0, 1]^N$. If we divide every edge of a unit n-cube into m equal parts, there would exist a unique small n-cube for which the length of each side equals $\frac{1}{m}$, which includes s. Since we have already defined $v^F(s)$ for each point of this small cube, define $v^F = zv$ by

$$zv(s) = v(x) + m^n \sum_{T \subseteq N} \prod_{j \in T} (s_j - x_j) \prod_{j \notin T} (x_j - s_j + \frac{1}{m}) \left[v\left(x + \frac{e^T}{m} \right) - v(x) \right],$$

where $x_i \in \{0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}\}$ is such that $x_i < s_j < x_{i+1/m}$ for all $i \in N$, and e^T is an n-tuple such that

$$e_i^T = \begin{cases} 1, & i \in T \\ 0. & i \notin T \end{cases}$$

We will call $v^F = zv$ a fuzzy game with the piecewise multilinear function extended from a multi-choice game$v$. Figure 1 illustrates the graphical image of the piecewise multilinear function derived from the two-person minimal effort game u_t, if their required level is $t = (\frac{2}{4}, \frac{3}{4})$ as an example.

To clarify the idea of piecewise multilinear functions, see the following three examples.

Example 1 If the number of participation levels is only 1, the multi-choice game v coincides with an ordinary cooperative game. The piecewise multilinear function extended from this game is shown as

$$zv(s) = \sum_{T \subseteq N} v(e^T) \prod_{j \in T} s_j \prod_{j \notin T} (1 - s_j),$$

which is equal to a multilinear extension of an ordinary cooperative game, defined by Owen (1972). Hence, the fuzzy value, which evaluates the gradient of v on the main diagonal of $[0, 1]^N$, coincides with the Shapley value of the cooperative game.

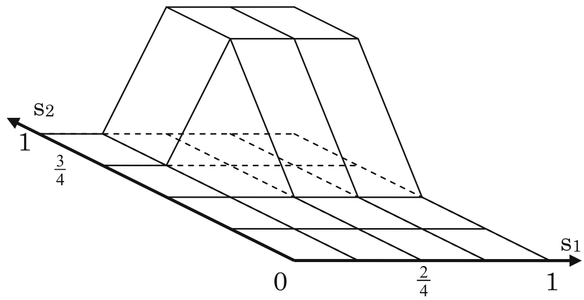

Fig. 1 Piecewise multilinear function derived from u_t, where $t = (\frac{2}{4}, \frac{3}{4})$

Example 2 Let us compute the fuzzy value of the fuzzy game extended from two-person $(m + 1)$-choice minimal-effort game u_t, where $t = (t_1, t_2)$. Let us first assume $t_1 > t_2$. There exists $(x_j, x_k) \in M^2$ such that $x_j < t_1 \leq x_{j+1/m}$ and $x_k < t_2 \leq x_{k+1/m}$ When we evaluate $\nabla zu_t = (\nabla_1 zu_t, \nabla_2 zu_t)$ on the main diagonal of $[0, 1]^2$, gradient $\nabla_1 zu_t$ equals m on $(x_j, x_{j+1/m}] \times (x_j, x_{j+1/m}]$; otherwise, it equals 0. On the other hand, $\nabla_2 zu_t$ equals 0 anywhere on the main diagonal of $[0, 1]^2$. Let us next assume $t_1 = t_2$. Then the elements of gradient $\nabla_1 zu_t$ and $\nabla_2 zu_t$ on $(x_j, x_{j+1/m}] \times (x_j, x_{j+1/m}]$ equal ms_1 and ms_2, respectively. Thus,

$$\Theta(zu_t) = \begin{cases} (1, 0) & t_1 > t_2, \\ (\frac{1}{2}, \frac{1}{2}) & t_1 = t_2, \\ (0, 1) & t_1 < t_2. \end{cases}$$

Example 3 The fuzzy value for n-person multi-choice minimal-effort game u_t is obtained analogously. Let $H(u_t)$ be the set of players who are required for the highest participation level in u_t: i.e.,

$$H(u_t) = \{i \in N | \forall k \in N, t_i \geq t_k\}.$$

Then the fuzzy value for player i is

$$\Theta_i(zu_t) = \begin{cases} \frac{1}{|H|}, & i \in H(u_t) \\ 0. & i \notin H(u_t) \end{cases}$$

Since Θ is a linear operator, the fuzzy value for games zv, extended from general multi-choice games v, is written as a linear combination of the values given in Example 3.

