Incomparability Graphs of Lattices II

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Abstract. In this paper, we study some graphs which are realizable and some which are not realizable as the incomparability graph (denoted by $\Gamma'(L)$) of a lattice L with at least two atoms. We prove that for $n \geq 4$, the complete graph K_n with two horns is realizable as $\Gamma'(L)$. We also show that the complete graph K_3 with three horns emanating from each of the three vertices is not realizable as $\Gamma'(L)$, however it is realizable as the zero-divisor graph of L. Also we give a necessary and sufficient condition for a complete bipartite graph with two horns to be realizable as $\Gamma'(L)$ for some lattice L.

Keywords: Incomparability graph, bipartite graph, horn, double star graph, zero-divisor graph.

1 Introduction

Filipov [5] discuses the comparability graphs of partially ordered sets by defining the adjacency between two elements of a poset by using the comparability relation, that is a, b are adjacent if either $a \leq b$ or $b \leq a$. Duffus and Rival [4] discuss the covering graph of a poset. The papers of Gadenova [6], Bollobas and Rival [2] discuss the properties of covering graphs derived from lattices. Nimbhorkar, Wasadikar and Pawar [10] defined the zero-divisor graphs of a lattice L with 0, by defining the adjacency of two elements $x, y \in L$ by $x \wedge y = 0$.

Also, the concept of the cozero divisor graph of a commutative ring was introduced by M. Afkhami and K. Khashyarmanesh in [1]. Let R be a commutative ring with identity and let $W(R)^*$ be the set of all nonzero and nonunit elements of R. Two distinct vertices a and b in $W(R)^*$ are adjacent if and only if $a \notin bR$ and $b \notin aR$.

Recently, Bresar et al. [3] introduced the cover incomparability graphs of posets and called these graphs as $C - I$ graphs of P. They defined the graph in which the edge set is the union of the edge sets of the corresponding covering graph and the corresponding incomparability graph.

In a lattice L, if a, b are incomparable then we write $a \parallel b$. Let L be a finite lattice and let $W(L) = \{x \mid \text{there exists}\, y \in L\, \text{such that } x \parallel y\}$. The incomparability graph of L, denoted by $\Gamma'(L)$, is a graph with the vertex set

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$W(L)$ and two distinct vertices $a, b \in W(L)$ are adjacent if and only if they are incomparable. Note that $\Gamma'(L)$ does not contain any isolated vertex.

Wasadikar and Survase [12] introduced the incomparability graph of a lattice.

Throughout this paper, L is a finite lattice with at least two atoms.

In this paper, we study some more properties of $\Gamma'(L)$. In section 2 we show that, if G is a graph on five vertices without any isolated vertex then G is realizable as $\Gamma'(L)$ for some lattice L if and only if G is not isomorphic to a member of a set of four graphs. Also we show when the zero-divisor graph and the incomparability graph of a lattice L are isomorphic. In section 3 we show that, the complete graph K_3 with exactly one pendant emanating from all the three vertices is not realizable as the incomparability graph of a lattice. However it is realizable as the zero-divisor graph of a lattice L.

The undefined terms are from West [14], Harary [8] and Gratzer [7].

A graph G is *connected* if there exists a path between any two distinct vertices. A graph G is *complete* if each pair of distinct vertices is joined by an edge. For a positive integer n, we use K_n to denote the complete graph with n vertices. A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent if and only if they belong to different partite sets. The complete bipartite graph is denoted by $K_{m,n}$. A graph in which one vertex is adjacent to every other vertex and no other adjacencies is called a *star graph.* A vertex of a graph G is called a *pendant vertex* if its degree is 1. A graph which is the union of two star graphs whose centers a and b are connected by a single edge is called a *double star* graph.

2 Some Realizable and Non Realizable Graphs

Nimbhorkar, Wasadikar and Pawar [10] associated a zero-divisor graph to a lattice L with 0, whose vertices are the elements of L and two distinct elements are *adjacent* if and only if their meet is 0. Similarly in [11] we define a graph of a lattice L with 0. We say that an element $x \in L$ is a *zero-divisor* if there exists a non zero $y \in L$ such that $x \wedge y = 0$. We denote by $Z(L)$ the set of zero-divisors of L. We associate a graph $\Gamma(L)$ to L with the vertex set $Z^*(L) = Z(L) - \{0\}$, the set of all nonzero zero-divisors of L and distinct $a, b \in Z^*(L)$ are adjacent if and only if $a \wedge b = 0$. We call this graph as the zero-divisor graph of L.

Wasadikar and Survase [12] have shown that all connected graphs with at most four vertices can be realized as $\Gamma'(L)$.

In this section we discuss graphs with five vertices. There are 34 graphs with five vertices (see [8] Appendix 1) out of which 19 are realizable as $\Gamma'(L)$.

Definition 1. *In a lattice L with 0, a nonzero element $a \in L$ is called an atom if there is no $x \in L$ such that $0 < x < a$.*

Lemma 1. If a lattice L contains n atoms, then these atoms induce a K_n in the incomparability graph.

We denote by $A^l = \{x \in L \mid x \leq y \text{ for all}\, y \in A\}$.

