Simultaneous Single-Item Auctions

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Abstract. In a combinatorial auction (CA) with item bidding, several goods are sold simultaneously via single-item auctions. We study how the equilibrium performance of such an auction depends on the choice of the underlying single-item auction. We provide a thorough understanding of the price of anarchy, as a function of the single-item auction payment rule.

When the payment rule depends on the winner's bid, as in a first-price auction, we characterize the worst-case price of anarchy in the corresponding CAs with item bidding in terms of a sensitivity measure of the payment rule. As a corollary, we show that equilibrium existence guarantees broader than that of the first-price rule can only be achieved by sacrificing its property of having only fully efficient (pure) Nash equilibria.

For payment rules that are independent of the winner's bid, we prove a strong optimality result for the canonical second-price auction. First, its set of pure Nash equilibria is always a superset of that of every other payment rule. Despite this, its worst-case POA is no worse than that of any other payment rule that is independent of the winner's bid.

1 Introduction

The problem of allocating multiple heterogeneous goods to a number of competing buyers is well motivated, notoriously difficult in practice, and, when buyers' preferences are private (i.e., unknown to the seller), central to the study of *algorithmic mechanism design*. More precisely, suppose there are m goods and each buyer i has a private *valuation* v_i that assigns a value $v_i(S)$ to each bundle (i.e., subset) S of goods. For example, each good could represent a license for exclusive use of a given frequency range in a given geographic area, buyers could correspond to mobile telecommunication companies, and valuations then describe a company's willingness to pay for a given collection of licenses [6]. One natural objective function, for example when the seller is the government, is *welfare maximization*: partition the goods into bundles $S_1, \ldots, S_n$, with S_i denoting the goods given to buyer i, to maximize the welfare $\sum_{i=1}^{n} v_i(S_i)$.

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A *combinatorial auction* is a protocol that elicits information from buyers about their private valuations, computes an allocation of the goods, and determines who pays what. There are at least three different types of obstacles to designing good combinatorial auctions. The first problem is information-theoretic: players' valuations have size exponential in m, so eliciting full valuations is not feasible unless m is small. The second problem is computational: the welfare-maximization problem is generally NP-hard, even to approximate, even when players' valuations have succinct representations. The third problem is game-theoretic: a player is happy to misreport its preferences to manipulate a poorly-designed auction to produce an outcome that favors the player. Thus designing combinatorial auctions requires compromises — on the welfare of the computed solution, the complexity of the mechanism, or the strength of the incentive-compatibility guarantee.

Most previous work on combinatorial auctions in the theoretical computer science literature focuses on *truthful approximation mechanisms* [3]. Such mechanisms run in time polynomial in n and m (with oracle access to players' valuations) and satisfy a very strong incentive-compatibility guarantee: for every player, reporting its true preferences in the auction is a dominant strategy (i.e., maximizes its utility, no matter what the other players do). The benefits of such mechanisms are clear: they require minimal work from and make minimal behavioral assumptions on the players, and are computationally tractable. They suffer from two major drawbacks, however. The first is that the strong requirement of a dominant-strategy implementation severely restricts what is possible: even for the relatively well-behaved class of submodular valuations,[1] no truthful approximation mechanism achieves a sub-polynomial approximation factor [7,9]. The second is that, even for settings where good truthful approximation mechanisms exist, these mechanisms are often quite complicated (see e.g. [8]).

The complexity and provable limitations of dominant-strategy implementations motivate the design of combinatorial auctions that have weaker incentive guarantees, in exchange for simpler formats or better approximation factors. One natural and practical auction format that has been studied recently is *combinatorial auctions (CA) with item bidding*. In a CA with item bidding, each player submits a single bid for each item, and each item is sold independently via a single-item auction. They were first studied in [5] and [4] with second-price single-item auctions. CAs with item bidding and first-price auctions were recently studied in [12].