4 Main Theorem

In this section, we discuss the limit property of the multi-choice value. Denote the multi-choice game with $m + 1$ participation levels by v^m, the multi-choice value of the game by $\varphi^m(v^m)$, and the total multi-choice value of the game by $\Phi^m(v^m)$, to clarify the number of participation levels.

Theorem 1 *Let v^m be an $(m + 1)$-choice game and zv^m be the related fuzzy game with the piecewise multilinear function. Then, for every $\varepsilon > 0$ there exists m_ε such that, for all $i \in N$,*

$$\sup_v |\Phi_i^m(v^m) - \Theta_i(zv^m)| < \varepsilon \text{ for all } m > m_\varepsilon.$$

Before we prove this theorem, let us consider two-person $(m + 1)$-choice minimal-effort game u_t, defined in Sect. 2.1, and the related fuzzy game zu_t. For $r, s \in M = \{0, 1/m, \ldots, 1\}$, define $f(r, s)$ as

$$f(r, s) = \frac{(m!)^2}{(2m)!} \cdot \frac{(mr + ms - 1)!}{(mr - 1)!(ms - 1)!} \cdot \frac{(2m - mr - ms)!}{(m - mr)!(m - ms)!}.$$

Note that both mr and ms are integers. Since the total multiple-choice value for the minimal-effort game u_t is written as

$$\Phi_i^m(u_t) = \frac{(m!)^n}{(nm)!} \sum_{s \in L(u_t)} \frac{(\sum_{k \in N} s_k - 1)!}{(\prod_{\substack{k \in N \\ k \neq i}} s_k!) \cdot (s_i - 1)!} \cdot \frac{(mn - \sum_{k \in N} s_k)!}{\prod_{k \in N} (m - s_k)!},$$

where $L(u_t) = \{s \in M^N : s_i = t_i, \text{and } s_j \geq t_j \quad \text{for all } j \neq i\}$, the total multi-choice value for two-person $(m + 1)$-choice minimal-effort game u_t is

$$\Phi_1^m(u_t) = \sum_{s_2 \geq t_2} f(t_1, s_2) \text{ and } \Phi_2^m(u_t) = \sum_{s_1 \geq t_1} f(s_1, t_2).$$

Lemma 2

$$\sum_{s=0}^{m} f(r, s) = 1$$

Proof Use the identity

$$(1 - x)^{-m-1} = (1 - x)^{-r}(1 - x)^{-m+r-1},$$

and compare the coefficients of x^m of both sides. □

We interpret $f(r,\cdot)$ as a probability density function. Denote a random variable according to this distribution by X. Note that X is a number in $[0, 1]$. We define $F(t) = \Pr(X \geq t)$. Then, $\Phi_1^m(u_t) = F(t_2)$ and $\Phi_2^m(u_t) = F(t_1)$ hold.

The following lemma is the two-person version of Theorem 1.

Lemma 3 *Consider a two-person* $(m+1)$*-choice game* v^m *and the related fuzzy game* zv^m*. For every* $\varepsilon > 0$ *there exists* m_ε *such that, for all* $i \in N$,

$$\sup_v |\Phi_i^m(v^m) - \Theta_i(zv^m)| < \varepsilon \text{ for all } m > m_\varepsilon.$$

Proof We will first prove this assertion for the class of minimal-effort games u_t defined in Sect. 2.1. Since we already know that the fuzzy value of a two-person minimal-effort game is given as in Example 2, we calculate the total multi-choice value for comparison. Let us calculate the total multi-choice value for player 1. For all $\varepsilon > 0$,

$$\Pr(|X - t_2| > \varepsilon) \leq \frac{E(|X - t_2|^2)}{\varepsilon^2}$$

holds from Chebyshev's inequality. The denominator of the right-hand value

$$E(|X - t_2|^2) = \frac{[2(1 - t_2)m + 2t_2 + 1]t}{(m + 1)(m + 2)} \to 0$$

as $m \to 0$, which implies $\Pr(|X - t| < \varepsilon) \to 1$ as $m \to 0$.

