What Is Fuzzy Logic – And Why It Matters to Us

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32.1 The Aim

The aim of this short note is twofold: recounting how our research group became interested in fuzzy logic, and briefly discussing a definition of fuzzy logic suggested by Běhounek and Cintula (see [1]). Lest the anecdotal *incipit* should be dismissed (perhaps deservedly) with a blunt *So what?*, we remind that prospective contributors to this volume are required to mention how they arrived to the field of fuzzy logic and to present their views and expectations 'on fuzziness'. Both aims, therefore, seem to sit comfortably within the scopes of this book, especially in view of the fact that Lofti Zadeh has always been concerned with the problem of delimiting the boundaries of the subject he pioneered (see e.g. his [16]).

32.2 Why Do We Care?

In the 1980s, the logic scene in the Philosophy Department at the University of Florence, where the two oldest members of our group were trained in the trade, was dominated by two charismatic figures, Ettore Casari and Maria Luisa Dalla Chiara. Neither the former nor the latter is a fuzzy logician *stricto sensu* — nor, so far as we can remember, did they ever devote to fuzzy logic more than a passing reference in the undergraduate courses we attended. Both of them, however, had research interests that bordered on fuzzy logic, and by sharing their own views with us they contributed in a decisive way to turn us to this kind of investigation.

Ettore Casari was fascinated by the project of building a formal model for comparison in natural language, a project he fleshed out in several papers published from the early 1980s onwards (see e.g. his [5]). He wanted to account for such comparative sentences as 'c is at most as P as d is Q', where c,d are names and P,Q are predicates. If we accept that sentences may admit of different 'degrees of truth', the aforementioned sentence can be considered true when 'c is P' is at most as true as 'd is Q'. To formalise his idea, Casari used an implication connective which comes out true exactly when its antecedent is at most as true as its consequent. Although fuzzy logics share the same basic assumptions, for a number of reasons Casari was dissatisfied with such an approach: for example, the use of bounded algebras as systems of truth degrees in mainstream fuzzy logics prevents a proper treatment of comparative

R. Seising et al. (Eds.): *On Fuzziness: Volume 1*, STUDFUZZ 298, pp. 211–215.
DOI: 10.1007/978-3-642-35641-4_32

sentences of the form 'c is less P than d' when both c and d are clear-cut instances of P, yet it makes sense to say that d is more P than c is. The vicinity between comparative logic and fuzzy logic is further highlighted by the fact that the equivalent variety semantics of Casari's propositional comparative logic is the variety of *ℓ-pregroups*, a common abstraction of Abelian ℓ-groups and MV algebras. Not surprisingly, a crucial influence on the definitive form that comparative logic assumed by the end of the 1980s was played (according to Casari himself) by Daniele Mundici, then at the Mathematics Department of the University of Florence (to which he recently returned), a leading figure in the research on fuzzy logic in general, and on MV algebras in particular.

Marisa Dalla Chiara has advocated and actively participated in the development of the so-called *unsharp approach* to quantum theory since the seminal contribution by Ludwig (see [13]). In a nutshell: in standard (sharp) quantum logic *à la* Birkhoff-von Neumann, propositions ascribing properties are represented by projection operators (or, equivalently, by closed subspaces of a Hilbert space). In this approach, vagueness and truth degrees play no rôle: the possible values of a given physical magnitude are expressed by the eigenvalues of the corresponding self-adjoint operator, and projection operators have eigenvalues in $\{0,1\}$ — meaning that either the property at issue definitely holds or it definitely does not hold. In unsharp quantum theory and in unsharp quantum logic, however, a more general notion of property has been suggested. Projections are replaced by *effects*, whose eigenvalues may range throughout the whole real interval $[0,1]$. Unsharp quantum theory, therefore, accommodates 'vague' properties as well, which are not an all-or-nothing matter but may hold to a given degree. True to form, the mathematical structures that arise within this research stream are, more often than not, either closely related to fuzzy logical structures or even plain generalisations of such. GLP (See e.g. [9].) More recently, Marisa also championed another brand of quantum logic, called *quantum computational logic* (see for instance [7], Chapter 17.), which departs even more drastically from the standard Birkhoff-von Neumann approach. Meanings of sentences are no longer formalised through closed subspaces of a Hilbert space, but by means of *quantum information units* acting as quantum analogues of classical bits and registers: qubits, quregisters, and qumixes. Somewhat unexpectedly, however, fuzzy-like structures appear in this setting, too. And it is precisely this interplay that triggered most of the joint research work subsequently done by our group.

