Search by Quantum Walks on Two-Dimensional Grid without Amplitude Amplification

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Abstract. We study search by quantum walk on a finite two dimensional grid. The algorithm of Ambainis, Kempe, Rivosh [AKR05] uses $O(\sqrt{N \log N})$ steps and finds a marked location with probability $O(1/\log N)$ for grid of size $\sqrt{N} \times \sqrt{N}$. This probability is small, thus [AKR05] needs amplitude amplification to get $\Theta(1)$ probability. The amplitude amplification adds an additional $O(\sqrt{\log N})$ factor to the number of steps, making it $O(\sqrt{N} \log N)$.

In this paper, we show that despite a small probability to find a marked location, the probability to be within $O(\sqrt{N})$ neighbourhood (at $O(\sqrt[4]{N})$ distance) of the marked location is $\Theta(1)$. This allows to skip amplitude amplification step and leads to $O(\sqrt{\log N})$ speed-up.

1 Introduction

Quantum walks are quantum counterparts of random walks [Amb03, Kem03]. They have been useful to design quantum algorithms for a variety of problems [CC+03, Amb04, Sze04, AKR05, MSS05, BS06]. In many of those applications, quantum walks are used as a tool for search.

To solve a search problem using quantum walks, we introduce marked locations corresponding to elements of the search space we want to find. We then perform a quantum walk on search space with one transition rule at unmarked locations and another transition rule at marked locations. If this process is set up properly, it leads to a quantum state in which marked locations have higher probability than unmarked ones. This method of search using quantum walks was first introduced in [SKW03] and has been used many times since then.

We study spatial search on a finite two-dimensional grid [Ben02, AA03, AKR05]. In this problem, we have a grid of size $\sqrt{N} \times \sqrt{N}$ on which some locations are marked. In one time step, we are allowed to examine the current location or move one step on the grid. The task is to find a marked location.

Ambainis et al. [AKR05] showed that this problem can be solved via quantum walk. Namely, after $O(\sqrt{N \log N})$ steps a quantum walk on 2D grid with one or two marked locations reaches a state that is significantly different from the

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state of a quantum walk with no marked location. If this state is measured, the probability to obtain a marked location is $O(1/\log N)$. This probability is small, thus [AKR05] uses amplitude amplification. Amplitude amplification adds an additional $O(\sqrt{\log N})$ factor to the number of steps, making it $O(\sqrt{N}\log N)$.

In case of two-dimensional grid it is logical to examine not only the marked location but also its close neighbourhood. We show that despite a small probability to find marked location, the probability to be within $O(\sqrt{N})$ neighbourhood, i.e. at $O(\sqrt[4]{N})$ distance from the marked location, is $\Omega(1)$. This allows us to skip amplitude amplification step and leads to $O(\sqrt{\log N})$ speed-up.

Similar speed-up has been already achieved by other research groups, by different methods. Their approaches to this problem are based on modification of the original algorithm [Tul08] or both the algorithm and the structure of the grid [KM+10].

Our result shows that the improvement of the running time to $O(\sqrt{N \log N})$ can be achieved without any modifications to the quantum algorithm, with just a simple classical post-processing.

2 Quantum Walks in Two Dimensions

Suppose we have N items arranged on a two dimensional lattice of size $\sqrt{N}\times\sqrt{N}$. We will also denote $n = \sqrt{N}$. The locations on the lattice are labelled by their x and y coordinate as (x, y) for $x, y \in \{0, \ldots, n-1\}$. We assume that the grid has periodic boundary conditions. For example, going right from a location $(n-1, y)$ on the right edge of the grid leads to the location $(0, y)$ on the left edge of the grid.

