

Fuzzy Implications: Some Recently Solved Problems

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Abstract. In this chapter we discuss some open problems related to fuzzy implications, which have either been completely solved or those for which partial answers are known. In fact, this chapter also contains the answer for one of the open problems, which is hitherto unpublished. The recently solved problems are so chosen to reflect the importance of the problem or the significance of the solution. Finally, some other problems that still remain unsolved are stated for quick reference.

1 Introduction

Fuzzy implications are a generalization of the classical implication. That they form an important class of fuzzy logic connectives is clear from the fact this is the second such monograph to be exclusively devoted to them. Despite the extensive research on these operations, a few problems have remained astutely unyielding - the so called "Open Problems". A list of open problems is a particularly important source of motivation, since by exposing the inadequacies of the tools currently available, it propels the researchers towards the creation of tools or approaches that further the advancement of the topic.

In this chapter, we discuss a few of the well-known problems that have been solved, either totally or partially, since the publication of our earlier monograph [2]. The problems we deal with have not been chosen with any particular bias. However, it could be said that the choice has been dictated either based on the importance

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of the problem or the significance of the solution. This choice can also be broadly classified into two types, viz., the recently solved problems that relate to

- (i) Interrelationships between the properties of fuzzy implications (Problems 1–3, Section 2),
- (ii) Properties or characterizations of specific families of fuzzy implications (Problems 4–8, Section 3).

Finally, in Section 4, we also list some open problems that are yet to be solved.

2 Fuzzy Implications: Properties and Their Interrelationships

In the literature, especially in the beginning, we can find several different definitions of fuzzy implications. In this chapter we will use the following one, which is equivalent to the definition proposed by KITAINIK [25] (see also FODOR, ROUBENS [14] and BACZYŃSKI, JAYARAM [2]).

Definition 2.1. A function $I: [0, 1]^2 \rightarrow [0, 1]$ is called a fuzzy implication if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, the following conditions:

$$\text{if } x_1 \leq x_2, \text{ then } I(x_1, y) \geq I(x_2, y), \text{ i.e., } I(\cdot, y) \text{ is non-increasing,} \quad (I1)$$

$$\text{if } y_1 \leq y_2, \text{ then } I(x, y_1) \leq I(x, y_2), \text{ i.e., } I(x, \cdot) \text{ is non-decreasing,} \quad (I2)$$

$$I(0, 0) = 1, \quad (I3)$$

$$I(1, 1) = 1, \quad (I4)$$

$$I(1, 0) = 0. \quad (I5)$$

While the above definition of a fuzzy implication is more or less accepted as the standard definition generalizing the classical implication operation, not all fuzzy implications possess many of the desirable properties satisfied by the classical implication on $\{0, 1\}^2$ to $\{0, 1\}$. Earlier definitions of fuzzy implications, assumed many of these desirable properties as part of the definition itself. For instance, TRILLAS and VALVERDE [40] also assumed the exchange principle (EP) (see Definition 2.2 below) as part of their definition of a fuzzy implication. Thus the study of the interrelationships between these properties is both interesting and imperative.

Various such properties of fuzzy implications were postulated in many works (see TRILLAS and VALVERDE [40]; DUBOIS and PRADE [11]; SMETS and MAGREZ [38]; FODOR and ROUBENS [14]; GOTTWALD [15]). The most important of them are presented below.

Definition 2.2. A fuzzy implication I is said to satisfy

- (i) the exchange principle, if

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1]; \quad (EP)$$

(ii) the ordering property, if

$$I(x, y) = 1 \iff x \leq y, \quad x, y \in [0, 1]. \quad (\text{OP})$$

The property (EP) is the generalization of the classical tautology known as the exchange principle:

$$p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \rightarrow r).$$

The ordering property (OP), called also the degree ranking property, imposes an ordering on the underlying set $[0, 1]$.

2.1 Are (EP) and (OP) Sufficient?

We start with the following lemma which shows that the exchange principle (EP) together with the ordering property (OP) are strong conditions.

Lemma 2.3 (cf. [14, Lemma 1.3]). *If a function $I: [0, 1]^2 \rightarrow [0, 1]$ satisfies (EP) and (OP), then I satisfies (I1), (I3), (I4) and (I5).*

The above result shows that (EP) and (OP) force any function $I: [0, 1]^2 \rightarrow [0, 1]$ to be almost a fuzzy implication. The only missing property of an I satisfying (EP) and (OP) is that of (I2). However, for long, the only examples of an $I: [0, 1]^2 \rightarrow [0, 1]$ with (EP) and (OP) that satisfied (I2) was also right-continuous in the second variable. This led to the following conjecture:

Solved Problem 1 ([2, Problem 2.7.2]). *Prove or disprove by giving a counter example:*

Let $I: [0, 1]^2 \rightarrow [0, 1]$ be any function that satisfies both (EP) and (OP). Then the following statements are equivalent:

- (i) I satisfies (I2).
- (ii) I is right-continuous in the second variable.

One can also trace the origin of the above open problem from a different but related topic. It also arises from the characterization studies of the family of R-implications (see Definition 3.1 below and the discussion in Section 3.1).

ŁUKASIK in [31] presented two examples (see Table 1) which finally show that the above properties are independent from each other.

2.2 Fuzzy Implication and Different Laws of Contraposition

One of the most important tautologies in the classical two-valued logic is the law of contraposition:

Table 1 The mutual independence for Problem 1

Function F	(I2)	(EP)	(OP)	Right-continuity
$F(x,y) = \begin{cases} 1, & \text{if } 0 \leq x \leq y \leq 1 \\ 1-x+y, & \text{if } 0 < y < x \leq 1 \\ 0, & \text{if } x > 0 \text{ and } y = 0 \end{cases}$	✓	✓	✓	×
$F(x,y) = \begin{cases} 1, & \text{if } 0 \leq x \leq y \leq 1 \\ \frac{1-x-3y}{1-4y}, & \text{if } 0 \leq y < x < \frac{1}{4} \\ \frac{3}{4}, & \text{if } 0 \leq y < \frac{1}{4} \leq x \leq \frac{3}{4} \\ (4y-1)x+1-3y, & \text{if } 0 \leq y < \frac{1}{4} \text{ and } \frac{3}{4} < x \leq 1 \\ y, & \text{if } \frac{1}{4} \leq y < \frac{3}{4} \text{ and } y < x \leq 1 \\ \frac{3x+y-3}{4x-3}, & \text{if } \frac{3}{4} \leq y < x \leq 1 \end{cases}$	×	✓	✓	✓

$$p \rightarrow q \equiv \neg q \rightarrow \neg p,$$

which is necessary to prove many results by contradiction. Its natural generalization to fuzzy logic is based on fuzzy negations and fuzzy implications. In fuzzy logic, contrapositive symmetry of a fuzzy implication I with respect to a fuzzy negation N (see Definition 2.4 below) plays an important role in the applications of fuzzy implications, viz., approximate reasoning, deductive systems, decision support systems, formal methods of proof, etc. (cf. [13] and [23]). Since the classical negation satisfies the law of double negation, the following laws are also tautologies in the classical logic

$$\neg p \rightarrow q \equiv \neg q \rightarrow p,$$

$$p \rightarrow \neg q \equiv q \rightarrow \neg p.$$

Consequently we can consider different laws of contraposition in fuzzy logic.

Definition 2.4 (see [14, p. 3], [27, Definition 11.3], [15, Definition 5.2.1]). A non-increasing function $N: [0, 1] \rightarrow [0, 1]$ is called a fuzzy negation if $N(0) = 1, N(1) = 0$. A fuzzy negation N is called

- (i) strict if it is strictly decreasing and continuous;
- (ii) strong if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0, 1]$.