Combinatorial auctions with item bidding are interesting for many reasons. First, they are one of the simplest auction formats that could conceivably admit performance guarantees for non-trivial combinatorial auction problems. By construction, they do not suffer from the informational problems of most combinatorial auctions — each player is forced to summarize its entire (exponential-size) valuation for the mechanism in the form of m bids — nor from the computational problems, since the auction outcome is as trivial to compute as in a

[1] A valuation v is *submodular* if, for every pair $S \subseteq T$ of goods and good $j \notin T$, $v(T \cup \{j\}) - v(T) \leq v(S \cup \{j\}) - v(S)$.

single-item auction. Of course, there is no hope for a"truthful" implementation — players are not even granted the vocabulary to express fully their preferences — and the incentive properties of CAs with item bidding will be weaker than in dominant-strategy implementations. Second, CAs with item bidding naturally arise "in the wild". They were first studied in the AI literature [4] because trading agents are often forced to participate in them — imagine, for example, an automated travel agent responsible for acquiring a vacation package by negotiating simultaneously with hotels, airlines, and tour guides. Similarly, single-item auction sites like eBay are presumably used by some buyers to acquire several goods in parallel, even when there are non-trivial substitutes or complements among the goods [5]. Third, the recent strong lower bounds on the performance of dominant-strategy CAs [7,9] imply that further progress in algorithmic mechanism design requires the systematic study of mechanisms with weaker incentive guarantees. CAs with item bidding are a natural and well-motivated starting point for this exploration. Fourth, as discussed in [12], equilibria in CAs with item bidding can be thought of as generalizations of price equilibria in settings with indivisible goods, where a conventional (i.e., Walrasian) price equilibrium need not exist.

The properties of a CA with item bidding depend on the format choice for the underlying single-item auctions. For example, CAs with item bidding and first-price auctions have Nash equilibria (in pure strategies) in strictly fewer settings than with second-price auctions; but Nash equilibria with first-price auctions are always welfare-maximizing, while those with second-price auctions are not [5,12].

The goal of this paper is to understand how the equilibrium set of a CA with item bidding depends on the format choice for its constituent single-item auctions.

(Q1) *How does the equilibrium performance of a combinatorial auction depend on the choice of the underlying single-item auction?*

(Q2) *Is there an "optimal" single-item auction for CAs with item bidding? Is there a single-item auction that shares the benefits of both the first- and second-price auctions?*

1.1 Our Results

We provide a thorough understanding of the price of anarchy of pure Nash equilibria, when such equilibria exist, in CAs with item bidding, as a function of the single-item auction payment rule. When the payment rule depends on the winner's bid (like in a first-price auction), we characterize the worst-case price of anarchy in the corresponding CAs with item bidding in terms of a "sensitivity measure" of the payment rule. As a corollary, we derive the following "undominated" property of the first-price payment rule: the *only way* to have broader equilibrium existence guarantees is to sacrifice the property of having only fully efficient equilibria.

For payment rules that are independent of the winner's bid, we prove a strong optimality result for the canonical second-price auction. First, its set of pure

Nash equilibria is always a superset of that of every other payment rule. Despite this, its worst-case POA is no worse than that of any other payment rule that is independent of the winner's bid.

1.2 Related Work

The literature on combinatorial auctions is too big to survey here; see the book [6] and book chapter [3] for general information on the topic. Related work on combinatorial auctions with item bidding, also mentioned above, are [2,5,21] for second-price auctions and [12] for first-price auctions. An alternative simple auction format is sequential (rather than simultaneous) single-item auctions; the price of anarchy in such auctions was studied recently in [16,20]. Most other work in theoretical computer science on combinatorial auctions has focused on truthful, dominant-strategy implementations (see [3]), with [1] being a notable exception.

A less obviously related paper is by Fu et al. [10]. This paper introduces the concept of a conditional equilibrium. Lavi (personal communication) showed that a conditional equilibrium exists for a valuation profile if and only if a "conservative" equilibrium (defined below) exists in the corresponding CA with item bidding with the second-price payment rule. The paper shows that, for every valuation profile, every conditional equilibrium has welfare at least 1/2 times that of an optimal allocation.

Finally, several previous works [13,19,18] consider the independent private values model and study how the Bayes-Nash equilibrium of a single-item auction varies with the choice of payment rule.

2 Preliminaries

Combinatorial Auctions. In a combinatorial auction (CA), there is a set of n players and a set M of m goods (or items). Each player i has a *valuation* $v_i : 2^M \to \mathbb{R}^+$ that describes its value for each subset of the goods. We always assume that $v_i(\emptyset) = 0$ and $v_i(S) \leq v_i(T)$ for all $S \subseteq T$. The *social welfare* $SW(\mathbf{X})$ of an allocation $\mathbf{X} := \{X_1, X_2, \ldots, X_n\}$ of the goods to the players is $\sum_{i=1}^n v_i(X_i)$.

