A Review of Some Recent Results on Power Indices

J. M. Alonso-Meijide, B. Casas-Méndez and M. G. Fiestras-Janeiro

1 Introduction

In principle, any democratic institution is designed as systems of governance that reflect the influential effect of each member on any decision making process. Thus, it can be seen as a decision support system in which voters choose among different alternatives. In this chapter we consider the case of binary decision rules in a double sense: each voter chooses between two alternatives; and, once the decision making process ends, voters are divided into two groups (agents are in favor of or against the proposal). Such situations can be represented by simple games where coalitions are classified in two groups: winning or losing in accordance with the success or failure of passing a proposal when all members of a coalition are involved in the decision making process.

The Shapley-Shubik index (Shapley and Shubik 1954) and the non-normalized Banzhaf index (Banzhaf 1965) are the most popular measures in this context. Other measures have been proposed such as the Deegan-Packel index (Deegan and Packel 1978) and the Public Good Index (Holler 1982). While the Shapley-Shubik index and the non-normalized Banzhaf index take into account all coalitions, the

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Deegan-Packel and the Public Good Index take only into account the minimal winning coalitions. In the literature there is no consensus about which power index is the most suitable. The study of the mathematical properties of a power index highlights similarities and differences in relation to other indices. Laruelle (1999) provides a good survey on the topic of properties of power indices. In this chapter, we provide some characterizations of the mentioned indices. There are some reasons for getting characterizations of the power indices: first, for a mathematically elegant and pleasant spirit; second, because a set of basic (and assumed independent and hence minimal) properties is a tool to decide on the use of the index. Finally, such a set allows researchers to compare a given value with others and to select the most suitable one for a given problem.

In this chapter we present some results on the study of the Deegan-Packel and the Public Good Index that the research group SaGaTh[1] (Galicia, Spain) has obtained and some results on a new power index, the Shift Power Index, proposed in Alonso-Meijide and Freixas (2010). This research group has also worked in other areas related with power indices. Results and discussion will be presented in Sect. 6. The chapter is organized as follows. Section 2 is devoted to recall some basic definitions on simple games and the study of power indices in order to make the chapter self-contained. Section 3 presents characterizations of the Shapley-Shubik, non-normalized Banzhaf, Deegan-Packel, and Public Good indices. Section 4 is devoted to the Shift Power Index, a new measure based on a desirability relation. In Sect. 5, we propose extensions of the Deegan-Packel and Public Good Index by taking into consideration winning coalitions without null agents.

2 Preliminaries

2.1 Simple Games

A **simple game** is a pair (N, W) where N is a coalition and W is a family of subsets of N satisfying:

(i) $N \in W$, $\emptyset \notin W$ and
(ii) the monotonicity property, i.e.,

$$S \subseteq T \subseteq N \text{ and } S \in W \text{ implies } T \in W.$$

[1] In this research program more colleagues were involved. We specially want to thank our coauthors Mikel Álvarez-Mozos, Mario Bilbao, Carlos Bowles, Francesc Carreras, Joseph Freixas, Julio R. Fernández, Flavio Ferreira, Manfred J. Holler, Silvia Lorenzo-Freire, Stefan Napel, Andreas Nohn, Guillermo Owen, and Alberto Pinto for the discussions and contributions to this research topic. More information about members and working areas of the group is available in the website http://eio.usc.es/pub/io/xogos/.

This representation of simple games follows the approach by Felsenthal and Machover (1998) and by Peleg and Sudhölter (2003). In a simple game, N is the set of members of a committee and W is the set of winning coalitions.

In a simple game (N, W), a coalition $S \subseteq N$ is **winning** if $S \in W$ and is **losing** if $S \notin W$. We denote by $SI(N)$ the set of simple games with player set N.[2]

A winning coalition $S \in W$ is a **minimal winning coalition** (MWC) if every proper subset of S is a losing coalition, that is, $S \subseteq N$ is a MWC in (N, W) if $S \in W$ and $T \notin W$ for any $T \subset S$. We denote by M^W the set of MWC of the simple game (N, W). Given a player $i \in N$ we denote by M_i^W the set of MWC such that i belongs to, that is, $M_i^W = \{S \in M^W / i \in S\}$. Two simple games (N, W) and (N, V) are **mergeable** if for all pair of coalitions $S \in M^W$ and $T \in M^V$, it holds that $S \not\subseteq T$ and $T \not\subseteq S$.

