Intuitionistic Fuzzy Preference Relations and Hypergroups

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Abstract. In this paper we present a connection between intuitionistic fuzzy relations and hypergroups. In particular, we construct a hypergroup associated with a binary relation naturally induced by an intuitionistic fuzzy relation. We present some of its properties, investigating when it is a join space or a reduced hypergroup, in the framework of the intuitionistic fuzzy preference relations.

Keywords: Hypergroup, Join space, Intuitionistic fuzzy set, Intuitionistic fuzzy preference relation.

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1 Introduction

In the real life, a lot of problems takes place in an environment in which the goals, the costraints, and the consequences of possible actions are not precisely known (Bellman and Zadeh [3]). The concept of intuitionistic fuzzy set (IFS) introduced by Atanassov [1,2] is one of the mathematical tools higly used to deal with imprecision, vagueness, uncertainty in diverse areas as Computer Science, Social Science, Decision Making, Management Science, Neural Networks, Medicine, Engineering, etc. The application of IFS theory in Decision Making, for example, is very useful to overcome and model the ambiguity generated by diverse factors: a decision maker may not posses a precise or sufficient level of knowledge of the problem, or is unable to discriminate the degree to which one alternative is better than others; it could also happen that the decision maker provides the degree of preference for alternatives, without being sure about it [21,32]. An updated review of the role of IFS theory in decision-making problems, supplemented with a rich bibliography, is presented in [21].

Correspondences between objects are suitable described by relations, that can be crisp or fuzzy. Remaining in the decision-making area, the most frequently used and thus investigated type of relation is that of preference relation, for the first time generalized from the fuzzy case to the intuitionistic fuzzy one by Szimdt and Kacprzyk [32]. A preference relation P on a discrete finite set X of alternatives is characterized by a function $\mu_P : X \times X \longrightarrow D$, where D is the

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domain of representation of preference degrees, and therefore can be expressed by meaning of a square matrix. The preference relations can be mainly classified into the following categories: the multiplicative preference relations [20], fuzzy preference relations [25], intuitionistic fuzzy preference relations [32,33], and interval-valued intuitionistic fuzzy preference relations [34].

On the other hand, fuzzy set theory has interesting applications also in algebra, in particular in algebraic hyperstructure theory, where the connections between the classical structures and fuzzy sets (or their generalizations) determined new crisp hyperstructures, fuzzy subhyperstructures, or fuzzy hyperstructures. A well-known method to obtain new algebraic hyperstructures is to define hyperproducts generated by relations. The most studied such constructions are those of Rosenberg [28] and Corsini [8], investigated later by Corsini and Leoreanu [10], Spartalis et al. [29,30,31], Cristea et al. [11,12,13,14], De Salvo and Lo Faro [17,18], etc. This connection has been extended to n-ary hyperstructures by Davvaz and Leoreanu-Fotea [15,27]. Another way to obtain hyperstructures is given by Chvalina [6] and called "Ends Lemma", used in [22]. Feng [19] obtained fuzzy hypergroups from fuzzy relations, while Jančić-Rašović in [23] constructed hyperrings from fuzzy relations defined on a semigroup. In this article, we continue in the same direction, proposing a method for defining hyperoperations from intuitionistic fuzzy relations.

The rest of the paper is organized as follows. After a short description of the main properties of the intuitionistic fuzzy relations (IFRs), emphasizing those of the intuitionistic fuzzy preference relations (IFPRs), covered in Preliminaries, a brief introduction to the theory of hypergroups associated with binary relations follows in Section 3. We recall the Rosenberg's method and the notion of reduced hypergroup introduced by Jantosciak [24]. Section 4 is dedicated to the construction of hypergroups associated with IFRs, giving examples of IFPRs and discussing their properties connected with join spaces and reduced hypergroups. We end this article with some concluding remarks and possible new lines of research.

2 Preliminaries Concerning Intuitionistic Fuzzy Relations

We recall some definitions concerning intuitionistic fuzzy relation theory and we fix the notations used in this paper.