To define a quantum walk, we add an additional "coin" register with four states, one for each direction: $|\Uparrow\rangle$, $|\Downarrow\rangle$, $|\Leftarrow\rangle$ and $|\Rightarrow\rangle$. At each step we perform a unitary transformation on the extra register and then evolve the system according to the state of the coin register. Thus, the basis states of quantum walk are $|i, j, d\rangle$ for $i, j \in \{0, \ldots, n-1\}$, $d \in \{\Uparrow, \Downarrow, \Leftarrow, \Rightarrow\}$ and the state of quantum walk is given by:

$$|\psi(t)\rangle = \sum_{i,j}(\alpha_{i,j,\Uparrow}|i, j, \Uparrow\rangle + \alpha_{i,j,\Downarrow}|i, j, \Downarrow\rangle + \alpha_{i,j,\Leftarrow}|i, j, \Leftarrow\rangle + \alpha_{i,j,\Rightarrow}|i, j, \Rightarrow\rangle) \tag{1}$$

A step of the coined quantum walk is performed by first applying $I \times C$, where C is unitary transform on the coin register. The most often used transformation on the coin register is the Grover's diffusion transformation D:

$$D = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \tag{2}$$

Then, we apply the shift transformation S:

$$\begin{array}{ll} |i,j,\Uparrow\rangle & \to |i,j-1,\Downarrow\rangle \\ |i,j,\Downarrow\rangle & \to |i,j+1,\Uparrow\rangle \\ |i,j,\Leftarrow\rangle & \to |i-1,j,\Rightarrow\rangle \\ |i,j,\Rightarrow\rangle & \to |i+1,j,\Leftarrow\rangle \end{array} \tag{3}$$

Notice that after moving to an adjacent location we change the value of the direction register to the opposite. This is necessary for the quantum walk algorithm of [AKR05] to work.

We start quantum walk in the state

$$|\psi(0)\rangle = \frac{1}{2\sqrt{N}} \sum_{i,j} \big(|i,j,\Uparrow\rangle + |i,j,\Downarrow\rangle + |i,j,\Leftarrow\rangle + |i,j,\Rightarrow\rangle\big)$$

It can be easily verified that the state of the walk stays unchanged, regardless of the number of steps. To use quantum walk as a tool for search, we "mark" some locations. In unmarked locations, we apply the same transformations as above. In marked locations, we apply $-I$ instead of D as the coin flip transformation. The shift transformation remains the same in both marked and unmarked locations.

If there are marked locations, the state of this process starts to deviate from $|\psi(0)\rangle$. It has been shown [AKR05] that after $O(\sqrt{N \log N})$ steps the inner product $\langle\psi(t)|\psi(0)\rangle$ becomes close to 0.

In case of one or two marked locations [AKR05] algorithm finds a marked location with $O(1/\log N)$ probability. For multiple marked locations this is not always the case. There exist marked location configurations for which quantum walk fails to find any of marked locations [AR08].

3 Results

In this paper we examine a single marked location case only. However, we note that numerical experiments give very similar results in the case of multiple marked locations.

Suppose we have an $\sqrt{N}\times\sqrt{N}$ grid with one marked location. The [AKR05] algorithm takes $O(\sqrt{N \log N})$ steps and finds the marked location with $O(1/\log N)$ probability. The algorithm then uses amplitude amplification to get $\Theta(1)$ probability. The amplitude amplification adds an additional $O(\sqrt{\log N})$ factor to the number of steps, making it $O(\sqrt{N}\log N)$.

Performing numerical experiments with [AKR05] algorithm, we have noticed that probability to be close to the marked location is much higher than probability to be far from the marked location. Figure 1 shows probability distribution by distance from the marked location for 1024×1024 grid on logarithmic scale.

We have measured the probability within $O(\sqrt{N})$ neighbourhood of the marked location (at $O(\sqrt[4]{N})$ distance)[1] for different grid sizes (figure 2) and have made the following conjecture:

[1] Another logical choice of the size of the neighbourhood would be $O(\sqrt{N \log N})$ - the number of steps of [AKR05] algorithm.