Thus we conclude that, for any $\varepsilon > 0$, there exists m_ε such that

$$m > m_\varepsilon \Rightarrow \Pr(X < t_2) = 1 - F(t_2) < \varepsilon,$$

which means $t_1 < t_2$ ($t_1 > t_2$, resp.) implies that $F(t_2)$ converges to 1 (0, respectively), i.e., the total multi-choice value Φ_1^m converges to 0 (1, respectively), as $m \to 0$.

When $t_1 = t_2$, the efficiency and symmetry property of the total multi-choice value imply

$$\Phi_1^m(u_t) = \Phi_2^m(u_t) = \frac{1}{2},$$

for all m. Comparing with the fuzzy value in Example 2, we obtain $\Phi^m(u_t)$ converges to $\Theta(zu_t)$ as $m \to 0$.

Since z, Φ^m, and Θ are linear operators, and since the dividend $\Delta_v(s) < \infty$, the statement holds for any multi-choice game v^m. □

As we mention in the previous section, a multi-choice game with the set of choices $M = \{0, \frac{1}{m}, \ldots, 1\}$ is defined only on the grid points in n-cube $[0, 1]^N$. The total multi-choice value evaluates each player's contribution on all the paths from the origin to $(1, \ldots, 1)$ to consider his/her expected contribution, while the fuzzy value evaluates the contribution only on the main diagonal of n-cube $[0, 1]^N$. The essential part of the proof is that most of the paths from the origin to $(1, \ldots, 1)$

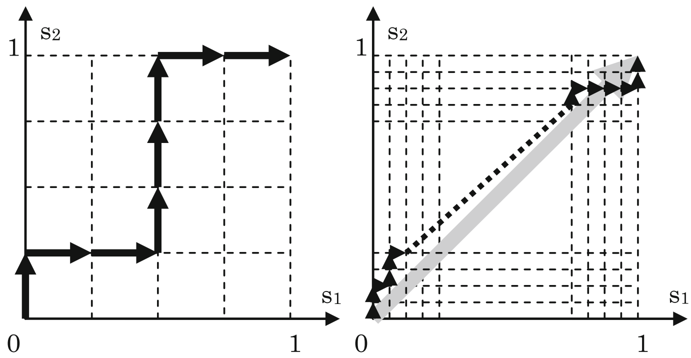

Fig. 2 Evaluation of the total multi-choice value

are close to the main diagonal when m is sufficiently large. Figure 2 provides an overview of these paths for different m.

Denote the number of ways of picking s_1 from s objects by

$$\binom{s}{s_1} = \frac{s!}{s_1!(s - s_1)!}.$$

Let us assume s objects, numbered n boxes ($n \le s$), and $s_1 + \cdots + s_n = s$. Denote the number of ways of putting s_1 objects into the first box, putting s_2 objects into the second box…, and putting s_n objects into the n-th box by

$$\binom{s}{s_1, \ldots, s_n} = \frac{s!}{s_1! \ldots s_n!}.$$

To prove Theorem 1, it is useful to generalize the function f as

$$f(i : (s_1, \ldots, s_n)) = \frac{(m!)^2}{(nm)!} \cdot \binom{\sum_{k \in N} s_k - 1}{s_1, \ldots, s_i - 1, \ldots, s_n} \binom{mn - \sum_{k \in N} s_k}{m - s_1, \ldots, m - s_n}.$$

Proof of Theorem 1 First, let us consider minimal-effort games and show that the total multi-choice value converges to the fuzzy value as in Lemma 3. Recall that $H(u_t)$ is the set of players who are required for the highest participation level in u_t, and that $\Phi^m(u_t)$ for player $i \notin H(u_t)$ converges to 0. Without loss of generality, assume that player 1 is an element of $H(u_t)$. Then there exists at least one player, say player 2, who is required a higher participation level $t_2 > t_1$. Then,

