# Bounded-Distance Network Creation Games*

Davide Bilò[1], Luciano Gualà[2], and Guido Proietti[3,4]

[1] Dipartimento di Scienze Umanistiche e Sociali, Università di Sassari, Italy
[2] Dipartimento di Ingegneria dell'Impresa, Università di Roma "Tor Vergata", Italy
[3] Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, Italy
[4] Istituto di Analisi dei Sistemi ed Informatica, CNR, Rome, Italy
davide.bilo@uniss.it, guala@mat.uniroma2.it, guido.proietti@univaq.it

**Abstract.** A *network creation game* simulates a decentralized and non-cooperative building of a communication network. Informally, there are $n$ players sitting on the network nodes, which attempt to establish a reciprocal communication by activating, incurring a certain cost, any of their incident links. The goal of each player is to have all the other nodes as close as possible in the resulting network, while buying as few links as possible. According to this intuition, any model of the game must then appropriately address a balance between these two conflicting objectives. Motivated by the fact that a player might have a strong requirement about its centrality in the network, in this paper we introduce a new setting in which if a player maintains its (either *maximum* or *average*) distance to the other nodes within a given *bound*, then its cost is simply equal to the *number* of activated edges, otherwise its cost is unbounded. We study the problem of understanding the structure of pure Nash equilibria of the resulting games, that we call MaxBD and SumBD, respectively. For both games, we show that when distance bounds associated with players are *non-uniform*, then equilibria can be arbitrarily bad. On the other hand, for MaxBD, we show that when nodes have a *uniform* bound $R$ on the maximum distance, then the *Price of Anarchy* (PoA) is lower and upper bounded by 2 and $O\left(n^{\frac{1}{\lfloor \log_3 R \rfloor + 1}}\right)$ for $R \geq 3$ (i.e., the PoA is constant as soon as $R$ is $\Omega(n^{\epsilon})$, for some $\epsilon > 0$), while for the interesting case $R = 2$, we are able to prove that the PoA is $\Omega(\sqrt{n})$ and $O(\sqrt{n \log n})$. For the uniform SumBD we obtain similar (asymptotically) results, and moreover we show that the PoA becomes constant as soon as the bound on the average distance is $2^{\omega\left(\sqrt{\log n}\right)}$.

## 1 Introduction

Communication networks are rapidly evolving towards a model in which the constituting components (e.g., routers and links) are activated and maintained

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by different owners, which one can imagine as players sitting on the network nodes. When these players act in a selfish way with the final intent of creating a connected network, the challenge is exactly to understand whether the pursuit of individual profit is compatible with the attainment of an equilibrium status for the system (i.e., a status in which players are not willing to move from), and how the social utility for the system as a whole is affected by the selfish behavior of the players. While the former question is inherently game-theoretic and has been originally addressed in [10] by the economists (for further references see also Chapter 6 in [11]), the latter one involves also computational issues, since it can be regarded as a comparison between the performances of an uncoordinated distributed system as opposed to a centralized system which can optimally design a solution. Not surprisingly then, this class of games, which we refer to as *network creation games* (NCGs), received a significative attention also from the computer science community, starting from the paper of Fabrikant *et al.* [9], where the main computational aspects of a NCG have been initially formalized and investigated. More precisely, in [9] the authors focused on an Internet-oriented NCG, defined as follows: We are given a set of $n$ players, say $V$, where the strategy space of player $v \in V$ is the power set $2^{V \setminus \{v\}}$. Given a combination of strategies $S = (S_v)_{v \in V}$, let $G(S)$ denote the underlying undirected graph whose node set is $V$, and whose edge set is $E(S) = \big\{(v, v') \mid v \in V \wedge v' \in S_v\big\}$. Then, the *cost* incurred by player $v$ under $S$ is

$$cost_v(S) = \alpha \cdot |S_v| + \sum_{u \in V} d_{G(S)}(u, v) \tag{1}$$

where $d_{G(S)}(u, v)$ is the distance between nodes $u$ and $v$ in $G(S)$. Thus, the cost function implements the inherently antagonistic goals of a player, which on the one hand attempts to buy as little edges as possible, and on the other hand aims to be as close as possible to the other nodes in the outcoming network. These two criteria are suitably balanced in (1) by making use of the parameter $\alpha \geq 0$. Consequently, the *Nash Equilibria*[1] (NE) space of the game is heavily influenced by $\alpha$, and the corresponding characterization must be given as a function of it. The state-of-the-art for the *Price of Anarchy* (PoA) of the game, that we will call henceforth SumNCG, is summarized in [15], where the most recent progresses on the problem have been reported.

*Further NCG models.* A first natural variant of SumNCG was introduced in [7], where the authors redefined the player cost function as follows

$$cost_v(S) = \alpha \cdot |S_v| + \max\{d_{G(S)}(u, v) : u \in V\}. \tag{2}$$

This variant, named MaxNCG, received further attention in [15], where the authors improved the PoA of the game on the whole range of values of $\alpha$. However, a criticism made to both the aforementioned models is that usage and building cost are summed up together in the player's cost, and this mixing is reflected

[1] In this paper, we only focus on *pure* strategies Nash equilibria.

in the social cost of the resulting network. As a consequence, we have that in this game the PoA alone does not say so much about the structural properties of the network, such as density, diameter, or routing cost. Moreover, they both incorporate in the cost function the parameter $\alpha$, which is in a sense artificially introduced in order to suitably balance usage and building cost.