Given two simple games (N, W) and (N, V), we define the union game $(N, W \vee V)$ in such a way that for all $S \subseteq N, S \in W \vee V$ if $S \in W$ or $S \in V$, and we define the intersection game $(N, W \wedge V)$ in such a way that for all $S \subseteq N, S \in W \wedge V$ if $S \in W$ and $S \in V$. If two simple games (N, W) and (N, V) are mergeable, the set of minimal winning coalitions in game $(N, W \vee V)$ is precisely the union of the set of minimal winning coalitions of games (N, W) and (N, V).

A **null player** in a simple game (N, W) is a player $i \in N$ such that $i \notin S$ for all $S \in M^W$. Two players $i, j \in N$ are **symmetric** in a simple game (N, W) if $S \cup i \in W$ if and only if $S \cup j \in W$ for all $S \subseteq N \backslash \{i, j\}$ such that $S \notin W$. Given a player $i \in N$, the **set of swings** of i in the simple game (N, W), $\eta_i(W)$, is formed by the coalitions $S \subseteq N \backslash i$ such that $S \notin W$ and $S \cup i \in W$.

A simple game(N, W) is a weighted majority game if there is a set of weights $w_1, w_2, \ldots, w_n$ for players, with $w_i \geq 0, 1 \leq i \leq n$, and a quota $q \in \mathbb{R}^+$ such that $S \in W$ if and only if $w(S) \geq q$, where $w(S) = \sum_{i \in S} w_i$. A weighted majority game is represented by $[q; w_1, w_2, \ldots, w_n]$. An example of a weighted majority game is provided below.

Example 1 Consider the weighted majority game [14; 6, 5, 4, 4, 3], whose minimal winning coalitions are:

$$M^W = \{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{2,3,4,5\}\}.$$

Coalition $\{1,2,3,4\}$ is a winning coalition, but is not a minimal winning coalition. Coalition $\{1,2\}$ is a losing coalition. Players 3 and 4 are symmetric and there is no null player.

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2 We will use shorthand notation and write $S \cup i$ for the set $S \cup \{i\}$ and $S \backslash i$ for the set $S \backslash \{i\}$. We will denote by s the number of members in a finite set S.

2.2 Power Indices

A **power index** is a function f which assigns an n-dimensional real vector $f(N, W)$ to a simple game (N, W), where the i-th component of this vector, $f_i(N, W)$, is the power of player i in the game (N, W) according to f. The power index of a simple game can be interpreted as a measure of the ability of the different players to turn a losing coalition into a winning one. In what follows we discuss the Shapley-Shubik index (Shapley and Shubik 1954), the non-normalized Banzhaf index (Banzhaf 1965; Owen 1975), the Deegan-Packel index (Deegan and Packel 1978), and the Public Good Index (Holler 1982). The Shapley-Shubik index depends on the number of permutations on the set of players and of the size of each coalition in each swing. The non-normalized Banzhaf index considers that power depends only on the number of swings of each player and is not directly associated with the order of players. The Deegan-Packel index considers that only minimal winning coalitions will emerge victorious, each minimal winning coalition has an equal probability of forming, and, players in a minimal winning coalition divide the "spoils" equally among the members of this coalition. In a similar way, for the Public Good Index, only minimal winning coalitions are considered relevant when it comes to measuring power. Given a simple game, the Public Good Index of a player is equal to the total number of minimal winning coalitions containing this player divided by the sum of these numbers for all players.

Next, we provide the formal expressions of each one of these indices. Take a simple game (N, W).

I1. The Shapley-Shubik index (SSI) assigns to each player $i \in N$ the real number

$$\varphi_i(N, W) = \sum_{S \in \eta_i(W)} \frac{s!(n-s-1)!}{n!}.$$

I2. The non-normalized Banzhaf index (BI) assigns to each player $i \in N$ the real number

$$\beta_i(N, W) = \frac{|\eta_i(W)|}{2^{n-1}}.$$

I3. The Deegan-Packel index (DP) assigns to each player $i \in N$ the real number

$$\rho_i(N, W) = \frac{1}{|M^W|} \sum_{S \in M_i^W} \frac{1}{|S|}.$$

I4. The Public Good Index (PGI) assigns to each player $i \in N$ the real number

$$\delta_i(N, W) = \frac{|M_i^W|}{\sum_{j \in N} |M_j^W|}.$$

Example 2 Using the weighted majority game defined in Example 1, we illustrate the computation of these power indices.

