On Some Voting Paradoxes: A Fuzzy Preference and a Fuzzy Majority Perspective

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Abstract. Group decision making, as meant in this paper, is the following choice problem which proceeds in a multiperson setting. There is a group of individuals (decision makers, experts, ...) who provide their testimonies concerning an issue in question as individual preference relations over some set of option (alternatives, variants, ...). The problem is to find a solution, i.e., an alternative or a set of alternatives which best reflects the preferences of the group of individuals as a whole. First, we survey main developments in group decision making under fuzziness and outline some basic inconsistencies and negative results in group decision making and social choice, and show how they can be alleviated by some plausible modifications of underlying assumptions, mainly by introducing fuzzy preference relations and a fuzzy majority. We concentrate on how to derive solutions under individual fuzzy preference relations, and a fuzzy majority equated with a fuzzy linguistic quantifier (e.g., most, almost all, ...), and discuss a related issue of how to define a "soft" degree of consensus in the group. Finally, we show how fuzzy preferences can help alleviate some known voting paradoxes.

Keywords: fuzzy logic, linguistic quantifier, fuzzy preference relation, fuzzy majority, group decision making, social choice, consensus.

1 Introduction

The purpose of this paper is to briefly discuss an important issue of *voting paradoxes*, meant as some intuitively implausible, surprising, or even difficult to imagine, and generally unwelcome results that occur in voting type situations. In this paper we will deal with the voting paradoxes in the perspective of the fundamental works by Nurmi (1999) (cf. also Nurmi and Meskanen 1999). Basically, in those works one type of the voting paradoxes is formed by the classic, best known paradoxes which are known as those of Condorcet and Borda. The former

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is related to the intransitivity of collective preference relation that are employed in pairwise majority comparisons, and the latter just shows that it is may be possible that an intuitively implausible alternative is elected, the one defeated by all other alternatives in pairwise comparisons by a majority of votes (cf. also [2,5]). There is a rich literature on how to try to avoid those, and many other, voting paradoxes. In this paper we will focus on an approach that is based on the use of individual fuzzy preference relations and the fuzzy majority as proposed and developed by Kacprzyk, Zadrożny, Nurmi and Fedrizzi [27,28,29].

In the next sections, we will first briefly summarize the essence of a fuzzy preference relation and fuzzy majority based approach to group decision making (and voting), then provide a brief overview of main voting paradoxes, and then show some effective and efficient methods of avoiding (or alleviating) those paradoxes by employing elements of the fuzzy preference and fuzzy majority based approach.

2 Group Decision Making: A Fuzzy Preference and Fuzzy Majority Based Approach

We consider the case of multiperson decision making, more specifically group decision making, practically from the perspective of social choice (voting), under some fuzzification of preferences and majority. We assume that there is a set of alternatives and a set of individuals who provide their testimonies as *preferences* over the set of alternatives. The problem is to find a *solution*, i.e., an alternative (or a set of alternatives) which is best acceptable by the group of individuals as a whole.

Since its very beginning group decision making has been plagued by negative results, the essence of which is that no "rational" choice procedure satisfies all "natural", or plausible, requirements; a notable example is Arrow's impossibility theorem (cf. Arrow [1] or Kelly [30]), or some results by Gibbard and Satterthwaite, McKelvey, Schofield, etc. – cf. Nurmi [40].

A main direction is here based on the *individual* and *social fuzzy preference relation*. If we have a set of $n \geq 2$ alternatives, $S = \{s_1, \ldots, s_n\}$, and a set of $m \geq 2$ individuals, $E = \{1, \ldots, m\}$, then an individual's $k \in E$ individual fuzzy preference relation in $S \times S$ assigns a number in $[0, 1]$ to the preference of one alternative over another according to individual k; for some conditions, see Fodor and Roubens' [9].

Another basic element underlying group decision making is the concept of a *majority*. Some of the above-mentioned negative results in group decision making are closely related to too strict a representation of majority (e.g., at least a half, at least 2/3, ...). A natural way out is clearly to somehow make that strict concept of majority softer, closer to its real human perception.

A Natural manifestations of a soft (fuzzy) majority are the so-called *linguistic quantifiers* as, e.g., most, almost all, much more than a half, etc., which can be dealt with by fuzzy-logic-based calculi of linguistically quantified statements as proposed by Zadeh [52], and some other approaches, notably Yager's [49] ordered weighted averaging (OWA) operators (cf. Yager and Kacprzyk [50] and Yager, Kacprzyk and Beliakov [51]). For simplicity and brevity, we will employ here the classic Zadeh's [52] approach, and refer the reader to, for instance, Kacprzyk, Zadrożny, Fedrizzi and Nurmi [27] for information on the use of other approaches. The fuzzy linguistic quantifiers, which stand for fuzzy majorities, are basically tools for a linguistic quantifier driven aggregation.

