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Preface

This volume contains the papers presented at WINE 2012, the 8th Workshop on Internet and Network Economics, held on December 10–12, 2012 in Liverpool, UK.

Over the past decade, there has been a growing interaction between researchers in theoretical computer science, networking and security, economics, mathematics, sociology, and management sciences devoted to the analysis of problems arising from the Internet and the World Wide Web. The Workshop on Internet and Network Economics (WINE) is an interdisciplinary forum for the exchange of ideas and results arising from these various fields. At the time of writing, WINE 2012 had just been approved for "in cooperation" status with ACM SIGecom (ACM's special interest group on electronic commerce).

In the Call for Papers we solicited regular papers (14 pages) and short papers (7 pages). We received 112 submissions, from which we accepted 36 regular and 13 short papers. As for WINE 2011, we also allowed submissions to be designated as working papers. For these papers, the submission was assessed in the same way as other papers, but only the abstract has been published in the proceedings. This allows subsequent publication in journals that do not accept papers where full versions have previously appeared in conference proceedings. Of the 49 accepted papers, 3 are working papers. All papers were rigorously reviewed by the program committee members and/or external referees; each received at least 3 detailed reviews. Submissions were evaluated on the basis of their significance, novelty, soundness, and relevance to the workshop.

Besides the regular talks, the program also included three invited talks by Kamal Jain (eBay Research Labs, USA), Benny Moldovanu (University of Bonn, Germany) and David Parkes (Harvard University, USA). The conference organizers also hosted tutorials on the day before WINE, on topics of interest to the community: an introduction to the GAMBIT software by Rahul Savani and Ted Turocy; a talk entitled "An Overview of Matching Markets: Theory and Practice" by David Manlove, and an introduction to Judgement Aggregation by Ulle Endriss.

We are very grateful to Google Research and Microsoft Research for their generous financial contribution to the conference. We also thank the Department of Computer Science at the University of Liverpool for their financial contribution and organizational support.

We also acknowledge Easychair, a powerful and flexible system for managing all stages of the paper handling process, from the submission stage to the preparation of the final version of the proceedings.

October 2012

Paul W. Goldberg
Mingyu Guo

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An Introduction to the Algorithmic Game Theory of eBay's Buyer-Seller Matching (Invited Talk)

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Abstract

Buyer-seller matching is a widely used problem. It is a problem of Google's (ads) and Amazon's; and it is also a problem of traditional retailers such as Walmart and Costco. In the offline world a traditional retailer is trying to match the products supplied by the manufacturers to the interested buyers. In the case of a traditional retailer this matching is a static matching done once for all the buyers. In the online world, it is possible to do this matching for every potential buyer, perhaps based on their expressed (e.g., based on a search query) or implied (e.g., based on a browser cookie) intent. eBay is perhaps the first major company to start such a buyer-seller matching online; hence the title. The presentation is based on buyer-seller matching from a viewpoint of electronic commerce industry in general. This includes search ads, online retailers, and online marketplaces.

There are various issues arise in buyer-seller matching perhaps many of them could be captured by the trade off between Relevance and Revenue. *Relevance* is broadly defined as the expected net utility of a seller's offering (known as *listing* on eBay) to a potential buyer at a given price. Decreasing the price of an offering increases the relevance while increasing the price decreases it. So essentially selling any item at a very high price can make an offering completely irrelevant. *Revenue* is defined as the expected fee charged by the company doing the matching, e.g., by eBay. The company doing the match is henceforth called an *intermediary.*

There are two major strategic decisions an intermediary makes; 1. on what event(s) a fee is charged for doing the matching; and 2. what criteria to use to decide the order of listings to display to a potential buyer. There are many different choices being made in the industry. eBay charges a fee at the time of including a listing in its index and then again when the product listed is bought by a buyer. Google charges a fee when a potential buyer clicks on an advertisement and lands on a seller's page. Walmart charges its fee as a markup on top of the wholesale price it gets from its suppliers. Costco charges its fee when a potential buyer registers with it and also as a markup on top of the wholesale price it gets from its suppliers. How these fees are charged and on what order a potential buyer sees the listings have a tremendous influence on the selection of

products a buyer sees and as well as the prices a buyer sees. For an example, it can be proven that given risk-neutral sellers and given that the same amount of expected revenue is made by the intermediary, if the intermediary fee is charged as a sale's commission versus a statistically equivalent fee charged on a click, then the net price a buyer sees is higher in the former pricing structure. The reason being that a click fee is sunk cost for the seller while a sale's commission is marginal cost. This is not true if the sellers are risk averse, which is often the case with small sellers. Small sellers may not have know-how or may not be able to afford to hire help to manage their risk. So despite higher prices to potential buyers, fee charged as a sale's commission may offer a bigger selection to a potential buyer than a statistically equivalent fee charged on a click.

When a problem space is defined by two separate parameters, such as Relevance and Revenue in our context, then it is often the case that one could define various notions of optimality. One of the simplest notions is perhaps ignoring one of the parameters altogether. So one question we ask is how to optimize the expected revenue for the intermediary, given a strategic buyer and sellers. A paper with Chris Wilken [2] looks at this problem. Given that a buyer probably has a limited attention span, the paper considers various conceptual models of a buyer's attention. A full attention model is when a buyer considers all possible listings before deciding what to purchase. On the other end of the spectrum, a buyer considers only 1 listing and decides whether to purchase it or not. The paper shows, in a very general Bayesian setting, that if the attention model is known then finding a revenue optimal mechanism is essentially an algorithmic problem, since game-theoretic properties are automatically satisfied. In other words the paper proposes an optimal mechanism for a general setting given unlimited computation.This is not necessarily true for approximately optimal algorithms. This is because the optimal algorithms result in some kind of monotonicity properties which are often needed to prove incentive compatibility, but approximation often lose the monotonicity. The paper proposes incentive compatible approximately optimal mechanisms for a set of attention models.

Another practical generalization of this setting is to associate multiple sellers with the same listing. When an item is sold often there are multiple sellers behind the item who benefit from the sale. For an example, if Best Buy sells a computer made by Samsung having Intel processor and Windows OS then all 4 companies benefit. Currently the surplus of only the last agent, Best Buy in this example, is directly represented in the matching marketplace. In reality all these 4 sellers are bundled together, because a computer is a bundled product. Separately, there are also settings where the buyers are bundled, e.g., Groupon purchases are executed when a certain number of buyers commit to a purchase.

A paper with Darrell Hoy and Chris Wilkens [1] introduces an ad matching auction where an ad benefits multiple sellers. The industry seems to be evolving in the direction where it is the products whose ads are auctioned rather than just sellers' ads, e.g., Google's search pages now also show the ads of products, besides the ads of the webpages of sellers. In product auction setting, one can conceive that in future the interest of various parties who benefit from the sale of the product could be represented in the marketplace to enhance both the revenue and relevance. This is indeed quite feasible in a marketplace like eBay which anyway displays specific products.

In general, in an auction setting when there are complementary bidders, the revenue for the auctioneer could be as little as zero. This paper [1] demonstrates that the first price auction has a minimum revenue guarantee at equilibrium. Even newer results demonstrate a bidding language which allows pure strategy equilibria in the first price auction, thereby fixing a historic flaw when the first price auction was used by Overture in ad-auctions. Overture's first price ad-auction did not always have a pure strategy equilibrium, thereby causing a cyclic behavior by the bidders. Subsequent work also demonstrate how a first price auction could converge to an equilibrium.

References

1. Hoy, D., Jain, K., Wilkens, C.A.: Coopetitive ad auctions (2012), http://arxiv.org/pdf/1209.0832.pdf
2. Jain, K., Wilkens, C.A.: ebay's market intermediation problem (2012), http://arxiv.org/pdf/1209.5348.pdf

On the Equivalence of Bayesian and Dominant Strategy Implementation (Invited Talk)

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Abstract. We consider a standard social choice environment with linear utilities and independent, one-dimensional, private types. We prove that for any Bayesian incentive compatible mechanism there exists an equivalent dominant strategy incentive compatible mechanism that delivers the same interim expected utilities for all agents and the same ex ante expected social surplus. The short proof is based on an extension of an elegant result due to Gutmann et al. (Annals of Probability, 1991). We also show that the equivalence between Bayesian and dominant strategy implementation generally breaks down when the main assumptions underlying the social choice model are relaxed, or when the equivalence concept is strengthened to apply to interim expected allocations.

Joint work with A. Gershkov, J. Goeree, A. Kushnir and X. Shi.

New Applications of Search and Learning to Problems of Mechanism Design (Invited Talk)

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Abstract. When faced with a hard optimization problem, common approaches are to either design a polynomial-time approximation algorithm, or design a heuristic algorithm (perhaps search-based) that is fast enough, and generates solutions of high enough quality, to be of practical interest. But the main focus in algorithmic mechanism design has been on the first, "polynomial + approximation" direction, with the requirement of truthful mechanisms tending to impede progress in the second (heuristic search) direction. In this talk I describe two ways in which heuristic algorithms can be leveraged within mechanism design. One approach is to modify branch-and-bound search to make it monotone in the input, enabling search to be used as a building block for single-parameter, truthful mechanisms on NP-hard problems, and even without running to optimality. A second approach, which applies also to multi-parameter domains, takes as input a particular allocation algorithm. Given this algorithm, statistical machine learning is used to identify a payment rule that minimizes expected ex post regret for deviating from truthful reports. A direct connection is established between this "minimize ex post regret" problem and the problem of training a multi-class classifier to minimize generalization error. By relaxing truthfulness, this opens up a new direction in coupling "almost implementable" allocation algorithms with suitable payment rules.

This talk is based on two papers: Monotone Branch-and-Bound Search for Restricted Combinatorial Auctions, by John K. Lai and David C. Parkes, in Proc. 13th ACM Conference on Electronic Commerce (EC '12), 2012, and Payment Rules through Discriminant-Based Classifiers, Paul Duetting, Felix Fischer, Pichayut Jirapinyo, John K. Lai, Benjamin Lubin, and David C. Parkes, in Proc. 13th ACM Conference on Electronic Commerce (EC '12), 2012.

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A Game-Theoretic Analysis of a Competitive Diffusion Process over Social Networks*

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Abstract. We study a game-theoretic model for the diffusion of competing products in social networks. Particularly, we consider a simultaneous non-cooperative game between competing firms that try to target customers in a social network. This triggers a competitive diffusion process, and the goal of each firm is to maximize the eventual number of adoptions of its own product. We study issues of existence, computation and performance (social inefficiency) of pure strategy Nash equilibria in these games. We mainly focus on 2-player games, and we model the diffusion process using the known linear threshold model. Nonetheless, many of our results continue to hold under a more general framework for this process.

1 Introduction

A large part of research on social networks concerns the topic of diffusion of information (e.g., ideas, behaviors, trends). Mathematical models for diffusion processes have been proposed ever since [11, 19] and also later in [9]. Given such a model, some of the earlier works focused on the following optimization problem: find a set of nodes to target so as to maximize the spread of a given product (in the absence of any competitors). This problem was initially studied by Domingos and Richardson [8], Kempe et al. [13], and subsequently by [6, 18]. Their research builds on a "word-of-mouth" approach, where the initial adopters influence some of their friends, who in turn recommend it to others, and eventually a cascade of recommendations is created. Within this framework, finding the most influential set of nodes is NP-hard, and approximation algorithms as well as heuristics have been developed for various models.

Different considerations, however, need to be made in the presence of multiple competing products in a market. In real networks, customers end up choosing a product among several alternatives. Hence, one natural approach to model this competitive process is the use of game-theoretic analysis with the players being the firms that try to market their product. The game-theoretic approaches that have been proposed along this direction mainly split into two types. The first is to view the process as a Stackelberg game, where the competitors of a product first choose their strategy, and then a

* Work supported by the project ALGONOW of the research funding program Thales (co-financed by the European Social Fund-ESF and Greek national funds). A full version of this work with all missing proofs is available at the authors' websites.

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last mover needs to make a decision on the set of nodes to target [3–5, 14]. This approach essentially reduces to the algorithmic question of finding the best response for the firm that moves last. The main results that have been obtained along this direction is that, in certain cases, the algorithm of [13], for the case of a single product, can be applied in the competitive environment as well. For more models and related problems under this context see also [2]. A different approach is to capture the competition as a simultaneous game, where firms pick their initial set of nodes at the same time, and then the diffusion process follows (after first taking care of ties). This was first proposed in [1], and has also been studied very recently by [10]. The approach of [1] and [10] as a noncooperative normal-form game is the focus of our work as well.

1.1 Contributions

Our work is an attempt to further understand game theoretic aspects of viral marketing. To this end, we first define in Section 2 a general framework for a competitive diffusion process in a social network, generalizing the model of [1]. This corresponds to a class of non-cooperative games where firms target customers in order to maximize the spread of their own product. We study issues of existence, computation, and performance (social inefficiency) of pure Nash equilibria (PNE). The games we define are one-stage games, as in [1, 10], i.e., all firms spend their budget in one step, a fact that renders natural the analysis of PNE. We use as instantiations of the competitive diffusion process the well-known linear threshold model, however some of our results also hold for more general local interaction schemes.

In more detail, we mostly deal with 2-player games, as in [10], and in Section 3 we first illustrate that such games may not possess PNE, even for simple graphs. On top of that, we also prove that it is co-NP-hard to decide whether a PNE exists for a given game. We then move on to investigate conditions for the existence of PNE. In Section 4, we begin with studying the improvement paths induced by our games. We exhibit that networks with special in and out-degree distributions — e.g. power law — are not expected to be more stable than others, in the sense that all possible dynamics can be realized essentially by any graph. Motivated by all these, we then focus on sufficient conditions for the existence of PNE via generalized ordinal potential functions. We also consider ϵ-approximate generalized ordinal potentials, and we provide tight upper bounds on the existence of such approximations, as well as, polynomial time algorithms for computing approximate PNE. Finally, we study the Price of Anarchy and Stability for games with an arbitrary number of players, and we show that PNE (when they exist) can be quite inefficient. We conclude with a discussion of the effects on the payoff of a single player (or a coalition of players), as the number of competitors increases.

We view as one of the main contributions the fact that we unveil new decisive factors for the existence of PNE that are intertwined with structural characteristics of the underlying network. For example, some of the factors that play a role in our model for obtaining generalized ordinal potentials (exact, or approximate) involve *i)* the *diffusion depth* of a game (defined in Section 2 as the maximum possible duration of the diffusion process), *ii)* the *ideal spread* (defined as the maximum possible spread that a strategy can achieve) and *iii)* the *diffusion collision factor* (defined in Section 4.3 as a measure for comparing how two strategies of one player perform against a given strategy of another player). We advocate that our results motivate further empirical research on social networks for identifying a typical range of these quantities in real networks. Regarding the diffusion depth, some empirical research has already provided new insights for certain recommendation networks [15].

1.2 Related Work

Our work has been largely motivated by [1], (see also the erratum [21]). To the best of our knowledge, this was the first article to consider such games over networks with the players being the firms. The diffusion process of [1] is a special case of our model, in particular, it is a linear threshold model where each firm is allowed to target only one node as a seed, and the thresholds and the weights are equal to $1/|N(v)|$ (with $N(v)$ being the neighborhood of v). We consider the general class of linear threshold models, and in some cases our results hold even for arbitrary local interaction schemes beyond threshold models. In [1] the existence of equilibria is linked to bounding the diameter of the graph. In our model we find that the diameter is not much correlated to existence. Instead we identify other parameters that influence the existence of equilibria.

Besides [1], a very recent related work is [10]. One of the major differences between [10] and our work is that they study the set of *mixed* Nash equilibria of a similar diffusion game, and focus on the Price of Anarchy, and another measure denoted as the *Budget Multiplier*. We, on the other hand, focus on *pure* Nash equilibria. Another difference is that [10] is studying stochastic processes whereas our local interaction schemes induce deterministic processes, as in [1].

Other game-theoretic approaches have also been considered for social networks. One line of work concerns models of Stackelberg games as mentioned earlier [3–5, 14]. A different approach is to consider a game where the players are the individual nodes of the network, who have a utility function depending on their own choice, and that of their neighbors, see e.g., [17, 20]. This leads to very different considerations.

2 Preliminaries

2.1 Social Networks

The underlying structure of the social network is assumed *static*, and is modeled by a fixed finite directed graph $G = (V, E)$ with no parallel edges and no self-loops. Each node $v \in V$ represents an individual within the social network, while each directed edge $(u, v) \in E$ represents that v can be influenced by u. We assume that there are two competing products (or trends, ideas, behavioral patterns) produced by two different firms $\mathcal{M} = \{1, 2\}$, and to each such product we assign a distinct color. Throughout this work, we shall use the terms product, color, and firm interchangeably. Further, each node can have at most one color, and as with most of the literature, we assume that all decisions are *final*; i.e., no node that has adopted a particular product will later alter its decision. Moreover, if a node has adopted a product, we shall refer to it as **colored**, or **infected**, otherwise we will call it a **white** node.

We denote the (in-)neighbors of a node v as $N(v) = \{u \in V | (u, v) \in E\}$, i.e., $N(v)$ is the set of nodes that can influence v. Also, we denote as d_v^{in} and d_v^{out} the in-degree and out-degree of v. The way that a node v can be influenced by $N(v)$ is usually described by a **local interaction scheme** (LIS). Hence, a local interaction scheme is essentially a function that takes as input a node v, the status of its neighbors, a product c under consideration, and possibly other characteristics of the graph, and determines if node v is eligible to adopt this product. An example of a LIS, that was initially studied for the spread of a single product, is the **linear threshold model** (LTM) [11, 19]. Under LTM, there is a weight $w_{uv} \in [0, 1]$ for every edge (u, v) such that for every node v, it holds that $\sum_{u \in N(v)} w_{uv} \leq 1$. Every node v also has a threshold value $\theta_v \in (0, 1]$. The condition that needs to hold, under LTM, so that node v can adopt a product c is

$$\sum_{u \in N(v)} \mathbb{I}_u w_{uv} \geq \theta_v,$$

where $\mathbb{I}_u$ is 1 if u has already adopted product c, and zero otherwise. Note that in a local interaction scheme, the eligibility condition may hold for more than one product at a given time (e.g., under LTM this could happen if $\theta_v < 1/2$ for some node v).

Given a local interaction scheme, and a set of competing firms, we consider the following competitive diffusion process, which evolves over discrete time steps:

The diffusion process. Initially each firm tries to infect a set of "seeds". The number of seeds for each firm may depend on its budget for advertising and marketing. We assume here that the firms have the same power so that in the beginning they can target a set of k nodes each (we think of k as being much smaller than $|V|$ but not necessarily a constant).

– At time step $t = 0$: This is the **initiation** step. In the beginning, all nodes are colored white. If a node v was targeted by a single firm c, then v adopts product c. Since each firm may pick to target an arbitrary set of k nodes, some overlaps may also occur. Thus, we assume that a tie-breaking criterion TBC1 is applied to resolve such dilemmas. This may be a global rule, or a rule that depends on each node.
– At any time step $t > 0$: We look at each remaining white node and check if it is eligible to adopt any of the products, i.e., if the adoption condition, as determined by LIS, holds. For this, we take into account *only* the neighbors of v that were infected up until time step $t - 1$, hence the order with which we examine the white nodes does not matter. During this process, a white node v may be eligible to adopt more than one product. To resolve such dilemmas a second tie-breaking criterion TBC2 should be considered. The process terminates at a time step t, when no white node is eligible to adopt any product. We allow that TBC1 may differ from TBC2, since TBC2 may depend on specific features of the diffusion process, whereas TBC1 occurs only at the initiation step.

A particular instance of a tie-breaking criterion, that we shall often use, is the rule that is also used in [12, 17], where ties are resolved in favor of the "best quality" product: all the individuals within the social network share a *common* **reputation ordering**, say $R^{\prec} \equiv 1 \succ 2$, over the products and in case of ties they decide according to $R^{\prec}$. We shall also see later that some of our results are independent of the tie-breaking rules.

Note 1. All definitions above can be generalized in a straightforward manner to an arbitrary number of m firms, i.e., $\mathcal{M} = \{1, \ldots, m\}$. In Section 3 and Section 4 we focus mostly on the *2-player* case. Section 5 deals with arbitrary m-player games as well.

Definition 1. *A* ***social network*** $\mathscr{N}$ *is defined through the tuple* $(G, \texttt{LIS}, \texttt{TBC1}, \texttt{TBC2})$.

2.2 Strategic Games Induced by Diffusion Processes

A game $\Gamma = (\mathscr{N}, \mathcal{M}, k)$ is induced by a social network $\mathscr{N} = (G, \texttt{LIS}, \texttt{TBC1}, \texttt{TBC2})$ and the set of firms $\mathcal{M} = \{1, \ldots, m\}$, which we shall refer to as a **diffusion game**. In a diffusion game, all participating firms choose simultaneously a set of k seeds, which then triggers a diffusion process according to the interaction scheme and tie-breaking criteria of $\mathscr{N}$. We denote as $\mathcal{S} = \{S : |S| = k\}$ the set of available strategies, which is the same for each firm. We shall use the phrases *strategy* S and *subset* S interchangeably. A pure strategy profile is a vector $\mathbf{s} = (S_1, \ldots, S_m) \in \mathcal{S}^m$, where S_i corresponds to the strategy played by player $i \in \mathcal{M}$. Also, we set $\mathbf{s_{-i}} \equiv \{S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_m\}$.

Given a strategy profile $\mathbf{s} \in \mathcal{S}^m$, the utility of firm $i \in \mathcal{M}$, denoted by $u_i(\mathbf{s})$, is the total number of nodes that have been colored by firm i at the end of the competitive diffusion process. We denote the associated game matrix as $\Pi(\Gamma)$. Moreover, a pure strategy profile $\mathbf{s} \in \mathcal{S}^m$ is a **pure Nash equilibrium (PNE)** of game Γ if $u_i(S_i', \mathbf{s_{-i}}) \leq u_i(\mathbf{s})$, $\forall i$ and $\forall S_i'$.

An important parameter in our games is the so-called diffusion depth defined below.

Definition 2. *The* ***diffusion depth*** $D(\Gamma)$ *of a game* Γ *is defined as the maximum number of time steps that the competitive diffusion process may need, where the maximum is taken over all strategy profiles* $\mathbf{s} \in \mathcal{S}^m$.

Observe that the diffusion depth can take values either lower, equal, or greater than the diameter of the underlying graph G.

Another important notion in our analysis is defined below. Consider a hypothetical scenario where only one player participates in the game. Then his payoff will not be obstructed by anybody else, and any strategy that he chooses achieves its best possible performance. This is useful for quantifying the players' utilities as we shall see later on.

Definition 3. *Assume that only one player from* $\mathcal{M}$ *participates in the game, and let* $S \in \mathcal{S}$ *be one of his strategies. We define as* ***ideal spread of*** $\mathbf{S}$*, denoted by* H_S*, the set of nodes that have adopted by the end of the diffusion process the product of this player under strategy* S*. This includes the initial seed as well, i.e.,* $S \subseteq H_S$.

3 Existence: Examples and Complexity

We start with some remarks concerning the presentation. In Section 3 and Section 4 we consider mostly *2-player* games. Furthermore, our results mainly hold for the linear threshold model but some of them can be generalized to arbitrary models. Whenever in stating a theorem, we do not specify a parameter of the network, it means that it holds independent of its value (e.g. in some results we do not specify the tie-breaking criteria, or the local interaction scheme).

The games that we study do not always possess PNE and we present an example below to illustrate this. We note that this is independent of the tie-breaking criteria used. For any other choice of such criteria (deterministic or even randomized), we can construct analogous examples.

Example 1. Consider the game $(\mathscr{N}, \mathcal{M} = \{1,2\}, k = 1)$ over the graph of Figure 1, where $\mathscr{N} = (G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec})$. It is easy to generalize this to a line with an arbitrary number of nodes. We assume that all nodes have threshold 1. The game matrix is seen in Table 1 and it is easy to check that no PNE exists.

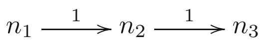

Fig. 1. A network with underlying structure a line

Table 1. The payoff matrix for the game of Figure 1

	n_1	n_2	n_3
n_1	$3,0$	$1,2$	$2,1$
n_2	$2,1$	$2,0$	$1,1$
n_3	$1,2$	$1,1$	$1,0$

The example reveals that even simple graph structures may fail to have PNE. This holds for larger values of k as well, and we have also found other examples with no PNE, where the graph G is a cycle, a clique, or belongs to certain classes of trees.

Given these examples, the next natural question is whether it is easy to decide if a given game has at least one PNE. We assume that the input to this problem is not the game matrix, which can be exponentially large, but simply the graph G and a description of the local interaction scheme. Note that for $k = O(1)$ the problem is easy, hence the challenge is for larger values of k. We establish the following hardness result.

Theorem 1. *Deciding whether a game* $((G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M}, k)$ *has a PNE is co-NP-hard and belongs to* Σ_2^p.

Remark 1. The reduction in the proof of Theorem 1 produces instances where the network is a directed acyclic graph (DAG) and the diffusion depth is $D = 3$.

The proof of Theorem 1 is based on a reduction from 3SAT. Note that we have not obtained membership in the class co-NP. This is because there seems to be no short certificate for checking that a game does not have any PNE (one would need to check all strategy profiles). It is an open problem to determine if the problem is complete for Σ_2^p. Another open problem would be to determine the complexity for games with diffusion depth $D = 1$, or $D = 2$.

4 Towards Characterizations

To understand better the issue of existence of PNE, we start with quantifying the utility functions $u_i : \mathcal{S}^2 \mapsto \mathbb{N}, i \in \mathcal{M} = \{1, 2\}$. For this we need to introduce some important notions. A convenient way to calculate the utility of a player under a profile $\mathbf{s}$, is by utilizing the definition of H_S in Section 2, which is the ideal spread of a product if the firm was playing on its own and used S as a seed. In the presence of a competitor, the firm will lose some of the nodes that belong to H_S. The losses happen due to three reasons. First, the competitor may have managed to infect a node at an earlier time step than the step that the firm would reach that node. Second, the firm may lose nodes due to the tie-breaking criteria, if both firms are eligible to infect a node at the same time step. Finally, there may be nodes that belong to H_S, but the firm did not manage to infect enough of their neighbors so as to color them as well. These nodes either remain white, or are eventually infected by the other player. All these are captured below:

Definition 4. *Consider a game* $((G, \texttt{LIS}, \texttt{TBC1}, \texttt{TBC2}), \mathcal{M}, k)$*, and a strategy profile* $\mathbf{s} = (S_1, S_2)$*. For* $i \in \{1, 2\}$*,*

i. we denote by $\alpha_i(\mathbf{s})$ *the number of nodes that belong to* H_{S_i}*, and under profile* $\mathbf{s}$*, player* i *would be eligible to color them at some time step* t *but the other player has already infected them at some earlier time step* $t' < t$ *(e.g., this may occur under the threshold model when* $\theta_v < 1/2$ *for some node* v*).*
ii. we denote by $\beta_i(\mathbf{s})$ *the number of nodes in* H_{S_i}*, such that under profile* $\mathbf{s}$*, both firms become eligible to infect them at the same time step, and due to tie-breaking rules, they get infected by the competitor of* i*.*
iii. we denote by $\gamma_i(\mathbf{s})$ *the number of nodes that belong to* H_{S_i}*, but under* $\mathbf{s}$*, firm* i *never becomes eligible to infect them (because* i *did not manage to color the right neighbors under* $\mathbf{s}$*).*

Finally, we set $\alpha_{i,max}$ (respectively $\beta_{i,max}, \gamma_{i,max}$) to be the maximum value of $\alpha_i(\mathbf{s})$ over all valid strategy profiles and also $\alpha_{max} = \max\{\alpha_{1,max}, \alpha_{2,max}\}$ (similarly for β_{max}, and γ_{max}). We refer the reader to an example in our full version for an illustration of these concepts.

When we use $R^{\prec}$ for tie-breaking, clearly $\beta_1(\mathbf{s}) = 0$. Hence for *2-player* games of the form $((G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M}, k)$, the utility functions of the players, given a strategy profile $\mathbf{s} = (S_1, S_2) \in \mathcal{S}^2$, are

$$u_1(\mathbf{s}) = |H_{S_1}| - \alpha_1(\mathbf{s}) - \gamma_1(\mathbf{s}), \quad (1)$$
$$u_2(\mathbf{s}) = |H_{S_2}| - \alpha_2(\mathbf{s}) - \beta_2(\mathbf{s}) - \gamma_2(\mathbf{s}). \quad (2)$$

4.1 Realizability of Improvement Paths

Following Section 3, we unwind further the richness and complexity of our games motivated by the study of their *improvement paths*. We establish that one of the main structural properties of social networks, their degree distribution, does not play a role on its own to the existence of equilibria. This fact motivates the search for other important parameters related to existence, which is the topic of the next subsections.

An **improvement path** is any sequence $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j, \ldots)$ of strategy profiles $\mathbf{x}_j \in \mathcal{S}^2$ such that for every j the strategy profiles $\mathbf{x}_j$ and $\mathbf{x}_{j+1}$ differ in exactly one coordinate, say the $i(j)$-th, i.e., only player $i(j)$ has switched to another strategy, and also $u_{i(j)}(\mathbf{x}_j) < u_{i(j)}(\mathbf{x}_{j+1})$, $\forall j \geq 1$. It is called a **best response path** if also $u_{i(j)}(\mathbf{x}_{j+1}) = max_{x \in \mathcal{S}} u_{i(j)}(x, (\mathbf{x}_{j+1})_{-i(j)})$. We can also define **improvement cycles** in a similar fashion.

A well-known sufficient condition for existence of PNE is the *Finite Improvement Property* (FIP), saying that all improvement paths are finite [16]. In our case, the FIP does not hold, but in order to find conditions for the existence of PNE, one could still try to understand how do cycles occur. For example, do the cycles have some particular form? Does the degree distribution affect the formation of cycles? We obtain a negative result in this direction, showing that essentially in any given graph, any possible set of cycles may be realized, independent of its structure.

We proceed with some more definitions. Given a finite 2-player game Γ, played on a $r \times r$ matrix, let $P(\Gamma)$ denote the set of all improvement paths (including infinite ones) that are induced by the game starting from any entry in the matrix. Let P denote any possible set of *consistent* improvement paths (including cycles) that can be created on a $r \times r$ matrix. By a consistent set we mean that if, e.g., there is a path with a move from entry (i, j) of the matrix to (i, l), then there cannot be another path in P that contains a move from (i, l) to (i, j). We say that P is **realizable** if there is a game Γ such that $P = P(\Gamma)$. We show that any such set P is realizable by the family of our games, essentially by any graph. Hence *all* possible dynamics can be captured by these games.

To prove our claim, we will argue about an appropriate submatrix of the games we construct, since some strategy profiles may need to be eliminated due to domination. Particularly, we need the following form of domination.

Definition 5. *Given a 2-player game, assume that $\mathcal{S} = X \cup Y$, where $X \cap Y = \emptyset$. We say that X is a* ***sink*** *in $\mathcal{S}$, if* at least one *of the following holds:*

i. $\forall (a, b) \in Y \times (Y \cup X)$, $\exists x \in X$ *such that* $u_1(x, b) > u_1(a, b)$, *and* $\forall (a, b) \in X \times Y$, $\exists x \in X$ *such that* $u_2(a, x) > u_2(a, b)$.
ii. $\forall (a, b) \in (Y \cup X) \times Y$, $\exists y \in X$ *such that* $u_2(a, y) > u_2(a, b)$, *and* $\forall (a, b) \in Y \times X$, $\exists x \in X$ *such that* $u_1(x, b) > u_1(a, b)$.

The definition says that any improvement path that is not a cycle, starting from the Y-region of the matrix, will eventually come to the X-region.

Given a 2-player game, we let $\mathcal{S}_{\mathcal{D}}$ denote a minimal sink in $\mathcal{S}$, and $\Pi(\mathcal{S}_{\mathcal{D}}, \mathcal{S}_{\mathcal{D}})$ be the restriction of the game matrix over this set of strategies. Furthermore, given a graph G, let P_{in} and P_{out} be the in and out-degree distributions of G, i.e., $P_{in}(i)$ is the number of nodes with in-degree equal to i. We can now state the following theorem.

Theorem 2. *Consider a graph $G' = (V, E)$ with in and out-degree distributions, P_{in}, and P_{out}. There exists a class of games $((G \in \mathcal{F}, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M}, k)$, where $\mathcal{F}$ is a family of graphs with the same set of nodes as G', such that:*

i. each $G \in \mathcal{F}$ has degree distributions P^G_{in}, P^G_{out} such that for all i, $|P_{in}(i) - P^G_{in}(i)|/|V| \to 0$, as $|V| \to \infty$, and the same holds for P_{out} and P^G_{out}.
ii. For any $r \geq 3$, all sets of consistent improvement paths (including cycles) formed on a $r \times r$ matrix are realizable over the games played on $\mathcal{F}$ in $\Pi(\mathcal{S}_{\mathcal{D}}, \mathcal{S}_{\mathcal{D}})$, where $\mathcal{S}_{\mathcal{D}}$ (for each $G \in \mathcal{F}$) is a minimal sink with $|\mathcal{S}_{\mathcal{D}}| = r$.

This result discloses the richness of our games, but above all it severely mitigates the role of the widely studied degree distribution of networks to the stability of the involved games. Hence, we advocate, in the following, that one needs to take into account the effects of other properties as well.

4.2 Conditions for the Existence of a PNE

In this subsection, we use the notion of ordinal potentials to argue about existence of PNE. A function $P : \mathcal{S}^2 \mapsto \mathbb{R}$ is a **generalized ordinal potential** [16] (**GOP**) for a game Γ if $\forall i \in \mathcal{M}$, $\forall \mathbf{s_{-i}} \in \mathcal{S}^{m-1}$, and $\forall x, z \in \mathcal{S}$,

$$u_i(x, \mathbf{s_{-i}}) > u_i(z, \mathbf{s_{-i}}) \Rightarrow P(x, \mathbf{s_{-i}}) > P(z, \mathbf{s_{-i}}).$$

If Γ admits a GOP and is also finite (as our games are), the FIP property holds (see Section 4.1), and all improvement paths terminate at a PNE [16]. On the other hand, in our games the existence of PNE is not equivalent with the FIP property; we can construct games that possess PNE, but do not admit a GOP. We omit these due to lack of space. Instead, we continue with a set of necessary conditions for the existence of a GOP. To this end, we shall say that a set X of nodes is **reachable** from a strategy S if and only if $X \subseteq H_S$, where H_S is the ideal spread of S.

Lemma 1. *The game* $((G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2}), \mathcal{M}, k)$ *cannot admit a generalized ordinal potential if*

i. $\exists (S_1, S_2) \in \mathcal{S}^2$, $S_1 \neq S_2$, *such that* S_1 *is reachable from* S_2, *and* S_2 *is reachable from* S_1.
ii. $\exists (S_1, S_2) \in \mathcal{S}^2$, $S_1 \neq S_2$, *such that* $|H_{S_1}| = |H_{S_2}|$, *and* S_1 *is reachable from* S_2, *or* S_2 *is reachable from* S_1.

The Lemma suggests that many classes of our games may not admit a GOP — in the next subsection we shall approximate how close to admitting a GOP these games are. The fact is elucidated further through the following corollary, where we assume $k = 1$, i.e., as in [1], each player has to pick a single node, therefore, the only reasonable strategies are the nodes u for which there is at least one edge (u, v) such that $w_{uv} \geq \theta_v$.

Corollary 1. *If the game* $((G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M}, k = 1)$ *admits a generalized ordinal potential, then*

i. G *contains a DAG that includes the set* $\{u | \exists v \in V, \text{such that } w_{uv} \geq \theta_v\}$.
ii. *If* $w_{uv} \geq \theta_v$ *for every edge* $(u, v) \in E$, *then* G *has to be a DAG.*

Corollary 1 shows that for the case of $k = 1$, the conditions on the ideal spreads implied by Lemma 1, enforce the graph to have the special structure of a DAG. But clearly not all DAGs admit a GOP as has been demonstrated in Example 1.

We now move on to derive a sufficient condition for the existence of a GOP.

Theorem 3. *Consider a game* $((G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2}), \mathcal{M}, k)$, *and suppose that we order the set of the available strategies so that* $|H_{S_1}| \geq \ldots \geq |H_{S_{|\mathcal{S}|}}|$. *If for all* $i \in \{1, \ldots, |\mathcal{S}| - 1\}$ *it holds that*

$$|H_{S_{i+1}}| \leq \left\lfloor \frac{|H_{S_i}| + max\{\gamma_1(S_i, S_{i+1}), \gamma_2(S_i, S_{i+1})\}}{2} \right\rfloor \quad (3)$$

then the game admits a generalized ordinal potential. Moreover, all its PNE have the form (S_{max}, S_2)*, where* $S_{max} \equiv argmax_{S \in \mathcal{S}} \{|H_S|\}$.

For an interpretation of Theorem 3, consider a game where the max term is zero in (3). Then, a GOP exists if all ideal spreads are well separated, and Player 2 can never hope to take more than half of the nodes that Player 1 would get ideally (we refer the reader to the introductory example for this theorem in our full version).

The condition of Theorem 3 can be relaxed so that not all ideal spreads need to be well separated, e.g., in certain cases where there is no overlap between the ideal spreads of some strategies. For example, when G is a full and complete d-ary tree, $d \geq 2$, then (3) does not hold but using similar arguments as in the proof of Theorem 3 we have:

Corollary 2. *The games of the form* $((G, \mathtt{LIS} = \mathtt{LTM}(w_{uv} \geq \theta_v, \forall (u,v) \in E), \mathtt{TBC1} = R^{\prec}, \mathtt{TBC2} = R^{\prec}), \mathcal{M}, k = 1)$*, where* G *is a full and complete* d*-ary tree, admit a GOP.*

4.3 Quantifying Instability

The previous sections on existence and complexity motivate our next discussion on approximate PNE. Overall, the main conclusion of this subsection is that even though PNE do not always exist, we do have in certain cases approximate equilibria with a good quality of approximation, and we can also compute them in polynomial time.

A strategy profile s is an **ϵ-PNE**, if no agent can benefit more than ϵ by unilaterally deviating to a different strategy, i.e., for every $i \in \mathcal{M}$, and $S'_i \in \mathcal{S}$ it holds that $u_i(S'_i, \mathbf{s_{-i}}) \leq u_i(\mathbf{s}) + \epsilon$. Recall that in our case, utilities are integers in $\{0, ..., |V|\}$, and ϵ also takes integer values[1]. Additionally, a function $P : \mathcal{S}^2 \mapsto \mathbb{R}$ is an **ϵ-generalized ordinal potential** (**ϵ-GOP**) for a game Γ (see [7]) if $\forall i \in \mathcal{M}$, $\forall \mathbf{s_{-i}} \in \mathcal{S}^2$, $\forall x, z \in \mathcal{S}$, $u_i(x, \mathbf{s_{-i}}) > u_i(z, \mathbf{s_{-i}}) + \epsilon \Rightarrow P(x, \mathbf{s_{-i}}) > P(z, \mathbf{s_{-i}})$. Such a function P yields directly the existence of ϵ-PNE. We first obtain such a potential function for games that have diffusion depth $D = 1$, based on the ideal spread of the players' strategies and on the quantification of the utility functions in the beginning of Section 4 (Definition 4).

Theorem 4. *Any game* $\Gamma = ((G, \mathtt{LIS}, \mathtt{TBC1} = R^{\prec}, \mathtt{TBC2} = R^{\prec}), \mathcal{M}, k)$*, where* $D(\Gamma) = 1$*, admits the function* $P(\mathbf{s}) = (1+\beta_{max}+\gamma_{max})|H_{S_1}|+|H_{S_2}|-\beta_2(\mathbf{s})-\gamma_2(\mathbf{s})$*, as a* k*-GOP. Moreover, a* k*-PNE can be computed in polynomial time.*

The last assertion of Theorem 4 is easy to see since the value of the function $P(\mathbf{s})$ is at most $O(|V|^2)$. Therefore, by following an improvement path (with improvements of more than k), we can find an approximate PNE quite efficiently.

Note that this holds for any local interaction scheme, and not just the linear threshold model. Theorem 4 implies that when $D(\Gamma) = 1$ and k is small, we can have a good quality of approximation. E.g., for $k = O(1)$, or $k = o(|V|)$, and as $|V| \to \infty$, we can have approximate equilibria where any node can additionally gain only a negligible fraction of the graph by deviating.

[1] We could normalize the utilities by dividing by $|V|$, and then ϵ would take values in the set $\{1/|V|, 2/|V|, ..., 1\}$. We present the theorems without the normalization so as to be consistent with all other sections.

For games with higher diffusion depth, we define below an important parameter that captures the quality of approximation we can achieve in worst case via ϵ-GOP.

Definition 6. *i. Given a 2-player game, and two strategy profiles* $\mathbf{s} = (S_1, S_2), \mathbf{s}' = (S_1', S_2)$*, the* ***diffusion collision factor*** *of player* 1 *for strategy* S_1' *compared to* S_1*, given* S_2*, is defined as* $DC_1(S_1', S_1|S_2) \equiv (\alpha_1(\mathbf{s}') + \gamma_1(\mathbf{s}')) - (\alpha_1(\mathbf{s}) + \gamma_1(\mathbf{s}))$.

ii. Similarly, for $\mathbf{s} = (S_1, S_2), \mathbf{s}' = (S_1, S_2')$*, the diffusion collision factor of Player 2 for* S_2' *compared to* S_2*, given* S_1*, is defined as* $DC_2(S_2', S_2|S_1) \equiv (\alpha_2(\mathbf{s}') + \gamma_2(\mathbf{s}')) - (\alpha_2(\mathbf{s}) + \gamma_2(\mathbf{s}))$.

In order to understand this new notion, recall from Equation (1) that, given a profile $\mathbf{s}$, $\alpha_1(\mathbf{s}) + \gamma_1(\mathbf{s})$ denotes the number of nodes that Player 1 does not infect due to the presence of Player 2 in the market; this fact directly yields some intuition for the definition of DC_1. This is not exactly the case for DC_2, as the β_2-term is missing (see Eq. (2)); nonetheless, it turns out that it suffices to define DC_2 in a uniform manner as DC_1, when using $R^{\prec}$ for ties. Finally, we set DC_{max} to be the maximum possible diffusion collision factor.

Theorem 5. *Any game* $\Gamma = ((G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M}, k)$*, where* $D(\Gamma) \geq 2$*, admits the function* $P(\mathbf{s}) = x_1|H_{S_1}| + |H_{S_2}| - \beta_2(\mathbf{s})$*, as a* DC_{max}*-GOP, where* x_1 *is any number satisfying* $x_1 > \beta_{max}$*. Moreover, a* DC_{max}*-PNE can be computed in polynomial time.*

The approximations of k and DC_{max} are *tight* for `LIS=LTM`, and we provide the corresponding examples in our full version.

5 Quantifying Inefficiency

5.1 Price of Anarchy and Stability

Given an m-player game, and a strategy profile $\mathbf{s}$, the sum $SW(\mathbf{s}) = \sum_{j=1}^{m} u_j(\mathbf{s})$ is the **social welfare** of $\mathbf{s}$. The *Price of Anarchy* (PoA), for a family of games, is the worst possible ratio of $SW(\mathbf{s})/SW(\mathbf{s}')$, where $\mathbf{s}$ is a social optimum, and $\mathbf{s}'$ is a Nash equilibrium. Similarly, the *Price of Stability* (PoS) is defined as the best such ratio.

Suppose now that $|V|$ is sufficiently large, so that players will never play overlapping strategies at a PNE, e.g., this is ensured if $|V| \geq mk$. In that case we would have $1 \leq PoA \leq |V|/(mk)$. The question of interest then is whether PoA can be much lower than this upper bound.

The following theorem exhibits that for diffusion depths greater than one, competition can severely hurt social welfare. This can be detrimental both to the firms, and the network users, since it implies that in worst case the firms will have a very low utility, and the service offered by these competing products will reach only a small fraction of the nodes. This is in agreement with the worst case scenario in the model of [10][2]. On the contrary, this is not always the case when the diffusion depth is one.

[2] In the stochastic process of [10], PoA can be very high when their so-called switching function is not concave.

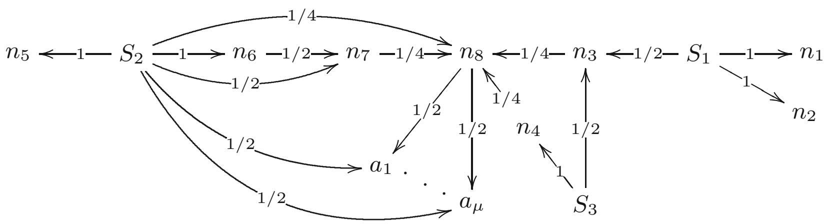

Fig. 2. The network for the proof of Theorem 7(ii): All nodes have threshold 1, except of node n_3 that has $\theta_{n_3} = 1/2$, and node n_8 that has $\theta_{n_8} = 1/2$

Theorem 6. *i. For the family of games* $((G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M}, k)$*,* $PoA = |V|/(mk)$*, and* $PoS \geq \frac{k}{k+1}\frac{|V|}{mk}$*, even for* $D = 2$*.*

ii. For the family of games $((G, \texttt{LIS} = \texttt{LTM}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2} = R^{\prec}), \mathcal{M} = \{1, 2\}, k = 1)$*, with* $D = 1$*, we have* $PoS = 1$*, and* $PoA \leq SW(\mathbf{s})/(SW(\mathbf{s}) - 1)$*, where* $\mathbf{s}$ *is a social optimum. Moreover, if there exist at least two nodes with nonzero out-degree, then* $PoA = 1$*.*

The negative effect of competition on the players' utilities is further illustrated in the next subsection from the perspective of the best quality player.

5.2 Worst-Case Scenarios for the Best Quality Player

We end our presentation with identifying a different form of inefficiency for PNE, which arises from the following question: Consider games with the reputation ordering $R^{\prec}$ as the tie-breaker. Does the firm with the best quality product ensure the maximum spread among all the players at any PNE? Theorem 7 illustrates that this may not always be the case for games with at least three players (but it is so for 2-player games). In fact, the payoff of the best quality player may be arbitrarily lower than the player with the highest market share at a PNE. We consider this as a form of inefficiency since in a socially desirable outcome, one would expect that the product with the best quality/reputation should have the largest market share. This surprising result dictates the necessity for quantifying such effects in PNE.

Theorem 7. *Consider the class of games* $((G, \texttt{LIS}, \texttt{TBC1} = R^{\prec}, \texttt{TBC2}), \mathcal{M}, k)$*.*

i. If $m = 2$*, then for all PNE* $\mathbf{s}$*, it is* $u_1(\mathbf{s}) \geq u_2(\mathbf{s})$*.*

ii. If $m \geq 3$*,* $\texttt{LIS} = \texttt{LTM}$*, and* $\texttt{TBC2} = R^{\prec}$*, then a game exists with a PNE* $\mathbf{s}$ *such that* $u_i(\mathbf{s}) < u_j(\mathbf{s})$*, although* $i \succ j$ *with regard to* $R^{\prec}$*.*

Proof. i. Assume that a PNE $\mathbf{s} = (S_1, S_2)$ exists such that $u_1(\mathbf{s}) < u_2(\mathbf{s})$. Then, Player 1 can deviate to $S_1' = S_2$, and obtain utility $u_1(S_2, S_2) \geq u_2(\mathbf{s}) > u_1(\mathbf{s})$. Thus, $\mathbf{s}$ cannot be a PNE.

ii. Note that $R^{\prec} = 1 \succ 2 \succ 3$, and consider the social network in Figure 2. As n_i, $\forall i \in \{1, \ldots, 8\}$, and as a_i, $\forall i \in \{1, \ldots, \mu\}$, where $\mu > k$, we denote single nodes.

We assume that all of them have threshold 1, except of nodes n_3, and n_8 that have $\theta_{n_3} = 1/2$, and $\theta_{n_8} = 1/2$. As S_i, $\forall i \in \{1, 2, 3\}$, we denote sets of k nodes. Finally, the edges between single nodes are annotated with their corresponding weight. On the other hand, the edges that emanate from a set S_i are annotated with the *accumulated* corresponding weight of the underlying edges between each of the nodes in S_i and the involved end-node (e.g., $\forall v \in S_1$, it is $w_{vn_3} = \theta_{n_3}/k$).
One can now verify that the profile $\mathbf{s} \equiv (S_1, S_2, S_3)$ constitutes a PNE, even though it is $u_2(\mathbf{s}) = k + \mu + 4$, $u_1(\mathbf{s}) = k + 3$, and $u_3(\mathbf{s}) = k + 1$ — i.e., $u_2(\mathbf{s}) > u_1(\mathbf{s}) > u_3(\mathbf{s})$. We omit the details.

In the network of Figure 2, observe that at the PNE $\mathbf{s} = (S_1, S_2, S_3)$, $u_2(\mathbf{s}) = k + \mu + 4 > u_1(\mathbf{s}) + u_3(\mathbf{s}) = 2k + 4$, since $\mu > k$ — note that μ can be arbitrarily large. Thereby, if firm 1 is affiliated with firm 3, while their products are marketed as competing and incompatible (e.g. airline merges), firm 1 is incentivized to withdraw firm 3 from the game: the resulting 2-player game between firm 1 and firm 2, has a unique PNE, namely (S_2, S_1), in which firm 1 achieves the maximum possible utility — $u_1(S_2, S_1) = k + \mu + 4$. Moreover, notice that in this 2-player game, firm 1 initiates only k nodes to achieve this utility. On the other hand, in the original 3-player game, firms 1 and 3 initiate k nodes *each*, and still they achieve a lower sum of utilities at $\mathbf{s}$.

Our discussion indicates the necessity to capture the motivation of a player to either merge with other players, or to divide itself to several new ones that, although affiliated, they are still non-cooperative within the induced game. For example, given the network of Figure 2 Player 1 faces the question: Should I play alone against the others, since I am the best firm, or should I merge even with the weakest? We believe this aspect of PNE is worth further investigation and we leave it as an open direction for future work.

6 Conclusions and Future Work

We have studied a competitive diffusion process from a non-cooperative game-theoretic viewpoint. We have investigated several aspects related to the stability of such games and we have unveiled some important parameters that have met no previous investigation. We believe that our work motivates primarily further empirical research on social networks with regard to the following questions: Can we identify a range of typical values for decisive structural features such as the diffusion depth, the ideal spread, and the maximum diffusion collision factor? This could quantify the instability of the induced games, in light of Theorems 4 and 5, as well as the results in Section 4.2.

Other interesting questions have to do with resolving some of the remaining open problems from our work. It is still open if the complexity of determining that a PNE exists is Σ_2^p-complete, or not. The Price of Anarchy is also not yet completely determined when $D = 1$ and k is arbitrary. Finally, additional compelling questions may concern the robustness to network changes.

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Agent Failures in Totally Balanced Games and Convex Games

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Abstract. We examine the impact of independent agents failures on the solutions of cooperative games, focusing on totally balanced games and the more specific subclass of convex games. We follow the reliability extension model, recently proposed in [1] and show that a (approximately) totally balanced (or convex) game remains (approximately) totally balanced (or convex) when independent agent failures are introduced or when the failure probabilities increase. One implication of these results is that any reliability extension of a totally balanced game has a non-empty core. We propose an algorithm to compute such a core imputation with high probability. We conclude by outlining the effect of failures on non-emptiness of the core in cooperative games, especially in totally balanced games and simple games, thereby extending observations in [1].

Keywords: Totally Balanced Games, Convex Games, Agent Failures, Cooperative Game Theory.

1 Introduction

Consider a communication network designed to transmit information from a source node to a target node, where selfish agents control the different links in the network. Suppose the utility generated by the network is proportional to the bandwidth it can achieve between the source and the target. Further suppose that the links are not fully reliable and may fail, and that these link failures are independent of each other, although the failure probability of each link may be different. The surviving links provide a certain bandwidth from the source to the target. Since it is not known a priori which links would fail, there is uncertainty regarding the revenue that the agents would generate.

In such a setting the agents owning the links typically need each other in order to generate revenue, but since they are selfish each of them attempts to maximize his own share of the revenue. Which agreements are these agents likely to make regarding sharing the revenue? How do the link failures affect the agents' ability to reach a stable agreement regarding distributing the gains amongst themselves? Can we compute such stable payment allocations?

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Interactions between selfish agents who must cooperate to achieve their goals are analyzed using cooperative game theory [18,7], where solution concepts attempt to characterize how such agents might agree to act and share the resulting gains among themselves. A prominent such concept is the core [11] which requires that no sub-coalition could defect and improve its utility by operating on its own.

Shapley and Shubik introduced the class of *totally balanced* games and showed that such games have a non-empty core [24]. Many interesting and practical classes of games have been shown to be totally balanced, such as the network flow game [13], Owen's linear production game [19], the permutation game [27], the assignment game [25], the minimum cost spanning tree game [12], etc. Our motivating example is a network flow game with independent agent failures.

Despite the wide coverage of cooperative interactions, most models ignore failures although it is hardly realistic to assume that all agents can always fill their roles. We use the reliability extension model [1] which formalizes independent agent failures in cooperative games, to investigate such failures in totally balanced games and in the more specific subclass of convex games.

Our Contribution: We first study the reliability extension of general cooperative games. We show how a game is transformed when failures are introduced or when the reliabilities (probabilities of agents not failing) change. Next we introduce the class of ϵ-totally balanced games, a natural generalization of totally balanced games, and investigate the effect of failures on such games. We prove that every reliability extension of an ϵ-totally balanced game is also ϵ-totally balanced. For $\epsilon = 0$, this implies that every reliability extension of a totally balanced game is also totally balanced. Further, we show that decreasing one or more reliabilities in an ϵ-totally balanced game keeps it ϵ-totally balanced.

Using Shapley's result that convex games are totally balanced [23], our results imply that every reliability extension of a convex game is also totally balanced. This strengthens a result by Bachrach et. al. [1] who prove that every reliability extension of a convex game has a non-empty core. Similarly to ϵ-totally balanced games, we also introduce ϵ-convex games and prove that every reliability extension of an ϵ-convex game is also ϵ-convex. For $\epsilon = 0$, this implies that every reliability extension of a convex game is not just totally balanced, but convex. Further, we show that decreasing reliabilities in an ϵ-convex game keeps it ϵ-convex. We then prove that any $\epsilon/(n-1)$-convex game is ϵ-totally balanced, generalizing Shapley's result that convex games are totally balanced. Additionally, we examine the computational aspects of a game's reliability extension, and provide an algorithm that computes a core solution of any reliability extension of a totally balanced game with high probability.

Our results show that both introducing failures and increasing failure probabilities preserve core non-emptiness in totally balanced games. We point out that neither of these preserve core non-emptiness in general cooperative games. Bachrach et. al. [1] observe that introducing failures preserves non-emptiness of the core in simple games (where every coalition has value either 0 or 1). Surprisingly, we show that this is not the case for increasing failure probabilities.

2 Related Work

Shapley and Shubik [24] introduced the notion of totally balanced games and proved their equivalence to the class of market games. Kalai and Zemel [13] later proved that they are also equivalent to two other classes of games: finite collections of simple additive games and network flow games. Owen [19] and Tijs et. al. [27] introduced two practical classes of totally balanced games: respectively, linear production games arising from linear programming problems and permutation games arising in sequencing and assignment problems. Deng et. al. [10] extended the analysis of total balancedness to various combinatorial optimization games, partition games and packing and convering games. These results suggest that the class of totally balanced games is elementary and practical.

Our analysis follows the reliability extension model of Bachrach et. al. [1] to examine the impact of independent agent failures in totally balanced games. A somewhat reminiscent model was proposed by Chalkiadakis et. al. [8] in which they consider the problem of coalition formation in a Bayesian setting. In their model, agents have types which are private information and agents have beliefs about the types of the other agents. In our setting, the failure probabilities can be viewed as types, but we focus on the specific case when these failure probabilities are public information and the failures are independent. Agent failures have also been widely studied in non-cooperative game theory. For example, Penn et. al. [20] study independent agent failures in congestion games, which are non-cooperative normal form games. Such failures have also been studied in other fields such as reliable network formation [4], non-cooperative Nash networks [5], sensor networks [14] etc. It is quite surprising that such an elementary notion of failure was only recently formalized in cooperative games.

3 Preliminaries

A transferable utility cooperative game $G = (N, v)$ is composed of a set of agents $N = \{1, 2, \ldots, n\}$ and a characteristic function $v : 2^N \to \mathbb{R}$ indicating the total utilities achievable by various coalitions (subsets of agents). By convention, $v(\emptyset) = 0$. For any agent $i \in N$ and coalition $S \subseteq N$, we denote $S \cup \{i\}$ by $S + i$ and $S \setminus \{i\}$ by $S - i$. For a game $G = (N, v)$ and coalition $S \subseteq N$, G_S denotes the subgame of G obtained by restricting the set of agents to S.

Convex Games: A characteristic function is called *supermodular* if for each $i \in N$ and for all S and T such that $S \subseteq T \subseteq N - i$, we have $v(S + i) - v(S) \leq v(T + i) - v(T)$ (i.e., increasing marginal returns). A game is called *convex* if its characteristic function is supermodular. Similarly, for any $\epsilon \geq 0$, a characteristic function is called ϵ-supermodular if for each $i \in N$ and for all S and T such that $S \subseteq T \subseteq N - i$, we have $v(S + i) - v(S) \leq v(T + i) - v(T) + \epsilon$. Define a game to be ϵ-convex if its characteristic function is ϵ-supermodular. Note that convex games are recovered as the special case of $\epsilon = 0$.

Imputation: The characteristic function defines the value that a coalition can achieve on its own, but not how it should *distribute* the value among its members. A payment vector $\boldsymbol{p} = (p_1, \dots, p_n)$ is called a *pre-imputation* if $\sum_{i=1}^{n} p_i = v(N)$. A payment vector $\boldsymbol{p} = (p_1, \dots, p_n)$ is called an imputation if it is a pre-imputation and also individually rational, i.e., $p_i \geq v(\{i\})$ for every $i \in N$. Here, p_i is the payoff of agent i, and the payoff of a coalition C is $p(C) = \sum_{i \in C} p_i$.

Core: A basic requirement for any good imputation is that the payoff to every coalition is at least as much it can gain on its own so that no coalition can gain by defecting. The *core* is the set of all imputations $\boldsymbol{p}$ such that $p(N) = v(N)$ and $p(S) \geq v(S)$ for all $S \subseteq N$. It may be empty or may contain more than one imputation. One closely related concept is that of ϵ-core. For any $\epsilon \in \mathbb{R}$, the ϵ-core is the set of all imputations $\boldsymbol{p}$ such that $p(N) = v(N)$ and for every $S \subseteq N$ such that $S \neq N$, we have $p(S) \geq v(S) - \epsilon$. When $\epsilon > 0$, it serves as a relaxation of the core and is useful in predicting behaviour in games where the core is empty. When $\epsilon < 0$, it serves as a stronger concept where every coalition requires at least an incentive of $|\epsilon|$ to defect. We denote the case of $\epsilon > 0$ as the approximate core and the case of $\epsilon < 0$ as the superstable core. For any game, it is easy to show that the set $\{\epsilon \mid \text{the } \epsilon\text{-core is non-empty}\}$ is compact and thus has a minimum element $\epsilon_{\min}$. The $\epsilon_{\min}$ is known as the *least core value* of the game and the $\epsilon_{\min}$-core is known as the *least core*.

Total Balancedness: As defined by Shapley and Shubik [24], a game is called *totally balanced* if every subgame of the game has a non-empty core. We define a natural generalization of totally balanced games. For any $\epsilon \geq 0$, a game is called *ϵ-totally balanced* if every subgame of the game has non-empty ϵ-core.

Reliability Game: As defined in [1], a *reliability game* $G = (N, v, \boldsymbol{r})$ consists of the set of agents $N = \{1, 2, \dots, n\}$, the *base characteristic function* v which describes the values of the coalitions in the absence of failures, and the reliability vector $\boldsymbol{r}$ where r_i is the probability of agent i not failing. After taking failures into account, the characteristic function of the reliability game, denoted by $v^{\boldsymbol{r}}$, is given by the following equation. For every coalition $S \subseteq N$,

$$v^{\boldsymbol{r}}(S) = \sum_{S' \subseteq S} \Pr(S'|S) \cdot v(S') = \sum_{S' \subseteq S} \left(\prod_{i \in S'} r_i \cdot \prod_{j \in S \setminus S'} (1 - r_j) \right) \cdot v(S'). \tag{1}$$

Here, $\Pr(S'|S)$ denotes the probability that every agent in S' survives and every agent in $S \setminus S'$ fails so $v^{\boldsymbol{r}}(S)$ is the expected utility S achieves under failures. The set S' is called the *survivor set* for the coalition S. For the *base game* $G = (N, v)$, the game $G^{\boldsymbol{r}} = (N, v, \boldsymbol{r})$ is called the *reliability extension* of G with the reliability vector $\boldsymbol{r}$. For a reliability vector $\boldsymbol{r}$, we denote by $\boldsymbol{r_{-i}}$ the vector of reliabilities of all agents except i and by $\boldsymbol{r'} = (p, \boldsymbol{r_{-i}})$ the reliability vector where $r'_i = p$ and $r'_j = r_j$ for $j \neq i$. For vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, define $\boldsymbol{x} \leq \boldsymbol{y}$ if $x_i \leq y_i$ for every i.

4 Reliabilities, Total Balancedness and Convexity

We examine how the value of a coalition changes as the reliability of an agent changes in a general game.

Lemma 1. *Let $G = (N, v, \boldsymbol{r})$ be a reliability game. Let $i \in N$ be an agent and let $p = r_i > 0$ be the reliability of agent i in G. Take $0 \leq p' \leq 1$ and let $G' = (N, v, \boldsymbol{r'})$ where $\boldsymbol{r'} = (p', \boldsymbol{r_{-i}})$. Let $v^{\boldsymbol{r}}$ and $v^{\boldsymbol{r'}}$ be the characteristic functions of G and G' respectively. Then the following holds.*[1]

1. *For any coalition $S \subseteq N$ such that $i \notin S$, we have $v^{\boldsymbol{r'}}(S) = v^{\boldsymbol{r}}(S)$.*
2. *For any coalition $S \subseteq N$ such that $i \in S$, we have*

$$v^{\boldsymbol{r'}}(S) = \frac{p'}{p} \cdot v^{\boldsymbol{r}}(S) + \left(1 - \frac{p'}{p}\right) \cdot v^{\boldsymbol{r}}(S - i).$$

Proof Sketch. Part 1 of the proof is trivial and follows directly from Equation (1). For the second part, for any coalition $S \subseteq N$ such that $i \in S$, we define $v_i^{rel}(S)$ as the value of S in the game $G_i^{rel} = (N, v, \boldsymbol{r_i^{rel}})$ where $\boldsymbol{r_i^{rel}} = (1, \boldsymbol{r_{-i}})$. Now we break the summation in Equation (1) into two parts: summation over the subsets containing i and summation over the subsets not containing i, and observe that $v^r(S) = p \cdot v_i^{rel}(S) + (1-p) \cdot v^{\boldsymbol{r}}(S - i)$ and similarly, $v^{\boldsymbol{r'}}(S) = p' \cdot v_i^{rel}(S) + (1 - p') \cdot v^{\boldsymbol{r'}}(S - i)$. Finally, observing that $v^{\boldsymbol{r'}}(S - i) = v^{\boldsymbol{r}}(S - i)$ (using part 1) and eliminating $v_i^{rel}(S)$ from the two equations, we get the desired result. ∎

The proof appears in the full version of the paper.[2] Note that by starting with $G = (N, v, \mathbf{1})$ where $\mathbf{1} = \langle 1, 1, \ldots, 1 \rangle$, we can use Lemma 1 to analyze the effect of introducing failures into a cooperative game as well.

4.1 Approximately Totally Balanced Games

We now analyze the reliability extension of ϵ-totally balanced games and prove that ϵ-total balancedness is preserved when the reliability of one agent decreases.

Theorem 1. *Let $\epsilon \geq 0$ and $G = (N, v, \boldsymbol{r})$ be a reliability game that is ϵ-totally balanced. Fix $i \in N$ and let $p = r_i > 0$ be the reliability of agent i in G. Take p' such that $0 \leq p' \leq p$ and define $G' = (N, v, \boldsymbol{r'})$ where $\boldsymbol{r'} = (p', \boldsymbol{r_{-i}})$. Then G' is ϵ-totally balanced.*

Proof. Let $v^{\boldsymbol{r}}$ and $v^{\boldsymbol{r'}}$ denote the characteristic functions of G and G' respectively. We want to prove that G' is ϵ-totally balanced, i.e., for any coalition

[1] The equation in part 2 of Lemma 1 really captures both parts 1 and 2. Part 1 is obtained by observing that $S - i = S$ when $i \notin S$. The two cases are separated for clarity and for the convenience of the reader.

[2] The full version is available from: `http://www.cs.cmu.edu/~nkshah/papers.html`.

$S \subseteq N$, the subgame G'_S has an ϵ-core imputation. Fix any coalition $S \subseteq N$. There are two cases: $i \in S$ and $i \notin S$.[3]

If $i \notin S$, then using Lemma 1 we see that $v^{\boldsymbol{r}'}(C) = v^{\boldsymbol{r}}(C)$ for every $C \subseteq S$, i.e., the subgames G'_S and G_S are equivalent. Further, since G is an ϵ-totally balanced game, its subgame G_S has an ϵ-core imputation $\boldsymbol{x}$. It is easy to see that $\boldsymbol{x}$ is also an ϵ-core imputation of G'_S.

Now let $i \in S$. Since G is an ϵ-totally balanced game, both its subgames G_S and G_{S-i} have ϵ-core imputations, say $\boldsymbol{x_S}$ and $\boldsymbol{x_{S-i}}$ (take $\boldsymbol{x_\phi} = \langle 0, 0, \ldots, 0\rangle$). Extend both vectors by setting the payments to agents in $N \setminus S$ (and payment to i in $\boldsymbol{x_{S-i}}$) to be zero. We prove that $\boldsymbol{x} = {}^{p'}\!/_{p} \cdot \boldsymbol{x_S} + (1 - {}^{p'}\!/_{p}) \cdot \boldsymbol{x_{S-i}}$ is an ϵ-core imputation of G'_S. First we show that $\boldsymbol{x}$ is a pre-imputation.

$$
\begin{aligned}
x(S) &= \frac{p'}{p} \cdot x_S(S) + \left[1 - \frac{p'}{p}\right] \cdot x_{S-i}(S) = \frac{p'}{p} \cdot x_S(S) + \left[1 - \frac{p'}{p}\right] \cdot x_{S-i}(S-i) \\
&= \frac{p'}{p} \cdot v^{\boldsymbol{r}}(S) + \left[1 - \frac{p'}{p}\right] \cdot v^{\boldsymbol{r}}(S-i) = v^{\boldsymbol{r}'}(S),
\end{aligned}
$$

Here, the second transition follows since the payment to agent i in $\boldsymbol{x_{S-i}}$ is 0, the third transition follows since $\boldsymbol{x_S}$ and $\boldsymbol{x_{S-i}}$ are ϵ-core imputations of G_S and G_{S-i} respectively and the last transition follows from Lemma 1.

Now for any coalition $C \subseteq S$ and $C \neq S$, we want to show that $x(C) \geq v^{\boldsymbol{r}'}(C) - \epsilon$. We again take two cases: $i \in C$ and $i \notin C$. Let $i \notin C$, i.e., $C \subseteq S - i$. Since $\boldsymbol{x_S}$ and $\boldsymbol{x_{S-i}}$ are ϵ-core imputations of G_S and G_{S-i} respectively, we have that $x_S(C) \geq v^{\boldsymbol{r}}(C) - \epsilon$ and $x_{S-i}(C) \geq v^{\boldsymbol{r}}(C) - \epsilon$. Therefore,

$$
x(C) = \frac{p'}{p} \cdot x_S(C) + \left[1 - \frac{p'}{p}\right] \cdot x_{S-i}(C) \geq \left[\frac{p'}{p} + 1 - \frac{p'}{p}\right] \cdot (v^{\boldsymbol{r}}(C) - \epsilon) = v^{\boldsymbol{r}'}(C) - \epsilon,
$$

where the second transition uses $p' \leq p$ and the third transition follows since $v^{\boldsymbol{r}'}(C) = v^{\boldsymbol{r}}(C)$ (part 1 of Lemma 1). Now let $i \in C$. Once again, we have that

$$
\begin{aligned}
x(C) &= \frac{p'}{p} \cdot x_S(C) + \left[1 - \frac{p'}{p}\right] \cdot x_{S-i}(C) = \frac{p'}{p} \cdot x_S(C) + \left[1 - \frac{p'}{p}\right] \cdot x_{S-i}(C-i) \\
&\geq \frac{p'}{p} \cdot (v^{\boldsymbol{r}}(C) - \epsilon) + \left[1 - \frac{p'}{p}\right] \cdot (v^{\boldsymbol{r}}(C-i) - \epsilon) = v^{\boldsymbol{r}'}(C) - \epsilon.
\end{aligned}
$$

The second transition follows since payment to agent i in $\boldsymbol{x_{S-i}}$ is 0. The third transition follows since $\boldsymbol{x_S}$ and $\boldsymbol{x_{S-i}}$ are ϵ-core imputations of G_S and G_{S-i} respectively and since $p' \leq p$. The last transition follows due to part 2 of Lemma 1.

Hence, we proved that $x(S) = v^{\boldsymbol{r}'}(S)$ and $x(C) \geq v^{\boldsymbol{r}'}(C) - \epsilon$ for every coalition $C \subseteq S$ where $C \neq S$. This proves that $\boldsymbol{x}$ is an ϵ-core imputation of G'_S. Since $S \subseteq N$ was selected arbitrarily, we have proved that every subgame of G' has non-empty ϵ-core, i.e., G' is ϵ-totally balanced. ∎

[3] It is possible to combine all the cases in the proof of Theorem 1 using Footnote 1. However, a case-wise analysis is presented to avoid any confusion.

Since ϵ-total balancedness is preserved when we decrease a single reliability, we can decrease multiple reliabilities one-by-one and repeatedly apply Theorem 1 to show ϵ-total balancedness in the resulting game, so we obtain the following.

Corollary 1. *Let $\epsilon \geq 0$. Let $G = (N, v, \boldsymbol{r})$ be an ϵ-totally balanced reliability game and $G' = (N, v, \boldsymbol{r'})$ where $\boldsymbol{r'} \leq \boldsymbol{r}$. Then G' is ϵ-totally balanced.*

Any reliability extension of a game can be obtained by starting from the base game (equivalent to its reliability extension with the reliability vector **1**) and decreasing reliabilities as required. Hence Corollary 1 implies that reliability extensions preserve ϵ-total balancedness. Conversely if a game G is not ϵ-totally balanced, it has a subgame G_S, which is also a reliability extension with reliability 1 for $i \in S$ and 0 otherwise, having empty ϵ-core. This proves the following.

Corollary 2. *For any $\epsilon \geq 0$, a game is ϵ-totally balanced if and only if every reliability extension of the game is ϵ-totally balanced.*

Shapley [23] showed that convex games are totally balanced. Thus Corollary 2 implies that every reliability extension of a convex game is totally balanced. This strengthens a theorem by Bachrach et. al. [1] which states that every reliability extension of a convex game has a non-empty core.[4] We strengthen this further and prove that every reliability extension of a convex game is in fact convex.

4.2 Approximately Convex Games

For the reliability extension of ϵ-convex games, the results are parallel to those for ϵ-totally balanced games, but require different proof techniques.

Theorem 2. *Let $\epsilon \geq 0$. Let $G = (N, v, \boldsymbol{r})$ be an ϵ-convex reliability game. Fix $i \in N$ and let $p = r_i > 0$ be the reliability of agent i in G. Take p' such that $0 \leq p' \leq p$ and define $G' = (N, v, \boldsymbol{r'})$ where $\boldsymbol{r'} = (p', \boldsymbol{r_{-i}})$. Then G' is ϵ-convex.*

Proof Sketch. Let $v^{\boldsymbol{r}}$ and $v^{\boldsymbol{r'}}$ be the characteristic functions of G and G' respectively. We want to prove that G' is ϵ-convex, i.e., for every $j \in N$ and for all $S \subseteq T \subseteq N - j$, $v^{\boldsymbol{r'}}(S+j) - v^{\boldsymbol{r'}}(S) \leq v^{\boldsymbol{r'}}(T+j) - v^{\boldsymbol{r'}}(T) + \epsilon$. We know that this is true for $v^{\boldsymbol{r}}$ since G is ϵ-convex. We analyze the marginal contributions of j to S and T in both $v^{\boldsymbol{r}}$ and $v^{\boldsymbol{r'}}$ and apply ϵ-convexity of G and Lemma 1 (wherever required) in order to prove ϵ-convexity of G'. ■

The proof appears in the full version of the paper. Similarly to totally balanced games, Theorem 2 can be extended to cover general decreases in reliabilities, including those starting from the base game.

Corollary 3. *Let $\epsilon \geq 0$. Let $G = (N, v, \boldsymbol{r})$ be an ϵ-convex reliability game and $G' = (N, v, \boldsymbol{r'})$ where $\boldsymbol{r'} \leq \boldsymbol{r}$. Then G' is ϵ-convex.*

Corollary 4. *For any $\epsilon \geq 0$, a game is ϵ-convex if and only if every reliability extension of the game is ϵ-convex.*

[4] The converse part of Theorem 3 in [1] is technically incorrect and only holds when the game is not totally balanced, which is again generalized by our results.

4.3 Relation between convexity and total balancedness

Shapley [23] proved that convex games are totally balanced. In the above results, we deal with the notions of ϵ-convexity and ϵ-total balancedness. We now prove a relation between the two concepts for any $\epsilon \geq 0$, extending Shapley's result.

Theorem 3. *For any $\epsilon \geq 0$, an $\epsilon/(n-1)$-convex game with n agents is ϵ-totally balanced.*

The proof of this theorem is along the same lines as the proof of Shapley's result (see, e.g., [7]) and appears in the full version of the paper. We also show that the sufficient condition in Theorem 3 cannot be improved by a factor of more than $n-1$. The proof again appears in the full version of the paper.

Lemma 2. *For any $\epsilon \geq 0$, $\delta > 0$ and $n \in \mathbb{N}$, there exists a game with n agents which is $\epsilon + \delta$-convex but not ϵ-totally balanced.*

There are several implications of this relation. First, Corollary 4 showed that any reliability extension of an ϵ-convex game is ϵ-convex. Using Theorem 3, we can see that such an extension is also $\epsilon \cdot (n-1)$-totally balanced. Second, the core has been well studied in the literature. For simple games, the core is non-empty if and only if a veto agent exists. In general games, convexity serves as a sufficient condition for non-emptiness of the core. However, conditions for non-emptiness of the approximate core are relatively less studied. Theorem 3 provides such a sufficient condition in terms of approximate convexity.

5 Computing a Core Imputation

In Section 4, we proved that every reliability extension of a totally balanced game is totally balanced and thus has a non-empty core. However, the proof was non-constructive. For several classes of totally balanced games without failures, elegant LP based approaches exist to compute a core imputation in polynomial time. But computing a core imputation in the reliability extension may have a different computational complexity. For example, Bachrach et. al. [1] note that although computing a core imputation is easy in connectivity games on networks, even computing the value of a coalition (and hence computing a core imputation) becomes computationally hard in the reliability extension.

Nevertheless, we show that it is possible to compute a core imputation in any reliability extension of a totally balanced game with high probability. In this section, we use ϵ-core for both $\epsilon \geq 0$ (core/approximate core) and $\epsilon < 0$ (superstable core). In literature, the approximate core is well studied in cases where the core is empty. When the core is not empty, typically only the least core, which corresponds to the $\epsilon_{\min}$-core ($\epsilon_{\min} < 0$) is studied.

First, we show how to compute the core (or the ϵ-core) in a reliability extension of a general game in terms of the core (or the ϵ-core) of the subgames of the base game. The latter is known to be a tractable problem for many domains.

Theorem 4. *Let $\epsilon \in \mathbb{R}$. Let $G = (N, v)$ be an ϵ-totally balanced game and $G^{\boldsymbol{r}} = (N, v, \boldsymbol{r})$ be its reliability extension. For any coalition $S \subseteq N$, let $\boldsymbol{x_S}$ be an ϵ-core imputation of the subgame G_S. Define $\boldsymbol{x}^* = \sum_{S \subseteq N} \Pr(S|N) \cdot \boldsymbol{x_S}$, where $\Pr(S|N) = \prod_{i \in S} r_i \cdot \prod_{i \in N \setminus S} (1 - r_i)$. Then the following holds.*

1. *If $\epsilon \geq 0$, then $\boldsymbol{x}^*$ is an ϵ-core imputation of $G^{\boldsymbol{r}}$.*
2. *If $\epsilon < 0$, then $\boldsymbol{x}^*$ is an $r_{\min} \cdot \epsilon$-core imputation of $G^{\boldsymbol{r}}$, where $r_{\min} = \min_{i \in N} r_i$.*

Proof. For every coalition S, by definition we have that $x_S(S) = v(S)$ and $x_S(C) \geq v(C) - \epsilon$ for every $C \subseteq S$. Let $v^{\boldsymbol{r}}$ be the characteristic function of $G^{\boldsymbol{r}}$. First, we prove that $\boldsymbol{x}^*$ is a pre-imputation of $G^{\boldsymbol{r}}$.

$$x^*(N) = \sum_{S \subseteq N} \Pr(S|N) \cdot x_S(S) = \sum_{S \subseteq N} \Pr(S|N) \cdot v(S) = v^{\boldsymbol{r}}(N),$$

where the first transition follows since payment to agents in $N \setminus S$ is zero in $\boldsymbol{x_S}$, the second transition follows since $x_S(S) = v(S)$ and the last transition follows due to Equation (1). Now, fix any coalition $C \subseteq N$ where $C \neq N$. For any $C' \subseteq C$, for all $S \subseteq N$ such that $S \cap C = C'$, we have $x_S(C) = x_S(C') \geq v(C') - \epsilon$ except when $S = C'$ where we have $x_{C'}(C) = x_{C'}(C') = v(C')$. Now,

$$\begin{aligned}
x^*(C) &= \sum_{S \subseteq N} \Pr(S|N) \cdot x_S(C) = \sum_{C' \subseteq C} \left[\sum_{S \subseteq N \; s.t. \; S \cap C = C'} \Pr(S|N) \cdot x_S(C) \right] \\
&\geq \sum_{C' \subseteq C} \left[\left(\sum_{\substack{S \subseteq N \; s.t. \\ S \cap C = C', S \neq C'}} \Pr(S|N) \cdot (v(C') - \epsilon) \right) + \Pr(C'|N) \cdot v(C') \right] \\
&= \sum_{C' \subseteq C} \left[(v(C') - \epsilon) \cdot \left(\sum_{S \subseteq N \; s.t. \; S \cap C = C'} \Pr(S|N) \right) + \Pr(C'|N) \cdot \epsilon \right] \\
&= \sum_{C' \subseteq C} \left[(v(C') - \epsilon) \cdot \Pr(C'|C) + \Pr(C'|N) \cdot \epsilon \right] \\
&= \sum_{C' \subseteq C} \left[\Pr(C'|C) \cdot v(C') \right] - \epsilon \cdot \left[\sum_{C' \subseteq C} \Pr(C'|C) - \sum_{C' \subseteq C} \Pr(C'|N) \right] \\
&= v^{\boldsymbol{r}}(C) - \epsilon \cdot \left(1 - \prod_{i \in N \setminus C} (1 - r_i) \right),
\end{aligned}$$

where the fourth transition follows by adding and subtracting $\Pr(C'|N) \cdot \epsilon$ in the outer summation and rearranging terms and the last transition follows from Equation (1). Formal proofs for intuitive substitutions used in the fifth and the last transitions appear in the full version of the paper.

Using this and that $\boldsymbol{x}^*$ is a pre-imputation, we know that $\boldsymbol{x}^*$ is an ϵ'-core imputation of $G^{\boldsymbol{r}}$ if $\epsilon' \geq \epsilon \cdot \left(1 - \prod_{i \in N \setminus C} (1 - r_i)\right)$, for every $C \subseteq N$ such that

$C \neq N$. If $\epsilon \geq 0$, then we need to maximize $1 - \prod_{i \in N \setminus C}(1 - r_i)$ else we need to minimize it. A trivial upper bound is $1 - \prod_{i \in N}(1 - r_i) \leq 1$. We use the loose upper bound of 1. For a lower bound, note that since $C \neq N$, there exists $j \in N \setminus C$, hence $\prod_{i \in N \setminus C}(1 - r_i) \leq 1 - r_j \leq 1 - r_{\min}$. Thus $r_{\min}$ is a lower bound (which is also attained when $C = N - t$ where $r_t = r_{\min}$). This proves that $\boldsymbol{x^*}$ is an ϵ-core imputation of $G^{\boldsymbol{r}}$ if $\epsilon \geq 0$, and an $r_{\min} \cdot \epsilon$-core imputation if $\epsilon < 0$. ■

For a game $G = (N, v)$, define ϵ^* as the maximum least core value over all subgames of G. That is, $\epsilon^*(G) = \max_{S \subseteq N} \epsilon_{\min}(G_S)$. Note that a subgame with a single agent has $\epsilon_{\min} = -\infty$ by definition. Thus, every subgame of G has non-empty ϵ^*-core and hence there exists an ϵ^*-core imputation for every subgame. Since an ϵ-core imputation is also an ϵ^*-core imputation for any $\epsilon \leq \epsilon^*$ (by definition), any least core imputation of any subgame of G is also an ϵ^*-core imputation of that subgame. Thus we obtain the following.

Corollary 5. *Let G, $G^{\boldsymbol{r}}$ and $r_{\min}$ be as defined in Theorem 4. Let ϵ^* denote the maximum least core value over all subgames of G. For any coalition $S \subseteq N$, let $\boldsymbol{x_S}$ be a least core imputation of G_S. Define $\boldsymbol{x^*} = \sum_{S \subseteq N} \Pr(S|N) \cdot \boldsymbol{x_S}$, where $\Pr(S|N)$ is as defined in Theorem 4. Then we have that*

1. *If $\epsilon^* \geq 0$, then $\boldsymbol{x^*}$ is an ϵ^*-core imputation of $G^{\boldsymbol{r}}$.*
2. *If $\epsilon^* < 0$, then $\boldsymbol{x^*}$ is an $r_{\min} \cdot \epsilon^*$-core imputation of $G^{\boldsymbol{r}}$.*

Consider a totally balanced game $G = (N, v)$. Assume that we have an oracle LC such that $LC(G_S)$ returns a least core imputation of the subgame G_S of the base game G. Such an oracle subroutine exists with polynomial time complexity for many classes of totally balanced games. For example, Solymosi et. al. [26] give a polytime algorithm to compute the nucleolus of assignment games which are totally balanced [25]. Nucleolus is a special (and unique) least core imputation that maximizes stability. Other examples of polytime algorithms to compute the nucleolus of totally balanced games include the algorithm by Kuipers [16] for convex games, the algorithm by Deng et. al. [9] for simple flow games and the algorithm by Kern et. al. [15] for matching games (which are totally balanced over bipartite graphs [10]). For technical reasons, we extend the payment vector $LC(G_S)$ to a payment vector over all agents by setting the payments to agents in $N \setminus S$ to be zero. For now, we assume that LC is deterministic, in the sense that it returns the same least core imputation every time it is called with the same subgame. We relax this assumption in Remark 2.

Observe that $\epsilon^* \leq 0$ for a totally balanced game. Corollary 5 implies that $\boldsymbol{x^*}$ is an $r_{\min} \cdot \epsilon^*$-core imputation of $G^{\boldsymbol{r}}$ (and hence a core imputation as well since $r_{\min} \cdot \epsilon^* \leq 0$) that can be computed using exponentially many calls to LC. We reduce the number of calls to LC to a polynomial in n, $\log(1/\delta)$ ($1 - \delta$ is the confidence level), $v(N)$, $1/r_{\min}$ and $1/|\epsilon^*|$ by sampling the subgames instead of iterating over them and using some additional tricks. However, both $r_{\min}$ and $|\epsilon^*|$ can be exponentially small (even zero) making this algorithm possibly an exponential time algorithm. Thus if the bound on k in Theorem 5 exceeds 2^n or if $\epsilon^* = 0$, we revert to the naïve exponential summation of Corollary 5. Section 7 discusses the issues with computation of ϵ^*. Note that even when ϵ^* is unknown, the algorithm can be used in practice by taking large number of samples. Also, the algorithm uses the value of $v^{\boldsymbol{r}}(N)$ which is easy to approximate by sampling and the additive error can be taken care of as in Footnote 5.

Algorithm CoreReliability: Computing a core imputation of a reliability extension of a totally balanced game.
Input: Totally balanced game $G = (N, v)$, subroutine LC to compute a least core imputation of subgames of G, reliability vector $\boldsymbol{r}$, $v^{\boldsymbol{r}}(N)$, δ and k.
Output: $\hat{\boldsymbol{x}}$, which is in the core of $G^{\boldsymbol{r}}$ with probability at least $1 - \delta$.

1. Set $\boldsymbol{y} = \boldsymbol{0}$.
2. For $t = 1$ to k do
   (a) For each agent $i \in N$, set $l_i = 1$ with probability r_i and $l_i = 0$ otherwise.
   (b) $\boldsymbol{y} = \boldsymbol{y} + LC(G_S)$ where $S = \{i \in N \mid l_i = 1\}$ (the survivor set).
3. Let $\boldsymbol{x} = \boldsymbol{y}/k$.
4. Return $\hat{\boldsymbol{x}} = \boldsymbol{x} - \gamma \cdot \boldsymbol{1}$, where $\boldsymbol{1} = \langle 1, 1, \ldots, 1\rangle$ and $\gamma = \frac{1}{n} \cdot (x(N) - v^{\boldsymbol{r}}(N))$.