The next theorem characterizes which graphs are realizable as the incomparability graph of a lattice.

Theorem 2. Let G be a graph on five vertices without any isolated vertex. Then G is realizable as $\Gamma'(L)$ for some L if and only if G is not isomorphic to any of the four graphs shown in Figures 1 to 4 given below.

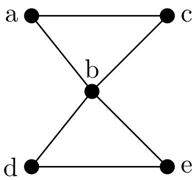


Fig. 1.

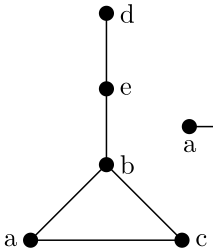


Fig. 2.

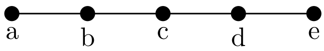


Fig. 3.

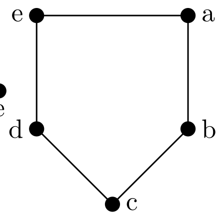


Fig. 4.

Proof. We know that, in a lattice the greatest lower bound of any nonempty finite subset of L exists. Here we show that, the greatest lower bound of some nonempty finite subset of L does not exist.

Consider the Figure 1. Suppose that $G = \Gamma'(L)$ for some lattice L. Since $\Gamma'(L)$ contains a 3 - cycle, L can contain two or three atoms and any two atoms are adjacent in $\Gamma'(L)$. we have the following cases.

Case (i) Suppose, without loss of generality, L has two atoms d, b. We show that $a \wedge c$ does not exist. Since from Figure 1, d and a are comparable and d is an atom hence $d \leq a$. Similarly $d \leq c$. Also a, e are comparable. If $a \leq e$, then $d \leq a$ implies $d \leq e$, a contradiction since d and e are adjacent. Hence $e \leq a$. Similarly $e \leq c$. Thus $\{a, c\}^l = \{0, d, e\}$ but $d \parallel e$ hence $a \wedge c$ does not exist.

Now suppose d, e are the two atoms in L then in a similar manner $\{a, c\}^l = \{0, d, e\}$ but $d \parallel e$. Thus $a \wedge c$ does not exist.

Case (ii) Suppose L has three atoms a, b and c. We show that $d \wedge e$ does not exist. We note that, $\{d, e\}^l = \{0, a, c\}$ but $a \parallel c$ hence $d \wedge e$ does not exist. So Figure 1 cannot be realizable as $\Gamma'(L)$.

Now for Figure 2 Suppose that $G = \Gamma'(L)$ for some lattice L. We have the following cases.

Case (i) Suppose L has two atoms a, b. We show that $d \wedge e$ does not exist. Since a is an atom we have $a \leq d$ and $a \leq e$. From Figure 2, c and d are comparable. If $d \leq c$, then $a \leq d$ implies $a \leq c$, a contradiction since a and c are adjacent. Hence $c \leq d$. Similarly $c \leq e$. Thus $\{d, e\}^l = \{0, a, c\}$ but $a \parallel c$. Hence $d \wedge e$ does not exist.

Suppose e, d are the two atoms in L then $e \leq a$, $e \leq c$ and $d \leq a$, $d \leq c$ that is $\{a, c\}^l = \{0, d, e\}$ but $e \parallel d$. Hence $a \wedge c$ does not exist.

Case (ii) Suppose L has three atoms a, b and c. Then by similar arguments as in the case (ii) of Figure 1, $d \wedge e$ does not exist. So Figure 2 is not realizable as $\Gamma'(L)$.

Consider the Figure 3. Suppose $P_5 = \Gamma'(L)$ for some lattice L. Then by Lemma 1 L has exactly two atoms.

Let b and c be the two atoms. Then we have $b \leq d$, $b \leq e$ and $c \leq a$, $c \leq e$.

Also we have $a \leq d$ or $d \leq a$.

If $a \leq d$, then $c \leq a$ implies $c \leq d$, a contradiction since c and d are adjacent.

If $d \leq a$, then $b \leq d$ implies $b \leq a$, a contradiction since a and b are adjacent.

Hence neither $a \leq d$ nor $d \leq a$, a contradiction since a and d are not adjacent.

Now let d and e be the two atoms in L. We show that $a \wedge b$ does not exist. We note that $\{a, b\}^l = \{0, d, e\}$ but $d \parallel e$ hence $a \wedge b$ does not exist. So the path P_5 cannot be realized as $\Gamma'(L)$.

Consider the Figure 4. Suppose that $G = \Gamma'(L)$ for some lattice L. By Lemma 1 L has exactly two atoms. Let, without any loss of generality, a and b be the two atoms. Then $a \leq c$, $a \leq d$ and $b \leq e$, $b \leq d$.

Also we have $c \leq e$ or $e \leq c$. If $c \leq e$, then $a \leq c$ implies $a \leq e$, a contradiction since a and e are adjacent.

If $e \leq c$, then $b \leq e$ implies $b \leq c$, a contradiction since b and c are adjacent. Neither $c \leq e$ nor $e \leq c$, a contradiction since c and e are nonadjacent. Hence $\Gamma'(L)$ cannot be a 5 - gon.