Over the last ten years or so, in fact, the Cagliari branch of the *équipe* led by Marisa has mainly focused on the algebraic models of quantum computational logics, as well as on the logics themselves but from the viewpoint of abstract algebraic logic. Most of the effort has gone into the investigation of *quasi-MV algebras* (see for instance [12]), generalisations of MV algebras connected with an irreversible disjunction connective arising in quantum computational logic, and their expansions by a genuinely quantum operator of square root of negation ($\sqrt{'}$ *quasi-MV algebras*: (see [10])). Since the 1-assertional logics of these varieties, or of some closely related quasivarieties (see [4] and [14].), are indeed weakenings or expansions, or expansions of weakenings, of infinite-valued Łukasiewicz logic, the belief that this domain and fuzzy logic are (to use a word cherished by quantum theorists) inextricably entangled is even more corroborated.

32.3 What Do We Think It Is(n't)?

The papers we devoted to quasi-MV algebras and their associated logics employ methods, concepts and tools that by any standard appear as closely related to the ones commonly adopted in present-day fuzzy logic. In spite of these evident similarities, these logics do not count as fuzzy according to the definition proposed by Běhounek and Cintula (see [1]). These authors confine themselves to what they call *weakly implicative logics*, i.e. (roughly speaking) propositional logics containing a connective $\rightarrow$ with properties that are reasonable for an implication (including modus ponens). In their opinion, a weakly implicative logic $\mathbf{L}$ is fuzzy iff it is strongly complete w.r.t. to the class of all *totally ordered* $\mathbf{L}$-matrices, where the order is so defined as to have $x \leq y$ just in case $x \rightarrow y$ is a designated value of the matrix [1].

To its advantage, this suggestion has a fair amount of liberality. Their propounders resist the temptation to relegate fuzzy logic into the safe territory of the $[0,1]$ closed real unit interval, because any possible way to formally specify this idea would lead to unreasonable verdicts: if we require from a fuzzy logic that it be complete w.r.t. a $[0,1]$-based semantics, then many fuzzy predicate logics would not come out fuzzy (because they fail to be standard complete), while if we require that algebras on $[0,1]$ generate the corresponding variety, a prototypically fuzzy logic like product Łukasiewicz logic (see [11]) would not count as such. On the other hand, when it comes to classifying individual logics as fuzzy or not fuzzy, this criterion seems to tally in most cases with the usual practice in the community, as Běhounek and Cintula observe.

Nonetheless, it remains to be seen whether the choice of restricting the domain of application of this definition to weakly implicative logics is reasonable. Should anyone suggest, perhaps in accord with a more conservative viewpoint, that a logic *with weakening* $\mathbf{L}$ is fuzzy iff it is strongly complete w.r.t. to the class of all totally ordered $\mathbf{L}$-matrices, Běhounek and Cintula would probably retort (and we would go with them) that such a delimitation is unjustified and even harmful, because it is precisely *outside* the class of logics with weakening that their own criterion makes the most interesting distinctions [2]. By parity of reasoning, one could rightly wonder if being able to decide whether some logic is fuzzy or not should really hinge on the presence of an implication satisfying modus ponens. Weakly implicative logics are

[1] This definition is mentioned in a slightly modified form in the more recent [2], written with Petr Hájek, but not as the proposed definition of fuzzy logic. Here, in fact, another definition is given, where Condition i) is replaced by a much more restrictive requirement: being an intuitionistic substructural logic, namely, having as algebraic counterpart a class of FL-algebras. Here we will discuss neither this stance, nor another position recently embraced by one of these authors in [6], where the plausibility of a sharp, formal definition of fuzzy logic is called into question.