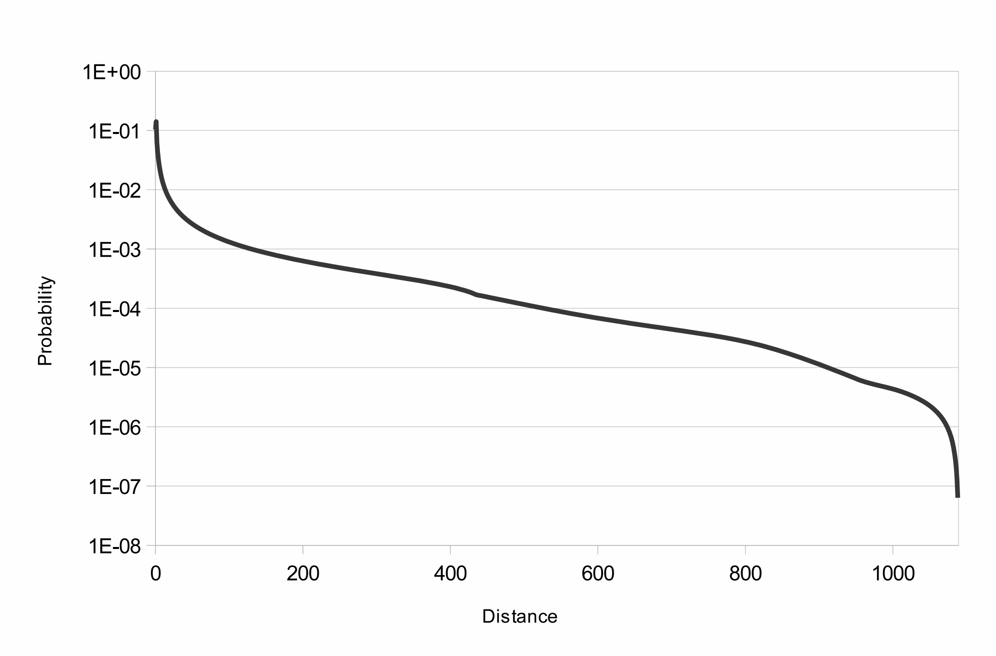

Fig. 1. Probability by distance, one marked location, grid size 1024×1024, logarithmic scale

Hypothesis 1. *The probability to be within $O(\sqrt{N})$ neighbourhood, i.e. at $O(\sqrt[4]{N})$ distance, of the marked location is $\Theta(1)$.*

In the next section we present a strict analytical proof of the conjecture.

This allows us to replace amplitude amplification with a classical post-processing step. After the measurement we classically check $O(\sqrt{N})$ neighbourhood of the outcome. This requires extra $O(\sqrt{N})$ steps but removes $O(\sqrt{\log N})$ factor. Therefore, the running time of the algorithm stays $O(\sqrt{N \log N})$.

Before going into details of the proof, we would like to give the reader some understanding of the final state of the algorithm (state before the measurement). Denote $Pr[0]$ the probability to find a marked location and $Pr[R]$ the probability to be at distance R from the marked location. For small R values ($R \ll \sqrt{N}$), the numerical experiments indicate that:

$$Pr[R] \approx \frac{Pr[0]}{R^2}$$

There are $4R$ points at the distance R from the marked location (we use Manhattan or L_1 distance). Thus, the total probability to be within $\sqrt{N}$ neighbourhood of the marked location is:

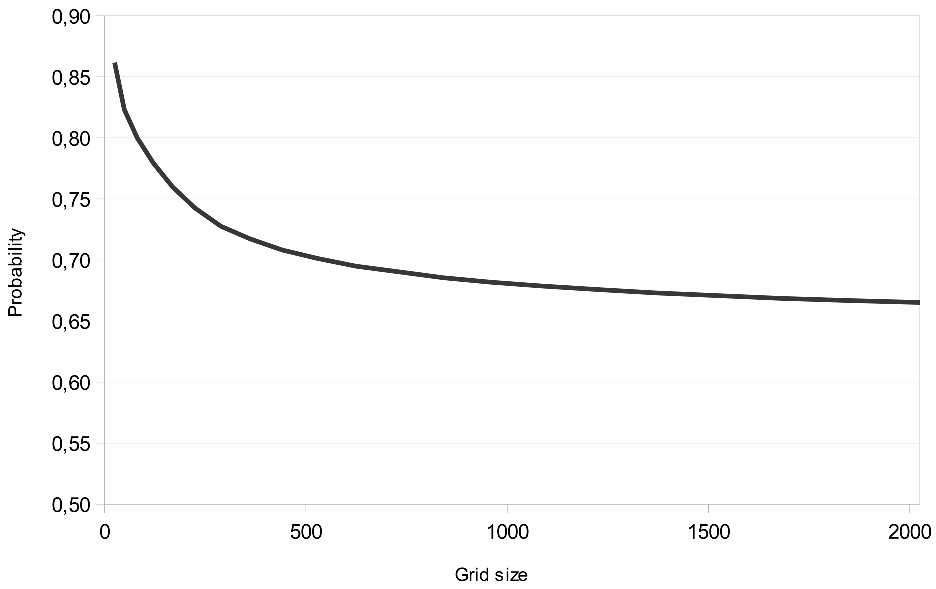

Fig. 2. Probability to be within $\sqrt{N}$ neibourghood from the marked location

