Monadic Social Choice

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Abstract. In this paper, we show how monads and substitutions allows for a separation between social choice and social 'choosing'. Choice as value and choosing as operation is modeled using underlying signatures and related term monads. These monads are arranged over Goguen's category $\mathtt{Set}(L)$, which provides the internalization of uncertainty both in choice as well as choosing.

Keywords: Choice function, monad, Kleisli category, substitution.

1 Introduction

The discipline of social choice originates from objective probability used in justice and as pioneered by French mathematicians Borda [4] and de Condorcet [5]. The balance between individual liberty and societal authority was related to the risk of innocent citizens being wrongly convicted and punished for crime. Social justice as well as social order required that particular risk to be minimized. Condorcet argued that judicial tribunals could manage probabilities and errors, taking into account also some minimum required plurality to guarantee the probability. Uncertainty based voting schemas then are just behind the corner, and is the historical prerequisite also for choice theory.

Objective probability eventually turns subjective, and probabilists believe they have keys to inference mechanisms as well. Some modern time improvements can be seen in these directions, but generally speaking, probability is not logical.

The subject of social choice was revived in the 20th century by Arrow [1] who, facing the inconsistencies of group decisions, put the discipline of social choice in a structured axiomatic framework leading to the birth of social choice theory in its modern form. As Sen [14] pointed out "Arrow's impossibility theorem is a result of breathtaking elegance and power, which showed that even some very mild conditions of reasonableness could not be simultaneously satisfied by any social choice procedure, within a very wide family". Accordingly, impossibility results in social choice theory have been seldom considered as being destructive of the possibility of social choice and welfare economics. Sen [14] argued against that view, claiming that formal reasoning about postulated axioms, as well as informal understanding of values and norms, both point in the productive direction of overcoming social choice pessimism and of avoiding impossibilities.

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Arrow's focused on individual values and ranking, together with impossibility theorems, and dealt with individual preferences and choice processes. Probabilities are not in the ingredients, but rather operators and functions, and properties about them. There are no counterparts in probability theory for these concepts. From a logical point of view, Arrow uses implicitly underlying signatures, even if they are never formalized, and since they are not formalized, it is never seen that these choice functions indeed could have been integrated into a logical framework. Arrow follows von Neumann and Morgenstern's "mathematical tradition" [13] in his success stories of economic and social sciences, but also without ending up in any logical framework.

In [6] we assumed that making a distinction between choice and mechanism for choice could advantageously enrich the theoretical framework of social choice theory opening the way to categorical approaches. The idea of generalizing the Arrow's paradigm through a new architecture of social choice procedure was introduced, e.g., by Bandyopadhyay [2] and then extended in [3] where a social choice procedure is proposed which depends both on the way a set of alternatives is broken up into the subsets and the sequence in which each of these subsets is taken up for consideration.

Our standpoint in this paper is that social choice functions must identify the difference between 'we choose' and 'our choice', the former being the operation of choosing, the latter being the result of that operation. We view this from a signature point of view, i.e., using formalism involving signatures and their algebras. Classically, and without consideration of underlying categories, a signature $\Sigma = (S, \Omega)$ consists of sorts, or types, in a set S, and operators in a set Ω. More precisely, Ω is a family of sets $(\Omega_n)_{n \leq k}$, where n is the arity of the operators in Ω_n. An operator $\omega \in \Omega_n$ is syntactically written as $\omega : s_1 \times \cdots \times s_n \longrightarrow s$, where $s_1, \ldots, s_n, s \in S$. Operators in Ω_0 are constants. Given a set of variables we may construct the set of all terms over the signature. This set is usually denoted $\mathsf{T}_\Omega X$, and its elements are denoted $(n, \omega, (t_i)_{i \leq n})$, $\omega \in \Omega_n$, $t_i \in \mathsf{T}_\Omega X, i = 1, \ldots, n$, or $\omega(t_1, \ldots, t_n)$.

