The Generalized Gini Welfare Function in the Framework of Symmetric Choquet Integration

Silvia Bortot and Ricardo Alberto Marques Pereira

Department of Economics and Management,
University of Trento, Via Inama, 5, 38122 Trento, Italy
{silvia.bortot,ricalb.marper}@unitn.it

Abstract. In the context of Social Welfare and Choquet integration, we briefly review, on the one hand, the classical Gini inequality index for populations of $n \geq 2$ individuals, including the associated Lorenz area formula, and on the other hand, the k-additivity framework for Choquet integration introduced by Grabisch, particularly in the additive and 2-additive symmetric cases. We then show that any 2-additive symmetric Choquet integral can be written as the difference between the arithmetic mean and a multiple of the classical Gini inequality index, with a given interval constraint on the multiplicity parameter. In the special case of positive parameter values this result corresponds to the well-known Ben Porath and Gilboa's formula for Weymark's generalized Gini welfare functions, with linearly decreasing (inequality averse) weight distributions.

Keywords: Social Welfare, Gini Inequality Index, Symmetric Capacities and Choquet Integrals, OWA Functions, 2-Additivity and Equidistant Weights.

1 Introduction

The Gini inequality index [24,25,21,15] plays a crucial role in Social Welfare Theory and the measurement of economic inequality [2,45]. In the literature several extensions of the Gini index have been proposed [14,47,48,49,16,9,4], in particular the generalized Gini inequality index and the associated welfare function introduced by Weymark [47] on the basis of Blackorby and Donaldson's correspondence formula [5,6],

$$A_G(\mathbf{x}) = \bar{x} - G_A(\mathbf{x})$$

where $G_A(\mathbf{x})$ denotes the (absolute) generalized Gini inequality index, $A_G(\mathbf{x})$ is the associated generalized Gini welfare function, and $\mathbf{x} = (x_1, \ldots, x_n)$ represents the income distribution of a population of $n \geq 2$ individuals. Recently, the extended interpretation of this formula in terms of the dual decomposition [19] of aggregation functions has been discussed in [20,1].

The generalized Gini welfare functions introduced by Weymark have the form

$$A(\mathbf{x}) = \sum_{i=1}^{n} w_i x_{(i)}$$

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where $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$ and $w_i \in [0,1]$ for $i = 1,\ldots,n$, with $\sum_{i=1}^{n} w_i = 1$. These welfare functions correspond to the ordered weighted averaging (OWA) functions introduced by Yager [50], which in turn correspond (see [17]) to the symmetric Choquet integrals. Moreover, the principle of inequality aversion for welfare functions requires non-increasing weights, $1 \geq w_1 \geq w_2 \geq \ldots \geq w_n \geq 0$, with $\sum_{i=1}^{n} w_i = 1$.

The use of non-additivity and Choquet integration [12] in Social Welfare and Decision Theory dates back to the seminal work of Schmeidler [43,44], Ben Porath and Gilboa [3], and Gilboa and Schmeidler [22,23]. In the discrete case, Choquet integration [41,10,13,26,27,36] corresponds to a generalization of both weighted averaging and ordered weighted averaging, which remain as special cases. For recent reviews of Choquet integration see [32,35,33,34].

The complex structure of Choquet capacities can be described in the k-additivity framework introduced by Grabisch [28,30,29,7,8,40]. The 2-additive case, in particular, has been examined in [40,37,38]. Due to its low complexity and versatility it is relevant in a variety of modeling contexts.

The characterization of symmetric Choquet integrals (OWA functions) has been studied in [7,8,18,40]. It is shown that in the k-additive case the generating function of the OWA weights is polynomial of degree $k-1$. In the symmetric 2-additive case, in particular, the generating function is linear and thus the weights are equidistant, in analogy with the classical Gini welfare function.

In this paper we examine explicitly the family of symmetric Choquet integrals (OWA functions) of the 2-additive type and show that any 2-additive OWA function can be written as the difference between the arithmetic mean and a multiple of the classical Gini inequality index, with a given interval constraint on the multiplicity parameter. In the special case of positive parameter values, this result corresponds to the well-known Ben Porath and Gilboa's formula [3] for Weymark's generalized Gini welfare functions with linearly decreasing (inequality averse) weight distributions.