For a valuation profile $\mathbf{v} = \{v_1, v_2, \ldots, v_n\}$, we denote the welfare-maximizing allocation by $OPT(\mathbf{v})$.

Item Bidding. In a CA with item bidding, each player i submits m bids, one for each good. Each good is allocated to the highest bidder at a price given by a payment rule p. We denote such a mechanism by $\mathcal{M}_p$.

For a fixed mechanism, we use $X_i(\mathbf{b})$ to denote the goods allocated to player i in the bid profile $\mathbf{b}$ and $SW(\mathbf{b}) = \sum_{i=1}^n v_i(X_i(\mathbf{b}))$ the social welfare of the resulting allocation. Player i's utility in a bid profile $\mathbf{b} = (b_1, b_2, \ldots b_n)$ is

$$u_i(\mathbf{b}) = v_i(X_i(\mathbf{b})) - \sum_{j \in X_i(\mathbf{b})} p_i(b_1(j), b_2(j), \ldots b_n(j)).$$

Payment Rules. We consider payment rules that meet the following natural conditions. We assume that the payment rule is anonymous. For such a payment rule p, the winner's payment when the bids are $x_1 \geq x_2 \geq \ldots \geq x_n$ is denoted by $p(x_1, x_2, \ldots, x_n)$. We further assume that the payment function is non-decreasing: raising bids can only increase the price charged to the winner. Finally, we assume that the payment function is continuous in every bid. For example, every payment rule given by a convex combination of the bids satisfies all of these assumptions.

For convenience, we also assume that the payment rule is not bounded or constant, and that the minimum price $p(0, 0, \ldots 0)$ is 0. As we show in the full version, payment rules that do not meet these assumptions are uninteresting — either there are never any equilibria, or such equilibria can be arbitrarily inefficient.

Auctions as Games. Players generally have no dominant strategies in a CA with item bidding, and we study the performance of an auction via the equilibria of the corresponding bidding game. In this paper, we focus on a full-information model, where players' valuations are publicly known, and on pure Nash equilibria. Recall that for a fixed valuation profile $\mathbf{v} = (v_1, v_2, \ldots v_n)$, a bid profile $\mathbf{b} = (b_1, b_2, \ldots b_n)$ is a *(pure) Nash equilibrium* if $u_i(\mathbf{b}) \geq u_i(b'_i, \mathbf{b}_{-i})$ for every player i and (feasible) deviation b'_i, where $(b'_i, \mathbf{b}_{-i})$ denotes the bid profile in which player i bids b'_i and all other players bid according to $\mathbf{b}$.

The *price of anarchy (POA)* is the ratio of the social welfare of an optimal allocation and that of the worst Nash equilibrium:

$$\text{POA} = \max_{\mathbf{b}\text{: a pure Nash eq.}} \frac{SW(OPT(\mathbf{v}))}{SW(\mathbf{b})}. \tag{1}$$

The POA is undefined when no equilibria exist.

3 Winner-Dependent Payment Rules

3.1 Overview

This section considers *winner-dependent* payment rules, such as the first-price rule, where the winner's payment is strictly increasing in its bid. The key property shared by such rules is that, in an equilibrium, the winner must bid the minimum amount required to win.

Are there winner-dependent payment rules that are "better" than the first-price rule? A drawback with CAs with item bidding and the first-price rule is that equilibria often fail to exist. Precisely, recall that a *Walrasian equilibrium* for a valuation profile is a set of prices $p_1, \ldots, p_m$ on the goods and a feasible allocation $(S_1, S_2, \ldots S_n)$ of the goods to the players so that each player obtains a bundle that maximizes its utility (i.e., value minus price). We say that a valuation profile is *Walrasian* if it admits a Walrasian equilibrium and *non-Walrasian* otherwise. Walrasian equilibria always exist when valuations meet the gross substitutes property, but not generally otherwise (see [11,14]). The pure Nash equilibria of a CA with item bidding and the first-price payment rule correspond to the Walrasian equilibria (if any) in a natural way, and are fully efficient when they exist [12].