Theorem 5. *The payment vector $\hat{\boldsymbol{x}}$ returned by Algorithm* CoreReliability *is in the core of $G^{\boldsymbol{r}}$ with probability at least $1 - \delta$ if*

$$k \geq \frac{2 \cdot v(N)^2 \cdot n^2 \cdot \log\left(\frac{2 \cdot n}{\delta}\right)}{r_{\min}^2 \cdot |\epsilon^*|^2}.$$

Proof. Let $\boldsymbol{x_S} = LC(G_S)$. In Step 2, every S is sampled with probability $\Pr(S|N)$ and the value added is $\boldsymbol{x_S}$, so $E[\boldsymbol{x}] = \sum_{S \subseteq N} \Pr(S|N) \cdot \boldsymbol{x_S} = \boldsymbol{x^*}$ (as in Corollary 5). For any $S \subseteq N$ and for any $i \in N$, the payment to agent i in $LC(G_S)$ is in $[0, v(N)]$. Using Hoeffding's inequality, for any $i \in N$,

$$\Pr\left(|x_i - x_i^*| \geq \frac{r_{\min} \cdot |\epsilon^*|}{2 \cdot n}\right) \leq 2 \cdot e^{-\frac{2 \cdot k}{v(N)^2} \cdot \left(\frac{r_{\min} \cdot |\epsilon^*|}{2 \cdot n}\right)^2}.$$

Substituting the value of k, we get that this probability is at most δ/n for every $i \in N$. Taking union bound over $i \in N$, we obtain that the probability that

$$\forall\, i \in N, \quad |x_i - x_i^*| \leq \frac{r_{\min} \cdot |\epsilon^*|}{2 \cdot n}, \tag{2}$$

holds is at least $1 - \delta$. Now we prove that $\hat{\boldsymbol{x}}$ is a core imputation of $G^{\boldsymbol{r}}$ assuming Equation (2) holds. First of all, we can see that $\sum_{i \in N} x_i \leq \sum_{i \in N} x_i^* + r_{\min} \cdot |\epsilon^*|/2$. But using Corollary 5, we know that $\boldsymbol{x^*}$ is an $r_{\min} \cdot \epsilon^*$-core imputation of $G^{\boldsymbol{r}}$ and hence $\sum_{i \in N} x_i^* = v^{\boldsymbol{r}}(N)$. Therefore γ in Step 4 of the algorithm follows $\gamma = \frac{1}{n} \cdot \left(\sum_{i \in N} x_i - v^{\boldsymbol{r}}(N)\right) \leq \frac{r_{\min} \cdot |\epsilon^*|}{2 \cdot n}$. Hence, we can see that for every $i \in N$,

$$\hat{x}_i = x_i - \gamma \geq x_i^* - \frac{r_{\min} \cdot |\epsilon^*|}{2 \cdot n} - \gamma \geq x_i^* - \frac{r_{\min} \cdot |\epsilon^*|}{n}, \tag{3}$$

where the second transition follows due to Equation (2). Hence for any $C \subseteq N$,

$$\hat{x}(C) = \sum_{i \in C} \hat{x}_i \geq \sum_{i \in C} \left(x_i^* - \frac{r_{\min} \cdot |\epsilon^*|}{n}\right) \geq x^*(C) - r_{\min} \cdot |\epsilon^*|. \tag{4}$$

Since $\boldsymbol{x^*}$ is an $r_{\min} \cdot \epsilon^*$-core imputation of $G^{\boldsymbol{r}}$, we know that $x^*(C) \geq v^{\boldsymbol{r}}(C) - r_{\min} \cdot \epsilon^* = v^{\boldsymbol{r}}(C) + r_{\min} \cdot |\epsilon^*|$ (since $\epsilon^* < 0$). Substituting this into Equation (4), we get that $\hat{x}(C) \geq v^{\boldsymbol{r}}(C)$ for every $C \subseteq N$ and $C \neq N$. Furthermore, $\hat{x}(N) = x(N) - n \cdot \gamma = v^{\boldsymbol{r}}(N)$ by definition of $\hat{\boldsymbol{x}}$ and γ. Hence $\hat{\boldsymbol{x}}$ is in the core of $G^{\boldsymbol{r}}$. ■

Remark 1. Note that the algorithm works so long as the subroutine LC can compute an ϵ^*-core imputation of every subgame of the base game. The reason why we have chosen to work with the least core is that in our case $\epsilon^* < 0$ and when it is possible to compute an ϵ^*-core imputation of every subgame, it is usually possible to compute a least core imputation of every subgame as well.[5]

Remark 2. Initially we assumed that LC returns a fixed least core imputation for every subgame. This is because Theorem 4 only works with fixed imputations. However, it is easy to check that if LC has any distribution over the set of all least core imputations of a subgame, then the expected payment vector returned by LC is a least core imputation of that subgame. Thus in Algorithm CoreReliability, $E[\boldsymbol{x}] = E[\boldsymbol{x^*}]$ is still an $r_{\min} \cdot \epsilon^*$-core imputation of the reliability extension, and the algorithm still works with the same bound on k.[6]

Remark 3. Note that Hoeffding's inequality is usually applied when the requirements for the result are somewhat fuzzy whereas the requirements of a core imputation are quite strict. The use of least core imputations of subgames of the base game provides us enough margin of error to be able to use Hoeffding's inequality and still satisfy the strict constraints with high probability.

6 Failures and Non-emptiness of the Core

While studying reliability extensions of totally balanced games, we saw that introducing failures in a totally balanced game without failures, and increasing failure probabilities in a totally balanced reliability game preserve total balancedness, and hence non-emptiness of the core. In this section, we outline the effect of these two operations on three classes of games: i) general cooperative games, ii) totally balanced games, and iii) simple games.

General Games: In many games introducing failures does not preserve non-emptiness of the core. Any game G that has a non-empty core but is not totally balanced is such a game since a subgame of G with an empty core is also a reliability extension of G. Introducing failures is a special case of increasing failure probabilities where we start with failure probabilities being zero, so increasing failure probabilities also does not preserve core non-emptiness in general games.

Totally Balanced Games: Our results show that for totally balanced games, both introducing failures and more generally increasing failure probabilities preserve non-emptiness of the core (in fact, they preserve total balancedness).

[5] LC can also be replaced by a subroutine LC' which returns an approximate least core imputation so long as the additive error in each component is less than $r_{\min} \cdot \epsilon^*/n$.

[6] Algorithm CoreReliability can be easily adapted to compute an approximate or superstable core imputation of any reliability game in general with high probability.

Simple Games: For simple games, Bachrach et. al. [1] observe that introducing failures preserves non-emptiness of the core. To analyze increasing failure probabilities, we performed simulations on a special class of simple games known as weighted voting games. A weighted voting game is defined by $G = (N, \boldsymbol{w}, t)$ where N is a set of agents where each agent $i \in N$ has a weight $w_i \geq 0$, $\boldsymbol{w}$ is the vector of these weights and t is the threshold; a coalition C with $\sum_{i \in C} w_i \geq t$ has $v(C) = 1$ and $v(C) = 0$ otherwise. Simulations revealed the following example where increasing failure probabilities does not preserve core non-emptiness.

Example: Consider a weighted voting game G with 5 agents with weight vector $\boldsymbol{w} = \langle 4, 3, 3, 2, 1 \rangle$ and threshold $t = 6$. Consider its reliability extension $G^{\boldsymbol{r}}$ with the reliability vector $\boldsymbol{r} = \langle 0.1, 0.6, 1, 1, 0.5 \rangle$. $G^{\boldsymbol{r}}$ has a non-empty core, but decreasing the reliability of agent 5 from 0.5 to 0.1 makes the core empty.

While the theme that decreasing reliability increases stability does not hold strictly for simple games, it appears to hold on average, at least for weighted voting games. We performed several simulations where we randomly generated weighted voting games with weights sampled from various distributions, e.g., uniform distribution, normal distribution, exponential distribution etc. We kept the reliabilities of all the agents equal, and observed that as this uniform reliability decreases (i.e., the failure probability increases), the probability of the core being non-empty increases. We also observed the same result for the ϵ-core.

7 Discussion and Future Work

We studied the reliability extension of totally balanced games. We proved that both ϵ-total balancedness and ϵ-convexity (generalizations of the respective concepts) are preserved when the reliabilities decrease. We proved a relation between these classes, generalizing a result by Shapley [23] that ties convexity and total balancedness. We also proposed an algorithm to compute a core imputaiton of any reliability extension of a totally balanced game with high probability.

This opens several possibilities for future research. First, Lemma 1 shows how the reliabilities affect the characteristic function of a game and we derived some useful results about totally balanced games and convex games building on it. Lemma 1 might also have other applications, e.g., in analyzing the effect of failures on power indices such as the Shapley value or the Banzhaf power index. It would also be interesting to examine how the reliability extension affects the external subsidy required to maintain stability, i.e. the Cost of Stability [3,21].

Next, the number of samples required in the algorithm presented in Section 5 depends on ϵ^*, the maximum least core value over all subgames of the game without failures. We are unable to settle the question of computing ϵ^* or obtaining a lower bound on it in polynomial time (and thus obtaining an upper bound on the number of samples required). Such an investigation may also lead to discoveries regarding the relative stabilities of different subgames of a cooperative game.

Lastly, our analysis is restircted to games where only one coalition can be formed. In contrast, cooperative games with coalitional structures [17] allow multiple coalitions to arise simultaneously, and are used successfully to model collaboration in multi-agent environments [22,6,2]. It would be interesting to extend the reliability extension model to games with coalitional structures.

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An Economic Analysis of User-Privacy Options in Ad-Supported Services

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Abstract. We analyze the value to e-commerce website operators of offering privacy options to users, *e.g.*, of allowing users to opt out of ad targeting. In particular, we assume that site operators have some control over the cost that a privacy option imposes on users and ask when it is to their advantage to make such costs low. We consider both the case of a single site and the case of multiple sites that compete both for users who value privacy highly and for users who value it less. One of our main results in the case of a single site is that, under normally distributed utilities, if a privacy-sensitive user is worth at least $\sqrt{2}-1$ times as much to advertisers as a privacy-insensitive user, the site operator should strive to make the cost of a privacy option as low as possible. In the case of multiple sites, we show how a Prisoner's-Dilemma situation can arise: In the equilibrium in which both sites are obliged to offer a privacy option at minimal cost, both sites obtain lower revenue than they would if they colluded and neither offered a privacy option.

1 Introduction

Advertising supports crucially important online services, most notably search. Indeed, more than 95% of Google's total revenue derives from advertising.[1] Other advertiser-supported websites provide a growing array of useful services, including news and matchmaking. Because of its essential role in e-commerce, online advertising has been and continues to be the subject of intensive study by diverse research communities, including Economics and Computer Science. In this paper, we focus on an aspect of online advertising that has received little attention to date: how website operators can maximize their revenue while permitting privacy-sensitive users to avoid targeted ads.

Targeted ads are those chosen to appeal to a certain group of users. In *contextual targeting*, ads are matched to search queries or other commands issued by users; because it does not entail the collection and mining of any information except that which the user provides voluntarily and explicitly at the time

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[1] http://investor.google.com/financial/tables.html

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the ad is placed, it is not usually viewed as intrusive, and few users try to avoid it. In *demographic targeting*, ads are matched to users' demographic categories such as gender, location, age, race, religion, profession, or income. Demographic targeting may involve considerable data collection and analysis, and it is more controversial than contextual targeting: Some users are uncomfortable about being categorized, feel more vulnerable to ads that make use of their demographic categories, and worry that the same demographic information may be used for purposes more consequential and nefarious than advertising; other users appreciate the fact that their membership in certain demographic categories, particularly age, gender, and location, can prevent their being shown numerous time-wasting ads that are provably irrelevant to them. In *behavioral targeting*, ads are matched to individual users' browsing histories. By definition, it involves the collection and analysis of sensitive information, and many users take steps to avoid it.

Unsurprisingly, targeted ads are more effective than generic, untargeted ads, and thus they fetch higher prices. Behavioral targeting, for example, has been shown to produce higher click-through rates than no targeting; estimates of the extent of improvement in click-through rates vary widely, however, from 20% in the work of Chen *et al.* [4] to a factor of six in the work of Yan *et al.* [13]. Conversion rate, *i.e.*, the fraction of those users who, after clicking through to the advertiser's site actually buy something, is also higher for targeted ads; see Beales [2] for a discussion of the effect of behavioral targeting on conversion rates and Jansen and Solomon [9] on the effect of demographic targeting in general and gender in particular. Although efforts to quantify the effect of ad targeting on website operators' revenues are ongoing, there is credible evidence that the effect is large enough to imply that the elimination of targeted ads could mean the end of the Web as we know it; Goldfarb and Tucker [6], for example, studied the effect of EU privacy regulation on ad revenue and concluded that, all else equal, advertisers would have to spend approximately $14.8B more annually to achieve the same effectiveness under a strict privacy regime (*i.e.*, one that is less friendly to targeted ads) and that the dependence on targeted ads is highest among general-audience websites, such as those that provide news or weather.

Because targeted ads are lucrative for website operators, users are being observed, categorized, and tracked ever more precisely. Understandably, some users fear loss of privacy, and various tools offered by website operators (*e.g.*, opt-outs and other customizable privacy settings) and by third parties (*e.g.*, anonymizing browser plug-ins such as Torbutton[2]) that promise online-privacy protection are proliferating. These tools allow users to avoid targeted ads, but of course people who use them can still be shown generic, untargeted ads. Although many such tools are available without charge, they can impose non-monetary costs on users, *e.g.*, time and effort spent figuring out the often obscure privacy options presented by a UI, time and effort spent on installation of new software such as a privacy-enhancing browser plug-in, and reduced ease of use, speed, and/or quality of service. To a considerable extent, these costs can be controlled by website operators.

[2] See `https://www.torproject.org/torbutton/`.

We ask when it is to website operators' advantage to make the cost of such privacy options low. Our major contributions include:

- Economic models in which to address the problem, both for the case of a single site and for that of multiple sites that compete both for users who value privacy very highly and for users who value it less.
- A complete analysis of the case of a single site with homogeneous users and normally distributed utilities. In this setting, if a privacy-sensitive user is worth at least $\sqrt{2} - 1$ times as much to advertisers as a privacy-insensitive user, the site operator should strive to make the cost of a privacy option as low as possible.
- A complete analysis of the case of two sites with user demand functions that denote their privacy preferences. In this setting, we show how a Prisoner's-Dilemma situation can arise: In the equilibrium in which both sites are obliged to offer a privacy option at minimal cost, both sites obtain lower revenue than they would if they colluded and neither offered a privacy option.

2 Related Work

To the best of our knowledge, we are the first to study the question of when it is to the advantage of website operators to minimize the cost of providing users with privacy options. However, several related aspects of web users' ability to control their personal information have been studied.

Riederer *et al.* [10] propose a market for personal information based on the notion of *transactional privacy*. Users decide what information about themselves should be for sale, and aggregators buy access to users' information and use it to decide what ads to serve to each user. The information market that connects users and aggregators, handles payments, and protects privacy achieves truthfulness and efficiency using an unlimited-supply auction.

Carrascal *et al.* [3] use *experience sampling* to study the monetary value that users place on difference types of personal information. They find, for example, that users place a significantly higher value on information about their offline behavior than they do on information about their browsing behavior. Among categories of online information, they value financial and social-network information more highly than search and shopping information.

Iyer, Soberman, and Villas-Boas [8] consider advertising strategies in segmented markets, where competing firms can target ads to different segments. They find that firms can increase profits by targeting more ads at consumers who have a strong preference for their product than at comparison shoppers who might be attracted to the competition. Interestingly, targeted advertising produces higher profits regardless of whether the firms can price discriminate. Moreover, the ability to target advertising can be more valuable to firms in a competitive environment than the ability to price discriminate.

Telang, Rajan, and Mukhopadhyay [12] address the question of why multiple providers of free, online search services can coexist for a long time. In standard models of vertical (or quality) differentiation, a lower-quality product or service must sell for a lower price than its higher-quality competitor if it is to remain in the marketplace; if the prices are equal, all consumers choose the higher-quality alternative. Similarly, in standard models of horizontal (or taste) differentiation, sustained differentiation among products or services occurs when users incur high transportation costs. Yet, neither price nor transportation cost is a strategic variable in online search. Telang, Rajan, and Mukhopadhyay point out that, although the quality of one search service may clearly be higher than that of its competitors *on average*, the quality of results of *a particular search by a particular user* is highly variable and inherently stochastic. Thus, there is a nontrivial probability that a user will be dissatisfied with the results of a particular search and wish to search again for the same information, using a different search service. It is precisely the zero-price, zero-transportation-cost nature of the user's task that may cause him to use more than one search service in a single session. In the aggregate, this feature creates residual demand for lower-quality search services, allowing them to coexist with their higher-quality competitor.

Acquisti and Varian [1] consider the conditions under which a merchant should price-discriminate based on consumers' past purchases. They find that it may be profitable to do so when anonymizing technologies are too costly for consumers to use. Conitzer *et al.* [5] study a similar setting, where the merchant has control over the anonymity option. They find that consumers will chose anonymity when it is costless, a behavior which also maximizes the merchant's profit. Similar to our results, they demonstrate at Prisoner's Dilemma: the consumers could have obtained higher welfare by jointly deciding to disclose their identities. Consequently, costly anonymity could be beneficial to all parties.

3 Single-Provider Case

We begin by presenting a general model of an ad-supported service with a single provider. Let n be the size of the market for this service (*i.e.*, the number of users), v be the revenue extracted by the service provider for each user that allows targeted ads (referred to below as a "targeted user"), and γv, with $0 < \gamma \leq 1$, the revenue extracted for each user that avoids targeted ads (referred to below as a "private user"). The total revenue extracted by the provider is given by:

$$r = nv(s + \gamma p), \tag{1}$$

where s is the fraction of the market that consists of targeted users, and p is the fraction that consists of private users.

In this setting, we model the users by way of their utilities. For a specific user, the random variables U^S and U^P denote the utilities that the provider derives from the targeted and private services, respectively. We discount the utility by a cost c, which can be thought of as fixed cost that the user pays to set up the privacy option. In our model, we assume c is under the control of the service provider; hence, the provider will choose the cost of the privacy option to optimize revenue.[3] We assume $c \geq 0$, but one could expand our analyses to include settings in which the provider "pays" users to use its service and thereby induces a negative cost c. Let $U = (U^S, U^P)$ be the corresponding joint distribution. Let $f(x, y) = \Pr[U^S = x, U^P = y]$ be the joint density, and similarly let $F(x, y) = \Pr[U^S \leqslant x, U^P \leqslant y]$ be the joint distribution function.

A user may:

1. use the targeted option and derive utility U^S;
2. use the privacy option and derive utility $U^P - c$;
3. abstain from using the service for a utility of 0.

Users choose among the above options to maximize their utility. Their choices determine the values of s and p.

From the standpoint of the provider, finding the revenue-maximizing cost c^* involves computing trade-offs between s and p. We have:

$$s = \Pr[U^P - U^S < c, U^S \geq 0] = \int_0^{\infty} \int_{-\infty}^{c+y} f(x, y) dx dy, \tag{2}$$

$$p = \Pr[U^P - U^S \geq c, U^P \geq c] = \int_0^{\infty} \int_{c+y}^{\infty} f(x, y) dx dy + \int_{-\infty}^{0} \int_{c}^{\infty} f(x, y) dx dy. \tag{3}$$

We emphasize that, in this model, $s + p$ may be less than 1, because users with negative utility from both targeted and privacy options will not use the service at all.

3.1 Normally Distributed User Utilities

We now explore this model by considering the case of normally distributed user utilities. Assume that $U = (U^S, U^P)$ follows a standard bivariate normal distribution with mean vector zero and covariance matrix $\Sigma = \{\{1, \rho\}, \{\rho, 1\}\}$; here, ρ is the correlation coefficient between U^S, and U^P. Use ϕ_2 to denote U's density and Φ_2 to denote its distribution function. The marginal distributions U^S and U^P are standard normal with mean 0, variance 1, density function $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$, and distribution function $\Phi(x) = \frac{1}{2}\left[1 + \text{erf}(x/\sqrt{2})\right]$. We first consider the case in which $\rho = 0$.

[3] One could imagine more elaborate models, where the cost was governed by a distribution, and the provider could, for example, control the mean of the distribution; for simplicity, we focus on the constant-cost model in this first paper.

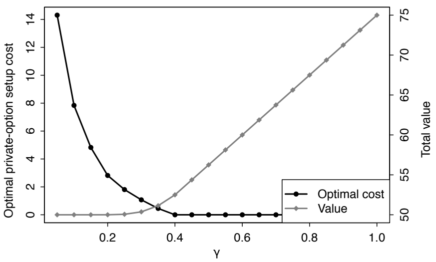

Fig. 1. Optimal privacy-option setup cost and value for various γ and $n = 100$, $v = 1$

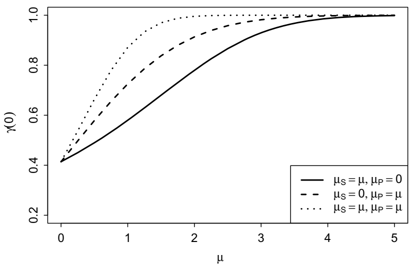

Fig. 2. The value of $\gamma(0)$ – where the privacy option takes on zero cost – for three settings: U^P's mean fixed at 0, U^S's mean fixed at 0, and U^P and U^S have the same mean

3.2 Uncorrelated User Utilities, $\rho = 0$.

The fraction of targeted users is

$$s = \Pr[U^P - U^S < c, U^S \geq 0] \tag{4}$$

$$= \int_0^\infty \int_{-\infty}^{c+y} \phi(x)\phi(y)dxdy. \tag{5}$$

Similarly, the fraction of private users is

$$p = \Pr[U^P - U^S \geq c, U^P \geq c] \tag{6}$$

$$= \int_0^\infty \int_{c+y}^\infty \phi(x)\phi(y)dxdy + \int_{-\infty}^0 \int_c^\infty \phi(x)\phi(y)dxdy, \tag{7}$$

where $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$. Observe that s is monotonically increasing in c, while p is monotonically decreasing in c. The rate of change of the fraction of targeted users as a function of c is:

$$\frac{\partial s}{\partial c} = \int_0^\infty \phi(y)\phi(c+y)\,dy \tag{8}$$

$$= \int_0^\infty \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}(c+y)^2}}{\sqrt{2\pi}}\,dy \tag{9}$$

$$= \int_0^\infty \frac{e^{-\frac{1}{2}(c+y)^2 - \frac{y^2}{2}}}{2\pi}\,dy \tag{10}$$

$$= \frac{e^{-\frac{c^2}{4}} \mathrm{erfc}\left(\frac{c}{2}\right)}{4\sqrt{\pi}}, \tag{11}$$

which is easily seen to be decreasing in c. Using similar calculations, we can compute the rate of change of the fraction of private users with respect to c:

$$\frac{\partial p}{\partial c} = \int_{-\infty}^{0} -\phi(c)\phi(y)\,dy + \int_{0}^{\infty} -\phi(y)\phi(c+y)\,dy \tag{12}$$

$$= -\frac{e^{-\frac{c^2}{2}}\left(e^{\frac{c^2}{4}}\operatorname{erfc}\left(\frac{c}{2}\right)+\sqrt{2}\right)}{4\sqrt{\pi}}, \tag{13}$$

which is similarly increasing in c. The provider is indifferent with respect to revenue earned between the two types of users when:

$$\frac{\partial s}{\partial c} = -\gamma\frac{\partial p}{\partial c}. \tag{14}$$

Denote by $\gamma(c)$ the value of c for which equality holds. Substituting and solving for $\gamma(c)$, we obtain:

$$\gamma(c) = 1 - \frac{2}{\sqrt{2}e^{\frac{c^2}{4}}\operatorname{erfc}\left(\frac{c}{2}\right)+2}. \tag{15}$$

We continue by proving an auxiliary lemma.

Lemma 1. *$\gamma(c)$ in decreasing in c.*

Proof. Consider the derivative

$$\frac{\partial e^{\frac{c^2}{4}}\operatorname{erfc}\left(\frac{c}{2}\right)}{\partial c} = \frac{1}{2}ce^{\frac{c^2}{4}}\operatorname{erfc}\left(\frac{c}{2}\right) - \frac{1}{\sqrt{\pi}}. \tag{16}$$

We show that it is negative for all $c > 0$; this suffices to prove the lemma. Note the following equivalences.

$$\frac{1}{2}ce^{\frac{c^2}{4}}\operatorname{erfc}\left(\frac{c}{2}\right) - \frac{1}{\sqrt{\pi}} < 0 \tag{17}$$

$$ce^{\frac{c^2}{4}}\operatorname{erfc}\left(\frac{c}{2}\right) < \frac{2}{\sqrt{\pi}} \tag{18}$$

$$ce^{\frac{c^2}{4}}\frac{2}{\sqrt{\pi}}\int_{c/2}^{\infty} e^{-t^2}\,dt < \frac{2}{\sqrt{\pi}} \tag{19}$$

$$ce^{\frac{c^2}{4}}\int_{c/2}^{\infty} e^{-t^2}\,dt < 1. \tag{20}$$

We prove the last line above, using the following bound of [11] (a tighter version of Komatsu's inequality [7]):

$$e^{x^2}\int_{x}^{\infty} e^{-t^2}dt < 2/\left(3x+\sqrt{x^2+4}\right). \tag{21}$$

At $x = c/2$, we obtain

$$e^{\frac{c^2}{4}}\int_{c/2}^{\infty} e^{-t^2}dt < 2/\left(3c/2+\sqrt{c^2/4+4}\right), \tag{22}$$

which yields

$$ce^{\frac{c^2}{4}} \int_{c/2}^{\infty} e^{-t^2} dt < 2c/\left(3c/2 + \sqrt{c^2/4+4}\right) < 1, \tag{23}$$

proving the lemma. □

We have that $\gamma(0) = \sqrt{2}-1$. Furthermore, because $\gamma(c)$ is decreasing in c, for any $\gamma \geq \gamma(0)$, it follows that the provider's best strategy is $c = 0$, *i.e.*, offering a free privacy option. Using the above, we are now ready to state the main theorem of this section.

Theorem 1. *If U follows a standard bivariate normal distribution with correlation $\rho = 0$, then the provider will offer a free privacy option whenever $\gamma \geq \sqrt{2}-1$.*

Remark 1. The specific value $\sqrt{2}-1$ arises from our assumption that U^P and U^S were distributed according to a standard normal distribution of mean 0. Similar results for other means and variances can be calculated in a similar fashion. Figure 2 demonstrates three variations: where the mean of U^P is fixed at 0 but the mean of U^S varies; where the mean of U^S is fixed at 0 but the mean of U^P varies; and where the means vary but are equal. (All variances remain 1.) For example, where U^S and U^P have equal means, we see that $\gamma(0)$ converges to 1 very quickly, as offering privacy cannibalizes more lucrative targeted users more readily than it garners new private users.

3.3 Correlated Utilities, $\rho \neq 0$.

Assume that $U = (U^S, U^P)$ follows a standard bivariate normal distribution with correlation ρ. Use ϕ_2 to denote its density function. We derive an indifference condition similar to the one in Equation 14:

$$\gamma \left(\int_{-\infty}^{0} \phi_2(c,y)\, dy + \int_{0}^{\infty} \phi_2(c+y,y)\, dy \right) = \int_{0}^{\infty} \phi_2(c+y,y)\, dy. \tag{24}$$

Substituting for the density function of the standard bivariate normal and integrating, we obtain an expression for γ in terms of c and ρ:

$$\gamma = \left(\frac{\sqrt{2-2\rho}\, e^{\frac{c^2(1-2\rho)}{4(\rho-1)}} \operatorname{erfc}\left(\frac{c\rho}{\sqrt{2-2\rho^2}}\right)}{\operatorname{erfc}\left(\frac{c}{2\sqrt{\rho+1}}\right)} + 1 \right)^{-1}. \tag{25}$$

As before, setting $c = 0$ yields

$$\gamma(0) = \frac{1}{1+\sqrt{2-2\rho}}. \tag{26}$$

We observe that, as the correlation coefficient increases (reps., decreases) the value of γ beyond which it makes sense to offer a privacy option at no cost also increases (reps., decreases). That is, greater correlation means one requires higher revenue from private users in order to offer privacy at no cost; in particular, when $U^P = U^S$, Equation 26 reasonably requires that $\gamma(0) = 1$. We can now state a generalized version Theorem 1, taking into account correlated user utilities.

Theorem 2. *If U follows a standard bivariate normal distribution with correlation ρ, then the provider will offer a free privacy option whenever* $\gamma \geq \frac{1}{1+\sqrt{2-2\rho}}$.

4 A Two-Player Game

We provide a general model of the two-player version of the game. We then explore the model by delving into a concrete example. Throughout this section, we use the terms "player" and "provider" interchangeably. As in the single-provider case, a "targeted user" is one who does not use the privacy option, and a "private user" is one who does.

The game proceeds in two periods $t = 1, 2$. To begin, at $t = 1$, we have two providers S_i, for $i = 1, 2$, that offer competing, advertising-supported, non-private services. We let S_0 denote a user's outside option, which in this case is to use neither service. Denote the fraction of users who choose S_i at time t by s_{it}. We have $\sum_{i=0}^{2} s_{i1} = 1$.

At $t = 2$, simultaneously, both providers can introduce private variants, *i.e.*, ones in which users avoid targeted ads; we denote these by P_i. The providers can determine an associated "cost" that controls the utility of the private variants, with the goal of tuning the market share for each service they provide. We denote these costs by c_i. The fraction of users that choose P_i at time t is given by p_{it}. We have $\sum_{i=0}^{2} s_{i2} + \sum_{j=1}^{2} p_{j2} = 1$. The fraction of users left using the non-private options (or neither option) at $t = 2$ is given by:

$$s_{i2} = s_{i1}(1 - F_i(c_1, c_2)) \text{ for } i = 0, 1, 2.$$

That is, $F_i(c_1, c_2)$ is the fraction of users who were using S_i or were not using either service but are now using one of the private variants. The users $s_{i1} - s_{i2}$ switching to a private variant are distributed among the two providers as follows:

$$p_{i2} = H_i(c_1, c_2) \sum_{j=0}^{2} (s_{j1} - s_{j2}) \text{ for } i = 1, 2.$$

Here, $H_i(c_1, c_2)$ is a function determining the split, among the competing private-service providers, of users switching to a private variant; note $H_1(c_1, c_2) = 1 - H_2(c_1, c_2)$. Also, recall that $p_{i1} = 0$.

Let v_i be the value that provider i derives from the standard service. Let $\gamma_i v_i$ be the value it derives from the private service, where $\gamma_i \in (0, 1]$. Let r_i denote the revenue function of provider i at the second stage of the game. We have:

$$\begin{aligned} r_i &= v_i s_{i2} + v_i \gamma_i p_{i2} \\ &= v_i s_{i1}(1 - F_i(c_1, c_2)) + v_i \gamma_i H_i(c_1, c_2) \sum_{j=0}^{2} (s_{j1} - s_{j2}) \\ &= v_i s_{i1}(1 - F_i(c_1, c_2)) + v_i \gamma_i H_i(c_1, c_2) \sum_{j=0}^{2} (s_{j1} - s_{j1}(1 - F_j(c_1, c_2))) \\ &= v_i \left(s_{i1}(1 - F_i(c_1, c_2)) + \gamma_i H_i(c_1, c_2) \sum_{j=0}^{2} s_{j1} F_j(c_1, c_2) \right) \end{aligned} \tag{27}$$

for $i = 1, 2$.

In equilibrium, neither provider can increase its revenue by unilaterally deviating. The first-order conditions (FOC) are given by

$$\frac{\partial r_i}{\partial c_i} = 0, \text{ for } i = 1, 2. \tag{28}$$

Let $\{\hat{c}_1, \hat{c}_2\}$ be a solution to this system of equations. Then, the second-order conditions (SOC) are given by:

$$\frac{\partial^2 r_i}{\partial c_i^2}(\hat{c}_1, \hat{c}_2) < 0, \text{ for } i = 1, 2. \tag{29}$$

Our questions revolve around the equilibrium of this game.

4.1 A Prisoners' Dilemma

The framework described above was designed to be very general; however, this makes it somewhat difficult to get a handle on the nature of the game. We consider a worked example in order to gain more insight. For simplicity, we will assume $F_i = F$ for $i = 0, 1, 2$. There are certain natural properties that we want for the function F: F should be decreasing in the costs c_i, and F should go to 0 as both c_i go to infinity. We simplify things further by assuming that, if either cost c_i goes to 0, all users will prefer the zero-cost privacy option, and thus F will go to 1. This may not be the case in all settings, but it is a reasonable and instructive place to start.

A relatively straightforward function with these properties is

$$F(c_1, c_2) = \begin{cases} 0 & \text{if } c_1 = c_2 = \infty, \\ 1 & \text{if } c_1 = 0, \text{ or } c_2 = 0, \\ \exp(-\frac{c_1 c_2}{c_1 + c_2}) & \text{otherwise.} \end{cases} \tag{30}$$

We define H_i so that the fraction that goes to P_i is proportional to its cost.

$$H_1(c_1, c_2) = \begin{cases} 0 & \text{if } c_1 = \infty, \\ \frac{1}{2} & \text{if } c_1 = c_2 = 0, \\ \frac{c_2}{c_1+c_2} & \text{otherwise.} \end{cases} \tag{31}$$

We define H_2 similarly. Also, for notational simplicity let $s_{i1} = s_i$. Under these assumptions, the payoff function for player 1 is:

$$r_1 = v_1 \left(e^{-\frac{c_1 c_2}{c_1+c_2}} \left(c_2\gamma_1 - (c_1 + c_2)\, s_1\right) / (c_1 + c_2) + s_1 \right), \tag{32}$$

and similarly, for player 2. Note that the payoffs are continuous. Furthermore, under the assumption that $F_i = F$, the first and second derivatives of the revenue function assume the following forms:

$$\frac{\partial r_i}{\partial c_i} = v_i \left(-s_{i1} \frac{\partial F}{\partial c_i} + \gamma_i \frac{\partial H_i}{\partial c_i} F(c_1, c_2) + \gamma_i H_i(c_1, c_2) \frac{\partial F}{\partial c_i} \right), \tag{33}$$

and

$$\frac{\partial^2 r_i}{\partial c_i^2} = v_i \left(-s_{i1} \frac{\partial^2 F}{\partial c_i^2} + \gamma_i \left(2 \frac{\partial H_i}{\partial c_i} \frac{\partial F}{\partial c_i} + \frac{\partial^2 H_i}{\partial c_i^2} F + \frac{\partial^2 F}{\partial c_i^2} H \right) \right). \tag{34}$$

4.2 Computation of Equilibria

We now assume that $\gamma_i > 0$, *i.e.*, that both players can derive some revenue from private users.

Theorem 3. *The game has two possible equilibria:*

1. *at* $\{c_1^* = 0, c_2^* = 0\}$, *with* $r_i = v_i\gamma_i/2$,
2. *at* $\{c_1^* = \infty, c_2^* = \infty\}$, *if* $s_i > \gamma_i$, *with* $r_i = s_i v_i$.

Remark 2. Theorem 3 demonstrates how the Prisoners' Dilemma arises naturally in this two-player game. For example, if $s_i = \frac{3}{4}\gamma_i$ for $i = 1, 2$, then, when neither provider offers a privacy option, their revenue is $\frac{3}{4}\gamma_i v_i$. However, this is not an equilibrium point; at equilibrium, both players offer zero-cost privacy options, and their revenue is reduced to $\frac{1}{2}\gamma_i v_i$.

Proof. The proof proceeds in a sequence of lemmas. We begin by considering cases in which one player offers a free privacy option, while the other charges a (possibly infinite) cost.

Lemma 2. *The game does not admit solutions of the form* $\{c_i = 0, c_{-i} > 0\}$.

Proof. Because the game is symmetric, it suffices to consider the case in which $c_1 = 0$, and $c_2 = c$, for some constant $c > 0$. From Equations 30 and 31, we have $F(0, c) = 1$, $H_1(0, c) = 1$, and $H_2(0, c) = 0$. From Equation 27, the revenue for player 1 is $r_1 = v_1\gamma_1$, and for player 2 it is $r_2 = 0$. Suppose player 2 unilaterally deviates by playing $c_2 = 0$. In this case, $H_2(0, 0) = 1/2$, and $r_2 = v_2\gamma_2/2$. Therefore, because $\gamma_2 > 0$, $c_1 = 0, c_2 = c$, does not constitute an equilibrium. □

Next, we consider settings in which both players offer the privacy option for a finite, non-zero cost.

Lemma 3. *The game does not admit solutions of the form* $\{c_i > 0, c_{-i} > 0\}$.

Proof. We consider candidate equilibria suggested by solutions to the FOC:

$$F(c_1, c_2)c_2v_1\left(c_2\left(c_1+c_2\right)s_1 - \left(c_2^2+c_2+c_1\right)\gamma_1\right)/\left(c_1+c_2\right)^3 = 0 \quad (35)$$
$$F(c_1, c_2)c_1v_2\left(c_1\left(c_1+c_2\right)s_2 - \left(c_1^2+c_1+c_2\right)\gamma_2\right)/\left(c_1+c_2\right)^3 = 0. \quad (36)$$

First, note that, because costs are finite, $F(c_1, c_2) > 0$. Therefore, the FOC can be simplified to the following equivalent conditions:

$$c_2\left(c_1+c_2\right)s_1 - \left(c_2^2+c_2+c_1\right)\gamma_1 = 0, \quad (37)$$
$$c_1\left(c_1+c_2\right)s_2 - \left(c_1^2+c_1+c_2\right)\gamma_2 = 0. \quad (38)$$

Solving the first equation above for c_1 and substituting into the second one, we obtain the following solution:

$$\left\{c_1 = \frac{\gamma_2}{s_2-\gamma_2}K, c_2 = \frac{\gamma_1}{s_1-\gamma_1}K\right\}, \quad (39)$$

where

$$K = \frac{\gamma_1 s_2 + \gamma_2 s_1 - 2\gamma_1\gamma_2}{\gamma_1 s_2 + \gamma_2 s_1 - \gamma_1\gamma_2}. \quad (40)$$

Next, we need to check the SOC for this solution. A long sequence of calculations yields:

$$\frac{\partial^2 r_1}{\partial c_1^2}(c_1, c_2) = \frac{\gamma_1^4 v_1 e^{-\frac{\gamma_1\gamma_2}{\gamma_1 s_2+\gamma_2(s_1-\gamma_1)}}\left(s_1-\gamma_1\right)\left(s_2-\gamma_2\right)^4\left(\gamma_1 s_2+\gamma_2 s_1-\gamma_1\gamma_2\right)}{\left(\gamma_1 s_2+\gamma_2 s_1-2\gamma_1\gamma_2\right)^5} < 0,$$
$$\frac{\partial^2 r_2}{\partial c_2^2}(c_1, c_2) = \frac{\gamma_2^4 v_2 e^{-\frac{\gamma_1\gamma_2}{\gamma_1 s_2+\gamma_2(s_1-\gamma_1)}}\left(s_2-\gamma_2\right)\left(s_1-\gamma_1\right)^4\left(\gamma_1 s_2+\gamma_2 s_1-\gamma_1\gamma_2\right)}{\left(\gamma_1 s_2+\gamma_2 s_1-2\gamma_1\gamma_2\right)^5} < 0,$$

which can be further simplified to the equivalent conditions

$$\left(s_1-\gamma_1\right)\frac{\gamma_1 s_2+\gamma_2 s_1-\gamma_1\gamma_2}{\gamma_1 s_2+\gamma_2 s_1-2\gamma_1\gamma_2} = \frac{s_1-\gamma_1}{K} < 0, \quad (41)$$
$$\left(s_2-\gamma_2\right)\frac{\gamma_1 s_2+\gamma_2 s_1-\gamma_1\gamma_2}{\gamma_1 s_2+\gamma_2 s_1-2\gamma_1\gamma_2} = \frac{s_2-\gamma_2}{K} < 0. \quad (42)$$

Notice that $(s_1-\gamma_1)/K$ has the same sign as c_1 in Equation 39. Similarly, $(s_2-\gamma_2)/K$ has the same sign as c_2. Because costs are non-negative, the second-order conditions are not met; this solution minimizes rather than maximizes the revenue of the players and is therefore not an equilibrium point. □

Next, we consider the case in which both players offer free privacy options.

Lemma 4. *Both players' offering free privacy options (i.e.,* $c_1 = c_2 = 0$*) constitutes an equilibrium of the game.*

Proof. In this case, users switch *en masse* to the private services and are distributed equally between the two providers. The revenue for player i is $v_i\gamma_i/2$. Furthermore, if a player unilaterally deviates and switches to a non-zero cost for privacy, his revenue instantly collapses to zero. Therefore, $c_1 = c_2 = 0$ constitutes an equilibrium to the game. □

Finally, we consider the case in which neither player offers a privacy option.

Lemma 5. *Neither player's offering a privacy option (i.e., $c_1 = c_2 = \infty$) constitutes an equilibrium of the game if $s_i < \gamma_i$, for $i = \{1, 2\}$.*

Proof. Suppose that neither player offers a privacy option; so $r_i = v_i s_i$. Now, consider the case in which player 1 wishes to deviate unilaterally. (The case for player 2 can be argued identically.) First, note that $\lim_{c_2\to\infty} F(c_1, c_2) = \exp(-c_1)$, and $\lim_{c_2\to\infty} H_1(c_1, c_2) = 1$. Therefore, if player 2 doesn't offer a privacy option, *i.e.*, $c_2 = \infty$, player 1 deviates and plays c_1; his new revenue will be

$$r_1 = s_1 v_1(1 - \exp(-c_1)) + v_1\gamma_1 \exp(-c_1) = s_1 v_1 + v_1 \exp(-c_1)(\gamma_1 - s_1). \quad (43)$$

If $\gamma_1 < s_1$, then player 1 cannot improve his position; therefore, not offering a privacy option constitutes an equilibrium. If $\gamma_1 > s_1$, player 1 can increase his revenue by decreasing his cost; therefore, not offering a privacy option is not an equilibrium. (If $\gamma_1 = s_1$, player 1 cannot strictly improve his revenue by deviating, and we have a weak equilibrium.) □

This concludes the proof of Theorem 3. □

This example demonstrates what we see as the potentially natural outcomes of this two-player dynamic that would extend to three or more players as well. It is possible that no service provider is incentivized to offer a privacy option in this game. In such a setting, one might expect a new entrant to enter the "game" and disrupt the status quo by offering a suitable privacy option, potentially forcing other providers to do so as well. It is also possible that all competitors are inclined to offer a privacy option. In our example, this led to all users' opting to offer privacy, but we could have utilized arguably more realistic functions F to account for the fact that some users might find more value in not offering privacy or might find the cost of doing so non-trivial regardless of the efforts they exert to drive the cost down. Under such settings, we would expect other equilibria to arise; in our example, there were other points at which the first-order conditions but not the second-order conditions were met, but, more generally (for other functional relationships), we could have other equilibria.

5 Conclusion

The results of Sections 3 and 4 suggest that website operators could have their cake and eat it, too. By carefully controlling the cost to users of opting out of targeted ads, they could maximize their revenue and respect their users' privacy concerns. Putting this approach into practice would require surmounting at least two major obstacles.

First, a service provider would need a good estimate of the parameter γ, *i.e.*, the fraction of the revenue derived from a targeted user that can be derived from a private user. The value of γ is closely related to the extent to which click-through and conversion rates are improved by various forms of ad targeting, which in turn is still the subject of intensive, ongoing research.

Second, the service provider would need to translate the abstract "cost" c_i of our economic analysis into a concrete privacy-enforcement tool that can be installed and used at a cost of c_i. It may suffice to be able to choose between two technological options, one of which is clearly more costly to users than the other, but even this would be nontrivial given the state of the art of privacy enforcement.

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Auctions with Heterogeneous Items and Budget Limits*

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Abstract. We study individual rational, Pareto optimal, and incentive compatible mechanisms for auctions with heterogeneous items and budget limits. For multi-dimensional valuations we show that there can be no *deterministic* mechanism with these properties for *divisible* items. We use this to show that there can also be no *randomized* mechanism that achieves this for either *divisible* or *indivisible* items. For single-dimensional valuations we show that there can be no *deterministic* mechanism with these properties for indivisible items, but that there is a *randomized* mechanism that achieves this for either divisible or indivisible items. The impossibility results hold for *public* budgets, while the mechanism allows *private* budgets, which is in both cases the harder variant to show. While all positive results are polynomial-time algorithms, all negative results hold independent of complexity considerations.

1 Introduction

A canonical problem in Mechanism Design is the design of economically efficient auctions that satisfy individual rationality and incentive compatibility. In settings with quasi-linear utilities these goals are achieved by the Vickrey-Clarke-Groves (VCG) mechanism. In many practical situations, including settings in which the agents have budget limits, the quasi-linear assumption fails to be true and, thus, the VCG mechanism is not applicable.

Ausubel [2] describes an ascending-bid auction for homogeneous items that yields the same outcome as the sealed-bid Vickrey auction, but offers advantages in terms of simplicity, transparency, and privacy preservation. In his concluding remarks he points out that "when budgets impair the bidding of true valuations in a sealed-bid Vickrey auction, a dynamic auction may facilitate the expression of true valuations while staying within budget limits" (p. 1469).

Dobzinski et al. [7] show that an adaptive version of Ausubel's "clinching auction" is indeed the unique mechanism that satisfies individual rationality, Pareto optimality, and incentive compatibility in settings with *public* budgets. They use this fact to show that there can be no mechanism that achieves those properties for *private* budgets.

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An important restriction of Dobzinski et al.'s impossibility result for private budgets is that it only applies to *deterministic* mechanisms. In fact, as Bhattacharya et al. [4] show, there exists a *randomized* auction that is individual rational, Pareto optimal, and incentive compatible with private budgets.

All these results assume that the items are homogeneous, although as Ausubel [3] points out, "situations abound in diverse industries in which heterogeneous (but related) commodities are auctioned" (p. 602). He also describes an ascending-bid auction, the "crediting and debiting auction", that takes the place of the "clinching auction" when items are heterogeneous.

Positive and negative results for *deterministic* mechanisms and public budgets are given in [8, 10, 9, 6]. We focus on *randomized* mechanisms, and prove positive results for private budgets and negative results for public budgets. *We thus explore the power and limitations of randomization in settings with heterogeneous items and budget limits.*

Model. There are n agents and m items. The items are either divisible or indivisible. Each agent has a valuation for each item and each agent has a budget. Agents can be assigned more than one item and valuations are additive across items. All valuations are private. We distinguish between settings in which budgets are public and settings in which budgets are private. A mechanism is used to compute assignments and payments based on the reported valuations and the reported budgets. An agent's utility is defined as valuation for the assigned items minus the payment if the payment does not exceed the budget and the utility is minus infinity otherwise. We assume that agents are utility maximizers and as such need not report their true valuations and true budgets.

Our goal is to design mechanisms with certain desirable properties or to show that no such mechanism exists. For deterministic mechanisms we require that the respective properties are always satisfied. For randomized mechanisms we either require that the properties hold for all outcomes or that they hold in expectation. In the former case we say that they are satisfied *ex post*, in the latter case we say that they are satisfied *ex interim*.

We are interested in the following properties:

(a) *Individual rationality (IR)*: A mechanism is IR if all outcomes it produces give non-negative utility to the agents *and* the sum of the payments is non-negative. (b) *Pareto optimality (PO)*: A mechanism is PO if it produces an outcome such that there is no other outcome in which all agents and the auctioneer are no worse off and at least one of the agents or the auctioneer is strictly better off. [1] (c) *No positive transfers (NPT)*: A mechanism satisfies NPT if it produces an outcome in which all payments are non-negative. (d) *Incentive compatibility (IC)*: A mechanism is IC if each agent maximizes his utility by reporting his true valuation(s) and true budget no matter what the other agents'

[1] If the outcome for which we want to establish PO is IR, then we only have to consider alternative outcomes that are IR. In the alternative outcome individual payments may be negative, even if the original outcome satisfied IR and NPT. See the arXiv version of [8] for a more detailed discussion.

reported valuations and reported budgets are. If the budget is public then the agents can only report their true budgets. Following prior work we focus on IR, PO, NPT, and IC for positive results and on IR, PO, and IC for negative results. Both the inclusion of NPT for positive results and the exclusion of NPT for negative results strengthens the respective results.

Results. We analyze two settings with heterogeneous items, one with multi-dimensional valuations and one with single-dimensional valuations. In the setting with multi-dimensional valuations, each agent has an arbitrary, non-negative valuation for each of the items. In the setting with single-dimensional valuations, which is inspired by sponsored search auctions, an agent's valuation for an item is the product of an item-specific quality and an agent-specific valuation. Our motivation for studying this setting is that an advertiser might want to show his ad in multiple slots on a search result page.

(a) For **multi-dimensional** valuations the impossibility result of [8] implies that there can be no deterministic mechanism for *indivisible* items that is IR, PO, and IC for public budgets. We show that there also can be no deterministic mechanism with these properties for *divisible* items. We use this to show that for both divisible and indivisible items there can be no randomized mechanism that is IR ex interim, PO ex interim, and IC ex interim. This is the first impossibility result for randomized mechanisms for auctions with budget limits. It establishes an interesting separation between randomized mechanisms for single-dimensional valuations, where such mechanisms exist (see below), and multi-dimensional valuations, where no such mechanism exists.

(b) For **single-dimensional** valuations the impossibility result of [7] implies that there can be no deterministic mechanism for indivisible items that is IR, PO, and IC for *private* budgets. We show that for heterogeneous items there can also be no deterministic mechanism for indivisible items that is IR, PO, and IC for *public* budgets. We thus obtain a strong separation between deterministic mechanisms, that do *not* exist for *public* budgets, and randomized mechanisms, that exist for *private* budgets (see below). This separation is stronger than in the homogeneous items setting, where a deterministic mechanism exists for public budgets [7]. Additionally, our impossibility result is tight in the sense that if any of the conditions is relaxed such a mechanism exists: (i) For *homogeneous*, indivisible items a deterministic mechanism is given in [7], (ii) we give a deterministic mechanism for heterogeneous, *divisible* items, and (iii) we give a *randomized* mechanism for heterogeneous, indivisible items.

(c) For **single-dimensional** valuations we give mechanisms that extend earlier work for homogeneous items to heterogeneous items. Specifically, we give a *randomized* mechanism that satisfies IR ex interim, NPT ex post, PO ex post, and IC ex interim for *divisible* or *indivisible* items and *public* or *private* budgets. Additionally, for the case of *divisible* items and *public* budgets we give a *deterministic* mechanism that is IR, NPT, PO, and IC.

We summarize our results and the results from related work described next in Table 1 and Table 2 below.

Related Work. The setting in which all items are identical was first studied by [7]. By adapting the "clinching auction" of [2] from settings without budgets to settings with budgets they obtain deterministic mechanisms that are IR, NPT, PO, and IC with public budgets for divisible and indivisible items. They also show that these mechanisms are the only mechanisms that are IR, PO, and IC, and that they are not IC for private budgets, implying that there can be no deterministic mechanism that is IR, PO, and IC when the budgets are private. However, [4] showed that there is such a mechanism for private budgets that is randomized. Note that both, [7] and [4] study only homogeneous items.

Impossibility results for general, non-additive valuations were given in [10, 6, 9]. Combined they show that there can be no deterministic mechanism for indivisible items that is IR, PO, and IC with public budgets for monotone valuations with decreasing marginals. These impossibility results do not apply to additive valuations, which is the case that we study.

Heterogeneous items were first studied in [8]. In their model each agent has the same valuation for each item in an agent-dependent interest set and zero for all other items. They give a deterministic mechanism for indivisible items that satisfies IR, NPT, PO, and IC when both interest sets and budgets are public. They also show that when the interest sets are *private*, then there can be no deterministic mechanism that satisfies IR, PO, and IC. This implies that for *indivisible* items and public budgets there can be no *deterministic* IR, PO, and IC mechanism for unconstrained valuations.

Settings with heterogeneous items were in parallel to this paper studied by [6] and [9]. The former study problems with multiple keywords, each having multiple slots. Agents have unit demand per keyword. They are either interested in a subset of the keywords and have identical valuations for the slots or they are interested in all keywords and have sponsored search like valuations for the slots. The latter study settings in which the agents have identical valuations and the allocations must satisfy polymatroidal or polyhedral constraints.

The settings studied in [6, 9] are more general than the single-dimensional valuations setting studied here. On the one hand this implies that their positive results apply to the single-dimensional valuations setting studied here, and show that there are deterministic mechanisms for divisible items and randomized mechanisms for both divisible and indivisible items that are IC with *public* budgets. On the other hand this implies that our negative result for the single-dimensional valuations setting applies to the settings studied in these papers, and shows that there can be no *deterministic* mechanisms that are IC with *public* budgets for *indivisible* items. Finally, the impossibility results presented in [6, 9] either assume that the valuations are non-additive or that the allocations satisfy arbitrary polyhedral constraints and have therefore no implications for the multi-dimensional valuations setting studied here.

Overview. We summarize the results from related work and this paper for indivisible items in Table 1 and for divisible items in Table 2. We use a plus (+ or $\oplus$) to indicate that there is an IR, PO, NPT, and IC mechanism. We use a minus ($-$ or $\ominus$) to indicate that there is no IR, PO, and IC mechanism. We use

+ and − for results from related work and ⊕ and ⊖ for results from this paper. A question mark (?) indicates that nothing is known for this setting. For the model of [8] the table has two entries, one for public and one for private interest sets. *While all positive results from this paper are polynomial-time algorithms, all negative results hold independent of complexity considerations.*

Table 1. Results for *Indivisible* Items from Related Work and this Paper

		homogeneous		heterogeneous & additive			
	budgets	add.	non-add.	interest set public/private	multi-keyword unit demand	single-dim.	multi-dim.
det.	public	+ [7]	−[10, 6]	+[8]/−[8]	⊖	⊖	− [8]
	private	− [7]	− [7]	−[7]/−[7]	− [7]	−[7]	− [7]
rand.	public	+ [7]	?	+[8]/?	+[6, 9]	⊕	⊖
	private	+ [4]	?	?/?	?	⊕	⊖

Table 2. Results for *Divisible* Items from Related Work and this Paper

		homogeneous		heterogeneous & additive			
	budgets	add.	non-add.	polymatroid constraints	multi-keyword unit demand	single-dim.	multi-dim.
det.	public	+ [7, 4]	−[9]	+[9]	+[6, 9]	⊕	⊖
	private	− [7]	−[7]	− [7]	− [7]	− [7]	− [7]
rand.	public	+ [7, 4]	?	+[9]	+[6, 9]	⊕	⊖
	private	+ [4]	?	?	?	⊕	⊖

Techniques. Our technical contributions are as follows:

(a) For multi-dimensional valuations we obtain a partial characterization of IC by generalizing the "weak monotonicity" (WMON) condition of [5] from settings *without budgets* to settings *with public budgets*. We obtain our impossibility result for deterministic mechanisms and divisible items by showing that in certain settings WMON will be violated. For this we use that multi-dimensional valuations enable the agents to lie in a sophisticated way: While all previous impossibility proofs in this area used agents that either only overstate or only understate their valuations, we use an agent that overstates his valuation for one item and understates his valuation for another.

(b) For single-dimensional valuations and both divisible and indivisible items we characterize PO by a simpler "no trade" (NT) condition. Although this condition is more complex than similar conditions in [7, 4, 8], we are able to show that an outcome is PO if and only if it satisfies NT. We also generalize the "classic"characterization results of IC mechanism of [11, 1] from settings *without budgets* to settings with *public budgets* by showing that a mechanism is IC with public budgets if and only if it satisfies "value monotonicity" (VM) and "payment identity" (PI). The characterizations of PO and IC with public budgets play a crucial role in the proof of our impossibility result for indivisible items, which uses NT and PI to derive lower bounds on the agents' payments that conflict with the upper bounds on the payments induced by IR.

(c) We establish the positive results for single-dimensional valuations and both divisible and indivisible items by giving a new reduction of this case to the case of a single and by definition homogeneous item. This allows us to apply the techniques that [4] developed for the single-item setting. This is a general reduction between the heterogeneous items setting and the homogeneous items setting, which is likely to have further applications.

(d) We give an explicit polynomial-time algorithm for the "adaptive clinching auction" for *divisible* items and an arbitrary number of agents. To the best of our knowledge we are the first ones to actually give a polynomial-time version of this auction for arbitrarily many agents.

Due to space constraints we omit some proofs and the description of the polynomial-time algorithm from this extended abstract, and refer the reader to the full version of the paper for details.

2 Problem Statement

We are given a set N of n agents and a set M of m items. We distinguish between settings with divisible items and settings with indivisible items. In both settings we use $X = \prod_{i=1}^{n} X_i$ for the allocation space. For divisible items the allocation space is $X_i = [0,1]^m$ for all agents $i \in N$ and $x_{i,j} \in [0,1]$ denotes the fraction of item $j \in M$ that is allocated to agent $i \in N$. For indivisible items the allocation space is $X_i = \{0,1\}^m$ for all agents $i \in N$ and $x_{i,j} \in \{0,1\}$ indicates whether item $j \in M$ is allocated to agent $i \in N$ or not. In both cases we require that $\sum_{i=1}^{n} x_{i,j} \leq 1$ for all items $j \in M$. We do *not* require that $\sum_{j=1}^{m} x_{i,j} \leq 1$ for all agents $i \in N$, i.e., we do *not* assume that the agents have unit demand.

Each agent i has a type $\theta_i = (v_i, b_i)$ consisting of a valuation function $v_i : X_i \to \mathbb{R}_{\geq 0}$ and a budget $b_i \in \mathbb{R}_{\geq 0}$. We use $\Theta = \prod_{i=1}^{n} \Theta_i$ for the type space. We consider two settings with heterogeneous items, one with multi- and one with single-dimensional valuations. In the first setting, each agent $i \in N$ has a valuation $v_{i,j} \in \mathbb{R}_{\geq 0}$ for each item $j \in M$ and agent i's valuation for allocation x_i is $v_i(x_i) = \sum_{j=1}^{m} x_{i,j} v_{i,j}$. In the second setting, which is inspired by sponsored search auctions, each agent $i \in N$ has a valuation $v_i \in \mathbb{R}_{\geq 0}$, each item $j \in M$ has a quality $\alpha_j \in \mathbb{R}_{\geq 0}$, and agent i's valuation for allocation $x_i \in X_i$ is $v_i(x_i) = \sum_{j=1}^{m} x_{i,j} \alpha_j v_i$. For simplicity we will assume that in this setting $\alpha_1 > \alpha_2 > \cdots > \alpha_m$ and that $v_1 > v_2 > \cdots > v_n > 0$.

A (direct revelation) mechanisms $M = (x, p)$ consisting of an allocation rule $x : \Theta \to X$ and a payment rule $p : \Theta \to \mathbb{R}^n$ is deployed to compute an outcome (x, p) consisting of an allocation $x \in X$ and payments $p \in \mathbb{R}^n$. We say that a mechanism is deterministic if the computation of (x, p) is deterministic, and it is randomized if the computation of (x, p) is randomized.

We assume that the agents are utility maximizers and as such need not report their types truthfully. We consider settings in which both the valuations and budgets are private and settings in which only the valuations are private and the budgets are public. When the valuations resp. budgets are private, then the other agents have no knowledge about them, not even about their distribution.

In the former setting a report by agent $i \in N$ with true type $\theta_i = (v_i, b_i)$ can be any type $\theta'_i = (v'_i, b'_i)$. In the latter setting agent $i \in N$ is restricted to reports of the form $\theta'_i = (v'_i, b_i)$. In both settings, if mechanism $M = (x, p)$ is used to compute an outcome for reported types $\theta' = (\theta'_1, \ldots, \theta'_n)$ and the true types are $\theta = (\theta_1, \ldots, \theta_n)$ then the utility of agent $i \in N$ is

$$u_i(x_i(\theta'), p_i(\theta'), \theta_i) = \begin{cases} v_i(x_i(\theta')) - p_i(\theta') & \text{if } p_i(\theta') \leq b_i, \text{ and} \\ -\infty & \text{otherwise.} \end{cases}$$

For deterministic mechanisms and their outcomes we are interested in the following properties:

(a) *Individual rationality (IR)*: A mechanism is IR if it always produces an IR outcome. An outcome (x, p) for types $\theta = (v, b)$ is IR if it is (i) *agent rational:* $u_i(x_i, p_i, \theta_i) \geq 0$ for all agents $i \in N$ and (ii) *auctioneer rational:* $\sum_{i=1}^n p_i \geq 0$. (b) *Pareto optimality (PO)*: A mechanism is PO if it always produces a PO outcome. An outcome (x, p) for types $\theta = (v, b)$ is PO if there is no other outcome (x', p') such that $u_i(x'_i, p'_i, \theta_i) \geq u_i(x_i, p_i, \theta_i)$ for all agents $i \in N$ and $\sum_{i=1}^n p'_i \geq \sum_{i=1}^n p_i$, with at least one of the inequalities strict.[2] (c) *No positive transfers (NPT)*: A mechanism satisfies NPT if it always produces an NPT outcome. An outcome (x, p) satisfies NPT if $p_i \geq 0$ for all agents $i \in N$. (d) *Incentive compatibility (IC)*: A mechanism satisfies IC if for all agents $i \in N$, all true types θ, and all reported types θ' we have $u_i(x_i(\theta_i, \theta'_{-i}), p_i(\theta_i, \theta'_{-i}), \theta_i) \geq u_i(x_i(\theta'_i, \theta'_{-i}), p_i(\theta'_i, \theta'_{-i}), \theta_i)$.

If a randomized mechanism satisfies any of these conditions in expectation, then we say that the respective property is satisfied *ex interim*. If it satisfies any of these properties for all outcomes it produces, then we say that it satisfies the respective property *ex post*.

3 Multi-dimensional Valuations

In this section we obtain a partial characterization of mechanisms that are IC with public budgets by generalizing the "weak monotonicity" condition of [5] from settings without budgets to settings with budgets. We use this partial characterization together with a sophisticated way of lying, in which an agent understates his valuation for some item and overstates his valuation for another item, to prove that there can be no *deterministic* mechanism for *divisible* items that is IR, PO, and IC with public budgets. Afterwards, we use this result to show that there can be no *randomized* mechanism for either *divisible* or *indivisible* items that is IR ex interim, PO ex interim, and IC ex interim for public budgets.

Partial Characterization of IC. For settings *without budgets* every mechanism that is incentive compatible must satisfy what is known as *weak monotonicity (WMON)*, namely if x'_i and x_i are the assignments of agent i for reports v'_i and v_i, then the difference in the valuations for the two assignments must

[2] Both IR and PO are defined with respect to the reported types, and are satisfied with respect to the true types only if the mechanism also satisfies IC.

be at least as large under v'_i as under v_i, i.e., $v'_i(x_i(\theta'_i, \theta_{-i})) - v'_i(x_i(\theta_i, \theta_{-i})) \geq v_i(x_i(\theta'_i, \theta_{-i})) - v_i(x_i(\theta_i, \theta_{-i}))$. We show that this is also true for mechanisms that respect the publicly known budget limits.[3]

Proposition 1. *If a mechanism $M = (x, p)$ for multi-dimensional valuations and either divisible or indivisible items that respects the publicly known budget limits is IC, then it satisfies WMON.*

Deterministic Mechanisms for Divisible Items. We prove the impossibility result by analyzing a setting with two agents and two items. This restriction is without loss of generality as the impossibility result for an arbitrary number of agents $n > 2$ and an arbitrary number of items $m > 2$ follows by setting $v_{i,j} = 0$ if $i > 2$ or $j > 2$. In our impossibility proof agent 2 is not budget restricted (i.e., $b_2 > v_{2,1} + v_{2,2}$). Agents can lie when they report their valuations, and it is not sufficient to study a single input to prove the impossibility. Hence, we study the outcome for three related cases, namely Case 1 where $v_{1,1} < v_{2,1}$ and $v_{1,2} < v_{2,2}$; Case 2 where $v_{1,1} > v_{2,1}$, $v_{1,2} < v_{2,2}$, and $b_1 > v_{1,1}$; and Case 3 where $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, and additionally, $b_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, and $v_{2,1} + v_{2,2} > b_1$. We give a partial characterization of those cases, which allows us to analyze the rational behavior of the agents.

Case 1 is easy: Agent 2 is not budget restricted and has the highest valuations for both items; so he will get both items. Thus the utility for agent 1 is zero. Based on this observation Case 2 can be analyzed: Agent 1 has the higher valuation for item 1, while agent 2 has the higher valuation for item 2. Thus, agent 1 gets item 1 and agent 2 gets item 2. Since the only difference to Case 1 is that in Case 2 $v_{1,1} > v_{2,1}$ while in Case 1 $v_{1,1} < v_{2,1}$, the critical value whether agent 2 gets item 1 or not is $v_{2,1}$. Thus, in every IC mechanism, agent 1 has to pay $v_{2,1}$ and has utility $v_{1,1} - v_{2,1}$. The details of these proofs can be found in the full version of the paper. Using these observations we are able to exactly characterize the allocation produced in Case 3 as follows: In Case 3 agent 1 has a higher valuation than agent 2 for both items, but he does not have enough budget to pay for both fully. First we show that if agent 1 does not spend his whole budget ($p_1 < b_1$) he must fully receive both items (specifically $x_{1,2} = 1$), since if not, he would buy more of them. Additionally, even if he spent his budget fully (i.e., $p_1 = b_1$) his utility u_i, which equals $x_{1,1}v_{1,1} + x_{1,2}v_{1,2} - b_1$, must be non-negative. Since $b_1 > v_{1,1}$ this implies that $x_{1,1}$ must be 1, i.e., he must receive item 1 fully, and $x_{1,2}$ must be non-zero.

Lemma 1. *Given $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $b_1 > v_{1,1}$, and $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, if $p_1 < b_1$ then $x_{1,1} = 1$ and $x_{1,2} = 1$, else if $p_1 = b_1$ then $x_{1,1} = 1$ and $x_{1,2} > 0$, in every IR and PO outcome.*

Then we show that actually $x_{1,2} < 1$, which, combined with the previous lemma, implies that $p_1 = b_1$. The fact that $x_{1,2} < 1$, i.e, that agent 1 does not fully get

[3] Without this restriction we could charge $p_i > b_i$ from all agents $i \in N$ to be IC. This restriction is satisfied by IR mechanisms to which we will apply this result.

item 1 *and* 2 is not surprising since he does not have enough budget to outbid agent 2 on both items as $b_1 < v_{2,1}+v_{2,2}$. However, we are even able to determine the exact value of $x_{1,2}$, which is $(b_1 - v_{2,1})/v_{2,2}$.

Lemma 2. *Given $b_2 > v_{2,1} + v_{2,2}$, $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $b_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, and $v_{2,1} + v_{2,2} > b_1$, then $p_1 = b_1$ and $x_{1,2} = (b_1 - v_{2,1})/v_{2,2} < 1$ in every IR and PO outcome selected by an IC mechanism.*

We combine these characterizations of Case 3 with (a) the WMON property shown in Proposition 1 and (b) a sophisticated way of the agent to lie: He *overstates* his value for item 1 by a value α and *understates* his value for item 2 by a value $0 < \beta < \alpha$, but by such small values that Case 3 continues to hold. Thus, by Lemma 1 $x_{2,1}$ remains 0 (whether the agent lies or does not), and thus, the WMON condition implies that $x_{2,2}$ does *not* increase. However, by the dependence of $x_{1,2}$ on $v_{2,1}$ and $v_{2,2}$ shown in Lemma 2, $x_{1,2}$, and thus also $x_{2,2}$ changes when agent 2 lies. This gives a contradiction to the assumption that such a mechanism exists.

Theorem 1. *There is no deterministic IC mechanism for divisible items which selects for any given input with public budgets an IR and PO outcome.*

Proof. Let us assume by contradiction that such a mechanism exists and consider an input for which $b_2 > v_{2,1} + v_{2,2}$, $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $b_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, and $v_{2,1} + v_{2,2} > b_1$ holds. Such an input exists, for example $v_{1,1} = 4$, $v_{1,2} = 5$, $v_{2,1} = 3$, and $v_{2,2} = 4$ with budgets $b_1 = 5$ and $b_2 = 8$ would be such an input. Lemma 1 and 2 imply that $x_{1,1} = 1$, $x_{2,1} = 0$, $x_{1,2} = \frac{b_1 - v_{2,1}}{v_{2,2}}$, $x_{2,2} = 1 - x_{1,2}$, and $p_1 = b_1$. Let us consider an alternative valuation by agent 2. We define $v'_{2,1} = v_{2,1} + \alpha$ and $v'_{2,2} = v_{2,2} - \beta$ for arbitrary $\alpha, \beta > 0$ and $\alpha > \beta$ which are sufficiently small such that $v_{1,1}v'_{2,2} > v_{1,2}v'_{2,1}$ holds. By Proposition 1, IC implies WMON, and therefore, $x'_{2,2}v'_{2,2} - x_{2,2}v'_{2,2} \geq x'_{2,2}v_{2,2} - x_{2,2}v_{2,2}$. It follows that $x_{2,2} \geq x'_{2,2}$, and by Lemma 2, $\frac{b_1 - v_{2,1}}{v_{2,2}} \leq \frac{b_1 - v'_{2,1}}{v'_{2,2}}$. Hence, the budget of agent 1 has to be large enough, such that $b_1 \geq \frac{v_{2,2}v'_{2,1} - v_{2,1}v'_{2,2}}{v_{2,2} - v'_{2,2}} = \frac{v_{2,1}\beta + v_{2,2}\alpha}{\beta} > v_{2,1} + v_{2,2}$, but $b_1 < v_{2,1} + v_{2,2}$ holds by assumption. Contradiction! □

Randomized Mechanisms for Divisible and Indivisible Items. We exploit the fact that randomized mechanisms for both divisible and indivisible items are essentially equivalent to deterministic mechanisms for divisible items.

We show that for agents with budget constraints every randomized mechanism $\bar{M} = (\bar{x}, \bar{p})$ for divisible or indivisible items can be mapped bidirectionally to a deterministic mechanism $M = (x, p)$ for divisible items with identical expected utility for all the agents and the auctioneer when the same reported types are used as input. To turn a randomized mechanism for *indivisible* items into a deterministic mechanism for *divisible* items simply compute the expected values of p_i and $x_{i,j}$ for all i and j and return them. To turn a deterministic mechanism for *divisible* items into a randomized mechanism for *indivisible* items simply pick values with probability $x_{i,j}$ and keep the same payment as the deterministic mechanism.

Proposition 2. *Every randomized mechanism $\bar{M} = (\bar{x}, \bar{p})$ for agents with finite budgets, a rational auctioneer, and a limited amount of divisible or indivisible items can be mapped bidirectionally to a deterministic mechanism $M = (x, p)$ for divisible items such that $u_i(x_i(\theta'), p_i(\theta'), \theta_i) = \mathrm{E}[u_i(\bar{x}_i(\theta'), \bar{p}_i(\theta'), \theta_i)]$ and $\sum_{i \in N} p_i(\theta') = \mathrm{E}[\sum_{i \in N} \bar{p}_i(\theta')]$ for all agents i, all true types $\theta = (v, b)$, and reported types $\theta' = (v', b')$.*

Proof. Let us map $\bar{M} = (\bar{x}, \bar{p})$ to $M = (x, p)$ that assigns for each agent $i \in N$ and item $j \in M$ a fraction of $\mathrm{E}[\bar{x}_{i,j}]$ of item j to agent i, and makes each agent $i \in N$ pay $\mathrm{E}[\bar{p}_i]$. The expectations exist since the feasible fractions of items and the feasible payments have an upper bound and a lower bound. For the other direction, we map $M = (x, p)$ to $\bar{M} = (\bar{x}, \bar{p})$ that randomly picks for each item $j \in M$ an agent $i \in N$ to which it assigns item j in a way such that agent i is picked with probability $x_{i,j}$, and makes each agent $i \in N$ pay p_i. Since $x = \mathrm{E}[\bar{x}]$ and $p = \mathrm{E}[\bar{p}]$, $\sum_{j \in M}(x_{i,j} v_{i,j}) - p_i = \mathrm{E}[\sum_{j \in M}(\bar{x}_{i,j} v_{i,j}) - \bar{p}_i]$ for all $i \in N$ and $\sum_{i \in N} p_i = \mathrm{E}[\sum_{i \in N} \bar{p}_i]$.

This proposition implies the non-existence of randomized mechanisms stated in Theorem 2.

Theorem 2. *There can be no randomized mechanism for divisible or indivisible items that is IR ex interim, PO ex interim, and IC ex interim, and that satisfies the public budget constraint ex post.*

Proof. For a contradiction suppose that there is such a randomized mechanism. Then, by Proposition 2, there must be a deterministic mechanism for divisible items and public budgets that satisfies IR, PO, and IC. This gives a contradiction to Theorem 1. □

4 Single-Dimensional Valuations

In this section we present exact characterizations of PO outcomes and mechanisms that are IC with public budgets. We characterize PO by a simpler "no trade" condition and, similar to Section 3, we extend the "classic" characterization results for IC mechanisms for single-dimensional valuations (see, e.g., [11, 1]) without budgets to settings with public budgets. We use these characterizations to show that there can be no deterministic mechanism for divisible items that is IR, PO, and IC with public budgets. We also present a reduction to the setting with a single (and thus homogeneous) item that allows us to apply the following proposition from [4]. The basic building block of the mechanisms mentioned in this proposition is the "adaptive clinching auction" for a single divisible item. It is described for two agents in [7], as a "continuous time process" for arbitrarily many agents in [4], and as an explicit polynomial-time algorithm for arbitrarily many agents in the full version of this paper.

Proposition 3 ([4]). *For a single divisible item there exists a deterministic mechanism that satisfies IR, NPT, PO, and IC for public budgets. Additionally,* *for a single divisible or indivisible item there exists a randomized mechanism that satisfies IR ex interim, NPT ex post, PO ex post, and IC ex interim for private budgets.*

Exact Characterizations of PO and IC. We start by characterizing PO outcomes through a simpler "no trade" condition. Outcome (x, p) for single-dimensional valuations and either divisible or indivisible items that respects the budget limits satisfies *no trade (NT)* if (a) $\sum_{i \in N} x_{i,j} = 1$ for all $j \in M$, and (b) there is no x' such that for $\delta_i = \sum_{j \in M}(x'_{i,j} - x_{i,j})\alpha_j$ for all $i \in N$, $W = \{i \in N \mid \delta_i > 0\}$, and $L = \{i \in N \mid \delta_i \leq 0\}$ we have $\sum_{i \in N} \delta_i v_i > 0$ and $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$.[4] This definition says that there should be no alternative assignment that overall increases the sum of the valuations, and allows the "winners" to compensate the "losers". It differs from the definitions in prior work in that it allows trades that involve both items *and* money. We will exploit this fact in the proof of our impossibility result.

Proposition 4. *Outcome (x, p) for single-dimensional valuations and either divisible or indivisible items that respects the budget limits is PO if and only if it satisfies NT.*

Next we characterize mechanisms that are IC with public budgets by "value monotonicity" and "payment identity". Mechanism $M = (x, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets satisfies *value monotonicity (VM)* if for all $i \in N$, $\theta_i = (v_i, b_i)$, $\theta'_i = (v'_i, b_i)$, and $\theta_{-i} = (v_{-i}, b_{-i})$ we have that $v_i \leq v'_i$ implies $\sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j \leq \sum_{j \in M} x_{i,j}(\theta'_i, \theta_{-i})\alpha_j$. Mechanism $M = (x, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets satisfies *payment identity (PI)* if for all $i \in N$ and $\theta = (v, b)$ with $c_{\gamma_t} \leq v_i \leq c_{\gamma_{t+1}}$ we have $p_i(\theta) = p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i})$, where $\gamma_0 < \gamma_1 < \ldots$ are the values $\sum_{j \in M} x_{i,j}\alpha_j$ can take and $c_{\gamma_s}(b_i, \theta_{-i})$ for $1 \leq s \leq t$ are the corresponding critical valuations. While VM ensures that stating a higher valuation can only lead to a better allocation, PI gives a formula for the payment in terms of the possible allocations and the critical valuations. In the proof of our impossibility result we will use the fact that the payments for worse allocations provide a lower bound on the payments for better allocations.

Proposition 5. *Mechanism $M = (x, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets is IC if and only if it satisfies VM and PI.*

Deterministic Mechanisms for Indivisible Items. The proof of our impossibility result uses the characterizations of PO outcomes and mechanisms that are IC with public budgets as follows: (a) PO is characterized by NT and NT induces a lower bound on the agents' payments for a *specific* assignment, namely

[4] For PO we only need that the outcome respects the *reported* budget limits. Hence our characterization also applies in *private* budget settings.

for the case that agent 1 only gets item m. (b) IC, in turn, is characterized by VM and PI. Now VM and PI can be used to extend the lower bound on the payments for the *specific* assignment to *all* possible assignments. (c) Finally, IR implies upper bounds on the payments that, with a suitable choice of valuations, conflict with the lower bounds on the payments induced by NT, VM, and PI.

Theorem 3. *For single-dimensional valuations, indivisible items, and public budgets there can be no deterministic mechanism $M = (x, p)$ that satisfies IR, PO, and IC.*

Proof. For a contradiction suppose that there is a mechanism $M = (x, p)$ that is IR, PO, and IC for all n and all m. Consider a setting with $n = 2$ agents and $m = 2$ items in which $v_1 > v_2 > 0$ and $b_1 > \alpha_1 v_2$.

Observe that if agent 1's valuation was $v_1' = 0$ and he reported his valuation truthfully, then since M satisfies IR his utility would be $u_1((0, b_1), \theta_{-1}, (0, b_1)) = -p_1((0, b_1), \theta_{-1}) \geq 0$. This shows that $p_1((0, b_1), \theta_{-1}) \leq 0$.

By PO, which by Proposition 4 is characterized by NT, agent 1 with valuation $v_1 > v_2$ and budget $b_1 > \alpha_1 v_2$ must win at least one item because otherwise he could buy any item from agent 2 and compensate him for his loss.

PO, respectively NT, also implies that agent 1's payment for item 2 must be strictly larger than $b_1 - (\alpha_1 - \alpha_2)v_2$ because otherwise he could trade item 2 against item 1 and compensate agent 2 for his loss.

By IC, which by Proposition 5 is characterized by VM and PI, agent 1's payment for item 2 is given by $p_1(\{2\}) = p_1((0, b_1), \theta_{-1}) + \alpha_2 c_{\alpha_2}(b_1, \theta_{-1})$, where c_{α_2} is the critical valuation for winning item 2. Together with $p_1(\{2\}) > b_1 - (\alpha_1 - \alpha_2)v_2$ this shows that $c_{\alpha_2}(b_1, \theta_{-1}) > (1/\alpha_2)[b_1 - (\alpha_1 - \alpha_2)v_2 - p_1((0, b_1), \theta_{-1})]$.

IC, respectively VM and PI, also imply that agent 1's payment for any non-empty set of items S in terms of the fractions $\gamma_t = \sum_{j \in S} \alpha_j > \cdots > \gamma_1 = \alpha_2 > \gamma_0 = 0$ and corresponding critical valuations $c_{\gamma_t}(b_1, \theta_{-1}) \geq \cdots \geq c_{\gamma_1}(b_1, \theta_{-1}) = c_{\alpha_2}(b_1, \theta_{-1})$ is $p_1(S) = p_1((0, b_1), \theta_{-1}) + \sum_{s=1}^{t} (\gamma_s - \gamma_{s-1}) c_{\gamma_s}(b_1, \theta_{-1})$. Because $c_{\gamma_s}(b_1, \theta_{-1}) \geq c_{\alpha_2}(b_1, \theta_{-1})$ for all s and $\sum_{s=1}^{t} (\gamma_s - \gamma_{s-1}) = \sum_{j \in S} \alpha_j$ we obtain $p_1(S) \geq p_1((0, b_1), \theta_{-1}) + (\sum_{j \in S} \alpha_j) c_{\alpha_2}(b_1, \theta_{-1})$.

Combining this lower bound on $p_1(S)$ with the lower bound on $c_{\alpha_2}(b_1, \theta_{-1})$ shows that $p_1(S) > (\sum_{j \in S} \alpha_j / \alpha_2)[b_1 - (\alpha_1 - \alpha_2)v_2]$.

For v_1 such that $(1/\alpha_2)[b_1 - (\alpha_1 - \alpha_2)v_2] > v_1 > v_2$ we know that agent 1 must win some item, but for any non-empty set of items S the lower bound on agent 1's payment for S contradicts IR. □

Randomized Mechanisms for Indivisible and Divisible Items. Interestingly, the impossibility result for deterministic mechanisms for indivisible items can be avoided by a randomized mechanism: (a) Apply the randomized mechanism for a single *indivisible* item of [4] to a single indivisible item for which agent $i \in N$ has valuation $\tilde{v}_i = \sum_{j \in M} \alpha_j v_i$. (b) Map the single-item outcome $(\tilde{x}, \tilde{p})$ into an outcome (x, p) for the multi-item setting by setting $x_{i,j} = 1$ for all $j \in M$ if and only if $\tilde{x}_i = 1$ and setting $p_i = \tilde{p}_i$ for all $i \in N$.

A similar idea works for divisible items. The only difference is that we use the mechanisms of [4] for a single *divisible* item, and map the single-item outcome

$(\tilde{x}, \tilde{p})$ into a multi-item outcome by setting $x_{i,j} = \tilde{x}_i$ for all $i \in N$ and all $j \in M$ and setting $p_i = \tilde{p}_i$ for all $i \in N$.

The main difficulty in proving that the resulting mechanisms inherit the properties of the mechanisms in [4] is to show that the resulting mechanisms satisfy PO (ex post). For this we argue that a certain structural property of the single-item outcomes is preserved by the mapping to the multi-item setting and remains to be sufficient for PO (ex post).

Proposition 6. *Let $(\bar{x}, \bar{p})$ be the outcome of our mechanism and let (x, p) be the outcome of the respective mechanism of [4], then $u_i(\bar{x}_i, \bar{p}_i) = u_i(x_i, p_i)$ for all $i \in N$ resp. $E[u_i(\bar{x}_i, \bar{p}_i)] = E[u_i(x_i, p_i)]$ for all $i \in N$.*

Theorem 4. *For single-dimensional valuations, divisible or indivisible items, and private budgets there is a randomized mechanism that satisfies IR ex interim, NPT ex post, PO ex post, and IC ex interim. Additionally, for single-dimensional valuations and divisible items there is a deterministic mechanism that satisfies IR, NPT, PO, and IC for public budgets.*

Proof. IR (ex interim) and IC (ex interim) follow from Proposition 6 and the fact that the mechanisms of [4] are IR (ex interim) and IC (ex interim). NPT (ex post) follows from the fact that the payments in our mechanisms and the mechanisms of [4] are the same, and the mechanisms in [4] satisfy NPT (ex post). For PO (ex post) we argue that the structural property of the outcomes of the mechanisms in [4] that (a) $\sum_{i \in N} \tilde{x}_{i,j} = 1$ for all $j \in M$ and (b) $\sum_{j \in M} \tilde{x}_{i,j} > 0$ and $\tilde{v}_{i'} > \tilde{v}_i$ imply $\tilde{p}_{i'} = b_{i'}$ is preserved by the mapping to the multi-item setting and remains to be sufficient for PO (ex post).

We begin by showing that the structural property is preserved by the mapping. For this observe that $\sum_{i \in N} \tilde{x}_{i,j} = 1$ for all $j \in M$ implies that $\sum_{i \in N} x_{i,j} = 1$ for all $j \in M$ and that $\sum_{j \in M} \tilde{x}_{i,j} > 0$ and $\tilde{v}_{i'} > \tilde{v}_i$ imply $\tilde{p}_{i'} = b_{i'}$ implies that $\sum_{j \in M} x_{i,j} > 0$ and $v_{i'} > v_i$ imply $p_{i'} = b_{i'}$.

Next we show that the structural property remains to be sufficient for PO (ex post). For this assume by contradiction that the outcome (x, p) is *not* PO (ex post). Then, by Proposition 4, there exists an x' such that $\sum_{i \in N} \delta_i v_i > 0$ and $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$, where $\delta_i = \sum_{j \in M} (x'_{i,j} - x_{i,j}) \alpha_j$, $W = \{i \in N \mid \delta_i > 0\}$, and $L = \{i \in N \mid \delta_i \leq 0\}$.

Because (x, p) satisfies condition (a), i.e., $\sum_{i \in N} x_{i,j} = 1$ for all $j \in M$, and x' is a valid assignment, i.e., $\sum_{i \in N} x'_{i,j} \leq 1$ for all $j \in M$, we have $\sum_{i \in N} \delta_i = \sum_{j \in M} \sum_{i \in N} (x'_{i,j} - x_{i,j}) \alpha_j \leq 0$. Because $\sum_{i \in N} \delta_i v_i > 0$ we have $\sum_{i \in W} \delta_i v_i \geq \sum_{i \in N} \delta_i v_i > 0$ and, thus, $\sum_{i \in W} \delta_i > 0$. We conclude that $\sum_{i \in L} \delta_i = \sum_{i \in N} \delta_i - \sum_{i \in W} \delta_i < 0$ and, thus, $\sum_{i \in L} \delta_i v_i < 0$.

