Pairwise Comparison Matrices over Abelian Linearly Ordered Groups: A Consistency Measure and Weights for the Alternatives

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Abstract. The pairwise comparison matrices play a basic role in multicriteria decision making methods such as the Analytic Hierarchy Process (AHP).

We provide a survey of results related to pairwise comparison matrices over a real divisible and continuous abelian linearly ordered group $\mathcal{G} = (G, \odot, \leq)$, focusing on a $\odot$-consistency measure and a weighting vector for the alternatives.

Keywords: Pairwise comparison matrices, consistency index, abelian linearly ordered group, weighting vector.

1 Introduction

Let $X = \{x_1, x_2, ..., x_n\}$ be a set of alternatives or criteria. A *Pairwise Comparison Matrix* (PCM)

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}, \tag{1}$$

with a_{ij} encoding the preference intensity of x_i over x_j, is a useful tool for determining a weighted ranking for the alternatives.

In the literature, several kinds of PCMs are proposed, as the entry a_{ij} may assume different meanings: in multiplicative PCMs it represents a preference ratio; in additive PCMs it is a preference difference; in fuzzy PCMs it is a preference degree in [0,1].

In an ideal situation, the PCM satisfies the consistency property, which, in the multiplicative case, is expressed as follows:

$$a_{ik} = a_{ij} \cdot a_{jk} \quad \forall\, i, j, k = 1, \ldots, n. \tag{2}$$

Under condition of consistency, the preference value a_{ij} can be expressed by means of the components of a suitable vector, called consistent vector for $A =$

A.G.S. Ventre et al. (Eds.): *Multicriteria & Multiagent Decision Making*, STUDFUZZ 305, pp. 49–64.
DOI: 10.1007/978-3-642-35635-3_4

(a_{ij}); for a multiplicative PCM, it is a positive vector $\underline{w} = (w_1, w_2, ..., w_n)$ verifying the condition

$$\frac{w_i}{w_j} = a_{ij} \quad \forall\, i, j = 1, \ldots, n.$$

Thus, if $A = (a_{ij})$ is a consistent PCM, then it is reasonable to choose a weighting vector in the set of consistent vectors, while, if $A = (a_{ij})$ is an inconsistent PCM, to look for a vector that is close to be a consistent vector. As an example, for the multiplicative case, we look for a vector such that:

$$\frac{w_i}{w_j} \approx a_{ij} \quad \forall\, i, j = 1, \ldots, n.$$

The multiplicative PCMs play a basic role in the well-known Analytic Hierarchy Process (AHP), the procedure developed by T.L. Saaty at the end of the 1970s [14], [15], [16]. In [2], [3], [4], [5] and [12], properties of multiplicative PCMs are analyzed in order to determine a qualitative ranking on the set of alternatives and find vectors representing this ranking. Additive and fuzzy PCMs are investigated for instance by [1] and [13].

The AHP provides a comprehensive and rational framework for structuring a decision problem, for representing and quantifying its elements, for relating those elements to overall goals, and for evaluating alternative solutions.
"*To make a decision in an organised way to generate priorities we need to decompose the decision into the following steps.*

1. *Define the problem and determine the kind of knowledge sought.*
2. *Structure the decision hierarchy from the top with the goal of the decision, then the objectives from a broad perspective, through the intermediate levels (criteria on which subsequent elements depend) to the lowest level (which usually is a set of the alternatives).*
3. *Construct a set of pairwise comparison matrices. Each element in an upper level is used to compare the elements in the level immediately below with respect to it.*
4. *Use the priorities obtained from the comparisons to weigh the priorities in the level immediately below. Do this for every element. Then for each element in the level below add its weighed values and obtain its overall or global priority. Continue this process of weighing and adding until the final priorities of the alternatives in the bottom most level are obtained.*" [17]

In order to unify the different approaches to the way of building the PCMs, in [7] the authors introduce PCMs whose entries belong to an abelian linearly ordered group (*alo-group*) $\mathcal{G} = (G, \odot, \leq)$. In this way, the consistency condition is expressed in terms of the group operation $\odot$. Under the assumption of divisibility of $\mathcal{G}$, for each $A = (a_{ij})$, a $\odot$-consistency measure $I_{\mathcal{G}}(A)$, expressed in terms of $\odot$-mean of $\mathcal{G}$-distances, is provided; furthermore a $\odot$-mean vector $\underline{w}_m(A)$, satisfying the independence of scale-inversion condition, is chosen as a weighting vector for the alternatives.

In this paper, we provide a survey of results related to PCMs on alo-groups, by focusing on properties of $I_{\mathcal{G}}(A)$ and $\underline{w}_m(A)$.

The paper is organized as follows: Section 2 focuses on alo-groups; Section 3 introduces PCMs on real divisible alo-groups; Section 4 provides concluding remarks and directions for future work.

2 Alo-groups

From now on, $\mathbb{R}$ will denote the set of real numbers, $\mathbb{Q}$ the subset of rational numbers, $\mathbb{Z}$ the subset of relative integers, $\mathbb{N}$ the subset of positive integers and $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$. $\mathcal{G} = (G, \odot, \leq)$ denotes an abelian linearly ordered group (alo-group), e its *identity*, $a^{(-1)}$ the *inverse* of $a \in G$ with respect to $\odot$, $\div$ the *inverse operation* of $\odot$, defined by:

$$a \div b = a \odot b^{(-1)} \quad \forall a, b \in G, \tag{3}$$

and $G^n = \{\underline{w} = (w_1, \ldots, w_n) | w_i \in G, \forall i \in \{1, \ldots, n\}\}$.

