Research in Social Sciences: Fuzzy Regression and Causal Complexity

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Abstract. Social science research is now primarily divided into two types: qualitative, or case-oriented research, focused on individual cases, which reviews all aspects of a few case studies and quantitative, or variable-oriented research, which considers only some quantitative aspects (variables) of a large number of cases and is looking for correlations between these variables.

The first type of research is based primarily on evidence, the second on theoretical models. The fundamental criticism to case-oriented research is that it does not lead to general theoretical models, while the most important criticisms to the variable-oriented research are the assumption of a population a priori and the hypothesis that the elements of the population are homogeneous.

A compromise between the two points of view is the diversity-oriented research, which takes into account the variables and the diversity of individual cases.

The fundamental purpose of our paper is to study the possibilities provided by fuzzy sets and algebra of fuzzy numbers for the study of social phenomena. We deepen some aspects of the fuzzy regression, and we present some operations between fuzzy numbers that are efficient alternatives to those based on Zadeh extension principle. Finally, we present some critical remarks about the causal complexity and logical limits of the assumption of linear relationship between variables. A solution of these problems can be obtained by the fuzzy sets that play a key role in diversity-oriented research.

Keywords: social science research, fuzzy regression, alternative fuzzy operations, case-oriented research, causal complexity.

1 Introduction and Motivation

Social science research is now primarily divided into two types: *qualitative* or *case-oriented research* and *quantitative* or *variable-oriented research* [BR1], [BR2], [RA].

The first research strategy, also called *intensive*, focuses on complexity of social phenomena. It is based on an in-depth study of individual cases and analyzes all aspects of a very small number of case studies. The second line of research, also

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called *extensive*, considers some quantitative aspects (variables) of a large number of cases and is looking for correlations between these variables.

Recently, some authors have highlighted that usually there is a very sharp distinction between the two categories of studies (see e.g., [RA]). The majority of studies of social phenomena is based or on an *intensive* study of a few cases, approximately not more than a dozen, or on an *extensive* study for many cases, approximately not less than one hundred. There are few papers on a number of case studies ranging a dozen to a hundred.

The intensive research is based primarily on *evidence*, the extensive on *theoretical models*. In the transition from an intensive study of the case studies in an extensive research, as the number of cases increases, we lose the characteristics of individual phenomena and we come to assumptions of *homogeneity* of individual cases, which often are not suitable for the study of social phenomena.

The fundamental criticism to *case-oriented* research is that it does not lead to theoretical and general models, while the most important criticisms to the *variable-oriented* research are the assumption of a population a priori and the hypothesis that the elements of the population are homogeneous.

A compromise between the two points of view is the *diversity-oriented research*, which takes into account the variables and the diversity of individual cases [BR1], [BR2], [RA]. It is also based on an idea of population not fixed from the beginning, but changing during the research according to a critical examination of the intermediate results. This characteristic of the diversity-oriented research allows us also to consider an intermediate number of cases that do not necessarily reduce to the extremes of a handful of cases or many cases.

A powerful tool in diversity-oriented research is provided by the theory of fuzzy sets and the opportunities they offer to replace relations belonging to linear relationships between variables and probabilistic assumptions.

In this sense, fuzzy logic and algebra of fuzzy numbers (see e.g. [DP1], [GO], [ZA1], [ZA2], [ZA3], [ZA4]) are proposed as an alternative to the study of the linear relationship between the dependent variable, to be explained, and the explanatory variables [RA].

In other contexts, however, keeping to extensive research, the algebra of fuzzy numbers works in accordance with the hypothesis that there is, at least in a first approximation, a linear relationship between dependent and independent variables. This leads to the fuzzy regression, in which, unlike the classical regression where the conclusions are expressed in probabilistic terms, the conclusions must be formulated in terms of degree of membership of the values of the dependent variable to fuzzy numbers calculated by the model.

The paper is structured as follows. In Section 2, we introduce a brief review of basic concepts on fuzzy regression for further development of some themes. In Section 3, we introduce operations that are efficient and mathematical consistent alternatives to those based on the Zadeh extension principle [ZA1], [ZA2], [ZA3], [ZA4], [YA], [KY], [RO] and we analyze how they can make new fuzzy regression tools. Finally, in Section 4, we analyze some concepts on the study of social phenomena based on *diversity-oriented research* and present some critical remarks upon some implicit assumptions of the variable-oriented research as the homogeneity of the case studies and the linear relationship among the variables.