Thus, in an effort of addressing these critical issues, in [14] the authors proposed an interesting variant in which a player $v$, when forming the network, has a limited *budget* $b_v$ to establish links to other players. This way, the player cost function restricts to the usage cost, namely either the maximum or the total distance to other nodes. For these *bounded-budget* versions of the game, that we call MaxBB and SumBB, respectively, the authors in [14] showed that determining the existence of a NE is NP-hard. On a positive side, they proved that for uniform budgets, say $k$, both variants always admit a NE, and that its *Price of Stability* (PoS) is $\Theta(1)$. Finally, they proved that the PoA of MaxBB is $\Omega\left(\frac{n}{k \log_k n}\right)$ and $O\left(\frac{n}{\log_k n}\right)$, while the PoA of SumBB is $\Omega\left(\sqrt{\frac{n}{k \log_k n}}\right)$, $O\left(\sqrt{\frac{n}{\log_k n}}\right)$. Notice that in both MaxBB and SumBB, links are seen as directed. Thus, a natural extension of the model was given in [8], were the undirected case was considered. For this, it was proven that both MaxBB and SumBB always admit a NE. Moreover, the authors showed that the PoA for MaxBB and SumBB is $\Omega(\sqrt{\log n})$ and $O(\sqrt{n})$, respectively, while in the special case in which the budget is equal to 1 for all the players, the PoA is $O(1)$ for both versions of the game.

In all the above models it must be noticed that, as stated in [9], for a player it is NP-hard to find a best response once that the other players' strategies are fixed. To circumvent this problem, in [4] the authors proposed a further variant, called *basic NCG* (BNCG), in which given some existing network, the only improving transformations allowed are *edge swaps*, i.e., a player can only modify a *single* incident edge, by either replacing it with a new incident edge, or by removing it. This naturally induces a weaker concept of equilibrium for which a best response of a player can be computed in polynomial time. In this setting, the authors were able to give, among other results, an upper bound of $2^{O(\sqrt{\log n})}$ for the PoA of SumBNCG, and a lower bound of $\Omega(\sqrt{n})$ for the PoA of MaxBNCG. However, as pointed out in [15], the fact that now an edge has not a specific owner, prevents the possibility to establish any implications on the PoA of the classic NCG, since a NE in a BNCG is not necessarily a NE of a NCG. Finally, another NCG model which is barely related to the NCG model we study in this paper has been addressed in [6].

*Our results.* In this paper, we propose a new NCG variant that complements the model proposed in [8]. More precisely, we assume that the cost function of each player only consists of the number of bought edges (without any budget on them), but with the additional constraint that each player $v$ needs to stay within a given (either *maximum* or *average*) *distance*, say (either $R_v$ or $D_v$), from the other players.

For this bounded-distance version of the NCG, we address the problem of understanding the structure of the NE associated with the two variants of the game, that we denote by MaxBD and SumBD. In this respect, we first show that both games can have an unbounded PoA as soon as players hold at least two different distance bounds. Moreover, in both games, computing a best response for a player is NP-hard. These bad news are counterbalanced by the positive results we get for *uniform* distance bounds. In this case, first of all, the PoS for MaxBD is equal to 1, while for SumBD it is at most 2. Then, as far as the PoA is concerned, let $R$ and $D$ denote the uniform bound on the maximum and the average distance, respectively. We show that

(i) for MaxBD, the PoA is lower and upper bounded by 2 and $O\left(n^{\frac{1}{\lfloor \log_3 R \rfloor + 1}}\right)$ for $R \geq 3$, respectively, while for $R = 2$ is $\Omega(\sqrt{n})$ and $O(\sqrt{n \log n})$;
(ii) for SumBD, the PoA is lower bounded by $2 - \epsilon$, for any $\epsilon > 0$, as soon as $D \geq 2 - 3/n$, while it is upper bounded as reported in Table 1.

**Table 1.** Obtained PoA upper bounds for uniform SumBD

| $D$ | $\in [2,3)$ | $\geq 3$ and $O(1)$ | $\omega(1) \cap O\left(3^{\sqrt{\log n}}\right)$ | $\omega\left(3^{\sqrt{\log n}}\right) \cap 2^{O\left(\sqrt{\log n}\right)}$ | $2^{\omega\left(\sqrt{\log n}\right)}$ |
|---|---|---|---|---|---|
| PoA | $O\left(\sqrt{n \log n}\right)$ | $O\left(n^{\frac{1}{\lfloor \log_3 D/4 \rfloor + 2}}\right)$ | $2^{O\left(\sqrt{\log n}\right)}$ | $O\left(n^{\frac{1}{\lfloor \log_3 D/4 \rfloor + 2}}\right)$ | $O(1)$ |