Diferent generalizations of fuzzy sets have been developed for a better modelling of ambigous problems. The concept of intuitionistic fuzzy set (called also Atanassov's intuitionistic fuzzy set) can be viewed as an alternative approach to define a fuzzy set whenever available information is not sufficient to describe an imprecise, vague concept by means of ordinary fuzzy sets [21].

Definition 2.1. [1,2] An *intuitionistic fuzzy set* (shortly IFS) on a universe X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, where $\mu_A(x) \in [0,1]$, called the *degree of membership* of x in A, $\nu_A(x) \in [0,1]$, called the *degree of non-membership* of x in A, verify, for any $x \in X$, the relation $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The class of IFSs on a universe X will be denoted by $\mathcal{IFS}(X)$.

It is clear that an IFS can be considered as a fuzzy set whenever $\nu_A(x) = 1 - \mu_A(x)$, for any $x \in X$, but conversely not.

Definition 2.2. [1,2] An *intuitionistic fuzzy relation* R (shortly IFR) from a universe X to a universe Y is an IFS in $X \times Y$, i.e. a set by the form $R = \{((x, y); \mu_R(x, y), \nu_R(x, y)) \mid (x, y) \in X \times Y\}$, where $\mu_R(x, y) + \nu_R(x, y) \leq 1$, for any $(x, y) \in X \times Y$.

Furthermore, the number $\pi_R(x, y) = 1 - \mu_R(x, y) - \nu_R(x, y)$, for $(x, y) \in X \times Y$, is called the *index* of the element (x, y) in IFR R and it is described as a degree of hesitation whether x and y are in the relation R on not.

The class of IFRs from X to Y will be denoted by $\mathcal{IFR}(X \times Y)$ and the class of IFRs on X will be denoted by $\mathcal{IFR}(X)$.

The *domain* of an IFR $R \in \mathcal{IFR}(X \times Y)$ is the IFS in X defined by $dom(R) = \{(x, \bigvee_{y \in Y} \mu_R(x, y), \bigwedge_{y \in Y} \nu_R(x, y)) \mid x \in X\}$ and the *range* of R is the IFS in Y defined by $rng(R) = \{(x, \bigvee_{x \in X} \mu_R(x, y), \bigwedge_{x \in X} \nu_R(x, y)) \mid y \in Y\}$.

In the following, we mention some basic operations between IFRs. For more details see [5,16].

Definition 2.3. *i*) Let R and S be in $\mathcal{IFR}(X \times Y)$. For every $(x, y) \in X \times Y$, we define

1. $R \subseteq S \Longleftrightarrow \mu_R(x, y) \leq \mu_S(x, y)$ and $\nu_R(x, y) \geq \nu_S(x, y)$
2. $R \preceq S \Longleftrightarrow \mu_R(x, y) \leq \mu_S(x, y)$ and $\nu_R(x, y) \leq \nu_S(x, y)$
3. $R \cup S = \{((x, y), \mu_R(x, y) \vee \mu_S(x, y), \nu_R(x, y) \wedge \nu_S(x, y))\}$
4. $R \cap S = \{((x, y), \mu_R(x, y) \wedge \mu_S(x, y), \nu_R(x, y) \vee \nu_S(x, y))\}$
5. $R^c = \{((x, y), \nu_R(x, y), \mu_R(x, y))\}$.

The family $(\mathcal{IFR}(X \times Y), \cup, \cap)$ is a complete, distributive lattice, with respect to the partially ordering $\preceq$.

ii) Let R in $\mathcal{IFR}(X \times Y)$ and S in $\mathcal{IFR}(Y \times Z)$. Then the composition between R and S is an IFR on $X \times Z$ defined as

$$R \circ S = \{((x, z), \bigvee_{y \in Y} (\mu_R(x, y) \wedge \mu_S(y, z)), \bigwedge_{y \in Y} (\nu_R(x, y) \vee \nu_S(y, z)))\}$$

whenever $0 \leq \bigvee_{y \in Y}(\mu_R(x, y) \wedge \mu_S(y, z)) + \bigwedge_{y \in Y}(\nu_R(x, y) \vee \nu_S(y, z)) \leq 1$.

Now we consider the IFRs defined on a set X.