Because (x, p) satisfies condition (b), i.e., $\sum_{j \in M} x_{i,j} > 0$ and $v_{i'} > v_i$ imply $p_{i'} = b_{i'}$, there exists a t with $1 \leq t \leq n$ such that (1) $\sum_{j \in M} x_{i,j} \geq 0$ and $p_i = b_i$ for $1 \leq i \leq t$, (2) $\sum_{j \in M} x_{i,j} \geq 0$ and $p_i \leq b_i$ for $i = t+1$, and (3) $\sum_{j \in M} x_{i,j} = 0$ and $p_i \leq b_i$ for $t + 2 \leq i \leq n$.

Case 1: $t = n$. Then $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) = 0$ and, thus, $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i < 0$.

Case 2: $t < n$ and $W \cap \{1, \ldots, t\} = \emptyset$. Then $\sum_{i \in W} \delta_i v_i \leq \sum_{i \in W} \delta_i v_{t+1}$ and $\sum_{i \in L} \delta_i v_i \leq \sum_{i \in L} \delta_i v_{t+1}$ and, thus, $\sum_{i \in N} \delta_i v_i = \sum_{i \in W} \delta_i v_i + \sum_{i \in L} \delta_i v_i \leq \sum_{i \in N} \delta_i v_{t+1} \leq 0$.

Case 3: $t < n$ and $W \cap \{1, \ldots, t\} \neq \emptyset$. Then $\sum_{i \in W} \min(p_i - b_i, \delta_i v_i) \leq \sum_{i \in W \setminus \{1..t\}} \delta_i v_{t+1}$ and $\sum_{i \in L} \delta_i v_i \leq \sum_{i \in L} \delta_i v_{t+1}$ and, thus, $\sum_{i \in W} \min(p_i - b_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \leq (\sum_{i \in N} \delta_i - \sum_{i \in W \cap \{1,\ldots,t\}} \delta_i) v_{t+1} < 0$. □

5 Conclusion and Future Work

In this paper we analyzed IR, PO, and IC mechanisms for settings with heterogeneous items. Our main accomplishments are: (a) An impossibility result for *randomized* mechanisms and *public* budgets for additive valuations. (b) *Randomized* mechanisms that achieve these properties for *private* budgets and a restricted class of additive valuations. We are able to circumvent the impossibility result in the restricted setting because our argument for the impossibility result is based on the ability of an agent to overstate his valuation for one and understate his valuation for another item, which is not possible in the restricted setting. A promising direction for future work is to identify other valuations for which this is the case.

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Bayesian Mechanism Design with Efficiency, Privacy, and Approximate Truthfulness*

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Abstract. Recently, there has been a number of papers relating mechanism design and privacy (e.g., see [1–6]). All of these papers consider a worst-case setting where there is no probabilistic information about the players' types. In this paper, we investigate mechanism design and privacy in the *Bayesian* setting, where the players' types are drawn from some common distribution. We adapt the notion of *differential privacy* to the Bayesian mechanism design setting, obtaining *Bayesian differential privacy*. We also define a robust notion of approximate truthfulness for Bayesian mechanisms, which we call *persistent approximate truthfulness*. We give several classes of mechanisms (e.g., social welfare mechanisms and histogram mechanisms) that achieve both Bayesian differential privacy and persistent approximate truthfulness. These classes of mechanisms can achieve optimal (economic) efficiency, and do not use any payments. We also demonstrate that by considering the above mechanisms in a modified mechanism design model, the above mechanisms can achieve actual truthfulness.

1 Introduction

One of the main goals in mechanism design is to design mechanisms that achieve a socially desirable outcome even if the players behave selfishly. Because of the revelation principle, mechanism design has focused on direct (revelation) mechanisms where each player simply reports his/her private type (or valuation). This leads to the issue of privacy, where the players may be concerned that the mechanism's output may leak information about their private types (even if the mechanism is trusted).

Mechanism Design and Privacy. Traditional mechanism design did not include the aspect of privacy. However, in the context of releasing information from databases, the issue of privacy has already been studied quite extensively. In this context, the current standard notion of privacy is *differential privacy* [7, 8]. A data release algorithm satisfies differential privacy if the algorithm's output distribution does not change much when one person's data is changed in

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the database. This implies that the algorithm does not leak much information about any person in the database.

Recently, there has been a number of papers that combine mechanism design with differential privacy. In [1], McSherry and Talwar develop a general mechanism called the *exponential mechanism* that is differentially private; they also show that any differentially private mechanism is *approximately* truthful. In [4], Nissim, Smorodinsky, and Tennenholtz modify the standard mechanism design model by adding a "reaction stage"; in this new model, the authors combine differentially private mechanisms with a "punishing mechanism" to obtain mechanisms that are *actually* truthful. However, the mechanisms in [4] might not protect the privacy of the players, due to the reaction stage.

The main goal of the above two papers was to use differential privacy as a tool for achieving some form of truthfulness, as opposed to achieving privacy for the players. However, there has been other papers that focus on designing mechanisms that protect the privacy of the players. In [6], Huang and Kannan show that a pricing scheme can be added to the exponential mechanism to make it *actually* truthful, resulting in a general mechanism that is both differentially private and truthful. In [2], Xiao provides a transformation that takes truthful mechanisms and transforms them into truthful and differentially private mechanisms. On the other hand, Xiao also shows that a mechanism that is truthful and differentially private might not be truthful in a model where the players are "privacy-aware", i.e., privacy is explicitly captured in the players' utility functions. In [3], Chen et al. construct mechanisms that are truthful even in a model where the players are privacy-aware. In [5], Nissim, Orlandi, and Smorodinsky construct mechanisms that are truthful in a different privacy-aware model.

Bayesian Mechanism Design. One desirable property of a mechanism is (economic) *efficiency*; in fact, it would be best if the mechanism always chooses a social alternative that is *optimal* with respect to some measure of efficiency, such as social welfare. However, such *optimal efficiency* is not achieved by any of the above results. In fact, it is not possible for a differentially private mechanism to achieve optimal efficiency (for a non-trivial problem), since the mechanism has to be randomized in order to satisfy differential privacy. However, all of the above results are in a worst-case setting where there is no probabilistic information about the players' types. If we consider a non-worst-case setting, then it may be possible for a mechanism to achieve differential privacy without using any randomization.

One such setting is the *Bayesian* setting, where the players' types are drawn from some common distribution. Such a setting follows the Bayesian approach that has been the standard in economic theory for many decades. Recently, mechanism design in the Bayesian setting has also been gaining popularity in the computer science community. Thus, it is interesting to consider the issue of privacy in the Bayesian setting as well. In particular, it may be possible for a Bayesian mechanism to achieve optimal efficiency while satisfying some form of differential privacy. Achieving optimal efficiency may be critical for certain problems, such as presidential elections and kidney transplant allocations, where it may be unethical and/or unfair to make a non-optimal choice. Although differentially private mechanisms in the worst-case setting may asymptotically achieve nearly optimal efficiency in expectation (or with reasonably high probability), there is no guarantee that the chosen outcome for a particular execution of the mechanism is actually close to optimal.

Bayesian Differential Privacy and Persistent Approximate Truthfulness. In this paper, we consider mechanism design in the Bayesian setting, and our main goal is to construct useful mechanisms that achieve optimal efficiency, some form of differential privacy, and some notion of truthfulness. Since differential privacy is a worst-case notion in the sense that no distributional assumptions are made on the input of the mechanism, we first adapt the notion of differential privacy to the Bayesian mechanism design setting. We call this new notion *Bayesian differential privacy*; this is the privacy notion that we use in this paper.

As mentioned above, Xiao [2] showed that a mechanism that is truthful and differentially private might not be truthful in a model where privacy is explicitly captured in the players' utility functions. In this paper, we do not use such a model, since there are many settings where the players would already be satisfied with differential privacy and would not report strategically in an attempt to further protect their privacy. Our results will be meaningful in these settings; furthermore, even in a setting where we want to explicitly capture privacy in the players' utility functions, our techniques and results can still be useful in constructing truthful mechanisms (similar to how the mechanisms in [3] and [5] are still based on differentially private mechanisms).

We also want our mechanisms to satisfy some form of truthfulness. The standard notion of truthfulness in Bayesian mechanism design is that the truthful strategy profile is a Bayes-Nash equilibrium. Similar to [1], we first relax truthfulness so that the truthful strategy profile only needs to be an *ϵ-Bayes-Nash equilibrium*, where an ϵ margin is allowed in the Nash conditions. However, we would like to obtain notions of truthfulness that are stronger than that provided by the ϵ-Bayes-Nash equilibrium. Thus, we strengthen the ϵ-Bayes-Nash equilibrium such that even if up to k players deviate from the equilibrium, everyone else's best-response is still to adhere to their part of the equilibrium. We call this new equilibrium concept the *k-tolerant ϵ-Bayes-Nash equilibrium*. We would also like our equilibrium concept to be resilient against coalitions. Thus, we further strengthen our notion of k-tolerant ϵ-Bayes-Nash equilibrium to *(k,r)-persistent ϵ-Bayes-Nash equilibrium*, which is resilient against coalitions of size r even in the presence of k deviating players. The notion of truthfulness we use requires that the truthful strategy profile is a (k,r)-persistent ϵ-Bayes-Nash equilibrium, which we will refer to as *persistent approximate truthfulness*.

1.1 Our Results

In this paper, we present three classes of mechanisms that achieve both Bayesian differential privacy and persistent approximate truthfulness:

Histogram Mechanisms. Roughly speaking, a histogram mechanism is a mechanism that first computes a histogram from the reported types, and then chooses a social alternative based only on the histogram. In Section 4.1, we show that if every bin of the histogram has positive expected count, then the histogram mechanism is both Bayesian differentially private and persistent approximately truthful.

Mechanisms for Two Social Alternatives. Roughly speaking, this class includes any mechanism that makes a choice between two social alternatives $\{A, B\}$ based on the difference between the sums of two functions $u(\cdot, A)$ and $u(\cdot, B)$ on the types. In Section 4.2, we show that as long as the random variable $u(t, A) - u(t, B)$ (where t is distributed according to the type distribution) has non-zero variance, then such a mechanism is both Bayesian differentially private and persistent approximately truthful.

Social Welfare Mechanisms. Roughly speaking, this class includes any mechanism that makes a choice based on the social welfare provided by each social alternative. An important subset of these mechanisms is the set of mechanisms that maximize social welfare. In Section 4.3, we show that if the players' valuations for each social alternative are normally distributed, then such a mechanism is both Bayesian differentially private and persistent approximately truthful. In our full paper, we generalize this result to the case where the players' valuations for each social alternative are arbitrarily distributed with non-zero variance.

The mechanisms in the above three classes are all deterministic and can achieve optimal efficiency. Furthermore, the mechanisms do not use any payments. All proofs, as well as additional examples, can be found in our full paper.

Obtaining Actual Truthfulness. Recall that in [4], the authors added a "reaction stage" to the standard mechanism design model in order to achieve actual truthfulness from approximate truthfulness (which is obtained via differential privacy). We can also use this model and their techniques to obtain actual truthfulness in our results. In our full paper, we also describe an alternative model where actual truthfulness can be obtained from approximate truthfulness. In this new model, the mechanism is given the ability to verify the truthfulness of a small number of players. This model is simple to use and is realistic in settings where the truthfulness of a player can be verified objectively (e.g., income, expenses, age, address).

2 Preliminaries and Definitions

For any $k \in \mathbb{N}$, we will use $[k]$ to denote the set $\{1, \ldots, k\}$. We consider a standard mechanism design environment consisting of the following components:

- A number n of *players*; we will often use $[n]$ to denote the set of n players.
- A *type space* T; each player has a private type from the type space T.

– A distribution $\mathcal{T}$ over the type space; the players' private types are independently drawn from this distribution.
– A set S of *social alternatives*; for convenience, we assume that S is finite.
– For each player i, a *utility function* $u_i : T \times S \to \mathbb{R}$; for $t \in T$ and $s \in S$, $u_i(t, s)$ represents the utility that player i receives if player i has type t and the social alternative s is chosen.

We will focus on direct revelation mechanisms where each player reports his/her type. Therefore, a *mechanism* is a function $M : T^n \to S$, and a (pure) strategy for player i is a function $\sigma_i : T \to T$ that maps true types to announced types. For convenience, whenever we refer to a mechanism $M : T^n \to S$, we assume that it is associated with an environment as described above.

2.1 Equilibrium Concepts

In this section, we will define several equilibrium concepts based on the standard Bayes-Nash equilibrium (see, e.g., [9]). These equilibrium concepts will be used to define various notions of truthfulness. Our definitions build on the *ϵ-Bayes-Nash equilibrium*, which is a relaxation of the Bayes-Nash equilibrium in the sense that an ϵ margin is allowed in the Nash conditions. This relaxation reflects the assumption that players will not deviate from the equilibrium if gains from deviation are sufficiently small. In this paper, we also refer to ϵ-Bayes-Nash equilibria as approximate Bayes-Nash equilibria. For more information about various notions of approximate equilibria, see [10–12].

Our equilibrium concepts strengthen the ϵ-Bayes-Nash equilibrium. We chose two strengthenings to address the following weaknesses of Nash equilibria. Firstly, a player's part of a Nash equilibrium is only guaranteed to be a best-response if all the other players are playing their parts of the equilibrium. In other words, a Nash equilibrium cannot tolerate players deviating from their equilibrium strategy — if there is one irrational person in the system, the equilibrium breaks down. Deviations are especially problematic in ϵ-equilibria, where there is less confidence that everyone would play their part of the equilibrium. Secondly, a Nash equilibrium is not resilient to deviations by more than one person; coalitions of players can have profitable deviations from the equilibrium.

To address the first problem, we strengthen the Nash conditions such that even if up to k players deviate from the equilibrium, everyone else's best-response is still to adhere to their part of the equilibrium. In other words, the equilibrium *tolerates* arbitrary deviations of k individuals.

Definition 1 (k-tolerant ϵ-Bayes-Nash equilibrium). *A strategy profile $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ is a* k-tolerant ϵ-Bayes-Nash equilibrium *if for every $I \subseteq [n]$ with $|I| \leq k$, every possible announced types $\boldsymbol{t}'_I \in T^{|I|}$ for I, every player $i \notin I$, and every pair of types t_i, t'_i for player i, we have*

$$\mathbb{E}_{\boldsymbol{t}_J}[u_i(t_i, M(\sigma_i(t_i), \boldsymbol{t}'_I, \boldsymbol{\sigma}_J(\boldsymbol{t}_J)))] \geq \mathbb{E}_{\boldsymbol{t}_J}[u_i(t_i, M(t'_i, \boldsymbol{t}'_I, \boldsymbol{\sigma}_J(\boldsymbol{t}_J)))] - \epsilon,$$

where $J = [n] \setminus (I \cup \{i\})$ and $\boldsymbol{t}_J \sim \mathcal{T}^{|J|}$.

We note that k-tolerance is distinct from the notion of *k-immunity* as defined in [13, 14], which guarantees that when up to k people deviate from the equilibrium, the utilities of the non-deviating players do not decrease.

The second problem mentioned above is addressed by *r-resilience* (see, e.g., [10, 14]). A Bayes-Nash equilibrium is r-resilient if for any group of size at most r, there does not exist a deviation of the group such that any member of the coalition has increased utility.

Definition 2 (r-resilient ϵ-Bayes-Nash equilibrium). *A strategy profile* $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ *is an* r-resilient ϵ-Bayes-Nash equilibrium *if for every coalition* $C \subseteq [n]$ *with* $|C| \leq r$*, every true types* $\boldsymbol{t}_C \in T^{|C|}$ *for* C*, every player* $i \in C$*, and every possible announced types* $\boldsymbol{t}'_C \in T^{|C|}$ *for* C*, we have*

$$\mathbb{E}_{\boldsymbol{t}_{-C}}[u_i(t_i, M(\boldsymbol{\sigma}_C(\boldsymbol{t}_C), \boldsymbol{\sigma}_{-C}(\boldsymbol{t}_{-C})))] \geq \mathbb{E}_{\boldsymbol{t}_{-C}}[u_i(t_i, M(\boldsymbol{t}'_C, \boldsymbol{\sigma}_{-C}(\boldsymbol{t}_{-C})))] - \epsilon,$$

where $\boldsymbol{t}_{-C} \sim \mathcal{T}^{n-|C|}$*.*

It is not hard to see that resilience and tolerance can be independently violated, and hence neither implies the other. Just as the authors in [13, 14] combine immunity and resilience, we consider the combination of tolerance and resilience. Roughly speaking, a *(k,r)-persistent Bayes-Nash equilibrium* is a Bayes-Nash equilibrium that is r-resilient (protects against coalitions of size r), even in the presence of up to k individuals that are deviating arbitrarily from the equilibrium.

Definition 3 ((k,r)-persistent ϵ-Bayes-Nash equilibrium). *A strategy profile* $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ *is a* (k,r)-persistent ϵ-Bayes-Nash equilibrium *if for every* $I \subseteq [n]$ *with* $|I| \leq k$*, every possible announced types* $\boldsymbol{t}'_I \in T^{|I|}$ *for* I*, every coalition* $C \subseteq [n] \setminus I$ *with* $|C| \leq r$*, every true types* $\boldsymbol{t}_C \in T^{|C|}$ *for* C*, every player* $i \in C$*, and every possible announced types* $\boldsymbol{t}'_C \in T^{|C|}$ *for* C*, we have*

$$\mathbb{E}_{\boldsymbol{t}_J}[u_i(t_i, M(\boldsymbol{\sigma}_C(\boldsymbol{t}_C), \boldsymbol{t}'_I, \boldsymbol{\sigma}_J(\boldsymbol{t}_J)))] \geq \mathbb{E}_{\boldsymbol{t}_J}[u_i(t_i, M(\boldsymbol{t}'_C, \boldsymbol{t}'_I, \boldsymbol{\sigma}_J(\boldsymbol{t}_J)))] - \epsilon,$$

where $J = [n] \setminus (I \cup C)$ *and* $\boldsymbol{t}_J \sim \mathcal{T}^{|J|}$*.*

2.2 Notions of Truthfulness

In this section, we define various notions of truthfulness based on the equilibrium concepts from the previous section. Recall that a mechanism is *Bayes-Nash truthful* if the truthful strategy profile is a Bayes-Nash equilibrium. Similarly, a mechanism is *ϵ-Bayes-Nash truthful* if the truthful strategy profile is an ϵ-Bayes-Nash equilibrium. By using the equilibrium concepts from the previous section, we can obtain stronger notions of truthfulness.

Definition 4 ([k-tolerant]/[r-resilient]/[(k,r)-persistent] ϵ-Bayes-Nash truthful). *A mechanism is* k-tolerant ϵ-Bayes-Nash truthful *if the truthful strategy profile is a k-tolerant ϵ-Bayes-Nash equilibrium. Similarly, a mechanism is* r-resilient *(resp.,* (k,r)-persistent*)* ϵ-Bayes-Nash truthful *if the truthful strategy profile is an r-resilient (resp., (k,r)-persistent) ϵ-Bayes-Nash equilibrium.*

It is easy to see that if a mechanism is (k, r)-persistent ϵ-Bayes-Nash truthful, then it is also k-tolerant ϵ-Bayes-Nash truthful and r-resilient ϵ-Bayes-Nash truthful. In many settings, it is reasonable to believe that players in an ϵ-Bayes-Nash truthful mechanism will be truthful, since (1) truth-telling is simple while computing a profitable deviation can be costly (see, e.g., [15]), and (2) lying can induce a psychological (morality) cost. Indeed, there are many results in mechanism design that assume that approximate truthfulness is enough to ensure that players will be truthful (see, e.g., [1, 16–18]).

3 Privacy for Bayesian Mechanism Design

In this section, we describe and define *Bayesian differential privacy*, which is a natural adaptation of *differential privacy* [7, 8] to the Bayesian mechanism design setting. Roughly speaking, differential privacy requires that when one person's input to the mechanism is changed, the output distribution of the mechanism changes very little (here, the mechanism is randomized).

We now describe Bayesian differential privacy. We first note that even though the players' true types are drawn from some distribution $\mathcal{T}$, if all the players are non-truthful and announce a type independently of their true type, then the input of the mechanism is no longer distributional and we are essentially in the same scenario as in (worst-case) differential privacy. Thus, it is necessary to make some assumptions on the strategies of the players, so that the input of the mechanism contains at least some randomness.

In our notion of Bayesian differential privacy, we assume that at least some players (e.g., a constant fraction of the players) are truthful so that their announced types have the same distribution as their true types. This assumption is not unreasonable, since we later show that if a mechanism is Bayesian differentially private, then the mechanism is automatically persistent approximately truthful, so we expect that most players would be truthful anyway. In particular, if we have an equilibrium where most players are truthful, then privacy is achieved at this equilibrium.

Roughly speaking, (k, ϵ, δ)-*Bayesian differentially privacy* requires that when a player i changes his/her announced type, the output distribution of the mechanism changes by at most an (ϵ, δ) amount, assuming that at most k players are non-truthful (possibly lying in an arbitrary way). This implies that the mechanism leaks very little information about each player's announced type, so each player's privacy is protected. The mechanism is assumed to be deterministic, so the randomness of the output is from the randomness of the types of the truthful players. (One can also consider randomized mechanisms, but we chose to focus on deterministic mechanisms in this paper.)

Definition 5 ((k, ϵ, δ)-Bayesian differential privacy). *A mechanism $M : T^n \to S$ is* (k, ϵ, δ)-Bayesian differentially private *if for every player $i \in [n]$, every subset $I \subseteq [n] \setminus \{i\}$ of players with $|I| \leq k$, every pair of types $t_i, t_i' \in T$*

for player i, and every $\boldsymbol{t}'_I \in T^{|I|}$*, the following holds: Let* $J = [n] \setminus (I \cup \{i\})$ *(the remaining players) and* $\boldsymbol{t}_J \sim \mathcal{T}^{|J|}$*; then, for every* $Y \subseteq S$*, we have*

$$\Pr[M(t_i, \boldsymbol{t}'_I, \boldsymbol{t}_J) \in Y] \leq e^{\epsilon} \cdot \Pr[M(t'_i, \boldsymbol{t}'_I, \boldsymbol{t}_J) \in Y] + \delta,$$

where the probabilities are over $\boldsymbol{t}_J \sim \mathcal{T}^{|J|}$*.*

The parameter k controls how many non-truthful players the mechanism can tolerate while satisfying privacy; k can be a function of n (the number of players), such as $k = \frac{n}{2}$. One can even view the non-truthful players as being controlled/known by an adversary that is trying to learn information about a player i's type; as long as the adversary controls/knows at most k people, player i's privacy is still protected. The parameters ϵ and δ bound the amount of information about each person's (announced) type that can be "leaked" by the mechanism. Since the above definition of Bayesian differential privacy is a natural adaptation of differential privacy to Bayesian mechanism design, and since differential privacy is a well-motivated and well-accepted notion of privacy, we will not further elaborate on the details of the above definition.

Our definition of (k, ϵ, δ)-Bayesian differential privacy has some similarities to the notion of (ϵ, δ)-*noiseless privacy* (for databases) introduced and studied in [19]. However, there are some subtle but significant differences between the two definitions, so the results in this paper do not follow from the theorems and proofs in [19]. Nevertheless, the ideas and techniques in [19], and for (ϵ, δ)-noiseless privacy in general, may be useful for designing Bayesian differentially private mechanisms.

It is known that differentially private mechanisms are approximately (dominant-strategy) truthful (see [1]). Similarly, Bayesian differentially private mechanisms are persistent approximate Bayes-Nash truthful.

Theorem 1 (Bayesian differential privacy $\Longrightarrow$ persistent approximate truthfulness). *Suppose the utility functions are bounded by* $\alpha > 0$*, i.e., the utility function for each player i is* $u_i : T \times S \to [-\alpha, \alpha]$*. Let* M *be any mechanism that is* (k, ϵ, δ)*-Bayesian differentially private. Then,* M *satisfies the following properties:*

1. M *is* k*-tolerant* $(\epsilon + 2\delta)(2\alpha)$*-Bayes-Nash truthful.*
2. *For every* $1 \leq r \leq k+1$*,* M *is* r*-resilient* $(r\epsilon + 2r\delta)(2\alpha)$*-Bayes-Nash truthful.*
3. *For every* $1 \leq r \leq k+1$*,* M *is* $(k-r+1, r)$*-persistent* $(r\epsilon + 2r\delta)(2\alpha)$*-Bayes-Nash truthful.*

4 Efficient Bayesian Mechanisms with Privacy and Persistent Approximate Truthfulness

In this section, we present three classes of mechanisms that achieve both Bayesian differential privacy and persistent approximate truthfulness.

4.1 Histogram Mechanisms

We first present a broad class of mechanisms, called *histogram mechanisms*, that achieve Bayesian differential privacy and persistent approximate truthfulness. Given a partition $P = \{B_1, \ldots, B_m\}$ of the type space T with m blocks (ordered in some way), a *histogram* with respect to P is simply a vector in $(\mathbb{Z}_{\geq 0})^m$ that specifies a count for each block of the partition. Given a partition P, let $\mathcal{H}_P$ denote the set of all histograms with respect to P; given a vector $\boldsymbol{t}$ of types, let $H_P(\boldsymbol{t})$ be the histogram formed from $\boldsymbol{t}$ by simply counting how many components (types) of $\boldsymbol{t}$ belong to each block of the partition P.

We now define what we mean by a *histogram mechanism*. Intuitively, a histogram mechanism is a mechanism that, on input a vector of types, computes the histogram from the types with respect to some partition P, and then applies any function $f : \mathcal{H}_P \to S$ to the histogram to choose a social alternative in S.

Definition 6 (Histogram mechanism). *Let P be any partition of the type space T. A mechanism $M : T^n \to S$ is a* histogram mechanism with respect to *P if there exists a function $f : \mathcal{H}_P \to S$ such that $M(\boldsymbol{t}) = f(H_P(\boldsymbol{t}))\ \forall\ \boldsymbol{t} \in T^n$.*

The following theorem states that any histogram mechanism with bounded utility functions and positive expected count for each bin is both Bayesian differentially private and persistent approximately truthful.

Theorem 2 (Histogram mechanisms are private and persistent approximately truthful). *Let $M : T^n \to S$ be any histogram mechanism with respect to some partition P of T. Let $p_{\min} = \min_{B \in P} \Pr_{t \sim \mathcal{T}}[t \in B]$, and suppose that $p_{\min} > 0$. Then, for every $0 \leq k \leq n-2$ and $\frac{4}{p_{\min} \cdot (n-k-1)} \leq \epsilon \leq 1$, M satisfies the following properties with $\delta = e^{-\Omega((n-k) \cdot p_{\min} \cdot \epsilon^2)}$:*

1. Privacy: *M is (k, ϵ, δ)-Bayesian differentially private.*
2. Persistent approximate truthfulness: *Suppose the utility functions are bounded by $\alpha > 0$, i.e., the utility function for each player i is $u_i : T \times S \to [-\alpha, \alpha]$. Then, for every $1 \leq r \leq k+1$, M is $(k-r+1, r)$-persistent $(r\epsilon + 2r\delta)(2\alpha)$-Bayes-Nash truthful.*

One possible partition of the type space is the one where there is a distinct block for each type. Thus, Theorem 2 covers the case where the choice of the mechanism depends only on the *number* of players that reported each type, and not their identities. In fact, given any partition, one can redefine the type space so that the new types are the blocks of the partition. This means we could always redefine the type space and simply use the partition where there is a distinct block for each type in the new type space. However, we believe it is more natural to preserve the original, natural type space, and to allow the histogram mechanism to use an appropriate partition of the type space.

In Theorem 2, since the histogram mechanism is not modified in any way to satisfy privacy and persistent approximate truthfulness, all properties of the mechanism (e.g., efficiency, truthfulness, individual rationality, etc.) are preserved. We now give a simple example to illustrate Theorem 2.

*Example 1 (***Voting with multiple candidates).** Suppose we are trying to select a winner from a finite set of candidates (e.g., political candidates) using the plurality rule (i.e., each voter casts one vote and the candidate with the most votes wins). The set of social alternatives is the set of candidates, and the natural type space is the set of all preference orders over the candidates. However, we can partition the type space such that each block b represents a candidate c_b, and all the types with c_b as their top choice belong to block b. Intuitively, announcing a type that belongs to block b can be understood as casting a vote for candidate c_b. Using this partition, we can define a histogram mechanism that implements the plurality rule. It is well known that the plurality rule is not strategy-proof when there are more than two candidates (see, e.g., [11]). However, by Theorem 2, this histogram mechanism is Bayesian differentially private and persistent approximate Bayes-Nash truthful.

4.2 Mechanisms for Two Social Alternatives

Although histogram mechanisms are useful in many settings, in order to apply Theorem 2 to get good parameters, the number of bins cannot be extremely large. We now present a class of mechanisms that do not require the partitioning of types into bins, but are still Bayesian differentially private and persistent approximately truthful. Roughly speaking, the following theorem states that any mechanism that makes a choice between two social alternatives $\{A, B\}$ based on the difference between the sums of two functions $u(\cdot, A)$ and $u(\cdot, B)$ on the types is Bayesian differentially private and persistent approximately truthful.

Theorem 3 (Private and persistent approximately truthful mechanisms for two social alternatives). *Let $S = \{A, B\}$ be any set of two social alternatives, let $T \subseteq \mathbb{R}$ be the type space, let $\mathcal{T}$ be any distribution over T, and let $u : T \times S \to [-\beta, \beta]$ be any function (e.g., a utility function for all the players). Suppose the random variable $u(t, A) - u(t, B)$, where $t \sim \mathcal{T}$, has non-zero variance and a probability density function.*

Let $M : T^n \to S$ be any mechanism such that

$$M(\boldsymbol{t}) = f\left(\sum_{i=1}^{n} u(t_i, A) - \sum_{i=1}^{n} u(t_i, B)\right)$$

for some function $f : \mathbb{R} \to S$. Then, for every $0 \leq k \leq n-2$ and $0 < \epsilon \leq 1$, M satisfies the following properties with $\epsilon' = \epsilon + O(\sqrt{\frac{\ln(n-k)}{n-k}})$ and $\delta = O(\frac{1}{\epsilon\sqrt{n-k}})$:

1. Privacy: *M is (k, ϵ', δ)-Bayesian differentially private.*
2. Persistent approximate truthfulness: *Suppose the utility functions are bounded by $\alpha > 0$, i.e., the utility function for each player i is $u_i : T \times S \to [-\alpha, \alpha]$. Then, for every $1 \leq r \leq k+1$, M is $(k-r+1, r)$-persistent $(r\epsilon' + 2r\delta)(2\alpha)$-Bayes-Nash truthful.*

The mechanism in Theorem 3 chooses a social alternative by applying some function f on the difference between $\sum_{i=1}^{n} u(t_i, A)$ and $\sum_{i=1}^{n} u(t_i, B)$. We note that the mechanism may already have certain properties, such as efficiency, truthfulness, individual rationality, etc.; by Theorem 3, this mechanism also satisfies privacy and persistent approximate truthfulness, in addition to the original properties that it already satisfies. One obvious application of Theorem 3 is to let u be a common utility function for the players, where the utility of player i with type t_i is $u(t_i, A)$ if A is chosen, and is $u(t_i, B)$ if B is chosen. If we define $f : \mathbb{R} \to S$ such that $f(x) = A$ if and only if $x > 0$, then the mechanism maximizes social welfare.

4.3 Social Welfare Mechanisms

In this section, we present a class of mechanisms that make choices based on the social welfare provided by each social alternative. An important subset of these mechanisms is the set of mechanisms that maximize social welfare.

In this section, a type $t \in T$ is a valuation function that assigns a valuation to each social alternative $s \in S$. In many settings, it is reasonable to assume that the players' valuations for each social alternative follow a normal distribution, since the normal distribution has been frequently used to model many natural and social phenomena. For convenience of presentation, we will use the *standard* normal distribution $\mathcal{N}(0, 1)$ in our theorems below. However, our theorems can be easily generalized to work with arbitrary normal distributions. In any case, it is easy to see that given any normal distribution over the valuations, the valuations can be translated and scaled to obtain the standard normal distribution.

For any reasonable mechanism, it is natural to have a bound on the set of possible valuations — it would be unreasonable to allow a player to report an arbitrarily high or low valuation (e.g. 2^{100}) and single-handedly influence the choice of the mechanism. Therefore, we will restrict the possible valuations to the interval $[-\alpha, \alpha]$ for some value $\alpha > 0$. As a result, our type space T will be the set of all valuation functions $t : S \to [-\alpha, \alpha]$. Furthermore, we will assume that the players' valuations for each social alternative follow the standard normal distribution. However, because of the bound on the set of valuations, we will use the truncated standard normal distribution obtained by conditioning $\mathcal{N}(0, 1)$ to lie on the interval $[-\alpha, \alpha]$. We denote this distribution by $\mathcal{N}(0, 1)_{[-\alpha,\alpha]}$.

For simplicity, we will first present the following theorem, which is a special case of our more general result (Theorem 5). The following theorem states that if each player's valuation for each social alternative is distributed as the truncated standard normal distribution $\mathcal{N}(0, 1)_{[-\alpha,\alpha]}$, then any mechanism that makes a choice based on the set of total valuations for each social alternative is Bayesian differentially private and persistent approximate Bayes-Nash truthful.

Theorem 4 (Social welfare mechanisms). *Let $S = \{s_1, \ldots, s_m\}$ be a set of m social alternatives. Let the type space T be the set of all valuation functions $t : S \to [-\alpha, \alpha]$ on S, where $\alpha = \Theta(\sqrt[4]{n})$. Let $\mathcal{T}$ be the distribution over T obtained by letting $t(s) \sim \mathcal{N}(0, 1)_{[-\alpha,\alpha]}$ for each $s \in S$ independently. For each player i, let the utility function for player i be $u_i(t_i, s) = t_i(s)$.*

Let $\mathsf{sw}_j(\boldsymbol{t}) = \sum_{i=1}^{n} t_i(s_j)$ *be the (reported) social welfare for the social alternative* s_j*. Let* $M : T^n \to S$ *be any mechanism such that*

$$M(\boldsymbol{t}) = f(\mathsf{sw}_1(\boldsymbol{t}), \ldots, \mathsf{sw}_m(\boldsymbol{t}))$$

for some function $f : \mathbb{R}^m \to S$*. Then, for every constant* $c < 1$*, every* $k \leq c \cdot n$*, and every* $0 < \epsilon \leq 1$*,* M *satisfies the following properties with* $\delta = O(e^{-\Omega(\frac{\epsilon^2}{m^2} \cdot \sqrt{n}) + \ln(m\sqrt{n})})$*:*

1. Privacy: M *is* (k, ϵ, δ)*-Bayesian differentially private.*
2. Persistent approximate truthfulness: *For every* $1 \leq r \leq k+1$*,* M *is* $(k-r+1, r)$*-persistent* $(r\epsilon + 2r\delta)(2\alpha)$*-Bayes-Nash truthful.*

In Theorem 4, $\mathsf{sw}_j(\boldsymbol{t})$ represents the social welfare that will be achieved if the players' types (i.e., valuation functions) are $\boldsymbol{t}$ and the social alternative s_j is chosen by the mechanism. Thus, Theorem 4 says that any mechanism whose choice depends only on the set $\{\mathsf{sw}_j(\boldsymbol{t})\}_{j \in [m]}$ of social welfare values satisfies Bayesian differential privacy and persistent approximate Bayes-Nash truthfulness, in addition to any properties that it may already satisfy (e.g., efficiency, truthfulness, individual rationality, etc.). In particular, a mechanism that chooses a social alternative to maximize social welfare satisfies this requirement and achieves optimal efficiency with respect to social welfare.

In Theorem 4, the value α at which the standard normal distribution is truncated is chosen so that the truncated distribution is very close to the untruncated one. This means that even if we had used the untruncated distribution instead, with high probability no valuation would fall outside the interval $[-\alpha, \alpha]$.

In the next theorem, we consider a setting where there is a set of available "options", and we allow the mechanism to choose any subset of these options. Thus, the set of social alternatives is the power set of the set of options. To keep the set of valuations tractable, instead of having a valuation for each social alternative, the players have a valuation for each option. Moreover, we allow for the flexibility where for each player, only certain options are relevant/applicable to that player. We capture this flexibility by having a binary weight for each player-option pair. Note that Theorem 4 is the special case where the set of social alternatives consists of the sets of single options (i.e., the singletons), and where all options are considered relevant to all players.

The binary weight $w_{i,j}$ associated with player i and option o_j indicates whether option o_j is relevant/applicable to player i. $w_{i,j} = 1$ means that option o_j is relevant/applicable to player i, so player i's announced valuation is taken into account in the social welfare for option o_j. On the other hand, $w_{i,j} = 0$ means that player i's valuation is ignored in the social welfare for option o_j. These weights are known to or set by the mechanism designer. For example, perhaps only people with low income should have a voice in decisions regarding subsidized housing, and only people with disabilities should have a say in decisions regarding building accessibility laws. We now state our next theorem, which generalizes Theorem 4 to this new setting.

Theorem 5 (Social welfare mechanisms with multiple options). *Let the set S of social alternatives be 2^O, where $O = \{o_1, \ldots, o_m\}$ is a set of m possible "options". Let the type space T be the set of all valuation functions $t : O \to [-\alpha, \alpha]$ on O, where $\alpha = \Theta(\sqrt[4]{n})$. Let $\mathcal{T}$ be the distribution over T obtained by letting $t(o) \sim \mathcal{N}(0,1)_{[-\alpha,\alpha]}$ for each option $o \in O$ independently. Suppose the weights $\{w_{i,j}\}_{i\in[n],j\in[m]}$ satisfy $\sum_{i=1}^n w_{i,j} \geq c_1 \cdot n$ for every option o_j, where $c_1 > 0$ is some constant.*

Let $\mathsf{sw}_j(\boldsymbol{t}) = \sum_{i=1}^n w_{i,j} \cdot t_i(o_j)$ be the (reported) social welfare for option o_j. Let $M : T^n \to S$ be any mechanism such that

$$M(\boldsymbol{t}) = f(\mathsf{sw}_1(\boldsymbol{t}), \ldots, \mathsf{sw}_m(\boldsymbol{t}))$$

for some function $f : \mathbb{R}^m \to S$. Then, for every constant $c_2 < c_1$, every $k \leq c_2 \cdot n$, and every $0 < \epsilon \leq 1$, M satisfies the following properties with $\delta = O(e^{-\Omega(\frac{\epsilon^2}{m^2}\cdot\sqrt{n})+\ln(m\sqrt{n})})$:

1. Privacy: *M is (k, ϵ, δ)-Bayesian differentially private.*
2. Persistent approximate truthfulness: *Suppose the utility functions are bounded by $\beta > 0$, i.e., the utility function for each player i is $u_i : T \times S \to [-\beta, \beta]$. Then, for every $1 \leq r \leq k+1$, M is $(k-r+1, r)$-persistent $(r\epsilon + 2r\delta)(2\beta)$-Bayes-Nash truthful.*

In Theorem 5, the requirement on the binary weights simply means that each option is relevant/applicable to at least some constant fraction of the players. Note that the persistent approximate truthfulness result of Theorem 5 requires the players' utility functions to be bounded by $\beta > 0$. This assumption is needed since the players' utility functions can actually be arbitrary functions. However, the most natural way to use Theorem 5 is to let player i's utility function be the following: if the chosen social alternative is a singleton $\{o_j\}$, then the utility for player i is $w_{i,j} \cdot t_i(o_j)$; if the chosen social alternative is a set s consisting of two or more options, then the utility for player i is the sum of the utilities for each singleton subset of s. Alternatively, a player i's utility for a social alternative s does not have to be *additive* in the options that s contains — the utility function for player i can capture complementarities and substitutabilities of the options as well. We now give a simple example that illustrates Theorem 5.

Example 2 (**Multiple public projects).** The municipal government would like to spend its budget surplus of 4 million on the community. There are four options that the government is considering, each costing 2 million to build: a senior home, a casino, a subsidized housing complex, and a library. The government would like to find out, on a scale from $-\alpha$ to α, how much each individual values each option. For each individual i, the government chooses the weights for each of the options as follows: the weight for the senior home is 1 if and only if individual i is over the age of 65; the weight for the casino is 1 if and only if individual i is over the age of 19; the weight for the subsidized housing complex is 1 if and only if individual i is classified as low-income; and the weight for the library is always 1.

After collecting the valuations from the individuals, the government can compute the social welfare provided by each option, or compute an average utility for each option by dividing its social welfare by the number of people who have weight 1 for that option. Finally, the government can choose two of the options to maximize social welfare or average utility. By Theorem 5, this mechanism is Bayesian differentially private and persistent approximately truthful.

In our full paper, we generalize Theorem 5 to the case where the players' valuations for each social alternative are arbitrarily distributed with non-zero variance.

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Bounded-Distance Network Creation Games*

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Abstract. A *network creation game* simulates a decentralized and non-cooperative building of a communication network. Informally, there are n players sitting on the network nodes, which attempt to establish a reciprocal communication by activating, incurring a certain cost, any of their incident links. The goal of each player is to have all the other nodes as close as possible in the resulting network, while buying as few links as possible. According to this intuition, any model of the game must then appropriately address a balance between these two conflicting objectives. Motivated by the fact that a player might have a strong requirement about its centrality in the network, in this paper we introduce a new setting in which if a player maintains its (either *maximum* or *average*) distance to the other nodes within a given *bound*, then its cost is simply equal to the *number* of activated edges, otherwise its cost is unbounded. We study the problem of understanding the structure of pure Nash equilibria of the resulting games, that we call MaxBD and SumBD, respectively. For both games, we show that when distance bounds associated with players are *non-uniform*, then equilibria can be arbitrarily bad. On the other hand, for MaxBD, we show that when nodes have a *uniform* bound R on the maximum distance, then the *Price of Anarchy* (PoA) is lower and upper bounded by 2 and $O\left(n^{\frac{1}{\lfloor \log_3 R \rfloor + 1}}\right)$ for $R \geq 3$ (i.e., the PoA is constant as soon as R is $\Omega(n^{\epsilon})$, for some $\epsilon > 0$), while for the interesting case $R = 2$, we are able to prove that the PoA is $\Omega(\sqrt{n})$ and $O(\sqrt{n \log n})$. For the uniform SumBD we obtain similar (asymptotically) results, and moreover we show that the PoA becomes constant as soon as the bound on the average distance is $2^{\omega\left(\sqrt{\log n}\right)}$.

1 Introduction

Communication networks are rapidly evolving towards a model in which the constituting components (e.g., routers and links) are activated and maintained

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by different owners, which one can imagine as players sitting on the network nodes. When these players act in a selfish way with the final intent of creating a connected network, the challenge is exactly to understand whether the pursuit of individual profit is compatible with the attainment of an equilibrium status for the system (i.e., a status in which players are not willing to move from), and how the social utility for the system as a whole is affected by the selfish behavior of the players. While the former question is inherently game-theoretic and has been originally addressed in [10] by the economists (for further references see also Chapter 6 in [11]), the latter one involves also computational issues, since it can be regarded as a comparison between the performances of an uncoordinated distributed system as opposed to a centralized system which can optimally design a solution. Not surprisingly then, this class of games, which we refer to as *network creation games* (NCGs), received a significative attention also from the computer science community, starting from the paper of Fabrikant *et al.* [9], where the main computational aspects of a NCG have been initially formalized and investigated. More precisely, in [9] the authors focused on an Internet-oriented NCG, defined as follows: We are given a set of n players, say V, where the strategy space of player $v \in V$ is the power set $2^{V \setminus \{v\}}$. Given a combination of strategies $S = (S_v)_{v \in V}$, let $G(S)$ denote the underlying undirected graph whose node set is V, and whose edge set is $E(S) = \big\{(v, v') \mid v \in V \wedge v' \in S_v\big\}$. Then, the *cost* incurred by player v under S is

$$cost_v(S) = \alpha \cdot |S_v| + \sum_{u \in V} d_{G(S)}(u, v) \tag{1}$$

where $d_{G(S)}(u, v)$ is the distance between nodes u and v in $G(S)$. Thus, the cost function implements the inherently antagonistic goals of a player, which on the one hand attempts to buy as little edges as possible, and on the other hand aims to be as close as possible to the other nodes in the outcoming network. These two criteria are suitably balanced in (1) by making use of the parameter $\alpha \geq 0$. Consequently, the *Nash Equilibria*[1] (NE) space of the game is heavily influenced by α, and the corresponding characterization must be given as a function of it. The state-of-the-art for the *Price of Anarchy* (PoA) of the game, that we will call henceforth SumNCG, is summarized in [15], where the most recent progresses on the problem have been reported.

Further NCG models. A first natural variant of SumNCG was introduced in [7], where the authors redefined the player cost function as follows

$$cost_v(S) = \alpha \cdot |S_v| + \max\{d_{G(S)}(u, v) : u \in V\}. \tag{2}$$

This variant, named MaxNCG, received further attention in [15], where the authors improved the PoA of the game on the whole range of values of α. However, a criticism made to both the aforementioned models is that usage and building cost are summed up together in the player's cost, and this mixing is reflected

[1] In this paper, we only focus on *pure* strategies Nash equilibria.

in the social cost of the resulting network. As a consequence, we have that in this game the PoA alone does not say so much about the structural properties of the network, such as density, diameter, or routing cost. Moreover, they both incorporate in the cost function the parameter α, which is in a sense artificially introduced in order to suitably balance usage and building cost.

Thus, in an effort of addressing these critical issues, in [14] the authors proposed an interesting variant in which a player v, when forming the network, has a limited *budget* b_v to establish links to other players. This way, the player cost function restricts to the usage cost, namely either the maximum or the total distance to other nodes. For these *bounded-budget* versions of the game, that we call MaxBB and SumBB, respectively, the authors in [14] showed that determining the existence of a NE is NP-hard. On a positive side, they proved that for uniform budgets, say k, both variants always admit a NE, and that its *Price of Stability* (PoS) is $\Theta(1)$. Finally, they proved that the PoA of MaxBB is $\Omega\left(\frac{n}{k \log_k n}\right)$ and $O\left(\frac{n}{\log_k n}\right)$, while the PoA of SumBB is $\Omega\left(\sqrt{\frac{n}{k \log_k n}}\right)$, $O\left(\sqrt{\frac{n}{\log_k n}}\right)$. Notice that in both MaxBB and SumBB, links are seen as directed. Thus, a natural extension of the model was given in [8], were the undirected case was considered. For this, it was proven that both MaxBB and SumBB always admit a NE. Moreover, the authors showed that the PoA for MaxBB and SumBB is $\Omega(\sqrt{\log n})$ and $O(\sqrt{n})$, respectively, while in the special case in which the budget is equal to 1 for all the players, the PoA is $O(1)$ for both versions of the game.

In all the above models it must be noticed that, as stated in [9], for a player it is NP-hard to find a best response once that the other players' strategies are fixed. To circumvent this problem, in [4] the authors proposed a further variant, called *basic NCG* (BNCG), in which given some existing network, the only improving transformations allowed are *edge swaps*, i.e., a player can only modify a *single* incident edge, by either replacing it with a new incident edge, or by removing it. This naturally induces a weaker concept of equilibrium for which a best response of a player can be computed in polynomial time. In this setting, the authors were able to give, among other results, an upper bound of $2^{O(\sqrt{\log n})}$ for the PoA of SumBNCG, and a lower bound of $\Omega(\sqrt{n})$ for the PoA of MaxBNCG. However, as pointed out in [15], the fact that now an edge has not a specific owner, prevents the possibility to establish any implications on the PoA of the classic NCG, since a NE in a BNCG is not necessarily a NE of a NCG. Finally, another NCG model which is barely related to the NCG model we study in this paper has been addressed in [6].

Our results. In this paper, we propose a new NCG variant that complements the model proposed in [8]. More precisely, we assume that the cost function of each player only consists of the number of bought edges (without any budget on them), but with the additional constraint that each player v needs to stay within a given (either *maximum* or *average*) *distance*, say (either R_v or D_v), from the other players.

For this bounded-distance version of the NCG, we address the problem of understanding the structure of the NE associated with the two variants of the game, that we denote by MaxBD and SumBD. In this respect, we first show that both games can have an unbounded PoA as soon as players hold at least two different distance bounds. Moreover, in both games, computing a best response for a player is NP-hard. These bad news are counterbalanced by the positive results we get for *uniform* distance bounds. In this case, first of all, the PoS for MaxBD is equal to 1, while for SumBD it is at most 2. Then, as far as the PoA is concerned, let R and D denote the uniform bound on the maximum and the average distance, respectively. We show that

(i) for MaxBD, the PoA is lower and upper bounded by 2 and $O\left(n^{\frac{1}{\lfloor \log_3 R \rfloor + 1}}\right)$ for $R \geq 3$, respectively, while for $R = 2$ is $\Omega(\sqrt{n})$ and $O(\sqrt{n \log n})$;
(ii) for SumBD, the PoA is lower bounded by $2 - \epsilon$, for any $\epsilon > 0$, as soon as $D \geq 2 - 3/n$, while it is upper bounded as reported in Table 1.

Table 1. Obtained PoA upper bounds for uniform SumBD

D	$\in [2, 3)$	≥ 3 and $O(1)$	$\omega(1) \cap O\left(3^{\sqrt{\log n}}\right)$	$\omega\left(3^{\sqrt{\log n}}\right) \cap 2^{O\left(\sqrt{\log n}\right)}$	$2^{\omega\left(\sqrt{\log n}\right)}$
PoA	$O\left(\sqrt{n \log n}\right)$	$O\left(n^{\frac{1}{\lfloor \log_3 D/4 \rfloor + 2}}\right)$	$2^{O\left(\sqrt{\log n}\right)}$	$O\left(n^{\frac{1}{\lfloor \log_3 D/4 \rfloor + 2}}\right)$	$O(1)$

Motivations and significance of the new model. Our model was originally motivated by the observation that, in a realistic scenario, a player might have a strong objective/requirement about its centrality in the under-costruction network. In fact, in daily life, people actively participate to the autonomous formation of (social) networks. In our experience, a user downplays any concerns about the number/cost of activated links. Rather, he initially pays attention only to the fact of remaining as close as possible to (a subset of) the other users, and only later on he tries to minimize his outdegree accordingly. Our model aims to (partially) address this dynamics. Actually, at this initial stage, we have relaxed this quite complicate setting, by associating with each user just a (uniform) single distance bound w.r.t. all the other users. Nevertheless, even in this simplified scenario, we can get some new insights as opposed to previous NCG models. Indeed, a closer inspection of our provided results suggests that the PoA becomes constant as soon as the maximum/average distance bound is $\Omega(n^{\epsilon})$, for some $\epsilon > 0$. This is quite interesting, since it implies that the autonomous network tends to be sparser as soon as the distance bounds grow. Notice that in the Fabrikant's model (and its variants), we cannot directly infer any information about network sparseness by just knowing that the PoA is constant. Furthermore, our model, as for those proposed in [14,8,4], does not rely on the α parameter, and this makes the proofs of the various bounds intimately related with some graph-theoretic properties of a stable network. For example, it is interesting to notice that in our setting the minimum degree and the size of a minimum dominating set play an important role. In this respect, in the concluding remarks of this paper, we pose an intriguing relationship between our problem and the well-known graph-theoretic *degree-diameter problem*, that we believe could help in solving some of the issues still left open, like the quite large gap between lower and upper bounds for the PoA. Finally, focusing on MaxBD, we observe that when $R = 2$, which should consistently model the scenario depicted by local-area networks, we obtain the meaningful result that the PoA is far to be constant. We also conjecture that this undesirable behavior can actually be extended to larger, still constant, values of R, although the generalization of the lower bounding argument seems likely technically involved.

The paper is organized as follows. After giving some basic definitions in Section 2, we provide some preliminary results in Section 3. Then, we study upper and lower bounds for uniform MaxBD and SumBD in Sections 4 and 5, respectively. Finally, in Section 6 we conclude the paper by discussing some intriguing relationships of our games with the famous graph-theoretic *degree-diameter* problem. Due to space limitations, some of the proofs are omitted here and will be given in the full version of the paper.

2 Problem Definition

Graph terminology. Let $G = (V, E)$ be an undirected (simple) graph with n vertices. For a graph G, we will also denote by $V(G)$ and $E(G)$ its set of vertices and its set of edges, respectively. For every vertex $v \in V$, let $N_G(v) := \{u \mid u \in V \setminus \{v\}, (u, v) \in E\}$. The *minimum degree* of G is equal to $\min_{v \in V} |N_G(v)|$.

We denote by $d_G(u, v)$ the *distance* in G from u to v. The *eccentricity* of a vertex v in G, denoted by $\varepsilon_G(v)$, is equal to $\max_{u \in V} d_G(u, v)$. The *diameter* and the *radius* of G are equal to the maximum and the minimum eccentricity of its nodes, respectively. A node is said to be a *center* of G if $\varepsilon_G(v)$ is equal to the radius of G. We define the *broadcast cost* of v in G as $B_G(v) = \sum_{u \in V} d_G(u, v)$, while the *average distance* from v to a node in G is denoted by $D_G(v) = B_G(v)/n$.

A *dominating set* of G is a subset of nodes $U \subseteq V$ such that every node of $V \setminus U$ is adjacent to some node of U. We denote by $\gamma(G)$ the cardinality of a minimum-size dominating set of G. Moreover, for any real $k \geq 1$, the kth power of G is defined as the graph $G^k = (V, E(G^k))$ where $E(G^k)$ contains an edge (u, v) if and only if $d_G(u, v) \leq k$. Let $F \subseteq \{(u, v) \mid u, v \in V, u \neq v\}$. We denote by $G + F$ the graph on V with edge set $E \cup F$. When $F = \{e\}$ we will denote $G + \{e\}$ by $G + e$.

Problem Statements. The *bounded maximum-distance* NCG (MaxBD) is defined as follows: Let V be a set of n nodes, each representing a selfish player, and for any $v \in V$, let $R_v > 0$ be an integer representing a bound on the eccentricity of v. The strategy of a player v consists of a subset $S_v \subseteq V \setminus \{v\}$. Denoting by S the strategy profile of all players, let $G(S)$ be the *undirected* graph with node set V, and with edge set $E(S) = \{(v, v') \mid v \in V \wedge v' \in S_v)\}$. When $u \in S_v$, we will say that v is buying the edge (u, v), or that the edge (u, v) is bought by v. Then, the cost of a player v in S is $cost_v(S) = |S_v|$ if $\varepsilon_{G(S)}(v) \leq R_v$, $+\infty$ otherwise.

The *bounded average-distance* NCG (SumBD) is defined analogously, with a bound D_v on the average distance of v from all the other nodes, and cost function $cost_v(S) = |S_v|$ if $D_{G(S)}(v) \leq D_v$, $+\infty$ otherwise. In the rest of the paper, depending on the context, we will interchangeably make use of the bound on the broadcast cost $B_v = D_v \cdot n$ when referring to SUMBD.

In both variants, we say that a node v is *within the bound in* S (or in $G(S)$) if $cost_v(S) < +\infty$. We measure the overall quality of a graph $G(S)$ by its social cost $SC(S) = \sum_{v \in V} cost_v(S)$. A graph $G(S)$ minimizing $SC(S)$ is called *social optimum.*

We use the *Nash Equilibrium* (NE) as solution concept. More precisely, a NE is a strategy profile S in which no player can decrease its cost by changing its strategy, assuming that the strategies of the other players are fixed. When S is a NE, we will say that $G(S)$ is *stable*. Conversely, a graph G is said to be stable if there exists a NE S such that $G = G(S)$. Notice that in both games, when S is a NE, all nodes are within the bound and, since every edge is bought by a single player, $SC(S)$ coincides with the number of edges of $G(S)$.

We conclude this section by recalling the definition of the two measures we will use to characterize the NE space of our games, namely the *Price of Anarchy* (PoA) [9] and the *Price of Stability* (PoS) [3], which are defined as the ratio between the highest (respectively, the lowest) social cost of a NE, and the cost of a social optimum.

3 Preliminary Results

First of all, observe that for MAXBD it is easy to see that a stable graph always exists. Indeed, if there is at least one node having distance bound 1, then the graph where all 1-bound nodes buy edges towards all the other nodes is stable. Otherwise, any spanning star is stable. Notice that any spanning star is stable for SUMBD as well, but only when every vertex has a bound $B_v \geq 2n-3$, while the problem of deciding whether a NE always exists for the remaining values of B_v is open. From these observations, we can derive the following negative result.

Theorem 1. *The PoA of* MAXBD *and* SUMBD *(with distance bounds* $B_v \geq 2n-3$*) is* $\Omega(n)$*, even for only two distance-bound values.*

Sketch of proof. We exhibit a graph G' with $\Omega(n^2)$ edges, and a strategy profile S such that $G(S) = G'$ and $G(S)$ is stable in both models for suitable distance bounds. We also show that the social optimum is $n-1$.

The graph G' is defined as follows. We have a clique of k nodes. For each node v of the clique, we add four nodes $v_1^1, v_2^1, v_1^2, v_2^2$ and four edges $(v_2^1, v_1^1), (v_1^1, v)$, (v_2^2, v_1^2), and (v_1^2, v). Clearly, G' has $n = 5k$ nodes and $\Omega(n^2)$ edges. Now, consider a strategy profile S with $G' = G(S)$ and such that (i) every edge is bought by a single player, and (ii) the edges $(v_2^j, v_1^j), (v_1^j, v)$ are bought by v_2^j and v_1^j, respectively, $j = 1, 2$.

For MAXBD, we set the bound of every node of the clique to 3, while all the other nodes have bound 5. For SUMBD, we set the bound of each node v of the clique to $\sum_{u \in V} d_G(v, u) = 11k - 5 > 2n - 3$, while we assign to all the other nodes bound n^2.

It is then not so hard to show that $G(S)$ is stable. To conclude the proof, observe that any spanning star (with cost $n-1$) is a social optimum for the two instances of MaxBD and SumBD given above. □

Given the above bad news, from now on we focus on the *uniform* case of the games, i.e., all the bounds on the distances are the same, say R and D (i.e., $B = D \cdot n$) for the maximum and the average version, respectively. Similarly to other NCGs, also here we have the problem of computing a best response for a player, as stated in the following theorem.

Theorem 2. *Computing the best response of a player in* MaxBD *and* SumBD *is* NP*-hard.* □

On the other hand, a positive result which clearly implies that SumBD always admits a pure NE is the following:

Theorem 3. *The PoS of* MaxBD *is 1, while for* SumBD *it is at most 2.* □

4 Upper and Lower Bounds to the PoA for MaxBD

We start by providing few results which will be useful to prove our upper bounds to the PoA for MaxBD.

Lemma 1. *Let $G(S) = (V, E(S))$ be stable and let H be a subgraph of $G(S)$. If for each node $v \in V$ there exists a set E_v of edges (all incident to v) such that v is within the bound in $H + E_v$, then $SC(S) \leq |E(H)| + \sum_{v \in V} |E_v|$.*

Proof. Let k_v be the number of edges of H that v is buying in S. If v buys E_v additionally to its k_v edges, then v will be within the bound in $H + E_v$. Hence, since S is a NE, we have that $cost_v(S) \leq k_v + |E_v|$, from which it follows that

$$SC(S) = \sum_{v \in V} cost_v(S) \leq \sum_{v \in V} k_v + \sum_{v \in V} |E_v| = |E(H)| + \sum_{v \in V} |E_v|. \qquad \square$$

Thanks to Lemma 1, we can prove the following lemma.

Lemma 2. *Let $G(S)$ be stable, and let γ be the cardinality of a minimum dominating set of $G(S)^{R-1}$. Then $SC(S) \leq (\gamma + 1)(n - 1)$.*

Proof. Let U be a minimum dominating set of $G(S)^{R-1}$, with $\gamma = |U|$. It is easy to see that there is a spanning forest F of $G(S)$ consisting of γ trees $T_1, \ldots, T_\gamma$, such that every T_j contains exactly one vertex in U, and when we root T_j at such vertex the height of T_j is at most $R - 1$.

For a node $v \in V$, let $E_v = \{(v, u) \mid u \in U \setminus \{v\}\}$. Clearly, v is within the bound in $F + E_v$, hence by using Lemma 1, we have

$$SC(S) \leq |E(F)| + \sum_{u \in U} |E_u| + \sum_{v \in V \setminus U} |E_v| = n - \gamma + (\gamma - 1)\gamma + \gamma(n - \gamma) \leq (\gamma + 1)(n - 1).$$

□

Let $G(S)$ be stable and let v be a node of $G(S)$. Since v is within the bound, the neigborhood of v in G is a dominating set of G^{R-1}. Therefore, thanks to Lemma 2 we have proven the following corollary.

Corollary 1. *Let $G(S)$ be stable, and let δ be the minimum degree of $G(S)$, then $SC(S) \le (\delta+1)(n-1)$.* □

We are now ready to prove our upper bound to the PoA for MaxBD.

Theorem 4. *The PoA of* MaxBD *is $O\left(n^{\frac{1}{\lfloor \log_3 R \rfloor + 1}}\right)$ for $R \ge 3$, and $O(\sqrt{n \log n})$ for $R = 2$.*

Proof. Let G be a stable graph, and let γ be the size of a minimum dominating set of G^{R-1}. We define the *ball* of radius k centered at a node u as $\beta_k(u) = \{v \in V \mid d_G(u,v) \le k\}$. Moreover, let $\beta_k = \min_{u \in V} |\beta_k(u)|$. The idea is to show that in G the size of any ball increases quite fast as soon as the radius of the ball increases.

Claim. For any $k \ge 1$, we have $\beta_{3k+1} \ge \min\{n, \gamma\beta_k\}$.

Proof. Consider the ball $\beta_{3k+1}(u)$ centered at any given node u, and assume that $|\beta_{3k+1}(u)| < n$. Let T be a maximal set of nodes such that (i) the distance from every vertex in T and u is exactly $2k+2$, and (ii) the distance between any pair of nodes in T is at least $2k+1$. We claim that for every node $v \notin \beta_{3k+1}(u)$, there is a vertex $t \in T$ with $d_G(t,v) < d_G(u,v)$. Indeed, consider the node t' in a shortest path in G between v and u at distance exactly $2k+1$ from u. If $t' \in T$ the claim trivially holds, otherwise consider the node $t \in T$ that is closest to t'. From the maximality of T we have that $d_G(t,v) \le d_G(t,t') + d_G(t',v) \le 2k + d_G(u,v) - (2k+1) < d_G(u,v)$.

As a consequence, we have that $T \cup \{u\}$ is a dominating set of G^{R-1}, and hence $|T| + 1 \ge \gamma$. Moreover, all the balls centered at nodes in $T \cup \{u\}$ with radius k are all pairwise disjoint. Then

$$|\beta_{3k+1}(u)| \ge |\beta_k(u)| + \sum_{t \in T} |\beta_k(t)| \ge \gamma\beta_k. \qquad \square$$

Now, observe that since the neighborhood of any node in G is a dominating set of G^{R-1}, we have that $\beta_1 \ge \gamma$. Then, after using the above claim x times, we obtain

$$\beta_{\frac{3^{x+1}-1}{2}} \ge \min\left\{n, \gamma^{x+1}\right\}.$$

Let us consider the case $R \ge 3$ first. Let U be a maximal independent set of G^{R-1}. Since U is also a dominating set of G^{R-1}, it holds that $|U| \ge \gamma$. We consider the $|U|$ balls centered at nodes in U with radius given by the value of the parameter $x = \lfloor \log_3 R - 1 \rfloor$. Every ball has radius at most $(R-1)/2$, and since U is an independent set of G^{R-1}, all balls are pairwise disjoint, and hence we have $n \ge |U|\gamma^{\lfloor \log_3 R - 1 \rfloor + 1} \ge \gamma^{\lfloor \log_3 R \rfloor + 1}$. As a consequence, we obtain $\gamma \le n^{\frac{1}{\lfloor \log_3 R \rfloor + 1}}$, and the claim now follows from Lemma 2.

Now assume $R = 2$. We use the bound given in [5] to the size $\gamma(G)$ of a minimum dominating set of a graph G with n nodes and minimum degree δ, namely $\gamma(G) \leq \frac{n}{\delta+1} H_{\delta+1}$, where $H_i = \sum_{j=1}^{i} 1/j$ is the i-th harmonic number. Hence, since a social optimum has cost $n-1$, from Lemma 2 and Corollary 1, we have $\frac{SC(S)}{n-1} \leq \min\left\{\delta+1, \frac{n}{\delta+1} H_{\delta+1} + 1\right\} = O\big(\min\{\delta, \frac{n}{\delta} \log n\}\big)$, for any stable graph $G(S)$ with minimum degree δ. Since this is asymptotically maximized when $\delta = \Theta\big(\sqrt{n \log n}\big)$, the claim follows. □

Now we focus on lower bounds to the PoA of MaxBD. We first prove a simple constant lower bound for $R = o(n)$, and then we show an almost tight lower bound of $\Omega(\sqrt{n})$ for $R = 2$. We postpone to the concluding section a discussion on the difficulty of finding better lower bounds for large values of R.

Theorem 5. *For any $\epsilon > 0$ and for every $1 < R = o(n)$, the PoA of* MaxBD *is at least $2 - \epsilon$.*

Proof. Assume we are given a set of $n = 2R + h$ vertices $\{u_1, \ldots, u_{2R}\} \cup \{v_1, \ldots, v_h\}$. The strategy profile S is defined as follows. Vertex u_j buys a single edge towards u_{j+1}, for each $j = 1, \ldots, 2R - 1$, while every v_i buys two edges towards u_1 and u_{2R}. It is easy to see that $G(S)$ has diameter R and is stable. The claim follows from the fact that $SC(S)$ goes to $2(n-1)$ as h goes to infinity, and since, as observed in Section 3, a spanning star (having social cost equal to $n-1$) is a social optimum. □

Theorem 6. *The PoA of* MaxBD *for $R = 2$ is $\Omega(\sqrt{n})$.*

Proof. We provide only the lower-bound construction due to lack of space. Let $p \geq 3$ be a prime number. We exhibit (see Figure 1) a graph G' of diameter 2 containing $O(p^2)$ vertices and $\Omega(p^3)$ edges, and a strategy profile S such that $G(S) = G'$ and $G(S)$ is stable. G' contains two vertex-disjoint rooted trees T and T' as subgraphs. T is a complete p-ary tree of height 2. We denote by r the root of T, by $C = \{c_0, \ldots, c_{p-1}\}$ the set of children of r, and by $V_i = \{v_{i,0}, \ldots, v_{i,p-1}\}$ the set of children of c_i. T' is a star with p^2 leaves rooted at the center r'. The leaves of T' are partitioned in p groups each having exactly p vertices. For every $i = 0, \ldots, p-1$, we denote by $U_i = \{u_{i,0}, \ldots, u_{i,p-1}\}$ the set of vertices of group i. $G' = (V, E)$ has vertex set $V = V(T) \cup V(T')$, and edge set

$$\begin{aligned} E = {} & E(T) \cup E(T') \cup \{(r, r')\} \cup \big\{(c, c') \mid c, c' \in C, c \neq c'\big\} \\ & \cup \bigcup_{i=0}^{p-1} \big\{(u, u') \mid u, u' \in U_i, u \neq u'\big\} \\ & \cup \big\{(u_{i,j}, v_{i',j'}) \mid i, i', j, j' \in [p-1], j + i'i \equiv j' \pmod{p}\big\}. \end{aligned}$$

In the strategy profile S, (i) r buys all edges of G' incident to it, (ii) each $v_{i',j'}$ buys all edges of G' incident to it, (iii) each edge $(u_{i,j}, r')$ of G' is bought by $u_{i,j}$, and (iv) each of the remaining edges in G' is bought by any of its two endpoint players. □

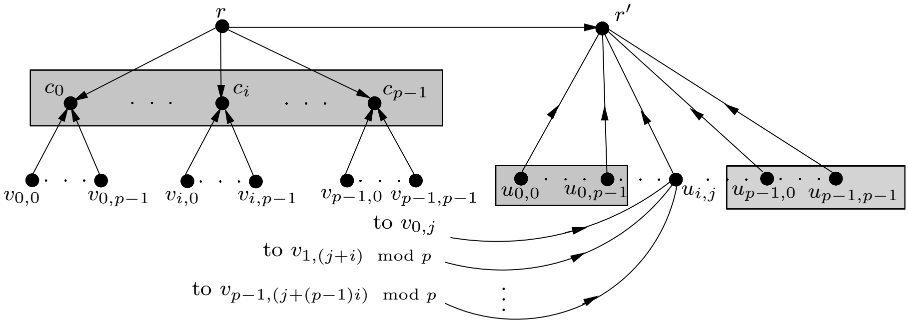

Fig. 1. The graph $G(S)$. Edges are bought from the nodes they exit from. Notice that nodes in grey boxes are clique-connected (with arbitrary orientations, i.e., ownership), and for the sake of readability we have only inserted edges leading to node $u_{i,j}$.

5 Upper and Lower Bounds to the PoA for SumBD

For SumBD, we start by giving an upper bound to the PoA similar to that obtained for MaxBD. For the remaining of this section we use D to denote the average bound of every node, namely $D = B/n$.

Theorem 7. *The PoA of* SumBD *is* $O\left(\sqrt{n \log n}\right)$ *when* $2 \leq D < 3$*, and* $O(n^{\frac{1}{\lfloor \log_3 D/4 \rfloor + 2}})$ *for* $D \geq 3$. □

From the above result, it follows that the PoA becomes constant when $D = \Omega(n^{\epsilon})$, for some $\epsilon > 0$. We now show how to lower such a threshold to $D = 2^{\omega(\sqrt{\log n})} = n^{\omega\left(\frac{1}{\sqrt{\log n}}\right)}$ (and we also improve the upper bound when $D = \omega(1) \cap o(3^{\sqrt{\log n}})$).

Lemma 3. *Let* $G(S)$ *be stable and let* v *be a node such that* $B_{G(S)}(v) \leq B - n$*, then* $SC(S) \leq 2(n-1)$.

Proof. Let T be a shortest path tree of G rooted at v. The claim immediately follows from Lemma 1 by observing that v is within the bound in T and every other node u is within the bound in $T + (u, v)$. □

Notice that the above lemma shows that when a stable graph G has diameter at most $D - 1$, then the social cost of G is at most twice the optimum. Now, the idea is to provide an upper bound to the diameter of any stable graph G as a function of δ, where δ is the minimum degree of G. Then we combine this bound with Lemma 3 in order to get a better upper bound to the PoA for interesting ranges of D.

Theorem 8. *Let* G *be stable with minimum degree* δ*. Then the diameter of* G *is* $2^{O(\sqrt{\log n})}$ *if* $\delta = 2^{O(\sqrt{\log n})}$*, and* $O(1)$ *otherwise.*

Proof. We start by proving two lemmas.

Lemma 4. *Let G be stable with minimum degree δ. Then either G has diameter at most $2\log n$ or, for every node u, there is a node x with $d_G(u,x) \leq \log n$ such that (i) x is buying δ/c edges (for some constant $c > 1$), and (ii) the removal of these edges increases the sum of distances from x by at most $2n(1+\log n)$.*

Proof. Assume that the diameter of G is greater than $2\log n$, and consider a node u. Let U_j be the set of nodes at distance exactly j from u, and let $n_j = |U_j|$. Moreover, denote by T a shortest path tree of G rooted at node u. Let i be the minimum index such that $n_{i+1} < 2n_i$ (i must exist since the height of T is greater than $\log n$). Consider the set of edges F of G having both endpoints in $U_{i-1} \cup U_i \cup U_{i+1}$ and that do not belong to T. Then, $|F| \geq \delta n_i/2 - 3n_i$. Moreover, we have that $n_{i-1} + n_i + n_{i+1} \leq n_i/2 + n_i + 2n_i = 7n_i/2$. As a consequence, there is a vertex $x \in U_{i-1} \cup U_i \cup U_{i+1}$ which is buying at least $\frac{n_i/2-3n_i}{7n_i/2} \geq \delta/c$ edges of F, for some constant $c > 1$. Moreover, when x removes these edges, the distance to any other node y increases by at most $2(1+\log n)$ because $d_T(x,y) \leq 2(1+\log n)$. The claim follows. □

Lemma 5. *In any stable graph G, there is a constant $c' > 1$ such that the addition of δ/c' edges all incident to a node u decreases the sum of distances from u by at most $5n\log n$.*

Proof. If G has diameter at most $2\log n$, then the claim trivially holds. Otherwise, let x be the node of the previous lemma and let c' be such that $\delta/c' \leq \delta/c - 1$. Moreover, assume by contradiction that the sum of distances from u decreases by more than $5n\log n$ when we add to G the set of edges $F = \{(u,v_1),\ldots,(u,v_h)\}$, with $h = \delta/c'$. Then, let $F' = \{(x,v_j) \mid j = 1,\ldots,h\}$. We argue that x can reduce its cost by saving at least an edge as follows: x deletes its δ/c edges and adds F'. Indeed, the sum of distances from x increases by at most $2n(1+\log n) \leq 4n\log n$, and decreases by at least $5n\log n - n\log n$, since for every node y such that the shortest path in $G+F$ from u to y passes through x, we have that $d_G(u,y) - d_{G+F}(u,y) \leq \log n$. Hence, x is still within the bound in $G+F'$ and is saving at least one edge, a contradiction. □

Recall that the ball of radius k centered at a node $u \in V$ is defined as $\beta_k(u) = \{v \in V \mid d_G(u,v) \leq k\}$, and that $\beta_k = \min_{u\in V} |\beta_k(u)|$. We claim that

$$\beta_{4k} \geq \min\left\{n/2+1, \frac{k\delta}{20c\log n}\beta_k\right\}, \tag{3}$$

for some constant $c > 1$. To prove that, let $u \in V$ be any node and assume that $|\beta_{4k}(u)| \leq n/2$. Let T be a maximal set of nodes such that (i) the distance from every vertex in T and u is exactly $2k+1$, and (ii) the distance between any pair of nodes in T is at least $2k+1$. From the maximality of T, for every node $v \notin \beta_{3k}(u)$ there is a node $t \in T$ such that $d_G(v,t) \leq d_G(u,v) - k$. Since $|\beta_{4k}(u)| \leq n/2$, at least $n/2$ nodes have a distance more than $3k$ from u. This implies the existence of a set Y of such vertices and a set $T' \subseteq T$ such that (i)

$|Y| \geq n\delta/(2|T|)$, (ii) $|T'| = \delta/c$, and (iii) for every $v \in Y$, there exists $v' \in T'$ such that $d_G(v, v') \leq d_G(u, v) - k$. If we add δ/c edges from u to nodes in T', the sum of distances from u decreases by at least $(k-1)n/(2|T|) \geq kn/(4|T|)$. By Lemma 5 this improvement is at most $5n \log n$ and, as a consequence, $|T| \geq \delta k/(20c \log n)$. Moreover, all the balls centered at nodes in T are disjoint, and this proves the recurrence (3). Now, the claim follows by solving such a recurrence. □

Next theorem provides an alternative upper bound to the PoA of SumBD.

Theorem 9. *The PoA of* SumBD *is* $2^{O(\sqrt{\log n})}$ *if* $D = \omega(1)$*, and* $O(1)$ *if* $D = 2^{\omega(\sqrt{\log n})}$.

Proof. Let $G(S)$ be stable, and let Δ be the diameter of $G(S)$. First of all, consider the case $\Delta = o(D)$, and observe that $B_{G(S)}(v) = o(B)$ for every v. Therefore, Lemma 3 implies that $\frac{SC(S)}{n-1} = O(1)$. This implies the second part of the claim since Theorem 7 implies that $\Delta = 2^{O\left(\sqrt{n}\right)}$.

Now, consider the case $\Delta = \Omega(D)$. Since $D = \omega(1)$, we have that $\Delta = \omega(1)$ and therefore, from Theorem 7, $\delta = 2^{O\left(\sqrt{n}\right)}$. To complete the proof, we show that $\frac{SC(S)}{n-1} \leq \delta + 1$. Let v be a node with degree δ, and let $N_{G(S)}(v) = \{u_1, \ldots, u_\delta\}$. Consider a shortest path tree T of $G(S)$ rooted at v. Clearly, v is within the bound in T, and if we define $E_x = \{(x, u_j) \mid 1 \leq j \leq \delta\}$ for any $x \neq v$, we have $B_{T+E_x}(x) \leq B_{G(S)}(v) \leq B$. Hence, from Lemma 1, it follows that $SC(S) \leq |E(T)| + (n-1)\delta \leq (\delta+1)(n-1)$. □

Then, by combining the results of Theorems 7 and 9, we get the bounds reported in Table 1. Finally, we can give the following

Theorem 10. *For any* $\epsilon > 0$ *and for* $2n - 3 \leq B = o(n^2)$*, the PoA of* SumBD *is at least* $2 - \epsilon$. □

6 Concluding Remarks

In this paper, we have introduced a new NCG model in which the emphasis is put on the fact that a player might have a strong requirement about its centrality in the resulting network, as it may well happen in decentralized computing (where, for instance, the bound on the maximum distance could be used for synchronizing a distributed algorithm). We developed a systematic study on the PoA of the two (uniform) games MaxBD and SumBD, which, however, needs to be continued, since a significant gap between the corresponding lower and upper bounds is still open. In particular, it is worth to notice that finding a better upper bound to the PoA would provide a better estimation about how much dense a network in equilibrium can be.

Actually, in an effort of reducing such a gap, we focused on MaxBD, and we observed the following fact: Recall that a graph is said to be *self-centered* if every node is a center of the graph (thus, the eccentricity of every node is equal to the radius of the graph, which then coincides with the diameter of the graph). An interesting consequence of Lemma 2 is that only stable graphs that are self-centered can be dense, as one can infer from the following

Proposition 1. *Let $G(S)$ be stable for* MaxBD*. If $G(S)$ is not self-centered, then $SC(S) \leq 2(n-1)$.*

Proof. Let v be a node with minimum eccentricity. It must be $\varepsilon_{G(S)}(v) \leq R-1$. Then, $U = \{v\}$ is a dominating set of G^{R-1}, and Lemma 2 implies the claim. □

Thus, to improve the lower bound for the PoA of MaxBD, one has to look to self-centered graphs. Moreover, if one wants to establish a lower bound of ρ, then a stable graph of minimum degree $\rho - 1$ (from Corollary 1) is needed. Starting from these observations, we investigated the possibility to use small and suitably dense self-centered graphs as *gadgets* to build lower bound instances for increasing values of R. To illustrate the process, see Figure 2, where using a self-centered cubic graph of diameter 3 and size 20, we have been able to obtain a lower bound of 3 (it is not very hard to see that the obtained graph is in equilibrium).

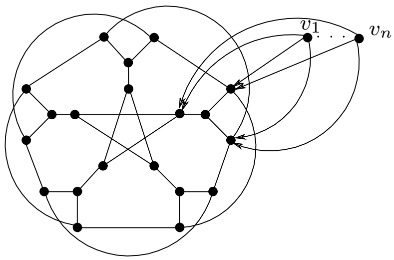

Fig. 2. A graph with $n + 20$ nodes and $3n + 30$ edges, showing a lower bound for the PoA of MaxBD for $R = 3$ approaching to 3, as soon as n grows. Edges within the gadget (on the left side) are bought by either of the incident nodes, while other edges are bought from the nodes they exit from.

Interestingly enough, the gadget is a famous extremal (i.e., maximal w.r.t. node addition) graph arising from the study of the *degree-diameter* problem, namely the problem of finding a largest size graph having a fixed maximum degree and diameter (for a comprehensive overview of the problem, we refer the reader to [1]). More precisely, the gadget is a graph of largest possible size having maximum degree $\Delta = 3$ and diameter $R = 3$. In fact, this seems not to be coincidental, since also *Moore graphs* (which are extremal graphs for $R = 2$ and $\Delta = 2, 3, 7, 57$), and the extremal graph for $R = 4$ and $\Delta = 3$ (see [1]), can be shown to be in equilibrium, and then they can be used as gadgets (clearly, the lower bounds implied by Moore graphs for $R = 2$ are subsumed by our result in Theorem 6). Notice that from this, it follows that we actually have a lower bound of 3 for the PoA of MaxBD also for $R = 4$. So, apparently there could be some strong connection between the equilibria for MaxBD and the extremal graphs w.r.t. to the degree-diameter problem, and we plan in the near future to explore such intriguing issue.

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Budget Optimization for Online Campaigns with Positive Carryover Effects

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Abstract. While it is relatively easy to start an online advertising campaign, proper allocation of the marketing budget is far from trivial. A major challenge faced by the marketers attempting to optimize their campaigns is in the sheer number of variables involved, the many individual decisions they make in fixing or changing these variables, and the nontrivial short and long-term interplay among these variables and decisions.

In this paper, we study interactions among individual advertising decisions using a Markov model of user behavior. We formulate the budget allocation task of an advertiser as a constrained optimal control problem for a Markov Decision Process (MDP). Using the theory of constrained MDPs, a simple LP algorithm yields the optimal solution. Our main result is that, under a reasonable assumption that online advertising has positive carryover effects on the propensity and the form of user interactions with the same advertiser in the future, there is a simple greedy algorithm for the budget allocation with the worst-case running time cubic in the number of model states (potential advertising keywords) and an efficient parallel implementation in a distributed computing framework like MapReduce. Using real-world anonymized datasets from sponsored search advertising campaigns of several advertisers, we evaluate performance of the proposed budget allocation algorithm, and show that the greedy algorithm performs well compared to the optimal LP solution on these datasets and that both show consistent 5-10% improvement in the expected revenue against the optimal baseline algorithm ignoring carryover effects.

1 Introduction

The Internet has become a major advertising medium, with billions of dollars at stake [21]. It has made it relatively easy even for small advertisers to quickly set up campaigns, track expenses, monitor effectiveness of the campaigns, and tinker with campaign parameters. Nonetheless, proper allocation of the marketing budget is far from trivial. A major challenge faced by the marketers attempting to optimize their campaigns is in the sheer number of variables they can possibly change. Even within a single advertising channel such as sponsored search ads on a particular search engine, the advertiser can optimize by reallocating the budget across different keywords, choosing a particular bidding strategy to use within a single ad auction, deciding on the daily advertising budget or what demographics of users to target. Each of these tasks can be solved reasonably well when considered as a standalone optimization problem,

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yet one can only wonder what fraction of social surplus (and advertising revenues) is lost by ignoring sophisticated dependencies and interaction patterns between individual optimization tasks, such as long-term effects of ads interacting with other ads.

In this paper, we study interactions among individual advertising decisions using a Markov model of user behavior, and develop optimization algorithms for budget allocation in this context. In particular, we focus on a potential *positive carryover effect* that online advertising has on the propensity and the form of user interactions with an advertiser in the future, and develop improved algorithms for the problem in this setting. We clarify these ideas on a simple scenario from the sponsored search area.

Example 1. A number of competing retailers are selling a single good with a certain brand name online. Every retailer has a choice of advertising only on the retailer specific keywords like the retailer's name or advertising on both the retailer specific keywords and the brand name of the good they sell. In this scenario, most of the users potentially interested in buying the good are initially uninformed about individual retailers' existence and therefore search directly with the brand name of the good. As the good is relatively expensive, they do not buy it from the first retailer found, instead clicking on multiple ads and comparing numerous offers. Once decided on the best offer, they often search with the retailer's name directly, proceed to the retailer's website and convert, i.e., make a purchase. Furthermore, a fraction of the converted users may become loyal customers that in the future skip the comparison shopping phase and go to the retailer's website directly without performing any brand related searches. Important property of this example is that analyzing profitability of retailer-specific keywords and brand-specific keywords separately improperly captures the influence of both on the retailer's revenue. Indeed, individual analysis in our example would suggest that brand-specific keywords provide significantly worse return-on-investment (ROI) than retailer-specific keywords due to both high CPC [1] values (heavy competition with other retailers) and low conversion rates (a lot of users clicking on multiple ads before converting). [2] Yet it would not be wise (and many advertisers know that) for the retailer to significantly cut spending on the brand-specific keywords as it is likely to reduce inflow of users to the retailer-specific keywords as well. One can say that there is a *carryover* from advertising on the brand-specific keywords to the ROI of advertising on the retailer-specific keywords. □

The above was only an example scenario and we emphasize that the model of carryover that we present in this paper is not restricted to only capture interactions between brand-specific and retailer-specific keywords, nor is it restricted to the domain of the sponsored search. Motivated by Markov models of user browsing behavior [20], in particular a previous study on mining advertiser-specific user behavior in sponsored search auctions [3], we model users using a Markov chain and advertising as not only affecting the current user action but also the future actions (through changing the state transition probabilities). Our contributions are as follows:

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[1] Cost per click.

[2] This is only a hypothetical scenario and its conclusions might not generalize to all settings. There are empirical findings that suggest that the presence of retailer-specific information in the keyword increases click-through rates, and the presence of brand-specific information in the keyword increases conversion rates [13].

– *(Problem)* In the Markovian user model, we formulate the budget allocation task of an advertiser as a constrained optimal control problem for a Markov Decision Process (MDP).
– *(Algorithm)* Using well-developed theory of constrained MDPs [2], we show that a simple LP algorithm yields the optimal policy. As the main contribution, we show that, under a reasonable assumption on the structure of carryover effects (see Section 5), there is a faster greedy algorithm for the optimal solution of the problem with the worst-case running time cubic in the number of model states (potential advertising keywords). This greedy algorithm is inspired by the Lagrangian relaxation of the optimization problem which is solvable using a combinatorial greedy algorithm in the presence of positive carryover effects. A major advantage of this algorithm is that it can be implemented efficiently in parallel using a distributed computing framework like MapReduce.
– *(Empirical Study)* Using real-world anonymized datasets from sponsored search advertising campaigns of several advertisers, we show that our greedy algorithm performs as well as the optimal LP solution, thus justifying our carryover assumption under which we can prove the optimality of our greedy algorithm. Furthermore, our budget allocation algorithm shows 5-10% improvement in revenues against the baseline, consistent across a wide range of different settings and budget constraints.