Example 2.5. The classical negation $N_C(x) = 1 - x$ is a strong negation, while $N_K(x) = 1 - x^2$ is only strict, whereas N_{D1} and N_{D2} - which are the least and largest fuzzy negations - are non-strict negations:

$$N_{D1}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0, \end{cases} \quad N_{D2}(x) = \begin{cases} 1, & \text{if } x < 1, \\ 0, & \text{if } x = 1. \end{cases}$$

Definition 2.6. Let I be a fuzzy implication and N be a fuzzy negation.

- (i) We say that I satisfies the law of contraposition (or in other words, the contrapositive symmetry) with respect to N , if

$$I(x, y) = I(N(y), N(x)), \quad x, y \in [0, 1]. \tag{CP}$$

- (ii) We say that I satisfies the law of left contraposition with respect to N , if

$$I(N(x), y) = I(N(y), x), \quad x, y \in [0, 1]. \tag{L-CP}$$

- (iii) We say that I satisfies the law of right contraposition with respect to N , if

$$I(x, N(y)) = I(y, N(x)), \quad x, y \in [0, 1]. \tag{R-CP}$$

If I satisfies the (left, right) contrapositive symmetry with respect to N , then we also denote this by CP(N) (respectively, by L-CP(N), R-CP(N)).

Firstly, we can easily observe that all the three properties are equivalent when N is a strong negation (see [2, Proposition 1.5.3]). Moreover we have the following result.

Proposition 2.7 ([2, Proposition 1.5.2]). *If $I: [0, 1]^2 \rightarrow [0, 1]$ is any function and N is a strict negation, then the following statements are equivalent:*

- (i) I satisfies L-CP with respect to N .
- (ii) I satisfies R-CP with respect to N^{-1} .

The classical law of contraposition (CP) has been studied by many authors (cf. TRILLAS and VALVERDE [40], DUBOIS and PRADE [11], FODOR [13]). It should be noted that in general it is required for N to be a strong negation and therefore it is not necessary to consider three different laws of contraposition. On the other hand, when N is only a fuzzy negation with no additional assumptions, then the different laws of contraposition may not be equivalent. In fact, only the following was known at the time of publication of [2], see Table 1.8 therein:

Table 2 Fuzzy implications and laws of contraposition

Fuzzy implication	(CP)	(L-CP)	(R-CP)
$I(x, y) = \begin{cases} \min(1, 1 - x^2 + y), & \text{if } y > 0, \\ 1, & \text{if } x \in [0, 0.25[\text{ and } y = 0, \\ 0.1, & \text{if } x \in [0.25, 0.75[\text{ and } y = 0, \\ 0, & \text{otherwise .} \end{cases}$	×	×	×
$I_{\mathbf{YG}}(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if } x > 0 \text{ or } y > 0 \end{cases}$	×	×	✓
$I(x, y) = \max(\sqrt{1 - x}, y)$	×	✓	✓
$I_{\mathbf{LK}}(x, y) = \min(1, 1 - x + y)$	✓	✓	✓

Please note that the positive cases in Table 2 are satisfied with the natural negation of I defined by $N_I(x) := I(x, 0)$, for all $x \in [0, 1]$. It can be easily observed that Table 2 is not fully complete and the following question naturally arises:

Solved Problem 2 (cf. [2, Problem 1.7.1]). Give examples of fuzzy implications I such that

- (i) I satisfies only CP(N),
- (ii) I satisfies only L-CP(N),
- (iii) I satisfies both CP(N) and L-CP(N) but not R-CP(N),
- (iv) I satisfies both CP(N) and R-CP(N) but not L-CP(N),

with some fuzzy negation N .

BACZYŃSKI and ŁUKASIK [6] analyzed this problem and they found examples for the first two points.

Fuzzy implication	(CP)	(L-CP)	(R-CP)
$I(x,y) = \begin{cases} 0 & \text{if } x = 1 \text{ and } y = 0 \\ \frac{1}{7}, & \text{if } (x,y) \in \{1\} \times]0, \frac{1}{2}] \cup]\frac{1}{2}, 1[\times \{0\} \\ \frac{2}{7}, & \text{if } (x,y) \in]\frac{1}{2}, 1[\times]0, \frac{1}{2}] \\ \frac{3}{7}, & \text{if } (x,y) \in \{1\} \times]\frac{1}{2}, 1[\cup]0, \frac{1}{2}] \times \{0\} \\ \frac{5}{7} - \frac{1}{7}e^{-\frac{2y}{x}}, & \text{if } (x,y) \in]0, \frac{1}{2}]^2 \\ \frac{5}{7} - \frac{1}{7}e^{-\frac{2-2x}{1-y}}, & \text{if } (x,y) \in]\frac{1}{2}, 1[^2 \\ \frac{6}{7}, & \text{if } (x,y) \in]0, \frac{1}{2}] \times]\frac{1}{2}, 1[\\ 1, & \text{if } x = 0 \text{ or } y = 1 \end{cases}$	✓	×	×
$I'_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x = 1 \text{ and } y = 1 \\ (1-x)^{1-y}, & \text{otherwise} \end{cases}$	×	✓	×

Surprisingly, it is easy to show that it is not possible to find examples for next two points.

Proposition 2.8. *If a fuzzy implication I satisfies CP(N) and L-CP(N) with some fuzzy negation N , then I satisfies also R-CP(N).*

Proof. Let us fix arbitrarily $x, y \in [0, 1]$. By (CP), (L-CP) and again by (CP) we get

$$I(x, N(y)) = I(N(N(y)), N(x)) = I(N(N(x)), N(y)) = I(y, N(x)),$$

so I satisfies also (R-CP) with the negation N .

In a similar way we can prove the next result.

Proposition 2.9. *If a fuzzy implication I satisfies CP(N) and R-CP(N) with some fuzzy negation N , then I satisfies also L-CP(N).*

In this way we have completely solved Problem 2.

2.3 The Law of Importation and the Exchange Principle

While the problems discussed so far arise from theoretical considerations, the problem to be discussed in this section stems from its practical significance. One of the desirable properties of a fuzzy implication, other than those listed in previous sections, is the importation law as given below:

$$I(x, I(y, z)) = I(T(x, y), z), \quad x, y, z \in [0, 1]. \quad (\text{LI})$$

where T is a t -norm, i.e., $T: [0, 1]^2 \rightarrow [0, 1]$ is monotonic non-decreasing, commutative, associative with 1 as its identity element.

Fuzzy implications satisfying (LI) have been found extremely useful in fuzzy relational inference mechanisms, since one can obtain an equivalent hierarchical scheme which significantly decreases the computational complexity of the system without compromising on the approximation capability of the inference scheme. For more on this, we refer the readers to the following works [17, 39].

It can be immediately noted that if a fuzzy implication I satisfies (LI) with respect to any t -norm T , by the commutativity of the t -norm T , we have that I satisfies the exchange principle (EP).

The following problem was proposed by the authors during the Eighth FSTA conference which later appeared in the collection of such open problems by KLEMENT and MESIAR [29].