Definition 2.4. An IFR R on a set X is

1. *reflexive* if $\mu_R(x, x) = 1$ (and consequently $\nu_R(x, x) = 0$), for any $x \in X$;
2. *symmetric* if $\mu_R(x, y) = \mu_R(y, x)$ and $\nu_R(x, y) = \nu_R(y, x)$, for any $x, y \in X$; in the opposite way we will say that it is *asymmetric*;
3. *transitive* if $R^2 = R \circ R \subseteq R$;
4. *antisymmetrical intuitionistic* if, for any $(x, y) \in X \times X$, $x \neq y$, then $\mu_R(x, y) \neq \mu_R(y, x)$, and $\nu_R(x, y) \neq \nu_R(y, x)$, but $\pi_R(x, y) = \pi_R(y, x)$.

5. *perfect antisymmetrical intuitionistic* if, for any $(x, y) \in X \times X$, $x \neq y$ and $\mu_R(x, y) > 0$ or ($\mu_R(x, y) = 0$ and $\nu_R(x, y) < 1$), then $\mu_R(y, x) = 0$ and $\nu_R(y, x) = 1$;
6. an *equivalence* if it is reflexive, symmetric and transitive.

Throughout this paper we focus on the intuitionistic fuzzy preference relations (shortly IFPRs), which are widely applied in decision-making theory, where we deal with the finite set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$ and a decision maker who needs to express his/her preferences over the alternatives, constructing thus a *preference relation* P on the set X. It is characterized by a function $\mu_P : X \times X \longrightarrow D$, where D is the domain of representation of preference degrees. If we pass to the fuzzy case, the definition changes; a *fuzzy preference relation* P on the set X is represented by a membership function $\mu_P : X \times X \longrightarrow [0, 1]$ satisfying several properties: taking $\mu_P(x_i, x_j) = \mu_{ij}$, then $\mu_{ij} + \mu_{ji} = 1$, $\mu_{ii} = 0.5$, for all $i, j = 1, 2, \ldots, n$, where μ_{ij} denotes the preference degree of the alternative x_i over x_j. Generalizing now to the intuitionistic fuzzy case, the definition is given as follows.

Definition 2.5. [32,33] An *intuitionistic fuzzy preference relation* R on the finite set X of cardinality n is represented by a matrix $R = (r_{ij})_{n \times n}$, with $r_{ij} = (\mu_{ij}, \nu_{ij}, \pi_{ij})$, for $i, j = 1, 2, \ldots, n$, where $\mu_{ij} = \mu_R(x_i, x_j)$ is the certainty degree to which x_i is preferred to x_j, $\nu_{ij} = \nu_R(x_i, x_j)$ is the certainty degree to which x_i is non-preferred to x_j, and $\pi_{ij} = \pi_R(x_i, x_j) = 1 - \mu_{ij} - \nu_{ij}$ describes the uncertainty degree (or the hesitation) to which x_i is preferred to x_j. Furthermore, μ_{ij} and ν_{ij} satisfy the following relations: $0 \leq \mu_{ij} + \nu_{ij} \leq 1$, $\mu_{ij} = \nu_{ji}$, $\mu_{ji} = \nu_{ij}$, $\mu_{ii} = \nu_{ii} = 0.5$, for all $i, j = 1, 2, \ldots, n$. It is clear that $\pi_{ij} = \pi_{ji}$, for all $i, j = 1, 2, \ldots, n$.

3 A Brief Introduction to Hypergroups Associated with Binary Relations

Several hyperproducts have been obtained by meaning of binary relations. We recall here that introduced by Rosenberg [28], which is the first one of this type and the most explored one. For a comprehensive overview of hypergroup theory, the reader is refereed to the fundamental books [7,9].

3.1 Rosenberg's Method

Let ρ be a binary relation defined on a nonempty set H. For any pair of elements $(a, b) \in \rho$, we call a a *predecessor* of b and b a *successor* of a.

We addopt the following notations: $L_a = \{b \in H \mid (a, b) \in \rho\}$ and $R_a = \{b \in H \mid (b, a) \in \rho\}$ for the *afterset* and, respectively, *foreset*, of the element a.