We use a standard notation and setting. A fuzzy set A in $X = \{x\}$, is characterized and equated with its membership function $\mu_A : X \longrightarrow [0,1]$ such that $\mu_A(x) \in [0,1]$ is the grade of membership of $x \in X$ in A, from full membership to full nonmembership, through all intermediate values. For a finite $X = \{x_1, \ldots, x_n\}$, we write $A = \mu_A(x_1)/x_1 + \cdots + \mu_A(x_n)/x_n$. Moreover, $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$.

A *linguistically quantified statement*, e.g., "most (Q) experts are convinced (F)", is generally written as

$$Qy\text{'s are } F \tag{1}$$

where Q is a linguistic quantifier (e.g., most), $Y = \{y\}$ is a set of objects (e.g., experts), and F is a property (e.g., convinced).

We can assign to the particular y's (objects) a different importance (relevance, competence, ...), B, which may be added to (1) yielding a *linguistically quantified statement with importance qualification*, e.g., "most (Q) of the important (B) experts (y's) are convinced (F)", written as

$$QBy\text{'s are } F \tag{2}$$

In Zadeh's [52] method, a (proportional, as assumed here) fuzzy linguistic quantifier Q is assumed to be a fuzzy set defined in $[0,1]$. For instance, $Q =$ "most" may be given as

$$\mu_Q(x) = \begin{cases} 1 & \text{for } x \geq 0.8 \\ 2x - 0.6 & \text{for } 0.3 < x < 0.8 \\ 0 & \text{for } x \leq 0.3 \end{cases} \tag{3}$$

which may be meant as that if at least 80% of some elements satisfy a property, then *most* of them certainly (to degree 1) satisfy it, when less than 30% of them satisfy it, then *most* of them certainly do not satisfy it (satisfy to degree 0), and between 30% and 80%—the more of them satisfy it the higher the degree of satisfaction by *most* of the elements.

Property F is defined as a fuzzy set in Y. For instance, if $Y = \{X, W, Z\}$ is the set of experts and F is a property "convinced", then $F =$ "convinced" $= 0.1/X + 0.6/W + 0.8/Z$ which means that expert X is convinced to degree 0.1, W to degree 0.6 and Z to degree 0.8. If now $Y = \{y_1, \ldots, y_p\}$, then it is assumed that truth(y_i is F) $= \mu_F(y_i)$, $i = 1, \ldots, p$.

Then, the truth of (1) is calculated as:

$$r = \frac{1}{p} \sum_{i=1}^{p} \mu_F(y_i) \tag{4}$$

$$\text{truth}(Qy\text{'s are } F) = \mu_Q(r) \tag{5}$$

With importance qualification, B is defined as a fuzzy set in Y, and $\mu_B(y_i) \in [0,1]$ is a degree of importance of y_i: from 1 for definitely important to 0 for definitely unimportant, through all intermediate values. We rewrite first "QBy's are F" as "$Q(B$ and $F)y'$s are B" which leads to the following counterparts of (4) and (5):

$$r' = \frac{\sum_{i=1}^{p}[\mu_B(y_i) \wedge \mu_F(y_i)]}{\sum_{i=1}^{p} \mu_B(y_i)} \tag{6}$$

$$\text{truth}(QBY'\text{s are } F) = \mu_Q(r') \tag{7}$$

The method presented is simple and efficient, and has proven to be useful in a multitude of cases, also in this paper.

Group decision making (equated here with social choice) proceeds as follows. We have a set of $n \geq 2$ alternatives, $S = \{s_1, \ldots, s_n\}$, and a set of $m \geq 2$ individuals, $E = \{1, \ldots, m\}$. Each individual $k \in E$ provides his or her testimony as to the alternatives in S, as individual fuzzy preference relations in $S \times S$.

An *individual fuzzy preference relation* of individual k, R_k, is given by its membership function $\mu_{R_k} : S \times S \longrightarrow [0,1]$ such that

$$\mu_{R_k}(s_i, s_j) = \begin{cases} 1 & \text{if } s_i \text{ is definitely preferred to } s_j \\ c \in (0.5, 1) & \text{if } s_i \text{ is slightly preferred to } s_j \\ 0.5 & \text{in the case of indifference} \\ d \in (0, 0.5) & \text{if } s_j \text{ is slightly preferred to } s_i \\ 0 & \text{if } s_j \text{ is definitely preferred to } s_i \end{cases} \tag{8}$$

Basically, two lines of reasoning may be followed here (cf. Kacprzyk [15]):

– a direct approach: $\{R_1, \ldots, R_m\} \longrightarrow$ solution, that is, a solution is derived directly (without any intermediate steps) just from the set of individual fuzzy preference relations, and
– an indirect approach: $\{R_1, \ldots, R_m\} \longrightarrow R \longrightarrow$ solution, that is, from the set of individual fuzzy preference relations we form first a social fuzzy preference relation, R (to be defined later), which is then used to find a solution.

