On Some Properties of Doughnut Graphs
(Extended Abstract)

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Abstract. The class doughnut graphs is a subclass of 5-connected planar graphs. It is known that a doughnut graph admits a straight-line grid drawing with linear area, the outerplanarity of a doughnut graph is 3, and a doughnut graph is k-partitionable. In this paper we show that a doughnut graph exhibits a recursive structure. We also give an efficient algorithm for finding a shortest path between any pair of vertices in a doughnut graph. We also propose a nice application of a doughnut graph based on its properties.

1 Introduction

A five-connected planar graph G is called a *doughnut* graph if G has an embedding Γ such that (a) Γ has two vertex-disjoint faces each of which has exactly p vertices, $p > 3$, and all the other faces of Γ has exactly three vertices; and (b) G has the minimum number of vertices satisfying condition (a). Figure 1(a) illustrates a doughnut graph where F_1 and F_2 are two vertex disjoint faces. Figure 1(b) illustrates a doughnut like embedding of G where F_1 is embedded as the outer face and F_2 is embedded as the inner face. A doughnut graph and their spanning subgraphs admit straight-line grid drawings with linear area [2,3]. The outerplanarity of this class is 3 [3], and it is k-partitionable [5].

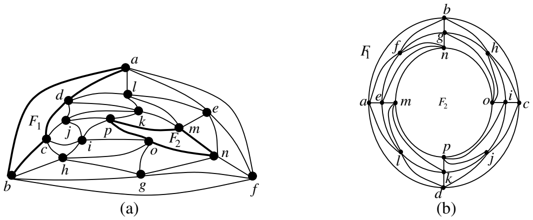

Fig. 1. (a) A doughnut graph G, and (b) a doughnut embedding of G

S. Arumugam and B. Smyth (Eds.): IWOCA 2012, LNCS 7643, pp. 60–64, 2012.

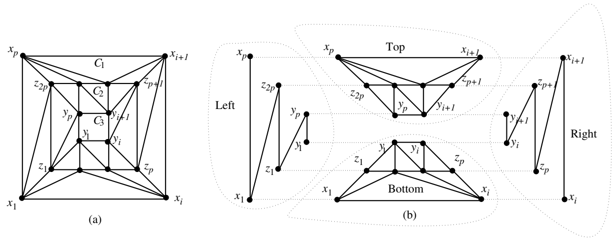

Fig. 2. (a) A straight-line drawing of a p-doughnut graph G where $p = 4$, and (b) illustration for four partition of edges of G

In this paper we present our results on recursive structure, shortest paths and topological properties of a doughnut graph.

2 Recursive Structure of Doughnut Graphs

A class of graphs has a recursive structure if every instance of it can be created by connecting the smaller instances of the same class of graphs. We now show that the doughnut graphs have a recursive structure. Let G be a p-doughnut graph. A doughnut graph G is 5-regular and has exactly $4p$ vertices. Furthermore, G has three vertex-disjoint cycles C_1, C_2 and C_3 with p, $2p$ and p vertices respectively, such that $V(C_1) \cup V(C_2) \cup V(C_3){=}V(G)$. Let z_1, z_2, ..., z_{2p} be the vertices of C_2 in counter clockwise order such that z_1 has exactly one neighbor on C_1. Let x_1 be the neighbor of z_1 on C_1, and let x_1, x_2, ..., x_p be the vertices of C_1 in the counter clockwise order. Let y_1, y_2, ..., y_p be the vertices on C_3 in counter clockwise order such that y_1 and y_p are the right neighbor and the left neighbor of z_1, respectively. Let D be a straight-line grid drawing of a p-doughnut graph G with linear area [2], as illustrated in Figure 2(a). We partition the edges of D as follows. The *left partition* consists of the edges - (i) (x_1, x_p), (ii) (z_1, z_{2p}), (iii) (y_1, y_p), (iv)(x_1, z_{2p}) and (v) (z_1, y_p); and the *right partition* consists of the edges - (i) (z_p, z_{p+1}), (ii) the edge between the two neighbors of z_p on C_1 if z_p has two neighbors on C_1 otherwise the edge between the two neighbors of z_{p+1} on C_1, (iii) the edge between the two neighbors of z_p on C_3 if z_p has two neighbors on C_3 otherwise the edge between the two neighbors of z_{p+1} on C_3, (iv) the edge between z_p and its right neighbor on C_1 if z_p has two neighbors on C_1 otherwise the edge between z_{p+1} and its left neighbor on C_1, and (v) the edge between z_p and its right neighbor on C_3 if z_p has two neighbors on C_3 otherwise the edge between z_{p+1} and its left neighbor on C_3. The graph G is divided into two connected components if we delete the edges of the left and the right partitions from G. We call the connected component that contains vertex x_p the *top partition* of edges and we call the connected component that contains vertex x_1 the *bottom partition* of edges.