Definition 1. *[11] A vector $\underline{w} \in G^n$ is called a $\odot$-*normal *vector if and only if*

$$w_1 \odot w_2 \ldots \odot w_n = e.$$

Definition 2. *[8] The vectors $\underline{w} = (w_1, \ldots, w_n)$ and $\underline{v} = (v_1, \ldots, v_n)$ are $\odot$-proportional if and only if there exists $c \in G$ such that $\underline{w} = c \odot \underline{v} = (c \odot v_1, \ldots, c \odot v_n)$.*

Proposition 1. *[7] If $\mathcal{G} = (G, \odot, \leq)$ is a non-trivial alo-group then it has neither a greatest element nor a least element.*

Proposition 2. *[7] The operation*

$$\begin{aligned} d_{\mathcal{G}} &: G \times G \to G \\ &(a, b) \mapsto d_{\mathcal{G}}(a, b) = (a \div b) \vee (b \div a) \end{aligned} \tag{4}$$

is a $\mathcal{G}$-distance that satisfies the following properties:

1. $d(a, b) \geq e$;
2. $d(a, b) = e \Leftrightarrow a = b$;
3. $d(a, b) = d(b, a)$;
4. $d(a, b) \leq d(a, c) \odot d(b, c)$.

Definition 3. *[7] Let $n \in \mathbb{N}_0$. The (n)-*power *$a^{(n)}$ of $a \in G$ is defined as follows:*

$$a^{(n)} = \begin{cases} e, & if \quad n = 0 \\ a^{(n-1)} \odot a, & if \quad n \geq 1. \end{cases}$$

Definition 4. *[10] Let $z \in \mathbb{Z}$. The (z)-*power *$a^{(z)}$ of $a \in G$ is defined as follows:*

$$a^{(z)} = \begin{cases} a^{(n)}, & if \quad z = n \in \mathbb{N}_0 \\ (a^{(n)})^{(-1)} & if \quad z = -n, \quad n \in \mathbb{N}. \end{cases}$$

2.1 Divisible Alo-groups

Definition 5. *$\mathcal{G} = (G, \odot, \leq)$ is divisible if and only if $(G, \odot)$ is divisible, that is, for each $n \in \mathbb{N}$ and each $a \in G$, the equation $x^{(n)} = a$ has at least a solution.*

If $\mathcal{G} = (G, \odot, \leq)$ is divisible, then the equation $x^{(n)} = a$ has a unique solution. Thus, we give the following definition:

Definition 6. *[7] Let $\mathcal{G} = (G, \odot, \leq)$ be divisible, $n \in \mathbb{N}$ and $a \in G$. Then, the (n)-*root *of a, denoted by $a^{(\frac{1}{n})}$, is the unique solution of the equation $x^{(n)} = a$, that is:*

$$(a^{(\frac{1}{n})})^{(n)} = a. \tag{5}$$

Definition 7. *[7] Let $\mathcal{G} = (G, \odot, \leq)$ be divisible.* $\odot$-mean *$m_{\odot}(a_1, a_2, ..., a_n)$ of the n elements $a_1, a_2, ..., a_n$ of G is the element $a \in G$ verifying the equality $a \odot a \odot ... \odot a = a_1 \odot a_2 \odot ... \odot a_n$; that is,*

$$m_{\odot}(a_1, a_2, ..., a_n) = \begin{cases} a_1 & if\ n = 1, \\ (\bigodot_{i=1}^{n} a_i)^{(\frac{1}{n})} & if\ n \geq 2. \end{cases}$$

Definition 8. *[10] Let $(G, \odot, \leq)$ be divisible. For each $q = \frac{m}{n}$, with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, and for each $a \in G$, the (q)-power $a^{(q)}$ is defined as follows:*

$$a^{(q)} = (a^{(m)})^{(\frac{1}{n})}.$$

2.2 Real Divisible Alo-groups

An alo-group $\mathcal{G} = (G, \odot, \leq)$ is a *real* alo-group if and only if G is a subset of the real line $\mathbb{R}$ and $\leq$ is the total order on G inherited from the usual order on $\mathbb{R}$. If G is an interval of $\mathbb{R}$ then, by Proposition 1, it has to be an open interval. Examples of real divisible continuous alo-groups are the following:

Multiplicative Alo-group. $]\mathbf{0,+\infty}[= (]0, +\infty[, \cdot, \leq)$, where $\cdot$ is the usual multiplication on $\mathbb{R}$. Then, $e = 1$ and for $a, b \in]0, +\infty[$ and $q \in \mathbb{Q}$:

$$a^{(-1)} = 1/a, \quad a \div b = \frac{a}{b}, \quad a^{(q)} = a^q,$$

$$d_{]\mathbf{0,+\infty}[}(a, b) = \frac{a}{b} \vee \frac{b}{a};$$

moreover, for $a_i \in]0, +\infty[, \ i \in \{1, \ldots n\}$, $m_{\cdot}(a_1, ..., a_n)$ is the geometric mean:

$$m_{\cdot}(a_1, ..., a_n) = \Big(\prod_{i=1}^{n} a_i\Big)^{\frac{1}{n}}.$$

Additive Alo-group. $\mathcal{R} = (\mathbb{R}, +, \leq)$, where $+$ is the usual addition on $\mathbb{R}$. Then, $e = 0$ and for $a, b \in \mathbb{R}$ and $q \in \mathbb{Q}$:

$$a^{(-1)} = -a, \quad a \div b = a - b, \quad a^{(q)} = qa,$$

$$d_{\mathcal{R}}(a, b) = |a - b| = (a - b) \vee (b - a);$$

moreover, for $a_i \in \mathbb{R},\ i \in \{1, \ldots n\}$, $m_+(a_1, ..., a_n)$ is the arithmetic mean:

$$m_+(a_1, ..., a_n) = \frac{\sum_i a_i}{n}.$$

Fuzzy group. $\mathbf{]0,1[}$=$(]0, 1[, \otimes, \leq)$, where $\otimes :]0, 1[^2 \to]0, 1[$ is the operation defined by

$$x \otimes y = \frac{xy}{xy + (1 - x)(1 - y)}.$$

Then, $e = 0.5$ and for $a, b \in]0, 1[$ and $q \in \mathbb{Q}$:

$$a^{(-1)} = 1 - a, \quad a \div b = \frac{a(1 - b)}{a(1 - b) + (1 - a)b}, \quad a^{(q)} = \frac{a^q}{a^q + (1 - a)^q},$$

$$d_{\mathbf{]0,1[}}(a, b) = \frac{a(1 - b)}{a(1 - b) + (1 - a)b} \vee \frac{b(1 - a)}{b(1 - a) + (1 - b)a};$$

moreover, for $a_i \in]0, 1[, i \in \{1, \ldots n\}$,

$$m_{\otimes}(a_1, ..., a_n) = \frac{\sqrt[n]{\prod_{i=1}^n a_i}}{\sqrt[n]{\prod_{i=1}^n a_i} + \sqrt[n]{\prod_{i=1}^n (1 - a_i)}}. \tag{6}$$

Two divisible continuous real alo-groups are isomorphic with respect to the group operations and the order relation; in particular for each real divisible continuous alo-group $\mathcal{G} = (G, \odot, \leq)$, there exists an isomorphism h between $\mathbf{]0,+\infty[}$ and $\mathcal{G}$. For instance:

$$l : x \in]0, +\infty[\mapsto \log x \in \mathbb{R} \tag{7}$$

is an isomorphism between $\mathbf{]0,+\infty[}$ and $\mathcal{R}$ and

$$\psi : x \in]0, +\infty[\mapsto \frac{x}{x + 1} \in]0, 1[\tag{8}$$

is an isomorphism between $\mathbf{]0,+\infty[}$ and $\mathbf{]0,1[}$.

Let $\mathcal{G} = (G, \odot, \leq)$ be a real divisible continuous alo-group. For each $a \in G$ and $r \in \mathbb{R}$, we set:

$$I_{a,r} = \{a^{(q)} : q \in Q \quad and \quad q < r\}, \quad S_{a,r} = \{a^{(q)} : q \in Q \quad and \quad q > r\}. \tag{9}$$

In [9], the authors extend the notion of (q)-power, with $q \in \mathbb{Q}$, in Definition 8, to the notion of (r)-power, with $r \in \mathbb{R}$, as follows:

Definition 9. *[9] Let $\mathcal{G} = (G, \odot, \leq)$ be a real divisible continuous alo-group. For each $a \in G$ and $r \in \mathbb{R}$, $a^{(r)}$ is the separation point of sets in* (9)*, thus the following holds:*

$$a^{(r)} = h((h^{-1}(a))^r),$$

with h an isomorphism between $]0, +\infty[$ and $\mathcal{G}$.

Proposition 3. *[9] Let $\mathcal{G} = (G, \odot, \leq)$ be a real divisible continuous alo-group a. For each $a, b \in G$ and $r, r_1, r_2 \in \mathbb{R}$, we have:*

1. $a^{(-r)} = (a^{(r)})^{(-1)} = (a^{(-1)})^{(r)}$;
2. $a^{(r_1)} \odot a^{(r_2)} = a^{(r_1+r_2)}$;
3. $(a^{(r_1)})^{(r_2)} = a^{(r_1 r_2)} = (a^{(r_2)})^{(r_1)}$;
4. $(a \odot b)^{(r)} = a^{(r)} \odot b^{(r)}$;
5. $e^{(r)} = e$.

Proposition 4. *[9] Let $\mathcal{G} = (G, \odot, \leq)$ be a real divisible continuous alo-group and $r \in \mathbb{R}$. Then, (r)-power function:*

$$f_{(r)} : a \in G \to a^{(r)} \in G$$

is strictly increasing if $r > 0$, strictly decreasing if $r < 0$ and is the constant function $f_{(0)} = e$ if $r = 0$.

Proposition 5. *[9] Let $\mathcal{G} = (G, \odot, \leq)$ be a real divisible continuous alo-group and $a \in G$, with $a \neq e$. Then, (r)-exponential function*

$$g : r \in \mathbb{R} \to a^{(r)} \in G$$

is strictly increasing if $a > e$ and strictly decreasing if $a < e$.

3 PCMs on Real Divisible Alo-groups

Let $X = \{x_1, x_2, ...x_n\}$ be a set of alternatives and $A = (a_{ij})$ in (1) the related PCM. We assume that $A = (a_{ij})$ is a PCM over a real continuous divisible alo-group $\mathcal{G} = (G, \odot, \leq)$, that is, $a_{ij} \in G$, $\forall i, j \in \{1, \ldots, n\}$ [7]. We assume that:

1. $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ are the rows of A;
2. $\underline{a}^1, \underline{a}^2, \ldots, \underline{a}^n$ are the columns of A;
3. $A_{(ijk)}$ is the sub-matrix $\begin{pmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{pmatrix}$;
4. A_{ijk} denotes $A_{(ijk)}$ if $i < j < k$ (see [7]);
5. $A^{(r)} = (a_{ij}^{(r)})$.