Partially Solved Problem 3 ([29, Problem 8.1]). *Let I be a fuzzy implication.*

- (i) *For a given (continuous) t -norm T , characterize all fuzzy implications which satisfy the law of importation with T , i.e., the pair (I, T) satisfies (LI).*
- (ii) *Since T is commutative, we know that the law of importation implies the exchange principle (EP).*
 - a. *Is the converse also true, i.e., does the exchange principle imply that there exists a t -norm such that the law of importation holds?*
 - b. *If yes, can the t -norm be uniquely determined?*
 - c. *If not, give an example and characterize all fuzzy implications for which such implication is true.*

A first partial answer to the above question, (Problem (ii) (a)), appeared in the monograph [2, Remark 7.3.1]. As the following example shows a fuzzy implication I may satisfy (EP) without satisfying (LI) with respect to any t -norm T . Consider the fuzzy implication

$$I_{\text{LI}}(x, y) = \begin{cases} \min(1 - x, y), & \text{if } \max(1 - x, y) \leq 0.5, \\ \max(1 - x, y), & \text{otherwise.} \end{cases}$$

If indeed there exists a T such that the above I satisfies (LI), then letting $x = 0.7, y = 1, z = 0.4$ we have

$$\begin{aligned} \text{LHS (LI)} &= I_{\mathbf{LI}}(T(0.7, 1), 0.4) = I_{\mathbf{LI}}(0.7, 0.4) = \min(1 - 0.7, 0.4) = 0.3, \\ \text{RHS (LI)} &= I_{\mathbf{LI}}(0.7, I_{\mathbf{LI}}(1, 0.4)) = I_{\mathbf{LI}}(0.7, 0) = 0 \neq 0.3. \end{aligned}$$

Hence $I_{\mathbf{LI}}$ does not satisfy (LI) with any t -norm T .

Further, it was also shown that the t -norm T with which an I satisfies (LI) need not be unique (Problem (ii) (b)). To see this, consider the Weber implication

$$I_{\mathbf{WB}}(x, y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases},$$

which satisfies (LI) with any t -norm T . To see this, let $x, z \in [0, 1]$. If $y = 1$, then $I_{\mathbf{WB}}(T(x, y), z) = I_{\mathbf{WB}}(x, z) = I_{\mathbf{WB}}(x, I(y, z))$. Now, let $y \in [0, 1)$. Since $T(x, y) \leq y < 1$, we have $I_{\mathbf{WB}}(T(x, y), z) = 1$, and so is $I_{\mathbf{WB}}(x, I_{\mathbf{WB}}(y, z)) = I_{\mathbf{WB}}(x, 1) = 1$.

Massanet and Torrens observed that though the fuzzy implication $I_{\mathbf{LI}}$ does not satisfy (LI) with any t -norm T , there exists a conjunctive commutative operator F with which it does satisfy (LI). In fact $I_{\mathbf{LI}}$ satisfies (LI) with the following uninorm

$$U(x, y) = \begin{cases} \min(x, y), & \text{if } x, y \in [0, \frac{1}{2}], \\ \max(x, y), & \text{otherwise.} \end{cases}$$

Thus they have further generalized the above problem in [34]. Note that an $F : [0, 1]^2 \rightarrow [0, 1]$ is said to be conjunctive if $F(1, 0) = 0$.

Definition 2.10. A fuzzy implication is said to satisfy the weak law of importation if there exists a non-decreasing, conjunctive and commutative $F : [0, 1]^2 \rightarrow [0, 1]$ such that

$$I(x, I(y, z)) = I(F(x, y), z), \quad x, y, z \in [0, 1]. \tag{WLI}$$

It is immediately obvious that (LI) implies (WLI) which in turn implies (EP). MASSANET and TORRENS [34] have studied the equivalence of the above 3 properties, which has led to further interesting characterization results.

It is well-known in classical logic that the unary negation operator \neg can be combined with any other binary operator to obtain the rest of the binary operators. This distinction of the unary \neg is also shared by the Boolean implication \rightarrow , if defined in the following usual way:

$$p \rightarrow q \equiv \neg p \vee q.$$

The tautology as given above was the first to catch the attention of the researchers leading to the following class of fuzzy implications.

Definition 2.11 (see [2, Section 2.4]). A function $I : [0, 1]^2 \rightarrow [0, 1]$ is called an (S,N)-implication if there exist a t -conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), y), \quad x, y \in [0, 1]. \tag{1}$$

If N is a strong fuzzy negation, then I is called a strong implication or S -implication. Moreover, if an (S,N) -implication is generated from S and N , then we will often denote this by $I_{S,N}$, while if N is equal to the classical negation N_C , then we will write I_S instead of I_{S,N_C} .

The following characterization of (S,N) -implications from continuous negations can be found in [2].

Theorem 2.12 ([2, Theorem 2.4.10]). *For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following statements are equivalent:*

- (i) I is an (S,N) -implication with a continuous fuzzy negation N .
- (ii) I satisfies (I1), (EP) and the natural negation $N_I = I(x, 0)$ is a continuous fuzzy negation.

Moreover, the representation of (S,N) -implication (1) is unique in this case.

One can easily replace (EP) in the above characterization with either (WLI) or (LI). However, in this case, the mutual independence and the minimality of the properties in the above characterization need to be proven. Note that if an I satisfies (EP) and is such that N_I is continuous still it need not satisfy (I1). We know that (WLI) is stronger than (EP), a fact, that is further emphasized in [34] by the following result which proves that (WLI) and the continuity of N_I imply (I1) of I .

Proposition 2.13 ([34, Proposition 6]). *Let $I: [0, 1]^2 \rightarrow [0, 1]$ be such that it satisfies (WLI) with a non-decreasing, conjunctive and commutative function F and let N_I be continuous. Then I satisfies (I1). Hence I is a fuzzy implication, in fact, an (S,N) -implication.*

Thus we have an alternative characterization of (S,N) -implications.

Theorem 2.14 ([34, Theorem 22]). *For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following statements are equivalent:*

- (i) I is an (S,N) -implication with a continuous fuzzy negation N .
- (ii) I satisfies (WLI) with a non-decreasing, conjunctive and commutative function F and the natural negation N_I is a continuous fuzzy negation.

The following result plays an important role in further analysis of the above equivalences.

Lemma 2.15 (cf. [2, Lemma A.0.6]). *If N is a continuous fuzzy negation, then the function $\mathfrak{N}: [0, 1] \rightarrow [0, 1]$ defined by*

$$\mathfrak{N}(x) = \begin{cases} N^{(-1)}(x), & \text{if } x \in]0, 1], \\ 1, & \text{if } x = 0, \end{cases}$$

is a strictly decreasing fuzzy negation, where $N^{(-1)}$ is the pseudo-inverse of N and is given by

$$N^{(-1)}(x) = \sup\{y \in [0, 1] \mid N(y) > x\}, \quad x \in [0, 1].$$

The next two results point to the equivalence of (WLI) and (LI) when the natural negation N_I of I is continuous.

Proposition 2.16 ([34, Proposition 9]). *An (S,N) -implication obtained from a t -conorm S and a continuous fuzzy negation N satisfies (WLI) with the function $F(x,y) = \mathfrak{N}(S(N(x),N(y)))$, which is non-decreasing, conjunctive and commutative.*

Proposition 2.17 ([34, Proposition 11]). *Let I be an (S,N) -implication obtained from a t -conorm S and a continuous fuzzy negation N . Then I satisfies (LI) with the following t -norm T defined as*

$$T(x,y) = \begin{cases} \mathfrak{N}(S(N(x),N(y))) & \text{if } \max(x,y) < 1, \\ \min(x,y) & \text{if } \max(x,y) = 1. \end{cases}$$

Summarizing the above discussion, we see that two important results emerge. Firstly, we have the following result showing that both (WLI) and (LI) are equivalent in a more general setting.

Theorem 2.18 ([34, Corollary 12]). *For a function $I: [0, 1]^2 \rightarrow [0, 1]$ whose natural negation N_I is continuous, the following statements are equivalent:*

- (i) I satisfies (WLI) with a non-decreasing, conjunctive and commutative function F .
- (ii) I satisfies (LI) with a t -norm T .

Secondly, when I is a fuzzy implication whose natural negation N_I is continuous, then all of (EP), (WLI) and (LI) are equivalent.

Theorem 2.19 ([34, Proposition 13]). *Let I be a fuzzy implication whose natural negation N_I is continuous. Then the following statements are equivalent:*

- (i) I satisfies (EP).
- (ii) I satisfies (WLI) with a non-decreasing, conjunctive and commutative function F .
- (iii) I satisfies (LI) with a t -norm T .