*Motivations and significance of the new model.* Our model was originally motivated by the observation that, in a realistic scenario, a player might have a strong objective/requirement about its centrality in the under-costruction network. In fact, in daily life, people actively participate to the autonomous formation of (social) networks. In our experience, a user downplays any concerns about the number/cost of activated links. Rather, he initially pays attention only to the fact of remaining as close as possible to (a subset of) the other users, and only later on he tries to minimize his outdegree accordingly. Our model aims to (partially) address this dynamics. Actually, at this initial stage, we have relaxed this quite complicate setting, by associating with each user just a (uniform) single distance bound w.r.t. all the other users. Nevertheless, even in this simplified scenario, we can get some new insights as opposed to previous NCG models. Indeed, a closer inspection of our provided results suggests that the PoA becomes constant as soon as the maximum/average distance bound is $\Omega(n^{\epsilon})$, for some $\epsilon > 0$. This is quite interesting, since it implies that the autonomous network tends to be sparser as soon as the distance bounds grow. Notice that in the Fabrikant's model (and its variants), we cannot directly infer any information about network sparseness by just knowing that the PoA is constant. Furthermore, our model, as for those proposed in [14,8,4], does not rely on the $\alpha$ parameter, and this makes the proofs of the various bounds intimately related with some graph-theoretic properties of a stable network. For example, it is interesting to notice that in our setting the minimum degree and the size of a minimum dominating set play an important role. In this respect, in the concluding remarks of this paper, we pose an intriguing relationship between our problem and the well-known graph-theoretic *degree-diameter problem*, that we believe could help in solving some of the issues still left open, like the quite large gap between lower and upper bounds for the PoA. Finally, focusing on MaxBD, we observe that when $R = 2$, which should consistently model the scenario depicted by local-area networks, we obtain the meaningful result that the PoA is far to be constant. We also conjecture that this undesirable behavior can actually be extended to larger, still constant, values of $R$, although the generalization of the lower bounding argument seems likely technically involved.

The paper is organized as follows. After giving some basic definitions in Section 2, we provide some preliminary results in Section 3. Then, we study upper and lower bounds for uniform MaxBD and SumBD in Sections 4 and 5, respectively. Finally, in Section 6 we conclude the paper by discussing some intriguing relationships of our games with the famous graph-theoretic *degree-diameter* problem. Due to space limitations, some of the proofs are omitted here and will be given in the full version of the paper.

## 2 Problem Definition

*Graph terminology.* Let $G = (V, E)$ be an undirected (simple) graph with $n$ vertices. For a graph $G$, we will also denote by $V(G)$ and $E(G)$ its set of vertices and its set of edges, respectively. For every vertex $v \in V$, let $N_G(v) := \{u \mid u \in V \setminus \{v\}, (u, v) \in E\}$. The *minimum degree* of $G$ is equal to $\min_{v \in V} |N_G(v)|$.

We denote by $d_G(u, v)$ the *distance* in $G$ from $u$ to $v$. The *eccentricity* of a vertex $v$ in $G$, denoted by $\varepsilon_G(v)$, is equal to $\max_{u \in V} d_G(u, v)$. The *diameter* and the *radius* of $G$ are equal to the maximum and the minimum eccentricity of its nodes, respectively. A node is said to be a *center* of $G$ if $\varepsilon_G(v)$ is equal to the radius of $G$. We define the *broadcast cost* of $v$ in $G$ as $B_G(v) = \sum_{u \in V} d_G(u, v)$, while the *average distance* from $v$ to a node in $G$ is denoted by $D_G(v) = B_G(v)/n$.

A *dominating set* of $G$ is a subset of nodes $U \subseteq V$ such that every node of $V \setminus U$ is adjacent to some node of $U$. We denote by $\gamma(G)$ the cardinality of a minimum-size dominating set of $G$. Moreover, for any real $k \geq 1$, the $k$th power of $G$ is defined as the graph $G^k = (V, E(G^k))$ where $E(G^k)$ contains an edge $(u, v)$ if and only if $d_G(u, v) \leq k$. Let $F \subseteq \{(u, v) \mid u, v \in V, u \neq v\}$. We denote by $G + F$ the graph on $V$ with edge set $E \cup F$. When $F = \{e\}$ we will denote $G + \{e\}$ by $G + e$.

*Problem Statements.* The *bounded maximum-distance* NCG (MaxBD) is defined as follows: Let $V$ be a set of $n$ nodes, each representing a selfish player, and for any $v \in V$, let $R_v > 0$ be an integer representing a bound on the eccentricity of $v$. The strategy of a player $v$ consists of a subset $S_v \subseteq V \setminus \{v\}$. Denoting by $S$ the strategy profile of all players, let $G(S)$ be the *undirected* graph with node set $V$, and with edge set $E(S) = \{(v, v') \mid v \in V \wedge v' \in S_v)\}$. When $u \in S_v$, we will say that $v$ is buying the edge $(u, v)$, or that the edge $(u, v)$ is bought by $v$. Then, the cost of a player $v$ in $S$ is $cost_v(S) = |S_v|$ if $\varepsilon_{G(S)}(v) \leq R_v$, $+\infty$ otherwise.

The *bounded average-distance* NCG (SumBD) is defined analogously, with a bound $D_v$ on the average distance of $v$ from all the other nodes, and cost function $cost_v(S) = |S_v|$ if $D_{G(S)}(v) \leq D_v$, $+\infty$ otherwise. In the rest of the paper, depending on the context, we will interchangeably make use of the bound on the broadcast cost $B_v = D_v \cdot n$ when referring to SumBD.

In both variants, we say that a node $v$ is *within the bound in* $S$ (or in $G(S)$) if $cost_v(S) < +\infty$. We measure the overall quality of a graph $G(S)$ by its social cost $SC(S) = \sum_{v \in V} cost_v(S)$. A graph $G(S)$ minimizing $SC(S)$ is called *social optimum.*

We use the *Nash Equilibrium* (NE) as solution concept. More precisely, a NE is a strategy profile $S$ in which no player can decrease its cost by changing its strategy, assuming that the strategies of the other players are fixed. When $S$ is a NE, we will say that $G(S)$ is *stable*. Conversely, a graph $G$ is said to be stable if there exists a NE $S$ such that $G = G(S)$. Notice that in both games, when $S$ is a NE, all nodes are within the bound and, since every edge is bought by a single player, $SC(S)$ coincides with the number of edges of $G(S)$.

