On Antimagic Labeling of Odd Regular Graphs

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Abstract. An antimagic labeling of a finite simple undirected graph with q edges is a bijection from the set of edges to the set of integers $\{1, 2, \cdots, q\}$ such that the vertex sums are pairwise distinct, where the vertex sum at vertex u is the sum of labels of all edges incident to such vertex. A graph is called antimagic if it admits an antimagic labeling. It was conjectured by N. Hartsfield and G. Ringel in 1990 that all connected graphs besides K_2 are antimagic. Another weaker version of the conjecture is every regular graph is antimagic except K_2. Both conjectures remain unsettled so far. In this article, certain classes of regular graphs of odd degree with particular type of perfect matchings are shown to be antimagic. As a byproduct, all generalized Petersen graphs and some subclass of Cayley graphs of $\mathbb{Z}_n$ are antimagic.

Keywords: antimagic labeling, regular graph, perfect matching, 2-factor, generalized Petersen graph, Cayley graph, circulant graph.

1 Introduction

All graphs in this paper are finite simple, undirected, and without loops unless otherwise stated. In 1990, N. Hartsfield and G. Ringel [9] introduced the concepts called antimagic labeling and antimagic graphs.

Definition 1. *For a graph $G = (V, E)$ with q edges and without any isolated vertex, an antimagic edge labeling is a bijection $f : E \to \{1, 2, \cdots, q\}$, such that the induced vertex sum $f^+ : V \to \mathbb{Z}^+$ given by $f^+(u) = \sum\{f(uv) : uv \in E\}$ is injective. A graph is called antimagic if it admits an antimagic labeling.*

N. Hartsfield and G. Ringel showed that paths, cycles, complete graphs K_n $(n \geq 3)$ are antimagic. They conjectured that all connected graphs besides K_2 are antimagic, which remains unsettled. In 2004 N. Alon et al [1] showed that the last conjecture is true for dense graphs. They showed that all graphs with $n(\geq 4)$ vertices and minimum degree $\Omega(\log n)$ are antimagic. They also proved that if G is a graph with $n(\geq 4)$ vertices and the maximum degree $\Delta(G) \geq n-2$, then G is antimagic and all complete partite graphs except K_2 are antimagic. In 2005, T.-M. Wang [15] studied antimagic labeling of sparse graphs, and showed that the toroidal grid graphs are antimagic. In 2008, T.-M. Wang et al. [16]

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showed various types of graph products are antimagic. In 2009, D. Cranston [7] proved that all regular bipartite graphs are antimagic. While many various types of graphs have been shown to be antimagic [2,3,4,5,6,10,11,17,18], the question of antimagic-ness of regular graphs still remains open. In this paper, we consider the antimagic labeling of certain classes of regular graphs with perfect matchings. For more conjectures and open problems on antimagic graphs and related type of graph labeling problems, please see the dynamic survey article of J. Gallian [8].

2 Antimagic Labeling of 3-Regular Graphs

In 2000, M. Miller and M. Bača studied antimagic labelings of arithmetic type for generalized Petersen graphs [2], which are referred as (a,d)-antimagic labelings. Note that (a,d)-antimagic labelings are requiring all vertex sums form an arithmetic progression, hence also antimagic. M. Miller and M. Bača showed (a,d)-antimagic-ness of $GP(n,2)$ for certain n, and also listed conjectures for other generalized Petersen graphs.

In this section we show the generalized Petersen graphs are antimagic by proving a more general theorem regarding 3-regular graphs with a particular type of perfect matchings, which contain generalized Petersen graphs as special cases. A r-factor of a graph is a r-regular spanning subgraph, and a 1-factor is a perfect matching. A factorization of a graph is a decomposition of the graph into union of factors so that the edge set is partitioned.