2 Fuzzy Numbers and Fuzzy Linear Regression: A Review and Critical Analysis of Some Fundamental Aspects

Fuzzy linear regression can be classified in "partially fuzzy" or "totally fuzzy" regression. In the first case, we have two possibilities: *fuzzy parameters with crisp data* or *fuzzy data with crisp parameters*. In the second case *data and parameters are both fuzzy* ([KY], [RO]).

The start point of a partially fuzzy linear regression is the individuation of an algebraic structure of fuzzy numbers (F, +, *) where F is a nonempty family of fuzzy numbers containing the set R of real numbers (e.g. the degenerate fuzzy numbers), + is an operation on F, called "addition", extension of the addition on R, and * is the multiplication of an element of F by a scalar, e.g., a function *: R×F →F, extension of the multiplication on R.

Usually, F is the set of triangular fuzzy numbers and the operations + and * are obtained by the "Zadeh extension principle". In this paper, we show that logical reasons and mathematical properties can lead to prefer other sets of fuzzy numbers or alternative fuzzy operations.

A total fuzzy linear regression needs a more complex algebraic structure of fuzzy numbers. Namely, we have to assign an algebraic structure (F, +, *, ·), where + is the addition, * is the multiplication of an element of F by a scalar, and · is a multiplication on F, extension to F×F of *.

Unfortunately, the multiplication defined by means of the Zadeh's extension principle presents some important drawbacks including the following:

(1) in general the Zadeh extension product of two triangular (resp. trapezoidal) fuzzy numbers is not a triangular (resp. trapezoidal) fuzzy number;
(2) if the supports of two triangular fuzzy numbers contain 0 the spread of the product of triangular fuzzy numbers does not have a simple expression;
(3) the distributive property applies only to particular triplets of fuzzy numbers.

For these reasons, in this section we consider especially the partial fuzzy linear regression. In the first subsection we recall and introduce some concepts and notations on fuzzy numbers necessary for the rest of the paper, in the second and third subsection we focus our interest on some fundamental aspects of partial fuzzy regression. Finally, in the fourth subsection we make a brief introduction and some critical comments on the totally fuzzy linear regression.

2.1 Fundamental Concepts and Notations on Fuzzy Numbers

Let us recall some fundamental concepts on fuzzy numbers and some related properties ([DP2], [KY], [YA], [RO], [ZA1], [ZA2], [ZA3], [ZA4], [YA], [MA1], [MA2], [CH]).

Definition 2.1. A fuzzy number is a function having as domain the set of real numbers and with values in [0, 1], u: R → [0, 1], such that:

(FN1) (*bounded support*) there are two real numbers a, b, with a ≤ b, called the *endpoints* of u, such that u(x) = 0 for x ∉ [a, b] and u(x) > 0 for x belonging to the open interval (a, b);

(FN2) (*normality*) there are two real numbers c, d, with a ≤ c ≤ d ≤ b such that u(x) = 1 if and only if x∈[c, d].

(FN3) (*convexity*) u(x) is a function increasing in the interval [a, c] and decreasing in the interval [d, b].

(FN4) (*compactness*) for every r∈(0, 1), the set {x∈R: u(x) ≥ r} is a closed interval.

The set of the real numbers x such that u(x) > 0 is said to be the *support* of u, denoted supp(u) or S(u), and the interval [c, d] is said to be the *core* or *central part* of u, noted core(u) or C(u). The intervals [a, c) and (d, b] are, respectively, the *left part* and the *right part* of u.

The fuzzy number u is said to be *simple* if c = d, i.e., C(u) is a singleton. Moreover, u is said to be *degenerate* if a = b, i.e., S(u) ={c}, c∈R.

The real numbers L(u) = c – a, M(u) = d – c, and R(u) = b – d are the *left, middle, and right spreads* of u, respectively. Their sum T(u) = b – a is the *total spread* of u.

For every r such that 0 ≤ r ≤ 1, the set of the x∈[a, b] such that u(x) ≥ r is denoted by $[u]^r$ and is said to be the *r-cut* of u. The left and right endpoints of $[u]^r$ are denoted, respectively, u_λ^r and u_ρ^r. In particular, $[u]^0$ is the closure of the support of u and $[u]^1$ is the core of u.