The set of swings of player 1 in this game is:

$$\eta_1(W) = \{\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\}.$$

Once the set of swings of each player is obtained, we compute the Shapley-Shubik index and the non-normalized Banzhaf index:

$$\varphi(N,W) = \left(\frac{20}{60},\frac{15}{60},\frac{10}{60},\frac{10}{60},\frac{5}{60}\right) \quad \text{and} \quad \beta(N,W) = \left(\frac{8}{16},\frac{6}{16},\frac{4}{16},\frac{4}{16},\frac{2}{16}\right).$$

The set of MWC such that player 1 belongs to is:

$$M_1^W = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\}\}.$$

After taking into account the sets M_i^W for every player $i = 1,2,3,4,5$, we obtain the Deegan-Packel index and Public Good Index:

$$\rho(N,W) = \left(\frac{16}{60},\frac{15}{60},\frac{11}{60},\frac{11}{60},\frac{7}{60}\right) \quad \text{and} \quad \delta(N,W) = \left(\frac{4}{16},\frac{4}{16},\frac{3}{16},\frac{3}{16},\frac{2}{16}\right).$$

3 Some Characterizations

For any power index, understood as a solution concept for simple games, it is always interesting, in both theory and practice, to have the above explicit formulae. Besides, to have a list of properties of an index is desirable with respect to the theoretical discussion and the application of the measures to real-world. Here, we review some of these properties and applied them to SSI, BI, DP, and PGI.

A1. A power index f satisfies **efficiency** if $\sum_{i \in N} f_i(N,W) = 1$, for every simple game (N,W).

A2. A power index f satisfies the **null player** property if $f_i(N,W) = 0$, for every simple game (N,W) and for every null player $i \in N$.

A3. A power index f satisfies **symmetry** if $f_i(N,W) = f_j(N,W)$, for every simple game (N,W), and for every symmetric players $i,j \in N$.

A4. A power index f satisfies the **strong monotonicity** property if $f_i(N,W) \geq f_i(N,V)$, for every simple games (N,W) and (N,V), and for every player $i \in N$ such that $\eta_i(V) \subseteq \eta_i(W)$.

A5. A power index f satisfies the **transfer** property if $f(N,W \vee V) + f(N,W \wedge V) = f(N,W) + f(N,V)$, for every simple games (N,W) and (N,V).

A6. A power index f satisfies the **total power** property if $\sum_{i \in N} f_i(N,W) = \overline{\eta}(W)/2^{n-1}$, for every simple game (N,W), where $\overline{\eta}(W) = \sum_{i \in N} |\eta_i(W)|$.

A7. A power index f satisfies the **DP-mergeability** property if

$$f(N, W \vee V) = \frac{|M^W|f(N, W) + |M^V|f(N, V)}{|M^{W \vee V}|},$$

for every pair of mergeable simple games (N, W) and (N, V).

A8. A power index f satisfies the **PGI-mergeability** property if

$$f(N, W \vee V) = \frac{f(N, W)\sum_{i \in N}|M_i^W| + f(N, V)\sum_{i \in N}|M_i^V|}{\sum_{i \in N}|M_i^{W \vee V}|},$$

for every pair of mergeable simple games (N, W) and (N, V).

A9. A power index f satisfies the **DP-minimal monotonicity** property if

$$|M^W|f_i(N, W) \geq |M^V|f_i(N, V),$$

for every pair of simple games (N, W) and (N, V), and for every player $i \in N$ such that $M_i^V \subseteq M_i^W$.

A10. A power index f satisfies the **PGI-minimal monotonicity** property if

$$f_i(N, W)\sum_{j \in N}|M_j^W| \geq f_i(N, V)\sum_{j \in N}|M_j^V|,$$

for every pair of simple games (N, W) and (N, V), and for every player $i \in N$ such that $M_i^V \subseteq M_i^W$.