A solution is here, unfortunately, not clearly understood – cf. Nurmi [36] – [40]. First, in the case of group decision making under fuzzy preferences only, i.e., under a conventional majority, we start with solution concepts that do not require any preference aggregation at all. One of them is that of a *core* or a *set of undominated alternatives*, under a nonfuzzy required majority be r (e.g., at least 50%).

Definition 1. An alternative $x \in S$ belongs to the *core* iff there is no other alternative $y \in S$ that defeats x by the required majority r.

We can extend the notion of a core to cover fuzzy individual preference relations by defining a *fuzzy α-core* as follows (cf. Nurmi [36]):

Definition 2. An alternative $s_i \in S$ belongs to the *fuzzy α-core* S_α iff there exists no other alternative $s_j \in S$ such that $r_{ji} > \alpha$ for at least r individuals.

The intuitive interpretation of the fuzzy α-core is obvious: an alternative is a member of S_α if and only if a sufficient majority of voters does not feel strongly enough against it.

Another nonfuzzy solution concept with much intuitive appeal is a *minimax set* defined in a nonfuzzy setting as:

Definition 3. If, for each $x, y \in S$, we denote the number of individuals preferring x to y by $n(x, y)$, and define $v(x) = \max_y n(y, x)$ and $n* = \min_x v(x)$, then the *minimax set* is

$$Q(n*) = \{x \mid v(x) = n*\}$$

Thus, $Q(n*)$ consists of those alternatives that in pairwise comparison with any other alternative are defeated by no more than $n*$ votes. Obviously, if $n* < m/2$, where m is the number of individuals, then $Q(n*)$ is singleton and $x \in Q(n*)$ is the core if the simple majority rule is being applied.

Analogously, we can define *the minimax degree set* $Q(\beta)$ as follows:

Definition 4. *Given:* $s_i, s_j \in S$ *and, for individuals* $k = 1, \ldots, m$, $v_D^k(x_j) = \max_i r_{ij}$, $v_D(x_j) = \max_k v_D^k(x_j)$, *and let* $\min_j v_D(x_j) = \beta$.

Then

$$Q(\beta) = \{x_j \mid v_D(x_j) = \beta\}$$

A more general solution concept, the α-*minimax set* (cf. Nurmi [36]) denoted $Q^\alpha(v_f^\alpha)$, is defined as follows:

Definition 5. *Let* $n_\alpha(x_i, x_j)$ *be the number of individuals for whom* $r_{ij} \leq \alpha$ *for some value of* $\alpha \in [0, 0.5)$. *We now define* $\forall x_i \in S : v_f^\alpha(x_i) = \max_j n_\alpha(x_i, x_j)$ *and* $\bar{v}_f^\alpha = \min_i v_f^\alpha(x_i)$.

Then the α*-minimax set is defined as*

$$Q^\alpha(v_f^\alpha) = \{x_i \mid v_f^\alpha(x_i) = \bar{v}_f^\alpha\}$$

It can be shown that $Q^\alpha(v_f^\alpha) \subseteq Q(n*)$ (see [36]).

Now, we will show some basic solution concepts based on a social fuzzy preference relation that is obtained by an aggregation of the individual fuzzy preference relations. We will concentrate on those derived along the lines of Nurmi [36].

Definition 6. The set S_α of α-*consensus winners* is defined as: $s_i \in S_\alpha$ iff $\forall s_j \neq s_i : r_{ij} \geq \alpha$, with $0.5 < \alpha \leq 1$

Whenever S_α is nonempty, it is a singleton, but it does not always exist.

Definition 7. Let $\bar{r}_j = \max_i r_{ij}$ and $\bar{r} = \min_j \max_i r_{ij}$. Then $s_i \in S_M$ is the set of *minimax consensus winners* if and only if $\bar{r}_i = \bar{r}$.

Clearly S_M is always nonempty, but not necessarily a singleton.

Now, we will consider some solution concepts of group decision making but when we both have fuzzy preference relations and a fuzzy majority.

We will first employ the direct approach, i.e., $\{R_1, \ldots, R_m\} \longrightarrow$ solution to derive two popular solution concepts: fuzzy cores and minimax sets.

Definition 8. *Conventionally, the core is defined as a set of undominated alternatives, i.e., those not defeated in pairwise comparisons by a required (strict!) majority $r \leq m$, i.e.*

$$C = \{s_j \in S : \neg \exists s_i \in S \text{ such that } r_{ij}^k > 0.5 \text{ for at least } r \text{ individuals}\} \tag{9}$$

As the first fuzzification attempt, Nurmi [36] who has extended it as follows:

Definition 9. *The fuzzy α-core is defined as*

$$C_\alpha = \{s_j \in S : \neg \exists s_i \in S \text{ such that } r_{ij}^k > \alpha \geq 0.5 \text{ for at least } r \text{ individuals}\} \tag{10}$$

that is, as a set of alternatives not sufficiently (at least to degree α) defeated by the required (still strict!) majority $r \leq m$.