For any two elements $x, y \in H$, we define the following hyperproduct

$$x \circ_\rho y = \{z \in H \mid (x, z) \in \rho \;\text{ or }\; (y, z) \in \rho\}.$$

We denote by $\mathbb{H}_\rho$ the hypergroupoid $(H, \circ_\rho)$.

An element $x \in \rho$ is called *outer element* of ρ if there exists $h \in H$ such that $(h, x) \notin \rho^2$. The following theorem states necessary and sufficient conditions for a binary relation ρ to generate a hypergroup $\mathbb{H}_\rho$ in the sense of Rosenberg.

Theorem 3.1. [28] $\mathbb{H}_\rho$ *is a hypergroup if and only if*

1. *ρ has full domain and full range.*
2. *$\rho \subset \rho^2$.*
3. *If $(a, x) \in \rho^2$ then $(a, x) \in \rho$, whenever x is an outer element of ρ.*

It follows immediately the following remark: If ρ is a preordering (reflexive and transitive), then $\mathbb{H}_\rho$ is a hypergroup.

Besides, Rosenberg gave a characterization of the hypergroup $\mathbb{H}_\rho$ in order to be a join space. Let us recall here this result.

Theorem 3.2. [28] *Let ρ be a binary relation with full domain. Then $\mathbb{H}_\rho$ is a join space if and only if*

1. *ρ has full range.*
2. *$\rho \subset \rho^2$.*
3. *If $(a, x) \in \rho^2$ then $(a, x) \in \rho$, whenever x is an outer element of ρ.*
4. *Every pair of elements of H with a common predecessor has a common successor.*
5. *For all $b, c, d \in H$, $a \in L_b$, $\{b, c\} \times L_a \subseteq \rho^2 \setminus \rho$, $L_b \cap L_a = \emptyset$ implies that $L_c \cap L_d \neq \emptyset$.*

We conclude this subsection with the following consequence.

Corollary 3.3. [28] *Let ρ be a binary relation on H with full domain and full range and such that either*

1. *$\rho = \rho^2$ or*
2. *ρ is reflexive and $(a, b) \in \rho^2 \Longrightarrow (a, b) \in \rho$, whenever b is an outer element of ρ.*

Then $\mathbb{H}_\rho$ is a join space if and only if every pair of elements of H with a common predecessor has a common successor.

3.2 Reduced Hypergroups

It may happen that a hyperproduct on a given set H does not discriminate between a pair of elements of H, when the elements play interchangeable roles with respect to the hyperoperation. Thus, a certain equivalence relation can be defined in order to identify the elements with the same properties. In order to explain better this situation, Jantosciak [24] defined on a hypergroup $(H, \circ)$ three equivalences, called *fundamental relations*: two elements x, y in H are called:

1. *operationally equivalent*, and write $x \sim_o y$, if $x \circ a = y \circ a$, and $a \circ x = a \circ y$, for any $a \in H$.

2. *inseparable*, and write $x \sim_i y$, if, for all $a, b \in H$, $x \in a \circ b \Longleftrightarrow y \in a \circ b$.
3. *essentially indistinguishable* if they are operationally equivalent and inseparable.

A *reduced hypergroup* has the equivalence class of each element with respect to the essentially indistinguishable relation a singleton. Therefore, the study of the hypergroups may be divided in the study of the reduced hypergroups and in that of the hypergroups with the same reduced form, as Jantosciak proved in [24]. Necessary and sufficient conditions such that a hypergroup associated with a binary relation is a reduced hypergroup have been presented in the papers [11,12].

A characterization of the fundamental relations for the Rosenberg hypergroup $\mathbb{H}_\rho$ is given in the following result.