$$S = \sum_{R=1}^{\sqrt[4]{N}} 4R \times O\left(\frac{Pr[0]}{R^2}\right) = Pr[0] \times \sum_{R=1}^{\sqrt[4]{N}} O\left(\frac{1}{R}\right) = Pr[0] \times O(\log N).$$

As probability to find the marked location is $O(1/\log N)$, we have

$$S = O\left(\frac{1}{\log N}\right) \times O(\log N) = const.$$

4 Proofs

In this section, we show

Theorem 1. *We can choose* $t = O(\sqrt{N \log N})$ *so that, if we run a quantum walk with one marked location* (i, j) *for* t *steps and measure the final state, the probability of obtaining a location* (i', j') *with* $|i - i'| \leq N^\epsilon$ *and* $|j - j'| \leq N^\epsilon$ *as the measurement result is* $\Omega(\epsilon)^2$.

[2] Here, $|i-i'| \leq N^\epsilon$ and $|j-j'| \leq N^\epsilon$ should be interpreted "modulo N": $|i-i'| \leq N^\epsilon$ if $(i-i') \bmod N \in \{-N^\epsilon, -N^\epsilon+1, \ldots, N^\epsilon\}$.

The proof of Theorem 1 consists of two steps. First, in Lemma 1, we derive an approximation for the state of quantum walk, at the time $t = O(\sqrt{N \log N})$ when the state of quantum walk has the biggest difference from the starting state. Then, in section 5, we use this approximation to derive our main result, via a sequence of algebraic transformations and approximations.

4.1 Approximation of the State of the Quantum Walk

Let

$$|\psi\rangle = \sum_{j=0}^{\sqrt{N}-1} \sum_{j'=0}^{\sqrt{N}-1} \sum_{d} \alpha^t_{j,j',d} |j, j', d\rangle$$

be the state of the quantum walk after t steps.

Lemma 1. *We can choose $t = O(\sqrt{N \log N})$ so that for any set*

$$S \subseteq \{0, ..., \sqrt{N} - 1\}^2,$$

we have

$$\sum_{(j,j') \in S} |\alpha^t_{j,j',\Uparrow}|^2 \geq C^2 \sum_{(j,j') \in S} (f(j, j') - f(j-1, j'))^2 + o(1)$$

where

$$f(j, j') = \sum_{(k,l) \neq (0,0)} \frac{1}{2 - \cos \frac{2k\pi}{\sqrt{N}} - \cos \frac{2l\pi}{\sqrt{N}}} \omega^{kj+lj'},$$

$\omega = e^{\frac{2\pi i}{\sqrt{N}}}$ *and* $C = \Theta(\frac{1}{\sqrt{N \log N}})$.

Proof. We will repeatedly use the following lemma.

Lemma 2. *[BV] Let $|\psi\rangle = \sum_{i=1}^{m} \alpha_i |i\rangle$ and $|\psi'\rangle = \sum_{i=1}^{m} \beta_i |i\rangle$. Then, for any set $S \subseteq \{1, 2, \ldots, m\}$,*

$$\sum_{i \in S} \left| |\alpha_i|^2 - |\beta_i|^2 \right| \leq 2 \|\psi - \psi'\|.$$

We recast the algorithm for search on the grid as an instance of an *abstract search algorithm* [AKR05]. An abstract search algorithm consists of two unitary transformations U_1 and U_2 and two states $|\psi_{start}\rangle$ and $|\psi_{good}\rangle$. We require the following properties:

1. $U_1 = I - 2|\psi_{good}\rangle\langle\psi_{good}|$ (in other words, $U_1|\psi_{good}\rangle = -|\psi_{good}\rangle$ and, if $|\psi\rangle$ is orthogonal to $|\psi_{good}\rangle$, then $U_1|\psi\rangle = |\psi\rangle$);
2. $U_2|\psi_{start}\rangle = |\psi_{start}\rangle$ for some state $|\psi_{start}\rangle$ with real amplitudes and there is no other eigenvector with eigenvalue 1;
3. U_2 is described by a real unitary matrix.