Other winner-dependent payment rules can yield CAs with item bidding that possess equilibria *even in non-Walrasian instances.* We give an explicit example in the full version, for the payment rule that averages the highest and third-highest bids. This observation motivates the question: is there a payment rule that strictly dominates the first-price rule? That is, is there a payment rule that induces an equilibrium in at least one non-Walrasian instance and has worst-case POA equal to 1?

We answer this question negatively in the following theorem (proved in Section 3.3).

Theorem 1. *If the worst-case POA for the mechanism $\mathcal{M}_p$ is 1, then pure Nash equilibria exists under this mechanism only in Walrasian instances.*

Thus, for every winner-dependent payment rule p, either there is an instance in which some pure Nash equilibrium of the mechanism $\mathcal{M}_p$ is not efficient, or every instance in which a pure Nash equilibrium exists is a Walrasian instance.

The main step in our proof of Theorem 1 is a characterization of the worst-case POA in CAs with item bidding and winner-dependent payment rules. For a payment rule p, we define a sensitivity measure ζ by

$$\zeta(p) = \sup_{\boldsymbol{b}:b_1=b_2\geq\ldots\geq b_n} \frac{p(b_1, \boldsymbol{b_{-n}})}{p(\boldsymbol{b})}, \tag{2}$$

where we interpret 0/0 as 1.

The denominator in (2) is the winner's payment with the bid vector $\mathbf{b}$. The numerator is the payment of the lowest bidder in $\mathbf{b}$, after it switches to bidding the minimum amount necessary to win (namely, b_1). We restrict attention to bid vectors $\mathbf{b}$ with $b_1 = b_2$ because this property is satisfied in every equilibrium under a winner-dependent rule. Because p is monotone, $p(b_1, b_{-n}) \geq p(\boldsymbol{b})$ and hence $\zeta(p) \geq 1$. Similarly, if a bidder other the lowest in $\mathbf{b}$ changes its bid to b_1, then its payment is at most the numerator in (2).

For a concrete example, consider the payment rule (first-price +2·third-price)/3. The numerator is $(b_1 + 2b_2)/3 = (b_1 + 2b_1)/3 = b_1$, while the denominator is $(b_1 + 2b_3)/3 \geq b_1/3$. In the worst case this ratio is 3, and hence $\zeta(p) = 3$.

We show in Theorem 2 that the parameter $\zeta(p)$ is exactly the worst-case POA in CAs with item bidding and the payment rule p. It follows that the POA is exactly 1 only when $\zeta(p) = 1$. We use this fact to prove Theorem 1, that a pure Nash equilibrium exists for such a payment rule only in Walrasian instances.

3.2 Characterization of Worst-Case POA

We now prove that for every winner-dependent payment rule p, the worst case POA of CAs with item bidding and rule p is exactly $\zeta(p)$. The upper bound applies to every valuation profile for which an equilibrium exists. The lower bound already applies to bidders with submodular (or even "budgeted additive") valuations.

Theorem 2. *For every winner-dependent payment rule p with $\zeta(p)$ finite, the worst-case POA of CAs with item bidding and payment rule p is precisely $\zeta(p)$. For winner-dependent payment rules with $\zeta(p) = +\infty$, there are CAs with item bidding with arbitrarily high POA.*

Proof. We first prove an upper bound of $\zeta(p)$ on the POA. Fix a valuation profile $\mathbf{v} = (v_1, v_2, \ldots v_n)$. Let $\mathbf{b} = (\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_n})$ denote an equilibrium bid profile and $\mathbf{X}(\mathbf{b}) = \{X_1(\mathbf{b}), X_2(\mathbf{b}), \ldots, X_n(\mathbf{b})\}$ the corresponding allocation. For each good j, we use $b_{j1}, b_{j2}, \ldots, b_{jn}$ to denote the sorted set of bids on that good and use $\mathbf{b}_{j,-i}$ to denote the same set with the ith bid removed. Since the payment rule is winner dependent, the winner of each good bids the minimum amount required to win, and thus $b_{j1} = b_{j2}$ for each good j. We use p_j to denote the payment $p(b_{j1}, b_{j2}, \ldots, b_{jn})$ of the winner of good j.