Proposition 3.4. [12] *Let $\mathbb{H}_\rho$ be the Rosenberg hypergroup associated with the binary relation ρ defined on H. For any $x, y \in H$, the following implications hold:*

1. $x \sim_o y \Longleftrightarrow L_x = L_y$.
2. $x \sim_i y \Longleftrightarrow R_x = R_y$.
3. *$\mathbb{H}_\rho$ is reduced if and only if, for any $x, y \in H$, $x \neq y$, either $L_x \neq L_y$ or $R_x \neq R_y$.*

4 Hypergroups Associated with IFRs

4.1 Main Construction

In this section, we present a method to construct a new hypergroupoid starting from an IFR. We will find connections with Rosenberg hypergroup, and thus we will investigate when the obtained hypergroupoid is a hypergroup, or a join space, or a reduced hypergroup. We will focuss more on the case of intuitionistic fuzzy preference relations.

IFRs can induce different binary relations in a universe X. We deal here with that introduced by Burillo and Bustince [4], in order to justify the definition given for an intuitionistic antisymmetrical relation on X.

Let H be an arbitrary finite nonempty set, endowed with an IFR $R = (\mu_R, \nu_R)$. It induces on H the crisp binary relation ρ, defined by

$$x \rho y \Longleftrightarrow \mu_R(y, x) \leq \mu_R(x, y) \wedge \nu_R(y, x) \geq \nu_R(x, y).$$

It is known that, if R is an intuitionistic order on H, then ρ is an ordinary ordering on H [4]. The definition of intuitionistic antisymmetry is fundamental for the proof of this implication. Moreover, if we replace the definition of intuitionistic antisymmetry given by Burillo and Bustince [4] by the one given by Kaufmann [26] for the fuzzy relations, we don't obtain this implication any more in the case of fuzzy relations.

Now we associate with ρ the hyperproduct defined on H in the sense of Rosenberg [28]

$$x \circ_\rho y = L_x \cup L_y,$$

where $L_x = \{z \in H \mid (x,z) \in \rho\}$ is the *afterset* of x, denoted also with $\rho(x)$, or with $x\rho$. As in the previous sections, $\mathbb{H}_\rho$ denotes the associated hypergroupoid $(H, \circ_\rho)$, called Rosenberg hypergroupoid.

Our primary aim is to determine conditions on the IFR R such that the induced crisp relation ρ satisfies the conditions from Rosenberg's theorem (Theorem 3.2). It is not difficult to notice that, for any IFR R, the induced crisp relation ρ has full domain and full range, it is always reflexive, so $\rho \subset \rho^2$, and has no outer element. Indeed, x is an outer element of ρ if there exists $h \in H$ such that $(h,x) \notin \rho^2$; this means that there exists $h \in H$ such that, for any $z \in H$, it holds $(\mu_R(z,h) > \mu_R(h,z)$ and $\nu_R(z,h) < \nu_R(h,z))$ or $(\mu_R(x,z) > \mu_R(z,x)$ and $\nu_R(x,z) < \nu_R(z,x))$, which is impossible for $z = h$, in the first case, and for $z = x$ in the second one. Concluding, it is clear that ρ always satisfies the conditions of Theorem 3.2, so $\mathbb{H}_\rho$ is always a hypergroup.

Two natural questions arrise:

1. *When $\mathbb{H}_\rho$ is a join space?*
2. *When $\mathbb{H}_\rho$ is a reduced hypergroup?*

Proposition 4.1. *For every IFR $R = (\mu_R, \nu_R)$ defined on a nonempty finite set H, the associated Rosenberg hypergroup $\mathbb{H}_\rho$ is a join space.*

Proof. This is an immediate consequence of Corollary 3.3, since the crisp relation ρ associated with R is reflexive, has no outer element and every pair of elements of H with a common predecessor has a common successor, because, for any $a, b \in H$, we have $(a,b) \in \rho$ or $(b,a) \in \rho$. Thus, if $(x,a), (x,b) \in \rho$, then there exists $y \in \{a,b\}$ such that $(a,y), (b,y) \in \rho$. □

Proposition 4.2. *If the IFR R on H is symmetric, then the associated crisp relation ρ is the total relation on H, thus $\mathbb{H}_\rho$is the total hypergroup, so it isn't reduced.*

In the following we consider only asymmetric IFRs on H.

