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Fuzziness in Automata Theory: Why? How?

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17.1 Introduction

The aim of this article is to explain why we study fuzzy automata and how we do it, i.e., to highlight the most efficient tools of the theory of fuzzy sets that we use in our research. In addition, we want to show how research in the theory of fuzzy automata affected our research in other areas of the theory of fuzzy sets.

We entered in the world of fuzziness when we crossed from the classical algebra and automata theory to the theory of fuzzy automata. Besides being considered as a natural generalization of ordinary automata and languages, fuzzy automata and related languages have also been studied as a means for bridging the gap between the precision of computer languages and vagueness and imprecision, which are frequently encountered in the study of natural languages (cf. [18]). During the decades, they have got a wide field of applications. However, many authors thought mainly about the properties of ordinary automata which can be transferred to fuzzy automata. We found that the theory of fuzzy automata is not only simple translation of the results from the classical automata theory to the language of fuzzy sets, but it is possible to use powerful tools of the theory of fuzzy sets in the study of fuzzy automata.

The key point is that a fuzzy automaton can be regarded as a fuzzy relational system. It can be specified by a family $\{\delta_x\}_{x\in X}$ of fuzzy transition relations on the set of states A, indexed by the input alphabet X, and fuzzy subsets σ and τ of A, the fuzzy subsets of initial and terminal states. Inductively we define the composite fuzzy transition relations $\{\delta_u\}_{u\in X^*}$ by putting that δ_ε is the crisp equality, and $\delta_{ux} = \delta_u \circ \delta_x$, for $u \in X^*$, $x \in X$. Now, the fuzzy language recognized by the fuzzy automaton $\mathscr{A}$ is defined as a fuzzy subset $L_{\mathscr{A}}$ of X^* given by $L_{\mathscr{A}}(u) = \sigma \circ \delta_u \circ \tau$, for $u \in X^*$.[1] This way of representing fuzzy automata, and fuzzy languages that they recognize, enables to study fuzzy automata using fuzzy relational calculus, and to express many problems through fuzzy relation equations and inequalities. Fuzzy relational calculus and fuzzy relation equations and inequalities have been widely used in our research.

Previously, fuzzy relational calculus and fuzzy relation equations and inequalities were used in the theory of fuzzy automata only by few authors – Peeva,

[1] Here X^* denotes the monoid of all words over X, $\varepsilon \in X^*$ is the empty word, and $\circ$ denotes the compositions of two fuzzy relations, of a fuzzy set and a fuzzy relation and two fuzzy sets, defined in the usual way over a residuated lattice or lattice-ordered monoid.

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Bělohlávek, and Li and Pedrycz (cf. [1, 19–22]). Surprisingly, such approach has not been used for ordinary nondeterministic automata, although their behavior can be expressed in terms of the calculus of two-valued relations. Probably, the reason for this is the fact that nondeterministic automata are predominantly considered from the perspective of the graph theory, and not from the perspective of the algebra of relations. A little bit similar approach has been used for weighted automata over a semiring, whose behavior is defined through the calculus of matrices with entries in the underlying semiring (cf. [8]). However, matrices over a semiring do not possess some very important properties of ordinary and fuzzy relations, and their use in the study of weighted automata is not as fruitful as the use of fuzzy relations in the study of fuzzy automata.

We will briefly explain how we used fuzzy relational calculus and fuzzy relation equations and inequalities in solving the fundamental problems of the theory of fuzzy automata: *determinization*, *equivalence* and *state reduction*.