Definition 10. *For each $A = (a_{ij})$ over $\mathcal{G} = (G, \odot, \leq)$, the* $\odot$-mean vector *associated to A is:*

$$\underline{w}_{m_\odot}(A) = (m_\odot(\underline{a}_1), m_\odot(\underline{a}_2), \cdots, m_\odot(\underline{a}_n)), \qquad (10)$$

where $m_\odot(\underline{a}_i) = m_\odot(a_{i1}, a_{i2}, \ldots, a_{in})$.

Definition 11. *$A = (a_{ij})$ is a $\odot$-reciprocal PCM if and only if verifies the condition:*

$$a_{ji} = a_{ij}^{(-1)} \quad \forall\, i,j \in \{1,\ldots,n\}, \qquad (\odot - reciprocity)$$

so $a_{ii} = e \ \ \forall i \in \{1,\ldots,n\}$.

$RM(n)$ will denote the set of $\odot$-reciprocal PCMs of order n. Let us assume $A \in RM(n)$, then we set:

$$x_i \succ x_j \Leftrightarrow a_{ij} > e, \qquad x_i \sim x_j \Leftrightarrow a_{ij} = e, \tag{11}$$

where $x_i \succ x_j$ and $x_i \sim x_j$ stand for "x_i is strictly preferred to x_j"and "x_i and x_j are indifferent,"respectively; the strict preference of x_i over x_j is expressed also by the equivalence:

$$x_i \succ x_j \Leftrightarrow a_{ji} < e. \tag{12}$$

Example 1. The matrix

$$A = \begin{pmatrix} 1 & 2 & \frac{1}{10} \\ \frac{1}{2} & 1 & 3 \\ 10 & \frac{1}{3} & 1 \end{pmatrix}$$

is a $\cdot$-reciprocal PCM on the multiplicative alo-group $(]0,+\infty[,\cdot,\leq)$.

Example 2. The matrix

$$B = \begin{pmatrix} 0 & 2 & -5 \\ -2 & 0 & 3 \\ 5 & -3 & 0 \end{pmatrix}$$

is a $+$-reciprocal PCM on the additive alo-group $(\mathbb{R},+,\leq)$.

Example 3. The matrix

$$C = \begin{pmatrix} 0.5 & 0.6 & 0.2 \\ 0.4 & 0.5 & 0.7 \\ 0.8 & 0.3 & 0.5 \end{pmatrix}$$

is a $\otimes$-reciprocal PCM on the fuzzy alo-group $(]0,1[,\otimes,\leq)$.

3.1 $\odot$-consistency

Definition 12. *A is a $\odot$-consistent PCM if and only if*

$$a_{ik} = a_{ij} \odot a_{jk} \quad \forall i,j,k \in \{1,\ldots n\}\ .$$

$CM(n)$ will denote the set of $\odot$-consistent PCMs of order n.

Example 4. The matrix

$$A = \begin{pmatrix} 1 & 2 & 6 \\ \frac{1}{2} & 1 & 3 \\ \frac{1}{6} & \frac{1}{3} & 1 \end{pmatrix}$$

is a $\cdot$-consistent PCM on the multiplicative alo-group $(]0, +\infty[, \cdot, \leq)$.

Example 5. The matrix

$$B = \begin{pmatrix} 0 & 2 & 5 \\ -2 & 0 & 3 \\ -5 & -3 & 0 \end{pmatrix}$$

is a $+$-consistent PCM on the additive alo-group $(\mathbb{R}, +, \leq)$.

Example 6. The matrix

$$C = \begin{pmatrix} 0.5 & 0.6 & 0.\overline{7} \\ 0.4 & 0.5 & 0.7 \\ 1 - 0.\overline{7} & 0.3 & 0.5 \end{pmatrix}$$

is a $\otimes$-consistent PCM on the fuzzy alo-group $(]0, 1[, \otimes, \leq)$.

Definition 13. *Let $A = (a_{ij}) \in CM(n)$. A vector $\underline{w} = (w_1, \ldots, w_n) \in G^n$, is a* $\odot$-consistent *vector for $A = (a_{ij})$ if and only if:*

$$w_i \div w_j = a_{ij} \quad \forall\, i, j \in \{1, \ldots, n\}.$$

Proposition 6. *[7] The following assertions related to $A = (a_{ij})$ are equivalent:*

1. *$A = (a_{ij}) \in CM(n)$;*
2. *there exists a $\odot$-consistent vector $\underline{w}$ for A;*
3. *each column $\underline{a}^k$ is a $\odot$-consistent vector;*
4. *the $\odot$-mean vector $\underline{w}_{m_\odot}(A)$ is a $\odot$-consistent vector.*

Proposition 7. *[8] Let $A \in RM(n)$. The following assertions are equivalent:*

1. *$A \in CM(n)$;*
2. *$a_{ik} = a_{ij} \odot a_{jk} \quad \forall\, i, j, k \in \{1, \ldots, n\} : i < j < k$;*
3. *$\underline{a}_i$ and $\underline{a}_{i+1}$ are $\odot$-proportional vectors ($\underline{a}_{i+1} = a_{i\,i+1}^{(-1)} \odot \underline{a}_i \quad \forall i < n$);*
4. *$\underline{a}^i$ and $\underline{a}^{i+1}$ are $\odot$-proportional vectors ($\underline{a}^{i+1} = a_{i+1\,i}^{(-1)} \odot \underline{a}^i \quad \forall i < n$);*
5. *$a_{ik} = a_{i\,i+1} \odot a_{i+1\,k} \quad \forall i, k : i < k$;*
6. *$a_{ik} = a_{i\,i+1} \odot a_{i+1\,i+2} \odot \ldots \odot a_{k-1\,k} \quad \forall i, k : i < k$.*

3.2 A $\odot$-consistency Measure

In order to measure how much a PCM is far from a consistent one, in [7], the following $\odot$-consistency index is provided:

Definition 14. *[7] Let $A \in RM(n)$ with $n \geq 3$, then the $\odot$-consistency index $I_{\mathcal{G}}(A)$ is defined as follows:*

$$I_{\mathcal{G}}(A) = \Big(\bigodot_{i<j<k} d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk})\Big)^{(\frac{1}{n_T})}$$

with $T = \{(i,j,k) : i < j < k\}$ and $n_T = |T| = \frac{n(n-2)(n-1)}{6}$.