The abstract search algorithm applies the unitary transformation $(U_2U_1)^T$ to the starting state $|\psi_{start}\rangle$. We claim that under certain constraints its final state $(U_2U_1)^T|\psi_{start}\rangle$ has a sufficiently large inner product with $|\psi_{good}\rangle$.

For the quantum walk on $\sqrt{N}\times\sqrt{N}$ grid,

$$|\psi_{good}\rangle = \frac{1}{2}|i,j,\Uparrow\rangle + \frac{1}{2}|i,j,\Downarrow\rangle + \frac{1}{2}|i,j,\Leftarrow\rangle + \frac{1}{2}|i,j,\Rightarrow\rangle,$$

where i, j is the marked location and

$$|\psi_{start}\rangle = \frac{1}{2\sqrt{N}}\sum_{i,j=0}^{\sqrt{N}-1}\left(|i,j,\Uparrow\rangle + |i,j,\Downarrow\rangle + |i,j,\Leftarrow\rangle + |i,j,\Rightarrow\rangle\right).$$

Since U_2 is described by a real-value matrix, its eigenvectors (with eigenvalues that are not 1 or -1) can be divided into pairs: $|\Phi_j^+\rangle$ and $|\Phi_j^-\rangle$, with eigenvalues $e^{i\theta_j}$ and $e^{-i\theta_j}$, respectively. In the case of the walk on the 2-dimensional grid, these eigenvalues were calculated in Claim 6 of [AKR05]:

Claim 1. *Quantum walk on the 2-dimensional grid with no marked locations has* $N-1$ *pairs of eigenvalues* $e^{-i\theta_j}$ *that are not equal to 1 or -1. These values can be indexed by pairs* (k,l), $k,l\in\{0,1,\ldots,\sqrt{N}-1\}$, $(k,l)\neq(0,0)$. *The corresponding eigenvalues are equal to* $e^{\pm i\theta_{k,l}}$, *where* $\theta_{k,l}$ *satisfies* $\cos\theta_{k,l} = \frac{1}{2}(\cos\frac{2\pi k}{\sqrt{N}}+\cos\frac{2\pi l}{\sqrt{N}})$.

We use $|\Phi_{k,l}^+\rangle$ and $|\Phi_{k,l}^-\rangle$ to denote the corresponding eigenvectors. According to [MPA10, pages 3-4], these eigenvectors are equal to $|\Phi_{k,l}^+\rangle = |\xi_k\rangle\otimes|\xi_l\rangle\otimes|v_{k,l}^+\rangle$, $|\Phi_{k,l}^-\rangle = |\xi_k\rangle\otimes|\xi_l\rangle\otimes|v_{k,l}^-\rangle$ where $|\xi_k\rangle = \sum_{i=0}^{\sqrt{N}-1}\omega^{ki}\frac{1}{\sqrt[4]{N}}|i\rangle$,

$$|v_{k,l}^+\rangle = \frac{i}{2\sqrt{2}\sin\theta_{k,l}}\begin{bmatrix} e^{-i\theta_{k,l}}-\omega^k \\ e^{-i\theta_{k,l}}-\omega^{-k} \\ e^{-i\theta_{k,l}}-\omega^l \\ e^{-i\theta_{k,l}}-\omega^{-l}\end{bmatrix}, \quad |v_{k,l}^-\rangle = \frac{i}{2\sqrt{2}\sin\theta_{k,l}}\begin{bmatrix} \omega^k - e^{i\theta_{k,l}} \\ \omega^{-k} - e^{i\theta_{k,l}} \\ \omega^l - e^{i\theta_{k,l}} \\ \omega^{-l} - e^{i\theta_{k,l}}\end{bmatrix}.$$

The order of directions for the coin register is: $|\Downarrow\rangle$, $|\Uparrow\rangle$, $|\Rightarrow\rangle$, $|\Leftarrow\rangle$. The sign of $|v_{k,l}^-\rangle$ has been adjusted so that

$$\frac{1}{\sqrt{2}}|\Phi_{k,l}^+\rangle + \frac{1}{\sqrt{2}}|\Phi_{k,l}^-\rangle = |\xi_k\rangle\otimes|\xi_l\rangle\otimes|\delta\rangle \tag{4}$$

where $|\delta\rangle = \frac{1}{2}|\Downarrow\rangle + \frac{1}{2}|\Uparrow\rangle + \frac{1}{2}|\Rightarrow\rangle + \frac{1}{2}|\Leftarrow\rangle$.