We first relate equilibrium payments to equilibrium welfare. Since the utility of every player in an equilibrium is non-negative, $\sum_{j \in X_i(\mathbf{b})} p_j \leq v_i(X_i(\mathbf{b}))$ for every player i. Summing over the players gives $\sum_j p_j \leq SW(\mathbf{X}(\mathbf{b}))$.

Next we relate the optimal welfare to the equilibrium utilities. Let $\mathbf{O} = (O_1, O_2, \ldots O_n)$ denote an optimal allocation. For each player i, define the bid vector a'_i as equal to $b_{j1} + \epsilon$ on each good $j \in O_i$ and zero otherwise. If player i bids a'_i, it wins at least the set O_i and pays $p(b_{j1} + \epsilon, b_{j,-i})$ on each good $j \in O_i$. Since $\mathbf{b}$ is an equilibrium bid profile, $u_i(\mathbf{b}) \geq u_i(a'_i, b_{-i}) \geq v_i(O_i) - \sum_{j \in O_i} p(b_{j1} + \epsilon, \mathbf{b}_{j,-i})$. Since this inequality holds for every $\epsilon > 0$ and the payment rule is continuous, $u_i(\mathbf{b}) \geq v_i(O_i) - \sum_{j \in O_i} p(b_{j1}, \mathbf{b}_{j,-i})$. By the definition of ζ in (2), $p(b_{j1}, \mathbf{b}_{j,-i}) \leq \zeta(p) \cdot p_j$ for every $j \in O_i$. Thus

$$u_i(\mathbf{b}) \geq v_i(O_i) - \zeta(p) \cdot \sum_{j \in O_i} p_j.$$

Next, since $v_i(X_i(\mathbf{b})) - \sum_{j \in X_i} p_j = u_i(\mathbf{b})$ for every player i, we can derive

$$\begin{aligned} SW(\mathbf{X}(\mathbf{b})) - \sum_j p_j &= \sum_{i=1}^{n} u_i(\mathbf{b}) \\ &\geq \sum_i v_i(O_i) - \zeta(p) \sum_i \sum_{j \in O_i} p_j \\ &= SW(\mathbf{O}) - \zeta(p) \sum_j p_j. \end{aligned}$$

Since $\zeta(p) \geq 1$, and $\sum_j p_j \leq SW(\mathbf{X}(\mathbf{b}))$, rearranging terms gives $\zeta(p) \cdot SW(\mathbf{X}(\mathbf{b})) \geq SW(\mathbf{O})$. This shows that the POA is at most $\zeta(p)$.

To establish the lower bound, fix $\epsilon > 0$ and set $\zeta' = \zeta(p) - \epsilon$. If $\zeta(p) = +\infty$ we can set ζ' to an arbitrarily large number. There must exist a bid vector $\boldsymbol{b}$ with $b_1 = b_2 \geq \ldots \geq b_n$ such that $\zeta' \leq p(b_1, \boldsymbol{b_{-n}})/p(\boldsymbol{b})$. Let $p_1 = p(b_1, \boldsymbol{b_{-n}})$ and let

$p_2 = p(\boldsymbol{b})$. Clearly $p_1 \geq p_2$. We construct an instance with n players where the equilibrium welfare is at most $p_2/p_1 \leq 1/\zeta'$ times that of the optimal allocation.

Consider an instance with n players and 2 goods denoted A, B. Player 1 values good A for p_1, good B for p_2 and both goods for p_1. Player 2 values good A for p_2, good B for p_1, and the two together for p_1. All other players value every subset of goods at 0. We show that the following bid profile is an equilibrium: player 1 bids (b_n, b_1), player 2 bids (b_1, b_n), and player i for $3 \leq i \leq n$ bids (b_{i-1}, b_{i-1}).

Fix a tie-breaking rule to favor player 2 over player 3 on good A and player 1 over player 3 on good B. (Note that the upper bound above is independent of the tie-breaking rule). In this bid profile, player 2 wins good A and player 1 wins good B. They both pay p_2 for the goods they win. If either of them tries to deviate to win the other good they have to pay p_1. Since their values for the good they currently win is p_2 and their value for the other good is p_1, these deviations are not profitable. No other player has an incentive to deviate.