[2] Some considerations along these lines are offered in [1], p. 608.

at least *protoalgebraic*, while some logics that are unanimously classified as fuzzy by the fuzzy logical community are not such, and therefore do not even fall within the scope of the criterion.

Consider the so-called 'infinite-valued Łukasiewicz logic that preserves degrees of truth', first introduced by Wójcicki [15] and deeply investigated by Josep Font and his collaborators.[3] This non-protoalgebraic logic uses the same valuations into the $[0,1]$ interval as the standard infinite-valued Łukasiewicz logic, but adopts a different consequence relation: whereas in Łukasiewicz logic a formula α follows from the set Γ just in case for all such valuations v, $v(\alpha) = 1$ whenever $v(\gamma) = 1$ for all γ in Γ, here a formula α follows from the set Γ just in case for all such valuations v, $v(\alpha)$ is greater or equal than the *minimum* of the set $\{v(\gamma) : \gamma \in \Gamma\}$. In other words, while valid inferences in standard Łukasiewicz logic preserve just *absolute truth* but allow degrees of truth to decrease from premisses to conclusion, here valid inferences always have a conclusion which is 'at least as true' as the 'falsest' premiss. If fuzzy logics are to be logics of truth degrees, it can be convincingly argued that not only this logic belongs to the class, but it also takes degrees of truth much more seriously than its absolute truth-preserving counterpart (see [8], p. 392). Furthermore, since in the times of Łukasiewicz it was commonplace to view logics as determined by a set of valid formulas, or theorems, rather than as consequence relations, it is not out of the question that Łukasiewicz himself, when thinking of his infinite-valued logic, had *this* logic in mind rather than the truth-preserving one that is nowadays associated with his name ([15, p. 279]).

The logics from [4] and [14] are in a similar situation. In general, they fail to be protoalgebraic and therefore their membership in the class of fuzzy logic cannot be determined by Běhounek's and Cintula's evaluation standard. Moreover, quantum computational logics are typically complete w.r.t. a class of *totally preordered* matrices, but this preordering may fail to be antisymmetric[4]. However, one could modify Běhounek's and Cintula's suggestion by relaxing in some way the precondition that the scope of the criterion is limited to weakly implicative logics, and, perhaps, by also loosening Condition ii) so as to let in logics that are complete w.r.t. totally *pre*ordered matrices. As regards both aspects, the framework suggested by Berman and Blok (see [3]) in their paper on algebras defined from ordered sets looks promising: one could simply require, for instance, that the ordering relation referred to in Condition ii) be equationally definable in the class of the algebra reducts of the matrix models of the logic at issue (but not necessarily through a condition of the type $x \to y \in D$, for some implication connective $\to$ and some equationally definable truth predicate D). In [4], a first attempt has been made to extend Berman's and Blok's framework to the case of equationally definable preorders. This is not the place, of course, to evaluate the merits of this proposal, nor to develop it further into an

[3] See [8] for a brief description and a philosophical assessment.

[4] There are important exceptions to this state of affairs. The quasivariety $\mathbb{C}$ of Cartesian $\sqrt{'}$ quasi-MV algebras is relatively 1-regular, and therefore its 1-assertional logic is regularly algebraisable with $\mathbb{C}$ as equivalent quasivariety semantics; $\mathbb{C}$, in turn, is generated by a single *lattice-ordered* algebra whose order is defined by the implication (generalised) connective of the logic.

alternative definition of fuzzy logic. What we wanted to point out is that Běhounek's and Cintula's criterion, though on the right track, is in need of some adjustment if it aims at a discrimination of many individual cases present in the logical landscape, of which quantum computational logics and logics preserving degrees of truth are interesting examples.

Acknowledgement. We thank Libor Běhounek and Petr Cintula for pointing out an inaccuracy in a first draft of this paper, and for their stimulating observations.

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