Property A1 states that the power sharing of the players is equal to 1. Following property A2, a null player gets nothing. Property A3 says that if two players play the same role, then they should have the same power. Property A4 was proposed by Young (1985), and it states that if a player “has more swings” in (N, W) than in (N, V), then, his power cannot be greater in the second game. We want to emphasize that each one of these first four properties use a single game in their definition. Property A5 was proposed by Dubey (1975) as a substitute of Shapley’s additivity property that makes no sense in the context of simple games. It establishes that the sum of the power of a player in two simple games coincides with the sum of the power of this player in the union and the intersection game. Property A6 was proposed by Dubey and Shapley (1979). It states that the power of players adds up to the total number of swings divided by the number of coalitions that player i can join. Properties A7 and A8 state that power in the union game is a weighted mean of the power of the two component games, where the weights are provided by taking into account the set of minimal winning coalitions. These two properties provide a link between the power of a player in the union game and the power of the player in the original games. That is, Properties A7 and A8 use three games in their definition.

The last two properties were presented in Alonso-Meijide et al. (2008) and Lorenzo-Freire et al. (2007). In this case, exactly two games are used in the definitions of the properties. These two properties do not establish two equalities, only two inequalities. In the formulation of DP-minimal monotonicity, a relation between a power index in two simple games (N, W) and (N, V) is given in terms of the cardinality of their sets of minimal winning coalitions. This property states that if every minimal winning coalition containing a player $i \in N$ in the game (N, V) is a minimal winning coalition in the game (N, W), then the power of this player in the game (N, W) is not less than his power in the game (N, V) (previously, we must weight this power by the number of minimal winning coalitions of games (N, W) and (N, V), respectively). Note that DP-minimal monotonicity implies

$$f_i(N, W)\left|M^W\right| = f_i(N, V)\left|M^V\right|,$$

for any two simple games (N, W) and (N, V), and for all $i \in N$ such that $M_i^W = M_i^V$. Similarly, the PGI-minimal monotonicity takes into account the relation between a power index in two simple games (N, W) and (N, V). For this property, the weights are given by the number of minimal winning coalitions of every player of games (N, W) and (N, V). The weight $\left|M_j^W\right|$ coincides with the number of swings for a player j in the game (N, W), if we consider minimal winning coalitions only. PGI-minimal monotonicity property implies

$$f_i(N, W)\sum_{j \in N}\left|M_j^W\right| = f_i(N, V)\sum_{j \in N}\left|M_j^V\right|,$$

for any two simple games (N, W) and (N, V), and for all $i \in N$ such that $M_i^W = M_i^V$. In Alonso-Meijide and Holler (2009) this property is discussed in detail.

To end this section, we present some characterizations that enable us to directly compare the power indices I1–I4.

C1. (Dubey 1975) The unique power index f that satisfies transfer, null player, symmetry, and efficiency is the Shapley-Shubik index.
C2. (Dubey and Shapley 1979) The unique power index f that satisfies transfer, null player, symmetry, and total power is the non-normalized Banzhaf index.
C3. (Deegan and Packel 1978) The unique power index f that satisfies DP-mergeability, null player, symmetry, and efficiency is the Deegan-Packel index.
C4. (Holler and Packel 1983) The unique power index f that satisfies PGI-mergeability, null player, symmetry, and efficiency is the Public Good Index.
C5. (Lorenzo-Freire et al. 2007) The unique power index f that satisfies DP-minimal monotonicity, null player, symmetry, and efficiency is the Deegan-Packel index.
C6. (Alonso-Meijide et al. 2008) The unique power index f that satisfies PGI-minimal monotonicity, null player, symmetry, and efficiency is the Public Good Index.

The null player property and symmetry appear in all characterizations presented above. Except the non-normalized Banzhaf index, that satisfies the properties of total power, all indices that we consider satisfy efficiency. In addition to these three properties, namely, symmetry, null player and efficiency (or total power in the case of the non-normalized Banzhaf index), that only use one game in their definitions, another property is necessary to characterize each of these indices. This last property must be one of the group of properties that established an inequality between two games (monotonicity properties) or an equality between the union game and the component games (transfer or mergeability properties). Depending on the property used, each one of the four indices is characterized.