Proposition 4.3. *If the IFR R is perfect antisymmetrical intuitionistic (and asymmetric), then $\mathbb{H}_\rho$ is a reduced hypergroup.*

Proof. We will prove that, for any $x \neq y$, we have that $L_x \neq L_y$, and then, by Proposition 3.4, it follows that $\mathbb{H}_\rho$ is reduced. In order to prove this, it is enough to note that, for $x \neq y$, $x \in L_y$ is equivalent with $y \notin L_x$, and then it is clear that $L_x \neq L_y$.

Let $x \in L_y$ and suppose that $y \in L_x$. Then we obtain that $\mu_R(x,y) \leq \mu_R(y,x) \leq \mu_R(x,y)$ and $\nu_R(x,y) \leq \nu_R(y,x) \leq \nu_R(x,y)$, that is R is symmetric, which is a contradiction of the hypothesis. Thus, it follows that $y \notin L_x$.

Conversely, let $y \notin L_x$, that is $\mu_R(y,x) > \mu_R(x,y)$ or $\nu_R(y,x) < \nu_R(x,y)$. Consider the first case. Since $\mu_R(y,x) > \mu_R(x,y)$, it follows that $\mu_R(y,x) > 0$, and since R is perfect antisymmetrical intuitionistic, we get $\mu_R(x,y) = 0$ and $\nu_R(x,y) = 1 \geq \nu_R(y,x)$. Thereby, $x \in L_y$.

Similarly, suppose that $\nu_R(y,x) < \nu_R(x,y)$. If $\mu_R(x,y) > \mu_R(y,x)$, then $\mu_R(x,y) > 0$ and, by the perfect antisymmetry property, it follows that $\mu_R(y,x) = 0$ and $\nu_R(y,x) = 1$, which is a contradiction with the inequality $\nu_R(y,x) < \nu_R(x,y)$. Therefore, $\mu_R(x,y) \leq \mu_R(y,x)$, which means that $x \in L_y$.

Now the equivalence $y \notin L_x \iff x \in L_y$ is completely proved and we can conclude that $L_x \neq L_y$, for any $x \neq y$. So $\mathbb{H}_\rho$ is a reduced hypergroup. □

Remark 4.4. It is worth to notice that there exist IFRs on a set H such that they are symmetric and perfect antisymmetrical intuitionistic. The identity Δ defined by

$$\mu_\Delta(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y \end{cases} \quad \nu_\Delta(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$

is a such relation. Moreover, all the relations of this type satisfy $\mu_R(x,y) = 0$ and $\nu_R(x,y) = 0$, for any $x \neq y$.

Proposition 4.5. *If the IFR R is antisymmetrical intuitionistic, then $\mathbb{H}_\rho$ is the total hypergroup or it is a reduced hypergroup.*

Proof. Let R be an antisymmetrical intuitionistic relation such that, for any $x \neq y$, $\mu_R(y,x) < \mu_R(x,y)$. Since $\pi_R(x,y) = \pi_R(y,x)$, it follows that $\nu_R(y,x) > \nu_R(x,y)$ and then $x\rho y$, for any $x \neq y$. Moreover, the associated crisp relation ρ is always reflexive. Thus, we conclude that $L_x = H$, for any $x \in H$, which means that $\mathbb{H}_\rho$ is the total hypergroup.

Let us suppose now that R is an antisymmetrical intuitionistic relation such that there exist $x \neq y$ with $\mu_R(x,y) < \mu_R(y,x)$. Since $\pi_R(x,y) = \pi_R(y,x)$, it follows that $\nu_R(x,y) > \nu_R(y,x)$. We obtain that $x \in L_y$, but $y \notin L_x$, so $L_x \neq L_y$ and thus $\mathbb{H}_\rho$ is a reduced hypergroup, accordingly with Proposition 3.4. □

4.2 The Case of IFPRs

This section is dedicated to the study of the hypergroup $\mathbb{H}_\rho$ associated with the IFPR R, insisting on the meaning of the related hyperoperation.