To show the converse, as mentioned earlier, $\Gamma'(L)$ cannot have any isolated vertex. There are 23 graphs on five vertices without isolated vertices. Hence there are 19 graphs other than the graphs shown in Figures 1 to 4. Each of these 19 graphs is realizable as the incomparability graph of a lattice. These graphs are shown in Figure 5 to Figure 23.

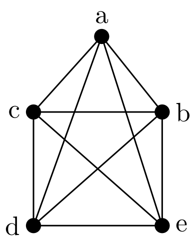


Fig. 5.

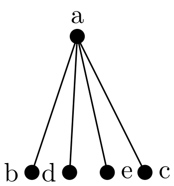
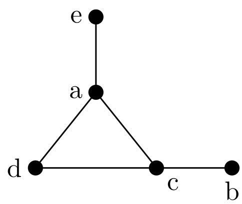


Fig. 6.



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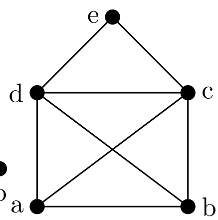

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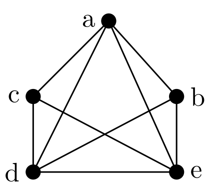


Fig. 9.

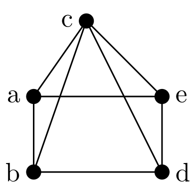
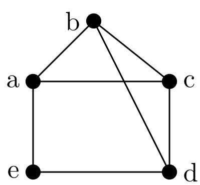


Fig. 10.



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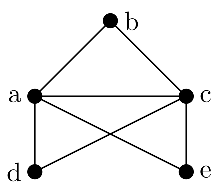


Fig. 12.

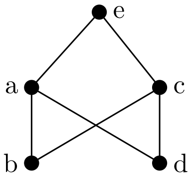


Fig. 13.

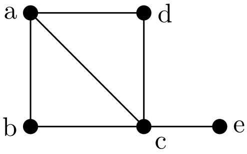
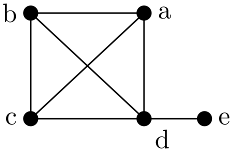


Fig. 14.



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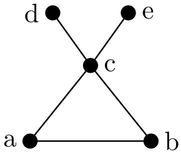

Fig. 16.

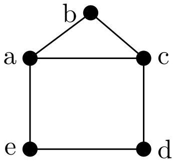

Fig. 17.

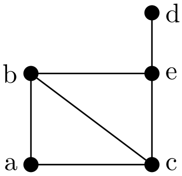

Fig. 18.

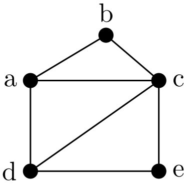

Fig. 19.

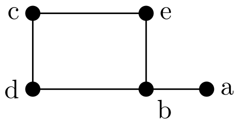

Fig. 20.

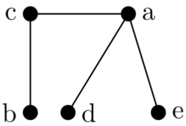

Fig. 21.

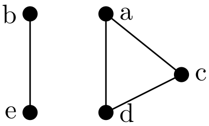

Fig. 22.

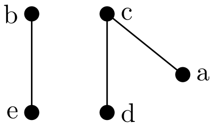

Fig. 23.

The following are examples of lattices corresponding to the above graphs respectively.

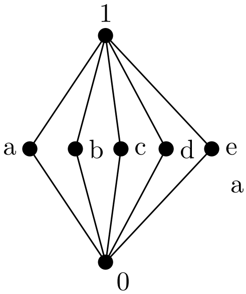

Fig. 5a.

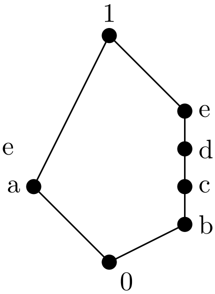

Fig. 6a.

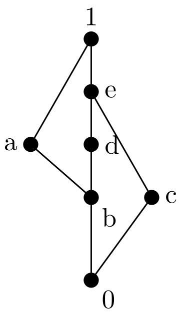

Fig. 7a.

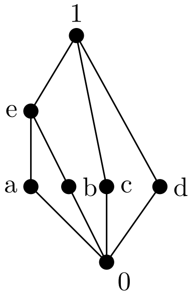

Fig. 8a.

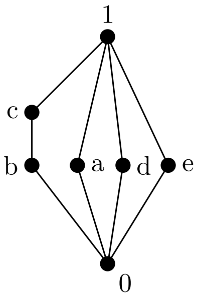

Fig. 9a.

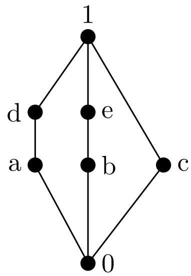

Fig. 10a.

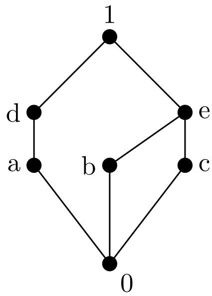

Fig. 11a.

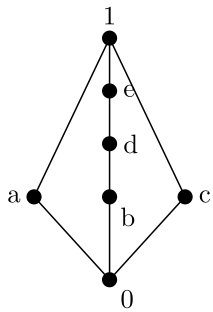

Fig. 12a.