Proof. (i) \implies (ii): From Theorem 2.12 we know that I is an (S,N) -implication obtained from a continuous negation N and Proposition 2.16 implies that I satisfies (WLI) with a non-decreasing, conjunctive and commutative function F .

(ii) \implies (iii): Follows from Theorem 2.18.

(iii) \implies (i): Obvious.

Finally, the following example shows that there exist infinitely many fuzzy implications that satisfy (EP) but do not satisfy (WLI) with any non-decreasing, conjunctive and commutative function F and hence do not satisfy (LI) too.

Example 2.20 ([34, Proposition 14]). Let S be a nilpotent t-conorm, i.e., $S(x, y) = \varphi^{-1}(\min(\varphi(x) + \varphi(y), 1))$ for some increasing bijection $\varphi: [0, 1] \rightarrow [0, 1]$, and N be a strict negation. Let $I: [0, 1]^2 \rightarrow [0, 1]$ be defined as follows:

$$I(x, y) = \begin{cases} 0, & \text{if } y = 0 \text{ and } x \neq 0, \\ S(N(x), y), & \text{otherwise.} \end{cases}$$

Then I is a fuzzy implication but does not satisfy (WLI) with any non-decreasing, conjunctive and commutative function F

3 Families of Fuzzy Implications

As already noted, fuzzy implications were introduced and studied in the literature as the generalization of the classical implication operation that obeys the following truth table:

Table 3 Truth table for the classical implication

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

There are many ways of defining an implication in the Boolean lattice (L, \wedge, \vee, \neg) . Many of these have been generalized to the fuzzy context, i.e., extended as functions on $[0, 1]$ instead of on $\{0, 1\}$. Interestingly, the different definitions are equivalent in the Boolean lattice (L, \wedge, \vee, \neg) . On the other hand, in the fuzzy logic framework, where the truth values can vary in the unit interval $[0, 1]$, the natural generalizations of the above definitions are not equivalent.

In the framework of intuitionistic logic the implication is obtained as the residuum of the conjunction as follows

$$p \rightarrow q \equiv \max\{t \in L \mid p \wedge t \leq q\}, \tag{2}$$

where $p, q \in L$ and the relation \leq is defined in the usual way, i.e., $p \leq q$ iff $p \vee q = q$, for every $p, q \in L$. In fact, (2) is often called as the pseudocomplement of p relative to q (see [7]).

Quite understandably then, one of the most established and well-studied classes of fuzzy implications is the class of R-implications (cf. [11, 14, 15]) that generalizes the definition in (2) to the fuzzy setting.

Definition 3.1. A function $I: [0, 1]^2 \rightarrow [0, 1]$ is called an R-implication, if there exists a t-norm T such that

$$I(x, y) = \sup \{t \in [0, 1] \mid T(x, t) \leq y\}, \quad x, y \in [0, 1], \quad (3)$$

If an R-implication is generated from a t-norm T , then we will often denote it by I_T . Obviously, due to the monotonicity of any t-norm T , if $T(x, y) \leq z$, then necessarily $x \leq I_T(y, z)$. Observe that, for a given t-norm T , the pair (T, I_T) satisfies the adjointness property (also called as residual principle)

$$T(x, z) \leq y \iff I_T(x, y) \geq z, \quad x, y, z \in [0, 1], \quad (\text{RP})$$

if and only if T is left-continuous (see, for instance, the monographs [15, 2]).

Most of the early research on fuzzy implications dealt largely with these families and the properties they satisfied. In fact, still newer families of fuzzy implications are being proposed and the properties they satisfy are explored, see for instance, [35, 37].

3.1 R-implications and the Exchange Principle

From Sections 2.1 and 2.3, it is clear that (EP) and (OP) are perhaps the most important properties of a fuzzy implication both from theoretical and applicational considerations. In fact, the only characterization of R-implications are known for those that are obtained from left-continuous t-norms and both (EP) and (OP) play an important role as the result stated below demonstrates:

Theorem 3.2. For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following statements are equivalent:

- (i) I is an R-implication generated from a left-continuous t-norm.
- (ii) I satisfies (I2), (EP), (OP) and I is right continuous with respect to the second variable.

The above characterization also gave rise to many important questions. Firstly, it is necessary to answer the mutual independence and the minimality of the properties in Theorem 3.2. It is in this context that the problem discussed in Section 2.1 arose. Secondly, can a similar characterization result be obtained for R-implications generated from more general t-norms? In other words, what is the role of the left-continuity of the underlying t-norm vis-à-vis the different properties. Note that since the I considered here is not any general fuzzy implication, but whose representation is known, it is an interesting task to characterize the underlying t-norm T whose residuals satisfy the different properties stated above.

It can be shown that for any t-norm T its residual I_T satisfies (I2), while left-continuity of T is important for I_T to be right continuous with respect to the second variable. Recently, it was shown in [3] that the left-continuity of a t-norm T is not required for its residual to satisfy (OP). In fact, the following result was proven giving the equivalence between a more lenient type of continuity than left-continuity.

Definition 3.3. A function $F: [0, 1]^2 \rightarrow [0, 1]$ is said to be border-continuous, if it is continuous on the boundary of the unit square $[0, 1]^2$, i.e., on the set $[0, 1]^2 \setminus]0, 1[^2$.

Proposition 3.4 ([3, Proposition 5.8], [2, Proposition 2.5.9]). For a t -norm T the following statements are equivalent:

- (i) T is border-continuous.
- (ii) I_T satisfies the ordering property (OP).

However, a similar characterization for the exchange principle, i.e., a characterization of those t -norms whose residuals satisfy (EP) is not known. Note that left-continuity of a t -norm T is sufficient for I_T to satisfy (EP), but is not necessary. Consider the non-left-continuous nilpotent minimum t -norm, which is border-continuous (see [33, p. 851]):

$$T_{\mathbf{nM}^*}(x, y) = \begin{cases} 0, & \text{if } x + y < 1, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

Then the R-implication generated from $T_{\mathbf{nM}^*}$ is the following Fodor implication (Figure 1(a))

$$I_{\mathbf{FD}}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ \max(1 - x, y), & \text{if } x > y, \end{cases}$$

which satisfies both (EP) and (OP). To note that (EP) and (OP) are mutually independent, consider the least t -norm, also called the drastic product, given as follows

$$T_{\mathbf{D}}(x, y) = \begin{cases} 0, & \text{if } x, y \in [0, 1[, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

Observe that it is a non-left-continuous t -norm. The R-implication generated from $T_{\mathbf{D}}$ is given by

$$I_{\mathbf{TD}}(x, y) = \begin{cases} 1, & \text{if } x < 1, \\ y, & \text{if } x = 1. \end{cases}$$

$I_{\mathbf{TD}}$ (see Figure 1(b)) satisfies (EP), but does not satisfy (OP). Thus the following problem appeared in [2].

Partially Solved Problem 4 ([2, Problem 2.7.3]). Give a necessary condition on a t -norm T for the corresponding I_T to satisfy (EP).

Note that the above problem also has relation to **Problem 4.8.1** in [2]. We will discuss this relation in detail after dealing with the solution of the above problem.

Recently, JAYARAM ET AL. [21] have partially solved the above problem for border-continuous t -norms. A complete characterization is not yet available. From the above work it can be seen that the left-continuous completion of a t -norm plays an important role in the solution. In fact, it can be seen that unless a t -norm can be embedded into a left-continuous t -norm, in some rather precise manner as presented in that work, its residual does not satisfy the exchange principle.

