Sufficient Condition for $\{C_4, C_{2t}\}$ - Decomposition of $K_{2m,2n}$ – An Improved Bound

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Abstract. In this paper, we have improved the bounds of the sufficient conditions obtained by C.C.Chou and C.M.Fu [J. Comb. Optim. 14, 205-218 (2007)] for the existence of decomposition of complete bipartite graph $K_{2m,2n}$ into cycles of length 4 and $2t$, $t > 2$. Further an algorithm is presented to provide such bound which in turn reduce the number of constructions for the existence of required decomposition.

Keywords: complete bipartite graph, cycle decomposition.

1 Introduction

All the graphs considered here are simple. Let $K_{m,n}$ denotes the complete bipartite graph with part sizes m, n and let C_k denotes the cycle of length k. By a decomposition of a graph G we mean a partition of G into edge-disjoint subgraphs $G_1, \ldots, G_n$ such that $\bigcup_{i=1}^{n} E(G_i) = E(G)$. If each $G_i \cong H$, for all i, then we say that H decomposes G, or G has an $H-$ *decomposition* and we denote it by $H|G$; If $H \cong C_k$, we say that G has a $C_k -$ *decomposition*. If G can be decomposed into p copies of C_{2t} and q copies of C_4 then we say that G has a $\{C_4, C_{2t}\}$ - decomposition and we write $G = p\,C_{2t} \oplus q\,C_4$ where $p, q \in \mathbb{N} \cup \{0\}$, the set of nonnegative integers. For the standard graph-theoretic terminology the reader is referred to [1].

For our convenience, we use some notations as in [3].
Let $D(G) = \{(p,q)|\, G = pC_{2t} \oplus qC_4$ where $p, q \in N \cup \{0\}\}$ and $S_r = \{(p,q)|2tp + 4q = r$ where $p, q \in N \cup \{0\}\}$. It is easy to see that $D(G) \subseteq S_r$ if G has r edges. For the two sets A, $B \subseteq S_r$ we define $A + B = \{(a_1 + b_1, a_2 + b_2)|(a_1, a_2) \in A, (b_1, b_2) \in B\}$ and $rA = A + A + \cdots + A$ (r times). Let U be the set of positive integers and for each $u, v \in U$ and $v \geq u$ we define $K_{u,\,U} = \bigoplus_{v \in U} K_{u,\,v}$, $D(K_{u,\,U}) = \bigcup_{v \in U} D(K_{u,\,v})$ and $S_{uU} = \bigcup_{v \in U} S_{uv}$.

1.1 Program Code

Program 1

The following MATHEMATICA program provides all posible p, q and its corresponding u, v such that $2t\,p + 4\,q = 4uv$, where t is even and $\frac{t}{2} \leq u,\, v < t$.

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```
t = input  even positive integer;
For[u = t/2, u < t, u++,  For[v = u, v < t, v++,
For[p = 0, p <= (4*u*v/2*t), p++, For[q = 0, q <= (u*v), q++,
If[(2*t*p) + (4*q) == (4*u*v),
Print["u=", u,"v=",v, 2*t,"-", p,"4-", q ]
]]]]]
```

Program 2

The following MATHEMATICA program provides required p, q and its corresponding u, v such that $2t\,p + 4\,q = 4uv$, where t is even and $\frac{t}{2} \leq u, v < t$

```
t = input  even positive integer; r = 0;
For[u = t/2, u < t, u++,  For[v = u, v < t, v++,
For[p = r, p <= ((4*u*v)/(2*t)), p++,
For[q = 0, q <= (4*u*v - 2*t*p)/4, q++,
If[((2*t*p) + (4*q)) == (4*u*v),
Print["u=", u,"v=", v, 2*t "-", p,"4-", q]; u = v + 1; v = v;
For[x = u, x < v, x++, For[y = x, y < v, y++,
For[s = 0, s < x*y, s++, If[((2*t*p) + (4*s)) == (4*x*y),
Print["v=", x,"v=", y, 2*t "-", p,"4-", s]; Break[]]];
If[((2*t*p) + (4*s)) == (4*x*y), Break[]]];
If[x == y || x + 1 == y, Break[]]];r += 1; Break[]
]]]]]
```

Program 3

The following MATHEMATICA program provides all posible p, q and its corresponding u, v such that $2t\,p + 4\,q = 4uv$, where t is odd and $\frac{t+1}{2} \leq u, v \leq \frac{3t-1}{2}$.