Let us assume the following notations:

- (*endpoints notation*) u ~ (a, c, d, b) stands u is a fuzzy number with endpoints a, b, and core [c, d]; u ~ (a, c, b) for a simple u with endpoints a, b, and core {c};
- (*spreads notation*) u ~ [c, d, L, R] denotes that u is a fuzzy number with core [c, d] and left and right spreads L and R, respectively; u ~ [c, L, R] denotes a simple u with core {c};
- (*r-cut spreads notation*) the numbers $L^r(u) = (c - u_\lambda^r)$ and $R^r(u) = (u_\rho^r - d)$ are called the *r-cut left spread* and the *r-cut right spread* of u, then we can write $[u]^r = [c - L^r(u), d + R^r(u)]$;
- (*sign*) the fuzzy number u ~ (a, c, d, b) is said to be *positive*, *strictly positive*, *negative*, or *strictly negative*, if a ≥ 0, a > 0, b ≤ 0, or b < 0, respectively;
- (*c-sign*) the fuzzy number u ~ (a, c, d, b) is said to be *c-positive*, *strictly c-positive*, *c-negative*, or *strictly c-negative*, if c ≥ 0, c > 0, d ≤ 0, or d < 0, respectively.

Definition 2.2. Let C be the set of the compact intervals of R. For every pair of intervals [a, b] and [c, d] in C, we assume:

$$[a, b] + [c, d] = [a + c, b + d]; \tag{2.1}$$

$$[a, b] \cdot [c, d] = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}]; \tag{2.2}$$

$$[a, b] \leq [c, d] \text{ if and only if } a \leq c \text{ and } b \leq d. \tag{2.3}$$

The subtraction and division are also defined on C by the formulae:

$$[a, b] - [c, d] = [a, b] + [-d, -c]; \tag{2.4}$$

$$\text{if } 0 \notin [c, d],\ [a, b] / [c, d] = [a, b] \cdot [1/d, 1/c]. \tag{2.5}$$

Remark 2.3. [KY, 103] The addition + defined by (2.1) is commutative, associative, having 0 = [0, 0] as neutral element. The multiplication defined by (2.2) is commutative, associative, having 1 = [1, 1] as neutral element. Moreover, for every compact intervals [a, b], [c, d], [e, f], the following subdistributive property holds:

$$([a, b] + [c, d])\,[e, f] \subseteq [a, b]\,[e, f] + [c, d]\,[e, f], \quad \text{(subdistributivity)} \tag{2.6}$$

The distributivity holds iff $[a, b] \cdot [c, d] \geq 0$ or [e, f] is a degenerate interval.

Definition 2.4. We say that the fuzzy number u ~ (a, c, d, b) is a *trapezoidal fuzzy number*, denoted u = (a, c, d, b), if:

$$\forall x \in [a, c),\quad a < c \Rightarrow u(x) = (x-a)/(c-a), \tag{2.7}$$

$$\forall x \in (d, b],\quad d < b \Rightarrow u(x) = (b-x)/(b-d). \tag{2.8}$$

In spread notation, if u ~ [c, d, L, R] is a trapezoidal fuzzy number, we write u = [c, d, L, R]. A simple trapezoidal fuzzy number u = (a, c, c, b) is said to be a *triangular fuzzy number* denoted u = (a, c, b). In spread notation, if u ~ [c, L, R] is a triangular fuzzy number we write u = [c, L, R]. A trapezoidal fuzzy number u = (c, c, d, d), with support equal to the core is said to be a *rectangular fuzzy number* and is identified with the compact interval [c, d] of R.

The necessary and sufficient conditions for u ~ [c, d, L, R] in order to be a trapezoidal fuzzy number, in terms of r-cut left and right spreads, are:

$$L^r(u) = (1-r)\,(c-a) = (1-r)\,L, \qquad R^r(u) = (1-r)\,(b-d) = (1-r)\,R. \tag{2.9}$$

Remark 2.5. For every real number x, let us denote with $\alpha(x)$ the sign of x. For every fuzzy number u ~ [c, d, L, R], we define +L = -R = L and +R =-L = R. Let + e * be the Zadeh extension addition and multiplication of a real number by a fuzzy number. If x is a real number and u and v are fuzzy numbers, then the following properties of the left and right spreads hold [MA1]:

$$L(u + v) = L(u) + L(v),\ L(x*u) = |x|\,(\alpha(x)\,L(u)); \tag{2.10}$$

$$R(u + v) = R(u) + R(v),\ R(x*u) = |x|\,(\alpha(x)\,R(u)). \tag{2.11}$$

2.2 Partial Fuzzy Linear Regression with Fuzzy Parameters

Let $\{K_i \sim [c_i, d_i, L_i, R_i],\ i = 1, 2, \ldots, n\}$ be a set of fuzzy numbers, $\{x_i,\ i=1, 2, \ldots, n\}$ the set of the independent variables, and y the dependent variable. Equation (2.12) shows a general fuzzy linear regression model with K_i fuzzy parameters:

$$Y = K_1 x_1 + K_2 x_2 + \ldots + K_n x_n, \tag{2.12}$$

where the addition and the multiplication of a real number by a fuzzy number are the Zadeh's extension operations.