While budget optimization problems have been studied previously in sponsored search, even in the setting of possible externalities, our paper is the first to consider the long term impacts of different ad instances on each other.

Due to space constraints, we do not include the proofs, and the details of the description of algorithms in this extended abstract.

2 Related Work

Advertising carryover in marketing refers to the well-known phenomenon that advertising messages affect consumers long after the initial exposure. Carryover effects have been extensively studied in marketing literature [7], including online settings [23]. The exact mechanism by which carryover works is often unspecified and the effect itself is usually modeled simply by assuming that a certain fraction of the advertising effects in the current period is retained in the next period. In our paper, we model carryover at the level of individual advertising decisions within the campaign. For instance, hypothetically, the decision of JetBlue to advertise on "cheap tickets" keyword may have carryover effect on the number of users that issue search queries with the airline's brand name in the future.

This paper focuses on positive externalities of ads for the *same* advertiser in different sessions, i.e., ads of an advertiser on multiple keywords reinforce each other in a positive way [18,8,14]. Certain empirical support for the presence of positive externalities in sponsored search can be found in [18]: a randomized controlled experiment, performed in cooperation between Yahoo! and a major retailer, found that the online advertising campaign had substantial positive impact not only on the users who clicked on the ads but also on those who merely viewed them. In another study, comScore [8] reported an incremental lift of 27% in the online sales after the initial exposure to an online ad, as well as lift in other important online behaviors, such as the brand site visitation and the trademark searches. Ghose and Yang [14] report positive interdependency between paid and organic search results: the presence of organic listings is associated with a higher probability of click-throughs on paid ads, and vice-versa.

In the world populated by Markov users, we consider the standard *budgeted campaign optimization problem* [12,19,10]: find an optimal bidding policy to maximize the number of user conversions subject to the budget constraint for the expected advertising cost. Our approach allows us to apply machinery from the familiar field of constrained MDPs, in particular reduce the budget optimization problem to a regular LP (an excellent review of constrained MDPs can be found in [2]). As the main contribution of the paper, under assumption of positive carryover effects we provide a simple, fast greedy algorithm for this problem based on the ideas of Lagrangian relaxation.

3 Our Model and Problem

The notation below is chosen to be consistent with [2], except that we consider the problem of maximizing the long-term total reward (conversion probability) while [2] considered the problem of minimizing the long-term total cost.

Let X be a finite state space representing possible user *states*. In one interpretation, the state can capture the last query issued by the user. For any $x \in X$, let $A(x)$ represents the finite set of possible *actions* (advertising levels) in state x. For instance, $A(x)$ can be $\{$advertise, do not advertise$\}$ but one can also consider more sophisticated possibilities with different levels of advertising, for instance, one can think of different slots on the search results page as possible advertising levels. Without loss of generality, we can always assume a common set of advertising levels $A(x) \equiv A$ available in all states. The user randomly "jumps" between states with transition probabilities depending on the level of the advertising the user is exposed to. Let $\mathcal{P}_{xay}$ be the probability of moving from state x to state y if advertising level $a \in A$ is chosen. Next, let $d(x, a) \geq 0$ be the immediate monetary cost of advertising at level a in state x. This cost will relate to the budget constraint (V) for our optimization problem.

We define three special states in the system: $x_c \in X$ representing the conversion state, $x_n \in X$ representing the non-conversion state, and x_f representing the final state. The final state x_f is absorbing. All transitions from the non-conversion state x_n and the conversion state x_c lead to the final state x_f. The initial flow of users to the system is given by measure $\beta(x)$ and the advertisers' optimization problem is to maximize the expected number of converted users subject to the budget constraint. Without loss of generality, we can assume that β is normalized to 1 and therefore represents a probability measure. In such normalization, V will represent a per-user budget constraint. We can recast the optimization problem as a particular case of constrained MDPs by defining the reward function that we are trying to maximize as $r(x, a) \equiv C$ when $x = x_c$ and $r(x, a) \equiv 0$ otherwise (i.e., we get a reward of C in the conversion state and zero everywhere else). We assume that the Markov process is absorbing, i.e, sooner or later we will end up in the final state in which we accumulate no reward and pay no cost, thus the optimization problem is well-defined. We formalize this as follows.

For $t = 1, \ldots, \infty$, let H_t be the set of all possible user histories of length t. Every element $h_t \in H_t$ is a history of states and chosen actions until time t, i.e. $h_t =$

$(x_1, a_1, x_2, a_2, ..., x_t)$ (note that the advertising exposure at time t is not included). A general *policy* u is a collection of functions

$$u_t : H_t \rightarrow \triangle(A),$$

where $\triangle(A)$ represents the set of probability distributions over A (policies can be randomized). Note that the general policy allows for the targeted advertising, i.e., choosing the advertising level for the user based on the complete history of the prior user searches and ads the user was exposed to.

Definition 1. *For every distribution over initial states β and a policy u, there is a unique measure on the space of trajectories H_∞. We can use P_β^u to denote this measure. Moreover, define*

$$p_\beta^u(t; x, a) = P_\beta^u(x_t = x, a_t = a),$$

i.e., $p_\beta^u(t; x, a)$ is the probability of observing the state x and the advertising level a at the step t of the process when following the policy u. Next, define the expected total reward *and the expected total* cost *for a policy u as $R(\beta, u) = \sum_{t=1}^{\infty} \sum_{X,A} p_\beta^u(t; x, a) r(x, a)$ and $D(\beta, u) = \sum_{t=1}^{\infty} \sum_{X,A} p_\beta^u(t; x, a) d(x, a)$ respectively.* □

Note that both are well-defined as we assume that the MDP is absorbing.

The *budget optimization problem* we face (for a single user) is simply

$$\max_{u \in U} R(\beta, u) \qquad \textbf{[P1]}, \text{ s.t. } D(\beta, u) \leq V$$

where U is the set of policies of interest.

Special Classes of Policies. There are three classes of policies of our interest:

- In *Markov policies*, u_t depends only on x_t, that is, we target users based only on their current state and the amount of time they spent in the system.
- In the special case of *stationary policies*, u_t does not depend on t, that is, we target users based on their current state only.
- Further special are *stationary deterministic policies*, for which the advertising level is chosen in each state deterministically. That is, we target users based on their current state only and all users in the same state are exposed to the same advertising level.

4 Classic Results for Constrained Markov Decision Processes

Below is a summary of well-known results for constrained MDPs that apply to our model. The proofs are in [2].

Fact 1. It is sufficient to restrict consideration to Markov policies only (see Theorem 2.1 of [2]) as for any general policy u, there exists some other Markov policy v such that $p_\beta^u(t; \cdot, \cdot) \equiv p_\beta^v(t; \cdot, \cdot)$.

Fact 2. Let $X' = X \setminus \{x_f\}$. An *occupation measure* is a "visit count" measure over the set of states and advertising levels ($\mu \in M(X' \times A)$) achievable by some Markov policy

u: $\mu(x,a) = \sum_{t=1}^{\infty} p_\beta^u(t;x,a)$. Let $L(\beta)$ be the set of all occupation measures, $L(\beta)_S$ be the set of all occupation measures achievable with a stationary policy and $L(\beta)_D$ be the set of all occupation measures achievable with a stationary deterministic policy. Theorem 3.2 from [2] gives characterization of the set of all occupation measures. It says that $L(\beta) = L(\beta)_S = coL(\beta)_D$ (convex hull). Moreover, it is equal to $Q(\beta)$, where $Q(\beta)$ is the set of all non-negative finite measures ρ on $X' \times A$ such that

$$\forall x \in X' \sum_{y \in X'} \sum_{a \in A} \rho(y,a)(\delta_x(y) - \mathcal{P}_{yax}) = \beta(x) \tag{1}$$

Note that Equation 1 is the basic "conservation of flow" statement, thus the result can be interpreted as "any measure satisfying the set of conservation of flow constraints is achievable with some stationary policy" (the reverse is obviously true as well).

Fact 3. The previous result means that we can only look for stationary policies or, even better, we can look for the solution in form of an occupation measure. Theorem 3.5 from [2] shows that there is one to one equivalence between feasible (and optimal) solutions of **P1** and feasible (and optimal) solutions of the following linear program:

$$\begin{aligned} \max_{\rho} \; & \sum_{x \in X'} \sum_{a \in A} r(x,a)\rho(x,a) & & \textbf{[P2]} \\ \text{s.t.} \; & \sum_{x \in X'} \sum_{a \in A} d(x,a)\rho(x,a) & & \leq V \\ & \sum_{y \in X'} \sum_{a \in A} \rho(y,a)(\delta_x(y) - \mathcal{P}_{yax}) & & = \beta(x) \; \forall x \in X' \\ & \rho(x,a) & & \geq 0 \; \forall x \in X', a \in A. \end{aligned}$$

In particular, if ρ^* is the optimal solution of **P2**, then the stationary policy u^* choosing the advertising level a with probability of $\frac{\rho^*(x,a)}{\sum_b \rho^*(x,b)}$ is the optimal randomized stationary policy (one can choose any advertising level if the denominator is zero).

Note that the linear program **P2** has $|X'| + 1$ constraints (the budget constraint and $|X'|$ consistency constraints) in addition to the non-negativity constraints. Thus, one can always find the optimal solution in which at most $|X'| + 1$ $\rho(y,a)$ values are positive. That implies that there is always an optimal advertising strategy with randomization in at most one state. **Fact 4.** In the following, it will also be useful to consider the dual program of **P2**:

$$\begin{aligned} \min_{\pi,\lambda} \quad & \sum_{x \in X'} \beta(x)\pi(x) + \lambda V & & \textbf{[P3]} \\ \text{s.t.} \quad & \lambda \geq 0 \\ & \pi(x) \geq r(x,a) - \lambda d(x,a) + \sum_{y \in X'} \mathcal{P}_{xay}\,\pi(y) \\ & \forall x \in X', \; a \in A \end{aligned}$$

Here λ is the Lagrange multiplier for the budget constraint and, for any fixed value of λ, $\pi(x)$ can be thought of as the optimal value function in the Markov model M_λ with adjusted rewards $r_\lambda(x) = r(x,a) - \lambda d(x,a)$. This intuition is captured by the following LP for a fixed λ:

$$\min_{\pi_\lambda} \quad \sum_{x \in X'} \beta(x)\pi(x) \qquad \textbf{[P3(}\lambda\textbf{)]}$$

$$\text{s.t.} \quad \pi_\lambda(x) \geq r_\lambda(x,a) + \sum_{y \in X'} \mathcal{P}_{xay}\,\pi_\lambda(y)$$

$$\forall x \in X',\ a \in A$$

Because the value of λ is fixed, **[P3(λ)]** is a classic infinite-horizon DP problem on a graph M_λ with rewards $r_\lambda(x,a)$, therefore it has a uniformly optimal stationary dual policy, which in every state x chooses the advertising level $a(x)$ deterministically and does not depend on the distribution of initial states β.

5 Budget Optimization with Positive Carryover Effects

The previous section shows that the budget optimization problem in Markovian world can be cast a simple linear program **P2** with $|X'| \times |A|$ variables and $|X'|+1$ constraints. In real world online advertising settings, in particular, in sponsored search, $|X'|$ represents the number of feasible keywords to advertise on and therefore can be as large as tens of thousands for a single advertiser. Number of advertising levels can be in the order of ten (different slots) or more. Considering the fact that the constraint matrix is not sparse, the direct LP approach presents significant practical computational challenges. In this section, we identify the structure in the problem and use that to design a simpler greedy algorithm which proceeds under assumption that the advertising carryover effects are positive, which is realistic. The algorithm is guaranteed to find the optimal solution of **P2** with the worst-case running time of $|X'|^3 \times |A|^2$ under this assumption. As the experimental section shows, the suggested algorithm performs very well in the real world settings even if the underlying assumptions are violated.

First, we impose that the set of advertising levels A is totally ordered $a_1 \preceq a_2 \preceq \ldots \preceq a_k$, with interpretation that if $a_i \prec a_j$ then a_j represents a more intense level of advertising than a_i. Next, we assume that the Markov user model satisfies the following conditions which are realistic (our empirical study will not make such assumptions):

- More advertising never hurts (Postive Carryover):

$$\forall x \in X',\ y \in X' \setminus \{x_n\},\ a \preceq b\ \mathcal{P}_{xay} \leq \mathcal{P}_{xby}$$

- More advertisting is more expensive:[3]

$$\forall x \in X',\ a \preceq b\ d(x,a) \leq d(x,b)$$

- Not advertising costs nothing: $d(x, a_1) \equiv 0$, i.e., we assume that the advertiser can always opt out of advertising in any state at no extra cost.

[3] This assumption is not essential and can be relaxed. Indeed, if there are two advertising levels a and b such that $a \preceq b$ but $d(x,a) > d(x,b)$ then the advertiser can safely drop level a from consideration (using b instead is always a better choice).

Now, for any fixed $\lambda \geq 0$ consider the optimization problem **P3(**λ**)**: and its dual **P4(**λ**)**:

$$\begin{aligned}
\min_{\rho} \quad & \sum_{x \in X'} \sum_{a \in A} (r(x,a) - \lambda d(x,a)) \rho(x,a) \ \textbf{[P4(}\lambda\textbf{)]} \\
\text{s.t} \quad & \forall x \in X' \ \sum_{y \in X'} \sum_{a \in A} \rho(y,a)(\delta_x(y) - \mathcal{P}_{yax}) = \beta(x) \\
& \forall x \in X', a \in A \ \rho(x,a) \geq 0.
\end{aligned}$$

We emphasize that because **P3(**λ**)** is a classic infinite-horizon DP problem on a graph, it has a uniformly optimal stationary policy. In case of $\lambda = 0$, this policy has a particularly simple structure due to positive externalities.

Lemma 1 (Solution of Unconstrained Problem). *For $\lambda = 0$ there is a uniformly optimal policy of* **P3(**λ**)** *in which we advertise with the highest possible intensity (a_k) in every state.*

The proofs are not included in this extended abstract.

Lemma 2 (Monotonicity of Dual Value Function). *Let $0 \leq \lambda_1 < \lambda_2$, u_1, u_2 be the corresponding uniformly optimal stationary policies and π_1, π_2 be the corresponding value functions for* **P3(**λ**)**. *Then, $\forall x \in X'$ $\pi_1(x) \geq \pi_2(x)$.*

Lemma 3 (Continuity of Dual Value Function). *Let $f_\beta(\lambda)$ be the value of the optimization problem* **P3(**λ**)** [4]. *$f_\beta(\lambda)$ is a continuous function of λ. In particular, taking $\beta \equiv \delta_x$, we obtain that $\pi^*_\lambda(x)$ is a continuous function of λ.*

Definition 2. *For any $\lambda \geq 0$ and $x \in X'$, define the set of active advertising levels $\mathcal{A}(\lambda, x)$ as*

$$\left\{ a \in A \text{ s. t. } \begin{array}{c} \exists \pi^* \text{ uniformly optimal for P3}(\lambda) \text{ and} \\ \pi^*(x) = r_\lambda(x,a) + \sum_{y \in X'} \mathcal{P}_{xay} \, \pi^*(y) \end{array} \right\}$$

Note that $\mathcal{A}(\lambda, x)$ is always non-empty.

Definition 3. *For any $\lambda \geq 0$ and $x \in X'$, define the lowest active advertising level $a_L(\lambda, x)$ and the highest active advertising level $a_H(\lambda, x)$ as*

$$a_L(\lambda, x) = \min \mathcal{A}(\lambda, x),$$

$$a_H(\lambda, x) = \max \mathcal{A}(\lambda, x).$$

Lemma 4 (Monotone Selection).
For any $x \in X$ and $0 \leq \lambda_1 < \lambda_2$, we have $a_L(\lambda_1, x) \succeq a_L(\lambda_2, x)$ and $a_H(\lambda_1, x) \succeq a_H(\lambda_2, x)$.

[4] Subscript β is used to indicate the dependence on the initial distribution β

Note. Proof of Lemma 4 looks like a standard single-crossing argument that can be used to prove monotone selection theorems for supermodular and quasisupermodular functions. While the primal optimization problem can indeed be written as a supermodular function, the Lagrangian relaxation of the dual is not supermodular, nor quasisupermodular (we omit the counterexamples due to space limitation), therefore Lemma 4 doesn't seem to be a corollary of Topkis's theorem [22] or monotone selection theorems for quasisupermodular functions.

Lemma 5 (Structure of Dual Value Function).
$f_\beta(\lambda)$ is a piecewise linear continuous function. Moreover, the slope of f_β at any particular λ is equal to $-\beta^T(I - \mathcal{P}_\lambda)^{-1}d_\lambda$, where β is the column vector of $\beta(x)$, d_λ is the column vector of $d(x, a_H(\lambda, x))$ and $\mathcal{P}_\lambda$ is the matrix of $\mathcal{P}_{xa_H(\lambda,x)y}$. [5]

5.1 Greedy BO Algorithm

Lemma 5 suggests a simple greedy algorithm for determining all breakpoint values λ_i for the dual value function $f_\beta(\lambda)$. Algorithm 2 keeps the the current set of highest active advertising levels $\mathbb{A}_{\lambda_i}$ as a part of its state. $\mathbb{A}_{\lambda_i}$ is stored as a simple set of numbers $m(x)$ for every state x, representing that $\mathbb{A}_{\lambda_i} = \{(x, a_{m(x)})|x \in X'\}$. At every step of the algorithm we choose one candidate node x^*. One way to define this node is to imagine that we freeze the current set of active advertising levels $\mathbb{A}_{\lambda_i}$ and start gradually increasing the value of λ. The first node, for which it will be locally optimal to decrease the advertising level $m(x)$, is the node x^*, the new value λ_{i+1} at which that happens is given by $\lambda_{i+1} = \lambda_i + \delta^*$ and the new advertising level at x^* will be m^*.

Theorem 1 (Greedy BO algorithm). *Algorithm 2 correctly constructs the dual value function $f_\beta(\lambda)$.*

5.2 Improved Greedy BO Algorithm

Number of iterations of Algorithm 2is bounded by $|X| \times |A|$. The most expensive operation inside a single iteration is solving a linear system with $|X|$ unknowns and $|X|$ variables. This can be done in $O(|X|^3)$ operations in practice or in $O(|X|^{2.376})$ asymptotically [9]. Fortunately, we can significantly improve the performance of the algorithm by noting that it proceeds one variable at a time, always adjusting advertising level in a single state only. Thus, we do not really need to solve the system $d\pi_{i+1} \Leftarrow -(I - P_{i+1})^{-1}d_{i+1}$ from the scratch each time. Instead, the algorithm can keep an LU decomposition of the matrix $I - P_i$, updating it in every step. Because only a single row is replaced in the matrix, updating the LU decomposition can be trivially done in a quadratic time by solving a system of equations with a triangular matrix. That results in the $O(|X|^2 \times |A|)$ worst-case performance of the inner loop and $O(|X|^3 \times |A|^2)$ worst-case performance of the whole algorithm assuming a sequential processing model. The improved version of the algorithm is given by Algorithm 3.

[5] Note that there is an equivalent representation in which $d_\lambda = d(x, a_L(\lambda, x))$ and $\mathcal{P}_\lambda = \mathcal{P}_{xa_L(\lambda,x)y}$.

5.3 Parallel Implementation of Greedy BO Algorithm

The most interesting property of Algorithm 3 is that it supports an efficient parallel implementation using a distributed programming framework like MapReduce [11]. This might be an important advantage for solving large-scale advertising campaigns with several thousands of keywords. This is in contrast to the original LP program **P2**, which is not a packing-covering linear program and, therefore, we are not aware of any distributed or parallel algorithm to solve it. Below, we give a brief description of the idea behind this parallel implementation.

The **Candidate Selection()** function can be parallelized to run on $|X|$ machines simply by distributing every iteration of the outer loop (**for** every $x \in X'$) to a separate machine and aggregating the results afterwards. Similarly, solution of a system of linear inequalities with a triangular matrix can be done in $|X|$ time on $|X|$ machines. Thus, we state that in a parallel processing framework with $|X|$ machines, Algorithm 3 worst-case performance is $O(|X|^2 \times |A|^2)$ plus the time needed to perform LU decomposition of the matrix $I - P_0$ in the initialization step. Details of the implementation are beyond the scope of this paper.

Table 1. Summary Statistics for CPC and number of keywords per campaign

	Min	**Max**	**Median**	**Mean**
CPC	1.43¢	$1.34	5.49¢	25.46¢
Keywords	285	998	933	782.33

6 Evaluation

We performed evaluation of our budget optimization algorithms on nine real world datasets containing data from nine different sponsored search campaigns. All datasets were advertiser-specific and included only user activities (such as ad clicks) related to a single search campaign of a single advertiser. The dataset was collected at user level and contained information on a random sample of users who converted with the advertiser within a period of two weeks in December 2009. For every anonymous user, the dataset recorded ad clicks of this user before the conversion. Every ad click event had associated timestamp and the keyword the user query was matched with. In this data set, we do not observe the events in which the users did not click on the ad. Moreover, since we focus on advertiser-specific information, user searches for which the advertiser's ad was not shown, such as searches with irrelevant search queries, keywords on which the advertiser bid too low, or keywords for which the advertiser was excluded from the auction due to a daily budget constraint, are *not* included in our dataset. While such extra data might be available in some form to the search engine, due to several privacy and competition issues, it would not be reported to the advertisers, therefore we intentionally focus on the restricted advertiser-specific data described above.

In addition to the above datasets, we compiled cost information for all keywords from our sample. To simplify the experiments, we used average CPC (cost per click) values, computed as the average cost of the clicks that the advertiser got for a particular

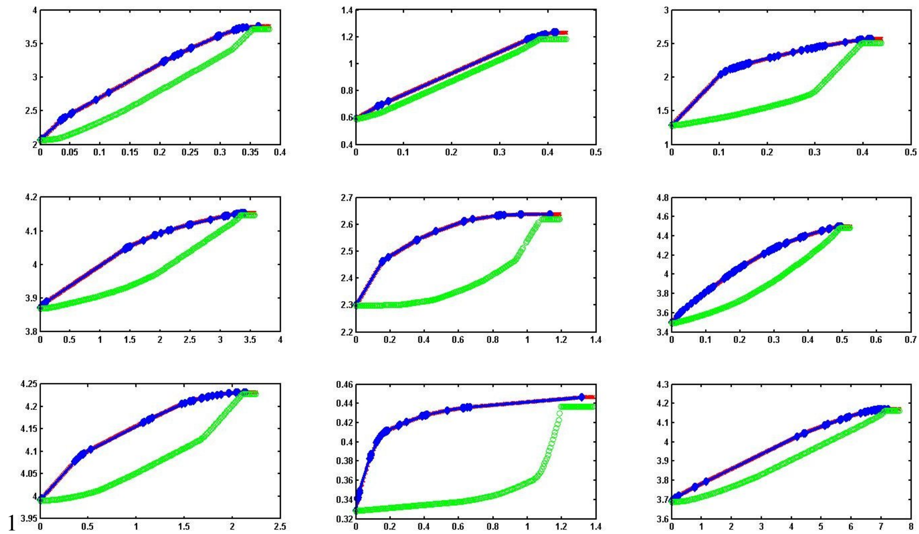

Fig. 1. Performance of budget optimization techniques: LP-based (red), greedy BO (blue) and baseline assuming no carryover (green). Horizontal axis: budget per user in cents. Vertical axis: expected value per user in cents.

keyword in a similar time period of two weeks. Summary statistics for the average CPC per keyword and the number of keywords per campaign are given in Table 1.

To represent user behavior by a Markov chain, we follow the approach of [3]. [3] suggests that from advertisers' perspective, user behavior can be reasonably approximated by a first order Markov chain. In such Markov Chain, state represents the last observed event for the user (for example, user searching for "Prada shirt") and transition probabilities between states are directly estimated from the data. Following [3], we model user state by the keyword that the last user search was matched with. In contrast with [3], we only include clicked ads as model states because pure impression information was missing from our sample.

Next, we add four special states: the begin state (x_b) representing a new user entering the system, the conversion state representing the user conversion event (x_c), the non-conversion state representing the user leaving the system without converting (x_n) [6] and the final state (x_f). The final state is absorbing and, by construction, conversion and non-conversion states always lead to the final state. The begin state has no incoming edges.

Due to the nature of our data, we only consider two possible advertising levels for every keyword, "advertise" and "do not advertise", and restrict consideration to the

[6] We never really know whether the user has dropped out or he is going to come back later and convert. As only small number of users converts, it is always reasonable to assume that the user, who hasn't converted so far, is not going to.

top 250 keywords in each campaign [7]. "Do not advertise" decisions cost nothing and "advertise" decisions cost the average CPC of the corresponding keyword. Consistent with our theoretical model, transition probabilities between states depend on whether the user was exposed to the advertisement or not. In order to distinguish between cases in which a user click influences his/her later activities, we apply the following model: assume that if the time gap between two consecutive user states (consecutive searches) is large enough (e.g., at least one day), the transition between these states was not due to influence of the online ad and therefore would have happened even if the ad was not shown to the user. Although this is an intuitive assumption, we acknowledge that this strategy may produce biased transition probability estimates. However, as our goal is only to evaluate performance of the budget optimization algorithm across multiple campaigns and wide range of parameters, such bias can be tolerated, and constructing approximate models should be enough to evaluate the performance of the algorithms and in particular, the comparison between the scalable greedy algorithms and LP solutoins. The graph construction algorithm has two configuration parameters that can be tuned. The first parameter α represents the probability that a user can leave the system at any moment of time. We follow a conservative approach and assume that this probability is unaffected by whether the user was exposed to the advertisement in the last step or not. The second parameter C represents the advertiser's value for a single converted user. As both parameters were unknown in our dataset, we have validated the model across a wide range of them. In the paper, we present results assuming $\alpha = 0.5$ and $C = \$5$.

In the following, we compare performance of three budget optimization algorithms. The baseline algorithm is a simple greedy solution of the fractional knapsack, in which the advertiser sorts all keywords by the immediate ROI value $\frac{\mathcal{P}_{xa_1y} - \mathcal{P}_{xa_0y}}{\text{CPC}}$ (ignoring the potential carryover effects to other keywords) and picks the keywords to advertise on in sequence starting from the keyword with the highest ROI. The process stops once we reach the expected allowed budget of the advertising campaign. As the expected campaign budget depends on the assumed model of user behavior, we still have to assume Markovian world when estimating the expected budget in the baseline algorithm. To reconcile this fact with the assumption that the advertiser is optimizing myopically, we assume that, in the baseline algorithm, we advertise to every user only the first time the user enters into the system. We compare the performance of the baseline algorithm with performance of the two alternative budget optimization algorithms:

– the direct approach which is based on solving the linear program **P2** and therefore is guaranteed to construct the optimal solution,
– the greedy budget optimization technique of Algorithm 3. [8]

We perform comparison across a range of possible budget values starting from zero budget (in which case the only feasible solution is not to advertise) up to the value V_{max}

[7] The main reasons for limiting the number of keywords to only 250 are slow performance of the LP algorithm with large number of variables (the greedy BO algorithm works fine) and presence of significant noise in transition probability estimates for infrequently used keywords.

[8] In fact, we use the alternative version of Algorithm 3in which we start from $\lambda = +\infty$ and reconstruct $f_\beta(\lambda)$ by gradually decreasing λ.

which represents the budget value for which the budget constraint does not bind the optimal solution anymore. Results of the three algorithms on all nine advertising campaigns are shown in Figure 1. As can be seen from the plot, there was no significant difference in performance of the LP algorithm and the greedy BO algorithm, confirming the positive carryover assumptions and the overall validity of our approach. Both algorithms consistently performed better than the baseline (the fractional knapsack) algorithm. If we measure the algorithm performance by AUC (area under the curve) in Figure 1, then the median gain in AUC was 5.79% and the mean gain in AUC was 9.14%. The largest observed difference in AUC was a gain of 27.14% and the smallest one was a gain of 1.77%. Furthermore, the difference in performance was particularly significant for medium values of the budget constraint, that are neither too small nor too large.

7 Conclusions

The Internet has become a major advertising medium. While it is relatively easy to start an online advertising campaign, proper allocation of the marketing budget is far from trivial. A major challenge faced by the marketers attempting to optimize their campaigns is in the sheer number of variables they can possibly change and nontrivial interactions between them. In this paper, we consider the important interaction effect between individual advertising decisions: a potential carryover effect that online advertising has on the propensity and the form of user interactions with an advertiser in the future. We adopt the Markov model of user browsing behavior and formulate the budget allocation task of an advertiser as a constrained optimal control problem for a Markov Decision Process (MDP). Using well-developed theory of constrained MDPs, we show that a simple LP algorithm yields the optimal policy. Furthermore, we show that, under reasonable assumptions on the structure of carryover effects, there is a simple greedy algorithm for the optimal solution of the problem that is faster and has an efficient implementation in a parallel processing framework. Using real-world anonymized datasets from sponsored search advertising campaigns of some large advertisers, we evaluate applicability of our model and performance of the proposed budget allocation algorithm. Our budget allocation algorithm shows 5-10% improvement in revenues against the optimal baseline algorithm ignoring carryover effects, consistent across a wide range of different settings and budget constraints.

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Choosing Products in Social Networks

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Abstract. We study the consequences of adopting products by agents who form a social network. To this end we use the threshold model introduced in [1], in which the nodes influenced by their neighbours can adopt one out of several alternatives, and associate with each such social network a strategic game between the agents. The possibility of not choosing any product results in two special types of (pure) Nash equilibria.

We show that such games may have no Nash equilibrium and that determining the existence of a Nash equilibrium, also of a special type, is NP-complete. The situation changes when the underlying graph of the social network is a DAG, a simple cycle, or has no source nodes. For these three classes we determine the complexity of establishing whether a (special type of) Nash equilibrium exists.

We also clarify for these categories of games the status and the complexity of the finite improvement property (FIP). Further, we introduce a new property of the uniform FIP which is satisfied when the underlying graph is a simple cycle, but determining it is co-NP-hard in the general case and also when the underlying graph has no source nodes. The latter complexity results also hold for verifying the property of being a weakly acyclic game.

1 Introduction

1.1 Background

Social networks are a thriving interdisciplinary research area with links to sociology, economics, epidemiology, computer science, and mathematics. A flurry of numerous articles and recent books, see, e.g., [2], testifies to the relevance of this field. It deals with such diverse topics as epidemics, analysis of the connectivity, spread of certain patterns of social behaviour, effects of advertising, and emergence of 'bubbles' in financial markets.

One of the prevalent types of models of social networks are the *threshold models* introduced in [3]. In such a setup each node i has a threshold $\theta(i) \in (0, 1]$ and adopts an 'item' given in advance (which can be a disease, trend, or a specific product) when the total weight of incoming edges from the nodes that have already adopted this item exceeds the threshold. One of the most important issues studied in the threshold models has been that of the spread of an item, see, e.g., [4,5,6]. From now on we shall refer to an 'item' that is spread by a more specific name of a 'product'.

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In this context very few papers dealt with more than one product. One of them is [7] with its focus on the notions of compatibility and bilinguality that result when one adopts both available products at an extra cost. Another one is [8], where the authors investigate whether the algorithmic approach of [5] can be extended to the case of two products.

In [1] the authors introduced a new threshold model of a social network in which nodes (agents) influenced by their neighbours can adopt one out of *several* products. This model allowed us to study various aspects of the spread of a given product through a social network, in the presence of other products. We analysed from the complexity point of view the problems of determining whether adoption of a given product by the whole network is possible (respectively, necessary), and when a unique outcome of the adoption process is guaranteed. We also clarified for social networks without unique outcomes the complexity of determining whether a given node has to adopt some (respectively, a given) product in some (respectively, all) final network(s), and the complexity of computing the minimum and the maximum possible spread of a given product.

1.2 Motivation

Our interest here is in understanding and predicting the behaviour of the consumers (agents) who form a social network and are confronted with several alternatives (products). To carry out such an analysis we use the above model of [1] and associate with each such social network a natural strategic game. In this game the strategies of an agent are products he can choose. Additionally a 'null' strategy is available that models the decision of not choosing any product. The idea is that after each agent chose a product, or decided not to choose any, the agents assess the optimality of their choices comparing them to the choices made by their neighbours. This leads to a natural study of (pure) Nash equilibria, in particular of those in which some, respectively all, constituent strategies are non-null.

Social network games are related to graphical games of [9], in which the payoff function of each player depends only on a (usually small) number of other players. In this work the focus was mainly on finding mixed (approximate) Nash equilibria. However, in graphical games the underlying structures are undirected graphs. Also, social network games exhibit the following ***join the crowd*** property: the payoff of each player depends only on his strategy and on the set of players who chose his strategy and weakly increases when more players choose his strategy.

Since these games are related to social networks, some natural special cases are of interest: when the underlying graph is a DAG, has no source nodes or a simple cycle which is a special case of a graph without source nodes. Such social networks correspond respectively to a hierarchical organization or to a 'circle of friends', in which everybody has a friend (a neighbour). Studying Nash equilibria of these games and various properties defined in terms of improvement paths allows us to gain better insights into the consequences of adopting products.

1.3 Related Work

There are a number of papers that focus on games associated with various forms of networks, see, e.g., [10] for an overview. A more recent example is [11] that analyses a strategic game between players being firms who select nodes in an undirected graph in order to advertise competing products via 'viral marketing'. However, in spite of the focus on similar questions concerning the existence and structure of Nash equilibria and on their reachability, from a technical point of view, the games studied here seem to be unrelated to the games studied elsewhere.

Still, it is useful to mention the following phenomenon. When the underlying graph of a social network has no source nodes, the game always has a trivial Nash equilibrium in which no agent chooses a product. A similar phenomenon has been recently observed in [12] in the case of their network formation games, where such equilibria are called degenerate. Further, note that the 'join the crowd' property is exactly the opposite of the defining property of the congestion games with player-specific payoff functions introduced in [13]. In these game the payoff of each player weakly decreases when more players choose his strategy. Because in our case (in contrast to [13]) the players can have different strategy sets, the resulting games are not coordination games.

2 Preliminaries

2.1 Strategic Games

Assume a set $\{1, \ldots, n\}$ of players, where $n > 1$. A ***strategic game*** for n players, written as $(S_1, \ldots, S_n, p_1, \ldots, p_n)$, consists of a non-empty set S_i of ***strategies*** and a ***payoff function*** $p_i : S_1 \times \ldots \times S_n \to \mathbb{R}$, for each player i.

Fix a strategic game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$. We denote $S_1 \times \cdots \times S_n$ by S, call each element $s \in S$ a ***joint strategy***, denote the ith element of s by s_i, and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i}. We also write (s_i, s_{-i}) instead of s. We call a strategy s_i of player i a ***best response*** to a joint strategy s_{-i} of his opponents if $\forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$. Next, we call a joint strategy s a ***Nash equilibrium*** if each s_i is a best response to s_{-i}, that is, if

$$\forall i \in \{1, \ldots, n\} \ \forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

Given a joint strategy s we call the sum $SW(s) = \sum_{j=1}^{n} p_j(s)$ the ***social welfare*** of s. When the social welfare of s is maximal we call s a ***social optimum***. Recall that, given a finite game that has a Nash equilibrium, its ***price of anarchy*** (respectively, ***price of stability***) is the ratio $\frac{SW(s)}{SW(s')}$ where s is a social optimum and s' is a Nash equilibrium with the lowest (respectively, highest) social welfare. For division by zero, we interpret the outcome as ∞.

Next, we call a strategy s_i of player i a ***better response*** given a joint strategy s if $p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i})$. Following the terminology of [14], a ***path*** in S is a sequence $(s^1, s^2, \ldots)$ of joint strategies such that for every $k > 1$ there is a player i

such that $s^k = (s'_i, s^{k-1}_{-i})$ for some $s'_i \neq s^{k-1}_i$. A path ξ is called an ***improvement path*** if it is maximal and for all k smaller than the length of ξ, $p_i(s^k) > p_i(s^{k-1})$, where i is the player who deviated from s^{k-1}. The last condition simply means that each deviating player selects a better response. A game has the ***finite improvement property*** (in short, ***FIP***) if every improvement path is finite. Obviously, if a game has the FIP, then it has a Nash equilibrium–the last element of each path.

Finally, recall that a game is called ***weakly acyclic*** (see [13]) if for every joint strategy there exists a finite improvement path that starts at it.

2.2 Social Networks

We are interested in specific strategic games defined over social networks. In what follows we focus on a model of social networks recently introduced in [1].

Let $V = \{1, \ldots, n\}$ be a finite set of ***agents*** and $G = (V, E, w)$ a weighted directed graph with $w_{ij} \in [0, 1]$ being the weight of the edge (i, j). We often use the notation $i \to j$ to denote $(i, j) \in E$ and write $i \to^* j$ if there is a path from i to j in the graph G. Given a node i of G we denote by $N(i)$ the set of nodes from which there is an incoming edge to i. We call each $j \in N(i)$ a ***neighbour*** of i in G. We assume that for each node i such that $N(i) \neq \emptyset$, $\sum_{j \in N(i)} w_{ji} \leq 1$. An agent $i \in V$ is said to be a ***source node*** in G if $N(i) = \emptyset$.

Let $\mathcal{P}$ be a finite set of alternatives or ***products***. By a ***social network*** (from now on, just ***network***) we mean a tuple $\mathcal{S} = (G, \mathcal{P}, P, \theta)$, where P assigns to each agent i a non-empty set of products $P(i)$ from which it can make a choice. θ is a ***threshold function*** that for each $i \in V$ and $t \in P(i)$ yields a value $\theta(i, t) \in (0, 1]$.

Given a network $\mathcal{S}$ we denote by $\mathit{source}(\mathcal{S})$ the set of source nodes in the underlying graph G. One of the classes of the networks we shall study are the ones with $\mathit{source}(\mathcal{S}) = \emptyset$.

2.3 Social Network Games

Fix a network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$. Each agent can adopt a product from his product set or choose not to adopt any product. We denote the latter choice by t_0.

With each network $\mathcal{S}$ we associate a strategic game $\mathcal{G}(\mathcal{S})$. The idea is that the nodes simultaneously choose a product or abstain from choosing any. Subsequently each node assesses his choice by comparing it with the choices made by his neighbours. Formally, we define the game as follows: the players are the agents, the set of strategies for player i is $S_i := P(i) \cup \{t_0\}$, for $i \in V$, $t \in P(i)$ and a joint strategy s, let $\mathcal{N}_i^t(s) := \{j \in N(i) \mid s_j = t\}$, i.e., $\mathcal{N}_i^t(s)$ is the set of neighbours of i who adopted in s the product t. The payoff function is defined as follows, where c_0 is some positive constant given in advance:

– for $i \in \mathit{source}(\mathcal{S})$, $p_i(s) := \begin{cases} 0 & \text{if } \ s_i = t_0 \\ c_0 & \text{if } \ s_i \in P(i) \end{cases}$

– for $i \notin source(\mathcal{S})$, $p_i(s) := \begin{cases} 0 & \text{if } s_i = t_0 \\ \sum_{j \in \mathcal{N}_i^t(s)} w_{ji} - \theta(i,t) & \text{if } s_i = t, \text{ for some } t \in P(i) \end{cases}$.

Let us explain the underlying motivations behind the above definition. In the first entry we assume that the payoff function for the source nodes is constant only for simplicity. In the last section of the paper we explain that the obtained results hold equally well in the case when the source nodes have arbitrary positive utility for each product.

The second entry in the payoff definition is motivated by the following considerations. When agent i is not a source node, his 'satisfaction' from a joint strategy depends positively from the accumulated weight (read: 'influence') of his neighbours who made the same choice as him, and negatively from his threshold level (read: 'resistance') to adopt this product. The assumption that $\theta(i,t) > 0$ reflects the view that there is always some resistance to adopt a product. So when this resistance is high, it can happen that the payoff is negative. Of course, in such a situation not adopting any product, represented by the strategy t_0, is a better alternative.

The presence of this possibility allows each agent to refrain from choosing a product. This refers to natural situations, such as deciding not to purchase a smartphone or not going on vacation. In the last section we refer to an initiated research on social network games in which the strategy t_0 is not present. Such games capture situations in which the agents have to take some decision, for instance selecting a secondary school for their children.

By definition the payoff of each player depends only on the strategies chosen by his neighbours, so the social network games are related to graphical games of [9]. However, the underlying dependence structure of a social network game is a directed graph and the presence of the special strategy t_0 available to each player makes these games more specific.

In what follows for $t \in \mathcal{P} \cup \{t_0\}$ we use the notation $\overline{t}$ to denote the joint strategy s where $s_j = t$ for all $j \in V$. This notation is legal only if for all agents i it holds that $t \in P(i)$. The presence of the strategy t_0 motivates the introduction and study of special types of Nash equilibria. A Nash equilibrium s is

– ***determined*** if for all i, $s_i \neq t_0$,
– ***non-trivial*** if for some i, $s_i \neq t_0$,
– ***trivial*** if for all i, $s_i = t_0$, i.e., $s = \overline{t_0}$.

3 Nash Equilibria: General Case

The first natural question that we address is that of the existence of Nash equilibria in the social network games. We establish the following result.

Theorem 1. *Deciding whether for a network $\mathcal{S}$ the game $\mathcal{G}(\mathcal{S})$ has a (respectively, non-trivial) Nash equilibrium is NP-complete.*

To prove it we first construct an example of a social network game with no Nash equilibrium and then use it to determine the complexity of the existence of Nash equilibria.

Example 1. Consider the network given in Figure 1, where the product set of each agent is marked next to the node denoting it and the weights are labels on the edges. The source nodes are represented by the unique product in the product set.

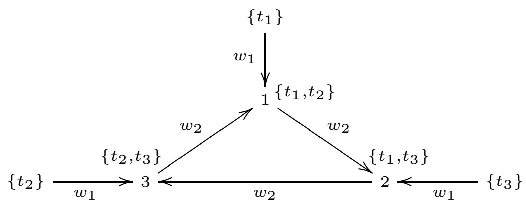


Fig. 1. A network with no Nash equilibrium

So the weights on the edges from the nodes $\{t_1\}, \{t_2\}, \{t_3\}$ are marked by w_1 and the weights on the edges forming the triangle are marked by w_2. We assume that each threshold is a constant θ, where $\theta < w_1 < w_2$. So it is more profitable to a player residing on a triangle to adopt the product adopted by his neighbour residing on a triangle than by the other neighbour who is a source node. For convenience we represent each joint strategy as a triple of strategies of players 1, 2 and 3.

It is easy to check that in the game associated with this network no joint strategy is a Nash equilibrium. Indeed, each agent residing on the triangle can secure a payoff of at least $w_1 - \theta > 0$, so it suffices to analyze the joint strategies in which t_0 is not used. There are in total eight such joint strategies. Here is their listing, where in each joint strategy we underline the strategy that is not a best response to the choice of other players: $(\underline{t_1}, t_1, t_2)$, $(t_1, t_1, \underline{t_3})$, $(t_1, t_3, \underline{t_2})$, $(t_1, \underline{t_3}, t_3)$, $(t_2, \underline{t_1}, t_2)$, $(t_2, \underline{t_1}, t_3)$, $(t_2, t_3, \underline{t_2})$, $(\underline{t_2}, t_3, t_3)$. □

Proof of Theorem 1. As in [1], to show NP-hardness, we use a reduction from the NP-complete PARTITION problem, which is: given n positive rational numbers $(a_1, \ldots, a_n)$, is there a set S such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$? Consider an instance I of PARTITION. Without loss of generality, suppose we have normalised the numbers so that $\sum_{i=1}^{n} a_i = 1$. Then the problem instance sounds: does there exist a set S such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i = \frac{1}{2}$?

To construct the appropriate network we employ the networks given in Figure 1 and in Figure 2, where for each node $i \in \{1, \ldots, n\}$ we set $w_{ia} = w_{ib} = a_i$, and assume that the threshold of the nodes a and b is constant and equal $\frac{1}{2}$.

We use two copies of the network given in Figure 1, one unchanged and the other in which the product t_1 is replaced by t'_1, and construct the desired network $\mathcal{S}$ by identifying the node a of the network from Figure 2 with the node marked

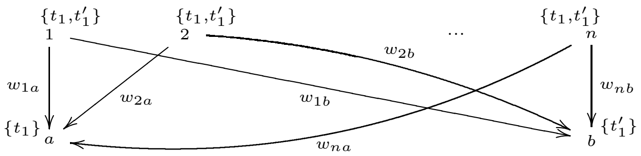


Fig. 2. A network related to the PARTITION problem

by $\{t_1\}$ in the network from Figure 1, and the node b with the node marked by $\{t'_1\}$ in the modified version of the network from Figure 1.

Suppose now that a solution to the considered instance of the PARTITION problem exists, i.e., for some set $S \subseteq \{1, \ldots, n\}$ we have $\sum_{i \in S} a_i = \sum_{i \notin S} a_i = \frac{1}{2}$. Consider the game $\mathcal{G}(\mathcal{S})$ and the joint strategy formed by the following strategies:

- t_1 assigned to each node $i \in S$ in the network from Figure 2,
- t'_1 assigned to each node $i \in \{1, \ldots, n\} \setminus S$ in the network from Figure 2,
- t_0 assigned to the nodes a, b and the nodes 1 in both versions of the network from Figure 1,
- t_3 assigned to the nodes 2, 3 in both versions of the networks from Figure 1 and the two nodes marked by $\{t_3\}$,
- t_2 assigned to the nodes marked by $\{t_2\}$.

We claim that this joint strategy is a non-trivial Nash equilibrium. Consider first the player (i.e, node) a. The accumulated weight of its neighbours who chose strategy t_1 is $\frac{1}{2}$, so its payoff after switching to the strategy t_1 is 0. Therefore t_0 is indeed a best response for player a. For the same reason, t_0 is also a best response for player b. The analysis for the other nodes is straightforward.

Conversely, suppose that a joint strategy s is a Nash equilibrium in the game $\mathcal{G}(\mathcal{S})$. Then it is also a non-trivial Nash equilibrium. We claim that the strategy selected by the node a in s is t_0. Otherwise, this strategy equals t_1 and the strategies selected by the nodes of the network of Figure 1 form a Nash equilibrium in the game associated with this network. This yields a contradiction with our previous analysis of this network.

So t_0 is a best response of the node a to the strategies of the other players chosen in s. This means that $\sum_{i \in \{1,\ldots,n\} | s_i = t_1} w_{ia} \leq \frac{1}{2}$. By the same reasoning t_0 is a best response of the node b to the strategies of the other players chosen in s. This means that $\sum_{i \in \{1,\ldots,n\} | s_i = t'_1} w_{ib} \leq \frac{1}{2}$.

But $\sum_{i=1}^{n} a_i = 1$ and for $i \in \{1, \ldots, n\}$, $w_{ia} = w_{ib} = a_i$, and $s_i \in \{t_1, t'_1\}$. So both above inequalities are in fact equalities. Consequently for $S := \{i \in \{1, \ldots, n\} \mid s_i = t_1\}$ we have $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$. In other words, there exists a solution to the considered instance of the PARTITION problem.

To prove that the problem lies in NP it suffices to notice that given a network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ with n nodes checking if a joint strategy is a non-trivial Nash equilibrium can be done by means of $n \cdot |\mathcal{P}|$ checks, so in polynomial time. □

4 Nash Equilibria: Special Cases

In view of the fact that in general Nash equilibria may not exist we now consider networks with special properties of the underlying directed graph. We consider first networks whose underlying graph is a directed acyclic graph (DAG). Intuitively, such networks correspond to hierarchical organizations.

Theorem 2. *Consider a network $\mathcal{S}$ whose underlying graph is a DAG.*

(i) $\mathcal{G}(\mathcal{S})$ always has a non-trivial Nash equilibrium.
(ii) Deciding whether $\mathcal{G}(\mathcal{S})$ has a determined Nash equilibrium is NP-complete.

Theorem 3. *Consider a network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ whose underlying graph is a simple cycle. There is a procedure that runs in time $\mathcal{O}(|\mathcal{P}| \cdot n)$, where n is the number of nodes in G, that decides whether $\mathcal{G}(\mathcal{S})$ has a non-trivial (respectively, determined) Nash equilibrium.*

Theorem 4. *The price of anarchy and the price of stability for the games associated with the networks whose underlying graph is a DAG or a simple cycle is unbounded.*

Finally, we consider the case when the underlying graph $G = (V, E)$ of a network $\mathcal{S}$ has no source nodes, i.e., for all $i \in V$, $N(i) \neq \emptyset$. Intuitively, such a network corresponds to a 'circle of friends': everybody has a friend (a neighbour). For such networks we prove the following result.

Theorem 5. *Consider a network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ whose underlying graph has no source nodes. There is a procedure that runs in time $\mathcal{O}(|\mathcal{P}| \cdot n^3)$, where n is the number of nodes in G, that decides whether $\mathcal{G}(\mathcal{S})$ has a non-trivial Nash equilibrium.*

The proof of Theorem 5 requires some characterization results that are of independent interest. The following concept plays a crucial role. Here and elsewhere we only consider subgraphs that are *induced* and identify each such subgraph with its set of nodes. (Recall that (V', E') is an induced subgraph of (V, E) if $V' \subseteq V$ and $E' = E \cap (V' \times V')$.)

We say that a (non-empty) strongly connected subgraph (in short, SCS) C_t of G is ***self sustaining*** for a product t if for all $i \in C_t$,

- $t \in P(i)$,
- $\sum_{j \in N(i) \cap C_t} w_{ji} \geq \theta(i, t)$.

An easy observation is that if $\mathcal{S}$ is a network with no source nodes, then it always has a trivial Nash equilibrium, $\overline{t_0}$. The following lemma states that for such networks every non-trivial Nash equilibrium satisfies a structural property which relates it to the set of self sustaining SCSs in the underlying graph. We use the following notation: for a joint strategy s and product t, $\mathcal{A}_t(s) := \{i \in V \mid s_i = t\}$ and $P(s) := \{t \mid \exists i \in V \text{ with } s_i = t\}$.

Lemma 1. *Let $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ be a network whose underlying graph has no source nodes. If $s \neq \overline{t_0}$ is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$ then for all products $t \in P(s) \setminus \{t_0\}$ and $i \in \mathcal{A}_t(s)$ there exists a self sustaining SCS $C_t \subseteq \mathcal{A}_t(s)$ for t and $j \in C_t$ such that $j \to^* i$.*

Lemma 2. *Let $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ be a network whose underlying graph has no source nodes. The joint strategy $\overline{t_0}$ is a unique Nash equilibrium in $\mathcal{G}(\mathcal{S})$ iff there does not exist a product t and a self sustaining SCS C_t for t in G.*

Proof. ($\Leftarrow$) By Lemma 1.
($\Rightarrow$) Suppose there exists a self sustaining SCS C_t for a product t. Let R be the set of nodes reachable from C_t which eventually can adopt product t. Formally, $R := \bigcup_{m \in \mathbb{N}} R_m$ where

- $R_0 := C_t$,
- $R_{m+1} := R_m \cup \{j \mid t \in P(j) \text{ and } \sum_{k \in N(j) \cap R_m} w_{kj} \geq \theta(j, t)\}$.

Let s be the joint strategy such that for all $j \in R$, we have $s_j = t$ and for all $k \in V \setminus R$, we have $s_k = t_0$. It follows directly from the definition of R that s satisfies the following properties:

(P1) for all $i \in V$, $s_i = t_0$ or $s_i = t$,
(P2) for all $i \in V$, $s_i \neq t_0$ iff $i \in R$,
(P3) for all $i \in V$, if $i \in R$ then $p_i(s) \geq 0$.

We show that s is a Nash equilibrium. Consider first any j such that $s_j = t$ (so $s_j \neq t_0$). By (P2) $j \in R$ and by (P3) $p_j(s) \geq 0$. Since $p_j(s_{-j}, t_0) = 0 \leq p_j(s)$, player j does not gain by deviating to t_0. Further, by (P1), for all $k \in N(j)$, $s_k = t$ or $s_k = t_0$ and therefore for all products $t' \neq t$ we have $p_j(s_{-j}, t') < 0 \leq p_j(s)$. Thus player j does not gain by deviating to any product $t' \neq t$ either.

Next, consider any j such that $s_j = t_0$. We have $p_j(s) = 0$ and from (P2) it follows that $j \notin R$. By the definition of R we have $\sum_{k \in N(j) \cap R} w_{kj} < \theta(j, t)$. Thus $p_j(s_{-j}, t) < 0$. Moreover, for all products $t' \neq t$ we also have $p_j(s_{-j}, t') < 0$ for the same reason as above. So player j does not gain by a unilateral deviation. We conclude that s is a Nash equilibrium. □

For a product $t \in \mathcal{P}$, we define the set $X_t := \bigcap_{m \in \mathbb{N}} X_t^m$, where

- $X_t^0 := \{i \in V \mid t \in P(i)\}$,
- $X_t^{m+1} := \{i \in V \mid \sum_{j \in N(i) \cap X_t^m} w_{ji} \geq \theta(i, t)\}$.

The following characterization leads to a direct proof of the claimed result.

Lemma 3. *Let $\mathcal{S}$ be a network whose underlying graph has no source nodes. There exists a non-trivial Nash equilibrium in $\mathcal{G}(\mathcal{S})$ iff there exists a product t such that $X_t \neq \emptyset$.*

Proof. Suppose $\mathcal{S} = (G, \mathcal{P}, P, \theta)$.
($\Rightarrow$) It follows directly from the definitions that if there is a self sustaining SCS C_t for product t then $C_t \subseteq X_t$. Suppose now that for all t, $X_t = \emptyset$. Then for all t, there is no self sustaining SCS for t. So by Lemma 2, $\overline{t_0}$ is a unique Nash equilibrium.
($\Leftarrow$) Suppose there exists t such that $X_t \neq \emptyset$. Let s be the joint strategy defined as follows:

$$s_i := \begin{cases} t & \text{if } i \in X_t \\ t_0 & \text{if } i \notin X_t \end{cases}$$

By the definition of X_t, for all $i \in X_t$, $p_i(s) \geq 0$. So no player $i \in X_t$ gains by deviating to t_0 (as then his payoff would become 0) or to a product $t' \neq t$ (as then his payoff would become negative since no player adopted t'). Also, by the definition of X_t and of the joint strategy s, for all $i \notin X_t$ and for all $t' \in P(i)$, $p_i(t', s_{-i}) < 0$. Therefore, no player $i \notin X_t$ gains by deviating to a product t' either. It follows that s is a Nash equilibrium. □

Proof of Theorem 5. On the account of Lemma 3, the following procedure can be used to check for the existence of a non-trivial Nash equilibrium.

found := **false**;
while $\mathcal{P} \neq \emptyset$ **and** $\neg$*found* **do**
  choose $t \in \mathcal{P}$;
  $\mathcal{P} := \mathcal{P} - \{t\}$;
  compute X_t;
  found := $(X_t \neq \emptyset)$
od
return *found*

To assess its complexity, note that for a network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ and a fixed product t, the set X_t can be constructed in time $\mathcal{O}(n^3)$, where n is the number of nodes in G. Indeed, each iteration of X_t^m requires at most $\mathcal{O}(n^2)$ comparisons and the fixed point is reached after at most n steps. In the worst case, we need to compute X_t for every $t \in \mathcal{P}$, so the procedure runs in time $\mathcal{O}(|\mathcal{P}| \cdot n^3)$. □

In fact, the proof of Lemma 3 shows that if a non-trivial Nash equilibrium exists, then it can be constructed in polynomial time as well.

5 The FIP and the Uniform FIP

A natural question is whether the games for which we established the existence of a Nash equilibrium belong to some well-defined class of strategic games, for instance, games with the finite improvement property (FIP). When the underlying graph of the network is a DAG, the game does indeed have the FIP. The following theorem shows that the result can be improved in the case of two player social network games.

Theorem 6. *Every two players social network game has the FIP.*

Proof. By the above comment on DAGs, we can assume that the underlying graph is a cycle, say $1 \to 2 \to 1$. Consider an improvement path ρ. Without loss of generality we can assume that the players alternate their moves in ρ. In what follows given an element of ρ (that is not the last one) we underline the strategy of the player who moves, i.e., selects a better response. We call each element of ρ of the type $(\underline{t}, t)$ or $(t, \underline{t})$ a *match*. Further, we shorten the statement "each time player i switches his strategy his payoff strictly increases and it never decreases when his opponent switches strategy" to "player i's payoff steadily goes up".

Consider two successive matches in ρ, based respectively on the strategies t and t_1. The corresponding segment of ρ is one of the following four types.
Type 1. $(\underline{t}, t) \Rightarrow^* (\underline{t_1}, t_1)$. The fragment of ρ that starts at $(\underline{t}, t)$ and finishes at $(\underline{t_1}, t_1)$ has the form: $(\underline{t}, t) \Rightarrow (t_2, \underline{t}) \Rightarrow^* (t_1, \underline{t_3}) \Rightarrow (\underline{t_1}, t_1)$. Then player 1's payoff steadily goes up. Additionally, in the step $(t_1, \underline{t_3}) \Rightarrow (\underline{t_1}, t_1)$ his payoff increases by w_{21}. In turn, in the step $(\underline{t}, t) \Rightarrow (t_2, \underline{t})$ player 2's payoff decreases by w_{12} and in the remaining steps his payoff steadily goes up. So $p_1(\bar{t}) + w_{21} < p_1(\overline{t_1})$ and $p_2(\bar{t}) - w_{12} < p_2(\overline{t_1})$.
Type 2. $(\underline{t}, t) \Rightarrow^* (t_1, \underline{t_1})$. Then player 1's payoff steadily goes up. In turn, in the first step of $(\underline{t}, t) \Rightarrow^* (t_1, \underline{t_1})$ the payoff of player 2 decreases by w_{12}, while in the last step (in which player 1 moves) his payoff increases by w_{12}. So these two payoff changes cancel against each other. Additionally, in the remaining steps player 2's payoff steadily goes up. So $p_1(\bar{t}) < p_1(\overline{t_1})$ and $p_2(\bar{t}) < p_2(\overline{t_1})$.
Type 3. $(t, \underline{t}) \Rightarrow^* (\underline{t_1}, t_1)$. This type is symmetric to Type 2, so $p_1(\bar{t}) < p_1(\overline{t_1})$ and $p_2(\bar{t}) < p_2(\overline{t_1})$.
Type 4. $(t, \underline{t}) \Rightarrow^* (t_1, \underline{t_1})$. This type is symmetric to Type 1, so $p_1(\bar{t}) - w_{21} < p_1(\overline{t_1})$ and $p_2(\bar{t}) + w_{12} < p_2(\overline{t_1})$.

Table 1 summarizes the changes in the payoffs between the two matches.

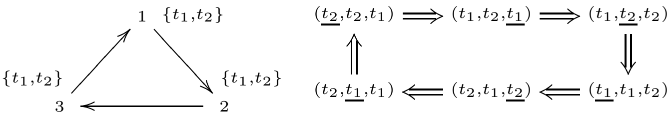
Table 1. Changes in p_1 and p_2

Type	p_1	p_2
1	increases by $> w_{21}$	decreases by $< w_{12}$
2, 3	increases	increases
4	decreases by $< w_{21}$	increases by $> w_{12}$

Consider now a match $(\underline{t}, t)$ in ρ and a match $(\underline{t_1}, t_1)$ that appears later in ρ. Let T_i denote the number of internal segments of type i that occur in the fragment of ρ that starts with $(\underline{t}, t)$ and ends with $(\underline{t_1}, t_1)$.
Case 1. $T_1 \geq T_4$. Then Table 1 shows that the aggregate increase in p_1 in segments of type 1 exceeds the aggregate decrease in segments of type 4. So $p_1(\bar{t}) < p_1(\overline{t_1})$.
Case 2. $T_1 < T_4$. Then analogously Table 1 shows that $p_2(\bar{t}) < p_2(\overline{t_1})$.

We conclude that $t \neq t_1$. By symmetry the same conclusion holds if the considered matches are of the form $(t, \underline{t})$ and $(t_1, \underline{t_1})$. This proves that each match occurs in ρ at most once. So in some suffix η of ρ no match occurs. But each step in η increases the social welfare, so η is finite, and so is ρ. □

The FIP ceases to hold when the underlying graph has cycles. Figure 3(a) gives an example. Take any threshold and weight functions which satisfy the condition that an agent gets positive payoff when he chooses the product picked by his unique predecessor in the graph. Figure 3(b) then shows an infinite improvement path. In each joint strategy, we underline the strategy that is not a best response to the choice of other players. Note that at each step of this improvement path a best response is used. On the other hand, one can check that for any initial joint strategy there exists a finite improvement path. This is an instance of a more general result proved below.



(a) (b)

Fig. 3. A social network with an infinite improvement path

By a ***scheduler*** we mean a function f that given a joint strategy s that is not a Nash equilibrium selects a player who did not select in s a best response. An improvement path $\xi = s^1, s^2, \ldots$ ***conforms*** to a scheduler f if for all k smaller than the length of ξ, $s^{k+1} = (s'_i, s^k_{-i})$, where $f(s^k) = i$. We say that a strategic game has the ***uniform FIP*** if there exists a scheduler f such that all improvement paths ρ which conform to f are finite. The property of having the uniform FIP is stronger than that of being weakly acyclic [15].

Theorem 7. *Let $\mathcal{S}$ be a network such that the underlying graph is a simple cycle. Then the game $\mathcal{G}(\mathcal{S})$ has the uniform FIP.*

Proof. We use the scheduler f that given a joint strategy s chooses the smallest index i such that s_i is not a best response to s_{-i}. So this scheduler selects a player again if he did not switch to a best response. Therefore we can assume that each selected player immediately selects a best response.

Consider a joint strategy s taken from a 'best response' improvement path. Observe that for all k if $s_k \in P(k)$ and $p_k(s) \geq 0$ (so in particular if s_k is a best response to s_{-k}), then $s_k = s_{k \ominus 1}$. So for all $i > 1$, the following property holds:

$Z(i)$: if $f(s) = i$ and $s_{i-1} \in P(i-1)$ then for all $j \in \{n, 1, \ldots, i-1\}$, $s_j = s_{i-1}$.

In words: if i is the first player who did not choose a best response and player $i-1$ strategy is a product, then this product is a strategy of every earlier player and of player n. Along each 'best response' improvement path that conforms to f the value of $f(s)$ strictly increases until the path terminates or at certain stage $f(s) = n$. In the latter case if $s_{n-1} = t_0$, then the unique best response for player n is t_0. Otherwise $s_{n-1} \in P(n-1)$, so on the account of property $Z(n)$ all players' strategies equal the same product and the payoff of player n is negative (since $f(s) = n$). So the unique best response for player n is t_0, as well.

This switch begins a new round with player 1 as the next scheduled player. Player 1 also switches to t_0 and from now on every consecutive player switches to t_0, as well. The resulting path terminates once player $n-2$ switches to t_0. $\square$

6 Concluding Remarks

In this paper we studied the consequences of adopting products by agents who form a social network. To this end we analysed a natural class of strategic games associated with the class of social networks introduced in [1]. The following table summarizes our complexity and existence results, where we refer to the underlying graph with n nodes.

property	arbitrary	DAG	simple cycle	no source nodes
Arbitrary NE	NP-complete	always exists	always exists	always exists
Non-trivial NE	NP-complete	always exists	$\mathcal{O}(\lvert\mathcal{P}\rvert \cdot n)$	$\mathcal{O}(\lvert\mathcal{P}\rvert \cdot n^3)$
Determined NE	NP-complete	NP-complete	$\mathcal{O}(\lvert\mathcal{P}\rvert \cdot n)$	NP-complete
FIP	co-NP-hard	yes	–	co-NP-hard
Uniform FIP	co-NP-hard	yes	yes	co-NP-hard
Weakly acyclic	co-NP-hard	yes	yes	co-NP-hard

In the definition of the social network games we took a number of simplifying assumptions. In particular, we stipulated that the source nodes have a constant payoff $c_0 > 0$. One could allow the source nodes to have arbitrary positive utility for different products. This would not affect any proofs. Indeed, in the Nash equilibria the source nodes would select only the products with the highest payoff, so the other products in their product sets could be disregarded. Further, the FIP, the uniform FIP and weak acyclicity of a social network game is obviously not affected by such a modification.

The results of this paper can be slightly generalized by using a more general notion of a threshold that would also depend on the set of neighbours who adopted a given product. In this more general setup for $i \in V$, $t \in P(i)$ and $X \subseteq N(i)$, the ***threshold function*** θ yields a value $\theta(i,t,X) \in (0,1]$ and satisfies the following ***monotonicity*** condition: if $X_1 \subseteq X_2$ then $\theta(i,t,X_1) \geq \theta(i,t,X_2)$. Intuitively, agent i's resistance to adopt a product decreases when the set of its neighbours who adopted it increases. We decided not to use this definition for the sake of readability.

This work can be pursued in a couple of natural directions. One is the study of social networks with other classes of underlying graphs. Another is an investigation of the complexity results for other classes of social networks, in particular for the equitable ones, i.e., networks in which the weight functions are defined as $w_{ij} = \frac{1}{|N(i)|}$ nodes i and $j \in N(i)$. One could also consider other equilibrium concepts like the strict Nash equilibrium.

Finally, we also initiated a study of slightly different games, in which the players are obliged to choose a product, so the games in which the strategy t_0 is absent. Such games naturally correspond to situations in which the agents always choose a product, for instance a subscription for their mobile telephone. These games substantially differ from the ones considered here. For example, Nash equilibrium may not exist when the underlying graph is a simple cycle.

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Efficiently Learning from Revealed Preference

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Abstract. In this paper, we consider the revealed preferences problem from a learning perspective. Every day, a price vector and a budget is drawn from an unknown distribution, and a rational agent buys his most preferred bundle according to some unknown utility function, subject to the given prices and budget constraint. We wish not only to find a utility function which rationalizes a finite set of observations, but to produce a hypothesis valuation function which accurately predicts the behavior of the agent in the future. We give efficient algorithms with polynomial sample-complexity for agents with linear valuation functions, as well as for agents with linearly separable, concave valuation functions with bounded second derivative.

1 Introduction

Consider the problem of a market-researcher attempting to divine the preferences of a population of consumers merely by observing their past buying behavior. Suppose, for example, that the researcher may observe a consumer each day: every day, the consumer is faced with the choice to buy some subset of goods, each of which may have a different price. The consumer is facing an optimization problem – each day he attempts to buy the subset of goods that maximizes his utility function, given his budget constraints. The market-researcher, on the other hand, is facing a learning problem. Based on his observations of the consumer, he would like to learn a model for the agent's utility function that can explain his behavior, and that can be used to predict (and therefore optimally exploit) his future behavior.

This is the "revealed preferences" problem, and it has received a great deal of attention in the economics literature (see, e.g., [1] for a nice survey). Typically, however, the work on the revealed preferences problem has focused on determining whether a set of observations is *rationalizable* or not – i.e. whether it is consistent with *any* utility function that is monotone increasing in each good. A classic result in this literature is Afriat's Theorem, which roughly states that any finite set of observations is rationalizable if and only if it is rationalizable by a monotone increasing, piecewise linear, concave utility function.

Note, however, that the problem of *rationalizing* is easier than the problem of *learning*. To rationalize a set of observations, it is sufficient to find a utility function which explains past behavior. Learning, however, requires finding a utility

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function which not only explains past behavior, but also will be *predictive* of future behavior! In particular, Afriat's theorem can be taken as showing that attempting to learn from the set of all monotone increasing, piecewise linear, concave utility functions is as hard (and as hopeless) as learning from the set of all utility functions. Indeed, Beigman and Vohra [2] have shown that this class of functions has infinite fat-shattering dimension, and so without further restricting the set of allowable utility functions, no accurate predictions can in general be made after any finite set of observations, even by inefficient learning algorithms!

In this paper, we initiate the study of *efficiently* (in terms of both computational complexity and sample complexity) learning utility functions which can accurately predict *future* purchases of a utility-maximizing agent, given access to past purchase behavior. We necessarily restrict the class of agent utility functions, and consider both linear utility functions, and linearly separable concave utility functions with bounded 2nd derivative. We give polynomial upper and lower bounds on the sample complexity (i.e. the number of observations) required for learning, as well as efficient algorithms that can learn predictive models from polynomially many observations.

1.1 Our Results

We consider a model in which an agent has an unknown utility function over a set of n divisible goods. We get to observe the behavior of the agent, who every day faces a set of prices for each good, together with a budget constraint, which is drawn from a fixed but unknown probability distribution. The agent selects a bundle of goods to buy so as to maximize his utility function subject to his budget constraint, and the goal of a learning algorithm is to impute a model for his utility function that correctly predicts his behavior with high probability on future price/budget instances drawn from the same distribution.

We consider both linear utility functions, and then more generally, linearly separable concave utility functions with bounded derivatives. For both of these cases, we give efficient learning algorithms with polynomially bounded sample complexity. We then consider a relaxed model in which our algorithm receives expanded feedback from the agent during the learning stage, and is permitted to predict bundles that are within a small additive error of the agent's optimal bundle. In this relaxed model, we give a polynomial time learning algorithm with improved sample complexity bounds.

1.2 Related Work

Work on the "revealed preferences problem" has a long history in economics, beginning with the seminal work of Samuelson [3]. Modern work on revealed preferences, in which explanatory utility functions are constructively generated from finitely many agent price/purchase observations began with Afriat [4, 5] who showed (via an algorithmic construction) that any finite sequence of observations is rationalizable if and only if it is rationalizable by a piecewise linear, monotone, concave utility function. We will not attempt to review the extremely large body of work on revealed preferences, and instead refer the reader to an excellent survey of Varian [1].

Algorithms that constructively generate utility functions given a finite set of observations can be viewed as *learning algorithms* for the set of all monotone increasing utility functions. These algorithms typically come with a caveat, however, that the hypothesis utility functions they generate have the same description length as the set of observations that they were generated from, and so tend to overfit the data – this observation is related to a recent paper of Echenique, Golovin, and Wierman [6], who gave a thought-provoking result: that any set of rationalizable observations can in fact be rationalized by a utility function which is computationally easy to optimize. However, such a utility function clearly *cannot* be predictive of the future behavior of an agent who is in fact making his decisions based on an intractable utility function[1], because the hypothesis produced by the learning algorithm would itself be witness to the existence of a polynomially sized circuit for optimizing the purportedly intractable utility function of the agent.

Most related to our work is the work of Beigman and Vohra [2] who first pose the revealed preferences problem in the model of computational learning theory, with a distribution over observations and the explicit goal of producing a predictive hypothesis. They show that the set of all monotone utility functions has infinite fat-shattering dimension, and therefore prove that (without restricting the class of allowable utility functions), there does not exist any algorithm (independent of computational efficiency) which can provide any non-trivial predictive guarantees from any finite number of samples, over every distribution over observations. They also show that if the agent utility functions satisfy a certain bounded-jump condition, then the resulting class in fact has finite fat-shattering dimension, and that predictive learning is therefore possible using a finite number of samples. We continue this line of work by considering specific, simple classes of utility functions, and give efficient learning algorithms together with small polynomial upper and lower bounds on the sample complexity necessary for learning.

A very nice recent line of work by Balcan and Harvey, and Balcan et al. [7, 8] considers a related problem of learning valuation functions. This is similar in motivation, but is orthogonal to the revealed preference setting considered here because it uses direct access to the valuation function evaluated on bundles, rather than only the "revealed" preference of the user, which is the maximum value bundle selected subject to some cost constraint.

2 Preliminaries

We consider the *revealed preferences problem* for an agent who when faced with a set of prices over n goods $[n]$ buys the most valued bundle available to him.

[1] We have two utility functions here, one utility function rationalizes the set of observations, and the other one is agent's actual utility function which could be intractable to optimize.

A *bundle* of goods is a vector of quantities $x \in [0,1]^n$, one for each good: x_i represents the fraction of the good i that is in the bundle. The goods are *divisible*: i.e. bundles can be arbitrary real valued vectors $x \in [0,1]^n$.

The agent has a *value function* $v : [0,1]^n \to \mathbb{R}$. His value for a bundle $x \in [0,1]^n$ is simply $v(x)$. Goods can also be paired with vectors of non-negative *prices* $p \in \mathbb{R}^n_+$, where p_i is the price for good i. The price of a bundle is linear in the goods in the bundle. The price of a bundle x with respect to prices p is therefore simply $x \cdot p$. Prices are important, because the agent may be faced with a budget constraint B: he can only buy bundles x such that $x \cdot p \leq B$.

The agent is a utility maximizer. When faced with a price vector p and a budget B, he will choose to buy the bundle that maximizes his value subject to his budget constraint: That is, he will choose the bundle:

$$x^*(v, p, B) = \underset{x \in [0,1]^n : x \cdot p \leq B}{\operatorname{argmax}} \; v(x)$$

We will consider several types of value functions in this paper. A *linear* value function v is defined by a vector $v \in \mathbb{R}^n_+$, where v_i is the marginal value of good i. In this case, $v(x) = v \cdot x$. More generally, we can consider linearly separable concave utility functions. A value function v is linearly separable and concave if it can be described using concave functions $v_1, \ldots, v_n$ where each $v_i : [0,1] \to \mathbb{R}_+$ is a one-dimensional real valued function, and we can evaluate $v(x) = \sum_{i=1}^n v_i(x_i)$.

The *revealed preferences problem* is to recover a value function that can explain a sequence of choices that the agent was observed to make. In this paper, we wish to recover a value function that cannot only rationalize observed behavior, but can help predict future behavior. In order for this to be a meaningful task, we must assume that the choices presented to the agent are drawn from some distribution.

Definition 1. *An* example *is a price vector $p \in \mathbb{R}^n_+$ paired with a budget $B \in \mathbb{R}_+$. A distribution over examples $\mathcal{D}$ is simply a distribution over $(p, B) \sim [0,1]^n \times \mathbb{R}_+$.*

Definition 2. *An observation of an agent with value function v, $(p, B, x^*(p, B, v)) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+$ is simply a triple consisting of a price vector p, a budget B, and a bundle $x^*(p, B, v)$ chosen by the agent given p and B: i.e. a bundle x that maximizes $v(x)$ subject to $x \cdot p \leq B$.*

Definition 3. *An algorithm A δ-learns a class of value functions $\mathcal{V}$ from $m = m(\delta)$ observations if for every distribution $\mathcal{D}$ over examples and for every value function $v \in \mathcal{V}$, given a set of m observations $\{(p_i, B_i, x^*(p_i, B_i, v))\}_{i=1}^m$ where examples (p_i, B_i) are drawn i.i.d. from $\mathcal{D}$, with probability $1 - \delta$ it produces a hypothesis $\hat{v}$ such that:*

$$\Pr_{(p,B)\sim\mathcal{D}}[v(x^*(p, B, v)) = v(x^*(p, B, \hat{v}))] \geq 1 - \delta.$$

We say that A is efficient *if both its run-time and its sample complexity $m(\delta)$ are bounded by some polynomial $p(n, 1/\delta)$. We say that the sample complexity of* *learning $\mathcal{V}$ is at most $m^* = m^*(\delta)$ if there is some algorithm A which δ-learns $\mathcal{V}$ from $m(\delta) \leq m^*(\delta)$ observations.*

Remark 1. Note that a learning algorithm must with high probability (over the choice of observations and coins of the mechanism) produce a value function which *most of the time* (over draws of examples) selects a bundle which is equal to the bundle that the agent would have selected.

In section 5 we relax our definition of learning to allow our learning algorithm to predict bundles which are only *approximately optimal* to the agent, rather than requiring that it select the exactly correct bundle. Note that such approximately optimal bundles might look very different from exactly optimal bundles, and so we will also need to allow our learning algorithms to receive richer feedback from the agent.

Definition 4. *An algorithm A (ϵ, δ)-learns a class of value functions $\mathcal{V}$ from $m = m(\delta)$ observations if for every distribution $\mathcal{D}$ over examples and for every value function $v \in \mathcal{V}$, given a set of m observations $\{(p_i, B_i, x^*(p_i, B_i, v))\}_{i=1}^m$ where examples (p_i, B_i) are drawn i.i.d. from $\mathcal{D}$, with probability $1-\delta$ it produces a hypothesis $\hat{v}$ such that:*

$$\Pr_{(p,B)\sim\mathcal{D}}[v(x^*(p, B, \hat{v})) \geq v(x^*(p, B, v)) - \epsilon] \geq 1 - \delta.$$

For this notion of additive approximation to be meaningful, we will typically normalize the target utility function v to lie in the range $[0, 1]$.

3 All Pairs Comparisons Algorithm: Learning Linear Valuation Functions

In this section, we present an algorithm that efficiently δ-learns the class of all linear valuation functions given a set of $m = O(\frac{n^2 \ln(n^2/\delta)}{\delta})$ observations. In particular, this provides a quadratic upper bound on the optimal sample complexity $m^*(\delta)$ for learning linear valuation functions. We note that a linear $\Omega(m)$ lower bound is immediate in this setting. We start by characterizing the optimal bundle for an agent maximizing a linear utility function, and give intuition for our learning algorithm.

Let v^* and p denote some fixed value and price vectors respectively, and let B denote some fixed budget. We denote the optimal bundle (according to the linear utility function defined by value vector v^*, price vector p, and budget B) by x^*. Recall that the value of the optimal bundle is $v^* \cdot x^*$, and its cost $p \cdot x^*$ is at most the budget B. Observe that in choosing bundle x^*, the agent is solving a divisible knapsack problem, and so the following structural lemma is immediate.

Lemma 1. *For any pair of goods $i, j \in [n]$ with $x_i^* > x_j^*$, it must be that:*

$$\frac{v_i^*}{p_i} \geq \frac{v_j^*}{p_j}$$

Equivalently, for any pair of goods with $\frac{v_i^*}{v_j^*} \geq \frac{p_i}{p_j}$, the optimal bundle "prefers" good i over good j (It will never buy any of good j until it has exhausted the supply of good i). Our algorithm is based on this structural characterization, and operates by maintaining upper and lower bounds on each of the n^2 ratios $\frac{v_i^*}{v_j^*}$ for $i \neq j \in [n]$. Based on this transitive relation, we can sort the goods, and find the optimal bundle by buying the goods one by one starting from high priority goods until the budget B is spent completely. In this optimal bundle, we have at most one fractional item. In our algorithm, we try to learn ratios $\frac{v_i}{v_j}$ accurately for all pair of goods with high probability.

Algorithm 1

AllPairsLearn(δ) *which takes as input an accuracy parameter* δ.

Training Phase:

Let E *be a set of* $m = O\left(\frac{n^2 \ln(n^2/\delta)}{\delta}\right)$ *observations* $(p, B, x^*(p, B, v))$.

Initialize *bounds* $(L_{i,j}, U_{i,j})$ *for each* $i \neq j \in [n]$. *Initially* $L_{i,j} = 0$ *and* $U_{i,j} = \infty$ *for all* i, j.

for *Each* $(p, B, x^*) \in E$ ***do***

  for *Each* $i \neq j \in [n]$ ***do***

    If $x_i^* > x_j^*$, ***Let*** $L_{i,j} = \max(L_{i,j}, \frac{p_i}{p_j})$

    If $x_j^* > x_i^*$, ***Let*** $U_{i,j} = \min(U_{i,j}, \frac{p_i}{p_j})$

  end for

end for

Classification Phase:

On a new example (p, B) *let* $v' \in [0,1]^n$ *be any vector such that for all* $i \neq j \in [n]$ $\frac{v'_i}{v'_j} \in [L_{i,j}, U_{i,j}]$. *Predict bundle* $x'(p, B, v')$ *that results from maximizing* v' *with respect to prices* p *and budget constraint* B.

end

The intuition is that in order to find the optimal bundle x^*, we need only know bounds on the ratios of the values of pairs of goods for which unequal quantities are purchased in the optimal bundle. So if we know that $\frac{v_i}{v_j} \geq \frac{p_i}{p_j}$ for any pair of goods with $x_i^* > x_j^*$, we can find the optimal bundle x^*. We need not know the values themselves – it is sufficient to bound these ratios. For example, if the lower bound $L_{i,j}$ is at least $\frac{p_i}{p_j}$, we can infer that good i is preferred to good j. If we can infer all these preferences for pairs of goods (i, j) with $x_i^* \neq x_j^*$, we can find the optimal bundle as well. Following we show that with high probability after observing $m = O(n^2 \ln(n^2/\delta)/\delta)$ i.i.d. examples we can find the optimal bundle.

Theorem 1. *AllPairsLearn*(δ) *efficiently* δ*-learns the class of linear valuation functions given* $m = O\left(\frac{n^2 \ln(n^2/\delta)}{\delta}\right)$ *observations.*

Proof. For each pair of goods (i, j), we define $a_{i,j}$ and $b_{i,j}$ as follows:

$$a_{i,j} = \inf \left\{ a | a \leq \frac{v_i}{v_j} \quad \& \quad Pr\left(x_i^* > x_j^* \quad \& \quad \frac{p_i}{p_j} \in [a, \frac{v_i}{v_j}] \right) \leq \frac{\delta}{n^2} \right\}$$
$$b_{i,j} = \sup \left\{ b | b \geq \frac{v_i}{v_j} \quad \& \quad Pr\left(x_j^* > x_i^* \quad \& \quad \frac{p_i}{p_j} \in [\frac{v_i}{v_j}, b] \right) \leq \frac{\delta}{n^2} \right\}$$

where p is the price vector drawn from the distribution $\mathcal{D}$, and x^* is its optimal bundle. The above equations are well defined while the right hand side sets are non-empty. If the first(second) right hand side set is empty in the above equation, we set the value of inf (sup) to be v_i/v_j. Every time an i.i.d. example is drawn, with probability δ/n^2, the lower bound $L_{i,j}$ becomes at least $a_{i,j}$, and the upper bound $U_{i,j}$ becomes at most $b_{i,j}$ for every pair (i, j). For each pair (i, j) after m observations, $L_{i,j}$ is less than $a_{i,j}$ with probability at most $(1-\delta/n^2)^m \leq e^{-\ln(n^2/\delta)} \leq \delta/n^2$. A similar argument holds for $U_{i,j}$. Using union bound, we can have that with probability $1 - \delta$, every $L_{i,j}$ is at least $a_{i,j}$, and every $U_{i,j}$ is at most $b_{i,j}$.

Now when a new example $(p', B', x'(p', B', v))$ arrives (x' is the optimal bundle), the probability that $x'_i \neq x'_j$ and we cannot imply which of these two items are preferred over the other one, i.e. $\frac{p_i}{p_j} \in [L_{i,j}, U_{i,j}]$ is at most $2\delta/n^2$, because we know that $[L_{i,j}, U_{i,j}] \subseteq [a_{i,j}, b_{i,j}]$. Using union bound, with probability $1 - \delta$ we can derive all preference relations for items with unequal fractions in the optimal bundle x'. In the other words, with probability $1 - \delta$, we can find the optimal bundle x'.

4 Learning Linearly Separable Concave Utility Function

In this section, we modify the algorithm presented in section 3 to learn the class of linearly separable concave utility functions. Recall that agents with linearly separable utility functions have a separate function $v_i : [0, 1] \to \mathbb{R}_+$ for each $1 \leq i \leq n$, and their utility for bundle x is $\sum_{i=1}^n v_i(x_i)$. We assume that each utility function v_i is a concave function with bounded second derivative. Concavity corresponds to a decreasing marginal utility condition: that buying an additional ϵ fraction of item i increases agent utility more when we have less of item i: $v_i(a+\epsilon) - v_i(a) \geq v_i(b+\epsilon) - v_i(b)$ for any $a \leq b$. Our bounded second derivative assumption states that the second derivative of each utility function has some supremum strictly less than ∞.

We first characterize optimal bundles, and then adapt our learning algorithm for linear valuation functions to apply to the class of linearly separable concave utility functions.

Fix a utility function $v^* = \{v_i^* : [0, 1] \to \mathbb{R}^+\}$ and a price/budget pair (p, B). The corresponding optimal bundle can be characterized as follows. For any threshold $\tau \geq 0$, define x_i^τ to be $Max\{f | f \in [0, 1] \& \frac{v'_i(f)}{p_i} \geq \tau\}$ where $v'_i(f)$ is the first derivative of function v_i at point f. We can now define p^τ to be $\sum_{i=1}^n p_i x_i^\tau$. We will show that the optimal bundle x^* for v^* in the face of price/budget pair (p, B) is the vector such that $x_i^* = x_i^\tau$ for each $1 \leq i \leq n$ where τ is the maximum value such that this bundle does not exceed the budget constraint.

Lemma 2. *The optimal bundle x^* for pair (p, B) is equal to x^τ where τ is $Max\{\tau | p^\tau \leq B\}$.*

The intuition for our algorithm now follows from the linear utility case. From each observation consisting of an example and its optimal bundle, we may infer some constraints on the derivatives of utility functions at various points. Just as in the linear utility case, these are the only pieces of information we need to infer the optimal bundle.