Theorem 2. *Let G be 3-regular with $2n$ vertices $\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\}$ and $M=\{u_iv_i|\ 1\le i\le n\}$ be a perfect matching of G. Assume additionally that $\{u_1,u_2,\cdots,u_n\}$ and $\{v_1,v_2,\cdots,v_n\}$ induce two 2-regular subgraphs of G respectively. Then G is antimagic.*

Proof. Let $G=M\bigoplus F=M\bigoplus(F_1\cup F_2)$, where the 2-factor F is a disjoint union of two 2-regular subgraphs F_1 and F_2, each with n vertices. Let $V(F_1)=\{u_1,u_2,\cdots,u_n\}$ and $V(F_2)=\{v_1,v_2,\cdots,v_n\}$. Now we give an edge labeling f by the following steps. First we label the edges of M via $f(u_iv_i)=3i$ for each $1\le i\le n$. Then labeling over edges of $F=F_1\cup F_2$ as follows. Since F_1 and F_2 are 2-regular graphs, we assign an orientation so that over each connected component (connected 2-cycle) the flow is either clockwise or counter-clockwise. We labeling over F by setting $f^{out}(w)$ and $f^{in}(w)$ respectively to be the outgoing edge label from the vertex w and the incoming edge label to the vertex w, according to the given orientation. Precisely we give the labeling as follows:

$$f^{out}(u_i)=3n+1-3i, f^{out}(v_i)=3n+2-3i$$

for each $1\le i\le n$. We claim this labeling f is antimagic. Note that the vertex sum $f^+(u_i)$ for each vertex u_i is $f^+(u_i)=f^{out}(u_i)+f(u_iv_i)+f^{in}(u_i)$, which is $(3n+1-3i)+(3i)+f^{in}(u_i)=3n+1+f^{in}(u_i)$ for each $1\le i\le n$. Also note that $f^{in}(u_i)=f^{out}(u_{k_i})$ for a unique k_i, where $1\le k_i\ne i\le n$, and $\{1,\cdots,n\}=\{k_1,\cdots,k_n\}$. Therefore $f^+(u_i)=3n+1+f^{in}(u_i)=6n+2-3k_i$

are pairwise distinct for $1 \leq i \leq n$. Similarly we obtain $f^+(v_i) = 3n+2+f^{in}(v_i)$ which are pairwise distinct for $1 \leq i \leq n$. Then f is antimagic since for each $1 \leq i \leq n$, we see $f^+(u_i) \equiv 2$ $(mod\ 3)$ and $f^+(v_i) \equiv 1$ $(mod\ 3)$. ■

Definition 3. *Let n, k be integers such that $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. The generalized Petersen graph $GP(n,k)$ is defined by $V(GP(n,k)) = \{u_i, v_i | \ 1 \leq i \leq n\}$, and $E(GP(n,k)) = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k} | \ 1 \leq i \leq n\}$ where the subscripts are taken modulo n.* **(See Figure 1.)** *We call $u_1, u_2, \cdots, u_n$ an outer cycle, and $v_1, v_2, \cdots, v_n$ an inner cycle.*

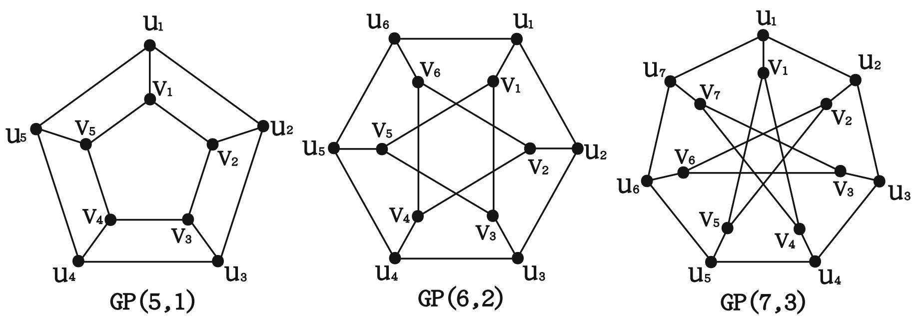

Fig. 1. Examples of generalized Petersen graphs

Note that all generalized Petersen graphs are 3-regular with $2n$ vertices, $3n$ edges, and admitting perfect matchings $\{u_iv_i | \ 1 \leq i \leq n\}$. Obviously $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ induce two 2-regular subgraphs respectively. Therefore, as a byproduct of the above Theorem 2:

Corollary 4. *Every generalized Petersen graph $GP(n,k)$ is antimagic.*

Example 5. *In the following Figure 2 antimagic labelings of $GP(5,2)$ and $GP(6,2)$ are given.*

3 Antimagic Labeling of Odd Regular Graphs

In this section, we extend previous Theorem 2 to a more general situation for regular graphs of odd degree. First we state a result we need here and also in later sections:

Theorem 6. (J. Petersen, 1891) *Let G be a $2r$-regular graph. Then there exists a 2-factor in G.*

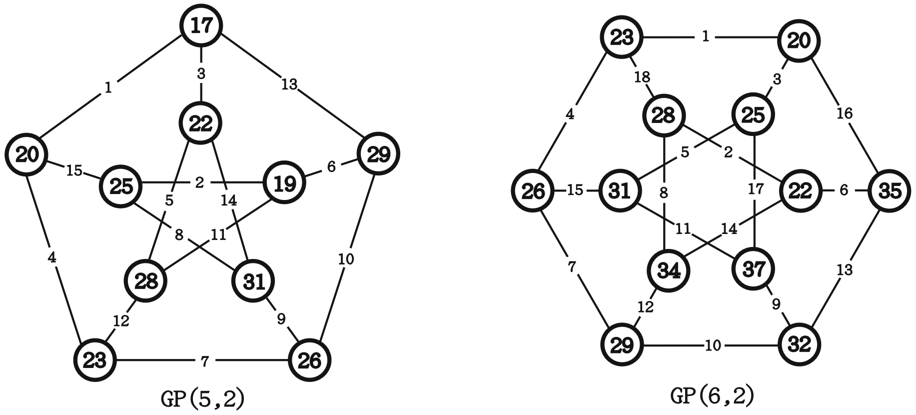

Fig. 2. GP(5,2) and GP(6,2) are antimagic

Notice that after removing edges of the 2-factor by the Petersen Theorem, we will get an even regular graph again and again. Thus an even regular graph has a 2-factorization.

Theorem 7. *Let $r \geq 1$ and let G be a $(2r+1)$-regular graph with $2n$ vertices $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and $M = \{u_i v_i | \ 1 \leq i \leq n\}$ be a perfect matching of G. Assume additionally that $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ induce two $2r$-regular subgraphs of G respectively. Then G is antimagic.*

Proof. Let $G = M \bigoplus (F_1 \cup F_2)$, where F_1 and F_2 are two $2r$-regular subgraphs F_1 and F_2, each induced by n vertices, $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ respectively. Note that by Petersen's Theorem 6, F_1 and F_2 can be factored as unions of 2-factors, say $F_1 = F_1^1 \bigoplus F_1^2 \bigoplus \cdots \bigoplus F_1^r$ and $F_2 = F_2^1 \bigoplus F_2^2 \bigoplus \cdots \bigoplus F_2^r$ respectively, where F_1^j and F_2^k are 2-factors for each $1 \leq j \leq r$ and each $1 \leq k \leq r$ respectively.

Now we give an antimagic labeling f by the following steps. Note that G has $(2r+1)n$ edges. First we split all edge labels $1, 2, \cdots, (2r+1)n$ into $2r+1$ groups as follows: $\{1, 2, \cdots, n\}, \{n+1, n+2, \cdots, 2n\}, \cdots\cdots \{2rn+1, 2rn+2, \cdots, (2r+1)n\}$. Then we will put these groups of labels in order over the edges of F_1^1, F_1^2, $\cdots$, F_1^r, F_2^1, F_2^2, $\cdots$, F_2^r, and M respectively in below.