Suppose we have the following sample:

Table 1.

Sample number, j	Output values, y_j	Input values, x_{ij}
1	y_1	$x_{11}, x_{21}, \ldots, x_{n1}$
2	y_2	$x_{12}, x_{22}, \ldots, x_{n2}$
m	y_m	$x_{1m}, x_{2m}, \ldots, x_{nm}$

The first step of the fuzzy linear regression is to replace in (2.12) the numerical vector $(x_{1j}, x_{2j}, \ldots, x_{nj})$ to the vector of independent variables $(x_1, x_2, \ldots, x_n)$ and then to obtain a fuzzy number Y_j, for every $j \in \{1, 2, \ldots, m\}$.

The second step is to calculate the degree to which y_j belongs to Y_j. A possible conclusion is to consider the fuzzy coefficients K_i adequate if for each j the degree of belonging of y_j to Y_j is "*sufficiently high*".

A key aspect is the calculation of the spreads of Y. The larger are the spreads, the greater the degree to which y_j belongs to Y_j.

From formulae (2.10), (2.11) we have:

$$L(Y) = (\alpha(x_1)\, L(K_1))\, |x_1| + (\alpha(x_2)\, L(K_2))\, |x_2| + \ldots + (\alpha(x_n)\, L(K_n))\, |x_n| \qquad (2.13)$$

$$R(Y) = (\alpha(x_1)\, R(K_1))\, |x_1| + (\alpha(x_2)\, R(K_2))\, |x_2| + \ldots + (\alpha(x_n)\, R(K_n))\, |x_n| \qquad (2.14)$$

Moreover, the core of Y is obtained by the following formula:

$$C(Y) = (\alpha(x_1)\, C(K_1))\, |x_1| + (\alpha(x_2)\, C(K_2))\, |x_2| + \ldots + (\alpha(x_n)\, C(K_n))\, |x_n|, \qquad (2.15)$$

where, for every interval [a, b] of the real line +[a, b] = [a, b], -[a, b] = [-b, -a].

If the K_i are simple fuzzy numbers with symmetric spreads, let $s_i = L(K_i) = R(K_i)$ be the bilateral spread of K_i and let $s(Y) = L(Y) = R(Y)$ be the bilateral spread of Y. Then we have the simpler formulae:

$$s(Y) = s_1\, |x_1| + s_2\, |x_2| + \ldots + s_n\, |x_n|, \qquad (2.16)$$

$$C(Y) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n. \qquad (2.17)$$

Usually in the scientific literature (see, e.g. [KY], [RO]), the K_i are symmetric and triangular fuzzy numbers. It is well-known that the sum of two triangular fuzzy numbers and the product of a real number for a triangular fuzzy number are triangular fuzzy numbers, too ([KY], [RO], [MA1], [MA2]). Then also Y is a triangular fuzzy number.

If the spread s(Y) is not null, then the membership function of Y is given by the formula:

$$\mu(y) = 1 - |y - C(Y)|/s(Y), \qquad y \in (C(Y) - s(Y), C(Y) + s(Y)), \qquad (2.18)$$

and $\mu(y)$ is null otherwise. If the spread $s(Y)$ is null, then Y reduces to the real number $C(Y)$, i.e., a degenerate fuzzy number.

Let us suppose $s(Y)$ is not null. Then, by considering the values of Table 1, we have:

$$\mu(y_j) = 1-|y_j-C(Y_j)|/s(Y_j) = 1-|y_j - c_1x_{1j}+c_2x_{2j}+\ldots+c_nx_{nj}|/(s_1|x_{1j}|+s_2|x_{2j}|+\ldots+s_n|x_{nj}|). \tag{2.19}$$

If $h\in(0, 1)$ is a number expressing (in the opinion of the decision maker) a sufficient degree of membership of y_j to Y_j, then the conditions $\mu(y_j) \geq h$, $j=1, 2, \ldots, m$ must be satisfied. Then from (2.19), we have the 2m constraints:

$$y_j \leq c_1x_{1j}+c_2x_{2j}+\ldots+c_nx_{nj} + (1-h)(s_1|x_{1j}|+s_2|x_{2j}|+\ldots+s_n|x_{nj}|), \quad j=1, 2, \ldots, m, \tag{2.20}$$

$$y_j \geq c_1x_{1j}+c_2x_{2j}+\ldots+c_nx_{nj} - (1-h)(s_1|x_{1j}|+s_2|x_{2j}|+\ldots+s_n|x_{nj}|), \quad j=1, 2, \ldots, m. \tag{2.21}$$

It is evident that whatever the sample (with the unique condition that $\forall j, \exists i: x_{ij} \neq 0$), and whatever the numbers c_i, conditions (2.20) and (2.21) are satisfied when the spreads s_i are sufficiently high. But high spreads mean an excessive vagueness, then it is necessary to introduce an objective function $u = f(s_1, s_2, \ldots, s_n)$, positive and increasing with respect to every variable, and seek a solution of the system (2.20), (2.21), with the unknowns c_i and s_i and minimizing u.