The optimal allocation in this instance is to allocate good A to player 1 and good B to player 2. This allocation has welfare $2p_1$ while the equilibrium allocation has welfare $2p_2$. Thus the POA is at least $p_1/p_2 \geq \zeta'$. □

3.3 Proof of Theorem 1

Consider a winner-dependent payment rule p with worst-case POA equal to 1. We show that every instance for which the mechanism $\mathcal{M}_p$ has an equilibrium is a Walrasian instance.

Fix a valuation profile and an equilibrium bid profile $\mathbf{b}$ for the mechanism $\mathcal{M}_p$ with some deterministic tie-breaking rule. Let $(S_1, S_2, \ldots S_n)$ denote an equilibrium allocation and $p_1, p_2, \ldots, p_m$ the prices paid by the winner on each good. We argue by contradiction that the S_i's and p_i's form a Walrasian equilibrium.

Suppose the equilibrium allocation with prices $p_1, p_2, \ldots p_m$ is not a Walrasian equilibrium. Then there must exist a player i and a set X of goods such that $u_i(S_i, p) < u_i(X, p)$, where $u_i(S, p)$ denotes the utility $v_i(S) - \sum_{j \in S} p_j$ of player i when receiving bundle S at prices p. Let δ satisfy $0 < \delta < u_i(X, p) - u_i(S_i, p)$.

Let $b_{j1} \geq b_{j2} \geq b_{j3} \ldots \geq b_{jn}$ denote the nondecreasing set of equilibrium bids on a good j. Since the payment rule is winner-dependent, $b_{j1} = b_{j2}$ for every good j. Since the payment rule p is assumed to induce only CAs with item bidding with fully efficient equilibria, Theorem 2 implies that $\zeta(p) = 1$. This fact and the monotonicity of p imply that $p(b_{j1}, \mathbf{b}_{j,-i}) = p_j$ for every j. By the continuity of p, we can identify an ϵ such that $\sum_{j \in X} p(b_{j1} + \epsilon, \mathbf{b}_{j,-i}) - p_j \leq \delta$. Then,

$$v_i(S_i) - \sum_{j \in S_i} p_j < v_i(X) - \sum_{j \in X} p(b_{j1} + \epsilon, \mathbf{b}_{j,-i}).$$

Player i can win set X by bidding $b_{j1} + \epsilon$ on each element $j \in X$ and bidding zero on the rest, and this deviation increases its utility. This contradicts the assumption that $\mathbf{b}$ is an equilibrium bid profile and completes the proof. □

We can sharpen Theorem 1 when there are only two players. Every winner-dependent payment rule p that depends only on the two highest bids satisfies $\zeta(p) = 1$. This holds, in particular, for every winner-dependent rule in a two-player setting. From the proof of Theorem 1, we conclude the following corollary.

Corollary 1. *For every winner-dependent payment rule p and two-player instance, $\mathcal{M}_p$ has an equilibrium only if it is a Walrasian instance.*

It is easy to construct non-Walrasian two-player instances. We conclude that no winner-dependent payment rule guarantees existence in all two-player instances.

4 Winner-Independent Payment Rules

This section focuses on *winner-independent* payment rules, for which the winner's payment does not depend on its bid. We prove that among all payment rules in this class, the second-price rule has the best worst-case POA while guaranteeing equilibrium existence most often.

First, we prove that there are more pure Nash equilibria under the second-price payment rule than under any other rule. This "maximal existence: guarantee has a possible drawback, however, in the form of a larger worst-case POA bound. We show that this drawback does not materialize: the second-price rule, despite the relative profusion of equilibria, leads to a worst-case POA that is as good as with any other winner-independent rule.

4.1 γ-Conservative Equilibria

To make meaningful statements about equilibrium efficiency in CAs with item bidding and winner-independent payment rules, we need to parameterize the equilibria in some way. The reason is that every winner-independent payment rule suffers from arbitrarily bad equilibria.[2]

We consider equilibria where the players' bids satisfy a certain "conservativeness" condition. This assumption is fairly standard in the POA of auctions literature [2,5,15,17]. The conservativeness condition assumes that the equilibrium bids guarantee each player positive utility on the set it wins, even when all other players bid the same as this player. More generally, we relax this idea in two ways: parameterizing it with a parameter $\gamma \geq 1$, and applying it only to the bundles that players win in the equilibrium (rather than to all bundles they might hypothetically win). Players have the freedom to bid as high as they want on the goods they lose and can contemplate arbitrary deviations.