We conclude this section by recalling the definition of the two measures we will use to characterize the NE space of our games, namely the *Price of Anarchy* (PoA) [9] and the *Price of Stability* (PoS) [3], which are defined as the ratio between the highest (respectively, the lowest) social cost of a NE, and the cost of a social optimum.

## 3 Preliminary Results

First of all, observe that for MaxBD it is easy to see that a stable graph always exists. Indeed, if there is at least one node having distance bound 1, then the graph where all 1-bound nodes buy edges towards all the other nodes is stable. Otherwise, any spanning star is stable. Notice that any spanning star is stable for SumBD as well, but only when every vertex has a bound $B_v \geq 2n-3$, while the problem of deciding whether a NE always exists for the remaining values of $B_v$ is open. From these observations, we can derive the following negative result.

**Theorem 1.** *The PoA of* MaxBD *and* SumBD *(with distance bounds* $B_v \geq 2n-3$*) is* $\Omega(n)$*, even for only two distance-bound values.*

*Sketch of proof.* We exhibit a graph $G'$ with $\Omega(n^2)$ edges, and a strategy profile $S$ such that $G(S) = G'$ and $G(S)$ is stable in both models for suitable distance bounds. We also show that the social optimum is $n-1$.

The graph $G'$ is defined as follows. We have a clique of $k$ nodes. For each node $v$ of the clique, we add four nodes $v_1^1, v_2^1, v_1^2, v_2^2$ and four edges $(v_2^1, v_1^1), (v_1^1, v)$, $(v_2^2, v_1^2)$, and $(v_1^2, v)$. Clearly, $G'$ has $n = 5k$ nodes and $\Omega(n^2)$ edges. Now, consider a strategy profile $S$ with $G' = G(S)$ and such that (i) every edge is bought by a single player, and (ii) the edges $(v_2^j, v_1^j), (v_1^j, v)$ are bought by $v_2^j$ and $v_1^j$, respectively, $j = 1, 2$.

For MaxBD, we set the bound of every node of the clique to 3, while all the other nodes have bound 5. For SumBD, we set the bound of each node $v$ of the clique to $\sum_{u \in V} d_G(v, u) = 11k - 5 > 2n - 3$, while we assign to all the other nodes bound $n^2$.

It is then not so hard to show that $G(S)$ is stable. To conclude the proof, observe that any spanning star (with cost $n-1$) is a social optimum for the two instances of MaxBD and SumBD given above. □

Given the above bad news, from now on we focus on the *uniform* case of the games, i.e., all the bounds on the distances are the same, say $R$ and $D$ (i.e., $B = D \cdot n$) for the maximum and the average version, respectively. Similarly to other NCGs, also here we have the problem of computing a best response for a player, as stated in the following theorem.

**Theorem 2.** *Computing the best response of a player in* MaxBD *and* SumBD *is* NP*-hard.* □

On the other hand, a positive result which clearly implies that SumBD always admits a pure NE is the following:

**Theorem 3.** *The PoS of* MaxBD *is 1, while for* SumBD *it is at most 2.* □

## 4 Upper and Lower Bounds to the PoA for MaxBD

We start by providing few results which will be useful to prove our upper bounds to the PoA for MaxBD.

**Lemma 1.** *Let $G(S) = (V, E(S))$ be stable and let $H$ be a subgraph of $G(S)$. If for each node $v \in V$ there exists a set $E_v$ of edges (all incident to $v$) such that $v$ is within the bound in $H + E_v$, then $SC(S) \leq |E(H)| + \sum_{v \in V} |E_v|$.*

*Proof.* Let $k_v$ be the number of edges of $H$ that $v$ is buying in $S$. If $v$ buys $E_v$ additionally to its $k_v$ edges, then $v$ will be within the bound in $H + E_v$. Hence, since $S$ is a NE, we have that $cost_v(S) \leq k_v + |E_v|$, from which it follows that

$$SC(S) = \sum_{v \in V} cost_v(S) \leq \sum_{v \in V} k_v + \sum_{v \in V} |E_v| = |E(H)| + \sum_{v \in V} |E_v|.$$ □

Thanks to Lemma 1, we can prove the following lemma.

**Lemma 2.** *Let $G(S)$ be stable, and let $\gamma$ be the cardinality of a minimum dominating set of $G(S)^{R-1}$. Then $SC(S) \leq (\gamma + 1)(n - 1)$.*

*Proof.* Let $U$ be a minimum dominating set of $G(S)^{R-1}$, with $\gamma = |U|$. It is easy to see that there is a spanning forest $F$ of $G(S)$ consisting of $\gamma$ trees $T_1, \ldots, T_\gamma$, such that every $T_j$ contains exactly one vertex in $U$, and when we root $T_j$ at such vertex the height of $T_j$ is at most $R - 1$.

For a node $v \in V$, let $E_v = \{(v, u) \mid u \in U \setminus \{v\}\}$. Clearly, $v$ is within the bound in $F + E_v$, hence by using Lemma 1, we have

$$SC(S) \leq |E(F)| + \sum_{u \in U} |E_u| + \sum_{v \in V \setminus U} |E_v| = n - \gamma + (\gamma - 1)\gamma + \gamma(n - \gamma) \leq (\gamma + 1)(n - 1).$$

□

Let $G(S)$ be stable and let $v$ be a node of $G(S)$. Since $v$ is within the bound, the neigborhood of $v$ in $G$ is a dominating set of $G^{R-1}$. Therefore, thanks to Lemma 2 we have proven the following corollary.