We define recursively that $G_k = M \bigoplus (F_1^1 \cup F_2^1) \bigoplus \cdots \bigoplus (F_1^k \cup F_2^k)$ for $1 \leq k \leq r$, and it is not hard to see $G = G_r$. Therefore $G_1 = M \bigoplus (F_1^1 \cup F_2^1)$, $G_2 = G_1 \bigoplus (F_1^2 \cup F_2^2)$, $\cdots$, till $G_r = G_{r-1} \bigoplus (F_1^r \cup F_2^r) = G$. Since F_1^j and F_2^k are 2-factors for each $1 \leq j \leq r$ and each $1 \leq k \leq r$ respectively, as before we assign an orientation so that over each connected component (connected 2-cycle) the flow direction is either clockwise or counter-clockwise. We set $f_k^{out}(w)$ and $f_k^{in}(w)$ respectively, for each $1 \leq k \leq r$, to be the outgoing edge label over the 2-factor $(F_1^k \cup F_2^k)$ from the vertex w and the incoming edge label to the vertex w according to the given orientation. On the other hand, we denote $f^+(w)$ to be the induced vertex sum at the vertex w, and we use $f_k^+(w)$ to stand for the partial vertex sum at w for G_k for each $1 \le k \le r$. Then we may start labeling recursively over G_1, G_2, $\cdots$, $G_r = G$.

Precisely we give the labeling in the following steps:

Step 1: For $G_1 = M \bigoplus (F_1^1 \cup F_2^1)$: first for the edges of the perfect matching M we set $f(u_iv_i) = 2rn + i$ for each $1 \le i \le n$. Then over $(F_1^1 \cup F_2^1)$ we set $f_1^{out}(u_i) = 1 + (2r+1)n - f(u_iv_i)$ and $f_1^{out}(v_i) = rn + 1 + (2r+1)n - f(u_iv_i)$ respectively for each $1 \le i \le n$. Therefore $f_1^+(u_i) = f_1^{in}(u_i) + f(u_iv_i) + f_1^{out}(u_i) = 1 + (2r+1)n + f_1^{in}(u_i)$. Also note that $f_1^{out}(u_i) = f_1^{in}(u_j)$ for a unique j, where $1 \le j \ne i \le n$. Therefore $f_1^+(u_i) = 1 + (2r+1)n + f_1^{in}(u_i) = i + 1 + (2r+1)n$, for $1 \le i \le n$, which form a sequence of consecutive integers. Similarly $f_1^+(v_i) = 1 + (3r+1)n + f_1^{in}(v_i) = i + 1 + (3r+1)n$, for $1 \le i \le n$, which form a sequence of consecutive integers.

Step 2: For G_2, G_3, ..., G_r we proceed recursively as follows: For $2 \le k \le r$, over $(F_1^k \cup F_2^k)$ we set $f_k^{out}(u_i) = (2r + 1 + k^2 - k)n + k - f_{k-1}^+(u_iv_i)$ and $f_k^{out}(v_i) = (2kr + r + k^2 - k + 1)n + k - f_{k-1}^+(v_i)$ respectively for each $1 \le i \le n$. Therefore $f_k^+(u_i) = f_k^{in}(u_i) + f_{k-1}^+(u_i) + f_k^{out}(u_i) = (2r+1+k^2-k)n + k + f_k^{in}(u_i)$. Also note that $f_k^{out}(u_i) = f_k^{in}(u_j)$ for a unique j, where $1 \le j \ne i \le n$. Therefore $f_k^+(u_i) = i + k + (2r + k^2)n$ for $1 \le i \le n$, which form a sequence of consecutive integers. Similarly $f_k^+(v_i) = i + k + (2kr + 2r + k^2)n$ for $1 \le i \le n$, which form a sequence of consecutive integers.