```
t = input  odd  positive integer;
For[u = ((t + 1)/2), u <= ((3*t - 1)/2), u++,
For[v = u, v <= ((3*t - 1)/2), v++,
For[p = 0, p <= (4*u*v/2*t), p++,
For[q = 0, q <= (u*v), q++,
If[(2*t*p) + (4*q) == (4*u*v),
Print["u=", u,"v=", v,2*t,"-", p,"4-", q ]
]]]]]
```

Program 4

The following MATHEMATICA program provides required p, q and its corresponding u, v such that $2t\,p + 4\,q = 4uv$, where t is odd and $\frac{t+1}{2} \leq u, v \leq \frac{3t-1}{2}$.

```
t = input  odd  positive integer; r = 0;
For[u = (t + 1)/2, u <= (3*t - 1)/2, u++,
For[v = u, v <= (3*t - 1)/2, v++,
For[p = r, p <= ((4*u*v)/(2*t)), p++,
For[q = 0, q <= (4*u*v - 2*t*p)/4, q++,
If[((2*t*p) + (4*q)) == (4*u*v),
Print["u=", u,"v=", v, 2*t "-", p,"4-", q]; u = u + 1; v = v;
For[x = u, x < v, x++, For[y = x, y < v, y++,
For[s = 0, s < x*y, s++, If[((2*t*p) + (4*s)) == (4*x*y),
Print["u=", x,"v=", y,  2*t "-", p,"4-", s]; Break[]]];
If[((2*t*p) + (4*s)) == (4*x*y), Break[]]];
If[x == y || x + 1 == y, Break[]]];
r += 2; Break[] ]]]]]
```

Let $X_t = \{(p,q)|p,q \in \mathbb{N} \cup \{0\}$ obtained from Program 1 $\}$, when t is even and $Y_t = \{(p,q)|p,q \in \mathbb{N} \cup \{0\}$ obtained from Program 3$\}$, when t is odd .

Let $P_t = \{(p,q)|p,q \in \mathbb{N} \cup \{0\}$ obtained from Program 2 $\}$, when t is even and $Q_t = \{(p,q)|p,q \in \mathbb{N} \cup \{0\}$ obtained from Program 4$\}$, when t is odd .

Sotteau [4] has shown that $K_{m,\,n}$ has a C_{2k}-decomposition if and only if (i) $m,n \geq k$ (ii) m and n are even and (iii) $mn \equiv 0 \pmod{2k}$.

C.C.Chou, C.M.Fu and W.C. Huang [2] have shown that G can be decomposed into p copies of C_4, q copies of C_6 and r copies of C_8 for each triple p,q,r of nonnegative integers such that $4p+6q+8r = |E(G)|$, in the following two cases: (a) $G = K_{m,n}$, if $m \geq 4, n \geq 6$, and m,n are even, (b) $G = K_{n,n}$ minus a $1-factor$, if n is odd.

C.C.Chou and C.M.Fu [3] have shown that the existence of $\{C_4, C_{2t}\}-$ decomposition of $K_{2u,\,2v}$, $\frac{t}{2} \leq u,v < t$ (i.e. for all $(p,q) \in X_t$) when t even (respectively $\frac{t+1}{2} \leq u,v \leq \frac{3t-1}{2}$,(i.e. for all $(p,q) \in Y_t$) when t odd) implies such decomposition in $K_{2m,\,2n}$, $m,\, n \geq t$ (respectively in $K_{2m,\,2n}$, $m,\, n \geq \frac{3t+1}{2}$).

In this paper, we show that the existence of $\{C_4, C_{2t}\}-$ decomposition of $K_{2u,\,2v}$, for all $(p,q) \in P_t$ when t even (respectively $(p,q) \in Q_t$ when t odd) implies such decomposition in $K_{2m,\,2n}$, $m,n \geq t$ (respectively in $K_{2m,\,2n}$, $m,n \geq \frac{3t+1}{2}$). Since $P_t \subseteq X_t$ and $Q_t \subseteq Y_t$, our result reduce the bounds given by C.C.Chou and C.M.Fu [3] which in turn reduce the number of constructions for the existence of such decomposition. Further the existence of $\{C_4, C_{2t}\}-$ decomposition of $K_{2u,\,2v}$ was assured by providing constructions for such decomposition in $K_{2u,\,2v}$.