**Corollary 1.** *Let $G(S)$ be stable, and let $\delta$ be the minimum degree of $G(S)$, then $SC(S) \leq (\delta+1)(n-1)$.* □

We are now ready to prove our upper bound to the PoA for MaxBD.

**Theorem 4.** *The PoA of* MaxBD *is $O\left(n^{\frac{1}{\lfloor \log_3 R \rfloor + 1}}\right)$ for $R \geq 3$, and $O(\sqrt{n \log n})$ for $R = 2$.*

*Proof.* Let $G$ be a stable graph, and let $\gamma$ be the size of a minimum dominating set of $G^{R-1}$. We define the *ball* of radius $k$ centered at a node $u$ as $\beta_k(u) = \{v \in V \mid d_G(u,v) \leq k\}$. Moreover, let $\beta_k = \min_{u \in V} |\beta_k(u)|$. The idea is to show that in $G$ the size of any ball increases quite fast as soon as the radius of the ball increases.

*Claim.* For any $k \geq 1$, we have $\beta_{3k+1} \geq \min\{n, \gamma\beta_k\}$.

*Proof.* Consider the ball $\beta_{3k+1}(u)$ centered at any given node $u$, and assume that $|\beta_{3k+1}(u)| < n$. Let $T$ be a maximal set of nodes such that (i) the distance from every vertex in $T$ and $u$ is exactly $2k+2$, and (ii) the distance between any pair of nodes in $T$ is at least $2k+1$. We claim that for every node $v \notin \beta_{3k+1}(u)$, there is a vertex $t \in T$ with $d_G(t,v) < d_G(u,v)$. Indeed, consider the node $t'$ in a shortest path in $G$ between $v$ and $u$ at distance exactly $2k+1$ from $u$. If $t' \in T$ the claim trivially holds, otherwise consider the node $t \in T$ that is closest to $t'$. From the maximality of $T$ we have that $d_G(t,v) \leq d_G(t,t') + d_G(t',v) \leq 2k + d_G(u,v) - (2k+1) < d_G(u,v)$.

As a consequence, we have that $T \cup \{u\}$ is a dominating set of $G^{R-1}$, and hence $|T| + 1 \geq \gamma$. Moreover, all the balls centered at nodes in $T \cup \{u\}$ with radius $k$ are all pairwise disjoint. Then

$$|\beta_{3k+1}(u)| \geq |\beta_k(u)| + \sum_{t \in T} |\beta_k(t)| \geq \gamma\beta_k. \qquad \square$$

Now, observe that since the neighborhood of any node in $G$ is a dominating set of $G^{R-1}$, we have that $\beta_1 \geq \gamma$. Then, after using the above claim $x$ times, we obtain

$$\beta_{\frac{3^{x+1}-1}{2}} \geq \min\left\{n, \gamma^{x+1}\right\}.$$

Let us consider the case $R \geq 3$ first. Let $U$ be a maximal independent set of $G^{R-1}$. Since $U$ is also a dominating set of $G^{R-1}$, it holds that $|U| \geq \gamma$. We consider the $|U|$ balls centered at nodes in $U$ with radius given by the value of the parameter $x = \lfloor \log_3 R - 1 \rfloor$. Every ball has radius at most $(R-1)/2$, and since $U$ is an independent set of $G^{R-1}$, all balls are pairwise disjoint, and hence we have $n \geq |U|\gamma^{\lfloor \log_3 R - 1 \rfloor + 1} \geq \gamma^{\lfloor \log_3 R \rfloor + 1}$. As a consequence, we obtain $\gamma \leq n^{\frac{1}{\lfloor \log_3 R \rfloor + 1}}$, and the claim now follows from Lemma 2.

Now assume $R = 2$. We use the bound given in [5] to the size $\gamma(G)$ of a minimum dominating set of a graph $G$ with $n$ nodes and minimum degree $\delta$, namely $\gamma(G) \leq \frac{n}{\delta+1} H_{\delta+1}$, where $H_i = \sum_{j=1}^{i} 1/j$ is the $i$-th harmonic number. Hence, since a social optimum has cost $n-1$, from Lemma 2 and Corollary 1, we have $\frac{SC(S)}{n-1} \leq \min\left\{\delta+1, \frac{n}{\delta+1} H_{\delta+1}+1\right\} = O\big(\min\{\delta, \frac{n}{\delta}\log n\}\big)$, for any stable graph $G(S)$ with minimum degree $\delta$. Since this is asymptotically maximized when $\delta = \Theta\big(\sqrt{n \log n}\big)$, the claim follows. □

Now we focus on lower bounds to the PoA of MaxBD. We first prove a simple constant lower bound for $R = o(n)$, and then we show an almost tight lower bound of $\Omega(\sqrt{n})$ for $R = 2$. We postpone to the concluding section a discussion on the difficulty of finding better lower bounds for large values of $R$.