In this algebraic formalism, ω corresponds to the operation of choosing, and $\omega(t_1, \ldots, t_n)$ is a result of choosing, i.e., a choice. Note that both the operator ω as well as the term $\omega(t_1, \ldots, t_n)$ are syntactic representations of mechanisms for choosing and choices. The semantics of ω is a mapping $A(\omega) : A(s_1) \times \cdots \times A(s_n) \longrightarrow A(s)$.

Social choice is basically seen as a mapping

$$f : X_1 \times \cdots \times X_n \longrightarrow X$$

where agents $i \in \{1, \ldots, n\}$ are choosing or arranging elements in sets X_i. The aggregated social choice related to $x_i \in X_i, i = 1, \ldots, n$ is then represented by $f(x_1, \ldots, x_n)$. In most cases $X_1 = \cdots = X_n = X$, and the social choice function is then

$$f : X \times \cdots \times X \longrightarrow X. \tag{1}$$

This can be seen either as a semantic representation which has an underlying choice operator in its signature, or it is syntactic and elements in X are basically constant operators, i.e., $X = \Omega_0$ in some operator domain.

In the view of 'choosing' we would replace X with the set of substitutions. More precisely, let $\mathtt{C}$ be the Kleisli category $\mathtt{Set}_{\mathbf{T}_\Omega}$, where $\mathbf{T}_\Omega$ is the term monad over $\mathtt{Set}$. Elements σ in $\mathrm{Hom}_{\mathtt{C}}(X, X)$ are then substitutions $\sigma : X \longrightarrow \mathsf{T}_\Omega X$, and $\mathcal{X} = \mathrm{Hom}_{\mathtt{C}}(X, X)$ is the corresponding set of substitutions capturing the notion of individual choice and choosing. The choice function

$$\varphi : \mathcal{X} \times \cdots \times \mathcal{X} \longrightarrow \mathcal{X} \tag{2}$$

therefore may consider and compute with not just the output, the choice, but also with all the operators, i.e., the whole mechanism of choosing, leading to that particular term.

We will expand these ideas to cover uncertainty modeling, and we will show how representation of uncertainty can be seen as related to an appropriate choice of an underlying category. Furthermore, we will see how all this can be embedded into a many-sorted framework.

In the literature there are some previous categorical approaches Keifing's [11] objective is similar to ours, namely a unification of framework, and indeed unification of concept, results, and theorem framework based on more or less formal methods. Keiding involves categories and Hom functors, but the categorical framework remains rather poor, as there is no use of operators. The set $\mathrm{Hom}(A, \mathsf{P}X)$ indeed comes with no structure. It is simply a set of mappings. In the end, we will have a $\mathrm{Hom}_{\mathtt{C}}$ functor, where $\mathtt{C}$ also can carry uncertainty once (many-sorted) term monads are constructed over Goguen's category $\mathtt{Set}(L)$. It then integrates both operators and uncertainties, and even more so, operators working internally over uncertainties. Eliaz [10], making no reference to [11], does not add any new formalism or formal methodology.

2 Monads and Underlying Categories

A monad (or triple, or algebraic theory) over a category $\mathtt{C}$ is denoted $\mathbf{F} = (\mathsf{F}, \eta, \mu)$, where $\mathsf{F} : \mathtt{C} \longrightarrow \mathtt{C}$ is a covariant functor, and $\eta\colon \mathsf{id} \longrightarrow \mathsf{F}$ and $\mu\colon \mathsf{F}\circ\mathsf{F} \longrightarrow \mathsf{F}$ are natural transformations satisfying $\mu\circ\mathsf{F}\mu = \mu\circ\mu\mathsf{F}$ and $\mu\circ\mathsf{F}\eta = \mu\circ\eta\mathsf{F} = \mathsf{id}_\mathsf{F}$. Any monad $\mathbf{F}$ over a category $\mathtt{C}$, gives rise to a Kleisli category $\mathtt{C}_\mathbf{F}$ whose objects are $\mathrm{Ob}(\mathtt{C}_\mathbf{F}) = \mathrm{Ob}(\mathtt{C})$, and morphisms are $\mathrm{Hom}_{\mathtt{C}_\mathbf{F}}(X, Y) = \mathrm{Hom}_\mathtt{C}(X, \mathsf{F}Y)$. Morphisms $f : X \longrightarrow Y$ in $\mathtt{C}_\mathbf{F}$ are morphisms $f : X \longrightarrow \mathsf{F}Y$ in $\mathtt{C}$, with $\eta_X : X \longrightarrow \mathsf{F}X$ being the identity morphism. Composition of morphisms in $\mathtt{C}_\mathbf{F}$ is defined as