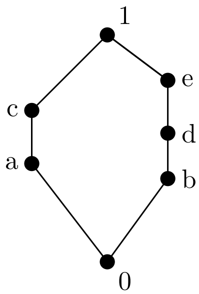

Fig. 13a.

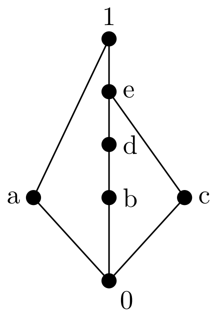

Fig. 14a.

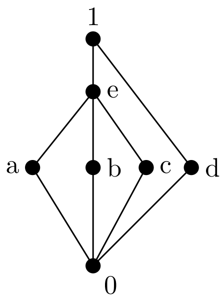

Fig. 15a.

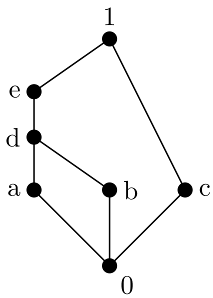

Fig. 16a.

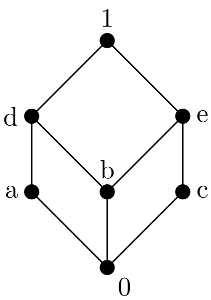

Fig. 17a.

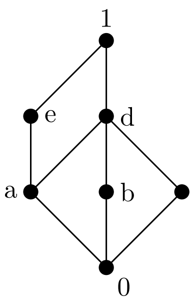

Fig. 18a.

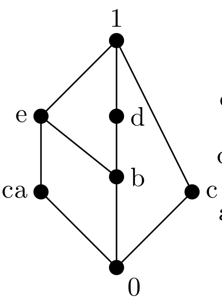

Fig. 19a.

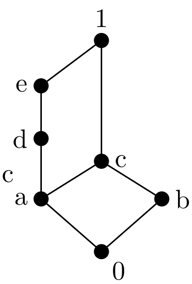

Fig. 20a.

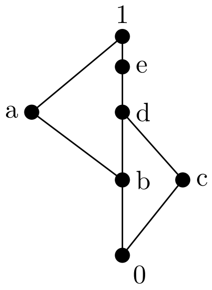

Fig. 21a.

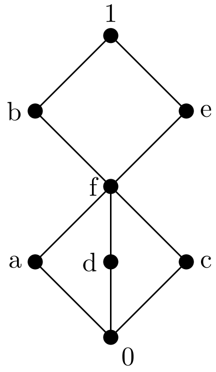

Fig. 22a.

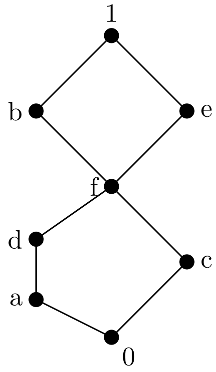

Fig. 23a.

Remark 1. However each graph shown in Figure 1 to Figure 4 can be realized as a subgraph of $\Gamma'(L)$ for some lattice L.

Definition 2. *Let L be a lattice then a non-zero element $a \in L$ is called meet-irreducible if $a = b \wedge c$ implies $a = b$ or $a = c$. Otherwise it is called meet-reducible.*

For example, in Figure 17(a), the elements a, c, d and e are meet-irreducible whereas the element b is meet-reducible.

Theorem 3. The zero-divisor graph and the incomparability graph of a lattice L are isomorphic if and only if L does not contain any meet-reducible element.

Proof. Suppose $\Gamma(L)$ and $\Gamma'(L)$ are isomorphic for some lattice L. We want to show that, L does not contain any meet-reducible element.

Suppose on the contrary L has a meet-reducible element say b then there exist $a, c \in L$ and $a, c \neq b$ such that $b = a \wedge c$. Hence a and c are incomparable. So there is an edge $a - c$ in $\Gamma'(L)$ but $a \wedge c \neq 0$. So a and c are not adjacent in $\Gamma(L)$, a contradiction to assumption that $\Gamma(L)$ and $\Gamma'(L)$ are isomorphic.

Conversely suppose L does not contain any meet-reducible element. We want to show that, $\Gamma(L)$ and $\Gamma'(L)$ are isomorphic. Since L does not contain any meet-reducible element the set of all zero-divisors and the set of all incomparable elements are equal hence $\Gamma(L)$ and $\Gamma'(L)$ are isomorphic.

Theorem 4. The complete graph K_n is realizable as the incomparability graph of a lattice.

Proof. Consider the complete graph K_n. Let a_i, $i = 1, 2, \dots, n$ be the vertices of K_n. The corresponding lattice is as shown in Figure 24.

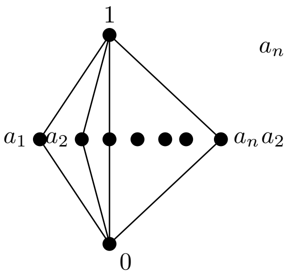

Fig. 24.

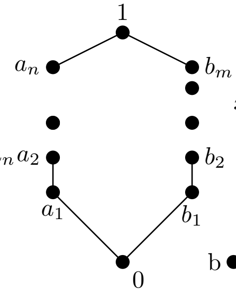

Fig. 25.