Thus:

$$I_{\mathcal{G}}(A) = \begin{cases} d_{\mathcal{G}}(a_{13}, a_{12} \odot a_{23}) & if\ n = 3, \\ \big(\bigodot_{i<j<k} I_{\mathcal{G}}(A_{ijk})\big)^{(\frac{1}{n_T})} & if\ n > 3. \end{cases} \tag{13}$$

$I_{\mathcal{G}}(A)$ has an intuitive meaning, because is a $\odot$-mean of $\mathcal{G}$-distances, and is suitable for several kinds of PCMs (e.g. multiplicative, additive and fuzzy).

Proposition 8. *[7] Let $A \in RM(n)$, then:*

$$I_{\mathcal{G}}(A) \geq e, \quad I_{\mathcal{G}}(A) = e \Leftrightarrow A \in CM(n).$$

Proposition 8 proves that there is a unique value of $I_{\mathcal{G}}(A)$ representing the $\odot$-consistency, that is, the identity element e (property that a consistency index must satisfy as required in [6]).

Example 7. Let

$$A = \begin{pmatrix} 1 & \frac{1}{7} & \frac{1}{7} & \frac{1}{5} \\ 7 & 1 & \frac{1}{2} & \frac{1}{3} \\ 7 & 2 & 1 & \frac{1}{9} \\ 5 & 3 & 9 & 1 \end{pmatrix}$$

be a PCM on the multiplicative alo-group $(]0,+\infty[,\cdot,\leq)$, then:

$$I_{]0,+\infty[}(A) = \sqrt[4]{I_{]0,+\infty[}(A_{123}) \cdot I_{]0,+\infty[}(A_{124}) \cdot I_{]0,+\infty[}(A_{134}) \cdot I_{]0,+\infty[}(A_{234})}$$
$$= \sqrt[4]{2 \cdot \frac{21}{5} \cdot \frac{63}{5} \cdot 6} = 5.02.$$

Example 8. Let

$$B = \begin{pmatrix} 0 & -\log 7 & -\log 7 & -\log 5 \\ \log 7 & 0 & -\log 2 & -\log 3 \\ \log 7 & \log 2 & 0 & -\log 9 \\ \log 5 & \log 3 & \log 9 & 0 \end{pmatrix}$$

be a PCM on the additive alo-group $(\mathbb{R},+,\leq)$, then:

$$I_{\mathcal{R}}(B) = \frac{I_{\mathcal{R}}(B_{123}) + I_{\mathcal{R}}(B_{124}) + I_{\mathcal{R}}(B_{134}) + I_{\mathcal{R}}(B_{234})}{4}$$
$$= \frac{0.6931 + 1.4350 + 2.5336 + 1.7917}{4} = 1.6134.$$

Example 9. Let

$$C = \begin{pmatrix} 0.5 & 0.3 & 0.4 & 0.4 \\ 0.7 & 0.5 & 0.1 & 0.2 \\ 0.6 & 0.9 & 0.5 & 0.8 \\ 0.6 & 0.8 & 0.2 & 0.5 \end{pmatrix}$$

be a PCM on the fuzzy alo-group $(]0,1[, \otimes, \leq)$, then:

$$I_{]0,1[}(C) = \frac{\sqrt[4]{\prod_{i<j<k} I_{]0,1[}(C_{ijk})}}{\sqrt[4]{\prod_{i<j<k} I_{]0,1[}(C_{ijk})} + \sqrt[4]{\prod_{i<j<k}(1 - I_{]0,1[}(C_{ijk}))}} = 0.833.$$

Invariance under Permutation of Alternatives

Let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a bijection, then $(\pi(1), \ldots, \pi(n))$ denotes the corresponding permutation of the n-tuple $(1, \ldots, n)$ and $\Pi : RM(n) \to RM(n)$ the function:

$$\Pi : A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \mapsto \Pi(A) = \begin{pmatrix} a_{\pi(1)\pi(1)} & a_{\pi(1)\pi(2)} & \ldots & a_{\pi(1)\pi(n)} \\ a_{\pi(2)\pi(1)} & a_{\pi(2)\pi(2)} & \ldots & a_{\pi(2)\pi(n)} \\ \ldots & \ldots & \ldots & \ldots \\ a_{\pi(n)\pi(1)} & a_{\pi(n)\pi(2)} & \ldots & a_{\pi(n)\pi(n)} \end{pmatrix}. \quad (14)$$

Proposition 9. *[9] Let $A \in RM(n)$ and Π the function in* (14), *then the following equality holds:*

$$I_{\mathcal{G}}(\Pi(A)) = I_{\mathcal{G}}(A).$$

By Proposition 9, the $\odot$-consistency index $I_{\mathcal{G}}(A)$ is independent from the order in which the alternatives are presented.