$$(X \xrightarrow{f} Y) \diamond (Y \xrightarrow{g} Z) = X \xrightarrow{\mu_Z \circ \mathsf{F}g \circ f} \mathsf{F}Z.$$

Let L is a completely distributive lattice, and let $\mathtt{Set}(L)$ be the (Goguen) category where objects are pairs (A, α) with $\alpha\colon A \longrightarrow L$, and morphisms $(A, \alpha) \xrightarrow{f} (B, \beta)$ are mappings $f\colon A \longrightarrow B$ such that $\beta(f(a)) \geq \alpha(a)$ for all $a \in A$. The category $\mathtt{Set}$ is not isomorphic to $\mathtt{Set}(2)$, where $2 = \{0, 1\}$.

For a set of sorts S, the many-sorted category of sets $\mathtt{Set}_S$ has objects $\{X_{\mathsf{s}}\}_{\mathsf{s}\in S}$, where X_{s}, $\mathsf{s} \in S$, are objects in $\mathtt{Set}$. Morphisms $f_{\mathsf{s}} : X_{\mathsf{s}} \longrightarrow Y_{\mathsf{s}}, \mathsf{s} \in S$, in $\mathtt{Set}$, produce morphisms $\{f_{\mathsf{s}}\}_{\mathsf{s}\in S} : \{X_{\mathsf{s}}\}_{\mathsf{s}\in S} \longrightarrow \{Y_{\mathsf{s}}\}_{\mathsf{s}\in S}$ in $\mathtt{Set}_S$. For a morphisms $\{g_{\mathsf{s}}\}_{\mathsf{s}\in S} : \{Y_{\mathsf{s}}\}_{\mathsf{s}\in S} \longrightarrow \{Z_{\mathsf{s}}\}_{\mathsf{s}\in S}$, composition with $\{f_{\mathsf{s}}\}_{\mathsf{s}\in S}$ is sort-wise, i.e., $\{g_{\mathsf{s}}\}_{\mathsf{s}\in S} \circ \{f_{\mathsf{s}}\}_{\mathsf{s}\in S} = \{g_{\mathsf{s}} \circ f_{\mathsf{s}}\}_{\mathsf{s}\in S}$. For objects in $\mathtt{Set}_S$, set operations are also defined sort-wise.

Functors $\mathsf{F}_{\mathsf{s}}, \mathsf{G}_{\mathsf{s}} : \mathtt{Set} \longrightarrow \mathtt{Set}$ can be lifted to functors $\mathsf{F}_S = \{\mathsf{F}_{\mathsf{s}}\}_{\mathsf{s}\in S}$ and $\mathsf{G}_S = \{\mathsf{G}_{\mathsf{s}}\}_{\mathsf{s}\in S}$ from $\mathtt{Set}_S$ to $\mathtt{Set}_S$, and composition is again sort-wise, i.e., $\mathsf{F}_S \circ \mathsf{G}_S = \{\mathsf{F}_{\mathsf{s}} \circ \mathsf{G}_{\mathsf{s}}\}_{\mathsf{s}\in S}$.

The product $\prod_{i\in I} \mathsf{F}_i$ and coproduct $\coprod_{i\in I} \mathsf{F}_i$ of covariant functors F_i over $\mathtt{Set}_S$ is defined as

$$(\prod_{i\in I} \mathsf{F}_i)\{X_{\mathsf{s}}\}_{\mathsf{s}\in S} = \prod_{i\in I} \mathsf{F}_i\{X_{\mathsf{s}}\}_{\mathsf{s}\in S}$$

and

$$(\coprod_{i\in I} \mathsf{F}_i)\{X_{\mathsf{s}}\}_{\mathsf{s}\in S} = \coprod_{i\in I} \mathsf{F}_i\{X_{\mathsf{s}}\}_{\mathsf{s}\in S}$$

with morphisms being handled accordingly.