**Theorem 5.** *For any $\epsilon > 0$ and for every $1 < R = o(n)$, the PoA of* MaxBD *is at least $2 - \epsilon$.*

*Proof.* Assume we are given a set of $n = 2R + h$ vertices $\{u_1, \ldots, u_{2R}\} \cup \{v_1, \ldots, v_h\}$. The strategy profile $S$ is defined as follows. Vertex $u_j$ buys a single edge towards $u_{j+1}$, for each $j = 1, \ldots, 2R-1$, while every $v_i$ buys two edges towards $u_1$ and $u_{2R}$. It is easy to see that $G(S)$ has diameter $R$ and is stable. The claim follows from the fact that $SC(S)$ goes to $2(n-1)$ as $h$ goes to infinity, and since, as observed in Section 3, a spanning star (having social cost equal to $n-1$) is a social optimum. □

**Theorem 6.** *The PoA of* MaxBD *for $R = 2$ is $\Omega(\sqrt{n})$.*

*Proof.* We provide only the lower-bound construction due to lack of space. Let $p \geq 3$ be a prime number. We exhibit (see Figure 1) a graph $G'$ of diameter 2 containing $O(p^2)$ vertices and $\Omega(p^3)$ edges, and a strategy profile $S$ such that $G(S) = G'$ and $G(S)$ is stable. $G'$ contains two vertex-disjoint rooted trees $T$ and $T'$ as subgraphs. $T$ is a complete $p$-ary tree of height 2. We denote by $r$ the root of $T$, by $C = \{c_0, \ldots, c_{p-1}\}$ the set of children of $r$, and by $V_i = \{v_{i,0}, \ldots, v_{i,p-1}\}$ the set of children of $c_i$. $T'$ is a star with $p^2$ leaves rooted at the center $r'$. The leaves of $T'$ are partitioned in $p$ groups each having exactly $p$ vertices. For every $i = 0, \ldots, p-1$, we denote by $U_i = \{u_{i,0}, \ldots, u_{i,p-1}\}$ the set of vertices of group $i$. $G' = (V, E)$ has vertex set $V = V(T) \cup V(T')$, and edge set

$$E = E(T) \cup E(T') \cup \{(r, r')\} \cup \big\{(c, c') \mid c, c' \in C, c \neq c'\big\}$$
$$\cup \bigcup_{i=0}^{p-1} \big\{(u, u') \mid u, u' \in U_i, u \neq u'\big\}$$
$$\cup \big\{(u_{i,j}, v_{i',j'}) \mid i, i', j, j' \in [p-1], j + i'i \equiv j' \pmod{p}\big\}.$$

In the strategy profile $S$, (i) $r$ buys all edges of $G'$ incident to it, (ii) each $v_{i',j'}$ buys all edges of $G'$ incident to it, (iii) each edge $(u_{i,j}, r')$ of $G'$ is bought by $u_{i,j}$, and (iv) each of the remaining edges in $G'$ is bought by any of its two endpoint players. □

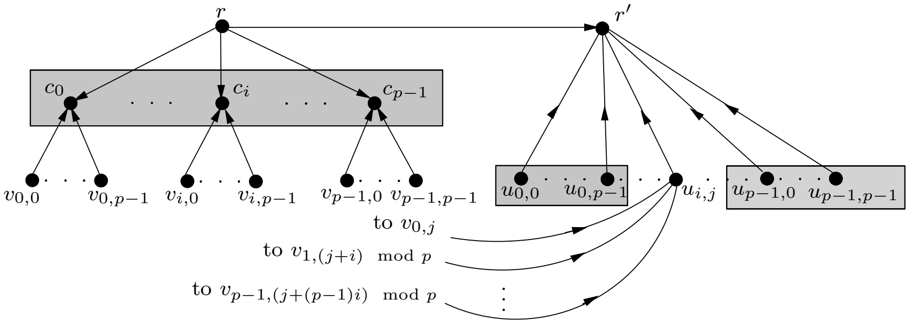


**Fig. 1.** The graph $G(S)$. Edges are bought from the nodes they exit from. Notice that nodes in grey boxes are clique-connected (with arbitrary orientations, i.e., ownership), and for the sake of readability we have only inserted edges leading to node $u_{i,j}$.

## 5 Upper and Lower Bounds to the PoA for SumBD

For SumBD, we start by giving an upper bound to the PoA similar to that obtained for MaxBD. For the remaining of this section we use $D$ to denote the average bound of every node, namely $D = B/n$.

**Theorem 7.** *The PoA of* SumBD *is* $O\left(\sqrt{n \log n}\right)$ *when* $2 \leq D < 3$*, and* $O(n^{\frac{1}{\lfloor \log_3 D/4 \rfloor + 2}})$ *for* $D \geq 3$. □

From the above result, it follows that the PoA becomes constant when $D = \Omega(n^{\epsilon})$, for some $\epsilon > 0$. We now show how to lower such a threshold to $D = 2^{\omega(\sqrt{\log n})} = n^{\omega\left(\frac{1}{\sqrt{\log n}}\right)}$ (and we also improve the upper bound when $D = \omega(1) \cap o(3^{\sqrt{\log n}})$).

**Lemma 3.** *Let* $G(S)$ *be stable and let* $v$ *be a node such that* $B_{G(S)}(v) \leq B - n$*, then* $SC(S) \leq 2(n-1)$.

*Proof.* Let $T$ be a shortest path tree of $G$ rooted at $v$. The claim immediately follows from Lemma 1 by observing that $v$ is within the bound in $T$ and every other node $u$ is within the bound in $T + (u, v)$. □

Notice that the above lemma shows that when a stable graph $G$ has diameter at most $D - 1$, then the social cost of $G$ is at most twice the optimum. Now, the idea is to provide an upper bound to the diameter of any stable graph $G$ as a function of $\delta$, where $\delta$ is the minimum degree of $G$. Then we combine this bound with Lemma 3 in order to get a better upper bound to the PoA for interesting ranges of $D$.