Let R be an IFPR on the set $H = \{x_1, x_2, \ldots, x_n\}$, represented by the matrix $R = (r_{ij})_{n\times n}$, with $r_{ij} = (\mu_{ij}, \nu_{ij}, \pi_{ij})$, for $i,j = 1,2,\ldots,n$. Then the induced crisp binary relation ρ (in the sense of [4]) is defined by the rule

$$x_i \rho x_j \iff \mu_R(x_j,x_i) \leq \mu_R(x_i,x_j) \iff \mu_{ji} \leq \mu_{ij}.$$

Since $\mu_{ji} = \nu_{ij}$, we can also write that

$$x_i \rho x_j \iff \nu_{ij} \leq \mu_{ij},$$

that is the degree of non-preference of the alternative x_i to the alternatives x_j is less than or equal to the degree of preference of the alternative x_i to x_j, and thus we can simply say that the alternative x_j is preferred less or equal with respect to x_i.

Let us see now which is the afterset L_{x_i} of the alternative x_i. By definition, $L_{x_i} = \{z \in H \mid (x_i, z) \in \rho\}$, i.e. it is the set of all alternatives $z \in H$ that the decision maker prefers less than or equal to the alternative x_i. And therefore the hyperproduct $x_i \circ_\rho x_j$ between two alternatives x_i and x_j is the set of all alternatives z that the decision maker prefers less than or equal to x_i or x_j.

Now we introduce three properties of the alternatives concerning the fundamental relations defined on a hypergroup.

Definition 4.6. We say that two alternatives x_i and x_j are

1. *operationally equivalent* if the elements x_i, x_j are operationally equivalent in the hypergroup $\mathbb{H}_\rho$, that is, for any alternative $a \in H$, the set of all alternatives that the decision maker prefers less than or equal to x_i or a coincide with the set of all alternatives that the decision maker prefers less than or equal to x_j or a.
2. *inseparable* if the elements x_i, x_j are inseparable in the hypergroup $\mathbb{H}_\rho$, that is, for any two alternatives $a, b \in H$, the decision maker prefers x_i less than or equal to a or b if and only if he/she prefers x_j less than or equal to alternatives a or b.
3. *essentially indistinguishable* if they are operationally equivalent and inseparable.

Proposition 4.7. *In a decision-making process, if two alternatives x_i and x_j are operationally equivalent or inseparable, then they are indifferent (one to respect to another) for the decision maker, that is $\mu_{ij} = \nu_{ij}$ (the degree of preference coincides with the degree of non-preference).*

Proof. Let us suppose that the alternatives x_i and x_j are operationally equivalent. A similar discussion can be done in the case they are inseparable. Since $x_i \sim_o x_j$ in the associated hypergroup $\mathbb{H}_\rho$, by Proposition 3.4, it follows that $L_{x_i} = L_{x_j}$. Thererfore $x_i \rho x_j$ and $x_j \rho x_i$, which is equivalent with $\mu_{ji} \leq \mu_{ij}$ and $\mu_{ij} \leq \mu_{ji}$, that is $\mu_{ij} = \mu_{ji} = \nu_{ij}$. □

The converse implication is not true, as we can notice from the following example.

Example 4.8. *Consider $H = \{x_1, x_2, x_3, x_4\}$ the set of four alternatives. Construct on H the IFPRs represented by the following matrices:*

$$R^{(1)} = \begin{pmatrix} (0.5,0.5,0) & (0.3,0.4,0.3) & (0.4,0.5,0.1) & (0.6,0.3,0.1) \\ (0.4,0.3,0.3) & (0.5,0.5,0) & (0.4,0.4,0.2) & (0.5,0.3,0.2) \\ (0.5,0.4,0.1) & (0.4,0.4,0.2) & (0.5,0.5,0) & (0.7,0.2,0.1) \\ (0.3,0.6,0.1) & (0.3,0.5,0.2) & (0.2,0.7,0.1) & (0.5,0.5,0) \end{pmatrix}$$

and

$$R^{(2)} = \begin{pmatrix} (0.5,0.5,0) & (0.3,0.4,0.3) & (0.4,0.5,0.1) & (0.6,0.3,0.1) \\ (0.4,0.3,0.3) & (0.5,0.5,0) & (0.4,0.4,0.2) & (0.3,0.4,0.3) \\ (0.5,0.4,0.1) & (0.4,0.4,0.2) & (0.5,0.5,0) & (0.7,0.2,0.1) \\ (0.3,0.6,0.1) & (0.4,0.3,0.3) & (0.2,0.7,0.1) & (0.5,0.5,0) \end{pmatrix}.$$