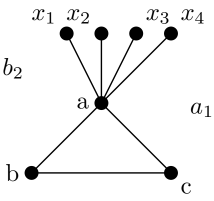

Fig. 26.

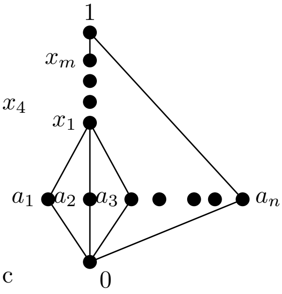

Fig. 27.

Theorem 5. Any complete bipartite graph $K_{m,n}$ is realizable as the incomparability graph of a lattice.

Proof. Consider the complete bipartite graph $K_{m,n}$. Let $V_1 = \{a_1, a_2, \ldots, a_n\}$ and $V_2 = \{b_1, b_2, \ldots, b_m\}$ be the two partitions. The corresponding lattice is as shown in Figure 25.

3 Graphs with Horns

Let G be a graph. All pendant vertices which are adjacent to the same vertex of G together with edges is called a *horn*.

For example, in Figure 26, $X = \{x_1, x_2, x_3, x_4\}$ together with the edges $a - x_1, a - x_2, a - x_3, a - x_4$ is a horn at a, and is denoted as $a - X$.

We denote the complete graph K_n together with m horns $X_1, X_2, \ldots, X_m$ by $K_n(m)$ where $a_1 - X_1, a_2 - X_2, \ldots, a_m - X_m$, $a_i \in V(K_n)$ and $0 \leq m \leq n$.

We note that $K_1(1)$, $K_2(1)$ and $K_2(0)$ are star graphs, $K_2(2)$ is a double star graph.

Theorem 6. The complete graph $K_n(1)$, $n \geq 3$ is realizable as the incomparability graph of a lattice.

Proof. Consider the complete graph K_n. Let X be a horn in K_n at the vertex a_n where $X = \{x_1, x_2, \ldots, x_m\}$ and let a_i, $i = 1, 2, \ldots, n$ be the vertices of K_n. The corresponding lattice is as shown in Figure 27.

Corollary 7. The complete graph $K_3(1)$ is realizable as the incomparability graph of a lattice.

Proof. Consider the complete graph K_3. Let a, b and c be the three vertices of K_3 and let X be horn at c. Let $X = \{x_1, x_2, \ldots, x_n\}$. The corresponding lattice is as shown in Figure 28.

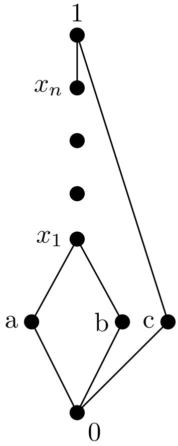

Fig. 28.

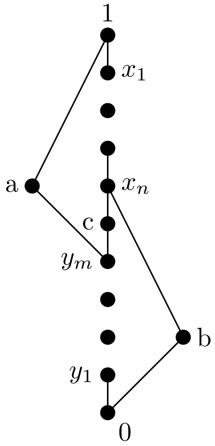

Fig. 29.

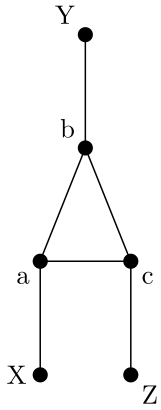

Fig. 30.

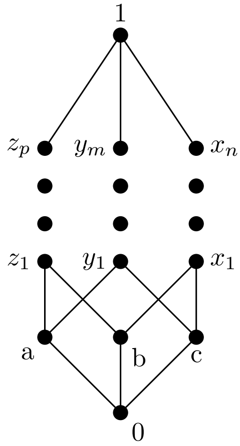

Fig. 31.

Lemma 8. The complete graph $K_3(2)$ is realizable as the incomparability graph of a lattice.

Proof. Consider the complete graph K_3. Let a, b and c be the three vertices of K_3 and let X and Y be horns at a and b respectively. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. The corresponding lattice is as shown in Figure 29.

Theorem 9. The complete graph $K_3(3)$ is not realizable as the incomparability graph of a lattice. However it is realizable as the zero-divisor graph of a lattice L.

Proof. Consider the complete graph K_3. Let a, b and c be the three vertices of K_3 and let X, Y and Z be horns at a, b and c respectively. Let $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_m\}$ and $Z = \{z_1, z_2, \ldots, z_p\}$.

Case (i) Suppose L has two atoms a and b. Then $a \leq z_j$ for $j = 1, 2, \ldots, p$, $a \leq y_k$ for $k = 1, 2, \ldots, m$ and $b \leq x_i$ for $i = 1, 2, \ldots, n$, $b \leq z_j$ for $j = 1, 2, \ldots, p$.

Also we have $y_k \leq x_i$ or $x_i \leq y_k$.

If $y_k \leq x_i$, then $a \leq y_k$ implies $a \leq x_i$, a contradiction since a and x_i are adjacent.

If $x_i \leq y_k$, then $b \leq x_i$ implies $b \leq y_k$, a contradiction since b and y_k are adjacent.