The many-sorted underlying category $\mathtt{Set}_S(\{L\}_{\mathsf{s}\in S})$ is defined sort-wise with respect to L. That is, objects are indexed sets of pairs $\{(A_{\mathsf{s}}, \alpha_{\mathsf{s}})\}_{\mathsf{s}\in S}$ with $\alpha_{\mathsf{s}} : A_{\mathsf{s}} \longrightarrow L_{\mathsf{s}}$ and morphisms $\{f_{\mathsf{s}}\}_{\mathsf{s}\in S} : \{(A_{\mathsf{s}}, \alpha_{\mathsf{s}})\}_{\mathsf{s}\in S} \longrightarrow \{(B_{\mathsf{s}}, \beta_{\mathsf{s}})\}_{\mathsf{s}\in S}$ are such that $\beta_{\mathsf{s}}(f_{\mathsf{s}}(a)) \geq_{\mathsf{s}} \alpha_{\mathsf{s}}(a)$ for all $\mathsf{s} \in S$ and $a \in A_{\mathsf{s}}$.

3 The Term Monad over $\mathtt{Set}_S(\{L\}_{\mathsf{s}\in S})$

A many-sorted signature $\Sigma = (S, \Omega)$ over $\mathtt{Set}_S$ consists of a set S of sorts considered as a set in ZF, and a set Ω of operators as an object in $\mathtt{Set}$. Operators in Ω are indexed by sorts and syntactically denoted $\omega : \mathsf{s}_1 \times \cdots \times \mathsf{s}_n \longrightarrow \mathsf{s}$, where n is the arity of the operation. We may write Ω_n for the set (as an object of $\mathtt{Set}$) of n-ary operations. Clearly $\Omega = \coprod_{n\leq k} \Omega_n$, where k is a cardinal number representing the 'upper bound of arities'.

Let now $\{(\Omega_n, \vartheta_n) \mid n \leq k\}$ be a family of objects in $\mathtt{Set}(L)$. Further, let $(\Omega, \vartheta) = \coprod_{n\leq k}(\Omega_n, \vartheta_n)$ be a fuzzy operator domain, i.e., $\vartheta_n : \Omega_n \longrightarrow L$. Note, we write $\Omega^{\mathsf{s}_1\times\cdots\times\mathsf{s}_n \longrightarrow \mathsf{s}}$ for the set of operations $\omega : \mathsf{s}_1 \times \cdots \times \mathsf{s}_n \longrightarrow \mathsf{s}$.

A many-sorted signature $\Sigma = (S, (\Omega, \vartheta))$ over $\mathtt{Set}(L)$ consists again of a set S of sorts considered as a set in ZF, and a pair (Ω, ϑ) (of operators) as an object in $\mathtt{Set}(L)$.

Let

$$\mathsf{T}^0_\Sigma = \mathrm{id}_{\mathtt{Set}_S(\{L\}_{\mathsf{s}\in S})}$$

and

$$\mathsf{T}^0_{\Sigma,\mathsf{s}}\{(X_{\mathsf{s}}, \xi_{\mathsf{s}})\}_{\mathsf{s}\in S} = (X_{\mathsf{s}}, \xi_{\mathsf{s}}).$$

For convenience, given an object A in a category $\mathtt{C}$, we will make use of the constant functor $A_{\mathtt{D}} : \mathtt{D} \longrightarrow \mathtt{C}$ which assigns any object in $\mathtt{D}$ to A, and morphisms in $\mathtt{D}$ to the identity morphism id_A in $\mathtt{C}$. Further, for $\mathsf{s}_1, \ldots, \mathsf{s}_n \in S$ we define a functor $\arg^{\mathtt{s}_1\times\cdots\times\mathtt{s}_n} : \mathtt{Set}_S(\{L\}_{\mathtt{s}\in S}) \longrightarrow \mathtt{Set}(L)$ by $\arg^{\varnothing}(\{(A_\mathtt{s},\alpha_\mathtt{s})\}_{\mathtt{s}\in S}) = (\{\varnothing\},\top)$ and