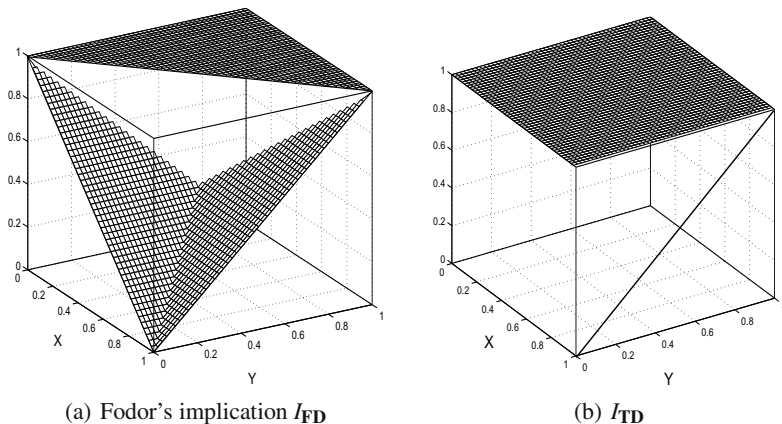


Fig. 1 Plots of I_{FD} and I_{TD} fuzzy implications

3.1.1 Conditionally Left-Continuous Completion

Definition 3.5. Let $F : [0, 1]^2 \rightarrow [0, 1]$ be monotonic non-decreasing and commutative. Then the function $F^* : [0, 1]^2 \rightarrow [0, 1]$ defined as below

$$F^*(x, y) = \begin{cases} \sup\{F(u, v) \mid u < x, v < y\}, & \text{if } x, y \in]0, 1[\\ F(x, y), & \text{otherwise,} \end{cases} \quad x, y \in [0, 1], \quad (4)$$

is called the *conditionally left-continuous completion* of F .

Lemma 3.6. If $F : [0, 1]^2 \rightarrow [0, 1]$ is monotonic non-decreasing and commutative, then the function F^* as defined in (4) is monotonic non-decreasing and commutative.

Proof. By the monotonicity of F we have

$$F^*(x, y) = \begin{cases} F(x^-, y^-), & \text{if } x, y \in]0, 1[\\ F(x, y), & \text{otherwise,} \end{cases}$$

for any $x, y \in [0, 1]$, where the value $F(x^-, y^-)$ denotes the left-hand limit. Clearly, $F^*(x, y) = F^*(y, x)$ and F^* is monotonic non-decreasing.

Remark 3.7. Let T be a t-norm.

- (i) T^* is monotonic non-decreasing, commutative, it has 1 as its neutral element and $T^*(0, 0) = 0$.
- (ii) If T is border-continuous, then T^* is left-continuous (in particular it is also border-continuous).
- (iii) One can easily check that I_{T^*} is a fuzzy implication.
- (iv) By the monotonicity of T we have $T^* \leq T$ and hence $I_{T^*} \geq I_T$.

- (v) If $x \leq y$, then $I_{T^*}(x, y) = I_T(x, y) = 1$.
- (vi) Also, if $x = 1$, then by the neutrality of T we have $I_{T^*}(x, y) = I_T(x, y) = y$.
- (vii) In general T^* may not be left-continuous. For example when $T = T_D$, the drastic t-norm, then $T^* = T$, but T_D is not left-continuous. This explains why T^* is called the *conditionally left-continuous completion* of T . Further, T^* may not satisfy the associativity (see Example 3.8).

Example 3.8 ([41, 42]). Consider the following non-left continuous but border-continuous Vicenik t-norm given by the formula

$$T_{VC}(x, y) = \begin{cases} 0.5, & \text{if } \min(x, y) \geq 0.5 \text{ and } x + y \leq 1.5, \\ \max(x + y - 1, 0), & \text{otherwise.} \end{cases}$$

Then the conditionally left-continuous completion of T_{VC} is given by

$$T_{VC}^*(x, y) = \begin{cases} 0.5, & \text{if } \min(x, y) > 0.5 \text{ and } x + y < 1.5, \\ \max(x + y - 1, 0), & \text{otherwise.} \end{cases}$$

One can easily check that T_{VC}^* is not a t-norm since it is not associative. Indeed, we have

$$T_{VC}^*(0.55, T_{VC}^*(0.95, 0.95)) = 0.5,$$

while

$$T_{VC}^*(T_{VC}^*(0.55, 0.95), 0.95) = 0.45.$$

For the plots of both the functions see Figure 2.

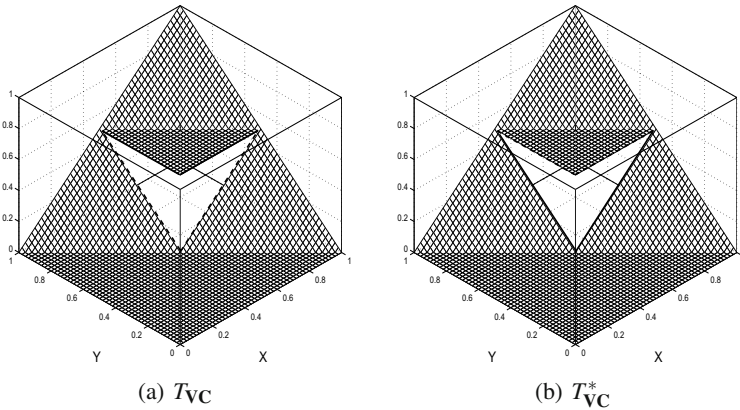


Fig. 2 The Vicenik t-norm T_{VC} and its conditionally left-continuous completion T_{VC}^* (see Example 3.8)

Definition 3.9 (cf. [24, Definition 5.7.2]). A monotonic non-decreasing, commutative and associative function $F: [0, 1]^2 \rightarrow [0, 1]$ is said to satisfy the (CLCC-A)-property, if its conditionally left-continuous completion F^* , as defined by (4), is associative.

3.1.2 Residuals of Border-Continuous T-norms and (EP)

Firstly, note that the t-norm $T_{\mathbf{B}^*}$ given below

$$T_{\mathbf{B}^*}(x, y) = \begin{cases} 0, & \text{if } x, y \in]0, 0.5[, \\ \min(x, y), & \text{otherwise,} \end{cases}$$

is a border-continuous but non-left-continuous t-norm whose residual does not satisfy (EP). Indeed, the R-implication generated from $T_{\mathbf{B}^*}$ is

$$I_{T_{\mathbf{B}^*}}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0.5, & \text{if } x > y \text{ and } x \in [0, 0.5[, \\ y, & \text{otherwise.} \end{cases}$$

Obviously, $I_{T_{\mathbf{B}^*}}$ satisfies (OP) but not (EP), since

$$I_{T_{\mathbf{B}^*}}(0.4, I_{T_{\mathbf{B}^*}}(0.5, 0.3)) = 0.5,$$

while

$$I_{T_{\mathbf{B}^*}}(0.5, I_{T_{\mathbf{B}^*}}(0.4, 0.3)) = 1.$$

The proof of main result is given in a series of lemmata. Firstly, it is shown that when T is a border-continuous t-norm and when I_T satisfies (EP), then the R-implication obtained from the conditionally left-continuous completion T^* of T is equivalent to I_T and hence also satisfies (EP).

Lemma 3.10. *Let T be a border-continuous t-norm such that I_T satisfies (EP). Then $I_T = I_{T^*}$.*

Further, it is shown that, under the above assumption, T does satisfy the (CLCC-A)-property, i.e., its conditionally left-continuous completion T^* is associative.

Lemma 3.11. *Let T be a border-continuous t-norm such that I_T satisfies (EP). Then T satisfies the (CLCC-A)-property, i.e., its conditionally left-continuous completion T^* is associative.*

The proof of the above result is given by showing that T^* is equal to the t-norm $T_{I_T^*}$ obtained from its residual I_{T^*} . Based on the above lemmata, we obtain the following partial characterization of R-implications that satisfy (EP).