Hence neither $y_k \leq x_i$ nor $x_i \leq y_k$, a contradiction since x_i and y_k are not adjacent.

Suppose a and x_1 are the two atoms in L. We have $a \leq z_j$ for $j = 1, 2, \ldots, p$, $a \leq y_k$ for $k = 1, 2, \ldots, m$ and $x_1 \leq x_i$ for $i = 2, \ldots, n$, $x_1 \leq z_j$ for $j = 1, 2, \ldots, p$, $x_1 \leq b$, $x_1 \leq c$, $x_1 \leq y_k$ for $k = 1, 2, \ldots, m$.

Also we have

(i) $y_k \leq c$ or $c \leq y_k$ (ii) $y_k \leq z_j$ or $z_j \leq y_k$ (iii) $b \leq z_j$ or $z_j \leq b$.

If $y_k \leq c$, then $a \leq y_k$ implies $a \leq c$, a contradiction since a and c are adjacent. Hence $c \leq y_k$.

If $y_k \leq z_j$, then $c \leq y_k$ implies $c \leq z_j$, a contradiction since c and z_j are adjacent. Hence $z_j \leq y_k$.

We have $b \leq z_j$ or $z_j \leq b$.

If $b \leq z_j$, then $z_j \leq y_k$ implies $b \leq y_k$, a contradiction since b and y_k are adjacent.

If $z_j \leq b$, then $a \leq z_j$ implies $a \leq b$, a contradiction since a and b are adjacent.

Hence neither $b \leq z_j$ nor $z_j \leq b$, a contradiction since b and z_j are not adjacent.

Case (ii) Suppose L has three atoms a, b and c. Then we have $a \leq z_j$, $a \leq y_k$, $b \leq x_i$, $b \leq z_j$, $c \leq x_i$ and $c \leq y_k$.

Also we have $y_k \leq x_i$ or $x_i \leq y_k$.

If $y_k \leq x_i$, then $a \leq y_k$ implies $a \leq x_i$, a contradiction since a and x_i are adjacent.

If $x_i \leq y_k$, then $b \leq x_i$ implies $b \leq y_k$, a contradiction since b and y_k are adjacent.

Hence neither $y_k \leq x_i$ nor $x_i \leq y_k$ since x_i and y_k are not adjacent. Hence $K_3(3)$ cannot be realized as $\Gamma'(L)$.

However it is realizable as the zero-divisor graph of a lattice L see Figure 31.

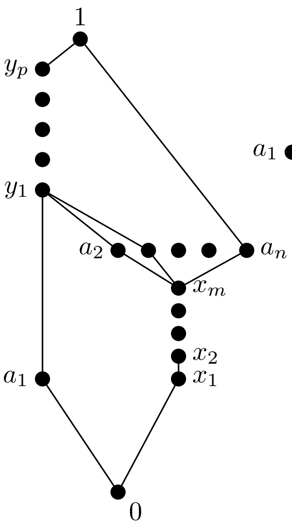

Fig. 32.

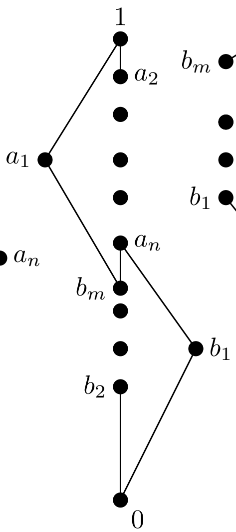

Fig. 33.

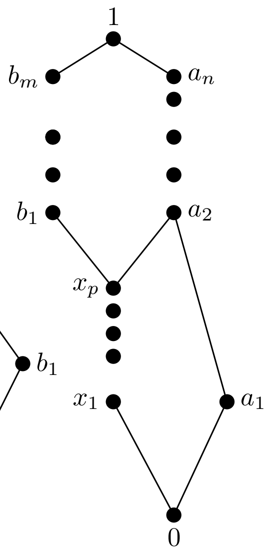

Fig. 34.

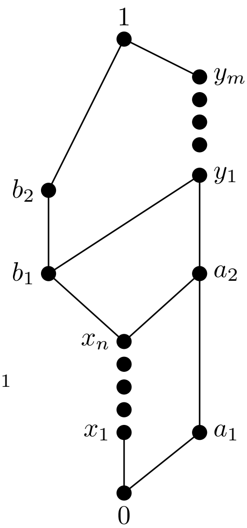

Fig. 35.

Theorem 10. The complete graph $K_n(2)$, $n \geq 4$ is realizable as the incomparability graph of a lattice.

Proof. Consider the complete graph K_n. Let a_i, $i = 1, 2, \ldots, n$ be the vertices of K_n and let X and Y be horns at a_1 and a_n respectively. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_p\}$. The corresponding lattice is as shown in Figure 32.

Theorem 11. A double star graph is realizable as the incomparability graph of a lattice.

Proof. Let $G = \Gamma'(L)$ be a double star graph with centers a_1, b_1 and end vertices b_j, $j = 2, 3, \ldots, m$ and a_i, $i = 2, \ldots, n$. The corresponding lattice is as shown in Figure 33.