$$\arg^{\mathtt{s}_1\times\cdots\times\mathtt{s}_n}(\{(A_\mathtt{s},\alpha_\mathtt{s})\}_{\mathtt{s}\in S}) = (\arg^{\mathtt{s}_1\times\cdots\times\mathtt{s}_n}(\{A_\mathtt{s}\}_{\mathtt{s}\in S}), \arg^{\mathtt{s}_1\times\cdots\times\mathtt{s}_n}(\{\alpha_\mathtt{s}\}_{\mathtt{s}\in S}))$$

where

$$\arg^{s_1\times\cdots\times s_n}(\{A_\mathtt{s}\}_{\mathtt{s}\in S}) = \prod_{i=1,\ldots,n} A_{\mathtt{s}_i} \text{ and}$$
$$\arg^{s_1\times\cdots\times s_n}(\{\alpha_\mathtt{s}\}_{\mathtt{s}\in S})(a_1,\ldots,a_n) = \bigwedge_{i=1,\ldots,n} \alpha_{s_i}(a_i).$$

The functor

$$(\Omega^{\mathtt{s}_1\times\cdots\times\mathtt{s}_m\longrightarrow\mathtt{s}},\vartheta_m)_{\mathtt{Set}_S(\{L\}_{\mathtt{s}\in S})} \times \mathsf{arg}^{\mathtt{s}_1\times\cdots\times\mathtt{s}_m} : \mathtt{Set}_S(\{L\}_{\mathtt{s}\in S}) \longrightarrow \mathtt{Set}(L)$$

now allows to define

$$\begin{aligned}\mathsf{T}^1_{\Sigma,\mathtt{s}}\{(X_\mathtt{s},\xi_\mathtt{s})\}_{\mathtt{s}\in S} &= \coprod_{\substack{\mathtt{s}_1,\ldots,\mathtt{s}_m\\0\le m\le k}} ((\Omega^{\mathtt{s}_1\times\cdots\times\mathtt{s}_m\longrightarrow\mathtt{s}},\vartheta_m)_{\mathtt{Set}_S(\{L\}_{\mathtt{s}\in S})} \\ &\qquad \times \mathsf{arg}^{\mathtt{s}_1\times\cdots\times\mathtt{s}_m}\{(X_\mathtt{s},\xi_\mathtt{s})\}_{\mathtt{s}\in S}) \\ &= \coprod_{\substack{\mathtt{s}_1,\ldots,\mathtt{s}_m\\0\le m\le k}} ((\Omega^{\mathtt{s}_1\times\cdots\times\mathtt{s}_m\longrightarrow\mathtt{s}},\vartheta_m) \\ &\qquad \times (\prod_{i=1,\ldots,m} X_{\mathtt{s}_i}, \bigwedge_{i=1,\ldots,m} \xi_{s_i})) \\ &= (T^1_{\Sigma,\mathtt{s}}\{X_\mathtt{s}\}_{\mathtt{s}\in S},\beta_\mathtt{s})\end{aligned}$$

where

$$\beta_\mathtt{s}(\omega : \mathtt{s}_1\times\cdots\times\mathtt{s}_m\longrightarrow\mathtt{s},(x_i)_{i\le m}) = \vartheta_m(\omega)\wedge\arg^{\mathtt{s}_1\times\cdots\times\mathtt{s}_m}(\{\xi_\mathtt{s}\}_{\mathtt{s}\in S})((x_i)_{i\le m}),$$

and $(x_i)_{i\le m}\in\prod_{i=1,\ldots,m} X_{\mathtt{s}_i}$. We then have

$$\mathsf{T}^1_\Sigma\{(X_\mathtt{s},\xi_\mathtt{s})\}_{\mathtt{s}\in S} = \{\mathsf{T}^1_{\Sigma,\mathtt{s}}\{(X_\mathtt{s},\xi_\mathtt{s})\}_{\mathtt{s}\in S}\}_{\mathtt{s}\in S}.$$