Monotonicity under Reciprocity Preserving Mapping

Let $A \in RM(n)$ and $r \in \mathbb{R}$, by Proposition 5, we have:

$$r > 1 \Rightarrow \begin{cases} a_{ij} > e \Rightarrow & e < a_{ij} < a_{ij}^{(r)}, \\ a_{ij} < e \Rightarrow & a_{ij}^{(r)} < a_{ij} < e; \end{cases}$$

$$0 < r < 1 \Rightarrow \begin{cases} a_{ij} > e \Rightarrow & e < a_{ij}^{(r)} < a_{ij}, \\ a_{ij} < e \Rightarrow & a_{ij} < a_{ij}^{(r)} < e; \end{cases} \quad (15)$$

$$r < 0 \Rightarrow \begin{cases} a_{ij} > e \Rightarrow & a_{ij}^{(r)} < e < a_{ij}, \\ a_{ij} < e \Rightarrow & a_{ij} < e < a_{ij}^{(r)}. \end{cases}$$

Thus, if $r > 1$ then $a_{ij}^{(r)}$ represents an intensification of the preference a_{ij}, if $0 < r < 1$ a weakening of the preference and if $r < 0$ a preference reversal.

Proposition 10. *[9] Let $r \in \mathbb{R}$, then the function:*

$$F_{(r)} : A \in RM(n) \mapsto A^{(r)} \in RM(n) \tag{16}$$

is $\odot$-consistency preserving and, if $r \in \mathbb{R}\backslash\{0\}$, it is a bijection.

We study how $I_{\mathcal{G}}(A)$ changes its value, when the function $F_{(r)}$ is applied to A.

Proposition 11. *[9] Let $A \in RM(n)$ and $r \in \mathbb{R}$, then:*

$$I_{\mathcal{G}}(A^{(r)}) = (I_{\mathcal{G}}(A))^{(|r|)} = \begin{cases} (I_{\mathcal{G}}(A))^{(r)} & if\ r \geq 0, \\ (I_{\mathcal{G}}(A))^{(-r)} & if\ r < 0. \end{cases}$$

Corollary 1. *[9] Let $A \in RM(n)\backslash CM(n)$. Then:*

$$I_{\mathcal{G}}(A^{(r)}) \begin{cases} > (I_{\mathcal{G}}(A)) & if\ |r| > 1, \\ < (I_{\mathcal{G}}(A)) & if\ |r| < 1. \end{cases}$$

By Corollary 1, Proposition 10 and Proposition 8, if $A \in RM(n)$, then the following inequality holds:

$$I_{\mathcal{G}}(A^{(r)}) \geq I_{\mathcal{G}}(A) \quad \forall r > 1. \tag{17}$$

Inequality (17) corresponds to the third characterizing property in [6].

Proposition 12. *[9] Let $A \in RM(n)$. Then the function:*

$$m : r \in \mathbb{R} \to I_{\mathcal{G}}(A^{(r)}) \in G$$

satisfies the following properties:

- *if $A \in CM(n)$ then m is the constant function $m : r \in \mathbb{R} \to e \in G$;*
- *if $A \notin CM(n)$ then m is strictly increasing in $[0, +\infty[$ and strictly decreasing in $]-\infty, 0]$.*

For multiplicative, additive and fuzzy cases, for some value of $I_{\mathcal{G}}(A)$, the graphics of $I_{\mathcal{G}}(A^{(r)})$ are shown in Figs 1, 2 and 3.

Strict Monotonicity on Single Entries

Let us consider $A = (a_{ij})$, a $\odot$-consistent PCM, and choose one of its non-diagonal entries a_{pq}. If we change a_{pq} in b_{pq}, by increasing or decreasing its value, and modify its reciprocal a_{qp} accordingly, while all the other entries remain unchanged, then the resulting PCM, $B = (b_{ij})$, is not anymore $\odot$-consistent and, by Proposition 8, $I_{\mathcal{G}}(B) > e$.

Proposition 13 proves that the more b_{pq} is far from a_{pq}, the more $B = (b_{ij})$ is $\odot$-inconsistent. This expresses a sort of monotonicity of the $\odot$-inconsistency with respect to a single entry of the PCM.

Proposition 13. *Let $A \in CM(n)$, $p, q \in \{1, \dots, n\}$, with $p \neq q$, and $B = (b_{ij})$, $C = (c_{ij}) \in RM(n)$ such that $a_{ij} = b_{ij} = c_{ij}$ for $i \neq p$, $j \neq q$ and for $i \neq q$, $j \neq p$. Then:*

$$\big(e < d_{\mathcal{G}}(b_{pq}, a_{pq}) < d_{\mathcal{G}}(c_{pq}, a_{pq})\big) \Rightarrow (I_{\mathcal{G}}(A) < I_{\mathcal{G}}(B) < I_{\mathcal{G}}(C)).$$

By Proposition 13, $\odot$-consistency index $I_{\mathcal{G}}(A)$ satisfies the fourth property provided in [6].

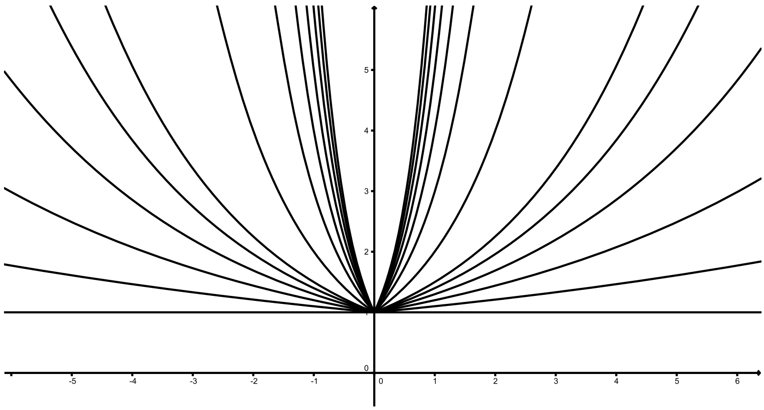

Fig. 1. Multiplicative case: $m : r \in \mathbb{R} \to I_{\mathcal{G}}(A^{(r)}) = I_{\mathcal{G}}((a_{ij}^r)) = (I_{\mathcal{G}}(A))^{|r|} \in [1, +\infty[$