2 $\{C_4, C_{2t}\}-$ Decompositions of $K_{2m,\,2n}$

Before proving our main results, we require the following properties of S_r.

Lemma 1 ([3]). *Let a, b and t be positive integers.*

(i) *If t is even and one of a, b is a multiple of t then $S_{2a} + S_{2b} = S_{2a+2b}$.*
(ii) *If t is odd and one of a, b is a multiple of t then $S_{4a} + S_{4b} = S_{4a+4b}$.*

Lemma 2. *Let $U = \{u \in \mathbb{Z}^+ | \frac{t}{2} \le u < t\}$, and $p, q, s \in \mathbb{Z}^+ \cup \{0\}$, the set of nonnegative integers, where t is even. If $P_t \subseteq D(K_{t,\,2U})$, then for each pair $(p,\, s) \in S_{2tU} \setminus P_t$, there exists a pair $(p,\, q) \in P_t$, $q < s$ such that $(p,\, s) \in D(K_{t,\,2U})$.*

Proof. Let $(p,\, q) \in P_t$ and $P_t \subseteq D(K_{t,\,2U})$. Then $K_{t,\,2u} = p\ C_{2t} \oplus q\ C_4$ for a positive integer $u \in U$ and hence $(p,\, q) \in S_{2tu}$. Suppose $(p,\, s) \in S_{2tU} \setminus P_t$ and $q < s$, i.e. $(p,\, s) \in S_{2tv}$, for a positive integer $v \neq u \in U$ then $s - q = \frac{t(v-u)}{2}$. We decompose $K_{t,2v}$ as follows $K_{t,2v} \cong K_{t,2u} \oplus K_{t,2(v-u)} \cong K_{t,2u} \oplus \dfrac{t(v-u)}{2} K_{2,2} \cong p\,C_{2t} \oplus s\,C_4$. Thus $(p, s) \in D(K_{t,\,2v})$, therefore $S_{2tU} \setminus P_t \subseteq D(K_{t,\,2U})$. Hence $D(K_{t,\,2U}) = S_{2tU}$. □

Lemma 3. *Let p be positive integer and let U be as defined in Lemma 2. If t is even and $P_t \subseteq D(K_{t,\,2U})$, then $D(K_{t,\,2p}) = S_{2tp}$ for all $p \ge \frac{3t+1}{2}$.*

Proof. Since t is even and $2p \ge 3t+1$, there is a nonnegative integer r such that $2p = rt + 2u$, $\frac{t}{2} \le u < t$. Therefore we can decompose $K_{t,\,2p}$ into $r\,K_{t,\,t}$ and $K_{t,\,2u}$ i.e. $K_{t,\,2p} \cong rK_{t,\,t} \oplus K_{t,\,2u}$. By the hypothesis, $P_t \subseteq D(K_{t,\,2U})$. Then by Lemmas 1 and 2, we have $D(K_{t,\,2p}) \supseteq r\,D(K_{t,\,t}) + D(K_{t,\,2u}) = r\,S_{t^2} + S_{2tu} = S_{2tp}$. Therefore $D(K_{t,\,2p}) = S_{2tp}$. □

Theorem 1. *Let m, n, u and v be positive integers and let U be defined as in Lemma 2. If t is even and $P_t \subseteq \bigcup_{u,\,v \in U} D(K_{2u,\,2v})$ then $D(K_{2m,2n}) = S_{4mn}$ for all $m, n \ge t$.*

Proof. For $2m$, $2n \ge t$, we can decompose $K_{2m,\,2n}$ as follows: $K_{2m,\,2n} \cong K_{2m-t,\,2n-t} \oplus K_{2m-t,\,t} \oplus K_{t,\,2n}$. $D(K_{2m,\,2n}) \supseteq D(K_{2m-t,\,2n-t}) + D(K_{2m-t,\,t}) + D(K_{t,\,2n})$. By the hypothesis, $D(K_{2m-t,\,2n-t}) = S_{(2m-t)(2n-t)}$. By Lemmas 2 and 3 we have $D(K_{2m-t,t}) = S_{t(2m-t)}$ and $D(K_{t,2n}) = S_{2nt}$. By Lemma 1 and the hypothesis, we have $D(K_{2m,2n}) \supseteq S_{(2m-t)(2n-t)} + S_{(2m-t)t} + S_{2nt} = S_{4mn}$. Thus $D(K_{2m,2n}) = S_{4mn}$. □