Theorem 3.12. *For a border-continuous t-norm T the following statements are equivalent:*

- (i) I_T satisfies (EP).
- (ii) T satisfies the (CLCC-A)-property (i.e., T^* is a associative), and $I_T = I_{T^*}$.

The sufficiency follows from Lemmas 3.11 and 3.10. To see the necessity, note that if T satisfies the (CLCC-A)-property, then T^* is a left-continuous t-norm. Therefore I_{T^*} satisfies (EP). But $I_T = I_{T^*}$, so I_T also satisfies (EP).

Based on the above results a further characterization of t-norms, whose residuals satisfy both the exchange principle and the ordering property can be given as follows:

Corollary 3.13. *For a t-norm T the following statements are equivalent:*

- (i) I_T satisfies (EP) and (OP).
- (ii) T is border-continuous, satisfies the (CLCC-A)-property and $I_T = I_{T^*}$.

3.1.3 (EP) of an I_T and the Intersection between (S,N)- and R-implications

The exchange principle (EP) also plays an important role in determining the intersection between (S,N)- and R-implications. Many results were obtained regarding the overlaps of the above two families. Still, the following question remains and appears in the monograph [2].

Problem 4.8.1 in [2]: *Is there a fuzzy implication I , other than the Weber implication I_{WB} , which is both an (S,N)-implication and an R-implication which is obtained from a non-left continuous t-norm and cannot be obtained as the residual of any other left-continuous t-norm, i.e., is the following equality true $\mathbb{I}_{S,N} \cap (\mathbb{I}_T \setminus \mathbb{I}_{T^*}) = \{I_{WB}\}$?*

Note that for an I_T to be an (S,N)-implication, it needs to satisfy (EP) and hence the above question roughly translates into finding t-norms T such that I_T satisfies (EP).

From the results above, it is clear that when T is a border-continuous t-norm, then the above intersection is empty, i.e., $\mathbb{I}_{S,N} \cap (\mathbb{I}_T \setminus \mathbb{I}_{T^*}) = \emptyset$.

3.2 R-implications and Their Continuity

In Section 3.1 we discussed the necessity of left-continuity of a t-norm T for I_T to have certain algebraic properties, viz., (EP) and (OP), an order-theoretic property (I2) and an analytic property, that of right-continuity of I_T in the second variable.

Yet another interesting question is the continuity of I_T in both variables. Note that, since I_T is monotonic, continuity in each variable separately is also equivalent to the joint continuity of I_T in both variables. The only known continuous R-implications are those that are isomorphic to the Łukasiewicz implication, i.e., those R-implications obtained as residuals of nilpotent t-norms. In fact, these are the only known class of R-implications obtained from *left-continuous* t-norms, that

are continuous. For R-implications generated from left-continuous many characterization results are available, see for example, [14, 2]. Now we state the following main characterization result whose generalization gives the requisite answers. In the following Φ denotes the family of all increasing bijections on $[0, 1]$.

Theorem 3.14 (cf. [13, Corollary 2] and [2, Theorem 2.5.33]). *For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following are equivalent:*

- (i) *I is a continuous R-implication based on some left-continuous t-norm.*
- (ii) *I is Φ -conjugate with the Łukasiewicz implication, i.e., there exists an increasing bijection $\varphi: [0, 1] \rightarrow [0, 1]$, which is uniquely determined, such that*

$$I(x, y) = \varphi^{-1}(\min(1 - \varphi(x) + \varphi(y), 1)), \quad x, y \in [0, 1]. \quad (5)$$

We would like to note that the proof of the above result is dependent on many other equivalence results concerning fuzzy implications, especially concerning R-implications and their contrapositivity, see, for instance, the corresponding proofs in [13, 2].

In the case of (S,N)-implications a characterization of continuous (S,N)-implications was given in [1]. However, a similar complete characterization regarding the continuous subset of R-implications was not known and the following problem remained open for long:

Solved Problem 5 ([2, Problem 2.7.4]). *Does there exist a continuous R-implication generated from non-(left)-continuous t-norm ?*

Recently, JAYARAM [18] answered the above poser in the negative, by showing that the continuity of an R-implication forces the left-continuity of the underlying t-norm and hence show that an R-implication I_T is continuous if and only if T is a nilpotent t-norm.

Before we proceed to give a sketch of this proof, let us look at some interesting consequences of the above result.

3.2.1 Importance of This Result

Firstly, using this result, one is able to resolve another question related to the intersections between the families of continuous R- and (S,N)-implications, which is also a generalization of an original result of SMETS and MAGREZ [38], see also [14, 15]. In particular, it can be shown that the only continuous (S,N)-implication that is also an R-implication obtained from any t-norm, not necessarily left-continuous, is the Łukasiewicz implication up to an isomorphism (see Section 3.2.3).

Note that this result also has applications in other areas of fuzzy logic and fuzzy set theory. For instance, in the many fuzzy logics based on t-norms, viz., BL-fuzzy logics [16], MTL-algebras [12] and their other variants, the negation is obtained from the t-norm itself and is not always involutive. However, the continuity of the residuum immediately implies that the corresponding negation is continuous, and hence involutive, see [5, Theorem 2.14].

It is well known that fuzzy inference mechanisms that use t-norms and their residual fuzzy implications as part of their inference scheme have many desirable properties (see, for instance, [26, 17]). Based on the results contained in this paper one can choose this pair of operations appropriately to ensure the continuity of the ensuing inference.

3.2.2 Sketch of the Proofs: Partial Functions of R-implications

We firstly note that though JAYARAM [18] answered the above problem by dealing with it exclusively, the answer could also have been obtained from some earlier works of DE BAETS and MAES [32, 8]. Interestingly, in both the proofs the partial functions of R-implications play an important role. In this section we detail the proof given in [18], since the proof is both independent and leads to what is perhaps - to the best of the authors’ knowledge - the first independent proof of Theorem 3.14 above.

As mentioned earlier, the partial functions of R-implications play an important role. Note that since $I_T(x, x) = 1$ and $I_T(1, x) = x$, for all $x \in [0, 1]$ the following definition is valid.

Definition 3.15. For any fixed $\alpha \in [0, 1[$, the non-increasing partial function $I_T(\cdot, \alpha): [\alpha, 1] \rightarrow [\alpha, 1]$ will be denoted by g_α^T .

Observe that g_α^T is non-increasing and such that $g_\alpha^T(\alpha) = 1$ and $g_\alpha^T(1) = \alpha$.

Remark 3.16. If the domain of g_α^T is extended to $[0, 1]$, then this is exactly what are called *contour lines* by MAES and DE BAETS in [32, 8]. If $\alpha = 0$, then g_0^T is the natural negation associated with the t-norm T (see [2]):

$$N_T(x) = I_T(x, 0) = \sup\{t \in [0, 1] \mid T(x, t) = 0\}, \quad x \in [0, 1].$$

In fact, the following result about these partial functions essentially states that, if the “generalized” inverse of a monotone function is continuous, then it is strictly decreasing (see [27, Remark 3.4(ii)], also [32, Theorem 11]).

Theorem 3.17. *Let T be any t-norm. For any fixed $\alpha \in [0, 1[$, if g_α^T is continuous, then g_α^T is strictly decreasing.*

The rest of the proof is given in a series of Lemmata in JAYARAM [18], which we club here into a single result:

Theorem 3.18. *Let T be a t-norm such that I_T is continuous. Then*

- (i) *T is border continuous.*
- (ii) *T is Archimedean, i.e., for any $x, y \in (0, 1)$ there exists an $n \in \mathbb{N}$ such that $x_T^{[n]} < y$, where $x_T^{[n]} = T(x, x_T^{[n-1]})$ and $x_T^{[1]} = x$.*

We note that based on Theorem 3.18 we can obtain an independent proof of Theorem 3.14. This result is based on some well-known results which we recall in the following remark.