17.2 Determinization of Fuzzy Automata

A deterministic fuzzy automaton is a fuzzy automaton having exactly one crisp initial state and a deterministic transition function, and the fuzziness is entirely concentrated in the fuzzy set of terminal states. The determinization of a fuzzy automaton is a procedure of constructing an equivalent deterministic fuzzy automaton[2]. Such procedure is usually required in most practical applications and implementation of automata. The first determinization algorithms for fuzzy automata, provided by Bělohlávek and Li and Pedrycz, generalize the well-known subset construction, and have the same shortcoming as its crisp counterpart: some states of the resulting automaton can be redundant (cf. [1, 19]). We have constructed the Nerode automaton associated with a fuzzy automaton, a deterministic fuzzy automaton which is equivalent to the original fuzzy automaton and has no redundant states. Its states are fuzzy sets of the form $\sigma_u = \sigma \circ \delta_u$, for $u \in X^*$, the single initial state is $\sigma = \sigma_\varepsilon$, the transition function δ_N is defined by $\delta_N(\sigma_u, x) = \sigma_{ux}$, for $u \in X^*$, $x \in X$, and the fuzzy set τ_N of terminal states is defined by $\tau_N(\sigma_u) = \sigma_u \circ \tau$, for $u \in X^*$. The Nerode automaton always has smaller number of states than automata constructed by the previous determinization methods, but nevertheless, in some cases it may be infinite. Its finiteness depends on certain local properties of the underlying structure of truth values, and necessary and sufficient conditions under which the Nerode automaton is finite have been determined. We have also provided an improved algorithm, which constructs the reduced Nerode automaton with even smaller number of states than the Nerode automaton (cf. [10, 13, 17]).

The Nerode automaton was originally constructed for fuzzy automata over a complete residuated lattice, but it was noted that the same construction can be applied to fuzzy automata over a lattice-ordered monoid, and even more, to weighted automata

[2] Two fuzzy automata are equivalent if they recognize the same fuzzy language.

over a semiring. All these structures have the multiplication which is distributive over the supremum (or addition), which ensures associativity of the composition of fuzzy relations. However, it was shown that the Nerode automaton and the reduced Nerode automaton can be constructed even if the composition is not associative, i.e., for automata with weights that are taken in a strong bimonoid, a structure which is not necessarily distributive. In particular, this includes fuzzy automata over arbitrary lattices (cf. [2, 17]).

17.3 Equivalence of Fuzzy Automata and Bisimulations

Another important problem of automata theory is to determine whether two given automata are equivalent. For deterministic automata this problem is solvable in polynomial time, but for nondeterministic and fuzzy automata it is computationally hard. It is also desirable to express the equivalence of automata as a relation between their states, if possible, or find some relation between states which implies the equivalence. The equivalence of two deterministic automata can be expressed in terms of relationships between their states, but in the case of nondeterministic and fuzzy automata the problem is more complicated, and we can only examine various relations which imply the equivalence.

It is generally agreed that the best way to model the equivalence of automata is the concept of bisimulation. They give a close enough approximation of the equivalence and are efficiently computable. Bisimulations were introduced in concurrency theory, and independently, in set theory and modal logic, and nowadays, they are successfully employed in many areas of computer science and mathematics. We have introduced two types of simulations for fuzzy automata, forward and backward simulations, and combining them, we have defined four types of bisimulations (cf. [5, 6]). Forward simulations between two fuzzy automata $\mathscr{A}$ and $\mathscr{A}'$ are defined as solutions to the system of fuzzy relation inequalities $\sigma \leq \sigma' \circ \varphi^{-1}$, $\varphi^{-1} \circ \delta_x \leq \delta'_x \circ \varphi^{-1}$ $(x \in X)$, $\varphi^{-1} \circ \tau \leq \tau'$, backward simulations as solutions to the system $\tau \leq \varphi \circ \tau'$, $\delta_x \circ \varphi \leq \varphi \circ \delta'_x$ $(x \in X)$, $\sigma \circ \varphi \leq \sigma'$, and bisimulations are defined by a combination of these two systems.[3] The greatest solutions to these systems, i.e., the greatest simulations and bisimulations between fuzzy automata, are computed by iterative procedures. Termination of these iterative procedures after a finite number of steps also depends on local properties of the underlying structure of truth values. Key role in the computation of the greatest simulations and bisimulations play the residuals of fuzzy relations, which we have introduced. To ensure the existence of these residuals, it is necessary that the underlying structure of truth values is also residuated, so our theory has been developed for fuzzy automata over a complete residuated lattice.[4]

[3] Here φ denotes an unknown fuzzy relation between the sets of states of $\mathscr{A}$ and $\mathscr{A}'$, and φ^{-1} denotes its inverse (converse, transpose) fuzzy relation.

[4] In fact, commutativity of the multiplication is not necessary, and analogous results can be obtained when the underlying structure of truth values is a quantale.

17.4 State Reduction

In contrast to deterministic automata, for which there are many fast minimization algorithms, the state minimization problem for nondeterministic and fuzzy automata is computationally hard. For these automata, a more practical problem is the *state reduction*, where we have to construct an automaton with as small as possible number of states, which is equivalent to a given automaton. This automaton need not be minimal, but must be efficiently computable.