Next we discuss some Theorems for complete bipartite graphs with a horn. We denote the complete bipartite graph $K_{m,n}$ together with P horns by $K_{m,n}(P)$.

Remark 2. Let $K_{m,n}$ be the complete bipartite graph with partitions $V_1 = \{a_1, a_2, \ldots, a_n\}$ and $V_2 = \{b_1, b_2, \ldots, b_m\}$. Then by Theorem 5, $K_{m,n}$ is realizable as $\Gamma'(L)$. Since the a_i are non-adjacent in $\Gamma'(L)$, they are comparable in L. So we can arrange them as $a_1 < a_2 < a_3 < \ldots < a_n$. Similarly, we can arrange b_j as $b_1 < b_2 < \ldots < b_m$.

Using this Remark we have the following Theorems.

Theorem 13. $K_{2,2}(2)$ is realizable as $\Gamma'(L)$ if and only if both the horns are at vertices a_1 and b_2.

Proof. Consider the complete bipartite graph $K_{2,2}$. Let $V_1 = \{a_1, a_2\}$ and $V_2 = \{b_1, b_2\}$ be the two partitions. Let $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_m\}$ be the two horns. If the two horns are at a_1 and b_2 respectively as shown in Figure (i), then the corresponding lattice is as shown in Figure 35.

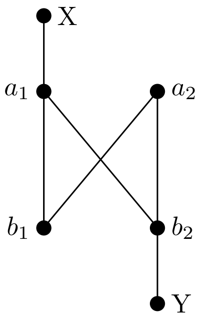

Figure (i)

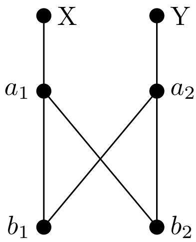

Figure (ii)

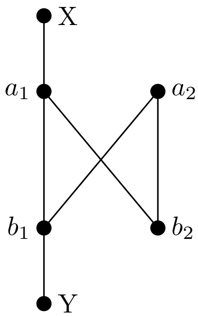

Figure (iii)

Conversely, we consider the two cases.

Case (i) Suppose the horns X and Y are at a_1 and a_2 respectively, see Figure (ii) and let this graph be realizable as $\Gamma'(L)$ for some lattice L. Clearly L does not contain three atoms as $K_{2,2}$ does not contain a 3 - cycle.

Subcase (i) Suppose a_1 and b_1 are the two atoms.

Then $a_1 \leq a_2$, $a_1 \leq y_k$, $b_1 \leq y_k$ for each k and $b_1 \leq x_i$ for each i.

Also we have $a_2 \leq x_1$ or $x_1 \leq a_2$.

If $a_2 \leq x_1$, then $a_1 \leq a_2$ implies $a_1 \leq x_1$, a contradiction since a_1 and x_1 are adjacent.

If $x_1 \leq a_2$, then $b_1 \leq x_1$ implies $b_1 \leq a_2$, a contradiction since a_2 and b_1 are adjacent.

Hence neither $a_2 \leq x_1$ nor $x_1 \leq a_2$, a contradiction since x_1 and a_2 are not adjacent.

Subcase (ii) Suppose a_1 and x_1 are the two atoms. Then $a_1 \leq a_2$, $x_1 \leq b_1$, $x_1 \leq b_2$, $x_1 \leq a_2$, $a_1 \leq y_k$ and $x_1 \leq y_k$ for each k .

We know that, in a lattice the greatest lower bound of any nonempty finite subset of L exists. We now show that the greatest lower bound of
$A = \{a_2, y_1, y_2, \ldots, y_m\}$ does not exist. The possible set of lower bounds of A is $\{0, a_1, x_1, \ldots, x_n\}$. If a_1 is the greatest lower bound, then $x_i \leq a_1$, a contradiction since a_1 is an atom.

If any x_i is the greatest lower bound then $a_1 \leq x_i$, a contradiction since $a_1 \parallel x_i$. Hence the greatest lower bound of A does not exist. So $K_{2,2}$ is not realizable as $\Gamma'(L)$ if both the horns are at vertices a_1 and a_2 respectively.

Case (ii) Suppose both the horns are at vertices a_1 and b_1 respectively see Figure (iii).

Subcase (i) Suppose a_1 and b_1 are the two atoms. Then by similar manner as in case (i) we get a contradiction.

Subcase (ii) a_1 and x_1 are the two atoms. Then $a_1 \leq a_2$, $x_1 \leq b_1$, $x_1 \leq b_2$, $x_1 \leq a_2$, $a_1 \leq y_k$ and $x_1 \leq y_k$ for each k .

By Remark 2 we have $b_1 \leq b_2$.

Also we have $b_2 \leq y_k$ or $y_k \leq b_2$.

If $b_2 \leq y_k$, then $b_1 \leq b_2$ implies $b_1 \leq y_k$, a contradiction since b_1 and y_k are adjacent.

If $y_k \leq b_2$, then $a_1 \leq y_k$ implies $a_1 \leq b_2$, a contradiction since a_1 and b_2 are adjacent.

Hence neither $b_2 \leq y_k$ nor $y_k \leq b_2$, a contradiction since b_2 and y_k are not adjacent.