Further,

$$\begin{aligned}\mathsf{T}^\iota_{\Sigma,\mathtt{s}}\{(X_\mathtt{s},\xi_\mathtt{s})\}_{\mathtt{s}\in S} &= \coprod_{\mathtt{s}_1,\ldots,\mathtt{s}_m} ((\Omega^{\mathtt{s}_1\times\cdots\times\mathtt{s}_m\longrightarrow\mathtt{s}},\vartheta_m)_{\mathtt{Set}_S(\{L_s\}_{s\in S})} \\ &\qquad \times \mathsf{arg}^{\mathtt{s}_1\times\cdots\times\mathtt{s}_m}\circ\bigcup_{\kappa<\iota}\mathsf{T}^\kappa_\Sigma\{(X_\mathtt{s},\xi_\mathtt{s})\}_{\mathtt{s}\in S})\end{aligned}$$

and

$$\mathsf{T}^\iota_\Sigma\{(X_\mathtt{s},\xi_\mathtt{s})\}_{\mathtt{s}\in S} = \{\mathsf{T}^\iota_{\Sigma,\mathtt{s}}\{(X_\mathtt{s},\xi_\mathtt{s})\}_{\mathtt{s}\in S}\}_{\mathtt{s}\in S},$$

for each positive ordinal ι. Finally, let $\mathsf{T}_\Sigma = \bigvee_{\iota<\bar{k}} \mathsf{T}^\iota_\Sigma$ where $\bar{k}$ is the least cardinal greater than k and $\aleph_0$. Terms of sort $\mathtt{s}$ are denoted $\mathsf{T}_{\Sigma,\mathtt{s}} = \mathsf{arg}^{\mathtt{s}} \circ \mathsf{T}_\Sigma$.

Clearly, each $\mathsf{T}_{\Sigma,\mathtt{s}} : \mathtt{Set}_S(\{L_{\mathtt{s}}\}_{\mathtt{s}\in S}) \longrightarrow \mathtt{Set}(L)$ is a functor and, by extension, so is

$$\mathsf{T}_\Sigma\{(X_{\mathtt{s}},\xi_{\mathtt{s}})\}_{\mathtt{s}\in S} = \{\mathsf{T}_{\Sigma,\mathtt{s}}\{(X_{\mathtt{s}},\xi_{\mathtt{s}})\}_{\mathtt{s}\in S}\}_{\mathtt{s}\in S}.$$

Note, it is easy to verify that

$$\mathsf{T}_{\Sigma,\mathtt{s}}\mathsf{T}_\Sigma\{X_{\mathtt{s}}\}_{\mathtt{s}\in S} = \mathsf{arg}^{\mathtt{s}}\ \mathsf{T}_\Sigma\{X_{\mathtt{s}}\}_{\mathtt{s}\in S}$$

and T_Σ is therefore idempotent.

The extension of T_Σ to a monad is enabled by the natural transformations

$$(\eta_s^{\mathsf{T}_\Sigma})_{(X_{\mathtt{s}},\xi_{\mathtt{s}})} : (X_{\mathtt{s}},\xi_{\mathtt{s}}) \longrightarrow \mathsf{T}_{\Sigma,\mathtt{s}}\{(X_{\mathtt{s}},\xi_{\mathtt{s}})\}_{\mathtt{s}\in S}, \text{ and}$$
$$(\mu_s^{\mathsf{T}_\Sigma})_{(X_{\mathtt{s}},\xi_{\mathtt{s}})} : \mathsf{T}_{\Sigma,\mathtt{s}}\mathsf{T}_\Sigma\{(X_{\mathtt{s}},\xi_{\mathtt{s}})\}_{\mathtt{s}\in S} \longrightarrow \mathsf{T}_{\Sigma,\mathtt{s}}\{(X_{\mathtt{s}},\xi_{\mathtt{s}})\}_{\mathtt{s}\in S}$$

that are simply defined, with the help of idempotency of T_Σ, by

$$(\eta_s^{\mathsf{T}_\Sigma})_{(X_{\mathtt{s}},\xi_{\mathtt{s}})}(x_{\mathtt{s}},\alpha_{\mathtt{s}}) = (x_{\mathtt{s}},\alpha_{\mathtt{s}}), \text{ and}$$
$$(\mu_s^{\mathsf{T}_\Sigma})_{(X_{\mathtt{s}},\xi_{\mathtt{s}})}(x_{\mathtt{s}},\alpha_{\mathtt{s}}) = (x_{\mathtt{s}},\alpha_{\mathtt{s}}).$$