We notice that for both relations we have $\mu_{23} = \nu_{23}$*, so the alternatives* x_2 *and* x_3 *are indifferent (one with respect to another) for the decision maker.*

On the other hand, the induced crisp binary relations are

$$\rho^{(1)} = \Delta \cup \{(x_1, x_4), (x_2, x_1), (x_2, x_3), (x_2, x_4), (x_3, x_1), (x_3, x_2), (x_3, x_4)\},$$

$$\rho^{(2)} = \Delta \cup \{(x_1, x_4), (x_2, x_1), (x_2, x_3), (x_3, x_1), (x_3, x_2), (x_3, x_4), (x_4, x_2)\},$$

where $\Delta = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4)\}$ *is the diagonal relation.*

For the first IFPR we obtain that the aftersets and foresets of the elements x_i*,* $i = \{1, 2, 3, 4\}$*, are:* $L_{x_1}^{(1)} = \{x_1, x_4\}$*,* $R_{x_1}^{(1)} = \{x_1, x_2, x_3\}$*;* $L_{x_2}^{(1)} = \{x_1, x_2, x_3, x_4\}$*,* $R_{x_2}^{(1)} = \{x_2, x_3\}$*;* $L_{x_3}^{(1)} = \{x_1, x_2, x_3, x_4\}$*,* $R_{x_3}^{(1)} = \{x_2, x_3\}$*;* $L_{x_4}^{(1)} = \{x_4\}$*,* $R_{x_4}^{(1)} = \{x_1, x_2, x_3, x_4\}$*.*

Because $L_{x_2}^{(1)} = L_{x_3}^{(1)}$ *and* $R_{x_2}^{(1)} = R_{x_3}^{(1)}$*, it follows, accordingly by Proposition 3.4, that* $x_2 \sim_e x_3$*, so the associated hypergroup* $\mathbb{H}_\rho$ *is not reduced.*

Regarding the second IFPR, the aftersets and foresets of the elements x_i*,* $i = \{1, 2, 3, 4\}$*, are:* $L_{x_1}^{(2)} = \{x_1, x_4\}$*,* $R_{x_1}^{(2)} = \{x_1, x_2, x_3\}$*;* $L_{x_2}^{(2)} = \{x_1, x_2, x_3\}$*,* $R_{x_2}^{(2)} = \{x_2, x_3, x_4\}$*;* $L_{x_3}^{(2)} = \{x_1, x_2, x_3, x_4\}$*,* $R_{x_3}^{(2)} = \{x_2, x_3\}$*;* $L_{x_4}^{(2)} = \{x_2, x_4\}$*,* $R_{x_4}^{(2)} = \{x_1, x_3, x_4\}$*.*

In this case, $x_2 \nsim_e x_3$*, and moreover* $x_i \nsim_e x_j$*, for any* $i \neq j, i, j \in \{1, 2, 3, 4\}$*. Thus the associated hypergroup* $\mathbb{H}_\rho$ *is reduced.*

As an immediate consequence of Proposition 4.7, we obtain the following algebraic property.

Corollary 4.9. *If a decision maker doesn't have any sort of indifference between any two distinct alternatives, then the associated hypergroup* $\mathbb{H}_\rho$ *is reduced.*

5 Conclusions and Future Work

In this paper we have started the study of the hypergroups associated with IFRs, considering the particular case of IFPRs. Any IFR induces several crisp binary relations. Here we have considered that one introduced by Burillo and Bustince [4]. Then, a hypergroupoid, in the sense of Rosenberg [28], is associated with the binary relation, and it is proved that it is always a join space. We have extended the fundamental equivalences of Jantosciak [24] to a decision-making process, investigating when the associated Rosenberg hypergroup is reduced.

In a future work, we will analyze this association in the general case of IFRs, considering other types of induced crisp binary relations, or associated hypergroupoids, making a comparison between these cases.

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