Figure 2(b) illustrates four partitions of edges (indicated by dotted lines) of a p-doughnut graph G in Figure 2(a) where $p = 4$. We now construct a $(p_1 + p_2)$-

doughnut graph G from a p_1-doughnut graph G_1 and a p_2-doughnut graph G_2. We first construct two graphs G_1' and G_2' from G_1 and G_2, respectively, as follows. We partition the edges of G_1 into left, right, top and bottom partitions. Then we identify the vertex x_{i+1} of the top partition to the vertex y_i of the right partition, the vertex z_{p_1+1} of the top partition to the vertex z_{p_1} of the right partition, and the vertex y_{i+1} of the top partition to the vertex x_i of the right partition. Thus we construct G_1' from G_1. Figure 3(c) illustrates G_1' which is constructed from G_1 in Figure 3(a) where $p_1 = 4$. In case of construction of G_2', after partitioning (left, right, top, bottom) the edges of G_2 we identify the vertex y'_{p_2} of left partition to the vertex x'_1 of the bottom partition, vertex z'_{2p_2} of the left partition to the vertex z'_1 of the bottom partition, and the vertex x'_{p_2} of left partition to the vertex y'_1. Figure 3(f) illustrates G_2' which is constructed from G_2 in Figure 3(d) where $p_2 = 5$. We finally construct a (p_1+p_2)-doughnut graph

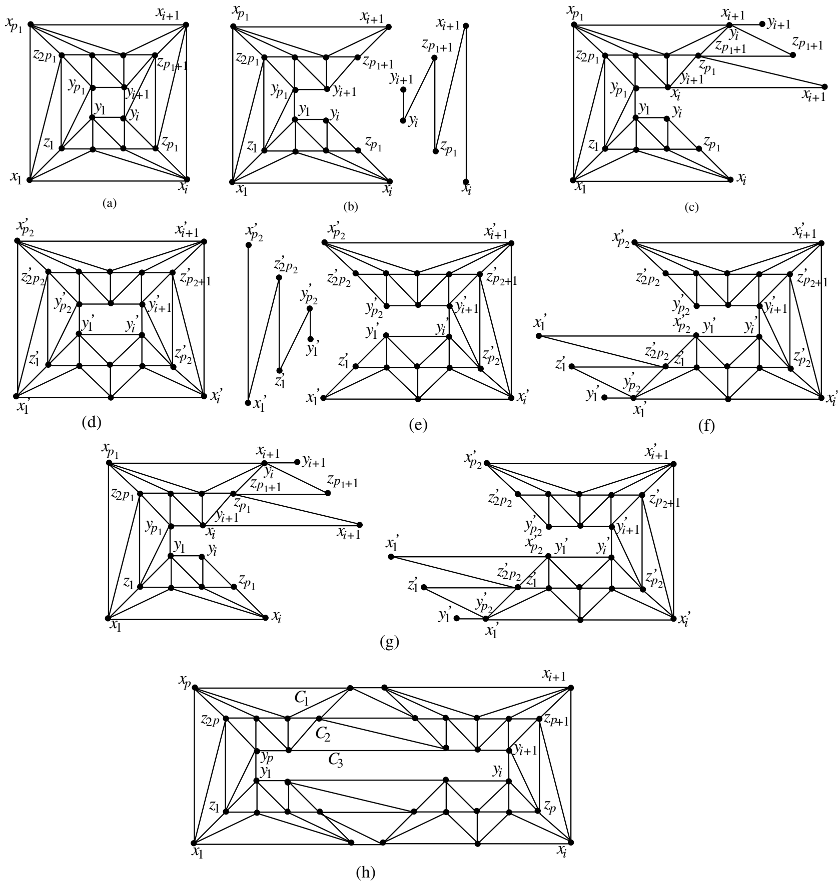

Fig. 3. Illustration for construction of a (p_1+p_2)-doughnut graph G from a p_1-doughnut graph G_1 and a p_2-doughnut graph G_2 where $p_1 = 4$ and $p_2 = 5$

G as follows. We identify the vertices y_{i+1}, z_{p_1+1}, x_{i+1} of G'_1 to the vertices of x'_{p_2}, z'_{2p_2}, y'_{p_2} of G'_2, respectively; and identify the vertices of y_i, z_{p_1}, x_i of G'_1 to the vertices of x'_1, z'_1, y'_1 of G'_2, respectively. Clearly the resulting graph G is a $(p_1 + p_2)$-doughnut graph as illustrated in Fig. 3(h).