The concept of a fuzzy majority has been here proposed by Kacprzyk [15] and has turned out to be useful and adequate.

We start by denoting

$$h_{ij}^k = \begin{cases} 1 \text{ if } r_{ij}^k < 0.5 \\ 0 \text{ otherwise} \end{cases} \tag{11}$$

where $i, j = 1, \ldots, n$ and $k = 1, \ldots, m$.

Thus, h_{ij}^k just reflects if alternative s_j defeats (in pairwise comparison) alternative s_i ($h_{ij}^k = 1$) or not ($h_{ij}^k = 0$). Then, we calculate $h_j^k = \frac{1}{n-1} \sum_{i=1, i \neq j}^{n} h_{ij}^k$ which is clearly the extent, from 0 to 1, to which individual k is not against alternative s_j, where 0 standing for definitely against to 1 standing for definitely not against, through all intermediate values. Next, we calculate $h_j = \frac{1}{m} \sum_{k=1}^{m} h_j^k$ which it to what extent *all* the individuals are not against alternative s_j. And, finally, we calculate $v_Q^j = \mu_Q(h_j)$ is to what extent, from 0 to 1 as before, Q (say, most) individuals are not against alternative s_j.

Then:

Definition 10. *The fuzzy Q-core is defined (cf. Kacprzyk [15]) as a fuzzy set,*

$$C_Q = v_Q^1/s_1 + \cdots + v_Q^n/s_n \tag{12}$$

i.e., as a fuzzy set of alternatives that are not defeated by Q (say, most) individuals.

Notice that in the above basic definition of a fuzzy Q-core, we do not take into consideration to what degrees those defeats of one alternative by another are. They can be accounted for in a couple of plausible ways.

First, the degree of defeat in (11) may be replaced by

$$h_{ij}^k(\alpha) = \begin{cases} 1 \text{ if } r_{ij}^k < \alpha \leq 0.5 \\ 0 \text{ otherwise} \end{cases} \tag{13}$$

where, again, $i, j = 1, \ldots, n$ and $k = 1, \ldots, m$. Thus, $h_{ij}^k(\alpha)$ just reflects if alternative s_j sufficiently (i.e., at least to degree $1 - \alpha$) defeats (in pairwise comparison) alternative s_i or not.

We can also explicitly introduce the strength of defeat into (11) via:

$$\hat{h}_{ij}^k = \begin{cases} 2(0.5 - r_{ij}^k) \text{ if } r_{ij}^k < 0.5 \\ 0 \qquad\qquad\quad \text{otherwise} \end{cases} \tag{14}$$

where, again, $i, j = 1, \ldots, n$ and $k = 1, \ldots, m$. Thus, $\hat{h}_{ij}^k$ just reflects how strongly (from 0 to 1) alternative s_j defeats (in pairwise comparison) alternative s_i.

Then, by following the same procedure we can derive an α/Q-fuzzy core and an s/Q-fuzzy core.

Another intuitively justified solution concept may be the minimax (opposition) set which may be defined for our purposes as follows.

Definition 11. *Let $w(s_i, s_j) \in \{1, 2, \ldots, m\}$ be the number of individuals who prefer alternative s_j to alternative s_i, i.e., for whom $r_{ij}^k < 0.5$.*

If now $v(s_i) = \max_{j=1,\ldots,n} w(s_i, s_j)$ and $v^* = \min_{i=1,\ldots,n} v(s_i)$, then the minimax (opposition) set is defined as

$$M(v^*) = \{s_i \in S : v(s_i) = v^*\} \tag{15}$$

i.e., as a (nonfuzzy) set of alternatives which in pairwise comparisons with any other alternative are defeated by no more than v^* individuals, hence by the least number of individuals.

Nurmi [36] extends the minimax set, similarly in spirit to his extension of the core (10), to the *α-minimax set* as follows:

Definition 12. *Let $w_\alpha(s_i, s_j) \in \{1, 2, \ldots, m\}$ be the number of individuals who prefer alternative s_j to alternative s_i at least to degree $1 - \alpha$, i.e., for whom $r_{ij}^k < \alpha \leq 0.5$.*

If now $v_\alpha(s_i) = \max_{j=1,\ldots,n} w_\alpha(s_i, s_j)$ and $v_\alpha^* = \min_{i=1,\ldots,n} v_\alpha(s_i)$, then the α-minimax set is defined as:

$$M_\alpha(v_\alpha^*) = \{s_i \in S : v_\alpha(s_i) = v_\alpha^*\} \tag{16}$$

i.e., as a (nonfuzzy) set of alternatives which in pairwise comparisons with any other alternative are defeated (at least to degree $1 - \alpha$) by no more than v^* individuals, hence by the least number of individuals.