Lemma 4. *Let $V = \{u \in \mathbb{Z}^+ | \frac{t+1}{2} \le u \le \frac{3t-1}{2}\}$, and $p, q, s \in \mathbb{Z}^+ \cup \{0\}$, the set of nonnegative integers where t is odd. If $Q_t \subseteq D(K_{2t,\,2V})$, then for each pair $(p,\, s) \in S_{4tV} \setminus Q_t$, there exists a pair $(p,\, q) \in Q_t$, $q < s$ such that $(p,\, s) \in D(K_{2t,\,2V})$.*

Proof. Let $(p,\, q) \in Q_t$ and $Q_t \subseteq D(K_{2t,\,2V})$. Then $K_{2t,\,2u} = p\ C_{2t} \oplus q\ C_4$ for a positive integer $u \in V$ and hence $(p,\, q) \in S_{4tu}$. Suppose $(p,\, s) \in S_{4tV} \setminus Q_t$ and $q < s$, i.e. $(p,\, s) \in S_{4tv}$, for a positive integer $v \neq u \in V$ then $s - q = t(v-u)$. We decompose $K_{2t,2v}$ as follows $K_{2t,\,2v} \cong K_{2t,\,2u} \oplus K_{2t,\,2(v-u)} \cong K_{2t,\,2u} \oplus t(v-u)K_{2,\,2} \cong p\,C_{2t} \oplus s\,C_4$. Thus $(p, s) \in D(K_{2t,\,2v})$, therefore $S_{4tV} \setminus Q_t \subseteq D(K_{2t,\,2V})$. Hence $D(K_{2t,\,2V}) = S_{4tV}$. □

Lemma 5. *Let p be positive integer and let V be defined as in Lemma 4. If t is odd and $Q_t \subseteq D(K_{2t,\,2V})$, then $D(K_{2t,\,2p}) = S_{4tp}$ for all $p \ge \frac{3t+1}{2}$.*

Proof. Since t is odd and $2p \geq 3t+1$, there is a nonnegative integer r such that $2p = 2rt + 2u$, $\frac{t+1}{2} \leq u \leq \frac{3t-1}{2}$. Therefore we can decompose $K_{2t,\,2p}$ into $r\,K_{2t,\,2t}$ and $K_{2t,\,2u}$ i.e. $K_{2t,\,2p} \cong rK_{2t,\,2t} \oplus K_{2t,\,2u}$. By the hypothesis, $Q_t \subseteq D(K_{2t,\,2U})$. Then by Lemmas 1 and 2, we have $D(K_{2t,\,2p}) \supseteq r\,D(K_{2t,\,2t}) + D(K_{2t,\,2u}) = r\,S_{4t^2} + S_{4tu} = S_{4tp}$. Therefore $D(K_{2t,\,2p}) = S_{4tp}$. □

Theorem 2. *Let m, n, u and v be positive integers and let V be defined as in Lemma 4. If t is odd and $Q_t \subseteq \bigcup_{u,\,v \in V} D(K_{2u,\,2v})$, then $D(K_{2m,2n}) = S_{4mn}$ for all $m, n \geq \frac{3t+1}{2}$.*

Proof. For $2m$, $2n \geq 3t+1$, we can decompose $K_{2m,\,2n}$ as follows: $K_{2m,\,2n} \cong K_{2m-2t,\,2n-2t} \oplus K_{2m-2t,\,2t} \oplus K_{2t,\,2n-2t} \oplus K_{2t,\,2t}$. $D(K_{2m,\,2n}) \supseteq D(K_{2m-2t,\,2n-2t}) + D(K_{2m-2t,\,2t}) + D(K_{2t,\,2n-2t}) + D(K_{2t,\,2t})$. By the hypothesis, $D(K_{2m-2t,\,2n-2t}) = S_{(2m-2t)(2n-2t)}$. By Lemmas 4 and 5 we have $D(K_{2m-2t,2t}) = S_{2t(2m-2t)}$, $D(K_{2t,2n-2t}) = S_{2t(2n-2t)}$ and $D(K_{2t,2t}) = S_{4t^2}$. By Lemma 1 and the hypothesis, we have $D(K_{2m,2n}) \supseteq S_{(2m-2t)(2n-2t)} + S_{(2m-2t)2t} + S_{2t(2n-2t)} + S_{4t^2} = S_{4mn}$. Thus $D(K_{2m,2n}) = S_{4mn}$. □

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