Further Results on the Mycielskian of Graphs

T. Kavaskar

Department of Mathematics, Bharathidasan University, Tiruchirappalli-620024, India
t_kavaskar@yahoo.com

Keywords: Mycielskian of a graph, acyclic chromatic number, dominator chromatic number, independent domination number.

1 Introduction

All graphs considered here are finite, undirected, connected and non-trivial. In the mid 20^{th} century there was a question regarding, triangle-free graphs with arbitrarily large chromatic number. In answer to this question, Mycielski [7] developed an interesting graph transformation as follows: For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with vertex set consisting of the disjoint union $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and edge set $E \cup \{x'y : xy \in E\} \cup \{x'u : x' \in V'\}$. We call x' the twin of x in $\mu(G)$ and vice versa and u, the root of $\mu(G)$. We can define the *iterative Mycielskian* of a graph G as follows: $\mu^m(G) = \mu(\mu^{m-1}(G))$, for $m \geq 1$. Here $\mu^0(G) = G$. It is well known [7] that if G is triangle free, then so is $\mu(G)$ and that the chromatic number $\chi(\mu(G)) = \chi(G) + 1$. There had been several papers on Mycielskian of graphs. Few of the references are [2], [3], [5], [7], [8]. Several graph parameters, especially in domination theory, on Mycielskian of graphs have been discussed in [2], [8].

We say that a set $S \subseteq V(G)$ is a *dominating set* of G if every vertex $x \in V(G)\backslash S$ has at least one neighbor in S. The *domination number* of G is defined by $\gamma(G) = \min\{|S| : S$ is a dominating set of $G\}$. A set $S \subseteq V(G)$ is said to be an *independent dominating set* of G [4] if S is a dominating set of G and the induced subgraph $\langle S \rangle$ has no edge in G. The *independent domination number* of G is defined by $i(G) = \min\{|S| : S$ is an independent dominating set of $G\}$. A set $S \subseteq V(G)$ is said to be *acyclic domination set* of G if it is a dominating set of G and $\langle S \rangle$ is a forest. The *acyclic dominating number* of G is defined by $\gamma_a(G) = \min\{|S| : S$ is an acyclic domination set of $G\}$.

A proper k-coloring, $V_1, V_2, \ldots, V_k$, of G is said to be *acyclic k-coloring* if for $1 \leq i < j \leq k$, $\langle V_i \cup V_j \rangle$ is a forest. The *acyclic chromatic number* of G is defined by: $\chi_a(G) = \min\{k : G$ has an acyclic k-coloring$\}$. A proper k-coloring of G is said to be *dominator k-coloring* [6] if for every vertex $v \in V(G)$, $N[v]$ contains at least one color class. The *dominator chromatic number* of G is defined by: $\chi_d(G) = \min\{k : G$ has a dominator k-coloring$\}$.

In this paper, we determine the independent domination number, acyclic chromatic number and dominator chromatic number of the Mycielskian and the iterated Mycielskian of a graph with respect to their parent graphs.

S. Arumugam and B. Smyth (Eds.): IWOCA 2012, LNCS 7643, pp. 73–75, 2012.

2 Main Results

We start with the following well-known result on the domination number of the Mycielskian.

Theorem 21 ([2]). *For any graph G, $\gamma(\mu(G)) = \gamma(G)$+1.*

We now show that similar results hold good for the independent domination number of $\mu(G)$.

Theorem 22. *For any graph G, $i(\mu(G)) = i(G) + 1$.*

Proof. Let S be an independent dominating set of G with $|S| = i(G)$. Then $S \cup \{u\}$ is an independent dominating set of $\mu(G)$ and hence $i(\mu(G)) \leq i(G)$+1. Suppose $i(\mu(G)) \leq i(G)$. Let T be an independent dominating set of $\mu(G)$ with $|T| = i(\mu(G)) \leq i(G)$. If $u \in T$, then $T \backslash \{u\}$ is an independent dominating set of G and hence $i(\mu(G)) - 1 = |T \backslash \{u\}| \geq i(G)$, a contradiction. Thus $u \notin T$, which implices that $T \cap V' \neq \emptyset$. Similarly we get a contradiction for this case also. Thus $i(\mu(G)) \geq i(G)$+1 and hence $i(\mu(G)) = i(G) + 1$.