The paper is organized as follows. In Section 2 we review the classical Gini index for populations of $n \geq 2$ individuals, including the Lorenz area formula in the discrete case. In Section 3 we present the basic definitions and results on capacities and Choquet integrals, particularly in the additive and 2-additive cases. In Sections 4 and 5 we consider symmetric Choquet integration and we present the main result of the paper, concerning the parametric expression of the 2-additive OWA functions in terms of the arithmetic mean and a multiple of the classical Gini inequality index.

2 Gini Inequality Index and Welfare Function

Consider a population of $n \geq 2$ individuals whose income distribution is represented by $\mathbf{x} = (x_1,\ldots,x_n)$. Typically the range of the income values is taken to be $[0,\infty)$ but in this paper, apart from the derivation of the Lorenz area formula below, it could be the whole real line.

We define the *(absolute) classical Gini inequality index* as

$$G_A^c(\mathbf{x}) = -\sum_{i=1}^{n} \frac{n-2i+1}{n^2} x_{(i)} \tag{1}$$

where $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$. This expression shows explicitly the coefficients of the ordered income variables and is the most convenient in our presentation. In what follows we will omit "classical" and refer only to "Gini inequality index."

The traditional form of the Gini inequality index $G_A^c(\mathbf{x})$ is given by

$$G_A^c(\mathbf{x}) = \frac{1}{2n^2} \sum_{i,j=1}^{n} |x_i - x_j| \tag{2}$$

which can be easily shown to be equivalent to (1). In fact, the double summation expression for $n^2 G_A^c(\mathbf{x})$ as in (2) corresponds to

$$\begin{array}{rcccccccl} (x_{(n)} - x_{(n-1)}) + & (x_{(n)} - x_{(n-2)}) & + \ldots + & (x_{(n)} - x_{(2)}) & + & (x_{(n)} - x_{(1)}) \\ & + (x_{(n-1)} - x_{(n-2)}) & + \ldots + & (x_{(n-1)} - x_{(2)}) & + & (x_{(n-1)} - x_{(1)}) \\ & & \vdots & & & \\ & & + & (x_{(3)} - x_{(2)}) & + & (x_{(3)} - x_{(1)}) \\ & & & & + & (x_{(2)} - x_{(1)}) \end{array} \tag{3}$$

which can be rewritten as

$$(n-1)x_{(n)} + ((n-2)-1)x_{(n-1)} + \ldots + (1-(n-2))x_{(2)} + (-(n-1))x_{(1)}\,. \tag{4}$$

It follows that

$$n^2 G_A^c(\mathbf{x}) = \frac{1}{2} \sum_{i,j=1}^{n} |x_i - x_j| = -\sum_{i=1}^{n} (n-2i+1)x_{(i)}\,. \tag{5}$$

In the discrete case, the Lorenz area formula can be derived as follows. Consider

$$V(\mathbf{x}) = \sum_{i=1}^{n} (x_{(1)} + \ldots + x_{(i)}) = nx_{(1)} + (n-1)x_{(2)} + \ldots + x_{(n)} \tag{6}$$

$$U(\mathbf{x}) = \sum_{i=1}^{n} (x_{(i)} + \ldots + x_{(n)}) = x_{(1)} + 2x_{(2)} + \ldots + nx_{(n)}\,. \tag{7}$$

We can easily express $U(\mathbf{x})$ in terms of $V(\mathbf{x})$,

$$\begin{aligned} U(\mathbf{x}) &= \sum_{i=1}^{n} (x_{(i)} + \ldots + x_{(n)}) \\ &= \sum_{i=1}^{n} [(x_{(1)} + \ldots + x_{(n)}) - (x_{(1)} + \ldots + x_{(i)}) + x_{(i)}] \\ &= n^2\bar{x} - V(\mathbf{x}) + n\bar{x} = n(n+1)\bar{x} - V(\mathbf{x}) \end{aligned} \tag{8}$$

where $\bar{x} = (x_{(1)} + \ldots + x_{(n)})/n$. Since

$$\begin{aligned} n^2 G_A^c(\mathbf{x}) &= -\sum_{i=1}^{n} (n-2i+1)x_{(i)} \\ &= -((n-1)x_{(1)} + (n-3)x_{(2)} + \ldots + (-n+1)x_{(n)}) \end{aligned} \tag{9}$$