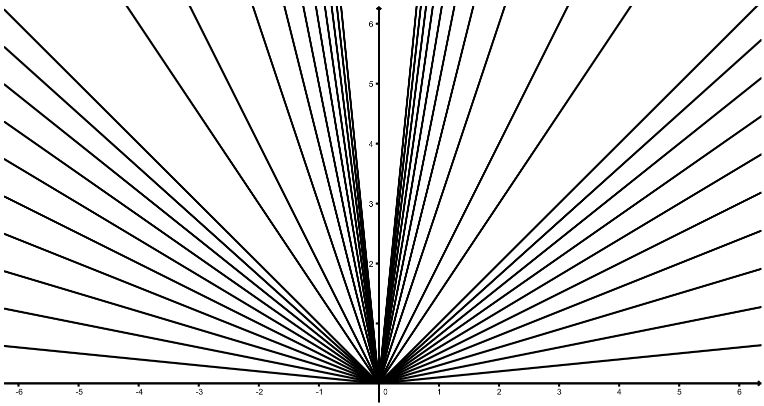

Fig. 2. Additive case: $m : r \in \mathbb{R} \to I_{\mathcal{G}}(A^{(r)}) = I_{\mathcal{G}}((r \cdot a_{ij})) = |r| \cdot I_{\mathcal{G}}(A) \in [0, +\infty[$

3.3 A Weighting Vector for the Alternatives

Proposition 14. *[11] The relation* $\succ$ *in* (11) *is asymmetric, the relation* $\sim$ *in* (11) *is reflexive and symmetric and, for each pair* (x_i, x_j)*, one and only one of the following conditions hold:*

$$x_i \succ x_j, \quad x_i \sim x_j, \quad x_j \succ x_i. \tag{18}$$

Let $\succeq$ denote the relation on X defined by

$$x_i \succeq x_j \Leftrightarrow x_i \succ x_j \text{ or } x_i \sim x_j. \tag{19}$$

Then, by Proposition 14:

$$x_i \sim x_j \Leftrightarrow (x_i \succeq x_j \text{ and } x_j \succeq x_i), \quad x_i \succ x_j \Leftrightarrow (x_i \succeq x_j \text{ and } x_j \succeq x_i). \tag{20}$$

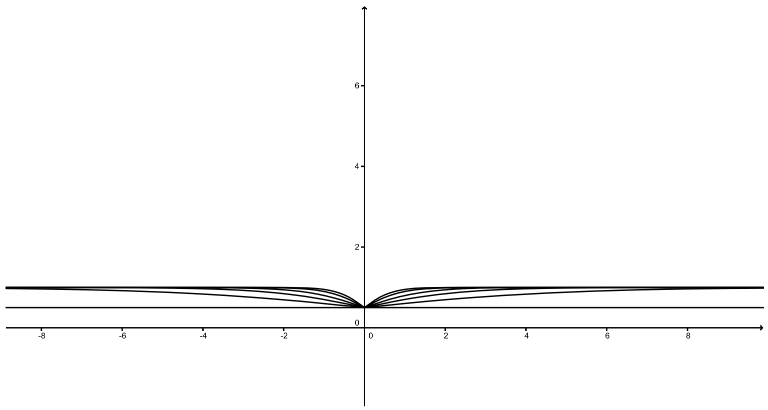

Fig. 3. Fuzzy case: $m : r \in \mathbb{R} \to I_{\mathcal{G}}(A^{(r)}) = I_{\mathcal{G}}((\frac{(a_{ij}^r)}{(a_{ij}^r)+(1-a_{ij}^r)})) = \frac{(I_{\mathcal{G}}(A))^r}{(I_{\mathcal{G}}(A))^r+(1-I_{\mathcal{G}}(A))^r} \in [0.5, 1[$

Proposition 15. *Let $A = (a_{ij}) \in CM(n)$. Then, the relations $\succ$ and $\sim$ are transitive, that is:*

1. $x_i \succ x_j$ *and* $x_j \succ x_k \Rightarrow x_i \succ x_k$,
2. $x_i \sim x_j$ *and* $x_j \sim x_k \Leftrightarrow x_i \sim x_k$.
   Moreover, $\succ$ and $\sim$ verify the following joint transitivity conditions:
3. $x_i \succ x_j$ *and* $x_j \sim x_k \Rightarrow x_i \succ x_k$,
4. $x_i \sim x_j$ *and* $x_j \succ x_k \Rightarrow x_i \succ x_k$.

Corollary 2. *[11] Let $A = (a_{ij}) \in CM(n)$. Then $\succ$ is a strict order, $\sim$ is an equivalence relation and $\succeq$ is a total weak order on X.*

By Corollary 2, if $A \in CM(n)$ then X is totally ordered by the relation $\succeq$. Hence, there is a permutation $(i_1, i_2, \cdots, i_n)$ of $(1, 2, \cdots, n)$ such that:

$$x_{i_1} \succeq x_{i_2} \succeq ..... \succeq x_{i_n}. \tag{21}$$

We say that the ranking in (21) is the *actual ranking* on X derived from A by means of the equivalences (19) and (11). Then, an *ordinal evaluation vector* for the actual ranking is a vector $\underline{w} = (w_1, w_2, ..., w_n) \in G^n$ verifying the equivalences:

$$x_i \succ x_j \Leftrightarrow w_i > w_j, \qquad x_i \sim x_j \Leftrightarrow w_i = w_j. \tag{22}$$

Proposition 16. *[11] Let $A = (a_{ij}) \in CM(n)$. Each $\odot$-consistent vector $\underline{w} = (w_1, w_2, ..., w_n)$ is an ordinal evaluation vector.*

In [11], the authors focus on the problem of deriving weights for the alternatives from a PCM over a divisible alo-group $(G, \odot, \leq)$, and deal with the following research questions:

RQ1 Let $A = (a_{ij})$ be a $\odot$-consistent PCM. Which vector can be chosen as a weighting vector?
RQ2 Let $A = (a_{ij})$ be a $\odot$-inconsistent PCM. Which vector can be chosen as a weighting vector?