Two alternative objective functions are:

(1) *the sum of spreads of coefficients* [KY], e.g.,

$$f(s_1, s_2, \ldots, s_n) = s_1 + s_2 +\ldots+ s_n; \tag{2.22}$$

(2) *the sum of spreads of the sample* [RO], e.g.,

$$f(s_1, s_2, \ldots, s_n) = \Sigma_{i,j} |x_{ij}|\, s_{ij}. \tag{2.23}$$

2.3 Partial Fuzzy Linear Regression with Fuzzy Data

Equation (2.24) shows a general fuzzy linear regression model with fuzzy data:

$$Y = k_1 X_1 + k_2 X_2 + \ldots + k_n X_n, \tag{2.24}$$

where coefficients k_i are crisp numbers and the values of variables are fuzzy numbers.

Let us have the following sample:

Table 2.

Sample number, j	Output values, Y_j	Input values, X_{ij}
1	Y_1	$X_{11}, X_{21}, \ldots, X_{n1}$
2	Y_2	$X_{12}, X_{22}, \ldots, X_{n2}$
m	Y_m	$X_{1m}, X_{2m}, \ldots, X_{nm}$

For every j = 1, 2, …, m, we have to calculate the fuzzy number:

$$Y_j^* = k_1 X_{1j} + k_2 X_{2j} + \dots + k_n X_{nj}, \tag{2.25}$$

and to compare Y_j^* with the sample value Y_j.

The logics of the algorithms used are fuzzy extensions of the previous case. The condition $\mu(y_j) \geq h$ is replaced by the condition of compatibility between Y_j^* and Y_j:

$$\forall j \in \{1, 2, \dots, m\},\ com(Y_j^*, Y_j) = \sup_{y \in R} (\min(Y_j^*(y), Y_j(y)) \geq h. \tag{2.26}$$

2.4 Total Fuzzy Linear Regression

Equation (2.27) shows a total fuzzy linear regression model:

$$Y = K_1 X_1 + K_2 X_2 + \dots + K_n X_n, \tag{2.27}$$

where the coefficients K_i and the values of variables are fuzzy numbers.

Also for the algorithms of the total fuzzy linear regression formula (2.26) holds. But some problems arise because, in general:

(1) the Zadeh extension product of triangular fuzzy numbers is not a triangular fuzzy number;
(2) the left and right spreads of the product of symmetric fuzzy numbers are not equal;
(3) there is not a general simple formula for the spreads of Y.

In the next section we propose a way to overcome those difficulties by proposing alternative fuzzy operations to those based on the extension principle.

3 Alternative Fuzzy Operations and Fuzzy Regression

Let us consider the total fuzzy regression model with fuzzy coefficients and fuzzy parameters:

$$Y = K_1 X_1 + K_2 X_2 + \dots + K_n X_n, \tag{3.1}$$

where $K_i \sim [c_i, s_i, s'_i]$ and $X_i \sim [x_i, t_i, t'_i]$ are fuzzy numbers.

3.1 Some Problems and Drawbacks of the Zadeh Extension Fuzzy Regression

If, as usually happens, the addition and the multiplication are the Zadeh's extension operations, then ([BB], [BF], [BG], [DP2], [GM], [MA1], [MA2]) the left and right spreads of the Zadeh's extension product $u \cdot_z v$ of two fuzzy numbers u and v have simple formulae if the factors are positive fuzzy numbers (at most one of the factors can be c-positive). In this case the formulae of core and r-cut spreads of $u \cdot_z v$ are [MA1]:

$$c(u \cdot_z v) = c(u)\, c(v) \tag{3.2}$$

$$\forall r \in [0, 1),\ L^r(u \cdot_z v) = u_\lambda^1 L^r(v) + v_\lambda^1 L^r(u) - L^r(u)L^r(v); \tag{3.3}$$

$$\forall r \in [0, 1),\ R^r(u \cdot_z v) = u_\rho^1 R^r(v) + v_\rho^1 R^r(u) + R^r(u)R^r(v). \tag{3.4}$$