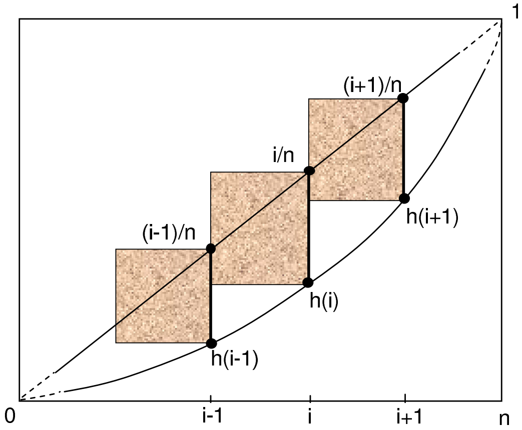

Fig. 1. Lorenz area in the discrete case

we can write $G_A^c(\mathbf{x})$ in terms of $\bar{x}$ and $V(\mathbf{x})$,

$$n^2 G_A^c(\mathbf{x}) = -(V(\mathbf{x}) - U(\mathbf{x})) = n(n+1)\bar{x} - 2V(\mathbf{x})\,. \tag{10}$$

Consider now the area illustrated in Fig. 1. The diagonal line and the Lorenz “curve” are hypothetical and are indicated only to suggest the analogy with the continuous case. In the discrete case we have just the vertical differences between the diagonal i/n values, associated with uniform cumulative income distribution, and the actual cumulative income distribution expressed by the $h(i)$ values,

$$h(i) = \frac{x_{(1)} + \ldots + x_{(i)}}{x_{(1)} + \ldots + x_{(n)}} \tag{11}$$

where we assume $x_{(i)} \geq 0$ for $i = 1, \ldots, n$ and $x_{(n)} > 0$, so that $\bar{x} > 0$.

The total area H in Fig. 1 is therefore given by

$$\begin{aligned} H = & \sum_{i=1}^{n} \left(\frac{i}{n} - h(i) \right) = \sum_{i=1}^{n} \left(\frac{i}{n} - \frac{x_{(1)} + \ldots + x_{(i)}}{x_{(1)} + \ldots + x_{(n)}} \right) \\ = & \frac{1}{n\bar{x}} \left[\sum_{i=1}^{n} \left(i\bar{x} - (x_{(1)} + \ldots + x_{(i)}) \right) \right] \\ = & \frac{1}{n\bar{x}} \left[\frac{n(n+1)}{2} \bar{x} - V(\mathbf{x}) \right] \\ = & \frac{1}{n\bar{x}} \left[\frac{n^2}{2} G_A^c(\mathbf{x}) \right] = \frac{n}{2\bar{x}} G_A^c(\mathbf{x})\,. \end{aligned} \tag{12}$$

Finally, we obtain

$$G_A^c(\mathbf{x}) = \frac{H}{n/2} \bar{x} \tag{13}$$

where the Lorenz area $H/(n/2)$ corresponds to the *relative Gini inequality index*.

The *welfare function* associated with the classical Gini inequality index is

$$A_G^c(\mathbf{x}) = \bar{x} - G_A^c(\mathbf{x}) \tag{14}$$

and it can be written as

$$A_G^c(\mathbf{x}) = \sum_{i=1}^{n} \frac{2(n-i)+1}{n^2} x_{(i)} = \sum_{i=1}^{n} \frac{1}{n} x_{(i)} + \sum_{i=1}^{n} \frac{n-2i+1}{n^2} x_{(i)} \tag{15}$$

where the coefficients of the Gini index sum up to zero, $\sum_{i=1}^{n}(n-2i+1) = 0$.

3 Capacities and Choquet Integrals

In this section we present a brief review of the basic facts on Choquet integration, focusing on the additive and 2-additive cases as described by their Möbius representations. For recent reviews on Choquet integration see [32,35,33,34] for the general case, and [40,37,38] for the 2-additive case.

Consider a finite set of interacting individuals $N = \{1, 2, \ldots, n\}$. The subsets $S, T \subseteq N$ with cardinalities $0 \leq s, t \leq n$ are usually called coalitions.

The concepts of capacity and Choquet integral in the definitions below are due to [12,46,13,26,27].

Definition 1. *A* capacity *on the set N is a set function $\mu : 2^N \longrightarrow [0,1]$ satisfying*

(i) $\mu(\emptyset) = 0$, $\mu(N) = 1$ (boundary conditions)
(ii) $S \subseteq T \subseteq N \quad \Rightarrow \quad \mu(S) \leq \mu(T)$ (monotonicity).