We write $\eta^{\mathsf{T}_\Sigma} = \{\eta_s^{\mathsf{T}_\Sigma}\}_{\mathtt{s}\in S}$ and $\mu^{\mathsf{T}_\Sigma} = \{\mu_s^{\mathsf{T}_\Sigma}\}_{\mathtt{s}\in S}$.

Proposition 1. $\mathbf{T}_\Sigma = (\mathsf{T}_\Sigma, \eta^{\mathsf{T}_\Sigma}, \mu^{\mathsf{T}_\Sigma})$ *is a monad over* $\mathtt{Set}_S(\{L_s\}_{\mathtt{s}\in S})$.

Remark 1. The many-sorted, many-valued, term monad specialized to a one-pointed set of sorts $S = \{\mathtt{s}\}$ collapses to the classical many-valued term monad.

Remark 2. Morphisms

$$\{f_{\mathtt{s}}\}_{\mathtt{s}\in S} : \{(X_{\mathtt{s}},\alpha_{\mathtt{s}})\}_{\mathtt{s}\in S} \longrightarrow \{(Y_{\mathtt{s}},\beta_{\mathtt{s}})\}_{\mathtt{s}\in S}$$

in $\mathtt{Set}_S(\{L_{\mathtt{s}}\}_{\mathtt{s}\in S})_{\mathbf{T}_\Sigma}$, the Kleisli category of $\mathbf{T}_\Sigma$, capture the notion of many-sorted and many-valued variables being substituted by many-sorted terms over many-sorted and many-valued variables.

4 Preference Relations

Arrow [1] studied social welfare functions, the arguments of which are named components of social states. These functions map n-tuples of individual preferences (orderings [1]) into a collective preference:

$$f : (X^m)^n \to X^m$$

Here the assumption is that X is an ordering $(X, \preceq)$ with suitable properties. The preference value in this case is an ordinal value and not a scale value. Clearly, choice functions can also involve scale values, so that

$$f : (\mathbb{R}^m)^n \to \mathbb{R}^m$$

i.e. using the real line, or some suitable closed interval within the real line, for the preference (scale) values. Note how the underlying signature handles this situation internally for $\mathcal{X} = \mathrm{Hom}_{\mathtt{C}}(X, X)$, where $\mathtt{C}$ is the Kleisli category $\mathtt{Set}(L)_{\mathbf{T}_\Sigma}$.

Computing with preferences is less transparent with orderings built into the set X of alternatives [6]. Also in this case there is a corresponding underlying signature capturing this situation.

5 Conclusion and Future Work

In the presentation above we still use only terms. Sentences, satisfaction $\models$ (based on the algebraic models of the signature), and entailment $\vdash$ are not yet included. Axioms of the logic and inference rules for entailment are then also missing, so we no 'logic of choice' at this point, and this has fallen outside the scope of this paper. See [7] for a treatment of generalized general logic.

Going beyond the distinction between choosing and choice, and entering rationality of choice, Mill [12] said that behavior is based on custom more than rationality. Custom is clearly based on particular algebras acting as models and used in $\models$, whereas rationality is based on representable sentences interrelated by $\vdash$. These aspects are investigated in future work.

In consensus reaching [8,9] we have a dynamic situation of aggregated choice, where individual preferences change within a consensus reaching mechanism. This opens up interesting perspectives as consensus reaching in our substitution model for social choice now also reaches the level of 'choosing', i.e., consensus is reached either on 'choice' level including dynamics for the 'choosing' level, or can even be a stronger consensus on 'choosing' levels as well. Similar situations appear in negotiation.

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