We can assume that $|\psi_{good}\rangle = |0\rangle\otimes|0\rangle\otimes|\delta\rangle$. This gives us an expression of $|\psi_{good}\rangle$ in terms of the eigenvectors of U_2:

$$|\psi_{good}\rangle = \frac{1}{\sqrt{N}}\sum_{k,l}|\xi_k\rangle\otimes|\xi_l\rangle\otimes|\delta\rangle$$

$$= \frac{1}{\sqrt{N}}|\psi_{start}\rangle + \sum_{(k,l)\neq(0,0)}\left(\frac{1}{\sqrt{2N}}|\Phi_{k,l}^+\rangle + \frac{1}{\sqrt{2N}}|\Phi_{k,l}^-\rangle\right).$$

Using the results from [AKR05], we can transform this into an expression for the final state of our quantum search algorithm. According to the first big equation in the proof of Lemma 5 in [AKR05], after $t = O(\sqrt{N \log N})$ steps, we get a final state $|\psi\rangle$ such that $\| |\psi\rangle - |\phi_{final}\rangle \| = o(1)$, where $|\phi_{final}\rangle = \frac{|\phi'_{final}\rangle}{\|\phi'_{final}\|}$ and

$$|\phi'_{final}\rangle = \frac{1}{\sqrt{N}}|\psi_{start}\rangle + \frac{1}{\sqrt{2N}} \sum_{(k,l)\neq(0,0)} a_{k,l}|\Phi^+_{k,l}\rangle + b_{k,l}|\Phi^-_{k,l}\rangle \tag{5}$$

and

$$a_{k,l} = 1 + \frac{i}{2}\cot\frac{\alpha+\theta_{k,l}}{2} + \frac{i}{2}\cot\frac{-\alpha+\theta_{k,l}}{2},$$
$$b_{k,l} = 1 + \frac{i}{2}\cot\frac{\alpha-\theta_{k,l}}{2} + \frac{i}{2}\cot\frac{-\alpha-\theta_{k,l}}{2}.$$

We now replace $\sum_{(j,j')\in S} |\alpha^t_{j,j',d}|^2$ by the corresponding sum of squares of amplitudes for the state $|\phi_{final}\rangle$. By Lemma 2, this changes the sum by an amount that is $o(1)$.

From [AKR05], we have $\alpha = \Theta(\frac{1}{\sqrt{N \log N}})$, $\min \theta_{k,l} = \Theta(\frac{1}{\sqrt{N}})$ and $\max \theta_{k,l} = \pi - \Theta(\frac{1}{\sqrt{N}})$. Hence, we have $\pm\alpha + \theta_{k,l} = (1+o(1))\theta_{k,l}$ and we get

$$|\phi'_{final}\rangle = \frac{1}{\sqrt{N}}|\psi_{start}\rangle + \sum_{(k,l)\neq(0,0)} \frac{1}{\sqrt{2N}}\left(1 + i(1+o(1))\cot\frac{\theta_{k,l}}{2}\right)|\Phi^+_{k,l}\rangle +$$
$$\frac{1}{\sqrt{2N}}\left(1 - i(1+o(1))\cot\frac{\theta_{k,l}}{2}\right)|\Phi^-_{k,l}\rangle. \tag{6}$$

This means that $\| |\psi_{final}\rangle - |\phi_{final}\rangle \| = o(1)$ where $|\psi_{final}\rangle = \frac{|\psi'_{final}\rangle}{\|\psi'_{final}\|}$ and

$$|\psi'_{final}\rangle = |\psi_{good}\rangle + \sum_{(k,l)\neq(0,0)} \frac{1}{\sqrt{2N}} i \cot\frac{\theta_{k,l}}{2}\left(|\Phi^+_{k,l}\rangle - |\Phi^-_{k,l}\rangle\right). \tag{7}$$

Again, we can replace a sum of squares of amplitudes for the state $|\phi_{final}\rangle$ by the corresponding sum for $|\psi_{final}\rangle$ and, by Lemma 2, the sum changes by an amount that is $o(1)$.