**Theorem 8.** *Let* $G$ *be stable with minimum degree* $\delta$*. Then the diameter of* $G$ *is* $2^{O(\sqrt{\log n})}$ *if* $\delta = 2^{O(\sqrt{\log n})}$*, and* $O(1)$ *otherwise.*

*Proof.* We start by proving two lemmas.

**Lemma 4.** *Let $G$ be stable with minimum degree $\delta$. Then either $G$ has diameter at most $2\log n$ or, for every node $u$, there is a node $x$ with $d_G(u,x) \le \log n$ such that (i) $x$ is buying $\delta/c$ edges (for some constant $c > 1$), and (ii) the removal of these edges increases the sum of distances from $x$ by at most $2n(1+\log n)$.*

*Proof.* Assume that the diameter of $G$ is greater than $2\log n$, and consider a node $u$. Let $U_j$ be the set of nodes at distance exactly $j$ from $u$, and let $n_j = |U_j|$. Moreover, denote by $T$ a shortest path tree of $G$ rooted at node $u$. Let $i$ be the minimum index such that $n_{i+1} < 2n_i$ ($i$ must exist since the height of $T$ is greater than $\log n$). Consider the set of edges $F$ of $G$ having both endpoints in $U_{i-1} \cup U_i \cup U_{i+1}$ and that do not belong to $T$. Then, $|F| \ge \delta n_i/2 - 3n_i$. Moreover, we have that $n_{i-1} + n_i + n_{i+1} \le n_i/2 + n_i + 2n_i = 7n_i/2$. As a consequence, there is a vertex $x \in U_{i-1} \cup U_i \cup U_{i+1}$ which is buying at least $\frac{n_i/2-3n_i}{7n_i/2} \ge \delta/c$ edges of $F$, for some constant $c > 1$. Moreover, when $x$ removes these edges, the distance to any other node $y$ increases by at most $2(1+\log n)$ because $d_T(x,y) \le 2(1+\log n)$. The claim follows. □

**Lemma 5.** *In any stable graph $G$, there is a constant $c' > 1$ such that the addition of $\delta/c'$ edges all incident to a node $u$ decreases the sum of distances from $u$ by at most $5n\log n$.*

*Proof.* If $G$ has diameter at most $2\log n$, then the claim trivially holds. Otherwise, let $x$ be the node of the previous lemma and let $c'$ be such that $\delta/c' \le \delta/c - 1$. Moreover, assume by contradiction that the sum of distances from $u$ decreases by more than $5n\log n$ when we add to $G$ the set of edges $F = \{(u,v_1),\ldots,(u,v_h)\}$, with $h = \delta/c'$. Then, let $F' = \{(x,v_j) \mid j = 1,\ldots,h\}$. We argue that $x$ can reduce its cost by saving at least an edge as follows: $x$ deletes its $\delta/c$ edges and adds $F'$. Indeed, the sum of distances from $x$ increases by at most $2n(1+\log n) \le 4n\log n$, and decreases by at least $5n\log n - n\log n$, since for every node $y$ such that the shortest path in $G+F$ from $u$ to $y$ passes through $x$, we have that $d_G(u,y) - d_{G+F}(u,y) \le \log n$. Hence, $x$ is still within the bound in $G+F'$ and is saving at least one edge, a contradiction. □

Recall that the ball of radius $k$ centered at a node $u \in V$ is defined as $\beta_k(u) = \{v \in V \mid d_G(u,v) \le k\}$, and that $\beta_k = \min_{u\in V} |\beta_k(u)|$. We claim that

$$\beta_{4k} \ge \min\left\{n/2+1, \frac{k\delta}{20c\log n}\beta_k\right\}, \quad (3)$$

for some constant $c > 1$. To prove that, let $u \in V$ be any node and assume that $|\beta_{4k}(u)| \le n/2$. Let $T$ be a maximal set of nodes such that (i) the distance from every vertex in $T$ and $u$ is exactly $2k+1$, and (ii) the distance between any pair of nodes in $T$ is at least $2k+1$. From the maximality of $T$, for every node $v \notin \beta_{3k}(u)$ there is a node $t \in T$ such that $d_G(v,t) \le d_G(u,v) - k$. Since $|\beta_{4k}(u)| \le n/2$, at least $n/2$ nodes have a distance more than $3k$ from $u$. This implies the existence of a set $Y$ of such vertices and a set $T' \subseteq T$ such that (i)

$|Y| \geq n\delta/(2|T|)$, (ii) $|T'| = \delta/c$, and (iii) for every $v \in Y$, there exists $v' \in T'$ such that $d_G(v, v') \leq d_G(u, v) - k$. If we add $\delta/c$ edges from $u$ to nodes in $T'$, the sum of distances from $u$ decreases by at least $(k-1)n/(2|T|) \geq kn/(4|T|)$. By Lemma 5 this improvement is at most $5n \log n$ and, as a consequence, $|T| \geq \delta k/(20c \log n)$. Moreover, all the balls centered at nodes in $T$ are disjoint, and this proves the recurrence (3). Now, the claim follows by solving such a recurrence. □

Next theorem provides an alternative upper bound to the PoA of SumBD.

**Theorem 9.** *The PoA of* SumBD *is* $2^{O(\sqrt{\log n})}$ *if* $D = \omega(1)$*, and* $O(1)$ *if* $D = 2^{\omega(\sqrt{\log n})}$.