A fuzzy majority was introduced into the above definitions of minimax sets by Kacprzyk [15] as follows.

We start with (11), i.e.,

$$h_{ij}^k = \begin{cases} 1 \text{ if } r_{ij}^k < 0.5 \\ 0 \text{ otherwise} \end{cases} \tag{17}$$

and $h_i^k = \frac{1}{n-1} \sum_{j=1, j \neq i}^n h_{ij}^k$ is the extent, between 0 and 1, to which individual k is against alternative s_i. Then $h_i = \frac{1}{m} \sum_{k=1}^m h_i^k$ is the extent, between 0 and 1, to which all the individuals are against alternative s_i. Next, $t_i^Q = \mu_Q(h_i)$ is the extent, from 0 to 1, to which Q (say, most) individuals are against alternative s_i, and $t_Q^* = \min_{i=1,\ldots,n} t_i^Q$ is the least defeat of any alternative by Q individuals.

Finally:

Definition 13. *The Q-minimax set is defined as*

$$M_Q(t^*_Q) = \{s_i \in S : t^Q_i = t^*_Q\} \tag{18}$$

And analogously as for the α/Q-core and s/Q-core, we can explicitly introduce the degree of defeat $\alpha < 0.5$ and s into the definition of the Q-minimax set.

In the case of an indirect derivation, we follow the scheme: $\{R_1, \ldots, R_m\} \longrightarrow R \longrightarrow$ solution, i.e., from the individual fuzzy preference relations we determine first a social fuzzy preference relation, R, and then find a solution from such a social fuzzy preference relation.

The indirect derivation involves two problems:

- how to find a social fuzzy preference relation from the individual fuzzy preference relations, i.e.,
$$\{R_1, \ldots, R_m\} \longrightarrow R$$
- how to find a solution from the social fuzzy preference relation, i.e.,
$$R \longrightarrow \text{ solution}$$

In this paper, we will not deal in more detail with the first step, i.e., $\{R_1, \ldots, R_m\} \longrightarrow R$, and assume a (most) straightforward alternative that the social fuzzy preference relation $R = [r_{ij}]$ is given by

$$r_{ij} = \begin{cases} \frac{1}{m} \sum_{k=1}^{m} a^k_{ij} & \text{if } i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

where $a^k_{ij} = \begin{cases} 1 \text{ if } r^k_{ij} > 0.5 \\ 0 \text{ otherwise} \end{cases}$. Notice that R obtained via (19) need not be reciprocal, i.e., $r_{ij} \neq 1 - r_{ji}$, but it can be shown that $r_{ij} \leq 1 - r_{ji}$, for each $i, j = 1, \ldots, n$.

In the second case, i.e., $R \longrightarrow$ solution, a solution concept of much intuitive appeal is here the consensus winner (cf. Nurmi [36]) which will be extended under a social fuzzy preference relation and a fuzzy majority.

We start with

$$g_{ij} = \begin{cases} 1 \text{ if } r_{ij} > 0.5 \\ 0 \text{ otherwise} \end{cases} \tag{20}$$

which expresses if s_i defeats (in the whole group's opinion!) s_j or not.

Next $g_i = \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} g_{ij}$ which is a mean degree to which s_i is preferred, by the whole group, over all the other alternatives. Then, $z^i_Q = \mu_Q(g_i)$ is the extent to which alternative s_i is preferred, by the whole group, over Q (e.g., most) other alternatives.

Finally:

Definition 14. *The fuzzy Q-consensus winner is defined as*

$$W_Q = z^1_Q/s_1 + \cdots + z^n_Q/s_n \tag{21}$$

i.e., as a fuzzy set of alternatives that are preferred, by the whole group, over Q other alternatives.

And analogously as in the case of the core, we can introduce a threshold $\alpha \geq 0.5$ and s into (20) and obtain a *fuzzy α/Q-consensus winner* and a *fuzzy s/Q-consensus winner*, respectively.

This concludes our brief exposition of how to employ fuzzy linguistic quantifiers to model the fuzzy majority. We did not present some other solution concepts as, e.g., minimax consensus winners (cf. Nurmi [36], Kacprzyk [15]) or those based on fuzzy tournaments which have been proposed by Nurmi and Kacprzyk [45]and are relevant in the voting context.