Algorithm 2

LinearSeparableLearn(ϵ, δ) *with accuracy and error parameters δ, and ϵ.*

Training Phase:

Let *E be a set of $m = O\left(\frac{(n(k+2))^2 \ln((n(k+2))^2/\delta)}{\delta}\right)$ observations $(p, B, x^*(p, B, v))$ where k is defined in Definition 5.*

Initialize *bounds $(L(i, r, j, s), U(i, r, j, s))$ for each $i \neq j \in [n]$ and $r, s \in [k]$ defined in Definition 5. Initially $L(i, r, j, s) = 0$ and $U(i, r, j, s) = \infty$.*

for *Each $(p, B, x^*) \in E$* ***do***

  for *Each $i \neq j \in [n]$* ***do***

    If *$x_i^* > x_j^*$,* ***Let*** *$L(i, \lfloor kx_i^* \rfloor, j, \lceil kx_j^* \rceil) = \max(L(i, \lfloor kx_i^* \rfloor, j, \lceil kx_j^* \rceil), \frac{p_i}{p_j})$*

    If *$x_i^* > x_j^*$,* ***Let*** *$U(i, \lceil kx_i^* \rceil, j, \lfloor kx_j^* \rfloor) = \min(U(i, \lceil kx_i^* \rceil, j, \lfloor kx_j^* \rfloor), \frac{p_i}{p_j})$*

  end for

end for

Classification Phase:

On a new example (p, B) find thresholds $\{l_i\}_{i=1}^n$ such that $\frac{v_i'(l_i/k)}{v_j'((l_j+1)/k)} \geq \frac{p_i}{p_j}$ for each pair $i, j \in [n]$, and $\sum_{i=1}^n \frac{p_i Max\{l_i, 0\}}{k} \leq B \leq \sum_{i=1}^n \frac{p_i Min\{l_i+1, k\}}{k}$. Buy l_i/k fraction of object i for every $i \in [n]$, and spend the remaining budget to buy equal fraction of all objects.

end

Unlike the linear utility setting, however, it is not possible to maintain bounds on all ratios of derivatives of utility functions at all relevant points, because there are a continuum of points and the derivatives may take a distinct value at each point. Instead, we discretize the interval $[0, 1]$ with $k+1$ equally distanced points $0, 1/k, 2/k, \cdots, 1$ for some positive integer value of k defined in Definition 5, and maintain bounds on the ratios of the derivatives at these points.

Definition 5. *We let k to be an integer at least $\left\lceil (2Q/\epsilon) \cdot \max_{(p,B)\sim\mathcal{D}, 1 \leq j \leq n} \{\frac{B}{p_j}\} \right\rceil$ where Q is an upper bound on $v_i''(x)$ over all i and $x \in [0, 1]$, and ϵ is the error with which we are happy learning to. We define $V(i, l) = v_i'(l/k)$ for item i, $1 \leq i \leq n$ and discretization step l, $0 \leq l \leq k$. For convenience, we define $V(i, k+1) = 0$. For any pairs $1 \leq i, j \leq n$, and $0 \leq r, s \leq l$, we define $L(i, r, j, s)$ and $U(i, r, j, s)$ to be the lower and upper bounds on the ratio $\frac{V(i,r)}{V(j,s)}$. The lower and upper bounds are intialized to zero and ∞ respectively.*

Analogously to the linear case, our algorithm will maintain upper and lower bounds on the pairwise ratios between each of these these $n(k+2)$ variables. Since the utilities are concave, we will also maintain the constraint that $V(i,l) \leq V(i,l-1)$ for any $1 \leq i \leq n$ and $1 \leq l \leq k+1$ throughout the course of the algorithm.

In the training phase, the algorithm selects $m = O((n(k+2))^2 log((n(k+2))^2/\delta)/\delta)$ observations. Note the similarity in the number of examples here as compared to the linear case: this is no coincidence. Instead of maintaining bounds on the pairwise ratios of n derivatives we are maintaining bounds on the pairwise ratios between $n(k+2)$ derivatives.

Consider the inequalities we can infer from each observation (p, B, x^*). By our optimality characterization, we know that for any pair of items i and j with $x_i^* > 0$ and $x_j^* < 1$, we must have: $\frac{v_i'(x_i^*)}{p_i} \geq \frac{v_j'(x_j^*)}{p_j}$. We therefore can obtain the following inequality:

$$\frac{V(i, \lfloor kx_i^* \rfloor)}{p_i} \geq \frac{v_i'(x_i^*)}{p_i} \geq \frac{v_j'(x_j^*)}{p_j} \geq \frac{V(j, \lceil kx_j^* \rceil)}{p_j}$$

The above inequality defines the update step that we can impose on the lower bound $L(i, l', j, l'')$ and upper bound $U(i, l', j, l'')$ on the ratios $\frac{V(i,l')}{V(j,l'')}$ where $l' = \lfloor kx_i^* \rfloor$, and $l'' = \lceil kx_j^* \rceil$, analogously to our algorithms update for the linear case. For each example, we update these bounds appropriately.

After the training phase completes, our algorithm uses these bounds to predict a bundle for a new example (p, B). The algorithm attempts to find some threshold $-1 \leq l_i \leq k$ for each item i such that the following two properties hold. We define $V(i, -1) = \infty$ for each $1 \leq i \leq n$.

- For each pair of items $i \neq j \in [n]$, upper and lower bounds imply that $\frac{V(i,l_i)}{p_i} \geq \frac{V(j,l_j+1)}{p_j}$.
- We have that: $\sum_{i=1}^{n} \frac{p_i Max\{l_i,0\}}{k} \leq B \leq \sum_{i=1}^{n} \frac{p_i Min\{l_i+1,k\}}{k}$. In other words, there is enough budget to buy $\max\{l_i, 0\}/k$ fraction of object i for all $1 \leq i \leq n$, and the total cost of buying $\min\{l_i+1, k\}/k$ fraction of each item i is at least B.

After finding these thresholds $l_1, l_2, \cdots, l_n$, our algorithm selects a bundle that contains $\max\{l_i, 0\}/k$ units of item i for each i, and then spend the rest of the budget (if there is any remaining) to buy an equal fraction of all objects with $0 \leq l_i < k$, i.e. if B' of the budget remains after the first step, we buy $\frac{B'}{\sum_{1 \leq i \leq n,\ 0 \leq l_i < k} p_i}$ units of each object i with $0 \leq l_i < k$. Intuitively, the objects with $l_i = 0$, represent very expensive objects (in comparison to their values) which we prefer not to buy at all. On the other hand, we have already exhausted the supply of objects with $l_i = 1$.

In the rest of this section, we show in Lemma 3 how to find these thresholds (the sequence l_i for $1 \leq i \leq n$) based on the learned upper and lower bounds on ratios if such thresholds exist. Then, we prove in Lemma 4 that after training on

m examples, with high probability (at least $1 - 2\delta$), this sequence of thresholds indeed exists. Finally we conclude that our algorithm is an (ϵ, δ)-learner.

Lemma 3. *Assuming there exists a sequence of thresholds $\{l_i\}_{i=1}^n$ with the two desired properties in our algorithm, there exists a polynomial time algorithm to find them.*

We now show that the required sequence of thresholds $\{l_i\}_{i=1}^n$ exist with high probability.

Lemma 4. *After updating the algorithm's upper and lower bounds using $m = O((n(k+2))^2 log((n(k+2))^2/\delta)/\delta)$ observations, when considering a new example (p, B), the sequence of thresholds $\{l_i\}_{i=1}^n$ exists with probability at least $1 - 2\delta$.*

To conclude, we just need to show that if we find the thresholds with the desired properties, the returned bundle is a good approximation of the optimum bundle.

Theorem 2. *For any $\epsilon > 0$, we can find some k (the discretization factor) such that with probability at least $1 - 2\delta$ over the choice of example (p, B), the bundle $\hat{x} = \hat{x}(p, B)$ returned by our mechanism admits at least one of the following properties:*

1. *For each item $1 \le i \le n$, we have that $\hat{x}_i \ge x_i^* - \epsilon$,*
2. *$v^*(\hat{x}) \ge v^*(x^*) - \epsilon$*

In other words, have that our mechanism is an efficient (ϵ, δ)-learning algorithm for the class of linearly separable concave utility functions with bounded range $v : [0, 1]^n \to [0, 1]$.

5 A Learning Algorithm Based on Sampling from a Convex Polytope

In this section, we present another learning algorithm for (ϵ, δ)-learning linear cost functions. We introduce a new model, that gets a stronger form of feedback from the agent, and as a result achieve an improved sample complexity bound that requires only $m = \tilde{O}\left(\frac{n\text{polylog}(n)}{\delta^3}\right)$ observations.

During the training phase of our algorithm, it will interact with the agent by adding constraints to a linear program and given a new example, propose a candidate bundle to the agent. The agent will either accept the candidate bundle (if it is approximately optimal), or else return to the algorithm a set of linear constraints witnessing the suboptimality of the proposed bundle. The main idea is that for each new example either our algorithm's bundle is almost optimal, or we receive a set of linear constraints to add to our linear program that substantially reduce the volume of the feasible polytope. If the set of constraints are restrictive enough, with high probability, we achieve an approximately optimum bundle on all new examples, and we can end the training phase. Otherwise each new example cuts off some constant fraction of the linear program polytope with high probability. After feeding a polynomial number of examples, and using some arguments to upper bound the volume of the polytope at the beginning and lower bound its volume at the end, we can prove with high probability, the algorithm finds an almost optimal bundle for future examples. First we explain the model, and then we present our algorithm.

Model: We consider agents with linear utility functions, here bounded so that $v \in [0,1]^n$. If we have that $v \cdot \hat{x} \geq v \cdot x - \epsilon$, we say bundle $\hat{x}$ is an ϵ-additive approximation to the optimal bundle $x^* = x^*(v^*, p, B)$, and it will be accepted by the agent if it is proposed. If a proposed bundle $\hat{x}$ is not ϵ-approximately optimal, the agent rejects the bundle if proposed, and instead returns a set of inequalities which are witness to the sub-optimality of our solution. The agent returns all valid inequalities of the following form for different pairs of objects $i, j \in [n]$: $\frac{v_i - \epsilon'}{p_i} > \frac{v_j + \epsilon'}{p_j}$ where $\epsilon' = \epsilon/nM$, and M is the maximum ratio of two different prices in the domain of the price distribution ($\mathcal{D}$).

Intuitively, for these pairs we have that $\frac{v_i}{p_i}$ is greater than $\frac{v_j}{p_j}$ by some non-negligible margin. In the following, we show that for any suboptimal bundle (not an ϵ-additive approximation) resulted from a value vector $\hat{v}$, there exists at least one of these inequalities for which we have that $\frac{\hat{v}_i}{p_i} \leq \frac{\hat{v}_j}{p_j}$. In other words, these set of inequalities that our algorithm returns could be seen as some evidence of suboptimality for any suboptimal bundle for example (p, B).

Lemma 5. *For any pair of price vector and budget* (p, B)*, and a suboptimal sampled value vector* $\hat{v}$ *(that does not generate an* ϵ*-approximately optimal bundle* $\hat{x}$*), there exists at least one pair of items* (i, j) *such that we have* $\frac{v_i - \epsilon'}{p_i} > \frac{v_j + \epsilon'}{p_j}$*, and* $\frac{\hat{v}_i}{p_i} \leq \frac{\hat{v}_j}{p_j}$.

Proof. Let x^* and $\hat{x}$ be the optimal bundle and the returned bundle based on $\hat{v}$ respectively. We note that since all objects have non-negative values, we have that $x^* \cdot p = \hat{x} \cdot p = B$ unless the budget B is enough to buy all objects in which case both x^* and $\hat{x}$ are equal to $(1, 1, \cdots, 1)$ which is a contradiction because we assumed $\hat{x}$ is suboptimal.

We can exchange v/p_i units of object i with v/p_j units of item j and vice versa without violating the budget constraint. We show that all the differences in entries of x^* and $\hat{x}$ can be seen as the sum of at most n of these simple exchanges between pairs of objects as follows. We take two entries i and j such that $x_i^* > \hat{x}_i$ and $x_j^* < \hat{x}_j$. We note that as long as two vectors x^* and $\hat{x}$ are not the same, we can find such a pair because we also have that $v \cdot p = \hat{v} \cdot p$. Without loss of generality, assume that $(x_i^* - \hat{x}_i)p_i \leq (\hat{x}_j - x_j^*)p_j$. Now we buy $x_i^* - \hat{x}_i$ more units of item i in bundle $\hat{x}$ to make the two entries associated with object i in bundles x^* and $\hat{x}$ equal. Instead we buy $(x_i^* - \hat{x}_i)p_i/p_j$ fewer units of object j to obey the budget limit B. This way, we decrease the number of different entries in x^* and $\hat{x}$, so after at most n exchanges we make $\hat{x}$ equal to x^*. By assumption, $v^*(\hat{x}) \leq v^*(x^*) - \epsilon$. Therefore, in at least one of these exchanges, the value of $\hat{x}$ is increased by more than ϵ/n.

Assume this increase happened in exchange of objects i and j. Let r be $(x_i^* - \hat{x}_i)p_i$. We bought r/p_i more units of i, and r/p_j fewer units of j. The increase in value is $r(v_i/p_i - v_j/p_j) = (x_i^* - \hat{x}_i)(v_i - v_j p_i/p_j) \geq \epsilon/n$. Since $x_i^* - \hat{x}_i$ is at most 1, we also have that $v_i - v_j p_i/p_j > \epsilon/n$ which can be rewritten as: $v_i - \epsilon/2n > v_j p_i/p_j + \epsilon/2n$. This is equivalent to $\frac{v_i - \epsilon/2n}{p_i} > \frac{v_j + (\epsilon/2n)(p_j/p_i)}{p_j}$. We can conclude that $\frac{v_i - \epsilon/(2nM)}{p_i} > \frac{v_j + (\epsilon/2nM)}{p_j}$ which is by definition of ϵ': $\frac{v_i - \epsilon'}{p_i} > \frac{v_j + \epsilon'}{p_j}$.

We also note that $\hat{x}_i < 1$ and $\hat{x}_j > 0$, so we can infer that $\frac{\hat{v}_i}{p_i} \leq \frac{\hat{v}_j}{p_j}$. Otherwise one could exchange some fraction of j with some fraction of i and gain more value with respect to value vector $\hat{v}$. This completes the proof of both inequalities claimed in this lemma.

Algorithm: We maintain a linear program with n variables representing a hypothesis value vector $\hat{v}$. Since v is in $[0,1]^n$, we initially have the constraints: $0 \leq v_i \leq 1$ for any $1 \leq i \leq n$. At any given time, our set of constraints forms a convex body K.

Our algorithm loops until we reach a desired property. At each step of the loop we sample $\frac{C \log(n) \log(1/\delta)}{\delta^2}$ examples, and for each of them we sample uniformly at random a vector $\hat{v}$ from the convex body K, and predict the optimal bundle based on this sampled vector. (Note that uniform sampling from a convex body can be done in polynomial time by [9]). At the end of the loop, we add the linear constraints that we obtained as feedback from the agent to our linear program, and get a more restricted version of K which we call K'.

If the volume of K' is greater than $1 - \delta$ times the volume of K, we stop the learning algorithm, and return K as the candidate convex body. Otherwise, we replace K with the new more constrained body K', and repeat the same loop again. To avoid confusion, we name the final returned convex body $\hat{K}$. After the training phase ends, for future examples, our algorithm samples a value vector $\hat{v}$ uniformly at random from this convex body $\hat{K}$, and predicts the optimal bundle based on $\hat{v}$. We explain what kinds of constraints we add at the end of each loop to find K'.

Each iteration of the training phase uses $\frac{C \log(n) \log(1/\delta)}{\delta^2}$ examples. Recall that for each one, the mechanism proposes a bundle to the agent, who either accepts or rejects it. For each rejected bundle, we are given a set of pairs of objects (i, j) such that $\frac{v_i - \epsilon'}{p_i} > \frac{v_j + \epsilon'}{p_j}$. For each inequality like this, we add the looser constraint $\frac{v_i}{p_i} > \frac{v_j}{p_j}$. At the end, we have a more restricted convex body K' which is formed by adding all of these constraints to K.

We must show that after the training phase of the algorithm terminates, we are left with a hypothesis which succeeds at predicting valuable bundles with high probability. We must also also bound the number of iterations (and therefore the number of examples used by the algorithm) before the training phase of the algorithm terminates. First we bound the total number of iterations of the training phase.

Lemma 6. *The total number of examples sampled by our algorithm is at most*

$$m = O\left(\frac{n\log(n)(\log(n) + \log(M))\log(1/\epsilon)\log(1/\delta)}{\delta^3}\right).$$

Finally, we argue that after the learning phase terminates, the algorithm returns a good hypothesis.

Theorem 3. *The algorithm (ϵ, δ)-learns from the set of linear utility functions.*

Proof. Given a new example (p, B), the algorithm samples a value vector $\hat{v}$ uniformly at random from the convex body $\hat{K}$, and returns an optimal bundle with respect to $\hat{v}$, p, and B.

Consider a price vector p and budget B. For some value vectors in $\hat{K}$, the returned bundle is suboptimal (not an ϵ-additive approximation). We call this subset the set of suboptimal value vectors with respect to (p, B), and the fraction of suboptimal value vectors in $\hat{K}$ is the probability that our algorithm does not return a good bundle, i.e. the error probability of our algorithm. We say a pair (p, B) is unlucky if for more than δ fraction of value vectors in $\hat{K}$, the returned bundle is suboptimal. We prove that with probability at least $1-\delta/2$, the convex body $\hat{K}$ we return, has this property that with at most probability $\delta/2$, the pair (p, B) drawn from $\mathcal{D}$ is unlucky. This way with probability at most $\delta/2+\delta/2 = \delta$, the pair (p, B) is unlucky which proves that our algorithm is (ϵ, δ)-learner.

We prove the claim by contradiction. Define A to be the event that "with probability more than $\delta/2$, the pair $(p, B) \sim \mathcal{D}$ is unlucky". We prove that the probability of event A is at most $\delta/2$. Let K_i be the convex body at the beginning of iteration i, and K'_i be the more restricted version of K_i that we compute at the end of iteration i. Event A holds if for some i we have these two properties: a) the probability that a pair (p, B) drawn i.i.d. from $\mathcal{D}$ is unlucky with respect to K_i is more than $\delta/2$, i.e. if we sample the value vector from K_i, the returned bundle for (p, B) is suboptimal with probability more than δ. b) the volume of K'_i is not less than $1 - \delta$ times volume of K_i.

We bound the probability of having both of these properties at iteration i. In this iteration, for every example we take, with probability more than $\delta/2$, the pair (p, B) is unlucky. For an unlucky pair (p, B), with probability more than δ, we return a suboptimal example, and then we get feedback from the agent. Using lemma 5, and the feedback we get from the agent, all of the suboptimal value vectors for pair (p, B) will be removed from K_i and will not exist in K'_i (by the new constraints we add in this loop). Since (p, B) is unlucky, more than δ fraction of the K_i will be deleted in this case. In other words, for each example in loop i with probability at least $\delta^2/2$, more than δ fraction of K_i will be removed. Clearly, since K'_i has volume at least $1-\delta$ fraction of K_i, this has not happened for any of the examples of loop i. Since we have $\frac{C\log(n)\log(1/\delta)}{\delta^2}$ examples in each loop, the probability of holding both these properties at loop i is at most $(1-\delta^2)^{\frac{C\log(n)\log(1/\delta)}{\delta^2}} < \delta/(2n^C)$ for $\delta \leq 1/2$. Since there are less than n^C number of loops for some large enough constant C, the probability of event A (which might happen in any of the loops) is less than $\delta/2$.

6 Discussion

In this paper we have considered the problem of efficiently learning predictive classifiers from revealed preferences. We feel that the revealed preferences problem is much more meaningful when the observed data must be rationalized with a *predictive* hypothesis, and of course much remains to be done in this study. Our work leaves many open questions:

1. What are tight bounds on the sample complexity for δ-learning linear valuation functions? There is a simple $\Omega(n)$ lower bound, and here we give an algorithm with sample complexity $\tilde{O}(n^2/\delta)$, but where does the truth lie?
2. Is there a general measure of sample complexity, akin to VC-dimension in the classical learning setting, that can be fruitfully applied to the revealed preferences problem? Beigman and Vohra [2] adapt the notion of fat-shattering dimension to this setting, but applied to the revealed preferences problem, fat shattering dimension is cumbersome and seems ill-suited to proving tight polynomial bounds.

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Funding Games: The Truth but Not the Whole Truth

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Abstract. We introduce the Funding Game, in which m identical resources are to be allocated among n selfish agents. Each agent requests a number of resources x_i and reports a valuation $\tilde{v}_i(x_i)$, which verifiably *lower*-bounds i's true value for receiving x_i items. The pairs $(x_i, \tilde{v}_i(x_i))$ can be thought of as size-value pairs defining a knapsack problem with capacity m. A publicly-known algorithm is used to solve this knapsack problem, deciding which requests to satisfy in order to maximize the social welfare.

We show that a simple mechanism based on the knapsack *highest ratio greedy* algorithm provides a Bayesian Price of Anarchy of 2, and for the complete information version of the game we give an algorithm that computes a Nash equilibrium strategy profile in $O(n^2 \log^2 m)$ time. Our primary algorithmic result shows that an extension of the mechanism to k rounds has a Price of Anarchy of $1 + \frac{1}{k}$, yielding a graceful tradeoff between communication complexity and the social welfare.

1 Introduction

Efficiently allocating resources among multiple potential recipients is a central problem in both computer science and economics. In the mechanism design literature it is customary to use the power of currency exchange to provide incentives for the agents to be truthful. However, it has been pointed out that assuming the existence of currency in the model is not always justified ([21]). In the present paper we initiate the study of mechanisms with verification, first introduced by Nisan and Ronen in [19] for the job scheduling problem, for resource allocation problems in a setting without currency. This reveals an unexplored middle ground area between the settings of the multiple choice knapsack problem and that of multi-item auctions, which has some obvious practical applications.

The knapsack problem and its variations model the setting where the supplier knows precisely what value the agents are getting from any number of items. This can be thought of as a perfect verification mechanism, and selfishness does not

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play a role. At the other extreme, work in algorithmic game theory has generally considered the case where the supplier knows nothing about the agents' valuation and must provide incentives, typically by imposing payments, for the agents to be truthful. In this paper we introduce the Funding Game, in which a supplier distributes m identical resources among n agents, each of whom has a private valuation function depending only on the number of items received. Each agent requests a number of items x_i, and specifies its value $\tilde{v}_i(x_i)$ for these items, which might be less that its real value $v_i(x_i)$. The supplier can verify that the valuations are not exaggerated, and uses a publicly known algorithm to allocate the items to the agents. The supplier's allocation algorithm has an impact on the requests made by agents, and in effect, on the instance of the allocation problem that must be solved. Therefore we desire a mechanism that encourages agents to be relatively abstemious, or *not too greedy* in choosing their requests, and thereby produces an allocation yielding near-optimal social welfare.

1.1 Motivation

Our model closely resembles a financing competition, where multiple contestants apply for funding provided by one supplier. Contestants must write an application, or proposal, showing how the resources requested are going to be used to acquire the said value. The supplier is able to verify the veracity of the proposals and disqualify any contestant that reports a higher valuation than justified. This is the verification mechanism, and motivates our assumption that agents cannot inflate the reported value. However, the supplier may not be able to verify that the reported value is the *maximum* a contestant could obtain, since it may not know the full capabilities of the contestant.

The verification mechanism can also be thought of as a set of laws or a reputation system. If an agent obtains the requested items and does not bring the reported value, the repercussions may outweigh any immediate gains. This understanding of the verification mechanism also justifies the assumption that agents cannot inflate their reported valuation.

1.2 Related Work

We show how related literature fits in our setting, categorizing it along two orthogonal dimensions: the power of the verification mechanism and communication complexity, or metaphorically, soundness and completeness. Fig. 1 classifies existing work within these dimensions.

No verification, full revelation. This is the most common assumption in the algorithmic mechanism design literature. Multi-unit auctions model the situation where a verification mechanism does not exist and thus agents must be assumed dishonest. Truthfulness can be achieved through VCG payments, but doing so depends on solving the allocation problem optimally, which may be intractable. Starting with the work of Nisan and Ronen [19], the field of algorithmic mechanism design has sought to reconcile selfishness with computational

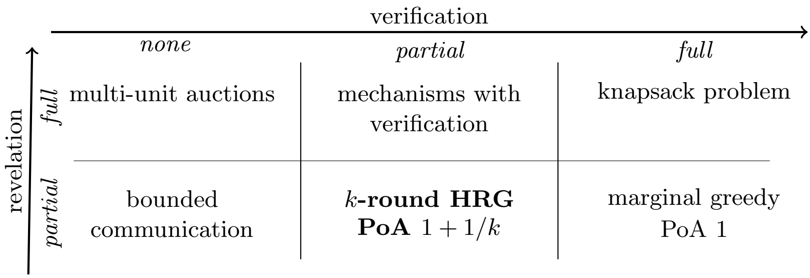

Fig. 1. Problem settings

complexity. Multi-unit auctions have been studied extensively in this context, including truthful mechanisms for single-minded bidders [18, 7], and k-minded bidders [16, 10, 11].

More recently Procaccia and Tennenholtz ([21]), initiated the study of strategy proof mechanisms without money, which was followed by the adaptation of many previously studied mechanism design problems to the non-monetary setting ([1], [2], [8], [12], [14], [17]).

No verification, partial revelation. The multi item allocation problem has also been studied in the setting where dishonest agents only partially reveal their valuation functions. The main question in this setting concerns the extent to which limiting communication complexity affects mechanism efficiency. In [5, 6], for example, bid sizes in a single-item auction are restricted to real numbers expressed by k bits. In [9], agent valuation functions are only partially revealed because full revelation would require exponential space in the number of items.

Partial verification, full revelation. Mechanisms with verification have been introduced in [19] for selfish settings of the task scheduling problem. The authors show that truthful mechanisms exist for this problem when the mechanism can detect some of the lies, which is very natural in this setting. More recently, this results were generalized to mechanisms that are collusion resistant ([20]), and to more general optimization functions ([4], [3], [13]), as well as multi parameter agents [23].

Full verification, full revelation. If the verification mechanism has full power to ensure agents' honesty and agents must report their full valuation functions, the supplier has complete information and selfishness on the part of the recipients is irrelevant. This setting can be modeled as a multiple-choice knapsack problem solvable by FPTAS [15].

1.3 Contributions

This paper extends the study of mechanisms with partial verification to multi unit resource allocation. Unlike the problems analyzed before, there are polynomial time truthful mechanisms for multi unit auctions. However, these mechanisms require both full revelation of the agent type, which may be hard to compute and communicate, and currency transfer, which may be impractical in some scenarios. Our work uses the added power of verification to provide an efficient approximation mechanism for scenarios where currency transfer cannot be modeled.

We propose the highest-ratio greedy (HRG) mechanism for the Funding Game, which provides a Bayesian PoA of 2 under the assumption that valuation functions give diminishing marginal returns (Theorem 1). We also provide an algorithm that computes the Nash equilibrium strategy profile in $O(n^2 \log^2 m)$ time and a best response protocol that converges to a Nash equilibrium profile. We show that an extension of HRG to multiple rounds can arbitrarily strengthen the pure PoA. In this extension, the supplier partitions the m items into k carefully-sized subsets, and allocates them successively over k consecutive Funding Games. We show that this mechanism has a pure PoA of $1 + \frac{1}{k}$, yielding a graceful tradeoff between communication complexity and the social welfare (Theorem 2).

2 Preliminaries

A single-round Funding Game is specified by a set of agents or *players* $\{1, ..., n\}$, a set of m identical resources or items, and for each agent i a valuation function $v_i : \{0, ..., m\} \rightarrow \mathbb{R}_0^+$ denoting the value i derives from receiving different numbers of items. We assume all valuation functions satisfy $v_i(0) = 0$, are nondecreasing, and exhibit diminishing marginal returns:

$$v_i(x) - v_i(x-1) \geq v_i(x+1) - v_i(x)$$

A strategy or *request* of agent i is a pair $s_i(x_i) = (x_i, \tilde{v}_i(x_i))$ specifying the number x_i of items requested, and its valuation for these items. A request is valid if $\tilde{v}_i(x) \leq v_i(x)$.

A *strategy profile* is an n-tuple of strategies $\mathbf{s} = (s_1(x_1), ..., s_n(x_n))$. We denote by $\mathcal{S}_i$ the set of valid strategies for agent i, and by $\mathcal{S} = \mathcal{S}_1 \times ... \times \mathcal{S}_n$ the set of valid strategy profiles. We denote by $X = (X_1, ..., X_n)$ an allocation of the items to the players where X_i is the number of items allocated to player i. Let $\mathcal{X}$ be the set of all valid allocations, i.e. all X such that $\sum_{i \in [n]} X_i \leq m$. A *mechanism* $M : \mathcal{S} \rightarrow \mathcal{X}$ is an allocation algorithm that takes as input a strategy profile $\mathbf{s}$ and outputs an allocation X of the items to the players. We will denote by $X^M(\mathbf{s}) = (X_1^M(\mathbf{s}), ..., X_n^M(\mathbf{s}))$ the output of mechanism M for strategy profile $\mathbf{s}$. The *payoff* of player i with valuation v_i is its valuation for the number of items it has been allocated: $u_i^M(v_i; \mathbf{s}) = v_i(X_i^M(\mathbf{s}))$. If $\mathbf{v} = (v_1, ..., v_n)$ is a valuation function profile we denote by $OPT^{\mathbf{v}}$ an *optimal allocation*, by $sw(OPT^{\mathbf{v}})$ the social welfare of the optimal allocation, and by $sw^M(\mathbf{v}; \mathbf{s}) = \sum_{i \in [n]} u_i^M(v_i; \mathbf{s})$ the social welfare of strategy profile $\mathbf{s}$. We use $(s_i', \mathbf{s}_{-i})$ to denote the strategy profile $\mathbf{s}$ in which player i^{th} strategy has been replaced by s_i'.

A strategy profile $\mathbf{s}$ is a *Nash equilibrium* for a Funding Game with valuation functions $\mathbf{v}$ if for any i and any $s'_i \in \mathcal{S}_i$, $u_i^M(v_i;\mathbf{s}) \geq u_i^M(v_i; s'_i, \mathbf{s}_{-i})$. The *Price of Anarchy* (PoA) bounds the ratio of the optimal social welfare and the social welfare of the worst Nash equilibrium in any Funding Game:

$$PoA^M = \sup_{\mathbf{v},\ \text{NE}\ \mathbf{s}} \frac{sw(OPT^{\mathbf{v}})}{sw^M(\mathbf{v};\mathbf{s})}$$

In incomplete information games we assume that player i's valuation function v_i is drawn from a set V_i of possible valuation functions, according to some distribution D_i. We denote by $D = D_1 \times ... \times D_n$ the product distribution of all players' valuation functions. A strategy σ_i in an incomplete information game is a mapping $\sigma_i : V_i \to S_i$ from the set of the possible valuation functions to the set of valid requests. Assuming that the distribution D is commonly known, the *Bayesian Nash equilibrium* is a tuple of strategies $\sigma = (\sigma_1, ..., \sigma_n)$ such that, for any player i, any valuation function $v_i \in V_i$ and any alternate pure strategy s'_i:

$$\mathbb{E}_{v_{-i}\sim D_{-i}}[u_i^M(v_i;\sigma_i(v_i),\sigma_{-i}(\mathbf{v}_{-i})] \geq \mathbb{E}_{v_{-i}\sim D_{-i}}[u_i^M(v_i; s'_i, \sigma_{-i}(\mathbf{v}_{-i})]$$

The *Bayesian Price of Anarchy* is defined as the ratio between the expected optimal social welfare and that of the worst bayesian Nash equilibrium:

$$BPoA = \sup_{D,\ \text{BNE}\ \sigma} \frac{\mathbb{E}_{\mathbf{v}\sim D}[sw^M(\mathbf{v};OPT^{\mathbf{v}}]}{\mathbb{E}_{\mathbf{v}\sim D}[sw^M(\mathbf{v};\sigma(\mathbf{v})]}$$

3 Single-Round Games

We first observe that the mechanism that solves the induced integer knapsack problem optimally has an unbounded PoA. This can be shown by the following simple example. Assume that n items are to be allocated to n players with valuation functions $v_i(x) = 1 + x * \epsilon$ for all i and $x > 0$. A Nash equilibrium of this game is when all players request all items. The mechanism allocates all items to one player resulting in a social welfare of $1+n\cdot\epsilon$. The optimal allocation will allocate one item to each player for a social welfare of n.

For the remainder of this section we analyze the performance of a simple greedy mechanism in a single shot game. The *Highest Ratio Greedy (HRG)* mechanism grants the requests in descending order according to the ratio $v_i(x_i)/x_i$, breaking ties in the favor of the player with lower index. If there are not enough items available to satisfy a request completely, the request is satisfied partially. This is exactly the greedy algorithm for the fractional knapsack problem. In this section we show that both the pure and Bayesian PoA are 2. An interesting open problem is whether a mechanism exist for the single round game that improves this PoA. We make use of the notion of smooth games ([22]) which we review below, cast to the Funding Games studied here. Since we are only considering the Highest Ratio Greedy mechanism we will omit the superscript M from all notations in this section.

Definition 1 (Smooth game [22]). *A Funding Game is* (λ, μ)*-smooth with respect to a choice function* $c^* : V_1 \times ... \times V_n \to \mathcal{S}$ *and the social welfare objective if, for any valuation function profiles* $\mathbf{v}$ *and* $\mathbf{w}$ *and any strategy profile* $\mathbf{s}$ *that is valid with respect to both* $\mathbf{v}$ *and* $\mathbf{w}$*, we have:*

$$\sum_{i=1}^{n} u_i(v_i; c_i^*(\mathbf{v}), \mathbf{s}_{-i}) \geq \lambda \cdot sw(\mathbf{v}; c^*(\mathbf{v})) - \mu \cdot sw(\mathbf{w}; \mathbf{s})$$

The choice function can be thought of as the optimal strategy profile, in our case the strategy profile in which each player requests the number of items received in an optimal allocation, when the valuation function profile is v.

Lemma 1. *Let* $OPT^{\mathbf{v}} = (o_1^{\mathbf{v}}, ..., o_n^{\mathbf{v}})$ *be an optimal allocation for valuation profile* $\mathbf{v}$ *and* $O : V_1 \times ... \times V_n \to \mathcal{S}$ *be the optimal strategy choice function, with* $O(\mathbf{v}) = ((o_i^{\mathbf{v}}, v_i(o_i^{\mathbf{v}}))_{i \in [n]})$*. The Funding Games are* $(1, 1)$*-smooth with respect to* O *and the social welfare objective.*

Proof. We will use o_i instead of either request $(o_i^{\mathbf{v}}, v_i(o_i^{\mathbf{v}}))$ or integer $o_i^{\mathbf{v}}$. It will be clear from context whether o_i stands for a request or an integer.

Fix valuation function profiles $\mathbf{v}$ and $\mathbf{w}$. For a strategy profile $\mathbf{s}$ valid with respect to both $\mathbf{v}$ and $\mathbf{w}$ we show that $\sum_{i=1}^{n} u_i(v_i; o_i, \mathbf{s}_{-i}) \geq sw(\mathbf{v}; O(\mathbf{v})) - sw(\mathbf{w}; \mathbf{s})$.

Let $A = \{i : u_i(v_i; o_i, \mathbf{s}_{-i}) < u_i(v_i; O(\mathbf{v}))\}$ be the set of players that are allocated more items in the optimal allocation than in profile $(o_i, \mathbf{s}_{-i})$. It is enough to show that $\sum_{i \in A} u_i(v_i; o_i, \mathbf{s}_{-i}) + sw(\mathbf{w}; \mathbf{s}) \geq \sum_{i \in A} u_i(v_i; O(\mathbf{v}))$.

For each player $i \in A$, the value per allocated item at profile $(o_i, \mathbf{s}_{-i})$ is at least $\frac{v_i(o_i)}{o_i}$ since by definition i is being allocated less than o_i items, and the valuation functions are concave. Then, $u_i(v_i; o_i, \mathbf{s}_{-i}) \geq \frac{v_i(x_i^*)}{x_i^*} \cdot X_i(o_i, \mathbf{s}_{-i})$. By definition, each player $i \in A$ would be allocated fewer items than o_i.

Therefore the requests in $\mathbf{s}_{-i}$ that have a better value per item ratio sum up to $m - X_i(c_i^*(\mathbf{v}), \mathbf{s}_{-i})$ items. Since the strategy profile $\mathbf{s}$ is assumed to be valid with respect to valuation function profile $\mathbf{w}$, the valuations expressed in $\mathbf{s}$ are at most equal to the valuations $\mathbf{w}$. We can conclude that for any $i \in A$

$$sw(\mathbf{w}; \mathbf{s}) \geq (m - X_i(o_i, \mathbf{s}_{-i})) \cdot \frac{v_i(o_i)}{o_i}$$

Then for any $i \in A$, $u_i(v_i; o_i, \mathbf{s}_{-i}) + sw(\mathbf{w}; \mathbf{s}) \geq m \cdot \frac{v_i(o_i)}{o_i}$. This is true in particular for player $j \in A$ with the highest value per item ratio $\frac{v_j(o_j)}{o_j}$. Therefore

$$\begin{aligned}\sum_{i \in A} u_i(v_i; o_i, \mathbf{s}_{-i}) + sw(\mathbf{w}; \mathbf{s}) &\geq u_j(v_j; o_j, \mathbf{s}_{-j}) + sw(\mathbf{w}; \mathbf{s}) \\ &\geq m \cdot \frac{v_j(o_j)}{o_j} \\ &\geq \sum_{i \in A} u_i(v_i; O(\mathbf{v}))\end{aligned}$$

which completes the proof. □

Theorem 1. *Both the pure and Bayesian Price of Anarchy for the Funding Games are equal to 2.*

Proof. Since the Funding Games are (1, 1)-smooth with respect to an optimal allocation, the extension theorem in [22] guarantees that the BPoA is bounded by 2. We now show that the pure PoA is arbitrarily close to 2. Consider the Funding Game with m items and two players with valuation functions $v_1(x) = m$ and $v_2(x) = x \; \forall x > 0$. One possible Nash equilibrium strategy is for both players to request all items. Since the value per item ratios are equal, only the first player will be allocated, for a social welfare of m. The optimal solution allocates one item to the first player and $m - 1$ items to the second player for a social welfare of $2m - 1$. Taking m large enough leads to a PoA arbitrarily close to 2. □

3.1 Complexity of Computing the Nash Equilibrium

We now present an algorithm that finds the Nash equilibrium in the full information setting in $O(n^2 \log^2 m)$ time. For each player i we use binary search to find the largest request $(\alpha_i, v_i(\alpha_i))$ that passes the isSatisfiable test. The isSatisfiable function below assures that regardless of the other players requests, there will be at least α_i items available when the request of player i is considered by the greedy algorithm. It is easy to see that for the resulting strategy profile each player receives exactly as many items as requested and that all items are allocated. We need to show that if player i increases its request then it will not receive more items. By the construction of α_j, for any player $j \neq i$, player j will receive at least α_j items regardless of the requests of the other players. Therefore player i cannot receive more than $\alpha_i = m - \sum_{j \neq i} \alpha_j$ by changing its request.

Algorithm 1. isSatisfiable (i, x_i)

```
for all j < i do
    x_j ← max{x ∈ [m] : v_j(x)/x ≥ v_i(α_i)/α_i}
end for
for all j > i do
    x_j ← max{x ∈ [m] : v_j(x)/x > v_i(α_i)/α_i}
end for
return true if Σ_{j≠i} x_j ≤ m − α_i else false
```

4 Multiple-Round Games

In this section we present our main algorithmic result. We extend the Funding Game introduced in the previous section to multiple rounds, and we show that the *PoA* of a k-round Funding Game is $1+\frac{1}{k}$, yielding a graceful tradeoff between mechanism complexity and the social welfare. In a k-round Funding Game, the supplier partitions the m items into k bundles, which are distributed among the n agents in k successive Funding Games or *rounds*. We assume that the supplier does not reveal the total number of available items m, nor the number of rounds

k a priori. In our analysis we assume that the agents play the Nash equilibrium strategy myopically, in each individual round. This assumption is in line with the maximin principle which states that rational agents will choose a strategy that maximizes their minimum payoff. If agents never know whether any additional items are going to be awarded in future rounds, they will try to maximize the utility in the current round. In the Funding Game, this is equivalent to playing the Nash equilibrium strategy.

As above, we use subscripts to indicate player index; we now use superscripts to indicate round index. Let $m^1, ..., m^k$ be the sizes of the bundles awarded in rounds $1, ..., k$ respectively, with $\sum_{t=1}^{k} m^t = m$. As before, the agents have valuation functions $v_i : \{0, ..., m\} \to \mathbb{R}_0^+$, which are normalized ($v_i(0) = 0$), are nondecreasing, and exhibit diminishing marginal returns.

Let x_i^t be the number of items requested by agent i in game t and let X^t be the allocation vector for round t. Let $\alpha_i^t = \sum_{j=1,..,t} X_i^j$ be the cumulative number of items allocated to agent i in the first t games, with $\alpha_i^0 = 0$ for all i. In round t, agent i's valuation function v_i^t is its *marginal valuation* given the number of items received in the earlier rounds:

$$v_i^t(x) = v_i(x + \alpha_i^{t-1}) - v_i(\alpha_i^{t-1})$$

Observe that these marginal valuations functions v_i^t are normalized, are nondecreasing and have diminishing marginal returns, just like the full valuation functions v_i. G^t will denote the Funding Game played at round t with m^t items and valuation functions v_i^t. Observe that these individual Funding Games agents are playing at each round depend on how items have been allocated in previous rounds, and indirectly, on players' strategies in previous rounds.

A strategy or request for agent i is a k-tuple $s_i(x_i^1, ..., x_i^k) = (s_i^1(x_i^1), ..., s_i^k(x_i^k))$ where $s_i^t(x_i^t) = (x_i^t, v_i^t(x_i^t))$ is the request of player i in game t. We use s_i as a shorthand to denote the strategy of player i in G, and s_i^t to denote the strategy of player i in game t. A *strategy profile* for a k-round Funding Game will refer to an n-tuple of strategies $\mathbf{s} = (s_1, ..., s_n)$ and a strategy profile for game G^t will refer to the n-tuple of requests of players in round t, $\mathbf{s}^t = (s_1^t, ..., s_n^t)$. For a strategy profile $\mathbf{s}$, we will write $sw(\mathbf{s}) = \sum_{i=1}^{n} v_i(\alpha_i^k)$ for the social welfare of $\mathbf{s}$. Let $sw(\mathbf{s}^t)$ be the social welfare of $\mathbf{s}^t$. Let $\Delta^t = \max_i v_i^t(1)$ be the highest marginal value for one item for any agent in round t. Observe that Δ^t is a nonincreasing function of t.

Definition 2. *Strategy profile* $\mathbf{s}$ *is a myopic equilibrium for the* k*-round Funding Game if for each* t*,* $\mathbf{s}^t$ *is a Nash equilibrium of round* t*. The myopic Price of Anarchy (PoA) bounds the ratio of the optimal social welfare and the social welfare of the worst myopic equilibrium in any* k*-round Funding Game:*

$$PoA = \sup_{\mathbf{v},\ myopic\ NE\ \mathbf{s}} \frac{sw(OPT^{\mathbf{v}})}{sw(\mathbf{s})}$$

Our goal is to analyze how a supplier should partition the m items into bundles in order to obtain as good a PoA as possible. Theorem 2 in this section shows how the PoA relates to the choices of bundle ratios, while in the next section we find the bundle ratios that give the best PoA guarantees.

Lemma 2. *For any myopic Nash equilibrium strategy profile* $\mathbf{s}$ *for a* k*-round game, we have* $\Delta^t \geq \frac{sw(\mathbf{s}^t)}{m^t} \geq \Delta^{t+1}$ *for each* t*.*

Proof: The first inequality follows from the definition of Δ^t and the diminishing returns assumption.

For the second inequality, suppose $\Delta^{t+1} > \frac{sw(\mathbf{s}^t)}{m^t}$. This would imply that either some items are not allocated at $\mathbf{s}^t$ (impossible since $\mathbf{s}^t$ is Nash equilibrium and by assumption $\Delta^{t+1} > 0$) or that some winning player i has valuation-per-item ratio $\frac{v_i^t(x_i^t)}{x_i^t} < \Delta^{t+1} = v_j^{t+1}(1)$, for some player j. But then j could have successfully requested another item in game G^t, meaning $\mathbf{s}^t$ is not Nash equilibrium, and so contradiction. □

Lemma 3. *For any myopic equilibrium* $\mathbf{s}$ *of a* k*-round Funding Game, we have:*

$$sw(OPT) \leq sw(\mathbf{s}) + \Delta^{k+1} \cdot \sum_{t=1}^{k} \left(m^t - \frac{sw(\mathbf{s}^t)}{\Delta^t} \right)$$

Theorem 2. *Let* $y_t = m^t/m^1$*. The PoA of the* k*-round Funding Game with bundle sizes* m^t *is bounded by:*

$$1 + \sup_{x_1,\ldots,x_n : x_i \geq 1} \frac{\sum_{t=1}^{k} y_t (1 - \frac{1}{x_t})}{\sum_{t=1}^{k} y_t \prod_{i=t+1}^{k} x_i} \tag{1}$$

Proof. Let $\mathbf{s}$ be a myopic equilibrium for a k-round game. We will show that there exist $x_1, ..., x_k$, $x_i \geq 1$, such that:

$$\frac{sw(OPT)}{sw(\mathbf{s})} \leq \frac{\sum_{t=1}^{k} y_t (1 - \frac{1}{x_t})}{\sum_{t=1}^{k} y_t \prod_{i=t+1}^{k} x_i}$$

From Lemma 3, we have:

$$sw(OPT) \leq sw(\mathbf{s}) + \Delta^{k+1} \cdot \sum_{t=1}^{k} \left(m^t - \frac{sw(\mathbf{s}^t)}{\Delta^t} \right)$$

Let $x_t = \frac{\Delta^t}{\Delta^{t+1}}$, which is at least 1 for each t. Since $\mathbf{s}^t$ is a Nash equilibrium for round t, $\Delta^{t+1} \leq \frac{sw(\mathbf{s}^t)}{m^t}$ for each t.

Then we have:

$$\begin{aligned} sw(OPT) - sw(\mathbf{s}) &\leq \Delta^{k+1} \cdot \sum_{t=1}^{k} \left(m^t - \frac{sw(\mathbf{s}^t)}{\Delta^t} \right) \\ &\leq \Delta^{k+1} \cdot \sum_{t=1}^{k} m^t \left(1 - \frac{\Delta^{t+1}}{\Delta^t} \right) \\ &\leq m^1 \Delta^{k+1} \cdot \sum_{t=1}^{k} y_t \left(1 - \frac{1}{x_t} \right) \end{aligned} \tag{2}$$

Observe that $\Delta^t = \Delta^{k+1} \prod_{i=t}^{k} x_i$. Therefore:

$$sw(\mathbf{s}) = \sum_{t=1}^{k} sw(\mathbf{s}^t) \geq \sum_{t=1}^{k} m^t \Delta^{t+1} \geq m^1 \Delta^{k+1} \cdot \sum_{t=1}^{k} y_t \cdot \prod_{i=t+1}^{k} x_i \tag{3}$$

From (2) and (3) it follows that for any k-round game with bundle sizes m^t, there exist $x_1, ..., x_k$ such that:

$$PoA = 1 + \sup \frac{sw(OPT) - sw(\mathbf{s})}{sw(\mathbf{s})} \leq 1 + \sup_{x_t \geq 1} \frac{\sum_{t=1}^{k} y_t \left(1 - \frac{1}{x_t} \right)}{\sum_{t=1}^{k} y_t \cdot \prod_{i=t+1}^{k} x_i}$$

□

5 Evaluating the PoA

In this section we present two results analyzing the expression (1) above. Theorem 3 shows that supremum of this expression taken over all valid choices of x_t but fixing $y_t = t$ is $1/k$. This corresponds to bundle sizes $m_1, 2 \cdot m_1, ..., k \cdot m_1$ for some m_1, indicating that the PoA *for such bundle sizes* equals $1 + 1/k$.

Second, we show that the min-sup of this expression, now also taken over choices of y_i, which corresponds to considering all possible choices of bundle sizes, equals the same value $1/k$, indicating that there is no better partition of the items.

Theorem 3. *Let*

$$F(x_1, ..., x_k) = \frac{\sum\limits_{i=1}^{k} i(1 - \frac{1}{x_i})}{\sum\limits_{i=1}^{k} i \prod\limits_{j=i+1}^{k} x_j} \quad x_i \geq 1,\ i = 1, ..., k$$

Then $\sup\limits_{\mathbf{x}} F(\mathbf{x}) = \frac{1}{k}$.

Proof. First observe that:

$$F(\mathbf{x}) = \frac{1 - \frac{1}{x_1} + \sum_{i=2}^{k} i(1 - \frac{1}{x_i})}{\sum_{i=1}^{k} i \prod_{j=i+1}^{k} x_j} < \lim_{x_1 \to \infty} F(\mathbf{x})$$

If we set $x_i = \frac{i}{i-1}$, $i = 2, ..., k$, we have $\lim_{x_1 \to \infty} F(\mathbf{x}) = \frac{1}{k}$. It remains to show that $\lim_{x_1 \to \infty} F(\mathbf{x}) \leq \frac{1}{k}$. We note that the following inequalities are equivalent:

$$\lim_{x_1 \to \infty} F(\mathbf{x}) \leq \frac{1}{k} \Leftrightarrow \lim_{x_1 \to \infty} \left(\sum_{i=1}^{k} i \prod_{j=i+1}^{k} x_j - k \sum_{i=1}^{k} i(1 - \frac{1}{x_i}) \right) \geq 0 \Leftrightarrow$$

$$\lim_{x_1 \to \infty} \left(\sum_{i=1}^{k} (iz_i + ik \cdot \frac{z_i}{z_{i-1}}) - \sum_{i=1}^{k} ik \right) \geq 0, \tag{4}$$

$$\text{where } z_i = \prod_{j=i+1}^{k} x_j,\ i = 1, ..., k-1; z_k = 1; z_0 = x_1 z_1$$

Now define a function $C : [0, \infty)^{k-1} \to \mathbb{R}$, $C(\mathbf{z}) = \sum_{i=1}^{k} (iz_i + ik \cdot \frac{z_i}{z_{i-1}}) - \sum_{i=1}^{k} ik$. Notice that C is a function of $k-1$ variables since z_0 and z_k are fixed. Also notice that the domain of C strictly includes the domain of $\mathbf{z}$ as defined in Eq. (4). To complete the proof, we show that $C(\mathbf{z}) \geq 0$ for any $\mathbf{z} \in [0, \infty)^{k-1}$. We will do this in two steps: (i) showing that $C(\mathbf{z})$ has a unique stationary point, and then (ii) showing that $C(\mathbf{z}) \geq 0$ at any of the domain boundaries and the stationary point.

$\mathbf{C}(\mathbf{z})$ has a unique stationary point. Let $\mathbf{a} = (a_1, ..., a_{k-1})$ be a stationary point for function C, and let $a_0 = z_0 = x_1 z_1$ and $a_k = z_k = 1$:

$$\frac{\partial C}{\partial z_i}(\mathbf{a}) = i + \frac{ik}{a_{i-1}} - k(i+1)\frac{a_{i+1}}{a_i^2} = 0,\ i = 1, ..., k-1 \tag{5}$$

We show now by induction that each a_i can be written as a function of a_1. For the base case, let $a_0 = x_1 \cdot a_1 = f_0(a_1)$ and $f_1(a_1) = a_1$.

Now assume that $a_{i-1} = f_{i-1}(a_1)$ and $a_i = f_i(a_1)$. Then we will define a_{i+1} as a function of a_1 as follows. From Eq. (5) we can infer:

$$a_{i+1} = \left(i + \frac{ik}{a_{i-1}} \right) \cdot \frac{a_i^2}{k(i+1)}$$

$$a_{i+1} = \left(i + \frac{ik}{f_{i-1}(a_1)} \right) \cdot \frac{f_i^2(a_1)}{k(i+1)} \triangleq f_{i+1}(a_1) \tag{6}$$

where $f_{i+1}(\cdot)$ is the name given to the expression in Eq. (6) as a function of a_1.

Therefore the equations $a_i = f_i(a_1)$, $i = 1, ..., k-1$ uniquely define a stationary point $\mathbf{a}$ with respect to a_1. To show that the stationary point $\mathbf{a}$ is unique, we only need to show that $f_k(a_1) = 1$ has a unique solution. For this it is sufficient to show that the derivative of f_k with respect to a_1 is always positive: $f_k'(a_1) > 0$. We show this by induction on $i = 0, ..., k$. Let $h_i = \frac{f_i}{f_{i-1}}$, $i = 2, ..., k-1$. The inductive hypothesis is that $f_i'(a_1) > 0$, $i = 1, ..., k$ and $h_j(a_1) > 0$ and $h_j'(a_1) > 0$, $j = 2, ..., k$.
For the base case, observe the following:

$$f_1(a_1) = a_1 > 0 \text{ and } f_1'(a_1) = 1 > 0$$
$$f_2(a_1) = \frac{x_1 a_1^2 + k a_1}{2kx_1} \text{ and } f_2'(a_1) = \frac{2x_1 a_1 + k}{2kx_1} > 0$$
$$h_2(a_1) = \frac{f_2(a_1)}{f_1(a_1)} = \frac{x_1 a_1 + k}{2kx_1} > 0 \text{ and } h_2'(a_1) = \frac{x_1}{2kx_1} > 0$$

Now assume that $f_i'(a_1) > 0$, $h_i(a_1) > 0$, and $h_i'(a_1) > 0$. We then observe that $f_{i+1}'(a_1)$, $h_{i+1}(a_1)$ and $h_{i+1}'(a_1)$ are all strictly positive:

$$f_{i+1}'(a_1) = h_i'(a_1) \cdot f_i(a_1) + h_i(a_1) \cdot f_i'(a_1) > 0$$
$$h_{i+1}(a_1) = \left(i + \frac{ik}{f_{i-1}(a_1)}\right) \cdot \frac{f_i(a_1)}{k(i+1)}$$
$$= \frac{i}{k(i+1)} f_i(a_1) + \frac{i}{i+1} \cdot h_i(a_1) > 0$$
$$h_{i+1}'(a_1) = \frac{i}{k(i+1)} f_i'(a_1) + \frac{i}{i+1} \cdot h_i'(a_1) > 0$$

This shows that the equation $f_k(a_1) = 1$ has a unique solution, and thus concludes step (i).

$\mathbf{C}(\mathbf{z}) \geq 0$ at all boundary points and at the unique stationary point. First observe that $a_i = \frac{k}{i}$ satisfies Eq. (6), $i = 1, ..., k$ and hence $\mathbf{a} = (a_1, ..., a_{k-1})$ is the unique stationary point for C. Now we show that $C(\mathbf{a}) \geq 0$:

$$C(\mathbf{a}) = \sum_{i=1}^{k} (ia_i + ik \cdot \frac{a_i}{a_{i-1}}) - \sum_{i=1}^{k} ik = \sum_{i=1}^{k} (k + k(i-1)) - \sum_{i=1}^{k} ik = 0$$

Let $\mathbf{b} = (b_1, ..., b_{k-1})$ be a boundary point. Then we must show that $C(\mathbf{b}) \geq 0$. Since $\mathbf{b}$ is a boundary point there must exist j such that $b_j = 0$ or $b_j = \infty$:

$$C(\mathbf{b}) = \sum_{i=1}^{k} (ib_i + ik \cdot \frac{b_i}{b_{i-1}}) - \sum_{i=1}^{k} ik$$

The only negative term is $\sum_{i=1}^{k} ik$, which is constant with respect to $\mathbf{b}$. If $b_j = 0$ for some j, then the positive term $(i+1)k \cdot \frac{b_{i+1}}{b_i}$ is infinite and $C(\mathbf{b}) > 0$. On the other hand, if $b_j = \infty$ for some j, then the positive term $ik \cdot \frac{b_i}{b_{i+1}}$ is infinite and again $C(\mathbf{b}) > 0$. Steps (i) and (ii) above show that $C(\mathbf{z}) \geq 0 \; \forall z \in [0, \infty)^{k-1}$ and therefore $C(\mathbf{z}) \geq 0$ on the restricted domain of equation (4), which completes the proof. □

Corollary 1. *The PoA for the k-round Funding Games with bundle ratios* $\frac{m_t}{m_1} = t$ *is* $1 + \frac{1}{k}$.

Theorem 4. *Let*

$$G(\mathbf{x}, \mathbf{y}) = \frac{\sum_{i=1}^{k} y_i (1 - \frac{1}{x_i})}{\sum_{i=1}^{k} y_i \prod_{j=i+1}^{k} x_j} \quad y_i \geq 0; \; x_i \geq 1, \; i = 1, ..., k$$

Then $\min_{\mathbf{y}} \sup_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) = \frac{1}{k}$.

6 Discussion

In this paper, we introduced the Funding Game, a novel formulation of resource allocation for agents whose valuation declarations can be verified, but reveal only partial information. We analyzed the *PoA* for the pure and Bayesian Nash equilibrium and showed that allocating the resources in multiple successive rounds can improve the pure *PoA* arbitrarily close to 1. There are two directions in which this work can be extended. First, our mechanism relies on the assumption that the valuation functions are concave. An interesting open problem is finding an efficient mechanism for general valuation functions. Second, it might be desirable to develop efficient verification mechanisms for combinatorial settings, where players' valuation functions are defined on subsets of items.

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Greedy Selfish Network Creation

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Abstract. We introduce and analyze *greedy equilibria* (GE) for the well-known model of selfish network creation by Fabrikant et al. [PODC'03]. GE are interesting for two reasons: (1) they model outcomes found by agents which prefer smooth adaptations over radical strategy-changes, (2) GE are outcomes found by agents which do not have enough computational resources to play optimally. In the model of Fabrikant et al. agents correspond to Internet Service Providers which buy network links to improve their quality of network usage. It is known that computing a best response in this model is NP-hard. Hence, poly-time agents are likely not to play optimally. But how good are networks created by such agents? We answer this question for very simple agents. Quite surprisingly, naive greedy play suffices to create remarkably stable networks. Specifically, we show that in the Sum version, where agents attempt to minimize their average distance to all other agents, GE capture Nash equilibria (NE) on trees and that any GE is in 3-approximate NE on general networks. For the latter we also provide a lower bound of $\frac{3}{2}$ on the approximation ratio. For the Max version, where agents attempt to minimize their maximum distance, we show that any GE-star is in 2-approximate NE and any GE-tree having larger diameter is in $\frac{6}{5}$-approximate NE. Both bounds are tight. We contrast these positive results by providing a linear lower bound on the approximation ratio for the Max version on general networks in GE. This result implies a locality gap of $\Omega(n)$ for the metric min-max facility location problem, where n is the number of clients.

1 Introduction

The area of Network Design is one of the classical and still very active fields in the realm of Theoretical Computer Science and Operations Research. But there is this curious fact: One of the most important networks which is increasingly shaping our everyday life – the Internet – cannot be fully explained by classical Network Design theory. Unlike centrally designed and optimized networks, the Internet was and still is created by a multitude of selfish agents (Internet Service Providers (ISPs)), who control and modify varying sized portions of the network structure ("autonomous systems"). This decentralized nature is an obstacle to approaching the design and analysis of the Internet as a classical optimization problem. Interestingly, each agent does face classical Network Design problems, i.e. minimizing the cost of connecting the *own* network to the rest of the Internet while ensuring a high quality of service. The Internet is the result of the interplay

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of such local strategies and can be considered as an equilibrium state of a game played by selfish agents.

The classical and most popular solution concept of such games is the (pure) Nash equilibrium [15], which is a stable state, where no agent unilaterally wants to change her current (pure) strategy. However, Nash equilibria (NE) have their difficulties. Besides their purely descriptive, non-algorithmic nature, there are two problems: (1) With NE as solution concept agents only care if there is a better strategy and would perform radical strategy-changes even if they only yield a tiny improvement. (2) In some games it is computationally hard to even tell if a stable state is reached because computing the best possible strategy of an agent is hard. Thus, for such games NE only predict stable states found by supernatural agents.

But what solutions are actually found by more realistic players, i.e. by agents who prefer smooth strategy-changes and who can only perform polynomial-time computations? And what impact on the stability has this transition from supernatural to realistic players?

In this paper, we take the first steps towards answering these questions for one of the most popular models of selfish network creation. This model, called the Network Creation Game (NCG), was introduced a decade ago by Fabrikant et al. [9]. In NCGs agents correspond to ISPs who create links towards other ISPs while minimizing cost and maximizing their quality of network usage. It seems reasonable that ISPs prefer greedy refinements of their current strategy (network architecture) over a strategy-change which involves a radical re-design of their infrastructure. Furthermore, computing the best strategy in NCGs is NP-hard. Hence, it seems realistic to assume that agents perform smooth strategy-changes and that they do not play optimally. We take this idea to the extreme by considering very simple agents and by introducing and analyzing a natural solution concept, called *greedy equilibrium*, for which agents can easily compute whether a stable state is reached and which models an ISP's preference for smooth strategy-changes.

1.1 Model and Definitions

In NCGs [9] there is a set of n agents V and each agent $v \in V$ can buy an edge $\{v, u\}$ to any agent $u \in V$ for the price of $\alpha > 0$. Here α is a fixed parameter of the game which specifies the cost of creating any link. The strategy S_v of an agent v is the set of vertices towards which v buys an edge. Let $G = (V, E)$ be the induced network, where an edge $\{x, y\} \in E$ is present if $x \in S_y$ or $y \in S_x$. The network G will depend heavily on the parameter α. To state this explicitly, we let (G, α) denote the network induced by the strategies of all agents V. In a NCG agents selfishly choose strategies to minimize their cost. There are basically two versions of NCGs, depending on the definition of an agent's cost-function. In the Sum *version* [9], agents try to minimize the sum of their shortest path lengths to all other nodes in the network, while in the Max *version* [7], agents try to minimize their maximum shortest path distance to any other network node. The precise definitions are as follows: Let S_v denote agent v's strategy in (G, α), then we have for the SUM version that the cost of agent v is $c_v(G,\alpha) = \alpha|S_v| + \sum_{w \in V(G)} d_G(v,w)$, if G is connected and $c_v(G,\alpha) = \infty$, otherwise. For the MAX version we define agent v's cost as $c'_v(G,\alpha) = \alpha|S_v| + \max_{w \in V(G)} d_G(v,w)$, if G is connected and $c'_v(G,\alpha) = \infty$, otherwise. In both cases $d_G(\cdot,\cdot)$ denotes the shortest path distance in the graph G. Note that both cost functions nicely incorporate two conflicting objectives: Agents want to pay as little as possible for being connected to the network while at the same time they want to have good connection quality. For NCGs we are naturally interested in networks where no agent unilaterally wants to change her strategy. Clearly, such outcomes are pure NE and we let SUM-NE denote the set of all pure NE of NCGs for the SUM version and MAX-NE denotes the corresponding set for the MAX version.

Another important notion is the concept of *approximate Nash equilibria*. Let (G,α) be any network in a NCG. For all $u \in V(G)$ let $c(u)$ and $c^*(u)$ denote agent u's cost induced by her current pure strategy in (G,α) and by her best possible pure strategy, respectively. We say that (G,α) is a β-approximate Nash equilibrium if for all agents $u \in V(G)$ we have $c(u) \leq \beta c^*(u)$, for some $\beta \geq 1$.

1.2 Related Work

The work of Fabrikant et al. [9] did not only introduce the very elegant model described above. Among other results, the authors showed that computing a best possible strategy of an agent is NP-hard.

To remove the quite intricate dependence on the parameter α, Alon et al. [3] recently introduced the BASIC NETWORK CREATION GAME (BNCG), in which a network G is given and agents can only "swap" incident edges to decrease their cost. Here, a swap is the exchange of an incident edge with a non-existing incident edge. The cost of an agent is defined like in NCGs but without the edge-cost term. The authors of [3] proposed the *swap equilibrium* (SE) as solution concept for BNCGs. A network is in SE, if no agent unilaterally wants to swap an edge to decrease her cost. This solution concept has the nice property that agents can check in polynomial time if they can perform an improving strategy-change. The greedy equilibrium, which we analyze later, can be understood as an extension of the swap equilibrium which has similar properties but provides agents more freedom to act. Note, that in BNCGs an agent can swap *any* incident edge, whereas in NCGs only edges which are bought by agent v can be modified by agent v. This problem, first observed by Mihalák and Schlegel [13], can easily be circumvented, as recently proposed by the same authors in [14]: BNCGs are modified such that every edge is owned by exactly one agent and agents can only swap *own* edges. The corresponding stable networks of this modified version are called *asymmetric swap equilibrium* (ASE). However, independent of the ownership, edges are still two-way. These simplified versions of NCGs are an interesting object of study since (asymmetric) swap equilibria model the local weighing of decisions of agents and despite their innocent statement they tend to be quite complicated structures. In [12] it was shown that greedy dynamics in a BNCG converge very quickly to a stable state if played on a tree. The authors of [5] analyzed BNCGs on trees with agents having communication interests.

However, simplifying the model as in [3] is not without its problems. Allowing only edge-swaps implies that the number of edges remains constant. Hence, this model seems too limited to explain the creation of rapidly growing networks.

A part of our work focuses on tree networks. Such topologies are common outcomes of NCGs if edges are expensive, which led the authors of [9] to conjecture that all (non-transient) stable networks of NCGs are trees if α is greater than some constant. The conjecture was later disproved by Albers et al. [1] but it was shown to be true for high edge-cost. In particular, the authors of [13] proved that all stable networks are trees if $\alpha > 273n$ in the SUM version or if $\alpha > 129$ in the MAX version. Experimental evidence suggests that this transition to tree networks already happens at much lower edge-cost and it is an interesting open problem to improve on these bounds.

Demaine et al. [6] investigated NCGs, where agents cannot buy every possible edge. Furthermore, Ehsani et al. [8] recently analyzed a bounded-budget version. Both versions seem realistic, but in the following we do not restrict the set of edges which can be bought or the budget of an agent. Clearly, such restrictions reduce the qualitative gap between simple and arbitrary strategy-changes and would lead to weaker results for our analysis. Note, that this indicates that outcomes found by simple agents in the edge or budget-restricted version may be even more stable than we show in the following sections.

To the best of our knowledge, approximate Nash equilibria have not been studied before in the context of selfish network creation. Closest to our approach here may be the work of Albers et al. [2], which analyzes for a related game how tolerant the agents have to be in order to accept a centrally designed solution. We adopt a similar point of view by asking how tolerant agents have to be to accept a solution found by greedy play.

Guylás et al. [10] recently published a paper having a very similar title to ours. They investigate networks created by agents who use the length of "greedy paths" as communication cost and show that the resulting equilibria are substantially different to the ones we consider here. Their term "greedy" refers to the distances whereas our term "greedy" refers to the behavior of the agents.

1.3 Our Contribution

We introduce and analyze greedy equilibria (GE) as a new solution concept for NCGs. This solution concept is based on the idea that agents (ISPs) prefer greedy refinements of their current strategy (network architecture) over a strategy-change which involves a radical re-design of their infrastructure. Furthermore, GE represent solutions found by very simple agents, which are computationally bounded. We show in Section 2 that such greedy refinements can be computed efficiently and clarify the relation of GE to other known solution concepts for NCGs.

Our main contribution follows in Section 3 and Section 4, where we analyze the stability of solutions found by greedily playing agents. For the SUM version we show the rather surprising result that, despite the fact that greedy strategy-changes may be sub-optimal from an agent's point of view, SUM-GE capture

SUM-NE on trees. That is, in any tree network which is in SUM-GE *no* agent can decrease her cost by performing *any* strategy-change. For general networks we prove that any network in SUM-GE is in 3-approximate SUM-NE and we provide a lower bound of $\frac{3}{2}$ for this approximation ratio. Hence, we are able to show that greedy play almost suffices to create perfectly stable networks.

For the MAX version we show that these games have a strong non-local flavor which yields diminished stability. Here even GE-trees may be susceptible to non-greedy improving strategy-changes. Interestingly, susceptible trees can be fully characterized and we show that their stability is very close to being perfect. Specifically, we show that any GE-star is in 2-approximate MAX-NE and that any GE-tree having larger diameter is in $\frac{6}{5}$-approximate MAX-NE. We give a matching lower bound for both cases. For non-tree networks in GE the picture changes drastically. We show that for GE-networks having a very small α the approximation ratio is related to their diameter and we provide a lower bound of 4. For $\alpha \geq 1$, we show that there are non-tree networks in MAX-GE, which are only in $\Omega(n)$-approximate MAX-NE. The latter result yields that the locality gap of uncapacitated metric min-max facility location is in $\Omega(n)$.

Regarding the complexity of deciding Nash-stability, we show that there are simple polynomial time algorithms for tree networks in both versions. Furthermore, greedy-stability represents an easy to check certificate for 3-approximate Nash-stability in the SUM version.

2 Greedy Agents and Greedy Equilibria

We consider agents which check three simple ways to improve their current infrastructure. The three operations are

- *greedy augmentation*, which is the creation of *one* new own link,
- *greedy deletion*, which is the removal of *one* own link,
- *greedy swap*, which is a swap of *one* own link.

Computing the best augmentation/deletion/swap for one agent can be done in $\mathcal{O}(n^2(n+m))$ steps by trying all possibilities and re-computing the incurred cost. Observe, that these smooth strategy-changes induce some kind of organic evolution of the whole network which seems highly adequate in modeling the Internet. This greedy behavior naturally leads us to a new solution concept:

Definition 1 (Greedy Equilibrium). (G, α) *is in* greedy equilibrium *if no agent in G can decrease her cost by buying, deleting or swapping* one *own edge.*

Note, that GE can be understood as solutions which are obtained by a distributed local search procedure performed by selfish agents.

The next theorem relates GE to other solution concepts in the SUM version. See Fig. 1 for an illustration. Relationships are similar in the MAX version.

Theorem 1. *For the* SUM *version it is true that $NE \subset GE \subset ASE$ and that $SE \subset ASE$. Furthermore, we have $NE \backslash SE \neq \emptyset$, $GE \backslash SE \neq \emptyset$, $(GE \backslash SE) \backslash NE \neq \emptyset$, $(GE \backslash NE) \cap SE \neq \emptyset$ and $NE \cap GE \cap SE \neq \emptyset$.*

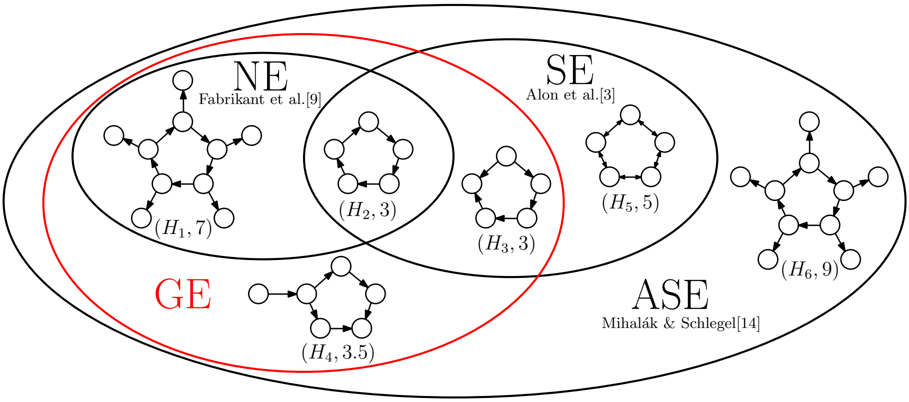

Fig. 1. Relations between solution concepts for NCGs in the SUM version. Edge-directions indicate edge-ownership, edges point away from its owner.

3 The Quality of Sum Greedy Equilibria

This section is devoted to discussing the quality of greedy equilibrium networks in the SUM version. We begin with a simple but very useful property.

Lemma 1. *If an agent v cannot decrease her cost by buying* one *edge in the* SUM *version, then buying $k > 1$ edges cannot decrease agent v's cost.*

3.1 Tree Networks in Sum Greedy Equilibrium

We show that in a NCG all stable trees found by greedily behaving agents are even stable against *any* strategy-change. Hence, in case of a tree equilibrium *no* loss in stability occurs by greedy play. This is a counter-intuitive result, since for each agent alone being greedy is clearly sub-optimal (the network in Fig. 2 with $\alpha = 6$ is an example). Thus, the following theorem shows the emergence of an optimal outcome out of a combination of sub-optimal strategies.

Theorem 2. *If (T, α) is in* SUM-*GE and T is a tree, then (T, α) is in* SUM-*NE.*

Before we prove Theorem 2, we first provide some useful observations. The well-known notion of a 1-*median* [11] is used: A 1-*median* of a connected graph G is a vertex $x \in V(G)$, where $x \in \arg\min_{u \in V(G)} \sum_{w \in V(G)} d(u, w)$.

Lemma 2. *Let (T, α) be a tree network in* SUM-*GE. If agent u owns edge $\{u, w\}$ in (T, α), then w must be a 1-median of its tree in the forest $T - \{u\}$.*

Let (T, α) be any tree network in SUM-GE and let T^u be the forest induced by removing all edges owned by agent u from T. Let F^u be the forest T^u without the tree containing vertex u. The above lemma directly implies the following:

Corollary 1. *Let (T, α) be in* SUM-*GE, and let F^u be defined as above. Agent u's strategy in (T, α) is the optimal strategy among all strategies that buy exactly one edge into each tree of F^u.*

Let $x \in V(T)$ be a 1-median of the tree T. Let $u \notin V(T)$ be a special vertex. We consider the network (G_T^u, α), which is obtained by adding vertex u and inserting edge $\{u, x\}$, which is owned by u, in T and by assigning the ownership of all other edges arbitrarily among the respective endpoints of any other edge in G_T^u. Furthermore, let $y_1, \ldots, y_l$ denote the neighbors of vertex x in T and let T_{y_i}, for $1 \leq i \leq l$, denote the maximal subtree of T which is rooted at y_i and which does not contain vertex x. See Fig. 2 (left) for an illustration. We consider a special

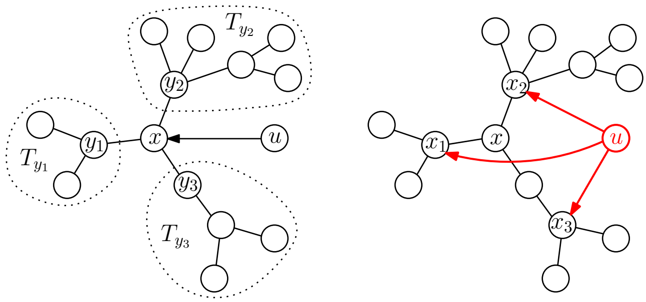

Fig. 2. The network (G_T^u, α) before and after agent u changes her strategy to S_u^*

strategy of agent u in (G_T^u, α): Let $S_u^* = \{x_1, \ldots, x_k\}$ be the best strategy of agent u which purchases *at least two* edges. The situation with agent u playing strategy S_u^* is depicted in Fig. 2 (right).

Lemma 3. *Let (G_T^u, α), $S_u^* = \{x_1, \ldots, x_k\}$ and the subtrees T_{y_i}, for $1 \leq i \leq l$ be specified as above. There is no subtree T_{y_i}, which contains all vertices $x_1, \ldots, x_k$.*

Next, let us consider two special strategies of agent u. Let S_u^1 be agent u's best strategy, which buys at least two edges including one edge towards vertex x. Furthermore, let S_u^2 be agent u's best strategy, which buys at least two edges, but no edge towards vertex x.

Lemma 4. *Let (G_T^u, α), S_u^1, S_u^2 and vertex x be specified as above. Let $x_j \in S_u^2$ be a vertex which has minimum distance to x among all vertices in S_u^2. If strategy S_u^2 yields less cost for agent u than strategy S_u^1, then x_j cannot be a leaf of G_T^u.*

Now we have all the tools we need to prove Theorem 2.

Proof (of Theorem 2). We will prove the contra-positive statement of Theorem 2. We show that if an agent u can decrease her cost by performing a strategy-change in a tree network (T, α) which is in Sum-GE, then there is an agent z in $V(T)$ who can decrease her cost by performing a *greedy* strategy-change. In that case we have a contradiction to (T, α) being in Sum-GE.

If agent u can decrease her cost by buying, deleting or swapping one own edge, then we have $u = z$ and we are done. Hence, we assume that agent u cannot decrease her cost by a greedy strategy-change but by performing an arbitrary strategy-change. We consider agent u's strategy-change towards the best possible *arbitrary* strategy S^* (if u has more than one such strategy, then we choose the one which buys the least number of edges). Clearly, agent u cannot remove any owned edge without purchasing edges, since T is a tree and the removal would disconnect T. Furthermore, since (T, α) is in SUM-GE and by Lemma 1, agent u cannot decrease her cost by purchasing $k > 0$ *additional* edges. Hence, the only way agent u can possibly decrease her cost is by removing j own edges and building k edges simultaneously. Clearly, $k \geq j$ must hold. Furthermore, by Corollary 1, it follows that $k > j$. Let F^u be the forest obtained by removing the j edges owned by agent u from T and let T^* be the tree in F^u which contains vertex u. Observe that among the k new edges, there cannot be edges having an endpoint in T^*. This is true because (T, α) is in SUM-GE and by Lemma 1. Any such edge would be a possible greedy augmentation which we assume not to exist. Hence, by the pigeonhole principle, we have that there must be at least one tree T_q in F^u into which agent u buys at least *two* edges with strategy S^*. We focus on T_q and will find agent z within.

Let $\{u, x\}$, with $x \in V(T_q)$, be the unique edge of T which connects u to the subtree T_q. Hence, agent u's strategy-change to S^* removes edge $\{u, x\}$ and buys $k_q > 1$ edges $\{u, x_1\}, \ldots, \{u, x_{k_q}\}$, with $x_j \in V(T_q)$ for $1 \leq j \leq k_q$. Let $X = \{x_1, \ldots, x_{k_q}\}$. By Lemma 1, we have $x_j \neq x$, for $x_j \in X$. Let $y_1, \ldots, y_l$ denote the neighbors of vertex x in T_q and let $T_{y_1}, \ldots, T_{y_l}$ be the maximal subtrees of T_q not containing vertex x, which are rooted at vertex $y_1, \ldots, y_l$, respectively. Let $x_a \in X$ be a vertex of X which has minimum distance to vertex x. Let $T_a \in \{T_{y_1}, \ldots, T_{y_l}\}$ be the subtree containing x_a. By Lemma 3, we have that there is a subtree $T_b \in \{T_{y_1}, \ldots, T_{y_l}\}$, with $T_b \neq T_a$, which contains at least one vertex of X. Let $B = \{x_{b_1}, \ldots, x_{b_p}\} = X \cap V(T_b)$. Furthermore, since no strategy which buys at least two edges including an edge towards x into T_q outperforms u's greedy strategy within T_q and by Lemma 4, we have that vertex x_a cannot be a leaf. That is, there is a vertex $z \in V(T_q)$, which is a neighbor of x_a, such that $d(z, x) > d(x_a, x)$. We show that agent z can decrease her cost by buying *one* edge in (T, α).

First of all, notice that by definition of S^*, we have that *each* edge $\{u, x_j\}$, with $x_j \in X$, must independently of the other bought edges yield a distance decrease of *more than* α for agent u. Otherwise agent u could remove this edge and obtain a strictly better (or smaller) strategy, which contradicts the fact that S^* is the best possible strategy (buying the least number of edges). Let $D_j \subset V(T_q)$ be the set of vertices to which edge $\{u, x_j\}$ is the first edge on agent u's unique shortest path. Since x_a has minimum distance to x, it follows that $D_r \subseteq V(T_b)$ for $r \in \{b_1, \ldots, b_p\}$. The main observation is that agent z faces in some sense the same situation as agent u with strategy S^* but without all edges $\{u, y\}$, where $y \in B$: Both have vertex x_a as neighbor and their shortest paths to any vertex in T_b all traverse x_a and x. Remember, that each edge $\{u, y\}$, for all $y \in B$, yields a distance decrease of more than α for agent u and that $D_r \subseteq V(T_b)$, for $r \in \{b_1, \ldots, b_p\}$. Furthermore, removing all those edges from S^* yields a strict cost increase for agent u. This implies that agent z can decrease her cost by buying all edges $\{z, y\}$, for $y \in B$, simultaneously. If $|B| = 1$, then this strategy-change is a greedy move by agent z which decreases z's cost. If $|B| > 1$, then, by the contra-positive statement of Lemma 1, it follows that there exists *one* edge $\{z, y^*\}$, with $y^* \in B$, which agent z can greedily buy to decrease her cost. □

3.2 Non-tree Networks in Sum Greedy Equilibrium

There exist non-tree networks in SUM-GE, since, as shown by Albers et al. [1], there exist non-tree networks in SUM-NE and we have SUM-NE $\subseteq$ SUM-GE. Having Theorem 2 at hand, one might hope that this nice property carries over to non-tree greedy equilibria. Unfortunately, this is not true.

Theorem 3. *There is a network in* SUM-*GE which is not in β-approximate* SUM-*NE for $\beta < \frac{3}{2}$.*

Fig. 3 shows the construction of a critical greedy equilibrium network.

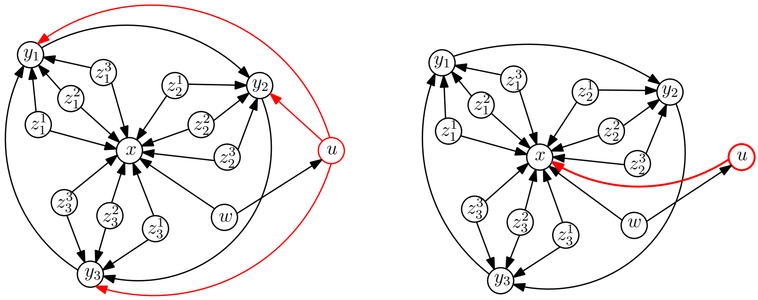

Fig. 3. The network $(G_k, k+1)$ for $k = 3$ and agent u's best response. Edges point away from its owner. For $k \to \infty$ agent u's improvement approaches a factor of $\frac{3}{2}$.

Now let us turn to the good news. We show that SUM-GEs cannot be arbitrarily unstable. On the contrary, they are very close to SUM-NEs in terms of stability.

Theorem 4. *Every network in* SUM-*GE is in* 3*-approximate* SUM-*NE.*

Proof. We prove Theorem 4 by providing a "locality gap preserving" reduction to the UNCAPACITATED METRIC FACILITY LOCATION problem (UMFL) [16]. Let u be an agent in (G, α) and let Z be the set of vertices in $V(G)$ which own an edge towards u. Consider the network (G', α), where all edges owned by agent u are removed. Observe, that the set Z is the same in (G, α) and (G', α). Let $\mathcal{S} = \{U \mid U \subseteq (V(G') \setminus \{u\}) \wedge U \cap Z = \emptyset\}$ denote the set of agent u's pure strategies in (G', α) which do not induce multi-edges or a self-loop. We transform (G', α) into an instance $I(G')$ for UMFL as follows:

Let $V(G') \setminus \{u\} = F = C$, where F is the set of facilities and C is the set of clients. For all facilities $f \in Z \cap F$ we define the opening cost to be 0, all other facilities have opening cost α. Thus, Z is exactly the set of cost 0 facilities in $I(G')$. For every $i, j \in F \cup C$ we define $d_{ij} = d_{G'}(i, j) + 1$. If there is no path between i and j in G', then we define $d_{ij} = \infty$. Clearly, since the distance in G' is metric we have that all distances d_{ij} in $I(G')$ are metric as well. See Fig. 4 for an example.

Now, observe that any strategy $S \in \mathcal{S}$ of agent u in (G', α) corresponds to the solution of the UMFL instance $I(G')$, where exactly the facilities in $F_S = S \cup Z$ are opened and where all clients are assigned to their nearest open facility. Moreover, every solution $F' = X \cup Z$, where $X \subseteq F \setminus Z$, for instance $I(G')$ corresponds to agent u's strategy $X \in \mathcal{S}$ in (G', α). Let $\mathcal{S}_{\text{UMFL}} = \{W \subseteq F \mid Z \subseteq$

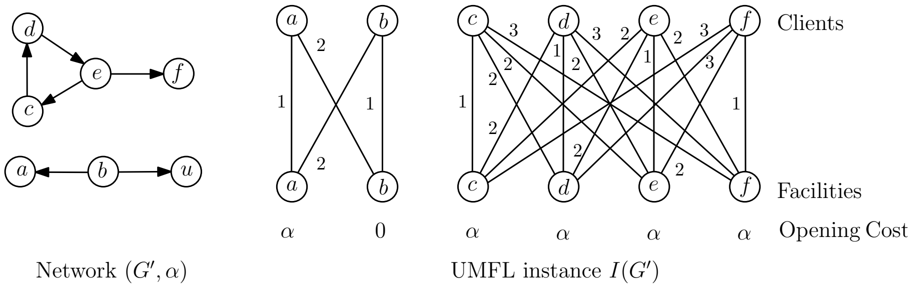

Fig. 4. Network (G', α) and its corresponding UMFL instance $I(G')$. Edges between clients and between facilities are omitted. All other omitted edges have length ∞.

$W\}$ denote the set of all solutions to instance $I(G')$, which open at least all cost 0 facilities. Hence, we have a bijection $\pi : \mathcal{S} \rightarrow \mathcal{S}_{\text{UMFL}}$, with $\pi(S) = S \cup Z$ and $\pi^{-1}(X) = X \setminus Z$. Let $\pi(S) = F_S$ and let (G_S, α) denote the network (G', α), where agent u has bought all edges towards vertices in S. Let $cost(F_S)$ denote the cost of the solution F_S to instance $I(G')$. We have that agent u's cost in (G_S, α) is equal to the cost of the corresponding UMFL solution F_S, since

$$\begin{aligned} c_u(G_S, \alpha) &= \alpha|S| + \sum_{w \in V(G_S)\setminus\{u\}} \left(1 + \min_{x \in S \cup Z} d_{G'}(x, w)\right) \\ &= \alpha|S| + 0|Z| + \sum_{w \in V(G_S)\setminus\{u\}} \min_{x \in S \cup Z} d_{xw} \\ &= \alpha|F_S \setminus Z| + 0|Z| + \sum_{w \in C} \min_{x \in F_S} d_{xw} \quad = cost(F_S). \end{aligned}$$

We claim the following: If agent u plays strategy $S \in \mathcal{S}$ and cannot decrease her cost by buying, deleting or swapping *one* edge in (G_S, α), then we have that the cost of the corresponding solution $F_S \in \mathcal{S}_{\text{UMFL}}$ to instance $I(G')$ cannot be strictly decreased by opening, closing or swapping *one* facility.