In particular, if u = [c(u), L(u), R(u)], v = [c(v), L(v), R(v)] are triangular fuzzy numbers, previous formulae (3.3) and (3.4) reduce to:

$$\forall r \in [0, 1),\ L^r(u \cdot_z v) = c(u)\, L^r(v) + c(v)\, L^r(u) - L^r(u)L^r(v); \tag{3.5}$$

$$\forall r \in [0, 1),\ R^r(u \cdot_z v) = c(u)\, R^r(v) + c(v)\, R^r(u) + R^r(u)R^r(v). \tag{3.6}$$

Some consequences are;

(a) the product of two non-degenerate triangular fuzzy numbers is not a triangular fuzzy number;
(b) the product of two non-degenerate symmetric fuzzy numbers is not a symmetric fuzzy number;
(c) the left and right spreads of the product depends not only by the spreads of the factors, but they are strongly increasing with the increase of the cores of the factors;
(d) the Zadeh's extension multiplication is subdistributive with respect to the Zadeh extension addition, i.e., for every fuzzy numbers u, v, w, we have:

$$(u + v)\, w \subseteq u\, w + v\, w, \qquad \text{(subdistributivity)} \tag{3.7}$$

where $\subseteq$ denotes inclusion between fuzzy sets. Equality in (3.7) holds if and only if u and v are both positive or both negative fuzzy numbers or w is a degenerate fuzzy number.

From previous formulae it follows that if $K_i \sim [c_i, s_i, s_i]$ and $X_i \sim [x_i, t_i, t_i]$ are positive, simple, and symmetric fuzzy numbers, then the spreads of Y are:

$$L(Y) = \Sigma_i\, (c_i\, t_i + x_i\, s_i - s_i\, t_i);\ R(Y) = \Sigma_i\, (c_i\, t_i + x_i\, s_i + s_i\, t_i), \tag{3.8}$$

It is worth noting that formula (3.8) reduces to (2.16) if every t_i is null and every x_i is positive.

The drawback (a) can be overcome by replacing the Zadeh's extension multiplication $u \cdot_z v$ with the approximate multiplication $u \cdot_a v$, defined by formulae:

$$c(u \cdot_a v) = c(u)\, c(v) \tag{3.9}$$

$$\forall r \in [0, 1),\ L^r(u \cdot_a v) = u_\lambda^{\ 1}\, L^r(v) + v_\lambda^{\ 1}\, L^r(u) - L^r(u)L^r(v)/(1\text{-}r); \tag{3.10}$$

$$\forall r \in [0, 1),\ R^r(u \cdot_a v) = u_\rho^{\ 1}\, R^r(v) + v_\rho^{\ 1}\, R^r(u) + R^r(u)R^r(v)/(1\text{-}r). \tag{3.11}$$

If u and v are triangular fuzzy numbers, then $u \cdot_a v$ is a triangular fuzzy number having the same core and the same spreads of $u \cdot_z v$. The limits of the approximation can be highlighted by a comparison of formulae (3.5), (3.6) with (3.10), (3.11).

The drawback (c) deserves careful consideration. Suppose the values of the independent variables are positive real numbers. If we change the origin of axes, i.e., increasing the value assumed by each variable of a positive number h, it would seem logical to expect an increase in the core of Y but not of left and right spreads of Y. It might not be appropriate, at least in some cases, define an addition and a multiplication in which spreads only depend on spreads of factors and the multiplication is distributive with respect to the addition?

A solution is given by the "bounded operations". In terms of spreads notation, the b-addition is defined by the formulae:

$$C(u +_b v) = C(u) + C(v); \tag{3.12}$$

$$\forall r \in [0, 1),\ L^r(u +_b v) = \max\{L^r(u), L^r(v)\};\ \ R^r(u +_b v) = \max\{R^r(u), R^r(v)\}. \tag{3.13}$$

Moreover, the b-multiplication is defined by the formulae:

$$C(u \cdot_b v) = C(u) \cdot C(v); \tag{3.14}$$

$$\forall r \in [0, 1),\ L^r(u \cdot_b v) = \max\{L^r(u), L^r(v)\};\ \ R^r(u \cdot_b v) = \max\{R^r(u), R^r(v)\}. \tag{3.15}$$

Some important properties of b-addition and b-multiplication are [MA1]:

(B1) b-sum and b-product of two trapezoidal fuzzy numbers are trapezoidal fuzzy numbers. Moreover, b-sum and b-product of simple fuzzy numbers are simple fuzzy numbers.

(B2) b-addition and b-multiplication are associative, commutative, and have neutral elements 0 and 1, respectively.

(B3) b-multiplication is subdistributive with respect to the b-addition. That is, for every fuzzy numbers u, v, w, we have

$$(u +_b v) \cdot_b w \subseteq u \cdot_b w +_b v \cdot_b w. \tag{3.16}$$

The equality holds iff $C(u)\, C(v) \geq 0$ or $C(w)$ is a real number.