One should also notice that in a number of recent papers by Kacprzyk and Zadrożny [25] it has been shown that the concept of Kacprzyk's [15] fuzzy Q-core can be a general (prototypical) choice function in group decision making and voting; for instance, those of: a "consensus solution", Borda's rule, the minimax degree set, the plurality voting, the qualified plurality voting, the approval voting-like, the "consensus + approval voting", Condorcet's rule, the Pareto rule, Copeland's rule, Nurmi's minimax set, Kacprzyk's Q-minimax, the Condorcet looser, the Pareto inferior alternatives, etc. This result, though of some relevance to the problem of dealing with voting paradoxes, is however beyond the scope of this paper.

To summarize this section, the fuzzy preferences and a fuzzy majority can be used to derive more flexible and human consistent versions of main solution concepts in group decision making, in a natural connection to voting.

3 Remarks on Some Voting Paradoxes and Their Alleviation

Voting paradoxes are an interesting and very relevant topic that has a considerable theoretical and practical relevance. In this paper we will just give some simple examples of well-known paradoxes and indicate some possibilities of how to alleviate them by using some elements of fuzzy preferences and a fuzzy majority. The paper is based on the works by Nurmi [43], [43], Nurmi and Kacprzyk [46], and Kacprzyk, Zadrożny, Fedrizzi and Nurmi [27,28,29].

In most case, one distinguishes between the so-called classic paradoxes, the so-called Condorcet and Borda paradoxes, and some other ones, which are interesting too but maybe less known. Basically, the former pertains to the non-transitivity of a collective preference relation if it is formed using pairwise majority comparisons, while the latter shows the possibility that an intuitively implausible alternative is elected even if it is defeated by all other alternatives in pairwise comparisons by a majority of voters (cf. Black [2], DeGrazia [5], Nurmi [43]).

3.1 Condorcet's Paradox

An example of Condorcet's paradox is shown in Table 1. There are 3 voter groups of equal size whose preferences over alternatives A, B and C are represented by the rank order indicated below each group. The equal size of the groups is not necessary but any two of them should constitute a majority. A collective preference relation formed by pairwise comparisons of alternatives using the majority rule results in a cycle: A is preferred to B, B is preferred to C and C is preferred to A.

Table 1. Condorcet's paradox

Group I	*Group II*	*Group III*
A	B	C
B	C	A
C	A	B

3.2 Borda's Paradox

An instance of Borda's paradox is given in Table 2 in which alternative A wins clearly by a plurality of votes and, yet, both B and C beat A under pairwise comparisons using the majority rule.

Table 2. Borda's paradox

voters 1-4	*voters 5-7*	*voters 8,9*
A	B	C
B	C	B
C	A	A

One can easily notice that a common characteristic in these classic paradoxes is the violation of intuitively plausible requirements. In the case of Condorcet's paradox, the result obtained by using the majority rule on a set of complete and transitive preferences is intransitive. In the case of Borda's paradox, the winner in the plurality sense is different from the winner in the sense that the winner is to beat all the other ones in pairwise comparisons.

3.3 Some other Paradoxes

Among other, less known paradoxes, presumably the most important in this class is the so-called *additional support paradox* which occurs whenever some additional support makes a winning alternative a non-winning one. Unfortunately, many commonly used voting procedures are plagued by this.

An example of an additional support paradox is shown in Table 3. Suppose that the voters vote according to their preferences, and in the first round alternative A gets 22 votes, B – 21 votes and C – 20 votes. Since no alternative gets more than 50% of the 63 votes, there will be a second round between A and B. Suppose that A wins as the 20 voters who have favored C will presumably vote for A rather than their lowest ranked B. Hence A is the winner. Suppose now that A obtains some more support, say, 2 out of those 21 voters with the preference ranking BCA. We now have 24 voters with the preference ranking ABC, 19 voters with the ranking BCA and 20 voters with the ranking CAB. A runoff is again needed, now between A and C. But now C wins by 39 votes against 22. Thus, an additional support would be disastrous for A.

Table 3. Additional support paradox

22 voters	*21 voters*	*20 voters*
A	B	C
B	C	A
C	A	B

Another type (maybe even a class) is the *choice set variance paradoxes* the crucial feature of which is a counter-intuitive variation in choice sets under certain types of modifications in the set of alternatives or the preference profile.

An example of a choice set variance paradox is given in Table 4 which shows the preference profile of a 100-voter electorate divided into two equal parts. Suppose that the plurality runoff system is used and that the votes are counted separately in each part of the electorate. In both parts A is the winner. In the left half there is a runoff between A and B yielding A as the winner. In the right half A is the winner with more than 50% of the votes. Suppose now that the whole electorate is taken as a whole. This implies C to be the winner. Thus, in spite of being the winner in both halves of the electorate, A is not the winner in the entire set of voters.