4 The Shift Power Index

Riker (1962, p. 100) claimed that "parties seek to increase votes only up to the size of a minimum coalition". This follows from the well–known "Size Principle" (Riker 1962, p. 32) which implies that, given a multi–member voting body or weighted majority game $[q; w_1, \ldots, w_n]$ a coalition S_0 will be formed provided that $w(S_0) = \min_{S \in W} w(S)$. Taking this principle into account, we could consider that not all the minimal winning coalitions will be formed. In Alonso-Meijide and Freixas (2010) a new power index is proposed. It contains elements of both the Public Good Index and Riker's principle; thus, to certain extent, it can be seen as an intermingled solution. The fundamental idea is the notion of desirability which we recall here.

Let (N, W) be a simple game, i and j be two voters. The *desirability relation* denoted by $\succsim$ is defined in N as follows. We say that i is at least as desirable as j (as a coalitional partner), denoted by $i \succsim j$ if $S \cup \{j\} \in W \;\Rightarrow\; S \cup \{i\} \in W$, for every coalition $S \subseteq N \setminus \{i,j\}$. Player i is said to be (strictly) more desirable than j, denoted by $i \succ j$ if $i \succsim j$ and there is a coalition $T \subseteq N \setminus \{i,j\}$ such that $T \cup \{i\} \in W$ and $T \cup \{j\} \notin W$. Players i and j are said to be equally desirable, denoted by $i \sim j$ if $i \succsim j$ and $j \succsim i$. It is not difficult to see that the desirability relation $(\succsim)$ is a pre-ordering. We say that a simple game (N, W) is **complete or linear** if the desirability relation is a complete pre-ordering. In particular, every weighted majority game is also a complete game, since $w_i \geq w_j$ implies $i \succsim j$. If voters act selfishly looking for the formation of a coalition with partners who are as weak as possible, then under the desirability relation the weaker voters tend to be the most crucial voters at the expense of the stronger ones. The following relative power measure intends to capture this idea.

The proposed index is to a certain extent similar to the Public Good Index and the Deegan-Packel index because it does not take into account surplus coalitions. If S is an arbitrary losing coalition in a simple game (N, W), and $i, j \in N \backslash S$ with $i \succ j$, then i joining S offers more chances to convert S into a winning coalition than j joining S. Precisely, if S is a losing coalition and $S \cup j$ wins, then $S \cup i$ also wins. Thus, if we regard all losing coalitions as a whole, we can assert that i is globally better positioned than j to be chosen as a coalitional partner. Roughly speaking the proposed power index is based on the following principles: (i) every voter wishes to form part of a minimal winning coalition and, (ii) every voter wishes to form part only of those minimal winning coalitions in which no player can be replaced by a weaker one and still winning.

These principles lead us to take a look at the notion of the shift minimal winning coalition which is at the core of the proposed power index. In practice, this notion implies that, for each player, an index will take into account the number of shift minimal winning coalitions that player belongs to, independently of the size of such coalitions.

Let (N, W) be a simple game and $\succsim$ be its desirability ordering. A coalition $S \in M^W$ is **shift minimal** if for every $i \in S$ and $j \notin S$ such that $i \succ j$ it holds $(S \setminus i) \cup j \notin W$. The set of shift minimal winning coalitions will be denoted by SM^W. SM_i^W is the set of shift minimal winning coalitions such that i belongs to, that is, $SM_i^W = \{S \in SM^W / i \in S\}$. Take a simple game (N, W).

I5. The Shift Power Index assigns to each player $i \in N$ the real number

$$\sigma_i(N, W) = \frac{|SM_i^W|}{\sum_{j=1}^{n} |SM_j^W|}.$$

The computation of the Shift Power Index involves a subset of the set of coalitions used in the computation of the Public Good Index, because each shift minimal winning coalition is a minimal winning coalition.

Example 3 Using the weighted majority game defined in Example 1 we illustrate the computation of the Shift Power Index. Notice that the complete desirability relation is $1 \succ 2 \succ 3 \sim 4 \succ 5$. Coalition $\{1, 2, 3\}$ is a minimal winning one, but is not shift minimal since $3 \succ 5$ and coalition $\{1, 2, 5\}$ still win. It is not difficult to check that the set of shift minimal coalitions is:

$$SM^W = \{\{1, 2, 5\}, \{1, 3, 4\}, \{2, 3, 4, 5\}\}.$$

The Shift Power Index is:

$$\sigma_i(N, W) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right).$$

We also introduce a property with the flavour of monotonicity written in terms of the shift minimal winning coalitions.