[2] Let $y > 0$ satisfy $p(y, y) > 0$ and consider an instance with two players and one good. Player 1 values the good at 0 and player 2 values it at $p(y, y)$. Then player 1 bidding y and player 2 bidding zero is an equilibrium and this equilibrium is clearly very inefficient.

Definition 1 (γ-conservative). *Suppose a player bidding $(b_1, b_2, \ldots b_m)$ wins a set S in the equilibrium. We say that the bid is* γ-conservative *if it satisfies*

$$\sum_{j \in S} p(b_j, b_j, \ldots, b_j) \leq \gamma \cdot v(S).$$

An equilibrium allocation is γ-conservative *if every player uses a γ-conservative bid in the equilibrium.*

4.2 The Second-Price Rule Has the Most Equilibria

Next we show that every γ-conservative equilibrium allocation for a payment rule p can also be realized as a γ-conservative equilibrium for the second-price rule. This transformation does not change the prices that the winners pay on the goods that they win. We use Σ_p^γ to denote the set of γ-conservative equilibrium allocations of the mechanism $\mathcal{M}_p$. In particular, $\Sigma_{s.p.}^\gamma$ denotes the set of γ-conservative equilibrium allocations of the item bidding mechanism with the second-price payment rule.

Theorem 3. *For every payment rule p, $\Sigma_p^\gamma \subseteq \Sigma_{s.p.}^\gamma$.*

Proof. We start with an equilibrium of the mechanism $\mathcal{M}_p$. Let $(S_1, S_2, \ldots S_n)$ denote the allocation. Focus on a good j, and let $b_{j1} \geq b_{j2} \geq \ldots \geq b_{jn}$ be the ordered bids on the good. While reasoning about individual goods we refer to the players by their rank in this ordering. The payment the winner (player 1) makes in this case is $p(b_{j1}, b_{j2}, \ldots b_{jn})$. Denote this as p_{j1}.

Let $p_{j2} = p(b_{j1}, b_{j1}, \ldots b_{j1})$. If any player i deviates, it will have to bid at least b_{j1} and pay at least $p(b_{j1}, b_{j,-i})$. Here $b_{j,-i}$ denotes the bids on good j by all players other than player i. Since the payment rule is monotone, this payment is at most that $p(b_{j1}, b_{j1}, \ldots b_{j1}) = p_{j2}$ when all players bid b_{j1}. By monotonicity, $p_{j2} \geq p_{j1}$.

Construct an equilibrium under the second-price rule as follows. Fix a player i. On good $j \in S_i$, player i bids p_{j2}, one other player bids p_{j1}, and all other players bid zero. Note that bidding p_{j2} is feasible for player i. This is because in the given equilibrium instance for payment rule p, the players' bids on the sets they win are γ-conservative. Hence for every player i, $\sum_{j \in S_i} p_{j2} \leq \gamma \cdot v_i(S_i)$. This is the same as the γ-conservativeness condition for the second price rule, as for the second price rule when all players bid p_{j2} the payment is p_{j2} as well.

In this construction the winner's payment on a good is the same as that in the equilibrium for payment rule p. Any player currently not winning a good has to pay at least p_{j2} if it deviates to win that good. Deviations are then not profitable, as in the equilibrium for payment rule p players do not find them profitable at even lower prices. The constructed bid profile is an equilibrium for the second-price rule. The equilibrium allocation and the prices paid by the winners remain the the same. □

Theorem 3 shows that the second-price payment rule has at least as large a set of γ-conservative equilibrium allocations as any other payment rule p. We include in the full version an example showing that this inclusion can be strict.

Theorem 3 has immediate implications, both positive and negative, for all winner-independent payment rules. On the negative side, it allows us to port equilibrium non-existence results for CAs with item bidding and the second-price rule — like the fact that with subadditive valuations (where $v_i(S \cup T) \leq v_i(S) + v_i(T)$ for every player i and bundles S, T), γ-conservative equilibria need not exist (see [2] and the full version) — to those with an arbitrary winner-independent rule. On the positive side, Theorem 3 implies that POA bounds for CAs with item bidding and the second-price rule carry over to all winner-independent rules. For example, we show in the full version, by modifying a result in [2], that the POA of γ-conservative equilibria with the second-price rule is at most $\gamma + 1$ (in instances where such an equilibrium exists). Using Theorem 3, this bound holds more generally for all winner-independent rules.