Capacities are also known as *fuzzy measures* [46] or *non-additive measures* [13]. Given two coalitions $S, T \subseteq N$, with $S \cap T = \emptyset$, the capacity μ is said to be

- additive for S, T if $\mu(S \cup T) = \mu(S) + \mu(T)$,
- subadditive for S, T if $\mu(S \cup T) < \mu(S) + \mu(T)$,
- superadditive for S, T if $\mu(S \cup T) > \mu(S) + \mu(T)$.

In general the capacity μ is additive over N if $\mu(S \cup T) = \mu(S) + \mu(T)$ for all coalitions $S, T \subseteq N$, with $S \cap T = \emptyset$. Otherwise, the capacity μ is subadditive over N if $\mu(S \cup T) \leq \mu(S) + \mu(T)$ for all coalitions $S, T \subseteq N$ with $S \cap T = \emptyset$, with at least two such coalitions for which μ is subadditive in the strict sense. Analogously, the capacity μ is superadditive over N if $\mu(S \cup T) \geq \mu(S) + \mu(T)$ for all coalitions $S, T \subseteq N$ with $S \cap T = \emptyset$, with at least two such coalitions for which μ is superadditive in the strict sense. In the additive case, $\sum_{i=1}^{n} \mu(i) = 1$.

Definition 2. *Let μ be a capacity on N. The* Choquet integral *of a point $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$ with respect to μ is defined as*

$$\mathscr{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})] x_{(i)} \tag{16}$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$. Moreover, $A_{(i)} = \{(i), \ldots, (n)\}$ and $A_{(n+1)} = \emptyset$.

In the additive case, since

$$\mu(A_{(i)}) = \mu(\{(i)\}) + \mu(\{(i+1)\}) + \ldots + \mu(\{(n)\}) = \mu(\{(i)\}) + \mu(A_{(i+1)}) \qquad (17)$$

the Choquet integral reduces to a weighted mean,

$$\mathscr{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)} = \sum_{i=1}^{n} \mu(\{(i)\}) x_{(i)} = \sum_{i=1}^{n} \mu(\{i\}) x_i \qquad (18)$$

where the weights are given by $w_i = \mu(\{i\})$, for $i = 1, \ldots, n$.

A capacity μ can be equivalently represented by its Möbius transform m_μ [42,29].

Definition 3. *Let* μ *be a capacity on* N*. The* Möbius transform *associated with the capacity* μ *is defined as*

$$m_\mu(T) = \sum_{S \subseteq T} (-1)^{t-s} \mu(S) \qquad T \subseteq N \qquad (19)$$

where s *and* t *denote the cardinality of the coalitions* S *and* T*, respectively.*

Conversely, given the Möbius transform m_μ, the associated capacity μ is obtained as

$$\mu(T) = \sum_{S \subseteq T} m_\mu(S) \qquad T \subseteq N\,. \qquad (20)$$

In the Möbius representation, the boundary conditions take the form

$$m_\mu(\emptyset) = 0 \qquad \sum_{T \subseteq N} m_\mu(T) = 1 \qquad (21)$$

and the monotonicity condition is expressed as follows [39,11]:

$$\sum_{S \subseteq T} m_\mu(S \cup i) \geq 0 \qquad i = 1, \ldots, n \qquad T \subseteq N \setminus i\,. \qquad (22)$$

This form of monotonicity condition derives from the original monotonicity condition in Definition 1, expressed as $\mu(T \cup i) - \mu(T) \geq 0$ for all $i \in N$ and $T \subseteq N \setminus i$.

The Choquet integral in Definition 2 can be expressed in terms of the Möbius transform in the following way [36,29]

$$\mathscr{C}_\mu(\mathbf{x}) = \sum_{T \subseteq N} m_\mu(T) \min_{i \in T}(x_i)\,. \qquad (23)$$

Defining a capacity μ on a set N of n elements requires $2^n - 2$ real coefficients, corresponding to the capacity values $\mu(T)$ for $T \subseteq N$. In order to control exponential complexity, Grabisch [28] introduced the concept of k-additive capacities.

A capacity μ is said to be *k-additive* [28] if its Möbius transform satisfies $m_\mu(T) = 0$ for all $T \subseteq N$ with $t > k$, and there exists at least one coalition $T \subseteq N$ with $t = k$ such that $m_\mu(T) \neq 0$.

We consider now, in particular, the 1-additive (or simply additive) case and the 2-additive case, and we revisit formulas (20) - (23).