Proving the above claim suffices to prove Theorem 4. This can be seen as follows: For UMFL, Arya et al. [4] have already shown that the locality gap of UMFL is 3, that is, that any UMFL solution in which clients are assigned to their nearest open facility and which cannot be improved by opening, closing or swapping *one* facility is a 3-approximation of the optimum solution. By construction of $I(G')$, we have that every facility $z \in Z$ is the unique facility which is nearest to some client $w \in C$. Thus, we have that in any locally optimal and any globally optimal UMFL solution to $I(G')$ all cost 0 facilities must be open, since otherwise such a solution can be improved by opening a cost 0 facility. Hence, every locally or globally optimal solution to $I(G')$ has a corresponding strategy of agent u which yields the same cost. Using the claim and the result by Arya et al. [4], it follows that if agent u cannot decrease her cost by buying, deleting or swapping an edge in (G_S, α) then we have $c_u(G_S, \alpha) \leq 3c_u(G_{S^*}, \alpha)$, where S^* is agent u's optimal (non-greedy) strategy in (G', α) and (G_{S^*}, α) the network induced by S^*.

Now we prove the claim. Let $\pi(S) = F_S$. We have already shown that $c_u(G_S, \alpha) = cost(F_S)$. Furthermore, we have $Z \subseteq F_S$. We prove the contra-positive statement of the claim. Assume that solution F_S can be improved by opening, closing or swapping *one* facility. Let F'_S be this locally improved solution and let $cost(F'_S) < cost(F_S)$. Note, that $Z \subseteq F'_S$ must hold. This is true, since by construction of $I(G')$ closing a cost 0 facility increases the cost of any solution to $I(G')$. Hence, no facility $z \in Z$ can be included in a closing or swapping operation. It follows that the strategy $S' := \pi^{-1}(F'_S)$ exists. Observe, that $S = F_S \setminus Z$ and $S' = F'_S \setminus Z$ must differ by one element. Furthermore, by cost-equality, we have that $c_u(G_{S'}, \alpha) = cost(F'_S) < cost(F_S) = c_u(G_S, \alpha)$. Hence, agent u can buy, delete or swap one edge in (G_S, α) to decrease her cost. □

4 The Quality of Max Greedy Equilibria

In this section, we discuss the stability of networks in Max-GE. We will start by showing that operations of buying, deleting and swapping edges each may have a strong non-local flavor. See Fig. 5 for an illustration.

Lemma 5. *For $k \geq 2$ there is a network (G, α), where an agent can decrease her cost by buying/deleting/swapping k edges but not by buying/deleting/swapping $j < k$ edges.*

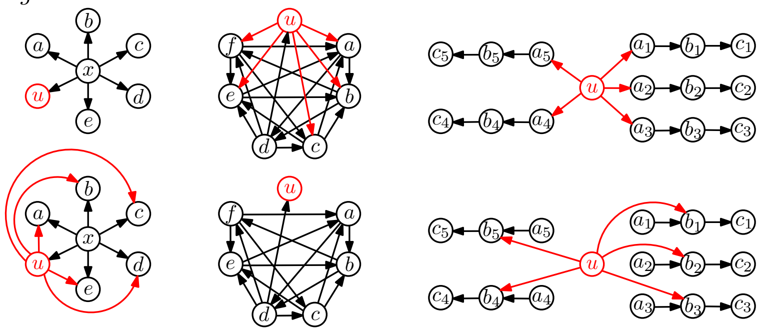

Fig. 5. The networks and strategy-changes for $k = 5$

Having seen Lemma 5, it should not come as a surprise that greedy local optimization may get stuck at sub-optimal states of the game.

4.1 Tree Networks in Max Greedy Equilibrium

The examples on the left and right side of Fig. 5 already show that there are tree networks, which are in Max-GE but not in Max-NE. In the following we show that this undesired behavior is restricted only to two families of tree networks in Max-GE. That is, we provide a characterization of all tree networks in Max-GE which are not in Max-NE. Furthermore, we show tight bounds on the stability for both mentioned families which are very close to the optimum. We start by introducing the main actors: *Cheap Stars* and *Badly Connected Trees.*

Definition 2 (Cheap Star). *A network (T, α) in* Max-*GE is called a* Cheap Star*, if T is a star having at least $n \geq 4$ vertices and $\alpha < \frac{1}{n-2}$. Furthermore, the ownership of all edges in T is arbitrary.*

Definition 3 (Badly Connected Tree). *A tree network* (T, α) *in* MAX-*GE is a* Badly Connected Tree *if there is an agent* $u \in V(T)$ *who can decrease her cost by swapping* $k > 1$ *own edges simultaneously.*

Intuitively, Cheap Stars owe their instability to a multi-buy operation, whereas Badly Connected Trees owe their instability to a multi-swap operation. Observe that Cheap Stars have diameter 2 and that Badly Connected Trees have diameter at least 3. Hence, these families are disjunct. The following theorem shows that Cheap Stars and Badly Connected Trees are the *only* tree networks in MAX-GE which are not in MAX-NE.

Theorem 5. *Let* (T, α) *be a network in* MAX-*GE, where* T *is a tree. The network* (T, α) *is in* MAX-*NE if and only if it is not a Cheap Star or a Badly Connected Tree.*

We can use the characterization provided by Theorem 5 to "circumvent" the hardness of deciding whether a tree network is in MAX-NE.

Theorem 6. *For every tree network* (T, α) *it can be checked in* $\mathcal{O}(n^4)$ *many steps whether* (T, α) *is in* MAX-*NE.*

We are interested in the stability of tree networks in MAX-GE. By Theorem 5, we only have to analyze the stability of Cheap Stars and Badly Connected Trees to get bounds on the stability on any tree network in MAX-GE.

Lemma 6. *Every Cheap Star is in* 2*-approximate* MAX-*NE. Furthermore, this bound is tight.*

Lemma 7. *Every Badly Connected Tree is in* $\frac{6}{5}$*-approximate* MAX-*NE. Furthermore, this bound is tight.*

Combining Theorem 5 with Lemma 6 and Lemma 7 we arrive at the following:

Theorem 7. *Let* (T, α) *be a tree network in* MAX-*GE. If* T *has diameter at most* 2*, then* (T, α) *is in* 2*-approximate* MAX-*NE. If* T *has diameter at least* 3*, then* (T, α) *is in* $\frac{6}{5}$*-approximate* MAX-*NE. Moreover, both bounds are tight.*

4.2 Non-tree Networks in Max Greedy Equilibrium

Fig. 5 (middle) shows that there are non-tree networks in MAX-GE, which are not in MAX-NE. We want to quantify the loss in stability of MAX-GEs versus MAX-NEs. For tree networks we have that Cheap Stars play a crucial role. These networks owe their instability to a multi-buy operation and to the fact that they are in MAX-GE for arbitrarily small α. We generalize this property of Cheap Stars to non-tree networks.

Definition 4 (Cheap Network). *A network* (G, α) *in* MAX-*GE, is called a* Cheap Network*, if* (G, α) *remains in* MAX-*GE when* α *tends to* 0*.*

Cheap Stars yield a lower bound on the stability approximation ratio which equals their diameter. We can generalize this observation:

Theorem 8. *If there is Cheap Network* (G, α) *having diameter* d*, then there is an* α^* *such that the network* (G, α^*) *is in* Max-*GE but not in* β*-approximate* Max-*NE for any* $\beta < d$*.*

Lemma 8. *There is a Cheap Network having diameter* 4*.*

Corollary 2. *For* $\alpha < 1$ *there is a network* (G, α) *in* Max-*GE, which is not in* β*-approximate* Max-*NE for any* $\beta < 4$*.*

Now we consider the case, where $\alpha \geq 1$. Quite surprisingly, it turns out that this case yields a very high lower bound on the approximation ratio.

Theorem 9. *For* $\alpha \geq 1$ *there is a* Max-*GE network* (G, α) *having* n *vertices, which is not in* β*-approximate* Max-*NE for any* $\beta < \frac{n-1}{5}$*.*

We give a family of networks in Max-GE each having an agent u who can decrease her cost by a factor of $\frac{n-1}{5}$ by a non-greedy strategy-change. The network (G_1, α) can be obtained as follows: $V(G_1) = \{u, v, l_1, l_2, a_1, a_2, b_1, b_2, x_1, y_1\}$ and agent u owns edges to a_1, a_2 and x_1. For $i \in \{1, 2\}$, agent b_i owns an edge to v and to a_i and agent l_i owns an edge to b_i. Finally, agent y_1 owns an edge to x_1 and to v. Fig. 6 (left) provides an illustration. To get the k-th member of the family, for $k \geq 2$, we simply add the vertices x_j, y_j, for $2 \leq j \leq k$, and let agent y_j own edges towards x_j and v. See Fig. 6 (right).

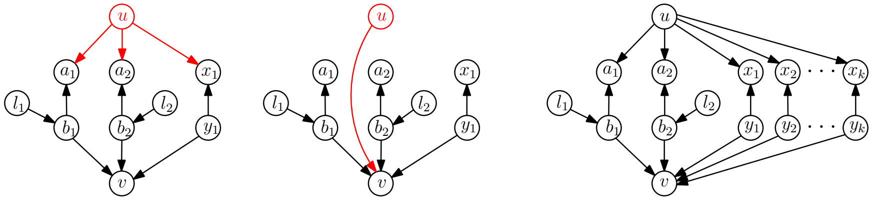

Fig. 6. (G_1, α) before (left) and after (middle) agent u's non-greedy strategy change and the network (G_k, α) (right)

Lemma 9. *Each of the networks* (G_i, α)*, as described above, is in* Max-*GE for* $1 \leq \alpha \leq 2$*.*

Proof (of Theorem 9). We focus on agent u in the network (G_k, α) and show that this agent can change her strategy in a non-greedy way and thereby decrease her cost by a factor of $\frac{n-1}{5}$, where n is the number of vertices of G_k. Let S_u be agent u's current strategy in (G_k, α) and let S_u^* be u's strategy which only buys one edge towards vertex v. See Fig 6 (left and middle). Let $cost(u)$ and $cost^*(u)$ denote agent u's cost induced by strategy S_u and S_u^*, respectively. For $\alpha = 2$, we have

$$\frac{cost(u)}{cost^*(u)} = \frac{\alpha(2+k)+3}{\alpha+3} = \frac{7}{5} + \frac{2k}{5} = \frac{n-1}{5},$$

where the last equality follows since $k = \frac{n-8}{2}$, by construction. □

Corollary 3. *Uncapacitated Metric Min-Max Facility Location has a locality gap of* $\frac{n-1}{5}$*, where* n *is the number of clients.*

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Group Activity Selection Problem

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Abstract. We consider a setting where one has to organize one or several group activities for a set of agents. Each agent will participate in at most one activity, and her preferences over activities depend on the number of participants in the activity. The goal is to assign agents to activities based on their preferences. We put forward a general model for this setting, which is a natural generalization of anonymous hedonic games. We then focus on a special case of our model, where agents' preferences are binary, i.e., each agent classifies all pairs of the form "(activity, group size)" into ones that are acceptable and ones that are not. We formulate several solution concepts for this scenario, and study them from the computational point of view, providing hardness results for the general case as well as efficient algorithms for settings where agents' preferences satisfy certain natural constraints.

1 Introduction

There are many real-life situations where a group of agents is faced with a choice of multiple activities, and the members of the group have differing preferences over these activities. Sometimes it is feasible for the group to split into smaller subgroups, so that each subgroup can pursue its own activity. Consider, for instance, a workshop whose organizers would like to arrange one or more social activities for the free afternoon.[1] The available activities include a hike, a bus trip, and a table tennis competition. As they will take place simultaneously, each attendee can select at most one activity (or choose not to participate). It is easy enough to elicit the attendees' preferences over the activities, and divide the attendees into groups based on their choices. However, the situation becomes more complicated if one's preferences may depend on the number of other attendees who choose the same activity. For instance, the bus trip has a fixed transportation cost that has to be shared among its participants, which implies that, typically, an attendee i is only willing to go on the bus trip if the number of other participants of the bus trip exceeds a threshold ℓ_i. Similarly, i may only be willing to play table tennis if the number of attendees who signed up for the tournament does *not*

[1] Some of the co-authors of this paper had to deal with this problem when co-organizing a Dagstuhl seminar.

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exceed a threshold u_i: as there is only one table, the more participants, the less time each individual spends playing.

Neglecting to take the number of participants of each activity into account may lead to highly undesirable outcomes, such as a bus that is shared by two persons, each of them paying a high cost, and a 48-participant table tennis tournament with one table. Adding constraints on the number of participants for each activity is a practical, but imperfect solution, as the agents' preferences over group sizes may differ: while some attendees (say, senior faculty) may be willing to go on the bus trip with just 4–5 other participants, others (say, graduate students) cannot afford it unless the number of participants exceeds 10. A more fine-grained approach is to elicit the agents' preferences over pairs of the form "(activity, group size)", rather than over activities themselves, and allocate agents to activities based on this information. In general, agents' preferences can be thought of as weak orders over all such pairs, including the pair "(do nothing, 1)", which we will refer to as the *void activity*. A simpler model, which will be the main focus of this paper, assumes that each agents classifies all pairs into ones that are acceptable to him and ones that are not, and if an agent views his current assignment as unacceptable, he prefers (and is allowed) to switch to the void activity (so the assignment is unstable unless it is acceptable to all agents).

The problem of finding a good assignment of agents to activities, which we will refer to as the *Group Activity Selection Problem* (GASP), may be viewed as a mechanism design problem (or, more narrowly, a voting problem) or as a coalition formation problem, depending on whether we expect the agents to act strategically when reporting their preferences. Arguably, in our motivating example the agents are likely to be honest, so throughout the paper we assume that the central authority knows (or, rather, can reliably elicit) the agents' true preferences, and its goal is to find an assignment of players to activities that, informally speaking, is stable and/or maximizes the overall satisfaction. This model is closely related to that of *anonymous hedonic games* [3], where, just as in our setting, players have to split into groups and each player has preferences over possible group sizes. The main difference between anonymous hedonic games and our problem is that, in our setting, the agents' preferences depend not only on the group size, but also on the activity that has been allocated to their group; thus, our model can be seen as a generalization of anonymous hedonic games. On the other hand, we can represent our problem as a general (i.e., non-anonymous) hedonic game [4,3], by creating a dummy agent for each activity and endowing it with suitable preferences (see Section 2.2 for details). However, our setting has useful structural properties that distinguish it from a generic hedonic game: for instance, it allows for succinct representation of players' preferences, and, as we will see, has several natural special cases that admit efficient algorithms for finding good outcomes.

In this paper, we initiate the formal study of GASP. Our goal is to put forward a model for this problem that is expressive enough to capture many real-life activity selection scenarios, yet simple enough to admit efficient procedures for finding good assignments of agents to activities. We describe the basic structure of the problem, and discuss plausible constraints of the number and type of available activities and the structure of agents' preferences. We show that even under a fairly simple preference model (where agents are assumed to approve or disapprove each available alternative) finding an assignment that maximizes the number of satisfied agents is computationally hard; however, we identify several natural special cases of the problem that admit efficient algorithms. We also briefly discuss the issue of stability in our setting.

We do not aim to provide a complete analysis of the group activity selection problem; rather, we view our work as a first step towards understanding the algorithmic and incentive issues that arise in this setting. We hope that our paper will lead to future research on this topic; to facilitate this, throughout the paper we highlight several possible extensions of our model as well as list some problems left open by our work.

2 Formal Model

Definition 1. *An instance of the* Group Activity Selection Problem (GASP) *is given by a set of* agents $N = \{1, \ldots, n\}$*, a set of* activities $A = A^* \cup \{a_\emptyset\}$*, where* $A^* = \{a_1, \ldots, a_p\}$*, and a* profile P*, which consists of* n votes *(one for each agent):* $P = (V_1, \ldots, V_n)$*. The vote of agent* i *describes his preferences over the set of* alternatives $X = X^* \cup \{a_\emptyset\}$*, where* $X^* = A^* \times \{1, \ldots, n\}$*; alternative* (a, k)*,* $a \in A^*$*, is interpreted as "activity* a *with* k *participants", and* $a_\emptyset$ *is the* void activity.

The vote V_i *of an agent* $i \in N$ *is (also denoted by* $\succeq_i$*) is a weak order over* X^**; its induced strict preference and indifference relations are denoted by* $\succ_i$ *and* $\sim_i$*, respectively. We set* $S_i = \{(a, k) \in X^* \mid (a, k) \succ_i a_\emptyset\}$*; we say that voter* i approves *of all alternatives in* S_i*, and refer to the set* S_i *as the* induced approval vote *of voter* i*.*

Throughout the paper we will mostly focus on a special case of our problem where no agent is indifferent between the void activity and any other alternative (i.e., for any $i \in N$ *we have* $\{x \in X^* \mid x \sim_i a_\emptyset\} = \emptyset$*), and each agent is indifferent between all the alternatives in* S_i*. In other words, preferences are* trichotomous*: the agent partitions* X *into three clusters* S_i*,* $\{a_\emptyset\}$ *and* $X \setminus (S_i \cup \{a_\emptyset\})$*, is indifferent between two alternatives of the same cluster, prefers any* (a, k) *in* S_i *to* $a_\emptyset$*, and* $a_\emptyset$ *to any* (a, k) *in* $X \setminus (S_i \cup \{a_\emptyset\})$*; we denote this special case of our problem by* a-GASP.

It will be convenient to distinguish between activities that are unique and ones that exist in multiple copies. For instance, if there is a single tennis table and two buses, then we can organize one table tennis tournament, two bus trips (we assume that there is only one potential destination for the bus trip, so these trips are identical), and an unlimited number of hikes (again, we assume that there is only one hiking route). This distinction will be useful for the purposes of complexity analysis: for instance, some of the problems we consider are easy when we have k copies of one activity, but hard when we have k distinct activities. Formally, we say that two activities a and b are *equivalent* if for every agent i and every $j \in \{1, \ldots, n\}$ it holds that $(a, j) \sim_i (b, j)$. We say that an activity $a \in A^*$ is k*-copyable* if A^* contains exactly k activities that are equivalent to a (including a itself). We say that a is *simple* if it is 1-copyable; if a is k-copyable for $k \geq n$, we will say that it is ∞*-copyable* (note that we would never need to organize more than n copies of any activity). If some activities in A^* are equivalent, A^* can be represented more succinctly by listing one representative of each equivalence class, together with the number of available copies. However, as long as we make the reasonable assumption that each activity exists in at most n copies, this representation is at most polynomially more succinct.

Our model can be enriched by specifying a set of *constraints* Γ. One constraint that arises frequently in practice is a *global cardinality* constraint, which specifies a bound K on the number of activities to be organized. More generally, we could also consider more complex constraints on the set of activities that can be organized simultaneously, which can be encoded, e.g., by a propositional formula or a set of linear inequalities. We remark that there can also be external constraints on the number of participants for each activity: for instance, a bus can fit at most 40 people. However, these constraints can be incorporated into agents' preferences, by assuming that all agents view the alternatives that do not satisfy these constraints as unacceptable.

2.1 Special Cases

We now consider some natural restrictions on agents' preferences that may simplify the problem of finding a good assignment. We first need to introduce some additional notation. Given a vote V_i and an activity $a \in A^*$, let $S_i^{\downarrow a}$ denote the projection of S_i onto $\{a\} \times \{1, \ldots, n\}$. That is, we set $S_i^{\downarrow a} = \{k \mid (a, k) \in S_i\}$.

Example 1. Let $A^* = \{a, b\}$ and consider an agent i whose vote V_i is given by $(a, 8) \succ_i (a, 7) \sim_i (b, 4) \succ_i (a, 9) \succ_i (b, 3) \succ_i (b, 5) \succ_i (a, 6) \succ_i (b, 6) \succ_i a_\emptyset \succ_i \ldots$ Then $S_i = \{a\} \times [6, 9] \cup \{b\} \times [3, 6]$ and $S_i^{\downarrow a} = \{6, 7, 8, 9\}$.

We are now ready to define two types of restricted preferences for a-GASP that are directly motivated by our running example, namely, *increasing* and *decreasing* preferences. Informally, under increasing preferences an agent prefers to share each activity with as many other participants as possible (e.g., because each activity has an associated cost, which has to be split among the participants), and under decreasing preferences an agent prefers to share each activity with as few other participants as possible (e.g., because each activity involves sharing a limited resource). Of course, an agent's preferences may also be increasing with respect to some activities and decreasing with respect to others, depending on the nature of each activity. We provide a formal definition for a-GASP only; however, it can be extended to GASP in a straightforward way.

Definition 2. *Consider an instance (N, A, P) of* a-GASP. *We say that the preferences of agent i are* increasing (INC) *with respect to an activity $a \in A^*$ if there exists a threshold $\ell_i^a \in \{1, \ldots, n + 1\}$ such that $S_i^{\downarrow a} = [\ell_i^a, n]$ (where we assume that $[n + 1, n] = \emptyset$). Similarly, we say that the preferences of agent i are* decreasing (DEC) *with respect to an activity $a \in A^*$ if there exists a threshold $u_i^a \in \{0, \ldots, n\}$ such that $S_i^{\downarrow a} = [1, u_i^a]$ (where we assume that $[1, 0] = \emptyset$).*

We say that an instance (N, A, P) of a-GASP *is* increasing *(respectively,* decreasing*) if the preferences of each agent $i \in N$ are increasing (respectively, decreasing) with respect to each activity $a \in A^*$. We say that an instance (N, A, P) of* a-GASP *is* mixed increasing-decreasing (MIX) *if there exists a set $A^+ \subseteq A^*$ such that for each agent $i \in N$ his preferences are increasing with respect to each $a \in A^+$ and decreasing with respect to each $a \in A^- = A^* \setminus A^+$.*

A recently proposed model which can be embedded into GASP with decreasing preferences is the ordinal version of *cooperative group buying* ([7], Section 6): the model has a set of buyers and a set of items with volume discounts; buyers rank all pairs (j, p_j) for any item j and any of its possible discounted prices, where the discounted price is a function of the number of buyers who are matched to the item.

For some activities, an agent may have both a lower and an upper bound on the acceptable group size: e.g., one may prefer to go on a hike with at least 3 other people, but does not want the group to be too large (so that it can maintain a good pace). In this case, we say that an agent has *interval* (INV) preferences; note that INC/DEC/MIX preferences are a special case of interval preferences.

Definition 3. *Consider an instance* (N, A, P) *of* a-GASP. *We say that the preferences of agent* i *are* interval (INV) *if for each* $a \in A^*$ *there exists a pair of thresholds* $\ell_i^a, u_i^a \in \{1, \ldots, n\}$ *such that* $S_i^{\downarrow a} = [\ell_i^a, u_i^a]$ *(where we assume that* $[i, j] = \emptyset$ *for* $i > j$*).*

Other natural constraints on preferences include restricting the size of S_i (or, more liberally, that of $S_i^{\downarrow a}$ for each $a \in A^*$), or requiring agents to have similar preferences: for instance, one could limit the number of agent *types*, i.e., require that the set of agents can be split into a small number of groups so that the agents in each group have identical preferences. We will not define such constraints formally, but we will indicate if they are satisfied by the instances constructed in the hardness proofs in Section 4.1.

2.2 GASP and Hedonic Games

Recall that a *hedonic game* [3,4] is given by a set of agents N, and, for each agent $i \in N$, a weak order $\geq_i$ over all coalitions (i.e., subsets of N) that include i. That is, in a hedonic game each agent has preferences over coalitions that he can be a part of. A coalition S, $i \in S$, is said to be *unacceptable* for player i if $\{i\} >_i S$. A hedonic game is said to be *anonymous* if each agent is indifferent among all coalitions of the same size that include him, i.e., for every $i \in N$ and every $S, T \subseteq N \setminus \{i\}$ such that $|S| = |T|$ it holds that $S \cup \{i\} \geq_i T \cup \{i\}$ and $T \cup \{i\} \geq_i S \cup \{i\}$.

At a first glance, it may seem that the GASP formalism is more general than that of hedonic games, since in GASP the agents care not only about their coalition, but also about the activity they have been assigned to. However, we will now argue that GASP can be embedded into the hedonic games framework.

Given an instance of the GASP problem (N, A, P) with $|N| = n$, where the i-th agent's preferences are given by a weak order $\succeq_i$, we construct a hedonic game $H(N, A, P)$ as follows. We create $n + p$ players; the first n players correspond to agents in N, and the last p players correspond to activities in A^*. The last p players are indifferent among all coalitions. For each $i = 1, \ldots, n$, player i ranks every non-singleton coalition with no activity players as unacceptable; similarly, all coalitions with two or more activity players are ranked as unacceptable. The preferences over coalitions with exactly one activity player are derived naturally from the votes: if S, T are two coalitions involving player i, x is the unique activity player in S, and y is the unique activity player in T, then i weakly prefers S to T in $H(N, A, P)$ if and only if $(x, |S| - 1) \succeq_i (y, |T| - 1)$, and i weakly prefers S to $\{i\}$ in $H(N, A, P)$ if and only if $(x, |S| - 1) \succeq_i a_\emptyset$. We emphasize that the resulting hedonic games are not anonymous. Further, while this embedding allows us to apply the standard solution concepts for hedonic games without redefining them, the intuition behind these solution concepts is not always preserved (e.g., because activity players never want to deviate). Therefore, in Section 3, we will provide formal definitions of the relevant hedonic games solution concepts adapted to the setting of a-GASP.

We remark that when A^* consists of a single ∞-copyable activity (i.e., there are n activities in A^*, all of them equivalent to each other), GASP become equivalent to anonymous hedonic games. Such games have been studied in detail by Ballester [2], who provides a number of complexity results for them. In particular, he shows that finding an outcome that is core stable, Nash stable or individually stable (see Section 3 for the definitions of some of these concepts in the context of a-GASP) is NP-hard. Clearly, all these complexity results also hold for GASP. However, they do not directly imply similar hardness results for a-GASP.

3 Solution Concepts

Having discussed the basic model of GASP, as well as a few of its extensions and special cases, we are ready to define what constitutes a solution to this problem.

Definition 4. *An* assignment *for an instance* (N, A, P) *of* GASP *is a mapping* $\pi : N \to A$*;* $\pi(i) = a_\emptyset$ *means that agent* i *does not participate in any activity. Each assignment naturally partitions the agents into at most* $|A|$ *groups: we set* $\pi^0 = \{i \mid \pi(i) = a_\emptyset\}$ *and* $\pi^j = \{i \mid \pi(i) = a_j\}$ *for* $j = 1, \ldots, p$*. Given an assignment* π*, the* coalition structure CS_π *induced by* π *is the coalition structure over* N *defined as follows:*

$$CS_\pi = \left\{\pi^j \mid j = 1, \ldots, p, \pi^j \neq \emptyset\right\} \cup \left\{\{i\} \mid i \in \pi^0\right\}.$$

Clearly, not all assignments are equally desirable. As a minimum requirement, no agent should be assigned to a coalition that he deems unacceptable. More generally, we prefer an assignment to be stable, i.e., no agent (or group of agents) should have an incentive to change its activity. Thus, we will now define several *solution concepts*, i.e., classes of desirable assignments. We will state our definitions for a-GASP only, though all of them can be extended to the more general case of GASP in a natural way. Given the connection to hedonic games pointed out in Section 2.2, we will proceed by adapting the standard hedonic game solution concepts to our setting; however, this has to be done carefully to preserve intuition that is specific to our model.

The first solution concept that we will consider is *individual rationality*.

Definition 5. *Given an instance* (N, A, P) *of* a-GASP*, an assignment* $\pi : N \to A$ *is said to be* individually rational *if for every* $j > 0$ *and every agent* $i \in \pi^j$ *it holds that* $(a_j, |\pi^j|) \in S_i$.

Clearly, if an assignment is not individually rational, there exists an agent that can benefit from abandoning his coalition in favor of the void activity. Further, an individually rational assignment always exists: for instance, we can set $\pi(i) = a_\emptyset$ for all $i \in N$. However, a benevolent central authority would usually want to maximize the number of agents that are assigned to non-void activities. Formally, let $\#(\pi) = |\{i \mid \pi(i) \neq a_\emptyset\}|$ denote the number of agents assigned to a non-void activity. We say that π is *maximum* *individually rational* if π is individually rational and $\#(\pi) \geq \#(\pi')$ for every individually rational assignment π'. Further, we say that π is *perfect*[2] if $\#(\pi) = n$. We denote the size of a maximum individually rational assignment for an instance (N, A, P) by $\#(N, A, P)$. In Section 4, we study the complexity of computing a perfect or maximum individually rational assignment for a-GASP, both for the general model and for the special cases defined in Section 2.1.

Besides individual rationality, there are a number of solution concepts for hedonic games that aim to capture stability against individual or group deviations, such as Nash stability, individual stability, contractual individual stability, and (weak and strong) core stability (see, e.g., [5]). In what follows, due to lack of space, we only provide the formal definition (and some results) for Nash stability. We briefly discuss how to adapt other notions of stability to our setting, but we leave the detailed study of their algorithmic properties as a topic for future work.

Definition 6. *Given an instance (N, A, P) of* a-GASP*, an assignment $\pi : N \to A$ is said to be* Nash stable *if it is individually rational and for every agent $i \in N$ such that $\pi(i) = a_\emptyset$ and every $a_j \in A^*$ it holds that $(a_j, |\pi^j| + 1) \notin S_i$.*

If π is not Nash stable, then there is an agent assigned to the void activity who wants to join a group that is engaged in a non-void activity, i.e., he would have approved of the size of this group and its activity choice if he was one of them. Note that a perfect assignment is Nash stable. The reader can verify that our definition is a direct adaptation of the notion of Nash stability in hedonic games: if an assignment is individually rational, the only agents who can profitably deviate are the ones assigned to the void activity. The requirement of Nash stability is much stronger than that of individual rationality, and there are cases where a Nash stable assignment does not exist (the proof is omitted due to space limits).

Proposition 1. *For each $n \geq 2$, there exists an instance (N, A, P) of* a-GASP *with $|N| = n$ that does not admit a Nash stable assignment. This holds even if $|A^*| = 1$ and all agents have interval preferences.*

In Definition 6 an agent is allowed to join a coalition even if the members of this coalition are opposed to this. In contrast, the notion of *individual stability* only allows a player to join a group if none of the existing group members objects. We remark that if all agents have increasing preferences, individual stability is equivalent to Nash stability: no group of players would object to having new members join.

A related hedonic games solution concept is *contractual individual stability*: under this concept, an agent is only allowed to move from one coalition to another if neither the members of his new coalition nor the members of his old coalition object to the move. However, for a-GASP contractual individual stability is equivalent to individual stability. Indeed, in our model no agent assigned to a non-void activity has an incentive to deviate, so we only need to consider deviations from singleton coalitions.

[2] The terminological similarity with the notion of perfect partition in a hedonic game [1] is not a coincidence; there a perfect partition assigns each agent to her preferred coalition; here a perfect assignment assigns each agent to one of her equally preferred alternatives.

The solution concepts discussed so far deal with individual deviations; resistance to group deviations is captured by the notion of the *core*. One typically distinguishes between *strong* group deviations, which are beneficial for each member of the deviating group, and *weak* group deviations, where the deviation should be beneficial for at least one member of the deviating group and non-harmful for others; these notions of deviation correspond to, respectively, *weak* and *strong* core. We note that in the context of a-GASP strong group deviations amount to players in π^0 forming a coalition in order to engage in a non-void activity. This observation immediately implies that every instance of a-GASP has a non-empty weak core, and an outcome in the weak core can be constructed by a natural greedy algorithm; we omit the details due to space constraints.

4 Computing Good Outcomes

In this section, we consider the computational complexity of finding a "good" assignment for a-GASP. We mostly focus on finding perfect or maximum individually rational assignments; towards the end of the section, we also consider Nash stability. Besides the general case of our problem, we consider special cases obtained by combining constraints on the number and type of activities (e.g., unlimited number of simple activities, a constant number of copyable activities, etc.) and constraints on voters' preferences (INC, DEC, INV, etc.). Note that if we can find a maximum individually rational assignment, we can easily check if a perfect assignment exists, by looking at the size of our maximum individually rational assignment. Thus, we will state our hardness results for the "easier" perfect assignment problem and phrase our polynomial-time algorithms in terms of the "harder" problem of finding a maximum individually rational assignment.

4.1 Individual Rationality: Hardness Results

We start by presenting four NP-completeness results, which show that finding a perfect assignment is hard even under fairly strong constraints on preferences and activities. We remark that this problem is obviously in NP, so in what follows we will only provide the hardness proofs.

Our first hardness result applies when all activities are simple and the agents' preferences are increasing.

Theorem 1. *It is* NP*-complete to decide whether* a-GASP *admits a perfect assignment, even when all activities in A^* are simple and all agents have increasing preferences.*

Proof (sketch). We provide a reduction from EXACT COVER BY 3-SETS (X3C). Recall that an instance of X3C is a pair $\langle X, \mathcal{Y}\rangle$, where $X = \{1, \dots, 3q\}$ and $\mathcal{Y} = \{Y_1, \dots, Y_p\}$ is a collection of 3-element subsets of X; it is a "yes"-instance if X can be covered by exactly q sets from $\mathcal{Y}$, and a "no"-instance otherwise. Given an instance $\langle X, \mathcal{Y}\rangle$ of X3C, we construct an instance of a-GASP as follows. We set $N = \{1, \dots, 3q\}$ and $A^* = \{a_1, \dots, a_p\}$. For each agent i, we define his vote V_i so that the induced approval vote S_i is given by $S_i = \{(a_j, k) \mid i \in Y_j, k \geq 3\}$, and let $P = (V_1, \dots, V_n)$. Clearly, (N, A, P) is an instance of a-GASP with increasing preferences. It is not hard to check that $\langle X, \mathcal{Y}\rangle$ is a "yes"-instance of X3C if and only if (N, A, P) admits a perfect assignment. □

Our second hardness result applies to simple activities and decreasing preferences, and holds even if each agent is willing to share each activity with at most one other agent.

Theorem 2. *It is* NP*-complete to decide whether* a-GASP *admits a perfect assignment, even when all activities in A^* are simple, all agents have decreasing preferences, and, moreover, for every agent $i \in N$ and every alternative $a \in A^*$ we have $S_i^{\downarrow a} \subseteq \{1, 2\}$.*

Proof (sketch). Consider the following restricted variant of the problem of scheduling on unrelated machines. There are n jobs and p machines. An instance of the problem is given by a collection of numbers $\{p_{ij} \mid i = 1, \ldots, n, j = 1, \ldots, p\}$, where p_{ij} is the running time of job i on machine j, and $p_{ij} \in \{1, 2, +\infty\}$ for every $i = 1, \ldots, n$ and every $j = 1, \ldots, p$. It is a "yes"-instance if there is a mapping $\rho : \{1, \ldots, n\} \to \{1, \ldots, p\}$ assigning jobs to machines so that the makespan is at most 2, i.e., for each $j = 1, \ldots, p$ it holds that $\sum_{i:\rho(i)=j} p_{ij} \leq 2$. This problem is known to be NP-hard (see the proof of Theorem 5 in [6]).

Given an instance $\{p_{ij} \mid i = 1, \ldots, n, j = 1, \ldots, p\}$ of this problem, we construct an instance of a-GASP as follows. We set $N = \{1, \ldots, n\}$, $A^* = \{a_1, \ldots, a_p\}$. Further, for each agent $i \in N$ we construct a vote V_i so that the induced approval vote S_i satisfies $S_i^{\downarrow a_j} = \{1\}$ if $p_{ij} = 2$, $S_i^{\downarrow a_j} = \{1, 2\}$ if $p_{ij} = 1$, and $S_i^{\downarrow a_j} = \emptyset$ if $p_{ij} = +\infty$. Clearly, these preferences satisfy the constraints in the statement of the theorem, and it can be shown that a perfect assignment for (N, A, P) corresponds to a schedule with makespan of at most 2, and vice versa. □

Our third hardness result also concerns simple activities and decreasing preferences. However, unlike Theorem 2, it holds even if each agent approves of at most 3 activities. The proof proceeds by a reduction from MONOTONE 3-SAT.

Theorem 3. *It is* NP*-complete to decide whether* a-GASP *admits a perfect assignment, even when all activities in A^* are simple, all agents have decreasing preferences, and, moreover, for every agent $i \in N$ it holds that $|\{a \mid S_i^{\downarrow a} \neq \emptyset\}| \leq 3$.*

Our fourth hardness result applies even when there is only one activity, which is ∞-copyable, and every agent approves at most two alternatives; however, the agents' preferences constructed in our proof do not satisfy any of the structural constraints defined in Section 2.1. The proof proceeds by a reduction from X3C.

Theorem 4. *It is* NP*-complete to decide whether* a-GASP *admits a perfect assignment, even when all activities in A^* are equivalent (i.e., A^* consists of a single ∞-copyable activity a) and for every $i \in N$ we have $|S_i^{\downarrow a}| \leq 2$.*

4.2 Individual Rationality: Easiness Results

The hardness results in Section 4.1 imply that if A^* contains an unbounded number of distinct activities, finding a maximum individually rational assignment is computationally hard, even under strong restrictions on agents' preferences (such as INC or DEC). Thus, we can only hope to develop an efficient algorithm for this problem if we assume that the total number of activities is small (i.e., bounded by a constant) or, more liberally, that the number of pairwise non-equivalent activities is small, and the agents' preferences satisfy additional constraints. We will now consider both of these settings, starting with the case where $p = |A^*|$ is bounded by a constant.

Theorem 5. *There exist an algorithm that given an instance of* a-GASP *finds a maximum individually rational assignment and runs in time* $(n + 1)^p\text{poly}(n)$.

Proof. We will check, for each $r = 0, \ldots, n$, if there is an individually rational assignment π with $\#(\pi) = r$, and output the maximum value of r for which this is the case. Fix an $r \in \{0, \ldots, n\}$. For every vector $(n_1, \ldots, n_p) \in \{0, \ldots, n\}^p$ that satisfies $n_1 + \cdots + n_p = r$ we will check if there exists an assignment of agents to activities such that for each $j = 1, \ldots, p$ exactly n_j agents are assigned to activity a_j (with the remaining agents being assigned to the void activity), and each agent approves of the resulting assignment. Each check will take poly(n) steps, and there are at most $(n+1)^p$ vectors to be checked; this implies our bound on the running time of our algorithm.

For a fixed vector $(n_1, \ldots, n_p)$, we construct an instance of the network flow problem as follows. Our network has a source s, a sink t, a node i for each player $i = 1, \ldots, n$, and a node a_j for each $a_j \in A^*$. There is an arc of unit capacity from s to each agent, and an arc of capacity n_j from node a_j to the sink. Further, there is an arc of unit capacity from i to a_j if and only if $(a_j, n_j) \in S_i$. It is not hard to see that an integral flow F of size r in this network corresponds to an individually rational assignment of size r. It remains to observe that it can be checked in polynomial time whether a given network admits a flow of a given size. □

Moreover, when A^* consists of a single simple activity a, a maximum individually rational assignment can be found by a straightforward greedy algorithm.

Proposition 2. *Given an instance* (N, A, P) *of* a-GASP *with* $A^* = \{a\}$, *we can find a maximum individually rational assignment for* (N, A, P) *in time* $O(s \log s)$, *where* $s = \sum_{i \in N} |S_i|$.

Proof. Clearly, (N, A, P) admits an individually rational assignment π with $\#(\pi) = k$ if and only if $|\{i \mid (a, k) \in S_i\}| \geq k$. Let $\mathcal{R} = \{(i, k) \mid (a, k) \in S_i\}$; note that $|\mathcal{R}| = s$. We can sort the elements of $\mathcal{R}$ in descending order with respect to their second coordinate in time $O(s \log s)$. Now we can scan $\mathcal{R}$ left to right in order to find the largest value of k such that $\mathcal{R}$ contains at least k pairs that have k as their second coordinate; this requires a single pass through the sorted list. □

Now, suppose that A^* contains many activities, but most of them are equivalent to each other; for instance, A^* may consist of a single k-copyable activity, for a large value of k. Then the algorithm described in the proof of Theorem 5 is no longer efficient, but this setting still appears to be more tractable than the one with many distinct activities. Of course, by Theorem 4, in the absence of any restrictions on the agents' preferences, finding a maximum individually rational assignment is hard even for a single ∞-copyable activity. However, we will now show that this problem becomes easy if we additionally assume that the agents' preferences are increasing or decreasing.

Observe first that for increasing preferences having multiple copies of the same activity is not useful: if there is an individually rational assignment where agents are assigned to multiple copies of an activity, we can reassign these agents to a single copy of this activity without violating individual rationality. Thus, we obtain the following easy corollary to Theorem 5.

Corollary 1. *Let* (N, A, P) *be an instance of* a-GASP *with increasing preferences where* A^* *contains at most* K *activities that are not pairwise equivalent. Then we can find a maximum individually rational assignment for* (N, A, P) *in time* $n^K \text{poly}(n)$.

If all preferences are decreasing, we can simply eliminate all ∞-copyable activities. Indeed, consider an instance (N, A, P) of a-GASP where some activity $a \in A^*$ is ∞-copyable. Then we can assign each agent $i \in N$ such that $(a, 1) \in S_i$ to his own copy of a; clearly, this will only simplify the problem of assigning the remaining agents to the activities.

It remains to consider the case where the agents' preferences are decreasing, there is a limited number of copies of each activity, and the number of distinct activities is small. While we do not have a complete solution for this case, we can show that in the case of a single k-copyable activity a natural greedy algorithm succeeds in finding a maximum individually rational assignment.

Theorem 6. *Given a decreasing instance* (N, A, P) *of* a-GASP *where* A^* *consists of a single* k*-copyable activity (i.e.,* $A^* = \{a_1, \ldots, a_k\}$*, and all activities in* A^* *are pairwise equivalent), we can find a maximum individually rational assignment in time* $O(n \log n)$.

Proof. Since all activities in A^* are pairwise equivalent, we can associate each agent $i \in N$ with a single number $u_i \in \{0, \ldots, n\}$, which is the maximum size of a coalition assigned to a non-void activity that he is willing to be a part of. We will show that our problem can be solved by a simple greedy algorithm. Specifically, we sort the agents in non-increasing order of u_is. From now on, we will assume without loss of generality that $u_1 \geq \cdots \geq u_n$. To form the first group, we find the largest value of i such that $u_i \geq i$, and assign agents $1, \ldots, i$ to the first copy of the activity. In other words, we continue adding agents to the group as long as the agents are happy to join. We repeat this procedure with the remaining agents until either k groups have been formed or all agents have been assigned to one of the groups, whichever happens earlier.

Clearly, the sorting step is the bottleneck of this procedure, so the running time of our algorithm is $O(n \log n)$. It remains to argue that it produces a maximum individually rational assignment. To show this, we start with an arbitrary maximum individually rational assignment π and transform it into the one produced by our algorithm without lowering the number of agents that have been assigned to a non-void activity. We will assume without loss of generality that π assigns all k copies of the activity (even though this is is not necessarily the case for the greedy algorithm).

First, suppose that $\pi(i) = a_\emptyset$, $\pi(j) = a_\ell$ for some $i < j$ and some $\ell \in \{1, \ldots, k\}$. Then we can modify π by setting $\pi(i) = a_\ell$, $\pi(j) = a_\emptyset$. Since $i < j$ implies $u_i \geq u_j$, the modified assignment is individually rational. By applying this operation repeatedly, we can assume that the set of agents assigned to a non-void activity forms a contiguous prefix of $1, \ldots, n$.

Next, we will ensure that for each $\ell = 1, \ldots, k$ the group of agents that are assigned to a_ℓ forms a contiguous subsequence of $1, \ldots, n$. To this end, let us sort the coalitions in π according to their size, from the largest to the smallest, breaking ties arbitrarily. That is, we reassign the k copies of our activity to coalitions in π so that $\ell < r$ implies $|\pi^\ell| \geq |\pi^r|$. Now, suppose that there exist a pair of players i, j such that $i < j$, $\pi(i) = a_\ell$, $\pi(j) = a_r$, and $\ell > r$ (and hence $|\pi^\ell| \leq |\pi^r|$). We have $u_j \geq |\pi^r| \geq |\pi^\ell|$, $u_i \geq u_j \geq |\pi^r|$, so if we swap i and j (i.e., modify π by setting $\pi(j) = a_\ell$, $\pi(i) = a_r$), the resulting assignment remains individually rational. Observe that every such swap increases the quantity $\Sigma = \sum_{t=1}^{k} \sum_{s\in\pi^t}(s \cdot t)$ by at least 1: prior to the swap, the contribution of i and j to Σ is $i\ell + jr$, ans after the swap it is $ir + j\ell > i\ell + jr$. Since for any assignment we have $\Sigma \leq kn(n+1)/2$, eventually we arrive to an assignment where no such pair (i, j) exists. At this point, each π^ℓ, $\ell = 1, \ldots, k$, forms a contiguous subsequence of $1, \ldots, n$, and, moreover, $\ell < r$ implies $i \leq j$ for all $i \in \pi^\ell$, $j \in \pi^r$.

Now, consider the smallest value of ℓ such that π^ℓ differs from the ℓ-th coalition constructed by the greedy algorithm (let us denote it by γ^ℓ), and let i be the first agent in $\pi^{\ell+1}$. The description of the greedy algorithm implies that π^ℓ is a strict subset of γ^ℓ and agent i belongs to γ^ℓ. Thus, if we modify π by moving agent i to π^ℓ, the resulting allocation remains individually rational (since i is happy in γ^ℓ). By repeating this step, we will gradually transform π into the output of the greedy algorithm (possibly discarding some copies of the activity). This completes the proof. □

The algorithm described in the proof of Theorem 6 can be extended to the case where we have one k-copyable activity a and one simple activity b, and the agents have decreasing preferences over both activities. For each $s = 1, \ldots, n$ we will look for the best solution in which s players are assigned to b; we will then pick the best of these n solutions. For a fixed s let $N_s = \{i \in N \mid (b, s) \in S_i\}$. If $|N_s| < s$, no solution for this value of s exists. Otherwise, we have to decide which size-s subset of N_s to assign to b. It is not hard to see that we should simply pick the agents in N_s that have the lowest level of tolerance for a, i.e., we order the agents in N_s by the values of u_i^a from the smallest to the largest, and pick the first s agents. We then assign the remaining agents to copies of a using the algorithm from the proof of Theorem 6. Indeed, any assignment can be transformed into one of this form by swapping agents so that the individual rationality constraints are not broken. It would be interesting to see if this idea can be extended to the case where instead of a single simple activity b we have a constant number of simple activities or a single k'-copyable activity.

We conclude this section by giving an $O(\sqrt{n})$-approximation algorithm for finding a maximum individually rational assignment in a-GASP with a single ∞-copyable activity.

Theorem 7. *There exists a polynomial-time algorithm that given an instance* (N, A, P) *of* a-GASP *where* A^* *consists of a single* ∞*-copyable activity* a*, outputs an individually rational assignment* π *with* $\#(\pi) = \Theta(\frac{1}{\sqrt{n}})\#(N, A, P)$.

Proof. We say that an agent i is *active* in π if $\pi(i) \neq a_\emptyset$; a coalition of agents is said to be *active* if it is assigned to a single copy of a. We construct an individually rational assignment π iteratively, starting from the assignment where no agent is active. Let $N^* = \{i \mid \pi(i) = a_\emptyset\}$ be the current set of inactive agents (initially, we set $N^* = N$). At each step, we find the largest subset of N^* that can be assigned to a single fresh copy of a without breaking the individual rationality constraints, and append this assignment to π. We repeat this step until the inactive agents cannot form another coalition.

Now we compare the number of active agents in π with the number of active agents in a maximum individually rational assignment π^*. To this end, let us denote the active coalitions of π by $B_1, \dots, B_s$, where $|B_1| \geq \dots \geq |B_s|$. If $|B_1| \geq \sqrt{n}$, we are done, so assume that this is not the case. Note that since B_1 was chosen greedily, this implies that $|C| \leq \sqrt{n}$ for every active coalition C in π^*.

Let $\mathcal{C}$ be the set of active coalitions in π^*. We partition $\mathcal{C}$ into s groups by setting $\mathcal{C}^1 = \{C \in \mathcal{C} \mid C \cap B_1 \neq \emptyset\}$ and $\mathcal{C}^i = \{C \in \mathcal{C} \mid C \cap B_i \neq \emptyset, C \notin \mathcal{C}^j \text{ for } j < i\}$ for $i = 2, \dots, s$. Note that every active coalition $C \in \pi^*$ intersects some coalition in π: otherwise we could add C to π. Therefore, each active coalition in π^* belongs to one of the sets $\mathcal{C}^1, \dots, \mathcal{C}^s$. Also, by construction, the sets $\mathcal{C}^1, \dots, \mathcal{C}^s$ are pairwise disjoint. Further, since the coalitions in $\mathcal{C}^i$ are pairwise disjoint and each of them intersects B_i, we have $|\mathcal{C}^i| \leq |B_i|$ for each $i = 1, \dots, s$. Thus, we obtain

$$\begin{aligned}\#(\pi^*) &= \sum_{i=1,\dots,s} \sum_{C \in \mathcal{C}^i} |C| \leq \sum_{i=1,\dots,s} \sum_{C \in \mathcal{C}^i} \sqrt{n} \\ &\leq \sum_{i=1,\dots,s} |\mathcal{C}^i| \sqrt{n} \leq \sum_{i=1,\dots,s} |B_i| \sqrt{n} \leq \#(\pi)\sqrt{n}. \qquad \square\end{aligned}$$

4.3 Nash Stability

We have shown that a-GASP does not not always admit a Nash stable assignment (Proposition 1). In fact, it is difficult to determine whether a Nash stable assignment exists. The proofs of the next two results are omitted due to space constraints.

Theorem 8. *It is* NP*-complete to decide whether* a-GASP *admits a Nash stable assignment.*

However, if agents' preferences satisfy INC, DEC, or MIX, a Nash stable assignment always exists and can be computed efficiently.

Theorem 9. *If (N, A, P) is an instance of* a-GASP *that is increasing, decreasing, or mixed increasing-decreasing, a Nash stable assignment always exists and can be found in polynomial time.*

Moreover, the problem of finding a Nash stable assignment that maximizes the number of agents assigned to a non-void activity admits an efficient algorithm if A^* consists of a single simple activity.

Theorem 10. *There exist a polynomial-time algorithm that given an instance (N, A, P) of* a-GASP *with $A^* = \{a\}$ finds a Nash stable assignment maximizing the number of agents assigned to a non-void activity, or decides that no Nash stable assignment exists.*

Proof. For each $k = n, \dots, 0$, we will check if there exists a Nash stable assignment π with $\#(\pi) = k$, and output the largest value of k for which this is the case.

For each $i \in N$, let $S'_i = S_i^{\downarrow a}$. For $k = n$ a Nash stable assignment π with $\#(\pi) = n$ exists if and only if $n \in S'_i$ for each $i \in N$. Assigning every agent to $a_\emptyset$ is Nash stable if and only if $1 \notin S'_i$ for each $i \in N$. Now we assume $1 \leq k \leq n-1$ and set

$U_1 = \{i \in N \mid k \in S'_i, k+1 \notin S'_i\}$, $U_2 = \{i \in N \mid k \notin S'_i, k+1 \in S'_i\}$, and $U_3 = \{i \in N \mid k \in S'_i, k+1 \in S'_i\}$. If $|U_1| + |U_3| < k$, there does not exist an individually rational assignment π with $\#(\pi) = k$. If $U_2 \neq \emptyset$, no Nash stable assignment π with $\#(\pi) = k$ can exist, since each agent from U_2 would want to switch. If $|U_3| > k$, no Nash stable assignment π with $\#(\pi) = k$ can exist, since at least one agent in U_3 would not be assigned to a and thus would be unhappy. Finally, if $|U_1| + |U_3| \geq k$, $|U_3| \leq k$, $U_2 = \emptyset$, we can construct a Nash stable assignment π by assigning all agents from U_3 and $k - |U_3|$ agents from U_1 to a. Since we have $\pi(i) = a_\emptyset$ for all i with $k \notin S'_i$ and $\pi(i) \neq a_\emptyset$ for all i with $k+1 \in S'_i$, no agent is unhappy. □

5 Conclusions and Future Work

We have defined a new model for the selection of a number of group activities, discussed its connections with hedonic games, defined several stability notions, and, for two of them, we have obtained several complexity results. A number of our results are positive: finding desirable assignments proves to be tractable for several restrictions of the problem that are meaningful in practice. Interesting directions for future work include exploring the complexity of computing other solution concepts for a-GASP and extending our results to the more general setting of GASP.

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Incentive Compatible Two Player Cake Cutting

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Abstract. We characterize methods of dividing a cake between two bidders in a way that is incentive-compatible and Pareto-efficient. In our cake cutting model, each bidder desires a subset of the cake (with a uniform value over this subset), and is allocated some subset. Our characterization proceeds via reducing to a simple one-dimensional version of the problem, and yields, for example, a tight bound on the social welfare achievable.

1 Introduction

The question of allocating resources among multiple people is one of the most basic questions that humans have been studying. At this level of generality one may say that most of the economic theory is devoted to this problem, as well as other fields of study. One class of scenarios of this form, with an enormous amount of literature, goes by the name of "cake cutting". In this type of scenario the goods are modeled as the (infinitely divisible) unit interval (the cake), the preferences as (measurable) valuation functions on the cake and the allocation as a partition of the cake. Many variants of this model have been considered and the usual goals are various notions of fairness and efficiency. See, e.g. [2] for an introduction.

Recently the research community has started looking at such models from a mechanism design point of view, i.e., considering the incentives of the players. From this perspective, players act *rationally* to maximize their utility and will thus "tell" the cake cutting algorithm whatever will make it maximize their own piece's value. In the simplest form[1] we would ask for an "incentive compatible" (equivalently, truthful or strategy-proof) cake cutting allocation mechanism where each bidder always maximizes his utility by reporting his true valuation.

Several recent papers have designed incentive-compatible cake cutting mechanisms. For example, in [4] an incentive-compatible, envy-free, Pareto-efficient,

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[1] Which by the revelation mechanisms is really without loss of generality since an arbitrary one, when analyzed at equilibrium, may be converted to an incentive-compatible one where truth is an equilibrium.

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and proportional cake cutting mechanism is obtained for the model where player valuations are "uniform": each player i desires a subset S_i of the items, and has a uniform value over this subset with the total value of each player normalized to 1.[2] In [8], an incentive-compatible, proportional, and Pareto-efficient mechanism is constructed for the case of arbitrary (not necessarily uniform) preferences. A "randomized" cake cutting mechanism that is truthful in expectation with better guarantees is also provided in that paper.

In this paper we seek to characterize incentive-compatible cake cutting mechanisms, and show bounds on possible performance measures. As our model has no "money" (i.e. no transferable utilities) the standard tools of mechanism design with quasi-linear utilities (such as Vickrey-Clarke-Groves [11,5,6] or Myerson [9]) do not apply. In this sense our work lies within the framework of approximate mechanism design without money, advocated, e.g., by [1,10]. As opposed to most of the cake cutting literature, we focus solely on incentive compatibility and efficiency and do not consider notions of fairness. As our results are mostly "negative", this only strengthens them. We should mention that the positive results that we provide, i.e. the mechanisms that have the "best" properties among all incentive-compatible ones, turn out to also be envy-free.

Our general model, following that of [4], considers an infinitely-divisible atomless cake and considers only the restricted class of uniform player valuations.[3] Formally, the "cake" is modeled as the real interval $[0, 1]$, each player desires a (measurable) set $A \subseteq [0, 1]$ and his valuation is uniform over that set (and normalized to 1): $V_A(S) = |S \cap A|/|A|$, where $|\cdot|$ specifies the usual Lebesgue measure. We restrict ourselves to "non-wasteful" mechanisms, where no piece that is desired by some player may be left unallocated and no piece is allocated to a player that does not want it (this is essentially equivalent to Pareto-efficiency of the outcome.[4]) We restrict ourselves to the case of two players. Thus a non-wasteful mechanism accepts as input the sets A and B desired by the two players and returns two disjoint (measurable) sets $C = C(A, B) \subseteq A$ and $D = D(A, B) \subseteq B$. In this case, the first player's utility is given by $V_A(C) = |C|/|A|$ and the second's by $V_B(D) = |D|/|B|$. A mechanism is called "incentive-compatible" if for every A, B and A' we get that $V_A(C(A, B)) \geq V_A(C(A', B))$ and similarly for the second player.

As a tool for studying this model, we introduce a simple, one-dimensional "aligned" model. In the aligned model we first restrict the possible player valuations: the first player desires the sub-interval $A = [0, a]$ and the second player desires the sub-interval $B = [1-b, 1]$. This is interesting when $1-b < a$ in which case the question is how to allocate the overlap $[1-b, a]$ between the players. We then also restrict the allowed allocation by the mechanism: the first player must be allocated an interval $C = [0, c]$ and the second an interval $D = [1 - d, 1]$.

[2] For purposes of efficient computation, it is also required that the sets would be given as a finite collection of intervals.

[3] Again, as our results are mostly "negative" this limited setting strengthens them.

[4] The inessential technical difference is detailed in the next section. While it does not *seem* that leaving pieces of the cake unallocated can be useful, whether this is really the case remains open.

Thus, in the aligned model the input is fully specified by its lengths a,b, the output by its lengths c,d, and a mechanism is a pair of real valued functions $f = (c(a, b), d(a, b))$. It turns out that these two restrictions offset each other in some sense, allowing us to convert mechanisms between the two models. As the aligned model is really single-dimensional, we are able to fully characterize incentive-compatible mechanisms in it, a characterization that then has strong implications in the general model as well.

Theorem 1. *(Characterization of Aligned Model) A non-wasteful deterministic mechanism for two-players in the aligned model is incentive-compatible if and only if it is from the following family, characterized by* $0 \leq \theta \leq 1$*: the allocation gives the first player the interval* $[0, \min\{a, \max\{1-b, \theta\}\}]$ *while the second player gets the interval* $[1 - \min\{b, \max\{1-a, 1-\theta\}\}, 1]$.

This characterization holds regardless of any issues of fairness, and the only mechanism in this family that is fair in any sense is that with $\theta = \frac{1}{2}$ which gives envy-freeness and turns out to be equivalent to the mechanism of [4] for the case of two players. This tight characterization in the aligned model allows the calculation of the best achievable results – under any desired performance measure – for incentive-compatible mechanisms. Specifically, we are interested in performance measures that depend on relative lengths of demands and allocations, formally on the set of 4-tuples $(\alpha, \beta, \gamma, \delta)$ where $\alpha = |A|/|A \cup B|$, $\beta = |B|/|A \cup B|$, $\gamma = |C|/|A \cup B|$, and $\delta = |D|/|A \cup B|$.[5] A typical performance measure of this form is the competitive ratio for social welfare: the worst case ratio between the social welfare achieved by the mechanism (which is $\gamma/\alpha + \delta/\beta$) and that achieved at the optimal allocation (which turns out to be $1 + (1 - \min\{\alpha, \beta\})/\max\{\alpha, \beta\}$). Many other variants can be considered, such as looking at other aggregations of the two players' utility (e.g. $\min\{\gamma/\alpha, \delta/\beta\}$ or $\log(\gamma/\alpha) + \log(\delta/\beta)$), assigning different weights to the different players, using a different comparison benchmark (e.g. the one splitting the intersection equally), using additive regret rather than multiplicative ratio, etc.

We prove the following reductions, which preserve the 4-tuples of ratios $(\alpha, \beta, \gamma, \delta)$, between these models.

Theorem 2. *(Reduction Between Models)*

1. *Let* $f = (c(a, b), d(a, b))$ *be an incentive-compatible and non-wasteful mechanism in the aligned model. There exists an incentive-compatible and non-wasteful mechanism* $F = (C(A, B), D(A, B))$ *in the general model such that for all* A, B*:* $|C(A, B)|/|A \cup B| = c(a, b)$ *and* $|D(A, B)|/|A \cup B| = d(a, b)$ *where* $a = |A|/|A \cup B|$ *and* $b = |B|/|A \cup B|$.
2. *Let* $F = (C(A, B), D(A, B))$ *be an incentive-compatible and non-wasteful mechanism in the general model. There exists an incentive-compatible and non-wasteful mechanism* $f = (c(a, b), d(a, b))$ *in the aligned model such that*

[5] Note that as $|A \cap B|/|A \cup B| = \alpha + \beta - 1$, $C \subseteq A$, $D \subseteq B$, $C \cap D = \emptyset$, and $A \cup B = C \cup D$ we have all the information regarding the sizes in the Venn diagram.

for all a, b *there exist* A, B *such that* $|A| = a$*,* $|B| = b$*,* $c(a,b) = |C(A,B)|$ *and* $d(a,b) = |D(A,B)|$*, and furthermore whenever* $a + b \geq 1$ *we have that* $A \cup B = [0,1]$*.*

These two reductions imply that while the general model may be (and actually is) richer, this richness cannot buy anything in terms of performance – for any notion of performance that depends on relative lengths of bids and allocations. For every mechanism with a certain performance level in the general model there exists a mechanism with the same performance level in the aligned model and vice-versa.

Thus our characterization in the aligned model implies the same bounds on performance in the general model as well. For example, in the aligned model, one may easily calculate that at most a fraction of $(8 - 4\sqrt{3})^{-1} \approx 0.93$ of social welfare can be extracted by any mechanism in the characterized family, and this competitive ratio is in fact obtained by the envy-free mechanism with $\theta = \frac{1}{2}$. The reductions imply that this same bound also applies to mechanisms in the general model. This ratio may thus be termed "the price of truthfulness" in this setting. A complementary result appears in [3], where the "price of fairness" is studied, comparing envy-free allocations to general ones, and obtaining the same numeric bound on the fraction of the optimal welfare that can be extracted by any *envy-free* allocation. Our results do not require any notion of fairness, but instead show that incentive-compatibility by itself implies this bound. In fact, for the special case of social welfare we also provide a direct proof for this bound, a proof that also applies to *randomized* mechanisms.

Theorem 3. *(Price of Truthfulness) Any deterministic or randomized incentive-compatible mechanism for cake cutting for two-players in the general model, achieves at most a* $(8 - 4\sqrt{3})^{-1} \approx 0.93$ *fraction of the optimal welfare for some player valuations.*

It should be noted that this is tight, as indeed the deterministic mechanism of [4] achieves this ratio when restricted to two players.

The paper is structured as follows: in section 2 we present our two models, the general one and the aligned one. Section 3 provides the characterization of the aligned model, and section 4 shows the reductions between the models. In section 5 we provide a direct proof of the price of truthfulness result for a randomized mechanism.

2 Models

2.1 The General Model

Our model has two players each desiring a measurable subset of $[0,1]$. We will denote by $A \subseteq [0,1]$ the set desired player I and by $B \subseteq [0,1]$ the set desired by the player II. We view A and B as private information. Everything else is common knowledge. The players will be assigned disjoint measurable subsets, $C \subseteq [0,1]$ to player I and $D \subseteq [0,1]$ to player II. We assume that player valuations are uniform over the subsets they desire and normalized to 1.

Definition 1. *The* valuation *of a player who desires subset $A \subseteq [0,1]$ for a subset $C \subseteq [0,1]$ is $V_A(C) = |C \cap A|/|A|$, where $|\cdot|$ specifies the Lebesgue measure.*

Definition 2. *A* mechanism *is a function which divides the cake between the two players. The function receives as inputs two measurable subsets of $[0,1]$: A and B (the demands of the players), and outputs two disjoint measurable subsets of $[0,1]$, C and D, where C is the subset that player I receives and D is the subset that player II receives.*

We denote a mechanism by $F(A,B) = (C(A,B), D(A,B))$, where $C(\cdot), D(\cdot)$ denote the functions that determine the allocations to the two players, respectively, and must satisfy $C(A,B) \cap D(A,B) = \emptyset$ for all A, B.

Our point of view is that the two players are strategic, aiming to maximize their valuation and since A and B are private information the players may "lie" to the mechanism regarding their real interest in the cake if that may give them an allocation with a higher valuation for them.

Definition 3. *$F = (C(A,B), D(A,B))$ is called* incentive-compatible *if none of the players can gain by declaring a subset which is different from the real subset he is interested in. Formally, for all A, B, A': $V_A(C(A,B)) \geq V_A(C(A',B))$ and similarly for the second player: for all A, B, B': $V_B(D(A,B)) \geq V_B(D(A,B'))$.*

Definition 4. *A mechanism $F = (C(A,B), D(A,B))$ is said to be* Pareto-efficient *if for every input A, B and the corresponding allocation made by the mechanism $C(A,B), D(A,B)$, any other possible allocation C', D' can not be strictly better for one of the players and at least as good for the other.*

Note that two possible allocations C, D and C', D', which differ only in the division of areas which none of the players is interested in, are equivalent in the eyes of the players. Therefore, we would use a specific Pareto-efficient allocation – a non-wasteful allocation, in which pieces of the cake that neither of the players demanded will not be allocated.

Definition 5. *A mechanism $F = (C(A,B), D(A,B))$ is called* non-wasteful *if for every A, B we have that $C(A,B) \subseteq A$, $D(A,B) \subseteq B$, and $C(A,B) \cup D(A,B) = A \cup B$.*

Proposition 1. *Every non-wasteful mechanism is Pareto-efficient. Every Pareto-efficient mechanism $F = (C(A,B), (D(A,B))$ can be converted to an equivalent non-wasteful one by defining $C'(A,B) = C(A,B) \cap A$ and $D'(A,B) = D(A,B) \cap B$.*

Thus any analysis of non-wasteful mechanisms directly implies a similar one for Pareto-efficient ones, as do all our results in this paper. For a non-wasteful mechanism the valuations of the players are simply $|C|/|A|$ for player I and $|D|/|B|$ for player II.

Although we do not deal directly with the envy-freeness of mechanisms, a mechanism that is described in this paper has this property, as described below.

Definition 6. *$F = (C(A,B), D(A,B))$ is called* envy-free *if each player weakly prefers the piece he received to the piece the other player received. Formally, for all A, B: $V_A(C(A,B)) \geq V_A(D(A,B))$ and similarly for the second player, for all A, B: $V_B(D(A,B)) \geq V_B(C(A,B))$.*

2.2 The Aligned Model

A special case of the above general model is called the *aligned* model. The model makes two specializing assumptions, one on player valuations, and the other on mechanism allocations:

1. The two players are interested in subsets of the form $[0, a]$ for player I and $[1 - b, 1]$ for player II.
2. The mechanism must divide te cake so that player I and player II would receive subsets of the form $[0, c]$ and $[1 - d, 1]$ respectively.

In the aligned model we denote a mechanism as $f(a, b) = (c(a, b), d(a, b))$, Where c, d are in fact functions c, d: $\mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that for all a, b: $c(a, b) + d(a, b) \leq 1$.

2.3 The Price of Truthfulness

As noted in the introduction, using the two reductions that will be proved in section 4, it is possible to study a family of performance measures for the aligned model and conclude from that implications for the general models. For example, one of these performance measures is the Price of Truthfulness.

Definition 7. *The* social welfare *of a mechanism $F = (C(A,B), D(A,B))$ on input A, B, denoted by $SW_F(A,B)$, is $SW_F(A,B) = V_A(C(A,B)) + V_B(D(A,B))$.*

Definition 8. *Denoted by $SW_{max}(A,B)$ is the sum of valuations of the two players in the allocation that maximizes social welfare: $SW_{max}(A,B) = \max_F SW_F(A,B)$.*

Definition 9. *The* competitive ratio *for social welfare of a mechanism F is $\eta_F = \min_{A,B} \eta_F(A,B)$, where $\eta_F(A,B) = \dfrac{SW_F(A,B)}{SW_{max}(A,B)}$.*

Similar to the price of anarchy, the *price of truthfulness* is the highest possible competitive ratio of a truthful mechanism. Formally:

Definition 10. *The price of truthfulness is $PoT \equiv \max_F \eta_F$, where F ranges over all non-wasteful truthful mechanisms.*

2.4 Randomized Mechanisms

In the last part of our paper we will also consider randomized mechanisms. For the purposes of this paper, one may either consider those as a probability distribution over deterministic mechanisms, or allow the mechanism's allocation (C, D) to be a random variable.

Definition 11. *For a randomized mechanism F, the above definitions are extended by replacing $SW_F(A,B)$ by $\mathbb{E}[SW_F(A,B)]$ where the expectation is over the random choices made by the mechanism.*

3 The Aligned Model

3.1 Characterization of the Aligned Model

Theorem 4. *(Characterization of Aligned Model) A non-wasteful deterministic mechanism for two-players in the aligned model is incentive-compatible if and only if it is from the following family, characterized by $0 \le \theta \le 1$: the allocation gives the first player the interval $[0, \min\{a, \max\{1-b, \theta\}\}]$ while the second player gets the interval $[1 - \min\{b, \max\{1-a, 1-\theta\}\}, 1]$.*

The remainder of this subsection is a proof of the above theorem.

Assume $f(a,b) = (c(a,b), d(a,b))$ is a non-wasteful incentive-compatible deterministic mechanism for two-players in the aligned model.

In case $a + b \le 1$, there is no overlap between the demands of the players which are aligned to the sides. Therefore, from non-wastefulness, the mechanism would have to give each player all of his demand (and that is clearly incentive-compatible and deterministic). We can also notice that this scenario matches the expressions for the pieces of the cake allocated to the players from the theorem, regardless of θ.

During the rest of this proof, we will assume that there is an overlap between the demands of the player, i.e. $a + b > 1$.

Definition 12. *For the mechanism $f(a,b) = (c(a,b), d(a,b))$ and a fixed demand b for player II, we will denote by $c_b(a)$ the function $c(a,b)$, which determines the size of the piece that the mechanism gives to player I according to his demands a. In a similar way $d_a(b)$ is also defined.*

Lemma 1. *For every b, the function $c_b(a)$ of the mechanism $f(a,b)$ is non decreasing and Lipschitz continuous (with a Lipschitz constant $K = 1$).*

Proof. For $a < a'$, say that $c_b(a) > c_b(a')$, then from non-wastefulness, $a' > a \ge c_b(a) > c_b(a')$. Therefore, if player I's real interest is a piece of size a', he can gain strictly more by demanding a instead. He would receive not only a larger piece of the cake, but also a larger piece of his interest, due to the alignment of the piece to the side. That stands in contradiction to the incentive-compatibility of the mechanism. Hence, $c_b(a) \le c_b(a')$, meaning that $c_b(a)$ is non decreasing.

Furthermore, for $a < a'$, $c_b(a') - c_b(a) \le a' - a$. Otherwise, if $c_b(a') - c_b(a) > a' - a$, this means that $c_b(a') - a' + a > c_b(a)$. Since the mechanism is non-wasteful, $c_b(a') \le a'$, and therefore $a > c_b(a)$. In such a case, if player I's real interest is of size a, he will not receive all of his demand. Therefore, he might lie and demand a' instead. By asking for a' he would receive a larger piece $(c_b(a') - a' + a > c_b(a) \Rightarrow c_b(a') > c_b(a))$, which because of the alignment, has a larger intersection with his real interest. Again, this contradicts the incentive-compatibility of the mechanism.

We have that $c_b(a)$ is Lipschitz continuous (with a Lipschitz constant $K = 1$).

Therefore $c_b(a)$ is continuous. Hence, in the interval $[0, 1]$ it must attain a maximum value, and the following quantities are well defined.

Definition 13. *$\mu(b)$ is the maximal piece size that player I can receive, when player II demands a piece of size b. Formally, $\mu(b) \equiv \max_a c_b(a)$.*

In the same way $\nu(a) \equiv \max_b d_a(b)$ is defined for player II.

Definition 14. *We will denote by a_m the minimal a for which $c_b(a_m) = \mu(b)$.*

Lemma 2. *For the mechanism $f(a, b)$ as mentioned, for every b:*

$$c_b(a) = \begin{cases} a & \text{for } a < \mu(b) \\ \mu(b) & \text{for } a \geq \mu(b) \end{cases} = \min\{a, \mu(b)\}$$

(see Figure 1)

Proof.

- For $a < a_m$, $c_b(a)$ can not be larger than a, because of the non-wastefulness of $f(a, b)$. If $c_b(a) < a$, then player I, whose real interest is of size a, does not receive all of his interest and therefore would prefer to lie and ask for a_m. Since $a < a_m$, by definition of a_m, $c_b(a) < c_b(a_m)$. Not only would player I receive a strictly larger piece by lying, since the piece is aligned to the side, he would also receive a strictly larger piece of his real interest. This stands in contradiction to the incentive-compatibility of f. Therefore, $c_b(a) = a$.
- For $a > a_m$, since $c_b(a)$ is non-decreasing, $c_b(a) \geq c_b(a_m)$. It is also known that $c_b(a_m) = \mu(b)$ is the maximal value of $c_b(a)$. Therefore, $c_b(a) = \mu(b)$.
- We showed that for $a < a_m$, $c_b(a) = a$, hence $c_b(a_m) = a_m$ by continuity. Since $c_b(a_m) = \mu(b)$, $a_m = \mu(b)$.

Putting everything together, we get that $c_b(a) = \min\{a, \mu(b)\}$.

Remark 1. Characterization of player II's piece size for a fixed a can be done in the same way to obtain $d_a(b) = \min\{b, \nu(a)\}$.

Now, we can continue to the characterization of the function $\mu(b)$.

Remark 2. We should notice that $\mu(1) + \nu(1) = 1$ (from non-wastefulness, in case both players want the whole cake we should divide the whole cake).

Lemma 3. *The function $\mu(b)$ must be of the form:*

$$\mu(b) = \begin{cases} 1 - b & \text{for } b < 1 - \theta \\ \theta & \text{for } b \geq 1 - \theta \end{cases} = \max\{1 - b, \theta\}$$

(For $\theta \in [0, 1]$). (see Figure 2)

Proof. According to the function $c_b(a)$, which we found earlier, the size of the piece that player I receives is $\min\{a, \mu(b)\}$. As mentioned in the beginning of the subsection, we assume that $a + b > 1$. As we are examining the aligned model, the mechanism should divide the whole interval $[0, 1]$. Therefore, player II would receive $1 - \min\{a, \mu(b)\}$. We also know that the form of the function $d_a(b)$ resembles the form of $c_b(a)$ and that means that the size of the piece that player II receives is $\min\{b, \nu(a)\}$. Combined together:

$$1 - \min\{a, \mu(b)\} = \min\{b, \nu(a)\} = \begin{cases} b & \text{for } b < \nu(a) \\ \nu(a) & \text{for } b \geq \nu(a) \end{cases}$$

$$\Rightarrow \min\{a, \mu(b)\} = \begin{cases} 1 - b & \text{for } b < \nu(a) \\ 1 - \nu(a) & \text{for } b \geq \nu(a) \end{cases}$$

Let us look at the last equation for $a = 1$:

$$\mu(b) = \min\{1, \mu(b)\} = \begin{cases} 1 - b & \text{for } b < \nu(1) \\ 1 - \nu(1) & \text{for } b \geq \nu(1) \end{cases}$$

We also know that $\mu(b)$ does not depend on a. Therefore, the last statement is true in general and not only for $a = 1$. We showed previously that $\mu(1)+\nu(1) = 1$. Let us denote $\theta \equiv \mu(1) = 1 - \nu(1)$, and rewrite $\mu(b)$ ($\nu(a)$ can be found in a similar way):

$$\mu(b) = \begin{cases} 1 - b & \text{for } b < 1 - \theta \\ \theta & \text{for } b \geq 1 - \theta \end{cases} = \max\{1 - b, \theta\}$$

$$\nu(a) = \begin{cases} 1 - a & \text{for } a < \theta \\ 1 - \theta & \text{for } a \geq \theta \end{cases} = \max\{1 - a, 1 - \theta\}$$

If we insert those $\mu(b)$ and $\nu(a)$ into the expressions for $c_b(a)$ and $d_a(b)$ that we found earlier, we would get that $c(a, b) = \min\{a, \max\{1 - b, \theta\}\}$ and $d(a, b) = \min\{b, \max\{1 - a, 1 - \theta\}\}$, as in the statement of the theorem.

In the opposite direction, it can be noticed that the allocation is deterministic. Furthermore, for all values of a, b and θ, each of the players either receives all of his demand, or a maximal value which depends only on the other player. Therefore, he cannot gain by lying. Moreover, $c(a, b)+d(a, b) = \min\{a+b, 1\}$, and because of the alignment of the interests and allocations, this type of allocation is non-wasteful.

We conclude that this is in fact the family of all possible mechanisms. We will denote by f_θ the mechanism with the parameter θ from that family.

3.2 Social Welfare in the Aligned Model

Theorem 5. *The non-wasteful and incentive-compatible deterministic mechanism $f_{\frac{1}{2}}$ for the aligned model achieves $\eta_{f_{\frac{1}{2}}} = (8 - 4\sqrt{3})^{-1} \approx 0.93$.*

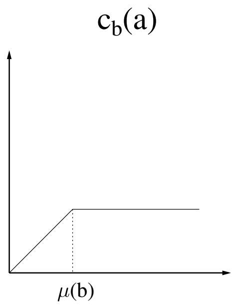

Fig. 1. The size of the allocation for player I as a function of his demand (for a constant b)

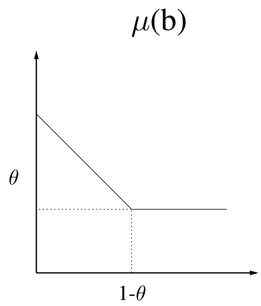

Fig. 2. The value in which the graph $c_b(a)$ (for a specific b) turns constant, as a function of that b)

This theorem is proved in the full version of the paper [7].

Remark 3. It is possible to prove that for every other mechanism f_θ from the family of mechanisms for the aligned model, $\eta_{f_\theta} < \eta_{f_{\frac{1}{2}}}$, by checking the value of η_{f_θ} for two possible sets of inputs: $\tilde{a} = 1, \tilde{b} = \sqrt{3} - 1$ and $\tilde{a} = \sqrt{3} - 1, \tilde{b} = 1$. We will prove a stronger theorem in the last section.

Moreover, it can be noticed that $\theta = \dfrac{1}{2}$ is the only θ for which f_θ is envy-free.

4 Reductions

4.1 Reduction From the Aligned to the General Model

Theorem 6. *Let $f = (c(a,b), d(a,b))$ be an incentive-compatible and non-wasteful mechanism for the aligned model. There exists an incentive-compatible and non-wasteful mechanism $F = (C(A,B), D(A,B))$ for the general model such that for all A, B: $|C(A,B)|/|A \cup B| = c(a,b)$ and $|D(A,B)|/|A \cup B| = d(a,b)$ where $a \equiv |A|/|A \cup B|$ and $b \equiv |B|/|A \cup B|$.*

Note that from the properties of f it has to be from the family of mechanisms described in the previous section. Therefore there is a θ such that f is f_θ.

For that f_θ, we will define mechanism $F(A,B)$ as follows:

– Use the mechanism f_θ to calculate the size of the players' allocations $(c(a,b), d(a,b))$ when:[6]
  - $a = \dfrac{|A|}{|A \cup B|}$, meaning player I demands the section $[0, \dfrac{|A|}{|A \cup B|}]$.
  - $b = \dfrac{|B|}{|A \cup B|}$, meaning player II demands the section $[1 - \dfrac{|B|}{|A \cup B|}, 1]$.

[6] The division by $|A \cup B|$ in this phase is a normalization of the original demands over a full $[0,1]$ interval.

– Calculate $|C(A,B)| \equiv c(a,b) \cdot |A \cup B|$ and $|D(A,B)| \equiv d(a,b) \cdot |A \cup B|$. [7]
– Give player I pieces in a total size of $|C(A,B)|$ and Player II pieces in a total size of $|D(A,B)|$. For each of them – start at first from giving the cake intervals that only he asked for, then move to intervals in the joint area.

The size of the piece that mechanism F would allocate to player I is: $|C(A,B)| = |A \cup B| \cdot \min\{\frac{|A|}{|A \cup B|}, \max\{1 - \frac{|B|}{|A \cup B|}, \theta\}\} = \min\{|A|, \max\{|A \cup B| - |B|, \theta \cdot |A \cup B|\}\} = \min\{|A|, \max\{|A \setminus B|, \theta \cdot |A \cup B|\}\}$. In a similar way we can get the expression for the size of player II's piece.

Lemma 4. *F is non-wasteful.*

Proof. The mechanism assigns two pieces with total size of $|C(A,B)|+|D(A,B)| = (c(a,b) + d(a,b)) \cdot |A \cup B| \underset{a+b \geq 1 \rightarrow c+d=1}{=} |A \cup B|$, meaning the total size that was assigned is equal to the total requested size. Moreover, $c(a,b) \leq a = \frac{|A|}{|A \cup B|}$, therefore $|C(A,B)| \leq |A|$ and in the same way $|D(A,B)| \leq |B|$. This means that the mechanism gives each player no more than his demand. Therefore, it is possible to construct the player's allocation only from intervals he has asked for. Since the allocation of those pieces starts with intervals that only one player asked for and because the total size allocated is $|A \cup B|$, the division is non-wasteful.

Lemma 5. *F is incentive-compatible.*

In this proof we examine a general subset A_1 which differs from the real interest of player I, A. We look at the symmetric difference between those two subsets, divide it into 4 disjoint sets, and one after the other show that zeroing the size of a set cannot damage the player. Therefore, he has no interest to lie. This theorem is proved in the full version of the paper [7].

Concluding, the mechanism F meets the demands of the theorem, thus completing the proof.

Say we choose f and examine the matching mechanism F, as described. If the inputs for mechanism F are A, B, we can look at the 4-tuple of ratios created by F: $\left(\frac{|A|}{|A \cup B|}, \frac{|B|}{|A \cup B|}, \frac{|C|}{|A \cup B|}, \frac{|D|}{|A \cup B|}\right)$. The above reduction shows that the inputs $a = \frac{|A|}{|A \cup B|}, b = \frac{|B|}{|A \cup B|}$ for mechanism f will result in the output $c = \frac{|C|}{|A \cup B|}, d = \frac{|D|}{|A \cup B|}$. Since $a + b = \frac{|A|}{|A \cup B|} + \frac{|B|}{|A \cup B|} \geq 1$ and since the requests are aligned to different sides, the total demand made by the two players is of size 1. Therefore, in this case, the 4-tuple of ratios is (a,b,c,d), which is identical to the 4-tuple that was obtained by F on the inputs A, B.

[7] A normalization of the results back to the original interval.

4.2 Reduction From the General to the Aligned Model

Theorem 7. *Let $F = (C(A,B), D(A,B))$ be an incentive-compatible and non-wasteful mechanism for the general model. There exists an incentive-compatible and non-wasteful mechanism $f = (c(a,b), d(a,b))$ for the aligned model, such that for all a, b there exist A, B such that $|A| = a$, $|B| = b$, $c(a,b) = |C(A,B)|$ and $d(a,b) = |D(A,B)|$, and furthermore whenever $a + b \geq 1$ we have that $A \cup B = [0,1]$.*

For mechanism $F(A,B)$ as mentioned, we will define mechanism $f(a,b)$ as follows:

- Find the division made by F in case both players want the whole cake: $F([0,1],[0,1]) = (\tilde{C}, \tilde{D})$. Since F is non-wasteful, $\tilde{C} \uplus \tilde{D} = [0,1]$.
- Denote $c(a,b) \equiv \min\left\{a, \max\left\{1-b, |\tilde{C}|\right\}\right\}$
- Denote $d(a,b) \equiv \min\left\{b, \max\left\{1-a, 1-|\tilde{C}|\right\}\right\}$
- Give players I and II pieces $[0, c(a,b)]$ and $[1-d(a,b), 1]$ respectively.

Lemma 6. $f = (c(a,b), d(a,b))$ *is non-wasteful and incentive-compatible.*

Proof. f receives an aligned players' input and divides the cake into aligned pieces. The size $|\tilde{C}|$ is between 0 and 1 (similar to θ). The sizes of the pieces that the players receive ($c(a,b)$ and $d(a,b)$) match the family of mechanisms that was mentioned in the section about the aligned model, for $\theta = |\tilde{C}|$. Therefore, f is, in fact, the mechanism $f_{|\tilde{C}|}$ from that family. We already know that for aligned players' valuation function (as in this case), mechanisms from that family are non-wasteful and incentive-compatible.

Lemma 7. *For $F = (C(A,B), D(A,B))$ and $f = (c(a,b), d(a,b))$ as defined, for all a, b there exists A, B such that $|A| = a$, $|B| = b$, $c(a,b) = |C(A,B)|$ and $d(a,b) = |D(A,B)|$, and furthermore whenever $a + b \geq 1$ we have that $A \cup B = [0,1]$.*

This lemma is proved in the full version of the paper [7].

Concluding, the mechanism f meets the demands of the theorem, completing the proof.

Say we choose F and examine the matching mechanism f, as described. Denote the inputs of mechanism f as a, b. If $a + b \leq 1$, choosing $A = [0,a], B = [1-b, 1]$ as inputs for F will result in each of the players receiving all of his demand, causing an identical 4-tuple of ratios for the two mechanisms: $\left(\frac{a}{a+b}, \frac{b}{a+b}, \frac{a}{a+b}, \frac{b}{a+b}\right)$. If $a + b > 1$, the union of the players' demands is of size 1. The theorem shows that there are A, B such that $|A| = a, |B| = b, |C(A,B)| = c(a,b), |D(A,B)| = d(a,b)$ and furthermore, $|A \cup B| = 1$. Therefore, the ratio 4-tuples obtained by $f(a,b)$ and $F(A,B)$ (for the specific A and B suggested in the theorem) are identical: (a,b,c,d).