We have reduced the state reduction problem for fuzzy automata to the problem of solving a particular system of fuzzy relation equations (cf. [7, 23]). For a given fuzzy automaton and a fuzzy equivalence on its set of states, we have defined the related factor fuzzy automaton. In general, these two fuzzy automata are not equivalent. Based on the fact that the fuzzy language recognized by a fuzzy automaton $\mathscr{A}$ can be expressed as $L_{\mathscr{A}}(u) = \sigma \circ \delta_u \circ \tau$, for $u \in X^*$, we have expressed the equivalence of a fuzzy automaton and its factor fuzzy automaton as a system of fuzzy relation equations, called the general system. Namely, we have shown that these two fuzzy automata are equivalent if and only if the fuzzy equivalence by which we perform factorization is a solution to the general system. However, the general system may consist of infinitely many equations, and finding its non-trivial solutions may be a very difficult task, so we have aimed our attention to some instances of this system which consist of finitely many equations and are easier to solve. The most interesting instances are those systems that define forward and backward bisimulations between the states of a single fuzzy automaton. We have provided effective procedures for computing the greatest forward and backward bisimulation fuzzy equivalences on a fuzzy automaton, which ensure the best reductions by fuzzy equivalences of these types. Moreover, we have shown that even better reductions can be achieved alternating reductions by forward and backward bisimulation fuzzy equivalences, and also, if we use fuzzy quasi-orders instead of fuzzy equivalences.

17.5 The Reverse Impact

As we have seen, fuzzy relational calculus and the theory of fuzzy relation inequalities and equations have had a tremendous impact on our research in the theory of fuzzy automata. However, this research has had a very strong reverse impact. Problems arising from the study of fuzzy automata have led to the launch of some new questions regarding various types of fuzzy relations. We have given many new results on fuzzy equivalences and fuzzy quasi-orders, and moreover, we have introduced a completely new concept of a uniform fuzzy relation (cf. [3, 4]). Our original intention was to introduce uniform fuzzy relations as a basis for defining such concept of a fuzzy function which would provide a correspondence between fuzzy functions and fuzzy equivalence relations, analogous to the correspondence between crisp functions and crisp equivalence relations. This was done, but also, it turned out that uniform fuzzy relations establish natural relationships between fuzzy partitions of two sets, some kind of "uniformity" between these fuzzy partitions. Roughly speaking, uniform fuzzy relations can be conceived as fuzzy equivalence relations which relate elements of two possibly different sets. They were employed to solve some systems of fuzzy relation equations that have important applications in approximate reasoning, and to define and study fuzzy homomorphisms and fuzzy relational morphisms of algebras (cf. [4, 11]). However, uniform fuzzy relations have shown their full strength in the study of equivalence between fuzzy automata, which has previously been discussed (cf. [5]).

Systems of fuzzy relation equations and inequalities that emerged from our research in the theory of fuzzy automata initiated the study of the systems of the same form from the general aspect. These systems are referred to as weakly linear systems (cf. [9, 12, 14]). There has been proved that every weakly linear system, with a complete residuated lattice as the underlying structure of truth values, has the greatest solution, and an algorithm has been provided for computing this greatest solution. This algorithm is based on the computing of the greatest post-fixed point, contained in a given fuzzy relation, of an isotone function on the lattice of fuzzy relations. The algorithm represents an iterative procedure whose each single step can be viewed as solving a particular linear system, and for this reason these systems were called weakly linear. This iterative procedure terminates in a finite number of steps whenever the underlying complete residuated lattice is locally finite, for example, when dealing with Boolean or Gödel structure. Otherwise, some sufficient conditions under which the procedure ends in a finite number of steps have been determined. If the underlying complete residuated lattice satisfies infinite distributive laws for the supremumand multiplication over infimum, for example, when dealing with a structure defined by a continuous t-norm on the real unit interval [0, 1] (an BL-algebra on [0, 1]), the greatest solution can be obtained as the infimum of fuzzy relations outputted after each single step of the iterative procedure.

It is worth noting that the methodology developed for solving weakly linear systems has been recently extended to an even broader context, and used for solving systems of inequalities and equations over partially ordered sets defined by residuated and residual functions (cf. [15]).

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