- In the additive case, the decomposition formula (20) takes the simple form

$$\mu(T) = \sum_{i \in T} m_\mu(\{i\}) \qquad T \subseteq N, \tag{24}$$

the boundary conditions (21) reduce to

$$m_\mu(\emptyset) = 0 \qquad \sum_{i \in N} m_\mu(\{i\}) = 1 \tag{25}$$

and the monotonicity condition (22) reduces to

$$m_\mu(\{i\}) \geq 0 \qquad i = 1, \ldots, n. \tag{26}$$

Moreover, for additive capacities, the Choquet integral in (23) reduces to

$$\mathscr{C}_\mu(x_1, \ldots, x_n) = \sum_{i \in N} m_\mu(\{i\}) x_i. \tag{27}$$

- In the 2-additive case, the decomposition formula (20) takes the form

$$\mu(T) = \sum_{\{i\} \subseteq T} m_\mu(\{i\}) + \sum_{\{i,j\} \subseteq T} m_\mu(\{ij\}) \qquad T \subseteq N, \tag{28}$$

the boundary conditions (21) reduce to

$$m_\mu(\emptyset) = 0 \qquad \sum_{\{i\} \subseteq N} m_\mu(\{i\}) + \sum_{\{i,j\} \subseteq N} m_\mu(\{ij\}) = 1 \tag{29}$$

and the monotonicity condition (22) reduces to

$$m_\mu(\{i\}) \geq 0 \qquad m_\mu(\{i\}) + \sum_{j \in T} m_\mu(\{ij\}) \geq 0 \quad i = 1, \ldots, n \quad T \subseteq N \setminus i. \tag{30}$$

Moreover, for 2-additive capacities, the Choquet integral in (23) reduces to

$$\mathscr{C}_\mu(\mathbf{x}) = \sum_{\{i\} \subseteq N} m_\mu(\{i\}) x_i + \sum_{\{i,j\} \subseteq N} m_\mu(\{ij\}) \min(x_i, x_j). \tag{31}$$

4 Symmetric Capacities and Choquet Integrals

We examine the basic definitions and results presented in the previous section in the particular case of symmetric capacities and Choquet integrals.

Definition 4. *A capacity μ is said to be* symmetric *if it depends only on the cardinality of the coalition considered*

$$\mu(T) = \mu(t) \qquad \textit{where} \qquad t = |T|. \tag{32}$$

Accordingly, for the Möbius transform m_μ associated with a symmetric capacity μ we use the notation

$$m_\mu(T) = m_\mu(t) \qquad \text{where} \qquad t = |T|\,. \tag{33}$$

Consider a Choquet integral with respect to a symmetric capacity μ. Then the Choquet integral reduces to an Ordered Weighted Averaging (OWA) function [50],

$$\mathscr{C}_\mu(\mathbf{x}) = \sum_{i\in N}[\mu(n-i+1)-\mu(n-i)]x_{(i)} = \sum_{i\in N} w_i x_{(i)} = A(\mathbf{x}) \tag{34}$$

where

$$w_i = \mu(n-i+1)-\mu(n-i) \tag{35}$$

correspond to the OWA weights. The traditional form of OWA functions as introduced by Yager [50] (OWA operators) is as follows:

$$A(\mathbf{x}) = \sum_{i\in N} \tilde{w}_i x_{[i]} \tag{36}$$

where $\tilde{w}_i = w_{n-i+1}$ and $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}$.

For a symmetric capacity μ, (25) - (26) and (29) - (30) take the following form:

- In the additive case the boundary conditions (25) reduce to

$$m_\mu(0) = 0 \qquad\qquad n\,m_\mu(1) = 1 \tag{37}$$

  and the monotonicity condition (26) reduces to

$$m_\mu(1) \geq 0\,. \tag{38}$$

  From the boundary conditions (37) we have $m_\mu(1) \geq 1/n$ and the OWA function is simply the arithmetic mean.
- In the 2-additive case the boundary conditions (29) reduce to

$$m_\mu(0) = 0 \qquad\qquad n\,m_\mu(1) + \frac{n(n-1)}{2}\,m_\mu(2) = 1 \tag{39}$$

  and the monotonicity condition (30) reduces to

$$m_\mu(1) \geq 0 \qquad m_\mu(1) + t\,m_\mu(2) \geq 0 \qquad\qquad 1 \leq t \leq n-1\,. \tag{40}$$

In the next section we present a detailed treatment of the 2-additive symmetric case.