Remark 3.19 (cf.[27]).

- (i) A left-continuous T that is Archimedean is necessarily continuous and hence either strict or nilpotent (see [27, Proposition 2.16]).
- (ii) A continuous Archimedean t-norm T is either strict or nilpotent.
- (iii) If a continuous Archimedean t-norm T has zero divisors, then it is nilpotent (see [27, Theorem 2.18]).
- (iv) A nilpotent t-norm T is a Φ -conjugate of the Łukasiewicz t-norm, i.e., there exists an increasing bijection $\varphi: [0, 1] \rightarrow [0, 1]$, which is uniquely determined, such that

$$T(x, y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1, 0)), \quad x, y \in [0, 1].$$

- (v) Notice that if $T(x, y) = 0$ for some $x, y \in [0, 1]$, then $y \leq N_T(x)$. Moreover, if any $z < N_T(x)$, then $T(x, z) = 0$. If T is left-continuous, then $T(x, y) = 0$ for some $x, y \in [0, 1]$ if and only if $y \leq N_T(x)$.
- (vi) If N_T is continuous, then it is strong (see [5, Theorem 2.14]).

Corollary 3.20. *Let T be a left-continuous t-norm and I_T the R-implication obtained from it. Then the following are equivalent:*

- (i) I_T is continuous.
- (ii) T is isomorphic to $T_{\mathbf{LK}}$.

Proof. (i) \implies (ii): Let T be left-continuous and I_T be continuous. Then, from Theorem 3.18 above, we see that T is Archimedean and hence by Remark 3.19(i) T is necessarily continuous. Further, by Remark 3.19(ii), T is either strict or nilpotent. Now, since I_T is continuous, by Remark 3.19(v) we have that $N_T = g_0^T$ is strict and strong and hence from Remark 3.19(iv) we see that T has zero divisors. Finally, from Remark 3.19(iii), it follows that T is nilpotent and hence is isomorphic to $T_{\mathbf{LK}}$.

- (ii) \implies (i): The converse is obvious, since the R-implication obtained from any nilpotent t-norm is a Φ -conjugate of the Łukasiewicz implication $I_{\mathbf{LK}}$. Since $I_{\mathbf{LK}}$ is continuous, any Φ -conjugate of it is also continuous.

Based on the above results, the main result in [18] shows that if I_T is continuous, then the left-continuity of T need not be assumed but follows as a necessity.

Theorem 3.21. *Let T be a t-norm and I_T the R-implication obtained from it. If I_T is continuous, then T is left-continuous.*

From Theorems 3.14 and 3.21 we obtain the following result.

Corollary 3.22. *For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following statements are equivalent:*

- (i) I is a continuous R-implication based on some t-norm.
- (ii) I is Φ -conjugate with the Łukasiewicz implication, i.e., there exists $\varphi \in \Phi$, which is uniquely determined, such that I has the form (5) for all $x, y \in [0, 1]$.

3.2.3 Intersection between Continuous R- and (S,N)-implications

The intersections between the families and subfamilies of R- and (S,N)-implications have been studied by many authors, see e.g. [10, 38, 14, 2]. As regards the intersection between their continuous subsets only the following result has been known so far.

Theorem 3.23. *The only continuous (S,N)-implications that are also R-implications obtained from left-continuous t-norms are the fuzzy implications which are Φ -conjugate with the Łukasiewicz implication.*

Now, from Corollary 3.22 and Theorem 3.23 the following equivalences follow immediately:

Theorem 3.24. *For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following statements are equivalent:*

- (i) *I is a continuous (S,N)-implication that is also an R-implication obtained from a left-continuous t-norm.*
- (ii) *I is a continuous (S,N)-implication that is also an R-implication.*
- (iii) *I is an (S,N)-implication that is also a continuous R-implication.*
- (iv) *I is Φ -conjugate with the Łukasiewicz implication, i.e., there exists an increasing bijection $\varphi: [0, 1] \rightarrow [0, 1]$, which is uniquely determined, such that I has the form (5).*

3.3 Characterization of Yager’s Class of Fuzzy Implications

As we have seen in earlier sections characterizations of different families of fuzzy implications are very important questions. One open problem has been connected with two families of fuzzy implications introduced by YAGER [43].

Definition 3.25. Let $f: [0, 1] \rightarrow [0, \infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I: [0, 1]^2 \rightarrow [0, 1]$ defined by

$$I(x, y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0, 1], \tag{6}$$

with the understanding $0 \cdot \infty = 0$, is called an f -generated implication. The function f itself is called an f -generator of the I generated as in (6). In such a case, to emphasize the apparent relation we will write I_f instead of I .

Definition 3.26. Let $g: [0, 1] \rightarrow [0, \infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I: [0, 1]^2 \rightarrow [0, 1]$ defined by

$$I(x, y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \quad x, y \in [0, 1], \tag{7}$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a g -generated implication, where the function $g^{(-1)}$ in (7) is the pseudo-inverse of g given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x), & \text{if } x \in [0, g(1)], \\ 1, & \text{if } x \in [g(1), \infty]. \end{cases}$$

Based on some works of BACZYŃSKI and JAYARAM [4] on the distributive equations involving fuzzy implications, a rather not-so-elegant and partial characterization of f - and g - generated fuzzy implications can be given. However, an axiomatic characterization was unknown during the preparation of the book [2], so the following problem has been presented.

Solved Problem 6 ([2, Problem 3.3.1]). *Characterize the families of f - and g -generated implications.*

Very recently MASSANET and TORRENS [36] solved the above problem by using law of importation (LI). Firstly notice in the case when $f(0) < \infty$, the generated f -implication is an (S,N)-implication obtained from a continuous negation N (see [2, Theorem 4.5.1]) and hence the characterization result in Theorem 2.14 is applicable.

Theorem 3.27 ([36, Theorem 6]). *For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following statements are equivalent:*

- (i) *I is an f -generated implication with $f(0) < \infty$.*
- (ii) *I satisfies (LI) with product t -norm $T_{\mathbf{P}}(x, y) = xy$ and the natural negation N_I is a continuous fuzzy negation.*

When $f(0) = \infty$, then f -generated implications are not (S,N)-implications, but still similar characterizations can be proved.

Theorem 3.28 ([36, Theorem 12]). *For a function $I: [0, 1]^2 \rightarrow [0, 1]$ the following statements are equivalent:*

- (i) *I is an f -generated implication with $f(0) = \infty$.*
- (ii) *I satisfies (LI) with product t -norm $T_{\mathbf{P}}(x, y) = xy$, I is continuous except at $(0, 0)$ and $I(x, y) = 1 \Leftrightarrow x = 0$ or $y = 1$.*

Similar characterizations have been obtained for g -generated fuzzy implications (see [36, Theorems 14, 17]).

3.3.1 Importance of This Result

While the (S,N)- and R-implications, dealt with in the earlier sections, are the generalizations of the material and intuitionistic-logic implications, there exists yet another popular way of obtaining fuzzy implications - as the generalization of the following implication defined in quantum logic:

$$p \rightarrow q \equiv \neg p \vee (p \wedge q).$$

Needless to state, when the truth values are restricted to $\{0, 1\}$ its truth table coincides with that of the material and intuitionistic-logic implications.

Definition 3.29. A function $I: [0, 1]^2 \rightarrow [0, 1]$ is called a *QL-operation* if there exist a t-norm T , a t-conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), T(x, y)), \quad x, y \in [0, 1]. \tag{8}$$

Note that not all QL-operations are fuzzy implications in the sense of Definition 2.1. A QL-operation is called a *QL-implication* only when it is a fuzzy implication. The set of all QL-implications will be denoted by \mathbb{I}_{QL} .