(B4) The set Δ of triangular fuzzy number is closed with respect to b-addition and b-multiplication. Moreover, in Δ b-multiplication is distributive with respect to b-addition.

(B5) (invariance for translation) for every real numbers (i.e., degenerate fuzzy numbers) h, k, if $u' = u + h$, $v' = v + k$, then:

$$\forall r \in [0, 1),\ L^r(u' +_b v') = L^r(u +_b v);\ \ R^r(u' +_b v') = R^r(u +_b v) \tag{3.17}$$

$$\forall r \in [0, 1),\ L^r(u' \cdot_b v') = L^r(u \cdot_b v);\ \ R^r(u' \cdot_b v') = R^r(u \cdot_b v) \tag{3.18}$$

Thus, unlike the Zadeh's extension operations, the b-product of triangular numbers is a triangular number, the distributive property of the b-multiplication w. r. to the b-addition holds in Δ and finally (3.17) and (3.18) imply that a change of the origin of axes do not change the spreads.

An extension of the bounded operations are the $\oplus$-operations, introduced in [MA1], where $\oplus$ is a t-conorm, i.e., an operation on the interval [0, 1], $\oplus$: $(a, b) \in [0, 1] \times [0, 1] \rightarrow a \oplus b \in [0, 1]$ associative, commutative, having 0 as neutral element and increasing with respect to every variable (see, e.g., [SU], [SV], [WE], [KY]).

We assume there exist two strictly positive real numbers, L_{max} and R_{max}, the maximum left and right spreads, respectively. Let S be the set of fuzzy numbers such that, for every $u \in S$, $L(u) \leq L_{max}$, and $R(u) \leq R_{max}$.

We define the ⊕-addition on S by formulae:

$$C(u +_{\oplus} v) = C(u) + C(v); \quad (3.19)$$

$$\forall r\in [0, 1),\ L^r(u +_{\oplus} v) = [(L^r(u)/L_{max})\oplus(L^r(v)/L_{max})]\ L_{max}; \quad (3.20)$$

$$\forall r\in [0, 1),\ R^r(u +_{\oplus} v) = [(R^r(u)/R_{max})\oplus(R^r(v)/R_{max})]\ R_{max}. \quad (3.21)$$

The ⊕-multiplication on S is defined by:

$$C(u \cdot_{\oplus} v) = C(u) \cdot C(v); \quad (3.22)$$

$$\forall r\in [0, 1),\ L^r(u \cdot_{\oplus} v) = [(L^r(u)/L_{max})\oplus(L^r(v)/L_{max})]\ L_{max}; \quad (3.23)$$

$$\forall r\in [0, 1),\ R^r(u \cdot_{\oplus} v) = [(R^r(u)/R_{max})\oplus(R^r(v)/R_{max})]\ R_{max}. \quad (3.24)$$

By previous definitions it follows:

(C1) The ⊕-addition and ⊕-multiplication are associative, commutative, having neutral elements 0 and 1, respectively.

(C2) If ⊕ is the fuzzy union, then the ⊕-addition and the ⊕-multiplication reduces to the bounded operations.

(C3) The left and right r-cut spreads of the sum $u +_{\oplus} v$ and the product $u \cdot_{\oplus} v$ are not greater than L_{max} and R_{max}, respectively.

(C4) the invariance for translations holds.

4 Remarks on Some Critical Points of the Variable-Oriented Research and Conclusions

The variable-oriented research presents some critical points (see e.g., [BR1], [BR2], [RA]). Among these are the following assumptions:

(1) homogeneity of the cases;
(2) a linear relationship among the variables;
(3) the additivity of the outcomes with respect to the variables input;
(4) the necessity and sufficiency of the causes for the outcomes.

The study of social phenomena based on *diversity-oriented research* put in evidence that these assumptions are often not justified by the evidence.

In fact, if the cases belong to different types (to define with suitable procedures) then the following circumstances may occur:

(a) the same causes give different outcomes;
(b) different causes may yield the same outcome;
(c) for some types a cause (or the intersection of a set of causes) is sufficient to produce an outcome, for other types it is not sufficient;
(d) for some types a cause (or the intersection of a set of causes) is necessary to produce an outcome, for other types it is not necessary;
(e) the aggregation of causes to obtain a sufficient condition is superadditive;
(f) the aggregation of necessary but not sufficient causes is subadditive.