Table 4. Consistency paradox

20	*20*	*10*	*26*	*4*	*20*
A	B	C	A	B	C
B	C	A	B	C	B
C	A	B	C	A	A

Two specific choice set variance paradoxes, the so-called *Ostrogorski's paradox* and the *referendum paradox* are related to the majority rule that is a foundation of all democratic systems. They occur also in contexts in which the choice sets, obtained by using the majority rule, are combined. Hence, they are often called the compound majority paradoxes.

Table 5 shows the essence of the Ostrogorski's paradox. It shows a distribution of support over two parties (X and Y) and three issues (issues $1, 2, 3$). Thus, for example, 20% of the electorate, denoted by group B, supports party X on issues 1 and 3 and party Y on issue 2. If all issues are of equal importance, then it may be assumed that they vote for that party which they support on more issues—cf. the last column.

Table 5. Ostrogorski's paradox

group	*issue 1*	*issue 2*	*issue 3*	*party supported*
A (20%)	X	X	Y	X
B (20%)	X	Y	X	X
C (20%)	Y	X	X	X
D (40%)	Y	Y	Y	Y

The result seems to be X since it is supported by 60% of the electorate. However, Ostrogorski argued that the legitimacy of X's victory could be challenged since, by voting on each issue separately, Y would win in each case by a majority of 60—40%.

In some countries where consultative non-binding referenda are being resorted to, a particular problem of great importance may be encountered, namely which result is more authoritative: the referendum outcome or the parliamentary voting outcome. This problem has certain similarities with Ostrogorski's paradox (see [41,42]).

Table 6. Referendum paradox

opinions	*MP's of A*	*MP's of B*	*vote total*
	1-6	*7-9*	
"yes"	5	11	63
"no"	6	0	36

Suppose that the parliament consists of 9 members and there are 99 voters. Assume, moreover, that the support for each elected member is the same, i.e., 11 votes for each member. Party A has 6 out of 9 or 2/3 of the parliament seats, while party B has 3 out of 9 or 1/3 of the seats. Suppose that the support of the parties corresponds to the seat distribution, that is, 2/3 of the electorate supports party A and 1/3 party B.

Now, suppose that a referendum is held in which the voters are asked to answer either *yes* or *no* to a question. Let the distribution of votes in both parliamentary elections and the referendum as shown in Table 6). Clearly, *yes* wins the referendum receiving 63 votes out of 99. Suppose now that the same issue is subjected to a parliamentary vote. Then, assuming that the members of parliament are aware of the distribution of opinions of their own supporters, it is plausible to predict that they vote in accordance with what they think is the opinion of the majority of their supporters. Thus, the members of party A would vote for *no* and those of party B for *yes*, and *no* wins by a handsome margin 6 to 3.

We have shown just a couple of voting paradoxes and in the next section we will show how one can eliminate them, or—better to say—alleviate them by using some elements of our approach to group decision making based on fuzzy preference relations and fuzzy majority.

4 Alleviating and Solving Some Voting Paradoxes

Now we will outline the essence of our approach on how to solve those voting paradoxes by using a setting of group decision making and voting with fuzzy preference relations. As an illustrative example we will consider Condorcet's and Borda's paradox only. A similar procedure, though more complicated, may be employed for other paradoxes.

We consider the set E of individuals and the set S of decision alternatives. Each individual $i \in E$ provides a fuzzy preference relation $R_i(x, y)$ over S. For each $x, y \in S$, the value $R_i(x, y)$ indicates the degree in which x is preferred to y by i with 1 indicating the strongest preference of x to y, 0.5 indifference between the two and value 0 the strongest preference of y to x.

To facilitate our discussion, let us briefly recall some issues related to fuzzy preference relations.

A fuzzy preference relation R is *connected* if and only if $R(x, y) + R(y, x) \geq 1, \forall x, y \in S$.

A fuzzy preference relation R is *reflexive* if and only if $R(x, x) = 1, \forall x \in S$.

A fuzzy connected and reflexive relation R is *max–min transitive* if and only if $R(x, z) \geq min[R(x, y), R(y, z)], \forall x, y, z \in S$.

For the case of Condorcet's paradox, a way out of cyclical collective preferences is to look at the sizes of majorities supporting various collective preferences. For example, if the number of voters preferring a to b is 5 out of 9, while that of voters preferring b to c is 7 out of 9, then, according to Condorcet, the latter preference is stronger than the former. By cutting the cycle of collective majority preferences at its weakest link, one ends up with a complete and transitive relation. Clearly, with a non-fuzzy preference relation, this method works only in cases where not all of the majorities supporting various links in the cycle are of same size.

With fuzzy preferences one can form the collective preference between any x and $y \in S$ using a variation of the average rule (cf. Intrilligator [13]), i.e.,

$$R(x, y) = \frac{\sum_i R_i(x, y)}{m} \tag{22}$$

where $R(x, y)$ is the degree of collective fuzzy preference of x over y.