Hence $K_{2,2}(2)$ is not realizable as $\Gamma'(L)$ if both the horns are at vertices a_1 and b_1 respectively.

Theorem 14. A complete bipartite graph with two horns, that is $K_{m,n}(2)$, $m > 2$ or $n > 2$ is realizable as $\Gamma'(L)$ for some lattice L if and only if the two horns are at vertices a_1, a_n or at vertices a_1, b_m.

Proof. Consider the complete bipartite graph $K_{m,n}$. Suppose, without loss of generality, $n > 2$. Let $V_1 = \{a_1, a_2, \ldots, a_n\}$ and $V_2 = \{b_1, b_2, \ldots, b_m\}$ be the two partitions. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_r\}$ be the two horns. If the horns are at a_1 and a_n respectively then the corresponding lattice is shown in Figure 36. If the horns are at a_1 and b_m respectively then the corresponding lattice is shown in Figure 37.

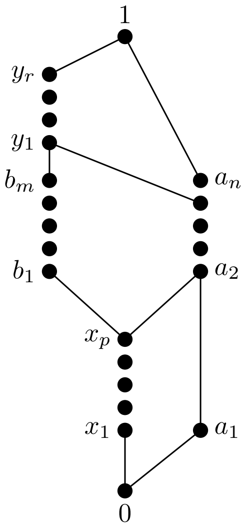

Fig. 36.

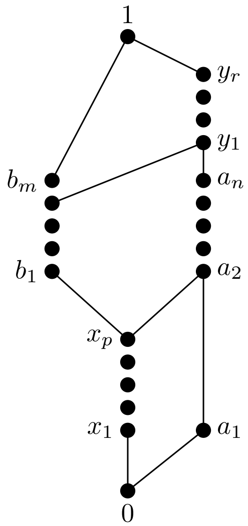

Fig. 37.

Conversely consider the complete bipartite graph $K_{m,n}$ and let both the horns be at vertices from the same partite set say V_1.

We have $V_1 = \{a_1, a_2, \ldots, a_n\}$and $V_2 = \{b_1, b_2, \ldots, b_m\}$. Let X and Y be the two horns at a_1 and a_i, $i \neq n$ respectively where $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_r\}$. Let this graph be realizable as $\Gamma'(L)$ for some lattice L. Clearly L does not contain three atoms as $K_{m,n}$ does not contain a 3 - cycle.

Case (i) Suppose a_1 and b_1 are the two atoms. We have $a_1 \leq y_j$ for $j = 1, 2, \ldots, r$, $b_1 \leq y_j$ for $j = 1, 2, \ldots, r$ and $b_1 \leq x_l$, $l = 1, 2, \ldots, p$.

Also we have $a_2 \leq x_1$ or $x_1 \leq a_2$.

If $a_2 \leq x_1$, then $a_1 \leq a_2$ implies $a_1 \leq x_1$, a contradiction since a_1 and x_1 are adjacent.

If $x_1 \leq a_2$, then $b_1 \leq x_1$ implies $b_1 \leq a_2$, a contradiction since b_1 and a_2 are adjacent.

Hence neither $a_2 \leq x_1$ nor $x_1 \leq a_2$, a contradiction since a_2 and x_1 are not adjacent.

Case (ii) Suppose that a_1 and x_1 are the two atoms. Since $x_1, x_2, \ldots, x_p$ are comparable we can arrange them as $x_1 < x_2 < \ldots < x_p$. Similarly we have $y_1 < y_2 < \ldots < y_r, a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_m$. Now $x_k \leq y_1$ or $y_1 \leq x_k$ for each k. $y_1 \leq x_k$ then $a_1 \leq y_1$ implies $a_1 \leq x_k$, a contradiction. Hence $x_k \leq y_1$ for each k. Thus we have $x_1 < x_2 < \ldots < x_p < y_1 < y_2 < \ldots < y_r$.

Now $y_r \leq a_{i+1}$ or $a_{i+1} \leq y_r$.

If $a_{i+1} \leq y_r$ then $a_i \leq y_r$, a contradiction. Hence $y_r \leq a_{i+1}$. Thus we have the chain $x_1 < x_2 < \ldots < x_p < y_1 < y_2 < \ldots < y_r < a_{i+1} < \ldots < a_n$.

Now for $k \leq i-1$, either $y_j \leq a_k$ or $a_k \leq y_j$ for each j. If $y_j \leq a_k$ then $a_k \leq a_i$ implies $y_j \leq a_i$, a contradiction. Hence $a_k \leq y_j$.

Now since y_1, b_1 are not adjacent, we have $y_1 \leq b_1$ or $b_1 \leq y_1$. If $y_1 \leq b_1$ then $a_2 \leq y_1$ implies $a_2 \leq b_1$, a contradiction since a_2 and b_1 are adjacent.

If $b_1 \leq y_1$ then $y_1 \leq a_{i+1}$ implies $b_1 \leq a_{i+1}$, a contradiction since b_1 and a_{i+1} are adjacent. Hence neither $y_1 \leq b_1$ nor $b_1 \leq y_1$and y_1 and b_1 are not adjacent, a contradiction.

Acknowledgment. The authors are thankful to the referees for many fruitful suggestions for improvement of the paper.

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