The characterization of Yager’s family of f -generated implications has helped us to know the answer for two other open problems. In fact, based on the above characterization the following question, originally posed in the monograph [2] has been completely solved.

Solved Problem 7 ([2, Problem 4.8.3]).

- (i) Is the intersection $\mathbb{I}_{F, \mathfrak{K}} \cap \mathbb{I}_{QL}$ non-empty?
- (ii) If yes, then characterize the intersection $\mathbb{I}_{F, \mathfrak{K}} \cap \mathbb{I}_{QL}$.

In [36] the authors have proven the following:

Theorem 3.30 ([36, Theorem 13]). Let $I_{T,S,N}$ be a QL-operation. Then the following statements are equivalent:

- (i) $I_{T,S,N}$ is an f -generated implication with $f(0) < \infty$.
- (ii) $I_{T,S,N}$ satisfies (LI) with $T_{\mathfrak{P}}$ and N is a strict negation.
- (iii) N is a strict negation such that

$$N(xy) = S(N(x), T(x, N(y))), \quad x, y \in [0, 1].$$

Moreover, in this case $f = N^{-1}$, up to a multiplicative positive constant.

3.4 R-implications and a Functional Equation

The following problem was posed by HÖHLE in KLEMENT ET AL. [28]. An interesting fallout of this problem is that, as the solution shows, it gives a characterization of conditionally cancellative t-(sub)norms.

Solved Problem 8 ([28, Problem 11]). Characterize all left-continuous t-norms T which satisfy

$$I(x, T(x, y)) = \max(N(x), y), \quad x, y \in [0, 1]. \tag{9}$$

where I is the residual operator linked to T given by (3) and

$$N(x) = N_T(x) = I(x, 0), \quad x \in [0, 1].$$

Further, U. Höhle goes on to the following remark:

Remark 3.31. "In the class of continuous t-norms, only nilpotent t-norms fulfill the above property."

It is clear that in the case T is left-continuous - as stated in **Problem 1** - the supremum in (3) actually becomes maximum. It is worth mentioning that the residual can be determined for more generalized conjunctions and the conditions under which this residual becomes a fuzzy implication can be found in, for instance, [9, 22, 30]. Hence JAYARAM [19] further generalized the statement of **Problem 8** by considering a t-subnorm instead of a t-norm and also dropping the condition of left-continuity.

Definition 3.32 ([27, Definition 1.7]). A t-subnorm is a function $M : [0, 1]^2 \rightarrow [0, 1]$ such that it is monotonic non-decreasing, associative, commutative and $M(x, y) \leq \min(x, y)$ for all $x, y \in [0, 1]$.

Note that for a t-subnorm 1 need not be the neutral element, unlike in the case of a t-norm.

Definition 3.33 (cf. [27, Definition 2.9 (iii)]). A t-subnorm M satisfies the conditional cancellation law if, for any $x, y, z \in]0, 1[$,

$$M(x, y) = M(x, z) > 0 \text{ implies } y = z. \tag{CCL}$$

In other words, (CCL) implies that on the positive domain of M , i.e., on the set $\{(x, y) \in (0, 1]^2 \mid M(x, y) > 0\}$, M is strictly increasing. See Figure 3 (a) and (b) for examples of a conditionally cancellative t-subnorm and one that is not.

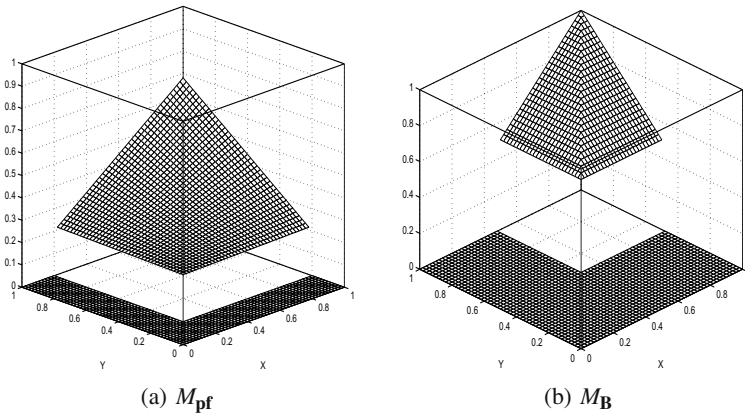


Fig. 3 M_{pf} is a conditionally cancellative t-subnorm, while M_{B} is not

Definition 3.34 (cf. [2, Definition 2.3.1]). Let M be any t -subnorm. Its natural negation n_M is given by

$$n_M(x) = \sup\{t \in [0, 1] \mid M(x, t) = 0\}, \quad x \in [0, 1].$$

Note that though $n_M(0) = 1$, it need not be a fuzzy negation, since $n_M(1)$ can be greater than 0. However, we have the following result.

Lemma 3.35 (cf. [2, Proposition 2.3.4]). Let M be any t -subnorm and n_M its natural negation. Then we have the following:

- (i) $M(x, y) = 0 \implies y \leq n_M(x)$.
- (ii) $y < n_M(x) \implies M(x, y) = 0$.
- (iii) If M is left-continuous, then $y = n_M(x) \implies M(x, y) = 0$, i.e., the reverse implication of (i) also holds.

It is interesting to note that the solution to the problem given below characterizes the set of all conditionally cancellative t -subnorms.

Theorem 3.36 ([19, Theorem 3.1]). Let M be any t -subnorm and I the residual operation linked to M by (3). Then the following are equivalent:

- (i) The pair (I, M) satisfies (9).
- (ii) M is a conditionally cancellative t -subnorm.

Example 3.37. Consider the product t -norm $T_P(x, y) = xy$, which is a strict t -norm and hence continuous and Archimedean, whose residual is the Goguen implication given by

$$I_{GG}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{y}{x}, & \text{if } x > y. \end{cases}$$

It can be easily verified that the pair (T_P, I_{GG}) does indeed satisfy (9) whereas the natural negation of T_P is the Gödel negation

$$n_{T_P}(x) = I_{GG}(x, 0) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0. \end{cases}$$

This example clearly shows that the remark of HÖHLE, Remark 3.31, is not always true. The following result gives an equivalence condition under which it is true.

Theorem 3.38 ([19, Theorem 3.2]). Let T be a continuous t -norm that satisfies (9) along with its residual. Then the following are equivalent:

- (i) T is nilpotent.
- (ii) n_T is strong.

4 Concluding Remarks

As can be seen since the publication of the monograph [2], there has been quite a rapid progress in attempts to solve open problems. However, there still remain many open problems involving fuzzy implications. In the following we list a few:

Problem 4.1. Give a necessary condition on a non-border continuous t-norm T for the corresponding I_T to satisfy (EP).

It should be mentioned that some related work on the above problem has appeared in [20]. Once again, as stated before, the above problem is also related to the following question regarding the intersection of (S,N)- and R-implications which still remains open:

Problem 4.2. (i) Is there a fuzzy implication I , other than the Weber implication I_{WB} , which is both an (S,N)-implication and an R-implication which is obtained from a *non-border continuous* t-norm and cannot be obtained as the residual of any other left-continuous t-norm?

(ii) If the answer to the above question is in the affirmative, characterize the above non-empty intersection.

The following problems originally appeared as open in [2] and still remain so:

Problem 4.3. What is the characterization of (S,N)-implications generated from non-continuous negations?

Problem 4.4. Characterize triples (T, S, N) such that $I_{T,S,N}$ satisfies (II).

Problem 4.5. (i) Characterize the non-empty intersection $\mathbb{I}_{S,N} \cap \mathbb{I}_{QL}$.

(ii) Is the Weber implication I_{WB} the only QL-implication that is also an R-implication obtained from a non-left continuous t-norm? If not, give other examples from the above intersection and hence, characterize the non-empty intersection $\mathbb{I}_{QL} \cap \mathbb{I}_T$.

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