A11. A power index f satisfies the **Shift-minimal monotonicity** property if

$$f_i(N,W)\sum_{j\in N}\left|SM_j^W\right|\geq f_i(N,V)\sum_{j\in N}\left|SM_j^V\right|,$$

for every pair of simple games (N,W) and (N,V), and for every player $i\in N$ such that $SM_i^V\subseteq SM_i^W$.

Using Property A11, and others defined previously, we obtain a characterization of the Shift Power Index.

C7. (Alonso-Meijide and Freixas 2010) The unique power index f that satisfies Shift-minimal monotonicity, null player, symmetry, and efficiency is the Shift Power Index.

5 Quasi-Minimal Coalition Indices

In Alonso-Meijide et al. (2011a), we follow a different point of view. Our intention was to consider as a measure of a player's power two indices similar to those proposed by Deegan and Packel (1978) and Holler (1982) using a set of coalitions bigger than the set of minimal winning coalitions and smaller than the set of winning coalitions. We are interested in that the new indices satisfy the null player property. Then, we define a new family of coalitions.

A winning coalition $S\subseteq N$ is a **quasi-minimal winning coalition** if there is not a null player $i\in S$. The set of quasi-minimal winning coalitions will be denoted by QM^W and the set of quasi-minimal winning coalitions such that i belongs to by QM_i^W, that is, $QM_i^W=\{S\in QM^W/i\in S\}$. It is clear that, for every simple game (N,W),

$$M^W\subseteq QM^W\subseteq W.$$

Example 4 Consider the simple game of four voters, $N=\{1,2,3,4\}$, defined by its set of minimal winning coalitions:

$$M^W=\{\{2,3\},\{2,4\},\{3,4\}\}.$$

Player 1 is the unique null player of the game and coalitions $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, and $\{1,2,3,4\}$ are winning coalitions. However, the set of quasi-minimal winning coalitions is:

$$QM^W=M^W\cup\{\{2,3,4\}\},$$

since $\{2,3,4\}$ is also a winning coalition and voter 1 does not belong to it.

We consider a modification of the Deegan-Packel index and a modification of the Public Good Index, defined as follows. Take a simple game (N,W).

I6. The modified Deegan-Packel index assigns to each player $i \in N$ the real number

$$\rho'_i(N, W) = \frac{1}{|QM^W|} \sum_{S \in QM_i^W} \frac{1}{|S|}.$$

I7. The modified Public Good Index assigns to each player $i \in N$ the real number

$$\delta'_i(N, W) = \frac{|QM_i^W|}{\sum_{j \in N} |QM_j^W|}.$$

Álvarez-Mozos (2012) proposes a characterization of each one of these two indices.

Example 5 Once again we use Example 1 to illustrate the computation of these power indices. The set of quasi-minimal winning coalitions such that player 1 belongs to is:

$$QM_1^W = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},$$

$$\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{1,2,3,4,5\}\}.$$

In this game, the set of null players is empty, thus $QM^W = W$.

After taking into account the sets of quasi-minimal winning coalitions of each one of the rest of players, we obtain the modified Deegan-Packel index and the modified Public Good Index:

$$\rho'(N, W) = \left(\frac{152}{600}, \frac{132}{600}, \frac{112}{600}, \frac{112}{600}, \frac{92}{600}\right) \quad \text{and} \quad \delta'(N, W) = \left(\frac{9}{37}, \frac{8}{37}, \frac{7}{37}, \frac{7}{37}, \frac{6}{37}\right).$$

6 Other Results in Related Research

Our research group has also worked in other areas related with power indices. Next, we briefly summarize these results.

In simple games, a member is considered critical when his elimination from a winning coalition turns this coalition into a losing coalition. Johnston (1978) argued that the non-normalized Banzhaf index, which is based on the idea of a removal of a critical voter from a winning coalition, does not take into account the total number of critical members in each coalition. Clearly, if a voter is the only critical agent in a coalition, this gives a stronger sign of power than in the case where all agents are critical. This is the focal idea underlying the Johnston index. In Lorenzo-Freire et al. (2007), we provide the first characterization of the Johnston index in the class of simple games, using a transfer property that involves the solution for unanimity games of certain coalitions.