We now estimate the amplitude of $|j,j',\Uparrow\rangle$ in $|\psi_{final}\rangle$. We assume that $(j,j') \neq (0,0)$. Then, the amplitude of $|j,j',\Uparrow\rangle$ in $|\psi_{good}\rangle$ is 0. Hence, we can evaluate the amplitude of $|j,j',\Uparrow\rangle$ in

$$\sum_{(k,l)\neq(0,0)} \frac{1}{\sqrt{2N}} i \cot\frac{\theta_{k,l}}{2}(|\Phi^+_{k,l}\rangle - |\Phi^-_{k,l}\rangle) \tag{8}$$

and then divide the result by $\Theta(\sqrt{\log N})$, because $\|\psi'_{final}\| = \Theta(\sqrt{\log N})$.

From the definitions of $|\Phi^\pm_{k,l}\rangle$ and $|v^\pm_{k,l}\rangle$,

$$\frac{1}{\sqrt{2}}|v^+_{k,l}\rangle - \frac{1}{\sqrt{2}}|v^-_{k,l}\rangle = \frac{i}{4\sin\theta_{k,l}}\begin{bmatrix} 2\cos\theta_{k,l} - 2\omega^k \\ 2\cos\theta_{k,l} - 2\omega^{-k} \\ 2\cos\theta_{k,l} - 2\omega^l \\ 2\cos\theta_{k,l} - 2\omega^{-l} \end{bmatrix}.$$

The amplitude of $|\Uparrow\rangle$ in this state is $\frac{i}{2\sin\theta_{k,l}}(\cos\theta_{k,l} - \omega^{-k})$. The amplitude of $|j\rangle$ in $|\xi_k\rangle$ is $\frac{1}{\sqrt[4]{N}}\omega^{kj}$. The amplitude of $|j'\rangle$ in $|\xi_l\rangle$ is $\frac{1}{\sqrt[4]{N}}\omega^{lj'}$. Therefore, the amplitude of $|j, j', \Uparrow\rangle$ in $\frac{1}{\sqrt{2}}|\Phi^+_{k,l}\rangle - \frac{1}{\sqrt{2}}|\Phi^-_{k,l}\rangle$ is

$$\frac{1}{\sqrt{N}}\omega^{kj+lj'}\frac{i}{2\sin\theta_{k,l}}(\cos\theta_{k,l} - \omega^{-k})$$

and the amplitude of $|j, j', \Uparrow\rangle$ in (8) is

$$\frac{1}{\sqrt{2}N}\sum_{(k,l)\neq(0,0)} i\cot\frac{\theta_j}{2}\cdot\frac{i}{2\sin\theta_{k,l}}(\cos\theta_{k,l} - \omega^{-k})\omega^{kj+lj'}.$$

By using $\sin\theta_{k,l} = 2\sin\frac{\theta_{k,l}}{2}\cos\frac{\theta_{k,l}}{2}$, we get that the amplitude of $|j, j', \Uparrow\rangle$ is

$$\frac{1}{\sqrt{2}}\sum_{(k,l)\neq(0,0)}\frac{1}{4N}\left(-\frac{\cos\theta_{k,l}}{\sin^2\frac{\theta_{k,l}}{2}}\omega^{kj+lj'} + \frac{1}{\sin^2\frac{\theta_{k,l}}{2}}\omega^{k(j-1)+lj'}\right) =$$

$$\frac{1}{\sqrt{2}}\sum_{(k,l)\neq(0,0)}\frac{1}{4N}\left(2\omega^{kj+lj'} - \frac{1}{\sin^2\frac{\theta_{k,l}}{2}}(\omega^{kj+lj'} - \omega^{k(j-1)+lj'})\right), \qquad (9)$$

with the equality following from $\cos 2x = 1 - 2\sin^2 x$.