By Proposition 16, $\odot$-consistent vectors are ordinal evaluation vectors for the actual ranking (21) and, by Definition 13 of $\odot$-consistent vector, they are the only ones such that the composition of its components w_i and w_j, by means of $\div$, returns the preference value a_{ij}. Hence it is reasonable to claim that the weighting vector has to be a $\odot$-consistent vector. Thus, the research question **RQ1** changes into:

RQ1'. Let $A = (a_{ij})$ be a $\odot$-consistent PCM. Which $\odot$-consistent vector can be chosen as a weighting vector?

In [11], the $\odot$-mean vector $\underline{w}_{m_\odot}(A) = (m_\odot(\underline{a}_1), m_\odot(\underline{a}_2), \cdots, m_\odot(\underline{a}_n))$ is chosen as weighting vector for the alternatives for the following reasons:

1. $m_\odot(\underline{a}_i)$ represents the $\odot$-mean of the preference intensities of x_i over all the elements x_j;
2. $\underline{w}_{m_\odot}(A)$ is the unique $\odot$-normal vector in the set of $\odot$-consistent vectors (see Definition 1);
3. each $\odot$-consistent vector $\underline{w}$ is $\odot$-proportional to $\underline{w}_{m_\odot}(A)$ (see Definition 2).

Example 10. Let

$$A = \begin{pmatrix} 1 & 2 & 6 \\ \frac{1}{2} & 1 & 3 \\ \frac{1}{6} & \frac{1}{3} & 1 \end{pmatrix}$$

be a multiplicative PCM on $(]0, +\infty[, \cdot, \leq)$, then $\underline{w}_{m_\cdot}(A) = (\sqrt[3]{12}, \sqrt[3]{\frac{3}{2}}, \sqrt[3]{\frac{1}{18}})$.

Example 11. Let

$$B = \begin{pmatrix} 0 & 2 & 5 \\ -2 & 0 & 3 \\ -5 & -3 & 0 \end{pmatrix}$$

be an additive PCM on $(\mathbb{R}, +, \leq)$, then $\underline{w}_{m_+}(B) = (\frac{7}{3}, \frac{1}{3}, -\frac{8}{3})$.

Example 12. Let

$$C = \begin{pmatrix} 0.5 & 0.6 & 0.\overline{7} \\ 0.4 & 0.5 & 0.7 \\ 1 - 0.\overline{7} & 0.3 & 0.5 \end{pmatrix}$$

be a fuzzy PCM on $(]0, 1[, \otimes, \leq)$, then $\underline{w}_{m_\otimes}(C) = (0.63, 0.54, 0.33)$.

If A is $\odot$-inconsistent, then the relation $\succeq$ defined in (19) may not provide a ranking on the set X of alternatives and, even if (19) provides a ranking, there is no $\odot$-consistent vector $\underline{w}$ such that $w_i \div w_j = a_{ij}$. Thus, in order to answer **RQ2**, in [10], [11], the authors look for a condition ensuring the existence of a vector $\underline{w} = (w_1, w_2, ..., w_n)$ such that $d_{\mathcal{G}}(w_i \div w_j, a_{ij}) \approx e$, for each $i, j = 1, 2, \ldots, n$, that is, $w_i \div w_j$ is very close to a_{ij}; they provide the following:

$$d_{\mathcal{G}}((m_{\odot}(\underline{a}_i) \div m_{\odot}(\underline{a}_j)), a_{ij}) \begin{cases} = I_{\mathcal{G}}(A)^{(\frac{1}{3})}, & n = 3 \\ \leq I_{\mathcal{G}}(A)^{(\frac{(n-2)(n-1)}{6})}, & n > 3 \; . \end{cases} \tag{23}$$

Formula (23) gives more validity to $I_{\mathcal{G}}(A)$ as $\odot$-consistency measure and more meaning to $\underline{w}_m(A)$; in fact, it ensures that if $I_{\mathcal{G}}(A)$ is close to the identity element then, from one side A is close to be a $\odot$-consistent PCM and from the other side $\underline{w}_m(A)$ is close to be a $\odot$-consistent vector.
Finally, $\underline{w}_m(A)$ satisfies the independence of scale inversion condition [11], that is, $\underline{w}_{m_{\odot}}(A^T)$ and $\underline{w}_{m_{\odot}}(A)$ provide the same ranking for the alternatives.

For these reasons, $\underline{w}_{m_{\odot}}(A)$ is the answer both to **RQ1** and **RQ2**; that is, we choose it as a weighting vector for the alternatives.

4 Final Remark

We consider PCMs on real divisible alo-groups; this approach allows us to unify several approaches proposed in the literature. We focus on properties of the $\odot$-consistency index $I_{\mathcal{G}}(A)$ and the weighting vector $\underline{w}_m(A)$.

In the future, we will investigate, among other things, conditions weaker than $\odot$-consistency that allow us to identify the actual qualitative ranking on X from A. Hence, the problem will be to find vectors agreeing with this ranking.

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