For representing social decision situations adequately, sophisticated models have been developed. One of them is the simple game endowed with a priori unions, that is, a partition of the player set which describes a pre-defined (exogenously given) coalition structure. The traditional power indices are not suitable for measuring the distribution of power in these situations because adequate measures of power should take the coalition structure into account. The so-called Owen value (Owen 1977) is an early extension of the Shapley-Shubik index to social decisions with a priori unions. Afterwards, several extensions of the Shapley-Shubik index and the non-normalized Banzhaf index have been proposed for the model of simple games with a priori unions. Alonso-Meijide et al. (2007) compare these different extensions and give arguments to defend the use of them that will depend on the context where they are to be applied. Alonso-Meijide et al. (2009c) introduce adaptations of the conventional monotonicity notions that are suitable for voting games with a priori unions. In Alonso-Meijide et al. (2011b) the Deegan-Packel index is extended to simple games with a priori unions and axiomatically characterized. Of course, this extension coincides with the Deegan-Packel index when each union is formed by only one player or there is only one union. Furthermore, this extension satisfies two types of symmetry, one among players of the same union, and the second one among unions in the game played among unions, and two properties with the same flavour as DP-mergeability.

Two similar versions of mergeability (PGI-mergeability) are quintessential to the two PGI extensions to a priori unions introduced in Alonso-Meijide et al. (2010a). The first one stresses the public good property which suggests that all members of a winning coalition derive equal power, irrespective of their possibility to form alternative coalitions. The second extension follows earlier work on the integration of a priori unions (see Owen 1977). It refers to essential subsets when allocating power shares, taking the outside options of the coalition members into consideration. In Alonso-Meijide et al. (2010b), axiomatizations for six variants of the Public Good Index for games with a priori unions are provided. Two such coalitional PGIs have been introduced and alternatively axiomatized in Alonso-Meijide et al. (2010a). The other four coalitional PGIs have been introduced in Holler and Nohn (2009). The first variant elaborates the original idea of the PGI that the coalitional value is a public good and only minimal winning coalitions of the so called quotient game played by the unions are relevant. The remaining three variants use a two-step distribution where, on the member stage, they take into account the possibilities of players to threaten their partners through leaving their union.

Alonso-Meijide and Casas-Méndez (2007) defined and characterized a modification of the Public Good Index for situations in which some players are incompatible, that is, some players cannot cooperate among them by political reasons. The incompatibilities among players are modelized by a graph in such a way that if two players are linked by an arc of the graph, these two players are incompatible. Alonso-Meijide et al. (2009a) propose a modification of the non-normalized Banzhaf value for this kind of situations, provide two characterizations of it and illustrate it with a real world example taken from the political field.

One of the main difficulties with power indices is that computation generally requires the sum of a very large number of terms. Owen (1972) defined the multilinear extension of a game. Owen (1975) proposed a procedure to compute the Shapley value and the non-normalized Banzhaf value based on the multilinear extension. The multilinear extension is useful in computing the power of large games such as the Presidential Election Game and the Electoral College Game studied by Owen (1972). The multilinear extension approach has two advantages: thanks to its probabilistic interpretation, the central limit theorem of probability can be applied, and, further, it is applied to composition of games. Alonso-Meijide et al. (2008) introduce procedures to calculate the Deegan-Packel index, the Public Good Index and the Johnston index by means of the multilinear extensions.

The generating functions are also efficient tools to compute power indices of weighted voting games. One of the strengths of these procedures is that they allow to get exact values of the indices, even in case we have a very large number of non-null voters. Furthermore, the time required for the computation is in practice much lower than the one required for the computation of the traditional Banzhaf and Shapley-Shubik indices. This method, drawn from Cantor's early work (see Lucas (1983)) and Brams and Affuso (1976), provides algorithms to compute the non-normalized Banzhaf and Shapley-Shubik power indices of weighted voting games. In Alonso-Meijide and Bowles (2005), a procedure based on generating functions is provided to compute power indices in weighted majority games restricted by a priori unions. The method is illustrated by an application to the International Monetary Fund. Alonso-Meijide et al. (2009b) apply the generating functions procedures to the distribution of power in the enlarged European Union, modelled as a weighted multiple majority game restricted by a priori unions.

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