4.3 POA Lower Bounds

The results of the previous section imply that, for every $\gamma \geq 1$, the POA of γ-conservative equilibria of CAs with item bidding is as bad with the second-price rule as with any other winner-independent rule. This section proves the converse, for every $\gamma \geq 1$.

Theorem 4. *For every winner-independent payment rule p, the worst-case POA of γ-conservative equilibria of $\mathcal{M}_p$ is at least $\gamma + 1$.*

We prove this theorem by establishing a stronger result: when there are only two players, the set of γ-conservative equilibrium allocations is the same for all winner-independent payment rules. The POA lower bound then follows from a lower bound construction for the second-price rule that uses only two players.

Lemma 1. *In a two-player CA with item bidding, every equilibrium of the second-price payment rule is an equilibrium of every winner-independent payment rule p.*

Proof. Consider an equilibrium under the second-price payment rule. Let S_1, S_2 denote the equilibrium allocation. Fix a player i, and suppose that on good $j \in S_i$ the player i bids b_j and pays p_j. Clearly $b_j \geq p_j$. The other player would have to bid at least b_j to win this good and would then pay b_j. The conservativeness condition for the second-price payment rule implies that for each player i, $\sum_{j \in S_i} b_j \leq \gamma \cdot v_i(S_i)$.

Since the given payment rule p is winner-independent and there are only two players, the payment only depends on the non-winning player's bid. To mimic the second-price equilibrium allocation with the mechanism $\mathcal{M}_p$, we first identify for each good a bid vector such that $p(b_{1j}, b_{1j}) = p_j$. This exists because the payment rule p is continuous and has full range. Similarly, we can identify a bid x_j such that $p(x_j, x_j) = b_j$. Since $b_j \geq p_j$, $x_j \geq b_{1j}$ Since the payment is independent of the highest bid it doesn't change if we raise the winner's bid to x_j. Hence, $p_j(x_j, b_{2j}) = p_j$.

Focus on a player i and set S_i. Set player i's bid on good j in S_i to x_j. Since x_j satisfies $p(x_j, x_j) = b_j$ and $\sum_{j \in S_i} b_j \leq \gamma \cdot v_i(S_i)$, these bids form a γ-conservative strategy for player i. The other player bids b_{1j} on each good $j \in S_i$. In case of a tie, we employ the same tie-breaking rule used in the second-price equilibrium, resulting in the tie being broken in favor of player i.

If the other player wishes to deviate to win good j it must bid at least x_j. By the choice of x_j, it would have to pay at least b_j. Since in the second-price equilibrium neither player wants to deviate when faced with the price b_j, no player wants to deviate in this constructed bid profile either. This bid profile is an equilibrium with the same allocation and payments as the given equilibrium under the second-price rule.

To complete the proof that the second-price rule has the best-possible worst-case POA of γ-conservative equilibria (for every fixed $\gamma \geq 1$), we give a two-player example with POA equal to $\gamma + 1$.

Example 1. There are two goods denoted A, B and two players. Player 1 values A for 1, B at $\gamma + 1$, and both for $\gamma + 1$. Player 2 values A for $\gamma + 1$, B for 1, and both for $\gamma + 1$.

The bid profile where player 1 bids $(\gamma, 0)$ and player 2 bids $(0, \gamma)$ is an equilibrium of the the CA with item bidding and the second-price payment rule. These bids are γ-conservative. The welfare of this equilibrium allocation is 2 while the optimal welfare is $2(\gamma + 1)$.

5 Conclusions

There are a number of opportunities for interesting further work. One important direction is to extend our study of CAs with item bidding to mixed-strategy Nash equilibria of the full-information model and to Bayes-Nash equilibria in incomplete information models. These more general equilibrium concepts are not well understood even for the second- and first-price payment rules [2,12]. A second topic is allocation rules different from the one studied here, where the highest bidder always wins. For example, can reserve prices improve the performance of CAs with item bidding in any sense? A third direction is to study systematically different single-item payment rules in sequential auctions, thereby extending the recent work in [16,20]. Finally, it would be very interesting to analyze restricted auction formats that extend simultaneous or sequential single-item auctions, such as combinatorial auctions with restricted package bidding.

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