5 Symmetric Capacities and Choquet Integrals: The 2-Additive Case and the Gini Inequality Index

Consider now the 2-additive symmetric case as discussed in the previus section. Let

$$\alpha = m_\mu(1) \qquad\qquad \beta = m_\mu(2)\,. \tag{41}$$

From the boundary conditions (39) we have

$$n\alpha + \frac{n(n-1)}{2}\beta = 1 \qquad \alpha = \frac{1}{n} - \frac{n-1}{2}\beta\,. \tag{42}$$

From the monotonicity condition (40) it follows that

$$\alpha \geq 0 \qquad \alpha + (n-1)\beta \geq 0 \tag{43}$$

where the second constraint corresponds to the dominating worst case $t = n-1$ in (40). Substituting α as in (42) in the two conditions (43) we obtain

$$-\frac{2}{n(n-1)} \leq \beta \leq \frac{2}{n(n-1)}\,. \tag{44}$$

Consider now the OWA operator as in (34) and (35),

$$A(\mathbf{x}) = \sum_{i=1}^{n} w_i x_{(i)} \qquad w_i = \mu(n-i+1) - \mu(n-i)\,. \tag{45}$$

In the 2-additive case we have that

$$\mu(n-i+1) = (n-i+1)\,\alpha + \frac{(n-i+1)(n-i)}{2}\beta \tag{46}$$

$$\mu(n-i) = (n-i)\,\alpha + \frac{(n-i)(n-i-1)}{2}\beta \tag{47}$$

and therefore we obtain

$$w_i = \alpha + (n-i)\beta = \frac{1}{n} + \frac{n-2i+1}{2}\beta \tag{48}$$

where β is subject to the constraints (44).

Introducing the notation $u_i = (n-2i+1)/2,\ i = 1,\ldots,n$, notice that $\sum_{i=1}^{n} u_i = 0$ and the coefficients $u_i,\ i = 1,\ldots,n$ are linearly decreasing $u_1 > u_2 > \ldots > u_n$ with $u_1 = (n-1)/2$ and $u_n = -(n-1)/2$.

The main result of the paper is then the following.

Proposition 1. *Any 2-additive OWA function can be written as*

$$A(\mathbf{x}) = \bar{x} - \frac{1}{2}\beta n^2 G_A^c(\mathbf{x}) \tag{49}$$

where β is a free parameter subject to the constraints $-\dfrac{2}{n(n-1)} \leq \beta \leq \dfrac{2}{n(n-1)}$.

The proof follows straightforwardly from (45) - (48), associated to the constraints (44), and the definition of the classical Gini inequality index (1).

Given that

$$A(\mathbf{x}) = \sum_{i=1}^{n} w_i x_{(i)} = \sum_{i=1}^{n} \frac{2 + \beta(n^2 - 2in + n)}{2n} x_{(i)} \tag{50}$$

we must have $\beta \geq 0$ in order to have non-increasing weights. In Proposition 1, the strict case $\beta > 0$ corresponds to the well-known Ben Porath and Gilboa's formula [3] for Weymark's generalized Gini welfare functions with linearly decreasing (inequality averse) weight distributions, see also [31].

In particular, with $\beta = 2/n^2$ we obtain the classical Gini welfare function

$$A(\mathbf{x}) = A_G^c(\mathbf{x}) \qquad \alpha = \frac{1}{n^2} \quad \beta = \frac{2}{n^2}. \tag{51}$$

Regarding the choice of the parameter values α and β, we introduce

$$a = n\alpha \qquad\qquad b = \frac{n(n-1)}{2}\beta \tag{52}$$

and then the boundary and monotonicity constraints (42) - (43) take the simple form

$$a + b = 1 \qquad\qquad a \geq 0 \qquad a + 2b \geq 0 \tag{53}$$

from which we obtain $a = 1 - b$ and $-1 \leq b \leq 1$. In this notation the general form (49) of a 2-additive OWA function is as follows:

$$A(\mathbf{x}) = \bar{x} - \frac{n}{n-1}\, b\, G_A^c(\mathbf{x}) \tag{54}$$

where $-1 \leq b \leq 1$. The classical Gini case $\alpha = 1/n^2$ and $\beta = 2/n^2$ corresponds to $a = 1/n$ and $b = (n-1)/n$.

Other interesting parameter choices for a, b could be $a = k/n$ and $b = (n-k)/n$ with $k = 0, \ldots, n$. In the case $k = 0$ the whole Choquet capacity structure lies in the edges, whereas the case $k = 1$ corresponds to the classical Gini inequality index; the remaining cases correspond to increasingly weak structure being associated to the edges, towards the additive case $k = n$.

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