Now, if a preference cycle is formed on the basis of collective fuzzy preferences, one could simply ignore the link with weakest degree of preference and thus possibly end up with a ranking. In general, one can proceed by eliminating weakest links in collective preference cycles until a ranking occurs.

The above method of successive elimination of the weakest links in the preference cycles works with the fuzzy and nonfuzzy preferences. When the individual preferences are fuzzy, each voter provides the his/her preferences so that the following matrix can be formed:

$$R_i = \begin{pmatrix} - & r_{12} & \dots & r_{1n} \\ r_{21} & - & \dots & r_{2n} \\ \dots & \dots & \dots & \dots \\ r_{n1} & r_{n2} & \dots & - \end{pmatrix} \tag{23}$$

where r_{ij} indicates the degree to which i prefers the i-th alternative to the j-th one.

By taking an average over the voters we obtain:

$$\bar{R} = \begin{pmatrix} - & \bar{r}_{12} & \dots & \bar{r}_{1n} \\ \bar{r}_{21} & - & \dots & \bar{r}_{2n} \\ \dots & \dots & \dots & \dots \\ \bar{r}_{n1} & \bar{r}_{n2} & \dots & - \end{pmatrix} \tag{24}$$

One can also compute first the row sums of the matrix: $\bar{r}_i = \sum_j \bar{r}_{ij}$ which represent the total fuzzy preference weight assigned to the i-th alternative in all pairwise preference comparisons, when the weight in each comparison is the average fuzzy preference value.

Let now $p_i = \frac{\bar{r}_i}{\sum_i \bar{r}_i}$, and, clearly $p_i \geq 0$ and $\sum_i p_i = 1$. Thus, p_i has the natural interpretation of a choice probability that can be used to form the collective preference ordering which is necessarily a complete and transitive relation.

For illustration, consider the example of Table 1 again and assume that each group consists of just one voter, and that the fuzzy preferences underlying the preference rankings are as shown in Table 7.

Table 7. Fuzzy Condorcet's paradox

voter 1				*voter 2*				*voter 3*			
	A	B	C		A	B	C		A	B	C
A	-	.6	.8	A	-	.9	.3	A	-	.6	.3
B	.4	-	.6	B	.1	-	.7	B	.4	-	.1
C	.2	.4	-	C	.7	.3	-	C	.7	.9	-

The $\bar{R}$- matrix is now: $\bar{R} = \begin{pmatrix} - & .7 & .5 \\ .3 & - & .5 \\ .5 & .5 & - \end{pmatrix}$ and $P_A = 0.4, P_B = 0.3, P_C = 0.3$.

Obviously, the solution is based on somewhat different fuzzy preference relations over the three alternatives. For identical, the preference relations we would necessarily end up with identical choice probabilities.

We can also resolve Borda's paradox by applying the same procedure. Suppose that Borda's paradox (exemplified by Table 2) in the fuzzy setting is as represented by the fuzzy preferences given in Table 8.

Table 8. A fuzzy Borda's paradox

4 voters				*3 voters*				*2 voters*			
	A	B	C		A	B	C		A	B	C
A	-	.6	.8	A	-	.9	.3	A	-	.2	.1
B	.4	-	.6	B	.1	-	.7	B	.8	-	.3
C	.2	.4	-	C	.7	.3	-	C	.9	.7	-

The matrix of average preference degrees is then: $\bar{R} = \begin{pmatrix} - & .6 & .5 \\ .4 & - & .6 \\ .5 & .4 & - \end{pmatrix}$. The choice probabilities of A, B and C are, thus, $0.37, 0.33, 0.30$. The choice probability of A is the largest. In a sense, then, the method does not solve Borda's paradox in the same way as the Borda count does since also the plurality method ends up with A being chosen instead of the Condorcet winner alternative B. Note, however, that the fuzzy preference relations give a richer picture of voter preferences than the ordinary preference rankings. In particular, A is strongly preferred to B and C by both the 4 and 3 voter groups, and its choice probability is the largest.

For additional information on voting paradoxes and some ways to solve them using fuzzy logic, we refer the reader to Nurmi and Kacprzyk [46].

5 Concluding Remarks

We have briefly outlined various ways to the derivation of group decision (voting) models under individual and social fuzzy preference relations and fuzzy majorities. In the first part we discussed issues related to their role as a tool to alleviate difficulties related to negative results in group decision making exemplified by Arrow's impossibility theorem. The second part has been focused on an important, sometimes dangerous phenomenon of so-called voting paradoxes which are basically intuitively implausible, surprising, counter-intuitive and generally unpleasant phenomena in voting contexts. We have provided some tools for finding a way of alleviating them. Our approach is based on the use of fuzzy preference relations. The results presented are of relevance for both social choice, voting, group decision making, etc. areas, but also for multi-agent systems in which some specific types of voting procedures are also employed.

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