We can decompose the sum into two sums, one over all the first components, one over all the second components. The first component of the sum in (9) is close to 0 and, therefore, can be omitted. Hence, we get that the amplitude of $|j, j', \Uparrow\rangle$ in the unnormalized state
$|\psi'_{final}\rangle$ can be approximated by

$$\frac{1}{\sqrt{2}}\sum_{(k,l)\neq(0,0)}\frac{1}{4N}\frac{1}{\sin^2\frac{\theta_{k,l}}{2}}(-\omega^{kj+lj'} + \omega^{k(j-1)+lj'}) = \Theta\left(\frac{1}{N}\right)\cdot(f(j-1,j') - f(j,j')).$$

To obtain the amplitude of $|j, j', \Uparrow\rangle$ in $|\psi_{final}\rangle$, this should be divided by $\|\psi'_{final}\|$ which is of the order $\Theta(\sqrt{\log N})$. This implies Lemma 1. □

5 Bounds on the Probability of Being Close to the Marked Location

We start by performing some rearrangements in the expression $f(j, j')$.

Let $n = \sqrt{N}$ and S be the set of all pairs (k, l) such as $k, l \in \{0, 1, \ldots, n-1\}$, except for $(0, 0)$. We consider

$$f(j, j') = \sum_{(k,l)\in S}\frac{1}{2 - \cos\frac{2k\pi}{n} - \cos\frac{2l\pi}{n}}\omega^{kj+lj'}$$

$$= \sum_{(k,l)\in S}\frac{\cos\frac{2(kj+lj')\pi}{n} + \sin\frac{2(kj+lj')\pi}{n}i}{2 - \cos\frac{2k\pi}{n} - \cos\frac{2l\pi}{n}}. \qquad (10)$$

Since the cosine function is periodic with period 2π, we have $\cos\frac{2l\pi}{n} = \cos\frac{2(l-N)\pi}{n}$. Hence, we can replace the summation over S by the summation over

$$S' = \left\{(k,l)|k,l \in \left\{-\left\lfloor\frac{n}{2}\right\rfloor, 1, \ldots, \left\lfloor\frac{n}{2}-1\right\rfloor\right\}\right\} \setminus \{(0,0)\}.$$

This implies that the imaginary part of (10) cancels out because terms in the sum can be paired up so that, in each pair, the imaginary part in both terms has the same absolute value but opposite sign. Namely:

- If none of $k, l, -k$ and $-l$ is equal to $\frac{n}{2}$, we pair up (k,l) with $(-k,-l)$.
- If none of k and $-k$ is equal to 0 or $\frac{n}{2}$, we pair up $(-\frac{n}{2}, k)$ with $(-\frac{n}{2}, -k)$ and $(k, -\frac{n}{2})$ with $(-k, -\frac{n}{2})$.
- The terms $(-\frac{n}{2}, 0)$, $(0, -\frac{n}{2})$ and $(-\frac{n}{2}, -\frac{n}{2})$ are left without a pair. This does not affect the argument because the imaginary part is equal to 0 in those terms.

Hence, we have

$$f(j,j') = \sum_{(k,l)\in S'} \frac{\cos\frac{2(kj+lj')\pi}{n}}{2-\cos\frac{2k\pi}{n}-\cos\frac{2l\pi}{n}}.$$

We define a function $g(j,j') = f(j,j') - f(j-1,j')$. By Lemma 1, $Cg(j,j')$ is a good approximation for the amplitude of $|j,j',\Uparrow\rangle$ in the state of the quantum walk after $t = O(\sqrt{N\log N})$ steps.

Lemma 3

$$\sum_{0<j',j<M} g^2(j,j') = \Omega(n^2 \ln M)$$

where $M = n^{\epsilon}$ *and* $\epsilon = \Omega(1)$, *and* $\epsilon = 1 - \Omega(1)$.

The proof of the lemma can be found in [AB+11]. Together with Lemma 1, this implies that the sum of amplitudes of $|j,j',\Uparrow\rangle$, $0 < j', j < M$ is $\Omega(\frac{\log M}{\log n}) - o(1)$. Since $\frac{\log M}{\log N} = \epsilon$, this would complete the proof of Theorem 1.

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