Then this labeling f is antimagic, since the vertex sum at the vertex u_i is $f^+(u_i) = i + r + (2r + r^2)n$ for $1 \le i \le n$, and similarly $f^+(v_i) = i + r + (2r + 3r^2)n$ for $1 \le i \le n$, which shows that the vertex sums form a strictly monotone sequence $f^+(u_1) < f^+(u_2) < \cdots < f^+(u_n) < f^+(v_1) < f^+(v_2) < \cdots < f^+(v_n)$. ■

To obtain more examples, we consider the Cayley graphs of $\mathbb{Z}_n$, which are also known as circulant graphs as follows:

Definition 8. *A circulant graph $CIR_n(S)$ with n vertices, with respect to $S \subset \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$, is a graph with the vertex set $V(CIR_n(S)) = \{0, 1, 2, \cdots, n-1\}$, and the edge set is formed by the following rule:*

$$E(CIR_n(S)) = \{ij : \ i - j \equiv \pm s \pmod n, \ s \in S\}.$$

Note that the circulant graph $CIR_n(S)$ is also called a Cayley graph of the finite cyclic group $\mathbb{Z}_n$ generated by S.

Example 9. *Note that for $n \ge 5$, the circulant graphs $CIR_{2n}(\{a, b, n\})$ (where $0 < a \ne b < n$, n odd, and $gcd(2n, a) = gcd(2n, b) = 2$) are 5-regular graphs with perfect matchings, which satisfy the assumption in Theorem 7. Therefore $CIR_{2n}(\{a, b, n\})$ are antimagic. See Figure 3 for the example $CIR_{14}(\{4, 6, 7\})$.*

In a similar fashion, we may construct an infinite class of circulant graphs which represent the class of odd $(2r + 1)$-regular graphs, for each $r \ge 2$, with perfect matchings, as stated in Theorem 7.

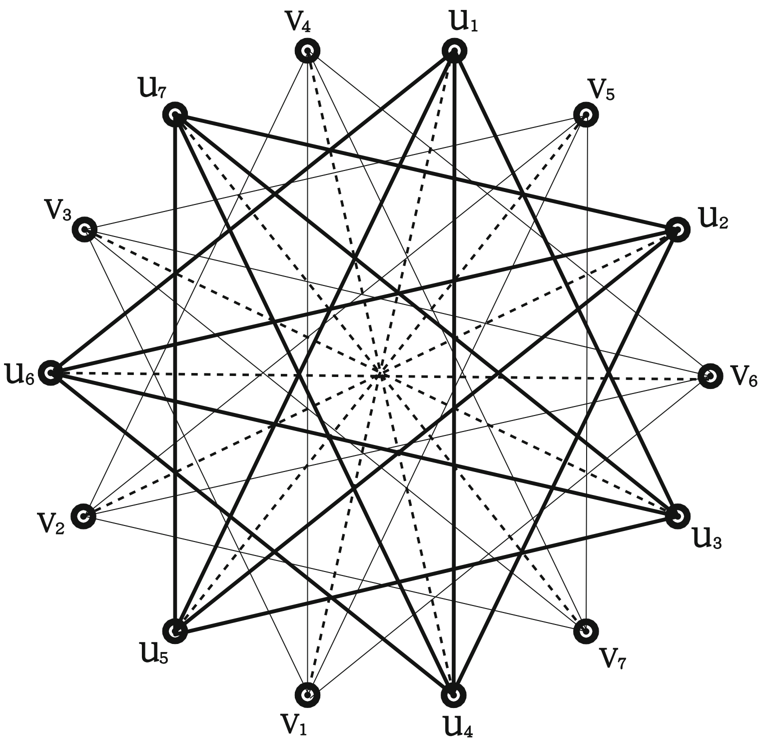

Fig. 3. Circulant graph $CIR_{14}(\{4,6,7\})$ (Cayley graph of $\mathbb{Z}_{14}$ generated by $\{4,6,7\}$)

4 Concluding Remark

In this article, we obtain antimagic labelings of a class of odd regular graphs with particular types of 1-factors, which contain the generalized Petersen graphs and certain circulant graphs as subclasses. Hopefully these results may be helpful to resolve more general situations regrading the conjecture that every regular graph except K_2 is antimagic, or helpful to resolve the Hartsfields-Ringel conjecture that every connected graph except K_2 is antimagic.

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