5 The Price of Truthfulness

As was mentioned in Remark 3, it is possible to show that for any $0 \leq \theta \leq 1$, $\theta \neq \frac{1}{2}$, the competitive ratio of the social welfare of the mechanism f_θ (marked as η_{f_θ}), is $< (8-4\sqrt{3})^{-1}$. Using the two reductions from the last section, we can conclude that there isn't a non-wasteful, incentive-compatible, deterministic mechanism for the general model with higher η. Moreover, Since $\eta_{f_{\frac{1}{2}}} = (8-4\sqrt{3})^{-1}$, there is an incentive-compatible, non-wasteful deterministic mechanism F for general model[8] which achieves $\eta_F = (8-4\sqrt{3})^{-1} \approx 0.93$.

We will now prove a stronger claim - this upper bound still holds even if the mechanism can be wasteful or randomized, as long as the valuation functions are of the same form which we defined in the general model (actually, the exact proof is even stronger and also works even if the players are limited only to the aligned model's valuation functions).

Theorem 8. *(Price of Truthfulness) Any deterministic or randomized incentive-compatible mechanism for cake cutting for two-players in the general model, achieves at most a* $(8-4\sqrt{3})^{-1} \approx 0.93$ *fraction of the optimal welfare for some player valuations.*

Proof. Say each of the two players' real demand is the whole cake: $[0,1]$. We will denote by p and q the expected sizes of the pieces of cake that the mechanism gives player I and player II in that case, respectively. W.l.o.g we assume that player I received the (weakly) smaller piece, $p \leq q$ and since $p+q \leq 1$, $p \leq \frac{1}{2}$.

Now, we will examine what happens if player I's demand is $A = [0, 1-\tau]$ for some $0 \leq \tau \leq 1$, and player II's demand remains unchanged. Intuitively, in order to maximize the social welfare, as a player demands a smaller piece, the mechanism needs to give him a larger allocation (in case he really asks for his real demand). However, from incentive-compatibility, the size of the piece that player I will receive can not be greater than p (if it did, it would have been better for him to lie in the previous case and ask for the smaller piece instead of the whole cake). We denote by p', q' the expected size of the pieces that the players receive in that case.

Since $\eta_F = \min_{A,B} \frac{\mathbb{E}\left[SW_F(A,B)\right]}{SW_{max}(A,B)}$, and we are checking only a specific subset of inputs (of the form $A = [0, 1-\tau], B = [0,1]$), we can say that for each of those A, B:

$$\eta_F \leq \frac{\mathbb{E}\left[SW_F(A,B)\right]}{SW_{max}(A,B)} = \frac{\frac{p'}{1-\tau} + \frac{q'}{1}}{\frac{1-\tau}{1-\tau} + \frac{\tau}{1}} \underset{\substack{p' \leq p, 1-\tau \leq 1 \\ q' \leq 1-p'}}{\leq}$$

[8] This mechanism is F that is generated by reduction 4.2 using the mechanism $f_{\frac{1}{2}}$.

$$\frac{\frac{p}{1-\tau}+1-p}{1+\tau} = \frac{1+p\left(\frac{1}{1-\tau}-1\right)}{1+\tau} \underset{\substack{\frac{1}{1-\tau}-1>0 \\ p\leq\frac{1}{2}}}{\leq} \frac{\frac{\frac{1}{2}}{1-\tau}+\frac{1}{2}}{1+\tau}$$

The minimal value of this expression is $(8-4\sqrt{3})^{-1}$ at $\tau = 2-\sqrt{3}$.

Therefore, $\eta_F \leq (8-4\sqrt{3})^{-1}$

We remark again – there exists a mechanism F, in the general model, which achieves the bound for a mechanism in that model, $(8-4\sqrt{3})^{-1} \approx 0.93$. This is the price of truthfulness.

On top of being incentive compatible, this mechanism is also deterministic, non-wasteful and envy-free.

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LP-Based Covering Games with Low Price of Anarchy

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Abstract. We design a new class of vertex and set cover games, where the price of anarchy bounds match the best known constant factor approximation guarantees for the centralized optimization problems for linear and also for submodular costs. This is in contrast to all previously studied covering games, where the price of anarchy grows linearly with the size of the game. Both the game design and the price of anarchy results are based on structural properties of the linear programming relaxations. For linear costs we also exhibit simple best-response dynamics that converge to Nash equilibria in linear time.

1 Introduction

Distributed algorithms is a developing field that tries to address the novel computational challenges of networked environments, such as the Internet. The purpose of these systems is to coordinate numerous computational resources so as to achieve a common goal, such as solving a large optimization problem. There is a wide spectrum of approaches within distributed algorithm design, depending on the level of coordination between different computational elements. On one extreme, complete coordination is equivalent to centralized algorithms, whereas in the other extreme, sometimes referred to as decentralized computation, there exists no common goal but only local ones that depend on the information available in the immediate neighborhood of each computational element. Such solutions, when they exist, are highly sought after since they can tolerate failures in individual subsystems, as well as evolving network topologies, including the arbitrary addition/deletion of computational elements.

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A fundamental trend in mechanism design, often called *implementation theory*, is to construct games where the Nash equilibria correspond to globally desirable outcomes (see e.g. [16,8,30]). By appropriately setting up the rules and payoffs, we can enforce the selfishly acting agents to arrive at an outcome realizing social goals. An important assumption in these works is the existence of an omnipotent central authority that collects the players' choices, determines and enforces the outcome. In contrast, we put emphasis on the distributed aspect: we do not allow computations performed by a central authority and use of global information, but require that the incentives of each agent/node need to be efficiently computable based on information available in its immediate neighborhood.

The standard measure of the efficiency in algorithmic game theory is the *price of anarchy* [26], the maximal ratio between the social cost of a Nash equilibrium and that of the global optimal configuration. Intuitively, a low price of anarchy implies that upon converging to a socially stable outcome, the quality of the acquired solution is almost optimal from a central optimization perspective.

Games for vertex cover and set cover problems have already been studied, however, all these approaches exhibit prohibitively high price of anarchy. Specifically, Cardinal and Hoefer in [6] define a vertex cover game where the edges of a network are owned by k agents. An agent's goal is to have each of his edges supplied by a service point at least one of its endpoints. There is a cost $c(v) \geq 0$ associated to building a service point at vertex v. The strategy of an agent is a vector consisting of offers to the vertices. Service points will be installed at vertices where the total offer exceeds the cost of the vertex. Similar games are defined by Buchbinder et al. [5] and by Escoffier et al. [11] for the more general set cover problem.

A different approach was followed by Balcan et al. [2]. Here the agents are the vertices of the graph, and their strategies are deciding whether they open a service point. If opening a service point, vertex v incurs a cost $c(v)$. If he decides not to open, he has to pay a penalty for all edges incident to v whose other endpoints are uncovered.

The price of anarchy is $\Theta(k)$ in [6] and $\Theta(n)$ in [2]. Indeed, if the underlying network is a star, and each edge is owned by a different agent in the first case, we get Nash equilibria with all leaves being service points. These guarantees are significantly worse than the ones available in the centralized setting, where simple factor 2-approximation algorithms exist [3].

We close this gap completely by designing a distributed game with low information burden (i.e. the utilities of the agents depends on the state of their local neighborhood/subnetwork) and high efficiency (i.e. low price of anarchy).

Specifically, we propose games (called "Mafia games") for covering problems with the price of anarchy being equal to the best constant factor approximation algorithms for the central optimization problems. We prove the following theorem which is the summary of Theorems 3, 4, 5 and 6.

Theorem 1. *The Mafia games for vertex cover, hitting set and submodular hitting set always have pure Nash equilibria, and the price of anarchy is 2 for vertex cover and d for (submodular) hitting set, where d is the maximum set cardinality.*

An important feature of our games is their distributed nature: we assume no central authority for computing the outcome, and the payoff of every player depends only on the decisions of their close neighbourhoods. We believe that this technique may have a wider applicability to extend approximation results from combinatorial optimization to the field of algorithmic game theory and decentralized algorithm design.

Let us give a brief informal description of our vertex cover game. The agents are the vertices, and the regulations delegate the responsibility of covering every edge of the network to its two endpoints: both incur a high penalty if the edge is left uncovered. The agents who open a service point can demand compensation from their neighbors. This is justified since if u opens a service point, every neighbor v benefits from this as the common responsibility of covering uv is taken over by u. In the description, we use intuitive terminology of a Mafia (service points) which provides "security" (covers edges). The vertices may choose to join Mafia or to remain civilians. Each edge of the graph has to be "secured", that is, at least one endpoint must be in Mafia. Mafiosi can collect ransoms as the price of security of the incident edges: if a vertex v chooses to be a mafioso, his strategy also includes a ransom vector, so that the total ransom he demands from his neighbors is $c(v)$. It is a one-shot game and mafiosi can ransom both their civilian and mafioso neighbors.

If v is a civilian, he has to pay to his neighbors in the Mafia all ransom they demand. Furthermore, if there is an incident uncovered edge uv, that is, u is also a civilian, both of them have to pay a huge penalty. In contrast, if v is a mafioso, he has to pay $c(v)$ for joining, and he receives whatever he can collect from ransoms. However, mafiosi ransomed excessively obtain a protected status: if the total demand from v is more than $c(v)$, he satisfies only a proportional fraction of the demands.

The model avoids bad Nash equilibria that are possible in [6] and [2]. As an example, consider a star with all vertices having cost 1. We cannot have a Nash equilibrium with the leaves forming the Mafia and the central vertex being the single civilian: all leaves would demand ransom from the central agent, who would then have a strong incentive to join the Mafia in order to obtain the protected status.

As a different interpretation of the game, consider a road network with the vertices representing cities. The maintenance of a road must be provided by a facility at one of the endpoints. The cost of opening the facility dominates the operating cost: if city v decides to open one at cost $c(v)$, it is able to maintain all incident roads. As a compensation, the cities can try to recollect the opening cost by asking contributions from their neighbors. A city without a facility has to pay all contributions he is asked to pay. However, if a city opens a facility, its liability is limited and has to satisfy demands only up to his opening cost, $c(v)$.

Our approach can be extended to the hitting set problem, which is equivalent to the set cover problem. We are given a hypergraph $G = (V, \mathcal{E})$, and a cost function $c : V \to \mathbb{R}_+$ on the vertices. Our aim is to find a minimum cost subset M of V intersecting every hyperedge in $\mathcal{E}$. This problem is known to be approximable within a factor of d, the maximum size of a hyperedge. In the corresponding Mafia game, the hyperedges shall be considered as clubs in need of security. A mafioso can assign ransoms to the clubs he is a member of, that will be distributed equally to all other members of the club.

We shall prove that for the vertex cover and hitting set games, the price of anarchy is 2 and d, respectively. Bar-Yehuda and Even gave a simple primal-dual algorithm with this guarantee in 1981 [3]. No better constant factor approximation has been given ever since. Furthermore, assuming the Unique Games Conjecture, Khot and Regev [20] proved that the hitting set problem cannot be approximated by any constant factor smaller than d.

As a further extension, we also investigate the submodular hitting set (or set cover) problem, that has received significant attention recently. The goal is to find a hitting set M of a hypergraph minimizing $C(M)$ for a submodular set function C on the ground set. Independently, Koufogiannakis and Young [25] and Iwata and Nagano [19] gave d-approximation algorithms. Our game approach extends even to this setting, with the same price of anarchy d. This involves a new agent, the Godfather, who's strategy consists of setting a budget vector in the submodular base polyhedron of C. Otherwise, the game is essentially the same as the (linear) hitting set game.

Convergence and Complexity of Dynamics. The above price of anarchy results imply that any Nash converging protocol, will reach an almost optimal cover. However, there exist no a priori convergence speed guarantees for such protocols in general games. So, in order to complete the picture we need to argue about the convergence properties and speed of reasonable game dynamics such as that of best response.

Indeed, in our covering games, we first show that even in simple instances, round robin best-response dynamics[1] may end in a loop. However, this can be simply fixed by a slight modification of the payoff. We introduce a tie-breaking rule for choosing amongst best responses, that does not affect the price of anarchy results, but merely instigates the mafiosi to use more fair (symmetric) ransoms. Given this breaking of ties, we show that actually a single round of best-response dynamics under a simple selection rule of the next agent results in a Nash-equilibrium. This dynamics in fact simulates the Bar-Yehuda–Even algorithm. An analogous dynamics is shown in the case of hitting set. Moreover, these dynamics can be interpreted in a distributed manner, enabling several agents to change their strategies at the same time.[2]

We also state the following theorem (and omit the proof due to paper size limits).

Theorem 2. *For the vertex cover and hitting set covering games, there is a best-response sequence of $\mathcal{O}(n)$ moves, such that we reach a Nash equilibrium.*

[1] These are the dynamics where each agent takes turn playing his best-response in a cyclic ordering according to some fixed permutation.

[2] In our games, the set of strategies is infinite as ransoms can be arbitrary real numbers. However, if the vertex weights are integers, we can restrict possible ransoms to be integers as well. All results of the paper straightforwardly extend to this finite game.

Related Work. There is a vast literature on implementation theory and on distributed algorithmic mechanism design. Here we only focus on literature related to covering games. The basic set cover games in [5], [11] and [2] fall into the class of congestion games [32]. In the models of [5], [11], in the hitting set terminology, the agents are the hyperedges that choose a vertex to cover them, and the cost of the vertex is divided among them according to some rule. [5] investigates the influence of a central authority that can influence choices by taxes and subsidies in a best-response dynamics; [11] studies different cost sharing rules of the vertices ("local taxes"). However, none of these methods achieve a constant price of anarchy. The model of [2] can achieve a good equilibrium by assuming a central authority that propagates information on an optimal solution to a fraction of the agents. In contrast to [5] and [2], our model is defined locally, without assuming a central authority.

Cardinal and Hoefer [7] define a general class of covering games, including the vertex cover game [6], and also the selfish network design game by Anshelevich et al. [1]. The game is based on a covering problem given by a linear integer program. Variables represent resources, and the agents correspond to certain sets of constraints they have to satisfy. An agent can offer money for resources needed to satisfy her constraints. From each variable, the number of units covered by the total offers of the agents will be purchased and can be used by all agents simultaneously to satisfy their constraints, regardless to their actual contributions to the resource. Further generalizations of this model were studied by Hoefer [15], and by Harks and Peis [14], investigating settings where the price of each resource may depend on the number of players using it.

In the vertex cover or hitting set game, the resources are the service points and the set of constraints belonging to the agents express that every (hyper)edge owned by them has to be covered. In the model of [1], agent i wants to connect a set of terminals S_i in a graph $G = (V, E)$ with edge costs c. Hence the variables represent the edges of the graph and the constraints belonging to agent i enforce the connectivity of S_i.

Our games can be seen as the *duals* of these coverings games. That is, the agents correspond to the variables, and are responsible for the satisfaction of the constraints containing them. If a constraint is left unsatisfied, the participating variables get punished. Also, a variable may require compensation (ransoms) from other variables participating in the same constraints. These compensations will correspond to a dual solution in a Nash equilibrium. We hope that our approach of studying dual covering games might be extended to a broader class of problems, with the price of anarchy matching the integrality gap.

Our result and the above papers are focused on noncooperative covering games. A different line study in mechanism design focuses on cost sharing mechanism, e.g. [9,10,17,12,27,28].

The performance of behavioral dynamics in games and specifically establishing fast convergence to equilibria of good quality has been the subject of intensive recent research [22,23,35]. The importance of such results that go beyond the analysis of performance of Nash equilibria has also been stressed in recent work

[21,18,31] where it has been shown that even in simple and well studied games, the performance of natural learning dynamics can be arbitrarily better than (any convex combination of) the payoffs of Nash equilibria. A mini review of this literature can be found here[29].

Recent work of Roughgarden et al. [33,4,34] has shown that the majority of positive results in price of anarchy literature can be reduced to a specific common set of structural assumptions. In contrast, in our work, we use a novel approach by exploring connections to the LP relaxations of the underlying centralized optimization problems. This connection raises interesting questions about the limits of its applicability.

The rest of this extended abstract is organized as follows. Section 2 defines the Mafia games for vertex cover, hitting set, and submodular hitting set. The proofs of existence of Nash equilibria and the price of anarchy bound is given only for vertex cover and omitted for the other two problems. Section 3 discusses results on dynamics, and Section 4 possible further research directions.

2 The Mafia Games and Price of Anarchy Bounds

2.1 Vertex Cover

Given a graph $G = (V, E)$, let $c : V \to \mathbb{R}^+$ be a cost function on the vertices. In the *vertex cover problem*, the task is to find a minimum cost set $M \subseteq V$ containing at least one endpoint of every edge in E. For a vertex $v \in V$, let $N(v) = \{u : uv \in E\}$ denote the set of its neighbors.

Game Definition. The *Mafia Vertex Cover Game* is a one-shot game on the agent set V. The basic strategy of an agent is to decide being a civilian or a mafioso. The set of civilians shall be denoted by C, the set of mafiosi (Mafia) by M. For civilians, no further decision has to be made, while for mafiosi, their strategy also contains a ransom vector. Each mafioso $m \in M$ can demand ransoms from his neighbors totaling $c(m)$. The ransom demanded from a neighbor $u \in N(m)$ is $r(m, u) \geq 0$, with $\sum_{u \in N(v)} r(m, u) = c(m)$. The strategy profile $\mathcal{S} = (M, C, r)$ thus consists of the sets of mafiosi and civilians, and the ransom vectors.

Let us call $c(v)$ the *budget* of an agent $v \in V$, and let $T > \sum_{v \in V} c(v)$ be a huge constant. Let $D(v) = \sum_{m \in M} r(m, v)$ be the demand asked from the agent $v \in V$.

Let us now define the payoffs for a given strategy profile $\mathcal{S}$. For a civilian $v \in C$, let $\text{Pen}(v) = T$ if v is incident to an uncovered edge, that is $C \cap N(v) \neq \emptyset$, and $\text{Pen}(v) = 0$ otherwise. The utility of $v \in C$ is $U_{\mathcal{S}}(v) = -D(v) - \text{Pen}(v)$.

If $v \in M$ and the total demand from v is $D(v) > c(v)$ (i.e. v is asked too much), we call v *protected* and denote the set of protected mafiosi by $P \subseteq M$. The real amount of money that the protected mafioso $p \in P$ pays to his neighbors is scaled down to $\frac{c(p)}{D(p)} r(u, p)$. Let $F^-(v) = \min\{D(v), c(v)\}$ be the total amount the mafioso v pays for ransom.

Let $F^+(v) = \sum_{u \in N(v) \setminus P} r(v,u) + \sum_{u \in N(v) \cap P} \frac{c(u)}{D(u)} r(v,u)$ denote the income of $v \in M$ from the ransoms. Then the utility of a mafioso $v \in M$ is defined as $U_{\mathcal{S}}(v) = -c(v) + F^+(v) - F^-(v)$.

This means v has his initial cost $c(v)$ for entering the Mafia, receives full payment from civilians and unprotected mafiosi, receives reduced payment from protected mafiosi, and pays the full demand to his neighboring mafiosi if v is unprotected, or reduced payment if v is protected.

The Existence of Pure Nash Equilibria. Pure Nash equilibria are (deterministic) strategy outcomes such that no agent can improve her payoff by unilaterally changing her strategy. We will start by establishing that our game always exhibits such states. The following is the standard linear programming relaxation of vertex cover along with its dual.

$$\min \sum_{v \in V} c(v)x(v) \qquad \text{(P-VC)}$$
$$x(u) + x(v) \geq 1 \quad \forall uv \in E$$
$$x \geq 0$$

$$\max \sum_{uv \in E} y(uv) \qquad \text{(D-VC)}$$
$$\sum_{uv \in E} y(uv) \leq c(u) \quad \forall u \in V$$
$$y \geq 0$$

For a feasible dual solution y we say that the vertex $v \in V$ is *tight* if $\sum_{uv \in E} y(uv) = c(v)$. We call the pair (M, y) a *complementary pair* if M is a vertex cover, y is a feasible dual solution, and each $v \in M$ is tight with respect to y. The following well-known claim states that a complementary solution provides good approximation.

Lemma 1. *If (M, y) is a complementary pair, then M is a 2-approximate solution to the vertex cover problem.*

The simple approximation algorithm by Bar-Yehuda and Even [3] returns a complementary pair, and therefore has approximation factor 2. We start from $y = 0$ and $M = \emptyset$. In each step, we pick an arbitrary uncovered edge uv, and raise $y(uv)$ until u or v becomes tight. We add the tight endpoint(s) to M and iterate with a next uncovered edge. It is straightforward that the algorithm returns a complementary pair (M, y). Our next lemma proves that a complementary pair provides a Nash equilibrium.

Lemma 2. *Let (M, y) be a complementary pair, and consider the strategy profile where the agents in M form the Mafia and $C = V \setminus M$ are the civilians. For $u \in M$, define $r(u, v) = y(uv)$ for every $v \in N(u)$. Then the strategy profile $\mathcal{S} = (M, C, r)$ is a Nash equilibrium.*

Proof. Since $D(v) \leq c(v)$ for all players, there are no protected mafiosi. If v is a civilian, his payoff is $-D(v)$. He would not get a protected status if he entered the Mafia as $D(v) \leq c(v)$, and thus his payoff would be $-c(v) + F^+(v) - D(v) \leq -D(v)$ by arbitrary choice of ransoms. If v is a mafioso, he has $F^+(v) = c(v)$ as none of his neighbors is protected. Thus his utility is $-D(v)$, the maximum he can obtain for any strategy. □

As an immediate consequence, we get the following.

Theorem 3. *The Mafia Vertex Cover Game always has a pure Nash equilibrium.*

The Price of Anarchy. For a strategy profile $\mathcal{S}$ with α vertices incident to uncovered edges, the sum of the utilities is $-c(M) - \alpha T$. The Price of Anarchy compares this sum in a Nash equilibrium at the worst case to the maximum value over all strategy profiles, that corresponds to a minimum cost vertex cover.

Consider a strategy profile $\mathcal{S}$ that encodes a Nash equilibrium. First, observe that Mafia M is a vertex cover due to the high penalties on uncovered edges. We shall prove that the cost $c(M)$ is at most twice the cost of an optimal vertex cover, consequently, the price of anarchy is at most 2.

Lemma 3. *Let the strategy profile $\mathcal{S} = (M, C, r)$ be a Nash equilibrium. Then there are no protected mafiosi.*

Proof. For a contradiction, suppose P is nonempty. First we show there exists an edge $mp \in E$ such that $m \in M \setminus P$, $p \in P$ and $r(m,p) > 0$. Indeed, if there were no such edges, then $\sum_{p\in P} D(p) \leq \sum_{p \in P} c(p)$ as the ransoms demanded from protected mafiosi are all demanded by others in P. However, by definition $D(p) > c(p)$ for all $p \in P$, giving $\sum_{p\in P} D(p) > \sum_{p\in P} c(p)$, a contradiction.

Consider the edge $mp \in E$ as above. If $N(m) \subseteq M$, then m could increase his utility by becoming a civilian, as $F^-(m) = D(m)$ and $F^+(m) < c(m)$, whereas he would receive $-D(m)$ as a civilian. If there is a $v \in C$, $mv \in E$, then m could increase his utility by decreasing $r(m,p)$ to 0 and increasing $r(m,v)$ by the same amount. □

Lemma 4. *Suppose the strategy profile $\mathcal{S} = (M, C, r)$ is a Nash equilibrium and let $v \in C$. Then $D(v) \leq 2c(v)$.*

Proof. Suppose the contrary: let $D(v) > 2c(v)$ and thus $U_{\mathcal{S}}(v) < -2c(v)$. If joining Mafia, v receives the protected status and thus gains utility at least $-2c(v)$ as $F^-(v) = c(v)$. □

Theorem 4. *The price of anarchy in the Mafia game is 2.*

Proof. Let $\mathcal{S} = (M, C, r)$ be a strategy profile in a Nash equilibrium. Using the convention $r(u,v) = 0$ if $u \in C$, let us define $y(uv) = r(u,v) + r(v,u)$ for every edge $uv \in E$. We show that $\sum_{u\in V} y(uv) \leq 2c(v)$ for every $v \in V$. Indeed, if $v \in C$, then $\sum_{u\in V} y(uv) = \sum_{u \in M} r(u,v) = D(v) \leq 2c(v)$ by Lemma 4. If $v \in M$, then $\sum_{u\in V} y(uv) = \sum_{u\in N(v)} r(v,u) + D(v) \leq 2c(v)$ by Lemma 3. Therefore $\frac{1}{2}y$ is a feasible solution to (D-VC) and

$$\sum_{uv\in E} \frac{1}{2} y(uv) = \frac{1}{2} \sum_{m\in M} \sum_{v\in V} r(m,v) = \frac{1}{2} \sum_{m\in M} c(m).$$

This verifies that the objective value for $\frac{1}{2}y$ is the half of the cost of the primal feasible vertex cover M, proving that M is a 2-approximate vertex cover. □

2.2 Set Cover and Hitting Set

In this section, we generalize our approach to the hitting set problem. Given a hypergraph $\mathcal{G} = (V, \mathcal{E})$ and a cost function $c : V \to \mathbb{R}_+$, we want to find a minimum cost $M \subseteq V$ intersecting every hyperedge. Let $d = \max\{|S| : S \in \mathcal{E}\}$.

The *set cover problem* is well-known to be equivalent to the hitting set problem. Also, without loss of generality we may define the hitting set game on a d-uniform hypergraph. The general case can be easily reduced to it by adding at most $d-1$ new dummy elements of high cost.

Game Definition. We define the *Mafia Hitting Set Game* on a d-uniform hypergraph $\mathcal{G} = (V, \mathcal{E})$. The set of agents is V, with $v \in V$ having a *budget* $c(v)$. We shall call the hyperedges *clubs*. For an agent $v \in V$, let $\mathcal{N}(v) \subseteq \mathcal{E}$ denote the set of clubs containing v. The agents again choose from the strategy of being a civilian or being a mafioso, denoting their sets by C and M, respectively. The strategies of the mafioso m incorporates the ransoms $r(m, S)$ for the clubs S containing m, with $\sum_{S \in \mathcal{N}(v)} r(m, S) = c(m)$.

We define the payoffs for the strategy profile $\mathcal{S} = (M, C, r)$ similarly to the vertex cover case. For a civilian $v \in C$, $\mathrm{Pen}(v) = T$ for a large constant T if v participates in a club containing no mafiosi, and 0 otherwise.

In each club S, the ransom $r(m, S)$ of a mafioso $m \in S \cap M$ has to be payed by all other members at equal rate, that is, everyone pays $\frac{r(m,S)}{(d-1)}$ to m. The demand from an agent is the total amount he has to pay in all clubs he is a member of, that is,

$$D(v) = \frac{1}{d-1} \sum_{S \in \mathcal{N}(v)} \sum_{m \in (M \cap S) \setminus \{v\}} r(m, S).$$

The utility of a civilian $v \in C$ is defined as $U_{\mathcal{S}}(v) = -D(v) - \mathrm{Pen}(v)$.

A mafioso v receives the protected status if $D(v) > c(v)$. The set of protected mafiosi is denoted by P, and they pay proportionally reduced ransoms. Let $F^-(v) = \min\{D(v), c(v)\}$ be the total amount v pays. The income is defined by

$$F^+(v) = \sum_{S \in \mathcal{N}(v)} \frac{r(v, S)}{d-1} \left(|S \setminus (P \cup \{v\})| + \sum_{u \in (S \cap P) \setminus \{v\}} \frac{c(u)}{D(u)} \right).$$

The utility of a mafioso $v \in M$ is then $U_{\mathcal{S}}(v) = -c(v) + F^+(v) - F^-(v)$.

Analogously to vertex cover, we show the following.

Theorem 5. *There exist pure Nash equilibria in the Mafia Hitting Set Game, and the Price of Anarchy is at most d. The output of the Bar-Yehuda–Even algorithm always gives a Nash equilbrium.*

The proof is omitted. It is similar to the case of vertex cover; the difficult part is showing the following analogue of Lemma 4.

Lemma 5. *Let the strategy profile $\mathcal{S} = (M, C, r)$ be a Nash equilibrium and let $v \in C$. Then $D(v) \le \frac{d}{d-1} c(v)$.*

2.3 Submodular Hitting Set

In the submodular hitting set problem, we are given a hypergraph $G = (V, \mathcal{E})$ with a submodular set function $C : 2^V \to \mathbb{R}_+$, that is, $C(\emptyset) = 0$, and

$$C(X) + C(Y) \geq C(X \cap Y) + C(X \cup Y) \quad \forall X, Y \subseteq V.$$

We shall assume also that C is monotone, that is, $C(X) \leq C(Y)$ if $X \subseteq Y$. Our aim is to find a hitting set M minimizing $C(M)$.

Koufogiannakis and Young [25], and Iwata and Nagano [19] obtained d-approximation algorithms for this problem, where d is the maximum size of a hyperedge. The primal-dual algorithm in [19] is a natural extension of the Bar-Yehuda–Even algorithm.

A notion needed to define our game is the *submodular base polyhedron*:

$$B(C) = \{z \in \mathbb{R}^V : z \geq 0, z(Z) \leq C(Z) \;\; \forall Z \subsetneq V, z(V) = C(V)\}.$$

Game Definition. The *Submodular Mafia Hitting Set Game* is defined on a hypergraph $\mathcal{G} = (V, \mathcal{E})$ and a monotone submodular set function $C : 2^V \to \mathbb{R}_+$. There are $|V| + 1$ agents, one for each vertex and a special agent g, called the *Godfather*.

The strategy of the Godfather is to return a budget vector $\tilde{c} \in B(C)$. The basic strategy of an agent $v \in V$ is to decide being a civilian or being a mafioso. The strategy of a mafioso m further incorporates normalized ransoms $r_0(m, S) \geq 0$ for clubs $S \in \mathcal{N}(m)$ with $\sum_{S \in \mathcal{N}(m)} r_0(m, S) = 1$, that is, $r_0(m, S)$ expresses the fraction of the budget of m he is willing to charge on S.

The sets of civilians and mafiosi will again be denoted by C and M, respectively. Hence a strategy profile is given as $\mathcal{S} = (M, C, \tilde{c}, r_0)$. The actual ransoms will be $r(m, S) = r_0(m, S) \cdot \tilde{c}(m)$.

The utility of the Godfather is the total budget of the Mafia: $U_{\mathcal{S}}(g) = C(M)$. The utility of the vertex agents is defined the same way as for the linear Mafia Hitting Set Game in Section 2.2, with replacing $c(v)$ by $\tilde{c}(v)$ everywhere.

For linear cost functions, we have $C(Z) = \sum_{v \in Z} c(z)$. Then the only vector in $B(C)$ is c, hence the Godfather has only one strategy to choose. Therefore we obtain the same game as described in Section 2.2. Our result can be summarized as follows.

Theorem 6. *There exist pure Nash equilibria in the Submodular Mafia Hitting Set Game, and the Price of Anarchy is at most d. The output of the primal-dual algorithm by Iwata and Nagano [19] always gives a Nash equilbrium.*

The proof reduces to the linear Mafia Hitting Set Game, exploiting the fact the whenever the Godfather has no incentive to change his strategy, from the perspective of the vertex players it is identical to a linear game with fixed budgets $\tilde{c}$.

3 Convergence to Nash Equilibrium

In this section, we investigate the Mafia Vertex Cover Game from the best-response dynamics perspective. There exists an example showing that a best-response dynamics can run into a loop. The problem is due to assymetric ransoms between mafiosi. We introduce secondary utilities motivated by this phenomenon.

For a strategy profile $\mathcal{S} = (M, C, r)$, $U_{\mathcal{S}}(v)$ is the utility as defined in Section 2.1. Let us define $\tilde{U}_{\mathcal{S}}(v) = 0$ if $v \in C$ and $\tilde{U}_{\mathcal{S}}(v) = -\sum_{uv \in E, u \in M} |r(u,v) - r(v,u)|$ if $u \in M$. The total utility is then $(U_{\mathcal{S}}(v), \tilde{U}_{\mathcal{S}}(v))$ in the lexicographic ordering: the agents' main objective is to maximize $U_{\mathcal{S}}(v)$, and if that is the same for two outcomes, they choose the one maximizing $\tilde{U}_{\mathcal{S}}(v)$.

$\tilde{U}_{\mathcal{S}}(v) \leq 0$ and equality holds if $r(u,v) = r(v,u)$ for every $uv \in M$, $u, v \in M$. Therefore all results in Section 2.1 remain valid: in Lemma 2 we define a strategy profile where $\tilde{U}_{\mathcal{S}}(v) = 0$ for all agents, hence it also gives a Nash equilibrium for the extended definition of utilities. The secondary utility term $\tilde{U}$ does not affect the proofs in Section 2.1.

Consider now the following simple dynamics: *Start from the strategy profile where all agents are civilians. In each step, take an agent who is incident to uncovered edge and subject to this, minimizes $c(v) - D(v)$, and give him the opportunity to change his strategy.*

Theorem 7. *After each agent changing his strategy at most once, we obtain a strategy profile in Nash equilibrium.*

Proof. By induction, we shall prove that in every step, $c(v) \geq D(v)$ and $\tilde{U}_{\mathcal{S}}(v) = 0$ for all $v \in V$. Consider the next move: let the player v on move be such that he is incident to some uncovered edges, and that minimizes $c(v) - D(v)$. He obviously has to enter the Mafia, and can achieve a maximal (primary and secondary) utility if he sets $r(v,u) = r(u,v)$ for any $u \in M \cap N(v)$, and distributes the rest of his ransoms arbitrarily to his civilian neighbors. Note that this can always be done because $c(v) \geq D(v)$. Also, note that the total ransom v will demand from other civilians is $c(v) - D(v)$. By the extremal choice of v, it follows that none of his civilian neighbors z will violate $c(z) \geq D(z)$. This also remains true if $z \in M$, as $D(z)$ is at most the total ransom z demands due to the symmetry of the ransoms.

Hence the induction hypothesis is maintained by an arbitrary best response of v. A mafioso who is not protected and has secondary objective 0 has no incentive to change his strategy. Also, a civilian v with $c(v) \geq D(v)$ has no incentive to join the Mafia if there are no uncovered edges incident to v. Consequently, the game ends after all uncovered edges are gone, and once an agent joins to Mafia, he would not change his strategy anymore. □

Observe that the dynamics is closely related to the Bar-Yehuda–Even algorithm: if the next agent always ransoms only one of its civilian neighbors, then it corresponds to a possible performance of the algorithm.

The above dynamics can be naturally interpreted in a distributive manner. In the proof of Theorem 7, we only use that the vertex v changing his strategy *is a local minimizer* of $c(v) - D(v)$. The simultaneous move of two agents u and v could interfere only if $uv \in E$ or they have a neighbor t in common. In this case, $c(t) < D(t)$ could result if both u and v start ransoming t simultaneously.

We assume that the agents have a hierarchical ordering $\prec$: $u \prec v$ expresses that v is more powerful than u. We call an agent v *a local minimizer* if $v \in C$, v is incident to some uncovered edges, and $c(v) - D(v) \leq c(u) - D(u)$ whenever $u \in C$, $uv \in E$. A local minimizer v is then called *eligible* if $u \prec v$ for all local minimizers u whose distance from v is at most 2.

We start from $C = V$. In each iteration of the dynamics, we let all eligible agents change their strategy to a best response simultaneously. As in the proof of Theorem 7, $c(v) - D(v) \geq 0$ is maintained for all $v \in V$, and thus the dynamics terminates after each agent changes his strategy at most once.

In contrast to efficient distributed algorithms for vertex cover in the literature (e.g. [24]), we cannot give good bounds on the number of iterations of our distributed dynamics. For example, if the graph is a path $v_1 \dots v_n$, and the budgets are $c(v_i) = i$, then only agent i will move in step i. Yet we believe that our dynamics could be practically efficient.

This result could be extended to the hitting set game, with a more sophisticated choice of the next player. There also exists a bad example for submodular hitting set.

4 Conclusions and Further Research

We have defined games whose Nash equilibria correspond to certain covering problems, with the price of anarchy matching the best constant factor approximations. The payoffs in these games are locally defined, and the analysis is based on the LP relaxations of the corresponding covering problems. An intriguing question is if similar mechanisms can be designed for further combinatorial optimization problems.

The first natural direction would be to extend our approach to a broader class of covering games. The most general approximation result on covering games is [25], giving a d-approximation algorithm for minimizing a submodular function under monotone constraints, each constraint dependent on at most d variables. As a first step, one could study hitting set with the requirement that each hyperedge S must be covered by at least $h(S) \geq 1$ elements; a simple primal-dual algorithm was given in [13]. However, extending our game even to this setting does not seem straightforward.

One could also try to formulate analogous settings for classical optimization problems such as facility location, Steiner-tree or knapsack. One inherent difficulty is that in our analysis, it seems to be crucial that any greedily chosen maximal feasible dual solution gives a good approximation. Also, we heavily rely on the fact that each constraint contains at most d variables.

In Section 3, we have shown that the best response dynamics rapidly converges for vertex cover and hitting set under certain assumptions. Stronger convergence results might hold: for example, it is open if arbitrary round robin best response dynamics converge to a Nash equilibrium. For the Submodular Mafia Hitting Set Game, no convergence result is known.

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Mechanism Design for a Risk Averse Seller

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Abstract. We develop efficient algorithms to construct approximately utility maximizing mechanisms for a risk averse seller in the presence of potentially risk-averse buyers in Bayesian single parameter and multi-parameter settings. We model risk aversion by concave utility function. Bayesian mechanism design has usually focused on revenue maximization in a *risk-neutral* environment, and while some work has regarded buyers' risk aversion, very little of past work addresses the seller's risk aversion.

We first consider the problem of designing a DSIC mechanism for a risk-averse seller in the case of multi-unit auctions. We give a poly-time computable pricing mechanism that is a $(1 - 1/e - \epsilon)$-approximation to an optimal DSIC mechanism, for any $\epsilon > 0$. Our result is based on a novel application of correlation gap bound, that involves *splitting* and *merging* of random variables to redistribute probability mass across buyers. This allows us to reduce our problem to that of checking feasibility of a small number of distinct configurations, each of which corresponds to a covering LP.

DSIC mechanisms are robust against buyers' risk aversion, but may yield arbitrarily worse utility than the optimal BIC mechanisms, when buyers' utility functions are assumed to be known. For a risk averse seller, we design a truthful-in-expectation mechanism whose utility is a small constant factor approximation to the utilty of the optimal BIC mechanism under two mild assumptions: (a) ex post individual rationality and (b) no positive transfers. Our mechanism simulates several rounds of sequential offers, that are computed using stochastic techniques developed for our DSIC mechanism. We believe that our techniques will be useful for other stochastic optimization problems with concave objective functions.

1 Introduction

Bayesian mechanism design has usually focused on maximizing expected revenue in a *risk-neutral* environment, *i.e.* where all the buyers and the seller have linear utility, and choose their strategy with the aim of maximizing their *expected payoff*. However, since the payoff is a random outcome that depends on other players' valuations and strategies, there is risk associated with it. A standard model [5,15] that captures risk aversion assumes that a player has a non-decreasing concave utility function $\mathbf{U} : (-\infty, \infty) \rightarrow (-\infty, \infty)$, so that when the payoff obtained is

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R, the player's utility is $\mathbf{U}(R)$. The player may choose to express various levels of risk aversion by specifying a suitable concave function as his utility, and then his aim becomes to maximize his expected utility. While mechanism design in a risk-neutral (linear utility) environment is well understood, many properties tend to break down in the presence of risk aversion (concave utility). In this paper, we develop efficient algorithms to compute mechanisms in the presence of risk-averse players. We primarily focus on the single-parameter setting of k-unit auctions, where each buyer wants at most one unit.

Our main results focus on maximizing expected utility for a risk-averse seller, in the presence of potentially risk-averse buyers. Very little past work has studied the effect of a seller's risk aversion. Even if the seller were risk-neutral, risk aversion among buyers is enough to violate the revenue equivalence principle established by Myerson [14], and an optimal *Dominant Strategy Incentive Compatible (DSIC)* mechanism may generate less expected utility for the seller compared to an optimal *Bayesian Incentive Compatible (BIC)* mechanism. This is because when buyers are risk-averse, the seller can extract greater expected revenue by offering a deterministic payment scheme to the buyers and charging extra for this *insurance* [11]. Further, the gap between optimal DSIC and BIC mechanisms for a risk averse seller is unbounded if buyers were risk neutral [9] (see Example 1).

DSIC mechanisms have the attractive property that truth-telling is an equilibrium for buyers as long as their utility functions are non-decreasing, thus they are independent of buyers' risk properties.[1] This property does not hold when computing optimal BIC mechanisms (or even truthful-in-expectation mechanisms). This motivates the study of two problems: (a) an approximately optimal DSIC mechanism for a risk averse seller, and (b) an approximately optimal BIC mechanism for a risk averse seller when buyers are also risk averse.

DSIC Mechanism for a Risk-Averse Seller: We reiterate that DSIC mechanisms are independent of buyers' risk properties as long as their utility functions are monotone. Myerson's mechanism is the optimal mechanism for a risk neutral seller, however it is not true for when the seller is risk averse. Further, a virtual value maximization approach does not apply for such seller when he has at least two units of inventory[2]. This is because the contributions of different buyers is not additive when the seller has a non-linear utility function. The following theorem summarizes our main result.

Theorem 1. *For a risk averse seller, there is a poly-time computable deterministic sequential posted pricing mechanism (SPM) for multi-unit auctions with*

[1] We say that a (possibly randomized) mechanism is DSIC if truth-telling maximizes every buyer's utility for any set of bids by other buyers and any realization of random bits used by the mechanism. It should be distinguished from the weaker notion of truthful-in-expectation (TIE), where truth-telling maximizes every buyer's *expected* utility, where expectation is taken over the random bits of the mechanism.

[2] With only one item to sell, at most one buyer pays in any realization, and the seller's utility can be maximized by scaling the bid values using the utility function.

expected utility at least $(1 - \frac{1}{e} - \epsilon)\mathrm{OPT}$*, for any* $\epsilon > 0$*, where* OPT *is the expected utility of an optimal DSIC mechanism.*

Our techniques extend to give a constant approximation to optimal deterministic DSIC mechanism in a multi-parameter setting, namely, when there are multiple distinct items and unit-demand buyers.

In a related work, Sundarajan and Yan [16] designed DSIC mechanisms for multi-unit auctions for a risk-averse seller when buyers' valuation functions are *regular*. They focus on designing mechanisms which do not even depend on the seller's own utility function, but simultaneously perform well with respect to every concave utility. While this stronger benchmark may be useful when the seller is unsure of his own utility, a mechanism that is simultaneously optimal for all concave utility functions may not even exist. The stronger benchmark also forces their approximation guarantees to be weaker – 1/8-approximation for regular distributions (1/2 when there is unlimited supply of items). They also exhibit a lower bound instance implying unbounded gap for general distributions (even neglecting computational constraints). Non-regular distributions are not uncommon – any multi-modal distribution is non-regular. Risk aversion is important in the presence of such high variance distributions, which motivates the design of mechanisms that are tailored to the seller's utility function.

BIC Mechanisms for Risk-Averse Seller and Buyers: We next design a BIC mechanism for multi-unit auctions, where buyers and seller may all be risk-averse, and all utility functions are public knowledge. Eso and Futo [9] designed optimal BIC mechanism for a risk-averse seller when buyers are risk-neutral: the mechanism obtains deterministic revenue by transferring all uncertainty to the buyers. However, such a strategy is infeasible when buyers are risk-averse too. Further, the gap between optimal DSIC mechanism and optimal BIC mechanism can be unbounded, as shown by the following example.

Example 1. Consider an instance with two buyers and a seller with two identical items. Each buyer has valuation 1 for the item w.p. ϵ and 0 otherwise, and the first buyer is risk-neutral. The seller's utility function $\mathbf{U}$ is as follows: $\mathbf{U}(t) = \min\{t, \epsilon\}$. The utility optimal DSIC mechanism sets a price of 1 for each buyer, and gets utility ϵ with probability less than 2ϵ, otherwise its utility is 0. So the expected utility of an optimal DSIC is at most $2\epsilon^2$. Consider the following BIC mechanism: charge the first buyer ϵ in every realization without giving the item (even when his value is zero), and set a price of 1 to the second buyer. If the second buyer pays up (which happens w.p. ϵ), then transfer this 1 dollar to the first buyer. The first buyer never gets the item, and makes zero expected payment. The seller gets a revenue of ϵ in every realization, so his expected utility is ϵ. Therefore the gap is unbounded as $\epsilon \to 0$. □

We design a BIC mechanism for a risk averse seller that is competitive against an optimal BIC mechanism that satisfies two reasonable conditions: (a) if $\mathcal{U}_i$ is the utility function of any buyer i, then $\mathcal{U}_i(t) = -\infty$ for any $t < 0$, which implies that the mechanism is restricted to be *ex post individually rational* for every buyer, and (b) there is no payment from the seller to buyer in any realization (no positive transfer). Our computed mechanism also satisfies these two conditions, and is in fact truthful-in-expectation (TIE). Our approximation factor is $(1 - 1/e)^3$ for $k = 1$, and approaches $(1-1/e)$ as k becomes large. As a corollary, this result *bounds the gap between TIE mechanisms and reasonable BIC mechanisms.* Let $\gamma(k) = (1 - \frac{k^k}{k!e^k})$; $\gamma(1) = 1 - 1/e$, and approaches $(1 - \frac{1}{\sqrt{2\pi k}})$ for large k. The following theorem summarizes our result.

Theorem 2. *There is a polynomial time algorithm to compute a TIE mechanism for a k-unit auction with expected utility at least $\left(1 - \frac{1}{e}\right)^2 \gamma(k)\mathrm{OPT}$ where* OPT *is the expected utility of an optimal BIC mechanism that respects the above two conditions. Moreover, for $k \geq 1/\epsilon^3$, there is a $\left(1 - \frac{1}{e} - \epsilon\right)$-approximation.*

For the case of IID buyers with identical utility functions, we show an improved approximation factor of $(1 - \frac{1}{e} - \epsilon)\gamma(k)$.

Other Related Work: Most of the past work on risk-averse mechanism design has focused on revenue maximization (risk-neutral seller) in presence of risk-averse buyers. In this setting, Maskin and Riley [11] characterized optimal BIC mechanism for selling a single item when buyers' value distributions are IID, again assuming that buyers' utility functions are public knowledge. Their result uses Border's inequality. We note that due to recent poly-time computable generalizations of Border's inequality to multiple items and non-identical distributions [3,7], the result of Maskin and Riley [11] easily extends to the same setting. Further, revenue maximization when buyers' preference are nonlinear and multi-dimensional, is considered by Alaei *et al* [4]. Another well-known result states that under some natural assumptions on the buyers' utility functions, (Bayes-Nash equilibrium of) first-price auction with reserve generates greater revenue than second-price auction with the same reserve [12,11,13].

Table 1 summarizes the best known poly-time approximation results for all the different risk averse settings and benchmarks discussed above.

1.1 Overview of Techniques

Every DSIC mechanism offers a price to each buyer as a function of other buyers' bids and the buyer is allocated an item when his valuation is more than the offered price. We argue that a $(1 - \frac{1}{e})$-approximate (randomized) SPM can be obtained by using the same price distribution as that offered to each buyer in the optimal mechanism, except that the prices are now set independently (see Lemma 4). The argument uses the *correlation gap* bound of Agrawal et. al. [1] for submodular objectives. Sampling from the randomized SPM yields a satisfactory deterministic SPM. However, this is only an existential result, since getting the randomized SPM requires oracle access to a utility-optimal DSIC mechanism. Our main technical contribution is to show that it suffices (with same loss factor of $(1-1/e)$) to match the optimal mechanism only in the *sum of sale probabilities over all buyers, and not the sale probability for each buyer*, at every price. That is,

Table 1. Summary of approximation results for k-unit auctions

Type of risk environment	Comp. with Opt. DSIC	Comparison with Optimal BIC	
	Poly-time DSIC [1]	Poly-time TIE [2]	Poly-time BIC [2]
risk-neutral seller, risk-neutral buyers	1 [Myerson '81]	1 [Myerson '81]	1 [Myerson '81]
risk-neutral seller, risk-averse buyers	1 [Myerson '81]	$\gamma(k)$ [3] [Full Version]	1 [Alaei *et al* '12]
risk-averse seller, risk-neutral buyers	$(1-1/e-\epsilon)$ [Theorem 1]	$(1-1/e)^2\gamma(k)-\epsilon$ [Theorem 2][4]	1 [Eso-Futo '99]
risk-averse seller, risk-averse buyers	$(1-1/e-\epsilon)$ [Theorem 1]	$(1-1/e)^2\gamma(k)-\epsilon$ [Theorem 2][4]	$(1-1/e)^2\gamma(k)-\epsilon$ [Theorem 2][4]

[a] Need to know seller's utility function. Independent of buyers' utility functions as long as they are non-decreasing.

[b] Need to know both seller and buyers' utility functions.

[c] $\gamma(k) = (1 - \frac{k^k}{k!e^k})$. $\gamma(1) = 1 - 1/e$, and it approaches $(1 - \frac{1}{\sqrt{2\pi k}})$ for large k.

[d] Improves to $(1-1/e)\gamma(k)-\epsilon$ for IID buyers. Further, the factor improves to $(1 - 1/e - \epsilon)$ if $k \geq 1/\epsilon^3$. Here, comparison is made only against optimal BIC satisfying: (i) ex-post IR, and (ii) no positive transfers.

any two mechanisms that match in this *coarse footprint* will have approximately equal expected utility. This property is a generalization of correlation gap bound in [1], which not only introduces independence but also redistributes probability mass across variables (see Lemma 8). The redistribution is achieved by *splitting* and *merging* random variables to transform one given mechanism to another that matches the coarse footprint. Using a careful classification of prices, we show that it suffices to match an even coarser footprint containing only constant number of parameters, which define a *configuration*. The algorithm finds a feasible solution for each configuration using a *covering LP*. Then, it simulates these SPMs, one for each feasible configuration, to choose one with the highest expected utility.

To design a BIC mechanism when the seller as well as the buyers are risk-averse, the techniques developed for DSIC mechanisms can be used to establish that if allocation and payment functions of the optimal mechanism across buyers are made independent, and inventory constraints removed, the utility will be at least $(1-1/e)\mathrm{OPT}_{\mathrm{BIC}}$. However, to convert such a *soft* mechanism into a mechanism that strictly satisfies the inventory constraint is not easy: if we restrict the allocation to buyers with *top* k payments in a realization of a soft mechanism, a function which is submodular, the resulting mechanism is no longer BIC. Further, distributions on the revenue from any two allocations in the mechanism are incomparable (as the seller's objective function is not linear), so restricting to *first* k allocations in a realization of a soft mechanism can be arbitrarily bad. To overcome this problem, we develop a mechanism with $L \to \infty$ *rounds*, such that in each round, each buyer is ignored with a high probability of $(1 - 1/L)$. We show that the revenue from each allocation in this mechanism has identical distribution, and the loss in the expected utility caused by imposing the hard inventory constraint is bounded.

Organization: In Section 2, we provide some background material. In Sections 3 and 4, we present our main DSIC and BIC mechanisms, respectively. Missing proofs and rest of the results are deferred to full version of the paper [6].

2 Preliminaries

Single Parameter Multi-unit Auctions: The seller provides a single type of item (or service), of which he has k identical copies. There are n buyers $\{1, 2, \ldots n\}$, who have some private value for that service. Let buyer i have a valuation of v_i for the item (and he can consume only one unit), which is drawn, independent of other buyers' valuations, from a known distribution with cdf $F_i(x) = \mathbf{Pr}[v_i \leq x]$. We refer to $\mathbf{v} = (v_1, v_2 \ldots v_n)$ as the *valuation vector.*

Revenue, Utility and Optimality: The *revenue* $\mathbf{Rev}(\mathcal{M}, \mathbf{v})$ of a mechanism $\mathcal{M}$, when the realized valuation vector is $\mathbf{v}$, is the sum of payments from each buyers. The expected revenue of a mechanism $\mathbf{Rev}(\mathcal{M})$ is $\mathbf{E_v}[\mathbf{Rev}(\mathcal{M}, \mathbf{v})]$. In this work, we assume that the seller has a monotonically increasing concave utility function $\mathbf{U}$, which also satisfies $\mathbf{U}(0) = 0$. The utility of the mechanism is $\mathbf{U}(\mathbf{Rev}(\mathcal{M}, \mathbf{v}))$, and the expected utility of the mechanism is $\mathbf{U}(\mathcal{M}) = \mathbf{E_v}[\mathbf{U}(\mathbf{Rev}(\mathcal{M}, \mathbf{v}))]$. Let OPT_{DSIC} and OPT_{BIC} denote the expected utility of a utility-optimal DSIC and BIC mechanisms respectively. A mechanism is said to be an *α-approximation* to optimal DSIC (or BIC) mechanism if $\mathbf{U}(\mathcal{M}) \geq \alpha\text{OPT}_{\text{DSIC}}$ (or $\mathbf{U}(\mathcal{M}) \geq \alpha\text{OPT}_{\text{BIC}}$).

DSIC Mechanisms: It is well-known (*eg.* [14]) that a DSIC mechanism sets a (possibly randomized) price for buyer i based on v_{-i} but independent of v_i, and buyer i gets an item if and only if his valuation exceeds this price. So as long as a buyer has a non-decreasing utility function, he will report truthfully in a DSIC mechanism, for any realization of valuation vector and random bits of the mechanism. Moreover, random bits do not help a DSIC mechanism obtain greater utility, since the definition of DSIC implies that truthfulness must hold even if the random bits were revealed prior to submitting bids. So there is a utility-optimal DSIC mechanism which is deterministic.

Buyer's Risk Aversion and BIC Mechanisms: Each buyer i is associated with a publicly known monotone concave utility function $\mathcal{U}_i$ (defined on the value of item received minus payment) with $\mathcal{U}_i(0) = 0$. A BIC mechanism is associated with two functions $h(\cdot, \cdot, \cdot)$ and $g(\cdot, \cdot, \cdot)$: $h(i, j, v)$ is the probability that for valuation v, buyer i is allocated an item for a payment of p_j, and $g(i, j, v)$ is the probability that he pays p_j and is not allocated an item for valuation v. We refer to these two functions as the *payment functions* of the mechanism. We note that the allocation and payment of a buyer is possibly correlated with other buyers' payments, allocations as well as their valuations. Thus, a mechanism is BIC if and only if for each i, v, v', we have

$$\sum_j \left(\mathcal{U}_i(v - p_j)\left(h(i,j,v) - h(i,j,v')\right) + \mathcal{U}_i(-p_j)\left(g(i,j,v) - g(i,j,v')\right)\right) \geq 0$$

We note, given any buyer i, we allow his payment to be randomized rather than a fixed value as a function of buyer i's valuation and whether he gets an item.

This strictly gives more power to a risk-averse seller maximizing his expected utility. This is in contrast to the setting considered by Maskin and Riley [11], where it suffices to assume that buyer i's payment for valuation v is a fixed value as a function of v and whether he gets the item.

We define a *soft randomized sequential mechanism* as a mechanism without inventory limit that arranges buyers in an arbitrary order, asks each buyer for his valuation one-by-one. If the buyer i's reported valuation is v, the mechanism allocates an item to him independently w.p. $\sum_j h(i,j,v)$. If he is allocated an item, then the seller charges him p_j w.p. $\frac{h(i,j,v)}{\sum_l h(i,l,v)}$. When he is not allocated an item, he pays p_j w.p. $\frac{g(i,j,v)}{\sum_l g(i,l,v)}$. *Randomized sequential mechanisms* are same as *soft randomized sequential mechanisms* with an exception that they stop after running out of inventory. We note that if a *soft randomized sequential mechanism* is BIC, then the corresponding *randomized sequential mechanism* is also BIC.

Stochastic Dominance: Given two non-negative distributions $\mathcal{D}_1$ and $\mathcal{D}_2$, we say $\mathcal{D}_1$ *stochastically dominates* $\mathcal{D}_2$, denoted by $\mathcal{D}_1 \succeq \mathcal{D}_2$, if $\forall a \geq 0, \mathbb{P}r_{X\sim\mathcal{D}_1}(X \geq a) \geq \mathbb{P}r_{X\sim\mathcal{D}_2}(X \geq a)$. We note an important property of concave functions in the following lemma.

Lemma 1. *Given any non-decreasing concave function* $\mathbf{U}$*, and three independent non-negative random variables* X, Y_1, Y_2*, let* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *be the distributions of* Y_1 *and* Y_2 *respectively. If* $\mathcal{D}_1 \succeq \mathcal{D}_2$*, then*
(a) $\mathbf{E}_{X,Y_1\sim\mathcal{D}_1}[\mathbf{U}(X+Y_1) - \mathbf{U}(Y_1)] \leq \mathbf{E}_{X,Y_2\sim\mathcal{D}_2}[\mathbf{U}(X+Y_2) - \mathbf{U}(Y_2)]$*, and*
(b) $\mathbf{E}_{X,Y_1\sim\mathcal{D}_1}[\mathbf{U}(X+Y_1) - \mathbf{U}(X))] \geq \mathbf{E}_{X,Y_2\sim\mathcal{D}_2}[\mathbf{U}(X+Y_2) - \mathbf{U}(X)]$

3 Risk-Averse Seller: DSIC Mechanism

In this section, we construct a DSIC mechanism for a risk-averse seller for k-unit auction, and prove Theorem 1. We first prove the existence of an SPM that achieves a $(1-1/e)$-approximation to the optimal expected utility (Section 3.1), however this result does not lead to an efficient implementation. We then identify a set of sufficient properties of $(1-1/e-\epsilon)$-approximate mechanisms that enables us to construct a poly-time algorithm (Section 3.2).

3.1 Existence of a $(1-1/e)$-Approximate SPM

Given a set $S = \{x_1, x_2 \dots x_n\}$ of non-negative real number, let $\max_i\{x_1, x_2 \dots x_n\}$ denote the i^{th} largest value in the set, and let it be zero if $i > n$. Let $\mathbf{U}_k : \mathbf{R^n} \to \mathbf{R}$ be the function defined as $\mathbf{U}_k(S) = \mathbf{U}(\sum_{i=1}^{k} \max_i\{x_1, x_2 \dots x_n\})$, *i.e.* utility of the sum of the k largest arguments. Let $\mathbf{U}(S)$ denote $\mathbf{U}_\infty(S) = \mathbf{U}_{|S|}(S)$, the utility of the sum of all variables. We note an important property of $\mathbf{U}_k$ in the following lemma; its proof is deferred to the full version of the paper.

Lemma 2. *For any concave utility function* $\mathbf{U}$*, and any* k *and* n*, the function* $\mathbf{U}_k : \mathbf{R^n} \to \mathbf{R}$ *is a symmetric, monotone and submodular.*

We shall use the following *correlation gap* bound established by Agrawal et. al. [1] for monotone submodular functions.

Lemma 3. *[1] Given n non-negative random variables $X_1, X_2, ..., X_n$ with distributions $D_1, D_2, ..., D_n$, let D be an arbitrary joint distribution over these n random variables such that the marginal distribution for each X_i remains unchanged. Let D_{ind} be the joint distribution where each X_i is sampled from D_i independent of X_{-i}. Then for any monotone submodular function f over $X_1, X_2, ..., X_n$, we have $\frac{\mathbf{E}_{X \sim D_{ind}} f(X)}{\mathbf{E}_{X \sim D} f(X)} \geq 1 - 1/e$.*

Let $\mathcal{M}_{\text{OPT}}$ be a utility optimal DSIC mechanism for a k-unit auction. It follows that in $\mathcal{M}_{\text{OPT}}$, every buyer i is offered a (random) price P_i as a function of other buyers' bids; he receives an item and pays the offered price if and only if his value exceeds the price. The following lemma uses the correlation gap to establish the existence of an SPM which is a $(1 - 1/e)$-approximation to $\mathcal{M}_{\text{OPT}}$.

Lemma 4. *Suppose that $\mathcal{M}_{\text{OPT}}$ offers a (random) price P_i to each buyer i (the prices P_i, $1 \leq i \leq n$, may be correlated). Let $\mathcal{M}'$ be a randomized SPM that selects an independent random price P_i' for each buyer, such that P_i' and P_i have the same marginal distribution, and offers items to buyers in decreasing order of prices, until the items run out. Then* $\mathbf{U}(\mathcal{M}') \geq (1 - 1/e)\text{OPT}$.

Proof. Let R_i be the payment obtained in $\mathcal{M}_{\text{OPT}}$ from buyer i. Note that P_i and R_i are correlated random variables that depend on the realization of the valuations, and $R_i = P_i$ if $v_i > P_i$, else $R_i = 0$. As at most k buyers can make a positive payment in any realization of $\mathcal{M}_{\text{OPT}}$, we have

$$\mathbf{U}(\mathcal{M}_{\text{OPT}}) = \mathbf{E}\left[\mathbf{U}(R_1, R_2 \dots R_n)\right] = \mathbf{E}\left[\mathbf{U}_k(R_1, R_2 \dots R_n)\right]$$

Let $R_i' = P_i'$ if $v_i > P_i'$, else $R_i' = 0$. Since the SPM $\mathcal{M}'$ orders buyers in decreasing order of offer prices, so it collects the k largest acceptable prices as payment. We have $\mathbf{U}(\mathcal{M}') = \mathbf{E}\left[\mathbf{U}_k(R_1', R_2' \dots R_n')\right]$. Note that R_i and R_i' have the same distribution for each i, except that $R_1, R_2 \dots R_n$ are correlated variables, while $R_1', R_2' \dots R_n'$ are mutually independent. Using the submodularity of $\mathbf{U}_k$ (Lemma 2) and the correlation gap (Lemma 3), we get

$$\begin{aligned}\mathbf{U}(\mathcal{M}') &= \mathbf{E}\left[\mathbf{U}_k(R_1', R_2' \dots R_n')\right] \geq (1 - 1/e)\mathbf{E}\left[\mathbf{U}_k(R_1, R_2 \dots R_n)\right] \\ &= (1 - 1/e)\mathbf{U}(\mathcal{M}_{\text{OPT}})\end{aligned}$$

This completes the proof. □

Correlation gap was used by Yan [18] to show the same approximation ratio for an SPM to expected revenue maximization. However, for revenue maximization, it suffices for the SPM to match a revenue-optimal mechanism only in the probability of sale to each buyer, which solely determines the buyer's contribution to expected revenue. In contrast, for the utility maximization result of Lemma 4, *the SPM should match a utility-optimal mechanism in the entire distribution of prices to each buyer*. Also, the SPM for revenue maximization is poly-time computable, since a revenue-optimal mechanism is known (Myerson's mechanism). To the best of our knowledge, the SPM designed in Lemma 4 is not poly-time computable: constructing it would need an oracle access to a utility-optimal mechanism. Further, as we have to match $\mathcal{M}_{\text{OPT}}$ for each buyer-price pair, guessing the entire price distribution would require time exponential in the number of buyers.

3.2 Algorithm to Compute a $(1 - 1/e - \epsilon)$-Approximate SPM

We now present a polynomial time algorithm to compute an SPM whose approximation guarantee essentially matches the existential result above. To simplify the exposition of our algorithm, we assume that prices offered by any truthful mechanism belong to some known set $\mathcal{P} = \{p_1, p_2, \ldots\}$ whose size is polynomial in n. Let π_{ij} be the probability that buyer i is offered price p_j in $\mathcal{M}_{\text{OPT}}$; and $q_j = \sum_i \pi_{ij}(1 - F_i(p_j))$, i.e. q_j is the total probability of sale of an item at price p_j summed over all buyers in $\mathcal{M}_{\text{OPT}}$.

We divide the prices in $\mathcal{P}$ into 3 classes, *small*, *large* and *huge*. Fix some $1 > \epsilon > 0$. Let $\mathcal{P}_{\text{hg}}$ be the set of *huge prices* defined as $p_j \geq \mathbf{U}^{-1}(\text{OPT}/\epsilon)$. The distinction between small and large prices depend more intricately on the optimal mechanism. Let p^* be the largest price such that $\sum_{\mathbf{U}^{-1}(\text{OPT}/\epsilon) > p_j \geq p^*} q_j \geq 1/\epsilon^4$, *i.e.* the threshold where the total sale probability of all large prices add up to at least $1/\epsilon^4$. If such a threshold does not exist, then let $p^* = 0$ (note that $p*$ must be zero if $k < 1/\epsilon^4$). Let $\mathcal{P}_{\text{sm}}$ be all prices less than p^*, so that $\mathcal{P}_{\text{lg}} = \{p_j \mid \mathbf{U}^{-1}(\text{OPT}/\epsilon) > p_j \geq p^*\}$.

In the following lemma, we present a key set of sufficient conditions for a $(1 - 1/e - \epsilon)$-approximate mechanism which forms the basis of our algorithm; we defer its proof to later in the section.

Lemma 5. *Consider any SPM $\mathcal{M}'$, that offers price p_j to buyer i w.p. π'_{ij}, such that (a) $\sum_{i,p_j \in \mathcal{P}_{sm}} p_j \pi'_{ij}(1 - F_i(p_j)) = \sum_{p_j \in \mathcal{P}_{sm}} p_j q_j$, (b) for each $p_j \in \mathcal{P}_{lg}$, $\sum_i \pi'_{ij}(1 - F_i(p_j)) = q_j$, (c) $\sum_{i,p_j \in \mathcal{P}_{hg}} \mathbf{U}(p_j)\pi'_{ij}(1 - F_i(p_j)) = \sum_{p_j \in \mathcal{P}_{hg}} \mathbf{U}(p_j) q_j$ and (d) $\sum_{i,p_j \in \mathcal{P}} \pi'_{ij}(1 - F_i(p_j)) \leq k$. Then we have $\mathbf{U}(\mathcal{M}') \geq (1 - \frac{1}{e} - O(\epsilon))\text{OPT}$.*

Lemma 5 states that instead of matching $\mathcal{M}_{\text{OPT}}$ in the probability mass of each <buyer, large-price> pair, *it suffices to match the total probability mass at each large price, summed over all buyers.* Thus the *probability mass can be redistributed across buyers* without much loss in utility.

Further, Lemma 5 effectively states that the contribution of the small prices and the large prices can be *linearized*. Intuitively, if the small prices make a significant contribution to utility, then the mechanism must be collecting many small prices, so the total revenue from small prices exhibits a concentration around its expectation. Moreover, whenever a huge price is obtained in a realization, we can neglect the contribution from all other buyers in that realization, without losing much of the expected utility. So the contribution of huge prices can be measured separately. *This separation of huge and small prices from large prices enables us to keep the number of distinct large prices to at most a constant.*

Algorithm: We give an outline of the algorithm; the details are deferred to the full version of the paper. From Lemma 5, it suffices to match $\mathcal{M}_{\text{OPT}}$ in (a) the expected revenue from the small prices ($\mathcal{R}$), (b) the expected contribution to utility from the huge prices H, and (c) the total sale probability at each large price (q_j). The values of these parameters define a *configuration*, and we guess the value of each parameter with appropriate discretization. The number of distinct configurations is bounded by $2^{poly(1/\epsilon)}$. For each configuration, we check if there exists an SPM satisfying the configuration, using the covering linear program (LP) below. In the LP, the variable x_{ij} denotes the probability that buyer i is offered price p_j.

$$
\begin{aligned}
\textstyle\sum_i (1 - F_i(p_j))\, x_{ij} &\geq q_j \qquad \forall p_j \in \mathcal{P}_{\text{lg}} \\
\textstyle\sum_{i, p_j \in \mathcal{P}_{\text{sm}}} (1 - F_i(p_j))\, p_j x_{ij} &\geq \mathcal{R} \\
\textstyle\sum_{i, p_j \in \mathcal{P}_{\text{hg}}} (1 - F_i(p_j))\, \mathbf{U}(p_j) x_{ij} &\geq H \\
\textstyle\sum_{i,j} (1 - F_i(p_j))\, x_{ij} &\leq k \\
\textstyle\sum_j x_{ij} &\leq 1 \qquad \forall j \\
x_{ij} &\in [0, 1]\ \forall i, j
\end{aligned}
$$

Any feasible solution to this linear program gives a distribution of prices for each buyer, which gives us an SPM that satisfies the guessed configuration. We iterate through all the configurations, and pick the best among these SPMs. A deterministic SPM with desired utility guarantees can be easily identified by sampling from this randomized SPM.

3.3 Proof of Lemma 5

We begin by introducing two operations on random variables, *split* and *merge*. Using these two operations, we prove two key properties of random variables in Lemmas 8 and 9; Lemma 5 would follow as a corollary of these two lemmas.

Split and Merge Operations: We now define two operations, *merge* and *split*, on non-negative random variables. In the *merge* operation, given a set S of independent non-negative random variables, let X_i, X_j be any two variables in S such that $\mathbf{Pr}[X_i \neq 0] + \mathbf{Pr}[X_j \neq 0] \leq 1$, then variables X_i, X_j are replaced by a new variable Y such that, for each $p > 0$, $\mathbf{Pr}[Y = p] = \mathbf{Pr}[X_i = p] + \mathbf{Pr}[X_j = p]$ and Y is independent of other variables in $S \backslash \{X_i, X_j\}$.

The *split* operation breaks a random variable into a set of independent variables. Formally, given a set S of non-negative (possibly correlated) random variables, first the variables in S are made mutually independent, and then each variable $X_i \in S$ is *split* into an arbitrary pre-specified set of independent random variables $\{X_{i1}, X_{i2}, ..., X_{it}\}$ such that for each $p > 0$, $\sum_{1 \leq j \leq t} \mathbf{Pr}[X_{ij} = p] = \mathbf{Pr}[X_i = p]$ and the sets of variables created are also made mutually independent. Intuitively, the merge operation introduces negative correlation. Analogously, the split operation introduces independence. In Lemmas 6 and 7, we establish useful properties of merge and the split operations for a concave non-decreasing function; their proofs are deferred to the full version of the paper.

Lemma 6. *Let S be a set of independent non-negative random variables, Let $X_1, X_2 \in S$, and let Y be the variable formed by merging X_1 and X_2. Then* $\mathbf{E}[\mathbf{U}_k(S)] \leq \mathbf{E}[\mathbf{U}_k((S \setminus \{X_1, X_2\}) \cup \{Y\})]$.

Lemma 7. *Consider a sequence of split operations on a set S of arbitrarily correlated non-negative random variables and let S' be the set of independent random variables at the end of the split operation. Then* $\mathbf{E}[\mathbf{U}_k(S')] \geq (1 - \frac{1}{e})\mathbf{E}[\mathbf{U}_k(S)]$.

Using these two operations, we establish an important property in the following lemma, that not only introduces independence across correlated random variables, but also allows to redistribute the probability mass across variables.

Lemma 8. *Given an arbitrarily correlated set $S = \{X_1, X_2, ..., X_n\}$of non-negative random variables, consider any set $S' = \{X'_1, X'_2, X'_3, ..., X'_m\}$ of independent non-negative random variables, such that for each value $p_j > 0$, we have $\sum_i \mathbf{Pr}[X_i = p_j] = \sum_i \mathbf{Pr}[X'_i = p_j]$. Then for any concave function $\mathbf{U}$ and any $k > 0$, we have* $\mathbf{E}[\mathbf{U}_k(S')] \geq (1 - \frac{1}{e})\mathbf{E}[\mathbf{U}_k(S)]$.

Proof. We perform the *split* operation on S to create a set $Y = \{Y_{ijl}\}$ of variables as follows: for each $1 \leq i \leq n$ and $p_j > 0$, create $L \to \infty$ variables $\{Y_{ijl} | 1 \leq l \leq L\}$ where Y_{ijl} takes value p_j w.p. $\frac{\mathbf{Pr}[X_i = p_j]}{L}$ and 0 otherwise. Using Lemma 7, we get that $\mathbf{E}[\mathbf{U}_k(Y)] \geq (1 - \frac{1}{e})\mathbf{E}[\mathbf{U}_k(S)]$

Now we perform merge operation repeatedly on variables in Y to simulate variables in S'. The condition in the lemma statement ensures that such merging is always possible, since $L \to \infty$. Then by Lemma 6, we get $\mathbf{E}[\mathbf{U}_k(S')] \geq \mathbf{E}[\mathbf{U}_k(Y)] \geq (1 - 1/e)\mathbf{E}[\mathbf{U}_k(S)]$. □

The following lemma effectively states that, given a set of independent random variables, the contribution to the utility of huge values can be separated, and for small values, the variables can be replaced by their expectations; we defer its proof to the full version of the paper.

Lemma 9. *Given any $\epsilon > 0$ and a set of independent non-negative random variables $S = \{X_1, X_2, X_3 \ldots\}$ such that X_i takes value p_i w.p. π_i and 0 otherwise, where $p_1 \geq p_2 \geq p_3 \geq \ldots \geq 0$. Also, suppose that $\sum_{X_i \in S} \pi_i \leq k$. Let $\hat{p}$ be a price that satisfies $\hat{p} \geq \mathbf{U}^{-1}(\mathbf{E}[\mathbf{U}_k(S)]/\epsilon)$, and let p^* be any price such that $\sum_{p_i \in [p^*, \hat{p})} \pi_i > \frac{1}{\epsilon^4}$ (p^* is 0, if no such price exists). Also, let $S_{sm} = \{X_i | p_i < p^*\}$, $S_{lg} = \{X_i | p^* \leq p_i < \hat{p}\}$, and $S_{hg} = \{X_i | p_i > p^*\}$. Then*

$$\sum_{X_i \in S_{hg}} \mathbf{E}[\mathbf{U}(X_i)] + \mathbf{E}\left[\mathbf{U}\left(\mathbf{E}[S_{sm}] + \sum_{X_i \in S_{lg}} X_i\right)\right] \in [1 \pm O(\epsilon)]\mathbf{E}[\mathbf{U}_k(S)]$$

Now we are ready to prove Lemma 5. The revenue from a buyer in a mechanism can be represented by a random variable, possibly correlated with other buyers' random variables. Let $\mathcal{M}'$ be a mechanism that matches $\mathcal{M}_{\text{OPT}}$ on the total sale probability for each price, and its sale probability for each <buyer, large-price> pair is same as $\mathcal{M}$. Using Lemma 8, we get $\mathbf{U}(\mathcal{M}') \geq (1 - 1/e)\mathbf{U}(\mathcal{M}_{\text{OPT}})$. As $\mathcal{M}$ and $\mathcal{M}'$ have (approximately) identical revenues from small prices and utilities from huge prices, we can invoke Lemma 9 to establish that $\mathbf{U}(\mathcal{M}) \geq (1 - \epsilon)\mathbf{U}(\mathcal{M}')$. This completes the proof.

4 BIC Mechanism for Risk Averse Seller and Buyers

We now prove Theorem 2 by designing a BIC mechanism which satisfies the two restrictions. Consider any mechanism $\mathcal{M}$: let $g(\cdot,\cdot,\cdot)$ and $h(\cdot,\cdot,\cdot)$ be its payment functions. Then we have $g(i,j,v) = 0$ for each i, v and payment p_j (from ex-post individual rationality assumption), thus it is not required to describe the mechanism. Further, we have $h(i,j,v) = 0$ for each i, v and payment $p_j < 0$ (from no positive transfer assumption). Let $\mathcal{M}_{\text{OPT}}$ be a utility optimal BIC mechanism satisfying the two restrictions, and $h_{\text{OPT}}(\cdot,\cdot,\cdot)$ be its payment function.

Overall Idea: Consider any *soft* randomized sequential mechanism $\mathcal{M}'$ that processes buyers independently (according to its payment function $h(\cdot,\cdot,\cdot)$ as follows: it asks buyer i for his valuation; if it is v, then the item is given to him w.p. $\sum_j h(i,j,v)$, and when he gets the item, buyer i makes a payment of p_j w.p. $\frac{h(i,j,v)}{\sum_k h(i,k,v)}$), and matches $\mathcal{M}_{\text{OPT}}$ for (a) the total probability of each large payment summed over all buyers, (b) the total revenue from small payments and (c) the utility from huge payments. Using stochastic techniques developed for DSIC mechanisms, we get $\mathbf{U}(\mathcal{M}') \geq (1-1/e)\text{OPT}$. However, converting such soft mechanism into a mechanism that strictly satisfies the inventory constraint while maintaining truthfulness is not easy. In the case of DSIC mechanisms, the buyers were arranged in a decreasing order of prices, noting that *top-k* is a sub-modular function. Here, if we allocate items to buyers with *top-k* payments in a realization of $\mathcal{M}'$, then the mechanism is no longer truthful. Further, as *first-k* is not a sub-modular function, the desired approximation guarantee cannot be proven if we process buyers according to a fixed order. We get around this problem by constructing a mechanism with $L \to \infty$ rounds, where in every round, each buyer is processed independently w.p. $1/L$. The revenue from each allocation in this mechanism has an identical distribution. This helps to limit the loss caused by imposing strict inventory constraints. We now describe our mechanism in detail.

The Mechanism: Our mechanism $\mathcal{M}_{\text{rounds}}$ consists of $L \to \infty$ rounds and $h_{\text{rounds}}(\cdot,\cdot,\cdot)$ is the payment function associated with it. In each round, buyers arrive according to a predefined order. When buyer i arrives, subject to availability of items, he is independently processed with probability $\frac{1}{L}$ as follows: if his reported valuation is v, then he is given an item w.p. $\sum_j h_{\text{rounds}}(i,j,v)$, and whenever he is given an item, he makes a payment of p_j w.p. $\frac{h_{\text{rounds}}(i,j,v)}{\sum_l h_{\text{rounds}}(i,l,v)}$. Once processed, buyer i is not considered for any future rounds. Further, the payment function $h_{\text{rounds}}(\cdot,\cdot,\cdot)$ satisfies following properties:

(a) $\sum_{i,v,p_j \in \mathcal{P}_{\text{sm}}} p_j h_{\text{rounds}}(i,j,v) f_i(v) = \sum_{i,v,p_j \in \mathcal{P}_{\text{sm}}} p_j h_{\text{OPT}}(i,j,v) f_i(v)$,
(b) for each $p_j \in \mathcal{P}_{\text{lg}}$, $\sum_{i,v} h_{\text{rounds}}(i,j,v) f_i(v) = \sum_{i,v} h_{\text{OPT}}(i,j,v) f_i(v)$,
(c) $\sum_{i,v,p_j \in \mathcal{P}_{\text{hg}}} \mathbf{U}(p_j) h_{\text{rounds}}(i,j,v) f_i(v) = \sum_{i,v,p_j \in \mathcal{P}_{\text{hg}}} \mathbf{U}(p_j) h_{\text{OPT}}(i,j,v) f_i(v)$,
(d) for each i, v, v', $\sum_j \mathcal{U}_i(v-p_j) h_{\text{rounds}}(i,j,v) \geq \sum_j \mathcal{U}_i(v-p_j) h_{\text{rounds}}(i,j,v')$, and
(e) $\sum_{i,j,v} h_{\text{rounds}}(i,j,v) f_i(v) \leq k$.

We draw a parallel between the properties of $h_{\text{rounds}}(\cdot,\cdot,\cdot)$ with the algorithm developed in the case of DSIC mechanisms: the first three properties are equivalent to designing a mechanism that matches $\mathcal{M}_{\text{OPT}}$ in the total probability for each large payment, the expected revenue from small payments and the expected utility from huge payments. The fourth constraint establishes the truthfulness of $\mathcal{M}_{\text{rounds}}$, and the last constraint ensures its feasibility in expectation. Further, $\mathcal{M}_{\text{rounds}}$ is a *TIE* mechanism: conditioned on processing buyer i in some round, the payment function ensures truthfulness in terms of his expected utility.

The following lemma bounds the utility of $\mathcal{M}_{\text{rounds}}$, we defer its proof to later in the section.

Lemma 10. *As $L \to \infty$, $\mathbf{U}(\mathcal{M}_{rounds}) \geq (1-\epsilon)\left(1-\frac{1}{e}\right)^2 \gamma(k)\text{OPT}$.*

Algorithm: To construct an algorithm, we guess the total probability for each large payment (q_j), the utility from huge payments (H) and the revenue from the small payments ($\mathcal{R}$). The feasibility of a configuration can be checked using a covering LP. There are $2^{\text{poly}(1/\epsilon)}$ configurations, and we select a feasible configuration with maximum expected utility. Further, the number of rounds can be limited to $O(n^2)$ with a small loss in the approximation factor. To establish our result, it remains to prove Lemma 10.

Proof of Lemma 10. We give an overview of the proof of Lemma 10; the detailed proofs of intermediate steps are deferred to the full version of the paper. Let $\mathcal{I}_{\text{copies}}$ be an instance of the problem where each buyer is split into L independent copies, the copies of buyer i are $i1, i2, ..., iL$, and the valuation for each copy is drawn independently from F_i. Consider a mechanism $\mathcal{M}_{\text{soft}}$ on $\mathcal{I}_{\text{copies}}$ with L iterations. The lth copy of every buyer is considered in the lth iteration; when buyer il arrives, $\mathcal{M}_{\text{soft}}$ discards him w.p. $(1-1/L)$, otherwise it processes him according to $h_{\text{rounds}}(i,\cdot,\cdot)$. In the following lemma, we lower bound the utility of $\mathcal{M}_{\text{soft}}$; its proof follows from Lemma 8 and Lemma 9.

Lemma 11. $\mathbf{U}(\mathcal{M}_{soft}) \geq (1 - 1/e - \epsilon)\text{OPT}$.

To simplify notation, in the rest of the proof, we refer to the payment function of $\mathcal{M}_{\text{soft}}$ by $h(\cdot,\cdot,\cdot)$. Further, let $k_{\text{exp}} = \sum_{i,j,v} h_{\text{rounds}}(i,j,v)f_i(v)$; note $k_{\text{exp}} \leq k$.

Observe that mechanisms $\mathcal{M}_{\text{rounds}}$ and $\mathcal{M}_{\text{soft}}$ are equivalent with two exceptions: (a) hard inventory constraint of $\mathcal{M}_{\text{rounds}}$, and (b) $\mathcal{M}_{\text{soft}}$ can process more than one copy of a buyer in a realization. We first address the issue of the inventory constraint. Using the correlation gap, we get that the expected number of allocations in $\mathcal{M}_{\text{soft}}$ after first k allocations is at most k_{exp}/e. This alone is not sufficient to prove the lemma as $\mathbf{U}$ is not linear. We note a crucial property of $\mathcal{M}_{\text{soft}}$ in Lemma 12, it establishes that *the revenue from any allocation in $\mathcal{M}_{soft}$ has an identical distribution.* Let $\mathcal{D}_i$ be the distribution on the revenue from first i allocations in $\mathcal{M}_{\text{soft}}$.

Lemma 12. *As $L \to \infty$, we have $\mathbb{P}r_{X_i \sim \mathcal{D}_i, X_{i-1} \sim \mathcal{D}_{i-1}}[(X_i - X_{i-1}) = p_j] = \frac{\sum_{i,v} h(i,j,v)f_i(v)}{k_{exp}}$.*

Consider a new mechanism $\mathcal{M}_{\text{hard}}$ on $\mathcal{I}_{\text{copies}}$ that is identical to $\mathcal{M}_{\text{soft}}$ with an exception that it stops after k allocations. We now bound its utility.

Lemma 13. *As* $L \to \infty$, $\mathbf{U}(\mathcal{M}_{hard}) \geq \gamma(k)\mathbf{U}(\mathcal{M}_{soft})$.

Now we address the issue that $\mathcal{M}_{\text{hard}}$ can process more than one copy of a buyer in a realization. Note that the distribution on the revenue from all copies of first i buyers in $\mathcal{M}_{\text{hard}}$ stochastically dominates the same in $\mathcal{M}_{\text{rounds}}$. Using correlation gap, the expected number of rounds in which buyer i is processed in $\mathcal{M}_{\text{hard}}$ after first processing is $1/e$. Thus the expected loss in the utility can be bounded by a factor $1/e$. This completes the proof. □

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Mechanism Design for Time Critical and Cost Critical Task Execution via Crowdsourcing

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Abstract. An exciting application of crowdsourcing is to use social networks in complex task execution. In this paper, we address the problem of a planner who needs to incentivize agents within a network in order to seek their help in executing an *atomic task* as well as in recruiting other agents to execute the task. We study this mechanism design problem under two natural resource optimization settings: (1) cost critical tasks, where the planner's goal is to minimize the total cost, and (2) time critical tasks, where the goal is to minimize the total time elapsed before the task is executed. We identify a set of desirable properties that should ideally be satisfied by a crowdsourcing mechanism. In particular, *sybil-proofness* and *collapse-proofness* are two complementary properties in our desiderata. We prove that no mechanism can satisfy all the desirable properties simultaneously. This leads us naturally to explore approximate versions of the critical properties. We focus our attention on approximate sybil-proofness and our exploration leads to a parametrized family of payment mechanisms which satisfy collapse-proofness. We characterize the approximate versions of the desirable properties in cost critical and time critical domain.

1 Introduction

Advances in the Internet and communication technologies have made it possible to harness the wisdom and efforts from a sizable portion of the society towards accomplishing tasks which are otherwise herculean. Examples include labeling millions of images, prediction of stock markets, seeking answers to specific queries, searching for objects across a wide geographical area, etc. This phenomenon is popularly known as *crowdsourcing* (for details, see [10] and [7]). *Amazon Mechanical Turk* is one of the early examples of online crowdsourcing platform. The other example of such online crowdsourcing platforms include *oDesk, Rent-A-Coder, kaggle, Galaxy Zoo,* and *Stardust@home.*

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In recent times, an explosive growth in online social media has given a novel twist to crowdsourcing applications where participants can exploit the *underlying social network* for inviting their friends to help executing the task. In such a scenario, the task owner initially recruits individuals from her immediate network to participate in executing the task. These individuals, apart from attempting to execute the task by themselves, recruit other individuals in their respective social networks to also attempt the task and further grow the network. An example of such applications include the DARPA *Red Balloon Challenge* [3], *DARPA CLIQR quest* [4], *query incentive networks* [8], and *multi-level marketing* [6]. The success of such crowdsourcing applications depends on providing appropriate incentives to individuals for both (1) executing the task by themselves and/or (2) recruiting other individuals. Designing a proper incentive scheme (*crowdsourcing mechanism*) is crucial to the success of any such crowdsourcing based application. In the red balloon challenge, the winning team from MIT successfully demonstrated that a crowdsourcing mechanism can be employed to accomplish such a challenging task (see [9]).