$$
\begin{aligned}
\varPhi_1^m(u_t) &= \sum_{s_j \geq t_j, j \geq 2} f(1 : (s_1, \ldots, s_n)) \\
&= \frac{(m!)^n}{(nm)!} \sum_{s_2 \geq t_2,} \binom{t_1 + s_2 - 1}{s_2} \binom{2m - t_1 - s_2}{m - s_2} \\
&\quad \times \sum_{s_3 \geq t_3, \ldots, s_n \geq t_3} \binom{t_1 + \sum_{j \geq 2} s_j - 1}{t_3, \ldots, t_n} \binom{nm - t_1 - \sum_{j \geq 2} s_j}{m - t_3, \ldots, m - t_n} \\
&\leq \frac{(m!)^n}{(nm)!} \sum_{s_2 \geq t_2,} \binom{t_1 + s_2 - 1}{s_2} \binom{2m - t_1 - s_2}{m - s_2} \\
&\quad \times \sum_{s_3 \geq 0, \ldots, s_n \geq 0} \binom{t_1 + \sum_{j \geq 2} s_j - 1}{t_3, \ldots, t_n} \binom{nm - t_1 - \sum_{j \geq 2} s_j}{m - t_3, \ldots, m - t_n}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(m!)^n}{(nm)!} \sum_{s_2 \geq t_2,} \binom{t_1 + s_2 - 1}{s_2} \binom{2m - t_1 - s_2}{m - s_2} \times \binom{nm}{2m, m, \ldots, m} \\
&= \frac{(m!)^2}{(2m)!} \sum_{s_2 \geq t_2,} \binom{t_1 + s_2 - 1}{s_2} \binom{2m - t_1 - s_2}{m - s_2} = F(t_2).
\end{aligned}
$$

Since Lemma 3 states that the value $F(t_2)$ for $t_2 > t_1$ converges to 0 as $m \to 0$, the total multi-choice value $\Phi_i^m(u_t)$ for $i \notin H(u_t)$ also converges to 0.

Meanwhile, players in $H(u_t)$ have the same total multi-choice value by symmetry. Using

$$\sum_{i \in N} \Phi_i^m(u_t) = 1,$$

known as the efficiency property, we obtain that the total multi-choice value $\Phi_i^m(u_t)$ for $i \in H(u_t)$ converges to $\frac{1}{h}$. Comparing this with the fuzzy value, calculated as in Example 3, the total multi-choice value converges to the fuzzy value for any minimal-effort game.

Because z, Φ^m, and Θ are linear operators, and the dividend is $\Delta_v(s) < \infty$, the statement of Theorem 1 holds for any multi-choice games v^m. $\square$

Example 4 Let us calculate the total multi-choice value and fuzzy value for two-person $(m + 1)$-choice minimal-effort game u_t, where $t = (\frac{2}{4}, \frac{3}{4})$. The fuzzy multi-

Table 1 Convergence of total multiple-choice value

m	$\Phi_1^m(u_t)$	$\Phi_2^m(u_t)$
4	0.2429	0.7571
8	0.1573	0.8427
20	0.0527	0.9473
100	0.0001	0.9999

choice value is $\Theta(zu_t) = (0, 1)$ as shown in Example 2. When $m = 4$, the graph of the piecewise multilinear function is shown in Fig. 1. As m becomes larger, the total multi-choice value converges to the fuzzy value (0, 1) as shown in Table 1.

5 Conclusion

We discussed the limit property of the multi-choice value proposed by van den Nouweland(1993) and van den Nouweland et al. (1995) and compared this value with the fuzzy value proposed by Aubin (1981, 1993). The sum of multi-choice values over choices, which we called the total multi-choice value, derives from combinatorial interpretation of the well-known Shapley value and depends on the number of levels m. We concluded that the total multiple-choice value is the consistent value in the sense that it connects the Shapley value and the fuzzy value. To obtain this result, we transformed multi-choice game into a fuzzy game, defined the piecewise multilinear function, and demonstrated that the total multiple-choice value converges to the fuzzy value for the extended fuzzy game as m increases.

Finally, I would like to briefly address Bolger's (1993) multi-alternative games and the generalized Shapley value. [1] In multi-choice games, we make admissible orders of both players and alternatives, where all players start by choosing level zero, and then advance step by step from each level to the next one after another, and finally choose level m. This is the key to connect the multi-choice value to the fuzzy value. In contrast, Bolger's generalization of the Shapley value does not assume such orders. However, Ono (2001, 2002) constructs the multilinear extension of the multi-alternative games, which is closely related with a combinatorial interpretation. It focuses on a certain alternative, and assumes the coalition of the players who choose this alternative becomes larger up to the grand coalition. The Bolger value for a player to choose this alternative is the expected contribution of this player over all such coalition-growing processes. The limit properties of this value will be studied in future research.

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