Moreover, often a set of causes produces an outcome only if they exceed a certain level (or a real, positive, increasing w. r. to every variable, function of the levels of the causes is greater than a positive real number h). These situations can be formalized in terms of fuzzy sets. If $K = \{k_1, k_2, \ldots, k_p\}$ is a set of causes and S is a population, the level of the cause k_i on S can be defined as a fuzzy set $\mu_i : x \in S \rightarrow [0, 1]$. If $h_i \in (0, 1]$ is the level at which the cause produces effect, then we have outcomes only in the h_i-cut of μ_i.

It follows a fundamental role of fuzzy sets and their aggregation to obtain necessary and/or sufficient conditions for outcomes. This is an alternative approach to the linear regression. This approach takes into account the complexity of the phenomena, i.e., the various features of each case study. In this frame of reference the relations between causes and effects are not linear, dependent on the diversity of cases, the level at which the cause produces effect on the elements of the population and so on. Insights into these aspects are in [BR1], [BR2], [RA].

References

[BB] Ban, A.I., Bede, B.: Cross product of L-R fuzzy numbers and properties. Annals of Oradea University, Fasc. Matem. 9, 95–108 (2003)

[BF] Bede, B., Fodor, J.: Product type operations between fuzzy numbers and their applications in Geology. Acta Polytechnica Hungarica 3(1), 123–139 (2006)

[BG] Bellman, R., Giertz, M.: On the analytic formalism of the theory of fuzzy sets. Information Sciences 5, 149–157 (1973)

[BR1] Boudon, R.: L'analyse mathématique des faits socials, Paris, Plon (1967)

[BR2] Boudon R.: Les méthods in Sociologie, Paris, P.U.F, (1969)

[CH] Cristea, I., Hošková, Š.: Fuzzy topological hypergroupoids. Iranian Journal of Fuzzy Systems 6(4), 13–21 (2009)

[DP1] Dubois, D., Prade, H.: Fuzzy Set and Systems. Academic Press, New York (1980)

[DP2] Dubois, D., Prade, H.: Fuzzy numbers: An overview. In: Bedzek, J.C. (ed.) Analysis of Fuzzy Information, vol. 2, pp. 3–39. CRC-Press, Boca Raton (1988)

[GO] Goguen J.: On fuzzy robot planning, Memo no. 1 on Artificial Intelligence. University of California, Los Angeles, (1974)

[GM] Grzegorzewski, P., Mrowska, E.: Trapezoidal approximations of fuzzy numbers. Fuzzy Sets and Systems 153, 115–135 (2005)

[KY] Klir, G., Yuan, B.: Fuzzy sets and fuzzy logic: Theory and Applications. Prentice Hall, New Jersey (1995)

[MM1] Mares, M.: Fuzzy Cooperative Games. Springer, New York (2001)

[MM2] Mares, M.: Weak arithmetic on fuzzy numbers. Fuzzy sets and Systems 91(2), 143–154 (1997)

[MA1] Maturo, A.: Alternative Fuzzy Operations and Applications to Social Sciences. International Journal of Intelligent Systems 24, 1243–1264 (2009)

[MA2] Maturo, A.: On Some Structures of Fuzzy Numbers. Iranian Journal of Fuzzy Systems 6(4), 49–59 (2009)

[MV] Maturo, A., Tofan, I., Ventre, A.: Fuzzy Games and Coherent Fuzzy Previsions. Fuzzy Systems & A. I. 10(3), 109–116 (2004)

[RA] Ragin, C.C.: Fuzzy-Set Social Science. University Chicago Press, Chicago (2000)

[RO] Ross, T.J.: Fuzzy logic with engineering applications. McGraw-Hill, New York (1995)

[SV] Squillante, M., Ventre, A.G.S.: Consistency for uncertainty measure. International Journal of Intelligent Systems 13, 419–430 (1998)

[SU] Sugeno, M.: Theory of fuzzy integral and its Applications, Ph. D. Thesis, Tokyo (1974)

[WE] Weber, S.: Decomposable measures and integrals for archimedean t-conorms. J. Math. Anal. Appl. 101(1), 114–138 (1984)

[YA] Yager, R.: A characterization of the extension principle. Fuzzy Sets Systems 18(3), 205–217 (1986)

[ZA1] Zadeh, L.: Fuzzy sets. Inf. Control 8, 338–353 (1965)

[ZA2] Zadeh, L.: The concept of a Linguistic Variable and its Application to Approximate Reasoning I. Information Sciences 8, 199–249 (1975)

[ZA3] Zadeh, L.: The concept of a Linguistic Variable and its Applications to Approximate reasoning II. Information Sciences 8, 301–357 (1975)

[ZA4] Zadeh, L.: The concept of a Linguistic Variable and its Applications to Approximate reasoning III. Information Sciences 9, 43–80 (1975)