*Proof.* Let $G(S)$ be stable, and let $\Delta$ be the diameter of $G(S)$. First of all, consider the case $\Delta = o(D)$, and observe that $B_{G(S)}(v) = o(B)$ for every $v$. Therefore, Lemma 3 implies that $\frac{SC(S)}{n-1} = O(1)$. This implies the second part of the claim since Theorem 7 implies that $\Delta = 2^{O\left(\sqrt{n}\right)}$.

Now, consider the case $\Delta = \Omega(D)$. Since $D = \omega(1)$, we have that $\Delta = \omega(1)$ and therefore, from Theorem 7, $\delta = 2^{O\left(\sqrt{n}\right)}$. To complete the proof, we show that $\frac{SC(S)}{n-1} \leq \delta + 1$. Let $v$ be a node with degree $\delta$, and let $N_{G(S)}(v) = \{u_1, \ldots, u_\delta\}$. Consider a shortest path tree $T$ of $G(S)$ rooted at $v$. Clearly, $v$ is within the bound in $T$, and if we define $E_x = \{(x, u_j) \mid 1 \leq j \leq \delta\}$ for any $x \neq v$, we have $B_{T+E_x}(x) \leq B_{G(S)}(v) \leq B$. Hence, from Lemma 1, it follows that $SC(S) \leq |E(T)| + (n-1)\delta \leq (\delta + 1)(n-1)$. □

Then, by combining the results of Theorems 7 and 9, we get the bounds reported in Table 1. Finally, we can give the following

**Theorem 10.** *For any* $\epsilon > 0$ *and for* $2n - 3 \leq B = o(n^2)$*, the PoA of* SumBD *is at least* $2 - \epsilon$. □

## 6 Concluding Remarks

In this paper, we have introduced a new NCG model in which the emphasis is put on the fact that a player might have a strong requirement about its centrality in the resulting network, as it may well happen in decentralized computing (where, for instance, the bound on the maximum distance could be used for synchronizing a distributed algorithm). We developed a systematic study on the PoA of the two (uniform) games MaxBD and SumBD, which, however, needs to be continued, since a significant gap between the corresponding lower and upper bounds is still open. In particular, it is worth to notice that finding a better upper bound to the PoA would provide a better estimation about how much dense a network in equilibrium can be.

Actually, in an effort of reducing such a gap, we focused on MaxBD, and we observed the following fact: Recall that a graph is said to be *self-centered* if every node is a center of the graph (thus, the eccentricity of every node is equal to the radius of the graph, which then coincides with the diameter of the graph). An interesting consequence of Lemma 2 is that only stable graphs that are self-centered can be dense, as one can infer from the following

**Proposition 1.** *Let $G(S)$ be stable for* MAXBD*. If $G(S)$ is not self-centered, then $SC(S) \leq 2(n-1)$.*

*Proof.* Let $v$ be a node with minimum eccentricity. It must be $\varepsilon_{G(S)}(v) \leq R-1$. Then, $U = \{v\}$ is a dominating set of $G^{R-1}$, and Lemma 2 implies the claim. □

Thus, to improve the lower bound for the PoA of MAXBD, one has to look to self-centered graphs. Moreover, if one wants to establish a lower bound of $\rho$, then a stable graph of minimum degree $\rho - 1$ (from Corollary 1) is needed. Starting from these observations, we investigated the possibility to use small and suitably dense self-centered graphs as *gadgets* to build lower bound instances for increasing values of $R$. To illustrate the process, see Figure 2, where using a self-centered cubic graph of diameter 3 and size 20, we have been able to obtain a lower bound of 3 (it is not very hard to see that the obtained graph is in equilibrium).

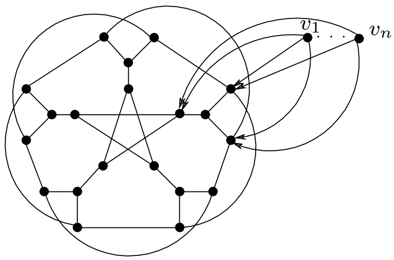


**Fig. 2.** A graph with $n + 20$ nodes and $3n + 30$ edges, showing a lower bound for the PoA of MAXBD for $R = 3$ approaching to 3, as soon as $n$ grows. Edges within the gadget (on the left side) are bought by either of the incident nodes, while other edges are bought from the nodes they exit from.

Interestingly enough, the gadget is a famous extremal (i.e., maximal w.r.t. node addition) graph arising from the study of the *degree-diameter* problem, namely the problem of finding a largest size graph having a fixed maximum degree and diameter (for a comprehensive overview of the problem, we refer the reader to [1]). More precisely, the gadget is a graph of largest possible size having maximum degree $\Delta = 3$ and diameter $R = 3$. In fact, this seems not to be coincidental, since also *Moore graphs* (which are extremal graphs for $R = 2$ and $\Delta = 2, 3, 7, 57$), and the extremal graph for $R = 4$ and $\Delta = 3$ (see [1]), can be shown to be in equilibrium, and then they can be used as gadgets (clearly, the lower bounds implied by Moore graphs for $R = 2$ are subsumed by our result in Theorem 6). Notice that from this, it follows that we actually have a lower bound of 3 for the PoA of MAXBD also for $R = 4$. So, apparently there could be some strong connection between the equilibria for MAXBD and the extremal graphs w.r.t. to the degree-diameter problem, and we plan in the near future to explore such intriguing issue.

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