Iteratively applying Theorem 22, we get

Corollary 23. *For any graph G, $i(\mu^m(G)) = i(G) + m$.*

Next we discuss the acyclic domination number of $\mu(G)$. Let S be an acyclic dominating set of G with $|S| = \gamma_a(G)$, then $S \cup \{u\}$ is an acyclic dominating set of $\mu(G)$. Hence we have the following result.

Proposition 24. *For any graph G, $\gamma_a(\mu(G)) \leq \gamma_a(G) + 1$.*

We have an example for $(\gamma_a(G)+1) - \gamma_a(\mu(G))$ is arbitrarily large. For example, consider a graph G is obtained by joining k pendent vertices to each vertex of the cycle C_n for any $k \geq 2$ and $n \geq 3$. One can easily check that $\gamma_a(G) = n + k - 1$ and $\gamma_a(\mu(G)) = n + 1$.

We now recall that the well-known result on the chromatic number of the Mycielskian.

Theorem 25 ([7]). *For any graph G, $\chi(\mu(G)) = \chi(G)$+1.*

Next we show that similar results hold good for the acyclic chromatic number and dominator chromatic number of $\mu(G)$.

Theorem 26. *For any graph G, $\chi_a(\mu(G)) = \chi_a(G) + 1$.*

Proof. Let $V_1, V_2, \ldots, V_k$ be a acyclic k-coloring, where $k = \chi_a(G)$. Now set $U_i = V_i \cup V_i'$, for i, $1 \leq i \leq k$ and $U_{k+1} = \{u\}$. Then $U_i's$ forms a acycile coloring of $\mu(G)$ using $k + 1$ colors and hence $\chi_a(\mu(G)) \leq \chi_a(G) + 1$. Next to prove $\chi_a(\mu(G)) \geq \chi_a(G) + 1$. Suppose $\chi_a(\mu(G)) < \chi_a(G) + 1$. Let $k = \chi_a(\mu(G))$. Then $\mu(G)$ has a acyclic k-colors. We have to show that G has a acyclic $(k - 1)$-coloring. Proof of this is similar to the Theorem 25 given in [7]. Thus $\chi_a(\mu(G)) = \chi_a(G) + 1$.

Iteratively applying Theorem 26, we get

Corollary 27. *For any graph G, $\chi_a(\mu^m(G)) = \chi_a(G) + m$.*

Finally we determine the dominator chromatic number of $\mu(G)$.

Theorem 28. *For any graph G, $\chi_d(\mu(G)) = \chi_d(G) + 1$.*

Iteratively applying Theorem 28, we get

Corollary 29. *For any graph G, $\chi_d(\mu^m(G)) = \chi_d(G) + m$.*

Acknowledgement. This research was supported by Dr.D.S. Kothari Post Doctoral Fellowship, University Grants Commission, Government of India grant F.4-2/2006(BSR)/13-511- 2011(BSR) dated 26-08-2011 at the Department of Mathematics, Bharathidasan University, Tiruchirappalli-620024, India.

References

1. Coekayne, E.J., Hedetniemi, S.T., Miller, D.J.: Properties of hereditary hypergraphs and middle graphs. Canad. Math. Bull. 21(4), 461–468 (1978)
2. Fisher, D.C., McKenna, P.A., Boyer, E.D.: Hamiltonicity, diameter, domination, packing, and biclique partitions of Mycielski's graphs. Discrete Applied Mathematics 84(1-3), 93–105 (1998)
3. Chang, G.J., Huang, L., Zhu, X.: Circular chromatic numbers of Mycielski's graphs. Discrete Mathematics 205(1-3), 23–37 (1999)
4. Halldorsson, M.M.: Approximating the minimum maximal independence number. Information Processing Letters 46(4), 169–172 (1993)
5. Larsen, M., Propp, J., Ullman, D.: The fractional chromatic number of Mycielski's graphs. J. Graph Theory 19, 411–416 (1995)
6. Chellali, M., Maffray, F.: Dominator Colorings in Some Classes of Graphs. Graphs and Combinatorics 28, 97–107 (2012)
7. Mycielski, J.: Sur le coloriage des graphs. Colloq. Math. 3, 161–162 (1955)
8. Lin, W., Wu, J., Lam, P.C.B., Gu, G.: Several parameters of generalized Mycielskians. Discrete Applied Mathematics 154, 1173–1182 (2006)