We thus have the following theorem.

Theorem 1. *Let G_1 be a p_1-doughnut graph and let G_2 be a p_2-doughnut graph. Then one can construct a $(p_1 + p_2)$-doughnut graph G from G_1 and G_2.*

3 Finding a Shortest Path

A shortest path between any pair of vertices of a doughnut graph can be found efficiently as stated in the following theorem.

Theorem 2. *Let G be a p-doughnut graph. Then a shortest path between any pair of vertices u and v of G can be found in $O(l_s)$ time, where l_s is the length of the shortest path between u and v.*

We have a constructive proof of Theorem 2. The detail is omitted in this extended abstract.

4 Topological Properties of Doughnut Graphs

Let G be a p-doughnut graph. The number of vertices of G is $4p$ where $p(> 3)$ is an integer. A p-doughnut graph is maximal fault tolerant since it is 5-regular. Every p-doughnut graph G has a doughnut embedding Γ where vertices of G lie on three vertex disjoint cycles C_1, C_2 and C_3 such that C_1 is the outer cycle containing p vertices, C_2 is the middle cycle containing $2p$ vertices and C_3 is the inner cycle containing p vertices. Then one can easily see that the diameter of a p-doughnut graph is $\lfloor p/2 \rfloor + 2$. Moreover, a doughnut graph admits a ring embedding since a doughnut graph is Hamilton-connected [5].

Table 1. Topological comparison of doughnut graphs with various Cayley graphs

Topology	number of nodes	diameter	degree	connectivity	Fault tolerance	Hamiltonian
n-cycle	n	$\lfloor n/2 \rfloor$	2	2	maximal	yes
Cube-connected -cycle [6]	$d2^d$	$\lfloor 5d/2 \rfloor - 2$	3	3	maximal	yes
Wrapped around butterfly graph [4]	$d2^d$	$\lfloor 3d/2 \rfloor$	4	4	maximal	yes
d-Dimensional hypercube [1]	2^d	d	d	d	maximal	yes
p-doughnut graphs [2]	$4p$	$\lfloor p/2 \rfloor + 2$	5	5	maximal	yes

5 Conclusion

We have shown that doughnut graphs exhibit recursive structure. We have proposed an efficient algorithm to find shortest path between any pair of vertices which exploit the structure of the graph. We have also found that doughnut graph has smaller diameter, higher degree and connectivity, maximal fault tolerance and ring embedding. There are several parameters like connectivity, degree, diameter, symmetry and fault tolerance which are considered for building interconnection networks [7]. Table 1 presents the topological comparison of various Cayley graphs, which are widely used as interconnection networks, with doughnut graphs. The table shows that topological properties of doughnut graphs are very much similar to interconnection networks. We may have an efficient routing scheme using shortest path finding algorithm.Thus doughnut graphs may find nice applications as interconnection networks.

Acknowledgement. This work is supported by Bangladesh Academy of Sciences.

References

1. Bhuyan, L., Agarwal, D.P.: Generalized hypercube and hyperbus structure for a computer network. IEEE Trans. Comput. 33, 555–566 (1984)
2. Karim, M.R., Rahman, M.S.: Straight-line grid drawings of planar graphs with linear area. In: Proceedings of Asia-Pacific Symposium on Visualisation 2007, pp. 109–112. IEEE (2007)
3. Karim, M.R., Rahman, M.S.: On a class of planar graphs with straight-line grid drawings on linear area. Journal of Graph Algorithms and Applications 13(2), 153–177 (2009)
4. Leighton, F.T.: Introduction to parallel algorithms and architectures: Arrays-trees-hypercubes. Morgan Kaufmann Publishers (1992)
5. Nahiduzaman, K.M., Karim, M.R., Rahman, M.S.: A linear-time algorithm for k-partitioning doughnut graphs. INFOCOMP 8(1), 8–13 (2009)
6. Preparata, F.P., Vuillemin, J.: The cube-connected-cycles: A versatile network for parallel computation. Communications of the ACM 24, 300–309 (1981)
7. Xu, J. (ed.): Topological Structure and Analysis of Interconnection Networks. Kluwer Academic Publishers (2001)