A major challenge in deploying such crowdsourcing mechanisms in realistic settings is their vulnerability to different kinds of manipulations (e.g. false name attacks, also known as *sybil attacks* in the literature) that rational and intelligent participants would invariably attempt. This challenge needs to be addressed in a specific manner for a specific application setting at the time of designing the mechanism. The application setting is characterized, primarily, by the nature of the underlying task and secondly, by the high level objectives of the designer. Depending on the nature of the underlying task, we can classify them as follows.

Viral Task. A viral task is the one where the designer's goal is to involve as many members as possible in the social network. This kind of tasks do not have a well defined stopping criterion. Examples of such a task include viral marketing, multi-level marketing, users of a social network participating in an election, etc.

Atomic Task. An atomic task is one in which occurrence of a particular event (typically carried out by a single individual) signifies the end of the task. By definition, it comes with a well defined measure of success or accomplishment. Examples of an atomic task include the DARPA Red Balloon Challenge, DARPA CLIQR quest, query incentive networks, and transaction authentication in Bitcoin system [1].

In this paper, we focus on the problem of designing crowdsourcing mechanisms for atomic tasks such that the mechanisms are robust to any kind of manipulations and additionally achieve the stated objectives of the designer.

2 Prior Work

Prior work can be broadly classified into two categories based on the nature of the underlying task - viral or atomic.

Viral Task. The literature in this category focuses, predominantly, on the problem of multi-level marketing. Emek *et al.* [6] and Drucker and Fleischer [5] have analyzed somewhat similar models for multi-level marketing over a social network.

In their model, the planner incentivizes agents to promote a product among their friends in order to increase the sales revenue. While [6] shows that the geometric reward mechanism uniquely satisfies many desirable properties except false-name-proofness, [5] presents a *capping reward mechanism* that is locally sybil-proof and collusion-proof. The collusion here only considers creating fake nodes in a collaborative way. In all multi-level marketing mechanisms, the revenue is generated *endogenously* by the participating nodes, and a fraction of the revenue is redistributed over the referrers. On slightly different kind of tasks, Conitzer *et al.*[2] proposes mechanisms that are robust to false-name manipulation for applications such as facebook inviting its users to vote on its future terms of use. Further, Yu *et al.*[11] proposes a protocol to limit corruptive influence of sybil attacks in P2P networks by exploiting insights from social networks.

Atomic Task. The red-bal loon challenge [3], query incentive networks [8], and transaction authentication in Bitcoin system [1] are examples of *atomic tasks*. The reward in such settings is *exogenous*, and hence the strategic problems are different from the viral tasks such as multi-level marketing. Sybil attacks still pose a problem here. Pickard *et al.*[9] proposed a novel solution method for Red Balloon challenge and can be considered as an early work that motivated the study of strategic aspects in crowdsourcing applications. [1] provides an *almost uniform* mechanism where sybil-proofness is guaranteed via iterated elimination of weakly dominated strategies. The work by Kleinberg and Raghavan [8] deals with a branching process based model for query incentive networks and proposes a decentralized reward mechanism for the nodes along the path from the root to the node who answers the query.

3 Contributions and Outline

In this paper, we propose design of crowdsourcing mechanisms for atomic tasks such that the mechanisms are robust to any kind of manipulations and additionally achieve the stated objectives of the designer. Our work is distinct from the existing body of related literature in the following aspects.

(1) Collapse-Proofness: We discover that agents can exhibit an important strategic behavior, namely *node collapse attack*, which has not been explored in literature. Though the sybil attack has been studied quite well, a sybil-proof mechanism cannot by itself prevent multiple nodes colluding and reporting as a single node in order to increase their collective reward. A node collapse behavior of the agents is undesirable because, (i) it increases cost to the designer, (ii) the distribution of this additional payment creates a situation of bargaining among the agents, hence is not suitable for risk averse agents, and (iii) it hides the structure of the actual network, which could be useful for other future purposes. A node collapse is a form of collusion, and it can be shown that the sybil-proof mechanisms presented in both [1] and [5] are vulnerable to collapse attack. In this paper, in addition to sybil attacks, we also address the problem of *collapse attacks* and present mechanisms that are collapse-proof.

(2) Dominant Strategy Implementation: In practical crowdsourcing scenarios, we cannot expect all the agents to be fully *rational and intelligent*. We, therefore, take a complementary design approach, where instead of satisfying various desirable properties (e.g. sybil-proofness, collapse-proofness) in the Nash equilibrium sense, [1] we prefer to address a *approximate versions* of the same properties, and design *dominant strategy* mechanisms. If a mechanism satisfies an approximate version of a cheat-proof property then it means the loss in an agents' utility due to him following a non-cheating behavior is bounded (irrespective of what others are doing).

(3) Resource Optimization Criterion: The present literature mostly focuses on the design of a crowdsourcing mechanism satisfying a set of desirable cheat-proof properties. The feasible set could be quite large in many scenarios and hence a further level of optimization of the resources would be a natural extension. In this paper, we demonstrate how to fill this gap by analyzing two scenarios - (1) cost critical tasks, and (2) time critical tasks.

A summary of our specific contributions in this paper is as follows.

1. We identify a set of desirable properties and prove that not even a subset of them are simultaneously satisfiable (Theorem 1).
2. We then prove a possibility result with one property relaxed, but the possibility yields a very restrictive mechanism (Theorem 2).
3. Next, we propose dominant strategy mechanisms for *approximate versions* of these properties, which is complementary to the solution provided by [1] that guarantees sybil-proofness in Nash equilibrium.
4. The approximate versions help expand the space of feasible mechanisms, leading us naturally to the following question: *Which mechanism(s) should be chosen from a bunch of possibilities*? We ask this question in two natural settings: (a) *cost critical tasks*, where the goal is to minimize the total cost, (b) *time critical tasks*, where the goal is to minimize the total time for executing the task [2]. The basic difference between these two scenarios is that in (b) the goal of the designer is not to save money from the budget, rather to dispense off the entire money satisfying the properties. Hope is that with this excess amount of money, the agents will act promptly, leading to a faster execution of the atomic task. Hence the name 'time-critical'. We provide characterization theorems (Theorems 4 and 5) in both the settings for the mechanisms satisfying approximate properties.

To the best of our knowledge, this is the first attempt at providing approximate sybil-proofness and exact collapse-proofness in dominant strategies with certain additional fairness guarantees.

[1] For example, the solution provided by Babaioff *et al.*[1] guarantees sybil-proofness only in Nash equilibrium and not in dominant strategies.

[2] Note, *query incentive networks* [8] and *multi-level marketing* [6] fall under the category of cost critical tasks, while search-and-rescue operations such as red balloon challenge [3] fall under that of time critical tasks.

4 The Model

Consider a planner (such as DARPA) who needs to get an atomic task executed. The planner recruits a set of agents and asks them to execute the task. The recruited agents can try executing the task themselves or in turn forward the task to their friends and acquaintances who have not been offered this deal so far, thereby recruiting them into the system. If an agent receives separate invitations from multiple nodes to join their network, she can accept exactly one invitation. Thus, at any point of time, the recruited agents network is a tree. The planner stops the process as soon as the atomic task gets executed by one of the agents and offers rewards to the agents as per a centralized monetary reward scheme, say R. Let $T = (V_T, E_T)$ denote the final recruitment tree when the atomic task gets executed by one of the recruited agents. In T, the agent who executes the atomic task first is referred to as the *winner*. Let us denote the winner as $w \in V_T$. The unique path from the winner to the root is referred to as the *winning chain*. We consider the mechanisms where only winning chain receives positive payments.

For our setting, we assume that the planner designs the centralized reward mechanism R, which assigns a non-negative reward to every node in the winning chain and zero to all other nodes. Hence, we can denote the reward mechanism as a mapping $R : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ where $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}^+$ is the set of nonnegative reals. In such a mechanism, $R(k,t),\ k \leq t$ denotes the reward of a node which is at depth k in the winning chain, where length of the winning chain is t. The payment is made only after completion of the task. Note, this reward mechanism is anonymous to node identities and the payment is solely dependent on their position in T. Throughout this paper, we would assume that the payment to all nodes of any *non-winning chain* is zero. Hence, all definitions of the desirable properties apply only to the winning chain.

An example of such a reward mechanism is the geometric payment used by [6] and [9]. These mechanisms pay the largest amount to the winner node and geometrically decrease the payment over the path to the root. This class of mechanisms are susceptible to sybil attacks. For example, the winning node can create a long chain of artificial nodes, $\{x_1, ..., x_m\}$, and report that x_i recruits x_{i+1} and x_m is the winner. Then each fake x_i would extract payment from the mechanism.

4.1 Desirable Properties

An ideal reward mechanism of our model should satisfy several desirable properties. In what follows, we have listed down a set of important properties that must be satisfied by an ideal mechanism under dominant strategy equilibrium.

Definition 1 (Downstream Sybilproofness, DSP). *Given the position of a node in a recruitment tree, a reward mechanism R is called* downstream sybilproof*, if the node cannot gain by adding fake nodes below itself in the current subtree (irrespective of what other are doing). Mathematically,*

$$R(k,t) \geq \sum_{i=0}^{n} R(k+i, t+n) \quad \forall k \leq t, \forall t, n. \tag{1}$$

Definition 2 (Budget Balance, BB). *Let us assume the maximum budget allocated by the planner for executing an atomic task is* $R_{\max}$. *Then, a mechanism* R *is budget balanced if,*

$$\sum_{k=1}^{t} R(k,t) \leq R_{\max}, \quad \forall t. \tag{2}$$

Definition 3 (Contribution Rationality, CR). *This property ensures that a node gets non-negative payoff whenever she belongs to the winning chain. We distinguish between strict and weak versions of this property as defined below. For all* $t \geq 1$, *(1)* ***Strict Contribution Rationality (SCR):***

$$R(k,t) > 0, \quad \forall k \leq t, \text{ if } t \text{ is length of the winning chain.} \tag{3}$$

(2) ***Weak Contribution Rationality (WCR):***

$$\begin{aligned} R(k,t) &\geq 0, \quad \forall k \leq t-1, \text{ if } t \text{ is the length of the winning chain.} \\ R(t,t) &> 0, \quad\quad \text{winner gets positive reward.} \end{aligned} \tag{4}$$

DSP ensures that an agent in the network cannot gain additional payment by creating fake identities and pretending to have recruited these nodes. SCR ensures that nodes have incentive to recruit, since all members of the winning chain are rewarded.

There exist some reward mechanisms that satisfy these three properties. For example, let us consider a mechanism that diminishes the rewards geometrically in both k and t, i.e. $R(k,t) = \frac{1}{2^{k+t}} \cdot R_{\max}$. This mechanism pays heavy to the nodes near the root and less near the leaf. We call this class of mechanisms as *top-down* mechanisms. This mechanism satisfies DSP, BB, and SCR properties for any finite t. However, the best response strategy of the agents in this type of mechanisms could introduce other kinds of undesirable behavior. For example, the agents of any chain would be better off by colluding among themselves and representing themselves as a single node in front of the designer, since if the winner emerges from that particular chain, they would gain more collective reward than they could get individually. We call this *node collapse problem.* This introduces a two-fold difficulty. First, the designer cannot learn the structure of the network that executed the task, and hence cannot use the network structure for future applications. Second, she ends up paying more than what she should have paid for a true network. Hence, in the scenario where designer is also willing to minimize the expenditure, she would like to have *collapse-proofness.*

Definition 4 (Collapse-Proofness, CP). *Given a depth* k *in a winning chain, a reward mechanism* R *is called* collapse-proof, *if the subchain of length* p *down below* k *collectively cannot gain by collapsing to depth* k *(irrespective of what others are doing). Mathematically,*

$$\sum_{i=0}^{p} R(k+i,t) \geq R(k,t-p) \quad \forall k+p \leq t, \forall t. \tag{5}$$

In the following section, we will show that some of these properties are impossible to satisfy together. To this end, we need to define a class of mechanisms, called *Winner Takes All (WTA)*, where the winning node receives a positive reward and all other nodes get zero reward.

Definition 5 (WTA Mechanism). *A reward mechanism R is called WTA mechanism if $R_{\max} \geq R(t,t) > 0$, and $R(k,t) = 0,\ \forall k < t$.*

5 Impossibility and Possibility Results

Theorem 1 (Impossibility Result). *For $t \geq 3$, no reward mechanism can satisfy DSP, SCR, and CP together.*

Proof: Suppose the reward mechanism R satisfies DSP, SCR, and CP. Then by CP, let us put $t \leftarrow t+n$ and $p \leftarrow n$ in Equation 5, and we get, $\sum_{i=0}^{n} R(k+i, t+n) \geq R(k, t+n-n) = R(k,t),\ \forall k \leq t, \forall t, n$. This is same as Equation 1 with the inequality reversed. So, to satisfy DSP and CP together, the inequalities reduce to the following equality.

$$R(k,t) = \textstyle\sum_{i=0}^{n} R(k+i, t+n), \forall k \leq t, \forall t, n. \tag{6}$$

Now we use the following substitutions, leading to the corresponding equalities.

$$\text{put } k \leftarrow t-2, t \leftarrow t-2, n \leftarrow 2, \text{ to get,}$$
$$R(t-2,t-2) = R(t-2,t) + R(t-1,t) + R(t,t) \tag{7}$$
$$\text{put } k \leftarrow t-1, t \leftarrow t-1, n \leftarrow 1, \text{ to get,}$$
$$R(t-1,t-1) = R(t-1,t) + R(t,t) \tag{8}$$
$$\text{put } k \leftarrow t-2, t \leftarrow t-2, n \leftarrow 1, \text{ to get,}$$
$$R(t-2,t-2) = R(t-2,t-1) + R(t-1,t-1) \tag{9}$$
$$\text{put } k \leftarrow t-2, t \leftarrow t-1, n \leftarrow 1, \text{ to get,}$$
$$R(t-2,t-1) = R(t-2,t) + R(t-1,t) \tag{10}$$

Substituting the value of Eq. 8 on the RHS of Eq. 9,

$$R(t-2,t-2) = R(t-2,t-1) + R(t-1,t) + R(t,t) \tag{11}$$

Substituting Eq. 11 on the LHS of Eq. 7 yields

$$R(t-2,t) = R(t-2,t-1) \tag{12}$$

From Eq. 12 and Eq. 10, we see that,

$$R(t-1,t) = 0. \tag{13}$$

which contradicts SCR. □

From the above theorem and the fact that additional properties reduce the space of feasible mechanisms, we obtain the following corollary.

Corollary 1. *For $t \geq 3$, it is impossible to satisfy DSP, SCR, CP, and BB together.*

Theorem 2 (Possibility Result). *For $t \geq 3$, a mechanism satisfies DSP, WCR, CP and BB iff it is a WTA mechanism.*

Proof: ($\Leftarrow$) It is easy to see that WTA mechanism satisfies DSP, WCR, CP and BB. Hence, it suffices to investigate the other direction.

($\Rightarrow$) From Equations 8 and 13, we see that, $R(t-1,t-1) = R(t,t)$, which is true for any t. By induction on the analysis of Theorem 1 for length $t-1$ in place of t, we can show that $R(t-2,t-1) = 0$. But, by Eq. 12, $R(t-2,t-1) = R(t-2,t)$. Hence, $R(t-2,t) = 0$. Inductively, for all t and for all $k < t$, $R(k,t) = 0$. It shows that for all non-winner nodes, the reward would be zero. So, we can assign any positive reward to the winner node and zero to all others, which is precisely the WTA mechanism. This proves that for WCR, the reward mechanism that satisfies DSP, CP and BB must be a WTA mechanism. □

6 Approximate Versions of Desirable Properties

The results in the previous section are disappointing in that the space of mechanisms satisfying desirable properties is extremely restricted (WTA being the only one). This suggests two possible ways out of this situation. The first route is to compromise on stronger equilibrium notion of dominant strategy and settle for a slightly weaker notion such as Nash equilibrium. The other route could be to weaken these stringent properties related to cheat-proofness and still look for a dominant strategy equilibrium. We choose to go by the later way because Nash equilibrium makes assumptions of all players being rational and intelligent which may not be true in crowdsourcing applications. Therefore, we relax some of the desirable properties to derive their approximate versions. We begin with approximation of the DSP property.

Definition 6 (ϵ - Downstream Sybilproofness, ϵ-DSP). *Given the position of the node in a tree, a payment mechanism R is called ϵ -* DSP*, if the node cannot gain by more than a factor of $(1+\epsilon)$ by adding fake nodes below herself in the current subtree (irrespective of what others are doing). Mathematically,*

$$(1+\epsilon) \cdot R(k,t) \geq \textstyle\sum_{i=0}^{n} R(k+i,t+n), \forall k \leq t, \forall t, n. \tag{14}$$

Theorem 3. *For all $\epsilon > 0$, there exists a mechanism that is ϵ-DSP, CP, BB, and SCR.*

Proof: The proof is constructive. Let us consider the following mechanism: set $R(t,t) = (1-\delta) \cdot R_{\max}, \forall\ t$, the reward to the winner, where $\delta \leq \frac{\epsilon}{1+\epsilon}$. Also, let $R(k,t) = \delta \cdot R(k+1,t) = \delta^{t-k} \cdot R(t,t) = \delta^{t-k}(1-\delta)R_{\max}, k \leq t-1$. By construction, this mechanism satisfies BB. It is also SCR, since $\delta \in (0,1)$. It remains to show that this satisfies ϵ-DSP and CP. Let us consider,

$$\begin{aligned}
&\textstyle\sum_{i=0}^{n} R(k+i,t+n) = \sum_{i=0}^{n} \delta^{t+n-k-i} \cdot R(t+n,t+n) \\
&= \delta^{t-k} \cdot (1+\delta+\cdots+\delta^{n}) \cdot (1-\delta)R_{\max} \\
&= R(k,t) \cdot (1+\delta+\cdots+\delta^{n}) \\
&\leq R(k,t) \cdot \frac{1}{1-\delta} \leq (1+\epsilon) \cdot R(k,t), \quad \text{since } \delta \leq \frac{\epsilon}{1+\epsilon}.
\end{aligned}$$

This shows that this mechanism is ϵ-DSP. Also,

$$\sum_{i=0}^{p} R(k+i,t) = \sum_{i=0}^{p} \delta^{t-k-i} \cdot R(t,t)$$
$$= \sum_{i=0}^{p-1} \delta^{t-k-i} \cdot R(t,t) + \underbrace{\delta^{t-k-p} \cdot R(t,t)}_{=R(k,t-p)} \geq R(k,t-p)$$

This shows that this mechanism is CP as well. □

Discussion: (a) The above theorem suggests that merely weakening the DSP property allows a way out of the impossibility result given in Theorem 1. One can try weakening the CP property analogously (instead of DSP) and check for the possibility/impossibility results. This we leave as an interesting future work. **(b)** One may argue that no matter how small is ϵ, as long we satisfy the ϵ-DSP property, an agent would always find it beneficial to add as many sybil nodes as possible. However, in real crowdsourcing networks, there would be a non-zero cost involved in creating fake nodes and hence there must be a threshold point so that the agent's net gain would increase till he creates that many sybil nodes but starts declining after that. Note, it is impossible for an agent to compute the threshold point a priori as his own reward is uncertain at the time of him getting freshly recruited by someone and he trying to create sybil nodes. Therefore, in the face of this uncertainty, the agent can assure himself of a bounded regret if he decides not to create any sybil nodes.

6.1 Motivation for δ-SCR and γ-SEC

As per the previous theorem, the class of mechanisms that satisfy ϵ-DSP, CP, BB, and SCR is non-empty. However, the exemplar mechanism of this class, which was used in the proof of this theorem, prompts us to think of the following undesirable consequence - the planner can assign arbitrarily low reward to the winner node and still manage to satisfy all these properties. This could discourage the agents from putting in effort by themselves for executing the task. Motivated by this considerations, we further extend the SCR property by relaxing it to δ-SCR and also introduce an additional property, namely *Winner's γ Security (γ-SEC)*.

Definition 7 (δ - Strict Contribution Rationality, δ-SCR). *This ensures that a node in the winning chain gets at least $\delta \in (0,1)$ fraction of her successor. Also the the winner gets a positive reward. For all $t \geq 1$,*

$$R(k,t) \geq \delta R(k+1,t), \forall k \leq t-1, \; t\text{: winning chain.}$$
$$R(t,t) > 0, \qquad \text{winner gets positive reward.} \tag{15}$$

Definition 8 (Winner's γ Security, γ-SEC). *This ensures that payoff to the winning node is at least γ fraction of the total available budget.*

$$R(t,t) \geq \gamma \cdot R_{\max}, \qquad t \text{ is the winning chain} \tag{16}$$

Discussion: (a) The δ-SCR property guarantees that recruiter of each agent on the winning chain gets a certain fraction of the agent's reward. This property will encourage an agent to propagate the message to her acquaintances even though she may not execute the task by herself. This would result in rapid growth of the network which is desirable in many settings.
(b) On the other hand, γ-SEC ensures that the reward to the winner remains larger than a fraction of the total reward. This works as a motivation for any agent to spend effort on executing the task by herself.

In what follows, we characterize the space of mechanisms satisfying these properties.

7 Cost Critical Tasks

In this section, we design crowdsourcing mechanisms for the atomic tasks where the planner's objective is to minimize total cost of executing the task.

Definition 9 (`MINCOST` over $\mathscr{C}$). *A reward mechanism R is called* `MINCOST` *over a class of mechanisms $\mathscr{C}$, if it minimizes the total reward distributed to the participants in the winning chain. That is, R is* `MINCOST` *over $\mathscr{C}$, if*

$$R \in \arg\min_{R' \in \mathscr{C}} \sum_{k=1}^{t} R'(k,t), \quad \forall t. \tag{17}$$

We will show that the `MINCOST` mechanism over the space of ϵ-DSP, δ-SCR, and BB properties is completely characterized by a simple geometric mechanism, defined below.

Definition 10 ((γ,δ)-Geometric Mechanism, (γ,δ)-GEOM). *This mechanism gives γ fraction of the total reward to the winner and geometrically decreases the rewards towards root with the factor δ. For all t, $R(t,t) = \gamma \cdot R_{\max}$; $R(k,t) = \delta \cdot R(k+1,t) = \delta^{t-k} \cdot R(t,t) = \delta^{t-k} \cdot \gamma R_{\max},\ k \leq t-1$.*

7.1 Characterization Theorem for MINCOST

Now, we will show that (γ,δ)-Geometric mechanism characterizes the space of `MINCOST` mechanisms satisfying ϵ-DSP, δ-SCR, γ-SEC, and BB. We start with an intermediate result.

Lemma 1. *A mechanism is δ-SCR, γ-SEC and BB only if $\gamma \leq 1-\delta$.*

Proof: Suppose $\gamma > 1-\delta$. Then by δ-SCR, we have,

$$\begin{aligned}\sum_{k=1}^{t} R(k,t) &\geq (1+\delta+\cdots+\delta^{t-1}) \cdot R(t,t) &(18)\\ &\geq (1+\delta+\cdots+\delta^{t-1}) \cdot \gamma R_{\max} &(19)\\ &> (1+\delta+\cdots+\delta^{t-1})(1-\delta) R_{\max}\end{aligned}$$

This holds for all $t \geq 1$. It must hold for $t \to \infty$. Hence, $\lim_{t\to\infty} \sum_{k=1}^{t} R(k,t) > \frac{1}{1-\delta} \cdot (1-\delta) R_{\max} = R_{\max}$. Which is a contradiction to BB. □

Theorem 4. *If $\delta \leq \min\{1-\gamma, \frac{\epsilon}{1+\epsilon}\}$, a mechanism is* `MINCOST` *over the class of mechanisms satisfying ϵ-DSP, δ-SCR, γ-SEC, and BB iff it is (γ,δ)-GEOM mechanism.*

Proof: ($\Leftarrow$) It is easy to see that (γ,δ)-GEOM is δ-SCR and γ-SEC by construction. It is also BB since $\delta \leq 1-\gamma$ or $\gamma \leq 1-\delta$. For the ϵ-DSP property, we see that the following expression,

$$\begin{aligned}
&\textstyle\sum_{i=0}^{n} R(k+i, t+n) = \sum_{i=0}^{n} \delta^{t+n-k-i} \cdot R(t+n, t+n) \\
&= \delta^{t-k} \cdot (1+\delta+\cdots+\delta^n) \cdot \gamma R_{\max} \\
&= R(k,t) \cdot (1+\delta+\cdots+\delta^n) \\
&\leq R(k,t) \cdot \frac{1}{1-\delta} \leq (1+\epsilon) R(k,t), \text{ as } \delta \leq \frac{\epsilon}{1+\epsilon}.
\end{aligned}$$

Also for a given δ and γ, this mechanism minimizes the total cost as it pays each node the minimum possible reward. Thus, δ-GEOM mechanism is `MINCOST` over ϵ-DSP, δ-SCR, γ-SEC, and BB.

($\Rightarrow$) Since $\delta \leq 1-\gamma$, from Lemma 1, we see that δ-SCR, γ-SEC, and BB are satisfiable. In addition the objective of the mechanism designer is to minimize the total reward (R_{total}) given to the winning chain.

$$R_{total} = \textstyle\sum_{k=1}^{t} R(k,t) \overset{\text{Eq. 19}}{\geq} (1+\delta+\cdots+\delta^{t-1}) \cdot \gamma R_{\max}$$

We require a mechanism that is also ϵ-DSP and minimizes the above quantity. Let us consider a mechanism R_1 that pays the leaf an amount of $\gamma R_{\max}$ and any other node at depth k, an amount $\delta^{t-k}\gamma R_{\max}$. We ask the question if this mechanism is ϵ-DSP. This is because if this is true, then there cannot be any other mechanism that minimizes the cost, as this achieves the lower bound of R_{total}. To check for ϵ-DSP of this mechanism, we consider the following expression.

$$\begin{aligned}
&\textstyle\sum_{i=0}^{n} R_1(k+i, t+n) = \sum_{i=0}^{n} \delta^{t+n-k-i} \cdot R_1(t+n, t+n) \\
&= \delta^{t-k} \cdot (1+\delta+\cdots+\delta^n) \cdot \gamma R_{\max} \\
&= R_1(k,t) \cdot (1+\delta+\cdots+\delta^n) \\
&\leq R_1(k,t) \cdot \frac{1}{1-\delta} \leq (1+\epsilon) R_1(k,t) \quad \text{ since } \delta \leq \frac{\epsilon}{1+\epsilon}
\end{aligned}$$

implying R_1 is also ϵ-DSP. Hence, R_1 is *the* `MINCOST` mechanism over ϵ-DSP, δ-SCR, γ-SEC, and BB. Note, R_1 is precisely the (γ,δ)-GEOM mechanism. □

Discussion: It can be shown that, (γ,δ)-GEOM mechanism additionally satisfies CP. The proof is omitted due to space constraint.

8 Time Critical Tasks

In applications where the faster growth of network is more important than maximizing the surplus, the designer can spend the whole budget in order to incentivize participants to either search for the answer or forward the information quickly among their acquaintances. In such settings, we can design mechanisms which aim to maximize reward of the leaf node of the winning chain. In this section, we show that such kind of mechanisms with the same fairness guarantees can also be characterized by a similar mechanism that exhausts the budget even for a finite length of the winning chain. In what follows, we define the design goal and a specific geometric mechanism.

Definition 11 (MAXLEAF over $\mathscr{C}$). *A reward mechanism R is called* MAXLEAF *over a class of mechanisms $\mathscr{C}$, if it maximizes the reward of the leaf node in the winning chain. That is, R is* MAXLEAF *over $\mathscr{C}$, if*

$$R \in \arg\max_{R' \in \mathscr{C}} R'(t,t), \quad \forall t. \tag{20}$$

Definition 12 (δ-Geometric mechanism, δ-GEOM). *This mechanism gives $\frac{1-\delta}{1-\delta^t}$ fraction of the total reward to the winner and geometrically decreases the rewards towards root with the factor δ, where t is the length of the winning chain. For all t, $R(t,t) = \frac{1-\delta}{1-\delta^t} \cdot R_{\max}$; $R(k,t) = \delta \cdot R(k+1,t) = \delta^{t-k} \cdot R(t,t),\ k \leq t-1$.*

8.1 Characterization Theorem for MAXLEAF

Theorem 5. *If $\delta \leq \frac{\epsilon}{1+\epsilon}$, a mechanism is* MAXLEAF *over the class of mechanisms satisfying ϵ-DSP, δ-SCR, and BB iff it is δ-GEOM mechanism.*

Proof: ($\Leftarrow$) By construction, the δ-GEOM mechanism is δ-SCR and BB for all t. It is also ϵ-DSP, as,

$$\begin{aligned}
&\textstyle\sum_{i=0}^{n} R(k+i,t+n) = \sum_{i=0}^{n} \delta^{t+n-k-i} \cdot R(t+n,t+n) \\
&= \delta^{t-k} \cdot (1+\delta+\cdots+\delta^n) \cdot R(t+n,t+n) \\
&= \delta^{t-k} R(t,t) \cdot \frac{R(t+n,t+n)}{R(t,t)} \cdot \frac{1-\delta^{n+1}}{1-\delta} \\
&= R(k,t) \cdot \frac{R(t+n,t+n)}{R(t,t)} \cdot \frac{1-\delta^{n+1}}{1-\delta} \\
&= R(k,t) \cdot \frac{\frac{1-\delta}{1-\delta^{t+n}} \cdot R_{\max}}{\frac{1-\delta}{1-\delta^{t}} \cdot R_{\max}} \cdot \frac{1-\delta^{n+1}}{1-\delta} = R(k,t) \cdot \frac{1-\delta^{n+1}}{1-\delta^{t+n}} \cdot \frac{1-\delta^{t}}{1-\delta}.
\end{aligned}$$

Since $\frac{1-\delta^{n+1}}{1-\delta^{t+n}} \uparrow n$ and $\frac{1-\delta^t}{1-\delta} \uparrow t$, we can take limits as $n \to \infty$ and $t \to \infty$ respectively to get an upper bound on the quantity of the RHS, which gives,

$$\sum_{i=0}^{n} R(k+i,t+n) = R(k,t) \cdot \frac{1}{1-\delta} \leq (1+\epsilon) \cdot R(k,t),$$

since $\delta \leq \frac{\epsilon}{1+\epsilon}$. Hence this is ϵ-DSP. Suppose this is not MAXLEAF. Then $\exists$ some other mechanism R' in the same class that pays $R'(t,t) > \frac{1-\delta}{1-\delta^t} \cdot R_{\max}$. Since, R' is also δ-SCR,

$$\sum_{k=1}^{t} R'(k,t) \geq (1+\delta+\cdots+\delta^{t-1}) \cdot R'(t,t)$$
$$= \frac{1-\delta^t}{1-\delta} \cdot R'(t,t) > \frac{1-\delta^t}{1-\delta} \cdot \frac{1-\delta}{1-\delta^t} \cdot R_{\max} = R_{\max},$$

which is a contradiction to BB. Hence proved.

($\Rightarrow$) Let R be a mechanism that is `MAXLEAF` over the class of mechanisms satisfying ϵ-DSP, δ-SCR, and BB. Hence,

$$R_{\max} \geq \sum_{k=1}^{t} R(k,t) \overset{\text{Eq. 18}}{\geq} \frac{1-\delta^t}{1-\delta} \cdot R(t,t)$$
$$\Rightarrow \quad R(t,t) \leq \frac{1-\delta}{1-\delta^t} \cdot R_{\max}, \quad \text{for all } t. \tag{21}$$

The first and second inequalities arise from BB and δ-SCR respectively. Now, from the ϵ-DSP condition of R, we get, for all $n, t, k \leq t$,

$$(1+\epsilon)R(k,t) \geq \sum_{i=0}^{n} R(k+i, t+n)$$
$$\geq \sum_{i=0}^{n} \delta^{t+n-k-i} \cdot R(t+n, t+n)$$
$$= \delta^{t-k} \cdot (1+\delta+\cdots+\delta^n) \cdot R(t+n,t+n),$$

where the second inequality comes from δ-SCR of R. Rearranging, we obtain,

$$1+\epsilon \geq \delta^{t-k} \cdot \frac{1-\delta^{n+1}}{1-\delta} \cdot \frac{R(t+n,t+n)}{R(k,t)} \tag{22}$$

Since this is a necessary condition for any $k \leq t$, it should hold for $k = t$ in particular. Using this in Equation 22 the necessary condition becomes,

$$1+\epsilon \geq \frac{1-\delta^{n+1}}{1-\delta} \cdot \frac{R(t+n,t+n)}{R(t,t)} \tag{23}$$

Now, we have two conditions on $R(t+n,t+n)$ as follows.

$$R(t+n,t+n) \leq (1+\epsilon) \cdot \frac{1-\delta}{1-\delta^{n+1}} \cdot R(t,t)$$
$$\overset{\text{Eq. 21}}{\leq} \underbrace{(1+\epsilon) \cdot \frac{1-\delta}{1-\delta^{n+1}} \cdot \frac{1-\delta}{1-\delta^t} \cdot R_{\max}}_{=:A(n,t)} \tag{24}$$

and using Eq. 21 directly on $R(t+n,t+n)$, we get,

$$R(t+n,t+n) \leq \underbrace{\frac{1-\delta}{1-\delta^{t+n}} \cdot R_{\max}}_{=:B(n,t)} \tag{25}$$

It is clear that to satisfy δ-SCR, ϵ-DSP and BB, it is necessary for R to satisfy,

$$R(t+n, t+n) \leq \min_{n,t}\{A(n,t), B(n,t)\}.$$

We can show the following bounds for the quantity $\frac{B(n,t)}{A(n,t)}$, which we skip due to space constraints.

$$\frac{1}{1+\epsilon} \leq \frac{B(n,t)}{A(n,t)} \leq \frac{1}{(1+\epsilon)(1-\delta)}. \tag{26}$$

Since $\delta \leq \frac{\epsilon}{1+\epsilon}$, we see that the upper bound $\frac{1}{(1+\epsilon)(1-\delta)} \leq 1$. Hence, $A(n,t)$ uniformly dominates $B(n,t)$, $\forall\ n,t$. Hence, $R(t+n,t+n) \leq B(n,t)$. Since R is also `MAXLEAF`, equality must hold and it must be true that,

$$R(t,t) = \frac{1-\delta}{1-\delta^t} \cdot R_{\max}, \forall\ t. \tag{27}$$

Also, since R is BB, it is necessary that,

$$R(k,t) = \delta^{t-k} \cdot R(t,t), \quad k \leq t-1. \tag{28}$$

This shows that R has to be δ-GEOM. □

Discussion: It can be proved that a δ-GEOM mechanism also satisfies CP.

9 Conclusions and Future Work

In this paper, we have studied the problem of designing manipulation free crowd-sourcing mechanisms for atomic tasks under the cost critical and time critical scenarios. We have motivated the need for having CP as an additional property of the mechanism beyond what already exists in the literature. Starting with an impossibility result, we have developed mechanisms for both cost and time critical scenarios which satisfy CP property along with weaker versions of other desirable properties (all under dominant strategy equilibrium). We find that there is a scope for further investigation in the cost-critical setting, but for the time-critical scenario, our results are tight and characterize the entire space of mechanisms. We would characterize the complementary scenarios of our results in the cost-critical setting in our future work.

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Non-redistributive Second Welfare Theorems*

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Abstract. The second welfare theorem tells us that social welfare in an economy can be maximized at an equilibrium given a suitable redistribution of the endowments. We examine welfare maximization without redistribution. Specifically, we examine whether the clustering of traders into k submarkets can improve welfare in a linear exchange economy. Such an economy always has a market clearing ε-approximate equilibrium. As $\varepsilon \to 0$, the limit of these approximate equilibria need not be an equilibrium but we show, using a more general price mechanism than the reals, that it is a "generalized equilibrium". Exploiting this fact, we give a polynomial time algorithm that clusters the market to produce ε-approximate equilibria in these markets of near optimal social welfare, provided the number of goods and markets are constants. On the other hand, we show that it is NP-hard to find an optimal clustering in a linear exchange economy with a bounded number of goods and markets. The restriction to a bounded number of goods is necessary to obtain any reasonable approximation guarantee; with an unbounded number of goods, the problem is as hard as approximating the maximum independent set problem, even for the case of just two markets.

1 Introduction

The fundamental theorems of welfare economics are considered "the most remarkable achievements of modern microeconomic theory" [9] and are the "central set of propositions that economists have to offer the outside world - propositions that are in a real sense, the foundations of Western capitalism" [5]. Informally, they state (under certain conditions that we will discuss later).

First Fundamental Welfare Theorem. A competitive equilibrium is Pareto efficient.

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Second Fundamental Welfare Theorem. Any Pareto efficient solution can be supported as a competitive equilibrium.

The First Welfare Theorem is widely viewed as "a mathematical statement of Adam Smith's notion of the invisible hand leading to an efficient allocation" [12]. The Second Welfare Theorem implies that we can separate out issues of economic efficiency from issues of equity. Specifically, by redistributing the initial endowments (by lump-sum payments), a set of prices exists that can sustain any Pareto solution. This second theorem has "fundamental implications for how we think about economic organization" [13] and is "arguably the theoretical result that has had the most dramatic effect on economic thinking" [4]. Despite this, "much of public economics takes as its starting point the rejection of the practical value of the second theorem" [11]. Why this discrepancy? To understand this, note that lump-sum transfers are theoretically considered a very desirable form of taxation as they do not distort incentives within the pricing mechanism. However, this is essentially accomplished by a massive distortion of the initial market! Moreover, these are *personalized* liabilities which in turn can be viewed as an extremely unfair form of taxation in that they don't depend upon the actions or behaviors of the agents, and are impractical for a myriad of implementational and political reasons (see, for example, [2], [3], [11] and [10]).

This observation motivates our work. Can the market mechanism be used to sustain Pareto allocations without redistribution? In particular, suppose that without redistribution a single market leads to low social welfare (or even has no competitive equilibrium at all). In these circumstances, can the market mechanism still be used to produce an allocation of high social welfare? We address this question under the classical model of exchange economy, and show that indeed this can often be achieved provided the single market can be clustered into submarkets.

1.1 The Exchange Economy

We consider the classical model of an *exchange economy* – an economy without production. We have n traders $i \in \{1, 2, \ldots, n\}$ and m goods $j \in \{1, 2, \ldots, m\}$. (To avoid any ambiguity between traders and goods we will often refer to good j as good g_j). Each trader i has an initial endowment $\mathbf{e_i} \in \mathbb{R}^m_+$, where e_{ij} is the quantity of good g_j that she owns, and a utility function $u_i : \mathbb{R}^m_+ \to \mathbb{R}$. The traders have no market power and so are *price-takers*. Given a set of prices $\mathbf{p} \in \mathbb{R}^m_+$, where p_j is the price of good g_j, trader i will demand the best bundle she can afford, that is, $\text{argmax}_{x_i}\, u_i(\mathbf{x_i})$ *s.t.* $\mathbf{p} \cdot \mathbf{x_i} \leq \mathbf{p} \cdot \mathbf{e_i}$. These prices and demand bundles form a *Walrasian (competitive) equilibrium* if all markets clear: demand does not exceed supply for any good g_j. That is, $\sum_i x_{ij} \leq \sum_i e_{ij}$. In this paper, we focus on the basic case of linear utility functions – the *linear exchange model.* Here the function $u_i(.)$ can be written as $u_i(\mathbf{x}_i) = \sum_{j=1}^m u_{ij}\, x_{ij}$ where $u_{ij} \geq 0$ is the utility per unit that trader i has for good g_j. (We denote by $\mathbf{u}_i$ the vector of utility coefficients for trader i.)

1.2 The Fundamental Welfare Theorems

An allocation is *Pareto efficient* if there is no feasible allocation in which some trader is strictly better off but no trader is worse off. The first welfare theorem states that any Walrasian equilibrium is Pareto efficient. It holds under very mild conditions, such as monotonic utilities or non-satiation. Clearly for this result to be of interest, though, we need this economy to possess Walrasian equilibria. In groundbreaking work, Arrow and Debreu [1] showed that this is indeed the case, under certain conditions such as concave utility functions and positive endowments.[1] Interestingly, equilibria need not exist even in a linear exchange economy. However, there is a combinatorial characterization for existence due to Gale [6], and we discuss this characterization and other properties of the linear exchange economy in detail in Section 2.

Observe that Pareto efficiency is not a particularly restrictive notion: an allocation is efficient unless there is an alternative that is *universally* agreed to be better (or at least as good). This requirement of unanimity has important implications. Allocations that may be viewed as societally better outcomes may be blocked by a single agent. For example, Pareto allocations can be extremely inequitable. The second welfare theorem attempts to address this concern: Any Pareto solution can be supported as a Walrasian equilibrium. Specifically, by redistributing the initial endowments via lump-sum payments, a set of prices exists that can sustain any Pareto solution. (The second theorem also requires concave utility functions.) Thus, the second welfare theorem implies that we can separate out issues of economic efficiency from issues of equity.

As stated, however, the second theorem is of limited practical value due to the infeasibility of direct transfer payments. Thus, our goal is to obtain non-redistributive second welfare theorems. Specifically, maximizing the social welfare, $\sum_{i=1}^{n} u_i(\mathbf{x}_i)$, is a fundamental question in economics; so, can we support at equilibrium an allocation with high social welfare? For example, in a linear exchange economy it is particularly easy to find an optimal social allocation. For each good g_j, simply give all of it to the trader i for whom it proffers the greatest utility per unit. However, even in this basic case, a Walrasian equilibrium may produce very low social welfare. Intuitively the reason is simple: a trader with a large utility coefficient for a good may not be able to afford many units of it. This may be because (a) the good is in high demand and thus has a high price and/or (b) the trader has a small budget because the goods she initially possesses are abundant and, thus, have a low price.

On the other hand, the second welfare theorem tells us that, with redistribution, it is possible to find prices that support an allocation of optimal welfare. Can any more practical, market-based mechanisms achieve this? To answer this, we consider a mechanism that is allowed to cluster the traders into trading groups.

[1] As well as the possibility of non-concave utility function, numerous other factors may affect the practicality of the welfare theorems: market power and the presence of price-makers; incomplete or asymmetric information; externalities; convergence issues for equilibria; the existence of multiple equilibria; economies of scale when production is added to the exchange economy, etc. Such issues are not our focus here.

1.3 Market Clustering

Suppose we partition the traders into k separate markets, for some integer k. In each market t, trade then proceeds as normal with a distinct set of Walrasian prices $\mathbf{p^t}$ and corresponding allocation generated. This partition and the collection of Walrasian equilibria (if they exist) form a *k-equilibrium*. The allocation induced by a k-equilibrium may be very different than one produced from a single market. So, first, can market clustering be used to improve social welfare? If so, second, can it be used to optimize social welfare?

The answer to the first question is `yes`. Trivially, the option to segment the market cannot hurt because we could simply place all the traders in the same market anyway. In fact, market clustering may dramatically improve social welfare; there is an example where the ratio between the social welfare with two markets and the social welfare with one market is unbounded.

The answer to the second question, however, is `no`. Not every Pareto solution can be supported by market clustering. In particular, there are cases where the optimal social solution cannot be obtained by clustering. Indeed, there is an example where the ratio between the optimal social welfare and the optimal welfare that can be generated by market clustering is also unbounded.

The main focus of this paper then becomes to efficiently obtain as large a welfare as possible under market clustering.

We remark that the basic idea underlying market clustering, i.e., the grouping and separation of traders, is a classical one in both economic theory and practice. In particular, it lies at the heart of the theory of trade. On the one hand, countries should trade together (grouping) to exploit the laws of comparative advantage; on the other hand, trade between countries may be restricted (separation) to protect the interests of certain subsections (e.g. specific industries or classes of worker). Interestingly, of course, whilst separation has a net negative effect on welfare in international trade models, our results show that it can have a large net positive effect in general equilibrium models. Other examples that can be viewed as market clustering arise in the regulation of oligopolies and in the issue of trading permits. A less obvious example concerns bandwidth auctions where participants are grouped into “large” (incumbent) and “small” (new-entrants). Trade, with the mechanism in the form of feasible bidding strategies, is then restricted depending upon the group.

1.4 Our Results

Our main result, in Section 6, is a polynomial time algorithm that finds an ε-approximate k-equilibrium, of almost optimal social welfare, provided the number of goods and markets are constants. The key to this result is a limit theorem in Section 5 showing that, in a single market, ε-approximate equilibria converge to what we call a *generalized equilibrium*.

On the other hand, in Section 4 we show that it is NP-hard to find an optimal k-equilibrium in a linear exchange economy with a bounded number of goods and markets. The restriction to a bounded number of goods is necessary to obtain any reasonable approximation guarantees; for linear exchange economies with an unbounded number of goods, the problem is as hard as approximating the maximum independent set problem, even for the case of just 2 markets.

Due to space constraints many proofs of theorems and lemmas are omitted but can be found in the full paper.

2 Walrasian Equilibria in the Linear Exchange Model

Take an equilibrium with prices $\mathbf{p}$ and allocations $\mathbf{x_i}$ for the Walrasian model with linear utilities. Recall, we may assume that the followings hold:

`Budget Constraints:` Trader i cannot spend more than she receives: $\mathbf{p} \cdot \mathbf{x_i} \leq \mathbf{p} \cdot \mathbf{e_i}$ (1)

`Optimality:` Each trader i optimizes the bundle of goods she buys: (2)
$\mathbf{u_i} \cdot \mathbf{x_i}$ is maximized subject to (1)

`Market Clearing:` Demand does not exceed supply, for any g_j: $\sum_i x_{ij} \leq \sum_i e_{ij}$ (3)

2.1 Properties of Equilibria

The following claims are well-known facts (see *e.g.* [8]).

Claim 1. *At equilibrium, the budget constraint (1) is tight for any trader.*

Claim 2. *At equilibrium, the market clearing condition (3) is tight for any g_j with $p_j > 0$.*

Claim 3. *At equilibrium, for any subset S of traders, there is a good g_j such that*

$$\sum_{i \in S} x_{ij} > \sum_{i \in S} e_{ij} \iff \text{there is a good } g_{j'} \text{ with } \sum_{i \notin S} x_{ij'} > \sum_{i \notin S} e_{ij'}$$

Claim 4. *At equilibrium, for any i with $P_i := \mathbf{p} \cdot \mathbf{e}_i > 0$ and any good g_j with price $p_j > 0$*

$$\frac{u_{ij}}{p_j} \leq \frac{\mathbf{u}_i \cdot x_i}{P_i} \tag{4}$$

Moreover, the inequality is tight for any i, j with $x_{ij} > 0$.

2.2 The Existence of Equilibria in a Single Market

Gale [6] gave a characterization for when linear exchange economies possess equilibria. Observe that the price of every good will be strictly positive provided that each good is owned by at least one trader, and at least one trader desires it.

We may assume this is the case as any good that does not satisfy this condition may be removed from the model; in this case, supply will exactly equal demand for each good. Gale also assumes that every trader is *non-altruistic* in that they each desire at least one good. (We say that a trader i is an *altruist* if $u_{ij} = 0$ for every good g_j.)

Theorem 5 ([6]). *An altruist-free linear exchange economy has a Walrasian equilibrium if and only if there is no super self-sufficient set of traders.*

Here, a subset S of traders is called *super self-sufficient* if

1. `Self-Sufficiency:` $\sum_{i \notin S} e_{ij} = 0$ for every good g_j such that $\sum_{i \in S} u_{ij} > 0$.
2. `Superfluity:` There is a good g_j such that $\sum_{i \in S} e_{ij} > 0$ and $\sum_{i \in S} u_{ij} = 0$.

It will be useful to reinterpret Gale's condition combinatorially using the market graph. The *market graph* G_M for a given market is a directed graph whose set of vertices is the set of goods in that market. There is an arc $j \to j'$ with label i if there is a trader i with $e_{ij} > 0$ and $u_{ij'} > 0$; thus, trader i has good g_j and, depending upon the prices, is willing to trade it for good $g_{j'}$. Let $h(S)$ denote the goods that are the heads of arcs with labels from traders in S, and let $t(\bar{S})$ denote the goods that are the tails of arcs corresponding to traders not in S. Then S is self-sufficient if $h(S) \cap t(\bar{S}) = \emptyset$. In this case, $h(S)$ induces a directed in-cut in the market graph. (Thus, a sufficient – but not necessary – condition for the existence of an equilibrium is the strong connectivity of the market graph. Moreover, any directed cut will correspond to a self-sufficient set.) If, in addition, $h(S)$ is a strict subset of $t(S)$, then S is super self-sufficient.

For example, the market graph shown in Figure 1 does not have an equilibrium. It represents a market with 6 traders and 5 goods $g_1, \dots g_5$: each arc $g \xrightarrow{i} h$ represents one trader i with $e_{ig} = 1$ and $u_{ih} = 1$; all other values are 0. Then traders $\{4, 5, 6\}$ form a super self-sufficient set, so this market does not have an equilibrium.

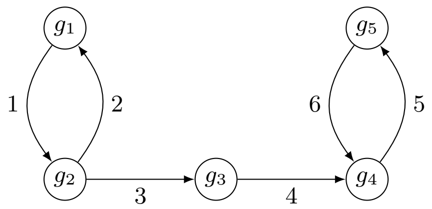

Fig. 1. A market graph with a super self-sufficient set and, therefore, has no equilibrium

We can use the market graph to test Gale's condition efficiently. Furthermore, Jain [8] gave a polynomial time algorithm to find an equilibrium when one exists.

2.3 The Existence of Equilibria in a Market Clustering

Recall that a trader i is an altruist if $u_{ij} = 0$ for every good g_j. An economy is *altruistic* if it is allowed to contain altruistic traders. It is important for us to understand the implications of altruism even for economies without altruists. This is because clustering may create de facto altruists in the submarkets. Moreover, such altruists are one of the factors that allow the equilibria to exist in a market clustering, even if the single market has no equilibrium. We can easily extend Gale's theorem to altruistic economies.

Theorem 6. *An altruistic, linear exchange economy has an equilibrium if and only if every super self-sufficient set of traders contains at least one altruist.*

So, altruistic economies need not have equilibria. However, they can always be clustered into markets with equilibria provided that the number of markets k is at least the number of goods m.

Theorem 7. *An altruistic, linear exchange economy with m goods has an m-equilibrium.*

Proof. We prove this by induction on the number of goods. An altruistic economy with one good g_j has a trivial equilibrium. Now take an altruistic economy with m goods. If it has no super self-sufficient set of traders consisting entirely of non-altruists then, by Theorem 6, it has an equilibrium. Otherwise, let S be a minimal super self-sufficient set of non-altruists. By minimality, the market induced by S contains an equilibrium as it has no super self-sufficient subset.

As they are not altruistic, each trader in S desires at least one good. By definition, however, traders in S desire no goods held by traders in $\bar{S}$. So, there is at least one good held by S that is not held by traders in $\bar{S}$. Thus, the market induced by the traders in $\bar{S}$ contains at most $m-1$ goods. By induction it can be partitioned into $m-1$ clusters that each has an equilibrium. Together with the cluster S, we obtain an m-equilibrium. □

For example, consider again the market in Figure 1. If we partition the traders into two, with trader 3 alone in the first market and traders $\{1, 2, 4, 5, 6\}$ in the second market, then both resulting markets have equilibria (with $x_{32} = 1$ in the first market, and $p_3 = 0$ in the second market).

3 Single Markets, Market Clustering and Welfare Redistribution

In this section, we examine the potential benefits of market clustering and the limits of its power as a tool. First, we have seen that equilibria may not exist in the single market case (i.e., when market clustering is prohibited). In such instances, by Theorem 7, market clustering can always be applied to produce equilibria. Furthermore, even when equilibria do exist in the single market case, market clustering may lead to huge improvements in social welfare in comparison.

On the other hand, market clustering is not as powerful as welfare redistribution; specifically, market clustering does not always support every Pareto allocation. To see this, we consider two measures regarding the social welfare function:

1. `The Clustering Ratio:` the ratio between the maximum social welfare under market clustering and the social welfare obtained in a single market.
2. `The Redistribution Ratio:` the ratio between the maximum achievable social welfare (with welfare redistribution) and the maximum welfare under market clustering.

Examples showing that both ratios can be unbounded can be found in the full paper.

4 The Hardness of Market Clustering

In this section, we consider the hardness of the k-market clustering problem. We show that the problem is NP-hard even if we only have a fixed number of goods and a fixed number of markets, that is, m and k are constant.

Theorem 8. *Given an instance of the* 2*-market clustering problem with five goods and linear utility functions, it is NP-hard to decide whether there is a clustering that yields a social welfare of value at least* Z*, for any* $Z > 0$*.*

The problem becomes much harder when the number of goods is unbounded.

Theorem 9. *For any constant* $\delta > 0$ *and maximum social welfare* Z*, unless* $\mathrm{NP} = \mathrm{ZPP}$*, it is hard to distinguish between the following two cases:*

– `Yes-Instance:` *There is a clustering that yields a social welfare of value at least* $Z^{1-\delta}$*.*
– `No-Instance:` *There is no clustering that yields a social welfare of value at least* Z^{δ}*.*

5 Approximate Walrasian Equilibria

We are interested in finding an *ε-approximate market equilibrium*; that is, for each market, our algorithm outputs a price $\mathbf{p}$ and an allocation $\mathbf{x}$ satisfying the following conditions.

– `Budget Constraints:` Trader i cannot spend more than she receives: $\mathbf{x}_i \cdot \mathbf{p} \leq \mathbf{e}_i \cdot \mathbf{p}$
– `Approximate Optimality:` Subject to the budget constraints, each trader i purchases a bundle $\mathbf{x}_i$ whose utility is similar to that of the optimal bundle $\mathbf{x}_i^*$: $\mathbf{u}_i \cdot \mathbf{x}_i \geq (1-\varepsilon)\mathbf{u}_i \cdot \mathbf{x}_i^*$
– `Market Clearing:` Demand never exceeds supply: for any g_j, $\sum_i x_{ij} \leq \sum_i e_{ij}$

5.1 Existence of Approximate Walrasian Equilibria

Compared to exact market equilibria, which do not always exist, there is always an approximate market equilibrium with arbitrary small approximation:

Theorem 10. *For $\varepsilon > 0$, every market has a market-clearing ε-approximate equilibrium.*

By *market-clearing ε-approximate equilibrium*, we mean an ε-approximate equilibrium for which the approximate market clearing inequality is tight. This theorem can be inferred from the algorithm of [7]. A direct proof of this fact can be found in the full paper.

5.2 Properties of Approximate Walrasian Equilibria

We now discuss some properties of equilibria that will later be very useful to us in designing efficient algorithms. Given a market, we use the following definitions:

- $u_{\max} = \max_{i,j} u_{ij}$ is the maximum coefficient of any utility function.
- $u_{\min} = \min_{i,j:u_{ij}>0} u_{ij}$ is the minimum *non-zero* coefficient of any utility function.
- $p_{\max} = \max_j p_j$ is the maximum price of any good in the market.
- $p_{\min} = \min_{j:p_j>0} p_j$ is the minimum *non-zero* price of any good in the market.
- $e_{\min} = \min_{i,j:e_{ij}>0} e_{ij}$ is the minimum *non-zero* endowment for any good and trader.

Assume wlog that $\sum_i e_{ij} = 1$, for every good g_j. We can connect the above values via the market graph. Recall that n denotes the number of traders and m denotes the number of goods; then we obtain

Lemma 1. *If the market graph is strongly connected then, at a market equilibrium,*

$$\frac{p_{\max}}{p_{\min}} \le e^{\frac{nm}{e}} \left(\frac{u_{\max}}{e_{\min} u_{\min}} \right)^{nm}$$

In particular, scaling so that $p_{\min} = 1$ gives $p_{\max} \le e^{\frac{nm}{e}} \left(\frac{u_{\max}}{e_{\min} u_{\min}} \right)^{nm}$.

Proof. We may assume (solely for the duration of this proof) that each trader has a positive endowment for exactly one good and no two traders have positive endowments for the same good [8]. To do this, consider a trader i with endowment $\mathbf{e_i}$. For each good g_j such that $e_{i,j} > 0$, we create a new trader i_j with $e_{i_j j} = e_{ij}$, $\mathbf{u}_{i_j} = \mathbf{u}_i$ and $e_{i_j j'} = 0$, for all $j' \neq j$. So, each trader now has only one good. Furthermore, if two traders have the same good, then we simply give the good two different names (and replicate the utility functions of other traders accordingly). Now, each trader represents a unique good, i.e., a trader i has a positive endowment for the unique good g_i. So, we have at most nm traders/goods.

These transformations maintain the strong connectivity of the market graph. Moreover, all the copies of the same original good will have the same price in an equilibrium. After these transformations, the number of units of each good will in general be less than one. Thus, we scale all the initial endowments so that each trader i has one unit of a good g_i. In addition, we must scale the coefficients of utility functions; otherwise, the scaling would effect the social welfare. Specifically, for each good g_i, we divide the initial endowments of trader i by e_{ii}, and we multiply the utilities of every trader for this good by e_{ii}, so as to keep the prices unchanged.

We may assume that no good has a price of zero. By Equation (4), we have:

$$\frac{u'_{ij}}{p_j} \le \frac{\sum_\ell u'_{i\ell} \cdot x_{i\ell}}{p_i}$$

For a pair i, j with $u'_{ij} > 0$, we get

$$\frac{p_i}{p_j} \le \frac{\sum_\ell u'_{i\ell} \cdot x_{i\ell}}{u'_{ij}} \le \frac{u'_{\max}}{u'_{\min}} \sum_\ell x_{i\ell}$$

Assume $p'_{\min} = p'_{i_0}$ and $p'_{\max} = p'_{i_s}$, where $s \le nm$. Because the market is strongly connected, there is a sequence of traders with indices $i_0, i_1 \dots, i_s$, $s \le nm$, such that $u'_{i_{j-1} i_j} > 0$ for all $i \in \{1, \dots, s\}$. Multiplying the previous inequalities for all consecutive terms of this sequence, we get

$$\frac{p_{\max}}{p_{\min}} = \prod_{j=0}^{s-1} \frac{p_{i_{j+1}}}{p_{i_j}} \le \left(\frac{u'_{\max}}{u'_{\min}}\right)^s \cdot \prod_{j=0}^{s} \left(\sum_\ell x_{j\ell}\right) \le \left(\frac{nm}{s}\right)^s \left(\frac{u'_{\max}}{u'_{\min}}\right)^{nm} \le e^{\frac{nm}{e}} \left(\frac{u'_{\max}}{u'_{\min}}\right)^{nm}$$

Here the second inequality follows from the Arithmetic-Geometric Mean Inequality and the fact that $\sum_{j=0}^{s} \sum_\ell x_{j\ell} \le \sum_{j=0}^{s} \sum_\ell e_{j\ell} \le nm$ by the market clearing constraint (3). Now, observe that there is a pair i, j such that $u'_{\min} = e_{jj} u_{ij} \ge e_{\min} u_{\min}$, and there is a (different) pair i, j such that $u'_{\max} = e_{jj} u_{ij} \le u_{\max}$. □

The same reasoning applied to approximate equilibria gives:

Lemma 2. *If the market graph is strongly connected, at an ε-approximate market equilibrium, we have* $\frac{p_{\max}}{p_{\min}} \le e^{\frac{nm}{e}} \left(\frac{u_{\max}}{(1-\varepsilon) u_{\min} e_{\min}}\right)^{nm}$.

It is possible for a market that is not strongly connected to have an equilibrium: the owners of the goods reachable from a strongly connected set induce a self-sufficient set but not necessarily a super self-sufficient set. However, in this case, we cannot bound the ratio $p_{\max}/p_{\min}$ as seen from the following lemma.

Lemma 3. *Consider a market with equilibrium* $\mathbf{p}, \mathbf{x}$. *Let W be a proper subset of goods such that for any trader i, if there is some good $g_j \in W$ with $e_{ij} > 0$, then $u_{ik} = 0$ for all $k \notin W$ (i.e., W is the shore of a directed cut in the market graph). Then, for any $B > 1$, $\mathbf{p}', \mathbf{x}$ is also an equilibrium where $p'_j = p_j$ if $g_j \notin W$ and $p'_j = B p_j$ if $g_j \in W$.*

Proof. The lemma follows from the following two facts. First, $x_{ij} = 0$ for any good $g_j \notin W$ and any trader i with $\sum_{k \in W} e_{ik} > 0$ since then $u_{ij} = 0$ by the definition of W. Second, $x_{ij} = 0$ for any trader i with $\sum_{k \in W} e_{ik} = 0$ and $g_j \in W$. Consequently, scaling the prices of goods in W does not effect the equilibrium. □

An implication of this lemma is that the strongly connected components have price allocations that are essentially independent of each other: for example one could decompose the problem, find local equilibria in each component, and then scale the prices accordingly to get a global equilibrium. Also, again by scaling the prices of W, we can assume that the minimum price in W is no more than $u_{\max}/u_{\min}$ times the maximum price outside W, as it does not change the optimality of the allocations (it would be a problem if there was a trader with $\sum_{j \notin W} u_{ij} = 0$ and $\sum_{j \notin W} e_{ij} = 0$, but then taking this trader plus $\{i : \sum_{j \in W} e_{ij} > 0\}$ would give a super self-sufficient set). This gives the following strengthening:

Lemma 4. *Any market having an equilibrium has one such that* $\frac{p_{\max}}{p_{\min}} \leq e^{\frac{nm}{e}} \left(\frac{u_{\max}}{e_{\min} u_{\min}}\right)^{nm}$.

5.3 Limits of Equilibria

By Theorem 10, for any $\varepsilon > 0$ there is a market-clearing ε-approximate equilibrium. When ε tends to 0, the prices of these approximate equilibria may diverge (if no exact equilibrium exists), but the allocations of goods to traders, as they are chosen from a compact set, admit at least one limit point, an allocation $\mathring{\mathbf{x}}$. We call such an allocation a *limit allocation.* In particular, if the market admits an exact equilibrium, then $\mathring{\mathbf{x}}$ is the allocation of an exact equilibrium (if $\mathring{\mathbf{x}}$ could not be obtained as a limit of approximate equilibria with converging prices, one could exhibit a super self-sufficient set and this would be a contradiction). In any case, $\mathring{\mathbf{x}}$ satisfies the market clearing constraints with equality.

The allocation $\mathring{\mathbf{x}}$ may not be supported by a set of real prices. For example, there is obviously no set of prices supporting $\mathring{\mathbf{x}}$ when the market does not have an exact equilibrium. We show that $\mathring{\mathbf{x}}$ can be supported by taking prices from a set more general than the real numbers. Consider the set $Q = \mathbb{N} \times \mathbb{R}_+$, our new set of "prices". We denote by π_1 and π_2 the first and second projection, *i.e.*, $\pi_1(x, y) = \pi_2(y, x) = x$. We extend these projections to vectors (and abuse notation) by: $\pi_i\left((v_j)_j\right) = \left(\pi_i(v_j)\right)_j$. We then redefine the notion of equilibrium in terms of Q. For $p \in Q^m$ and $\mathring{\mathbf{x}} \in \mathbb{R}_+^{m \times n}$, let the *rank* r_i of i be the maximum a such that $\sum_{j\,:\,\pi_1(p_j)=a} e_{ij} > 0$, for all i. The pair $\mathbf{p}, \mathbf{x}$ is a *generalized equilibrium* if

– `Budget Constraints:` For all $i \in \{1, \ldots, n\}$, for all $a \geq r_i$,

$$\sum_{j\,:\,\pi_1(p_j)=a} \pi_2(p_j) \cdot \mathbf{x}_{ij} \leq \sum_{j\,:\,\pi_1(p_j)=a} \pi_2(p_j) \cdot \mathbf{e}_{ij}$$

- `Optimality`: For each trader, $\mathbf{x}_i$ maximizes the utility $\mathbf{u}_i \cdot \mathbf{x}_i$ over all allocations satisfying the budget constraint.
- `Market Clearing`: No good is in deficit: $\sum_i x_{ij} \le \sum_i e_{ij}$ for all goods j with $\pi_2(p_j) > 0$.

This is indeed a generalization. If we force the prices to be in $\{0\} \times \mathbb{R}_+$ then a generalized equilibrium would give a Walrasian equilibrium. An *ε-approximate generalized equilibrium* is defined by replacing the optimality condition by: $\mathbf{u}_i \cdot \mathbf{x}_i$ is at least $(1-\varepsilon)$ times the utility of a best response of trader i, for all i.

Theorem 11. *For any market, each limit allocation $\mathring{\mathbf{x}}$ gives a generalized equilibrium.*

A generalized equilibrium can be approximated by an approximate Walrasian equilibrium with almost as high welfare. The converse is not true. An approximate equilibrium may achieve a welfare arbitrarily high compared to a generalized equilibrium; consider the market with two traders and two goods where $e_{11} = e_{22} = 1$, $u_{12} = L$, $u_{22} = 1$ and all the other values are zero. In this market, the only generalized equilibrium has welfare 1, but there is an approximate equilibrium with welfare $\varepsilon \cdot L + (1-\varepsilon)$, and this tends to $+\infty$ when L tends to $+\infty$.

Lemma 5. *Let $\mathring{\mathbf{x}}, \mathring{\mathbf{p}}$ be a generalized equilibrium. For any $\varepsilon > 0$, there is an approximate equilibrium with total welfare at least $1-\varepsilon$ times the welfare of $\mathring{\mathbf{x}}, \mathring{\mathbf{p}}$.*

6 A Fully Polynomial Time Approximation Scheme

In this section, we exploit the structure we have now developed to obtain a polynomial time algorithm to find an ε-approximate equilibrium for the k-market clustering problem where the number of goods m and the number of markets k are constant. Moreover, this equilibrium has a very strong welfare guarantee: it gives social welfare of at least $1-\varepsilon$ times the welfare of the optimal k-cluster generalized equilibrium.

Theorem 12. *For any $\varepsilon > 0$ and for fixed $k, n \in \mathbb{N}$, there is an algorithm that, given a market M with m goods, computes within time polynomial in $\frac{1}{\varepsilon}$ and the size of the M, an ε-approximate generalized k-equilibrium for M with welfare at least $1-\varepsilon$ the optimal welfare of a generalized k-equilibrium.*

6.1 The Dynamic Program

Our dynamic program takes as an input a set of generalized prices for every good in each market. This follows as we may try all possible prices selected from a finite set of prices in $\{1, \ldots, m\} \times P$ where $P = \{1, 1+1/b, (1+1/b)^2, \ldots, (1+1/b)^{\sigma-2}\}$. Here $b \in \mathbb{N}_+$ is a parameter to be set later, and σ is such that $(1+1/b)^{\sigma-3} \le$

$e^{nm/e}\left(\frac{m\cdot u_{max}}{e_{min}u_{min}}\right)^{nm} \le (1+1/b)^{\sigma-2}$. A generalized price (a,p) encodes a real price $\nu(a,p)$ given by $L^a \cdot p$, where L is an arbitrarily large constant. So, we have $m\cdot\sigma$ possible prices for each good in each market. Thus, the number of combinations of prices is $(m\sigma)^{km}$, where k is the number of markets and m is the number of goods.

Given the estimated prices, the dynamic program runs over each market to compute an approximate equilibrium that maximizes the total social welfare. We denote the estimated prices in market t by $\mathbf{p}^t \in (\{1,\ldots,m\}\times P)^m$, *i.e.*, p_j^t is the price of a good g_j in a market t. We denote an initial endowment and a final allocation in a clustered market by $\mathbf{e}_i^t$ and $\mathbf{x}_i^t$. (Thus, $\sum_{t=1}^k \mathbf{e}_i^t = \mathbf{e}_i$.)

The algorithm considers each trader iteratively. At each iteration, it assigns the trader to a market and gives a near-optimal bundle to this trader, according to her utility function and the prices in that market. Once the ith trader is assigned, the algorithm only remembers the deficit (or surplus) of each good in every market – this will be a key in obtaining an efficient algorithm. Once every trader is assigned, it selects the best possible solution that satisfies the approximate market clearing constraint, *i.e.*, the deficits must be small. Thus, we encode the state of each market t by a vector $\mathbf{y}^t$, where y_j^t denotes the surplus (or deficit) of the good g_j. Let I_t denote the set of traders already in the market t, and $\mathbf{x}_i, i\in I_t$ the bundles given to these traders. Ideally, we would like to have $y_j^t = \sum_{i\in I_t} e_{ij} - \sum_{i\in I_t} x_{ij}$. Hence, the value of y_j^t could be any real value between -1 and 1. However, as we cannot afford to store all possible values for y^t, we round these values into a set $\widetilde{W}$ of cardinality $\alpha\cdot 4n$, where α will be set later. To define $\widetilde{W}$, we first define a coarser set W.

$$W = \left\{\left(\frac{b}{b+1}\right)^{\alpha}, \left(\frac{b}{b+1}\right)^{\alpha-1}, \ldots, \frac{b}{b+1}, 1\right\}.$$

We then choose α to be minimal such that

$$\left(\frac{b}{b+1}\right)^{\alpha} \le \frac{e_{\min}}{2n(b+1)} \quad \text{and} \quad \left(\frac{b}{b+1}\right)^{\alpha} < \frac{u_{\min}\cdot e_{\min}}{m\cdot u_{\max}\cdot p_{\max}}\cdot\frac{1}{(b+1)^2}$$

Observe that W induces a set of intervals $\left[(b/(b+1))^{\ell}, (b/(b+1))^{\ell-1}\right]$, for $\ell = 1,\ldots\alpha$. We can now create the set $\widetilde{W}$ by dividing each interval of W and its negation into subintervals. Specifically, for each interval $\left[(b/(b+1))^{\ell}, (b/(b+1))^{\ell-1}\right]$ (resp., for each interval $\left[-(b/(b+1))^{\ell-1}, -(b/(b+1))^{\ell}\right]$), we divide W equally into $2n(b+1)$ subintervals and put the boundary points in $\widetilde{W}$. Thus,

$$\widetilde{W} = \bigcup_{\ell=1}^{\alpha-1}\left\{\left(\frac{b}{b+1}\right)^{\ell}\left(1+\frac{q}{2nb(b+1)}\right) : q\in\{0,1,\ldots,2n(b+1)\}\right\}\cup\{0\}\cup$$
$$\bigcup_{\ell=1}^{\alpha-1}\left\{\left(-\frac{b}{b+1}\right)^{\ell}\left(1+\frac{q}{2nb(b+1)}\right) : q\in\{0,1,\ldots,2n(b+1)\}\right\}$$

We insist that the algorithm selects allocations of goods that take values from W, and we then round down the surplus (or deficits) to values in $\widetilde{W}$. Formally, $\mathbf{x}^t \subseteq W^{n\times m}$ and $\mathbf{y}^t \subseteq \widetilde{W}^m$ for any market t. Towards this goal, let $\lfloor a \rfloor_{\widetilde{W}}$ denote the value of a rounded down to the closest value in $\widetilde{W}$, i.e., $a' = \lfloor a \rfloor_{\widetilde{W}}$ is the largest value in $\widetilde{W}$ such that $a' \leq a$.

We now need to ensure that these allocations correspond to an approximate generalized equilibrium (which in turn will correspond to an approximate equilibrium). To do this, for a market t and a trader i, we say that an allocation x_i of goods is *compatible* with i and t if, for $r_i = \max_{j:e_{ij}>0} \pi_1(p_j)$:

- $u_{ij} > 0 \implies \pi_1(p^t_j) \geq r_i$,
- $x_{ij} > 0 \implies \pi_1(p^t_j) \leq r_i$,
- $\sum_{j:\pi_1(p^t_j)\geq r_i} \pi_2(p^t_j) \cdot x_{ij} \leq \sum_{j:\pi_1(p^t_j)\geq r_i} \pi_2(p^t_j) \cdot e_{ij}$ (Budget constraint).

An allocation $\mathbf{x}_i$ compatible with i and t is *nearly-optimal* if $\mathbf{u}_i \cdot \mathbf{x}_i \geq (1 - \varepsilon) \max_z \mathbf{u}_i \cdot \mathbf{z}$ where $\mathbf{z}$ ranges over all the allocations compatible with i and t. By this definition, assuming L is large enough, a nearly-optimal allocation is an approximate best response for trader i in market t. The recurrence relation of our dynamic programming algorithm is then

$$f(0, \mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^k) = \begin{cases} 0 & \text{if } \mathbf{y}^1 = \mathbf{y}^2 = \ldots = \mathbf{y}^k = \mathbf{0} \\ -\infty & \text{otherwise} \end{cases}$$

$$f(i, \mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^k) = \max_{t\in\{1,2,\ldots,k\},\mathbf{x}_i} f(i-1, \mathbf{y}^1, \ldots, \lfloor \mathbf{y}^t - \mathbf{e}_i + \mathbf{x}_i \rfloor_{\widetilde{W}}, \ldots, \mathbf{y}^k) + \mathbf{u}_i\mathbf{x}_i$$

where $\mathbf{x}_i$ ranges over the nearly-optimal allocations compatible with i and t.

Finally, an allocation is *valid* if $0 \leq \mathbf{y}^t \leq 1-(b/(b+1))^3$ for all t, i.e., there is no market having a positive deficit, or a surplus greater than $1-(b/(b+1))^3$. Notice that some of the surplus may have been lost in the rounding steps; we will show later that these additional losses amount to a fraction at most $1/(b+1)$ of each good. Hence, if $f(n, \mathbf{y}^1, \ldots, \mathbf{y}^t)$ is finite for some valid allocation, and provided that b is sufficiently large, it gives an approximate *generalized* equilibrium: the budget constraints and approximate optimality constraints are guaranteed by the restriction on the choice of $\mathbf{x}_i$ at each step, and market clearing is guaranteed by the validity of the allocation.

This completes the description of the dynamic program. It remains to compare the total social welfare to that of the optimal clustering and to analyze the running time of the algorithm.

6.2 Quality Analysis

Because of the rounding step, our dynamic programming algorithm loses some fraction of each good g_j. We have to bound the number of units of the good g_j that we lose. By the scaling on $\widetilde{W}$, each time we round y^t_j, we lose at most $(2n(b+1))^{-1} \sum_{i\in I_t} e_{ij}$ units when y^t_j is rounded to a positive value, and we lose at most $(b/(b+1))^\alpha \leq e_{min}\,(2n(b+1))^{-1}$ units when y^t_j is rounded to zero. By the definition of e_{min}, we have $e_{min} \leq \sum_{i \in I_t} e_{ij}$ for all goods g_j. Furthermore, we round down y_j^t at most n times, once for each trader. Thus, summing them up, we lose at most $1/(2(b+1)) \cdot \sum_{i \in I_t} e_{ij} < 1/(2b+2)$ units of each good g_j.

Now, we compare the social welfare of the approximate generalized equilibrium obtained by our algorithm with that of some exact generalized equilibrium.

Lemma 6. *For any market $M = (n, m, \mathbf{u}, \mathbf{e})$ and any $\varepsilon > 0$, there is an exact generalized equilibrium such that*

$$\frac{\max_j \pi_2(p_j)}{\min_{j:\pi_2(p_j) \neq 0} \pi_2(p_j)} \leq e^{nm/e} \left(\frac{p_{\max} \cdot m}{e_{\min} \cdot p_{\min}} \right)^{nm}$$

Proof. Take an exact generalized equilibrium $\mathbf{x}, \mathbf{p}$ that maximizes the total welfare. Let $G_r := \{g_j \ : \ \pi_1(p_j) = r\}$, $I_r := \{i \ : \ r_i = r\}$ and $A_r := \{i \in I_r \ : \ \forall j \in G_r, u_{ij} = 0\}$. For each possible rank r, let M_r be the market obtained by restricting M to the set of goods G_r. Define $x_{ij}^r = x_{ij}$ and $p_j^r = \pi_2(p_j)$ for any trader i and good g_j in M_r. Fix some rank r.

Claim 13. *We may assume that $\sum_{j \in G_r} p_j^r x_{ij} = \sum_{j \in G_r} p_j^r e_{ij}$ for any i.*

Proof. By market-clearing for $\mathbf{x}, \mathbf{p}$, we have $\sum_i x_{ij} = \sum_i e_{ij}$ for any $j \in G_r$. By optimality, we also have $\sum_{j \in G_r} p_j^r e_{ij} = \sum_{j \in G_r} p_j^r x_{ij}$ for any $i \in I_r - A_r$. Subtracting the two equalities, we get $\sum_{j \in G_r} \left(\sum_{i \in A_r} p_j^r e_{ij} + \sum_{i:r_i > r} p_j^r e_{ij} \right) = \sum_{j \in G_r} \left(\sum_{i \in A_r} p_j^r x_{ij} + \sum_{i:r_i > r} p_j^r x_{ij} \right)$. Because $u_{ij}^r = 0$ for any i with $r_i > r$ or any $i \in A_r$, we can redistribute the goods of G_r allocated to these traders to fulfill the condition of the claim. □

Then $\mathbf{x}^r, \mathbf{p}^r$ is an equilibrium in M_r because of the previous claim (the optimality constraints and market-clearing constraints follow from the definition of generalized equilibrium). Let r be some rank and consider M_r. We want to compute an approximate equilibrium for M_r such that we can bound the prices. If there is no altruist in M_r, then we apply Lemma 4. Otherwise, for any altruist i, let j_i be such that $p_{j_i} x_{ij_i}^r$ is maximized. Define the utility vector $\mathbf{u}^r$

$$u_{ij}^r = \begin{cases} u_{ij} & \text{if } i \text{ is not an altruist in } M_r, \\ 1 & \text{if } i \text{ is altruist in } M_r \text{ and } j = j_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then the market defined by $\mathbf{e}$ and $\mathbf{u}^r$, $\mathbf{x}^r, \mathbf{p}^r$ is an approximate solution, where only the altruists of M_r do not follow a best response, and the approximation ratio is at most $(m-1)/m$ by the choice of $\mathbf{u}_i^r$ for altruist trader i (and because there are at most m goods in M_r). Hence, by applying Lemma 2, we get $\frac{p_{\max}}{p_{\min}} \leq e^{nm/e} \left(\frac{m u_{\max}}{u_{\min} e_{\min}} \right)^{nm}$. □

Fix any optimal clustering and consider any market t. Take a generalized equilibrium $\mathbf{p}^*, \mathbf{x}^*$ as in Lemma 6. We show that the dynamic program outputs an approximate generalized equilibria with total welfare $(1-\varepsilon)$ times the welfare of $\mathbf{p}^*, \mathbf{x}^*$. This is done by building from $\mathbf{p}^*, \mathbf{x}^*$ a set of generalized prices $\mathbf{p}'$ and allocations $\mathbf{x}'$ computable by the dynamic program.

We assume wlog that $p^*_{min} := \min_{j:p^*_j>0} \pi_2(p^*_j) = 1$. Denote $p_{\max} := \max_{j:p^*_j>0} \pi_2(p^*_j)$. $\mathbf{p}'$ is obtained by rounding down the second components of prices to values in P; hence, $b/(b+1)\cdot\pi_2(\mathbf{p}^*) \leq \pi_2(\mathbf{p}') \leq \pi_2(\mathbf{p}^*)$. Consequently, we have for each trader i and $a \geq r_i$:

$$\frac{b}{b+1}\sum_{j:\pi_1(p'_j)=a} \pi_2(p'_j)x^*_{ij} \leq \frac{b}{b+1}\sum_{j:\pi_1(p'_j)=a} \pi_2(p^*_j)e_{ij} \leq \sum_{j:\pi_1(p'_j)=a} \pi_2(p'_j)e_{ij}$$

Hence, $b/(b+1)\cdot\mathbf{x}^*$ satisfies the budget constraint for the prices $\mathbf{p}'$.

Next, we have to modify $b/(b+1)\cdot\mathbf{x}^*$ further so that it satisfies the condition in our dynamic programming algorithm. Namely, we round down the coefficients of $b/(b+1)\cdot\mathbf{x}^*$ to W. This gives an allocation $\mathbf{x}'$ with the properties:

$$\left(\frac{b}{b+1}\right)^2 x^*_{ij} \leq x'_{ij} \leq x^*_{ij} \quad \text{if } x^*_{ij} \geq \left(\frac{b}{b+1}\right)^\alpha, \text{ and } x'_{ij} = 0 \text{ otherwise.}$$

Clearly, $\mathbf{x}'$ also satisfies the budget constraint inequalities. It remains to show that $\mathbf{x}'_i$ is an almost optimal choice for trader i. To see this, consider any good g_j. We have

$$\left(\frac{b}{b+1}\right)^3 \pi_2(p^*_j)\cdot x^*_{ij} \leq \frac{b}{b+1}\pi_2(p^*_j)\cdot x'_{ij} \leq \pi_2(p'_j)\cdot x'_{ij} \leq \pi_2(p^*_j)\cdot x^*_{ij}$$

when $x^*_{ij} \geq \left(\frac{b}{b+1}\right)$ or $x^*_{ij} = 0$, otherwise $u_{ij}x'_{ij} = 0$ and $u_{ij}x^*_{ij} \leq \left(\frac{b}{b+1}\right)^\alpha u_{ij}$. We have to handle the latter special case, when $x'_{ij} = 0 < x^*_{ij}$. For this purpose, notice first that the welfare of a trader i with $\sum_{j:\pi_1(p^*_j)=r_i} u_{ij} > 0$ is lower bounded by $\frac{e_{\min}\cdot u_{\min}}{p_{\max}}$ as $e_{\min}$ is the minimum possible budget for a trader (other traders have welfare 0). The maximum ratio utility per unit of price achievable is $\frac{u_{\max}}{p_{\min}}$. Thus, any allocation $x^*_{ij} \leq (b/(b+1))^\alpha$ contributes to the total welfare at most $\left(\frac{b}{b+1}\right)^\alpha \frac{u_{\max}\cdot p_{\max}}{u_{\min}\cdot e_{\min}}$, which is less than $(1/m)(1/(b+1))^2 \leq (1/m)\left(1-(b/(b+1))^3\right)$ by the choice of α. Hence, over all goods, the fraction of welfare lost in rounding down $\mathbf{x}^*$ is at most $1-2(b/(b+1))^3$. This bounds our approximation ratio to $1-\varepsilon \leq 2\left(\frac{b}{b+1}\right)^3$, which is true for $b = \lceil 3/\varepsilon \rceil$.

With this choice, $\mathbf{x}'$ and $\mathbf{p}'$ satisfy the approximate market equilibrium constraints, thus the dynamic algorithm will find a solution with welfare at least $\mathbf{u}\cdot\mathbf{x}' \geq (b/(b+1))^2\mathbf{u}\cdot\mathbf{x}^*$.

6.3 Running Time Analysis

Now, consider the complexity of the dynamic program. It can be seen that the running of our algorithm for one set of prices is $O(nk\alpha^m(\alpha nb)^{km})$, and the running time for all possible price allocation is $(m\sigma)^{km}$. This implies the total running time of $O(n^{km+1}k\alpha^{(k+1)m}(mb\sigma)^{km})$. Thus, we have a $(1-\varepsilon)$-approximation algorithm with a running time of

$$O\left(n^{(3k+1)m+1}m^{(2k+1)m}k\left(1+\frac{3}{\varepsilon}\right)^{3km+m}\left(\frac{1}{e}+\log\frac{m\cdot u_{\max}}{e_{\min}u_{\min}}\right)^{(k+1)m}\left(\log\frac{m\cdot u_{\max}}{u_{\min}e_{\min}}\right)^{km}\right)$$

The input of the k-market clustering problem in a standard binary representation has a size of $\Omega(n(\log(1/e_{min})+\log(u_{max}/u_{min})))$. Thus, the running time of our algorithm is polynomial in the size of the input when m and k are constant.

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On Budget-Balanced Group-Strategyproof Cost-Sharing Mechanisms

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Abstract. A cost-sharing mechanism defines how to share the cost of a service among serviced customers. It solicits bids from potential customers and selects a subset of customers to serve and a price to charge each of them. The mechanism is group-strategyproof if no subset of customers can gain by lying about their values. There is a rich literature that designs group-strategyproof cost-sharing mechanisms using schemes that satisfy a property called *cross-monotonicity*. Unfortunately, Immorlica et al. showed that for many services, cross-monotonic schemes are provably not *budget-balanced*, i.e., they can recover only a fraction of the cost. While cross-monotonicity is a sufficient condition for designing group-strategyproof mechanisms, it is not necessary. Pountourakis and Vidali recently provided a complete characterization of group-strategyproof mechanisms. We construct a fully budget-balanced cost-sharing mechanism for the edge-cover problem that is not cross-monotonic and we apply their characterization to show that it is group-strategyproof. This improves upon the cross-monotonic approach which can recover only half the cost, and provides a proof-of-concept as to the usefulness of the complete characterization. This raises the question of whether all "natural" problems have budget-balanced group-strategyproof mechanisms. We answer this question in the negative by designing a set-cover instance in which no group-strategyproof mechanism can recover more than a (18/19)-fraction of the cost.

1 Introduction

In *cost-sharing* problems, a service provider faces a set of potential customers, each of which has a private value for the service. The provider must select a subset of customers to serve, and a price to charge each of them. To this end, he defines a mechanism that solicits bids from potential customers and, based on these bids, outputs the serviced subset and corresponding prices. To cover the cost of providing service, he looks for a mechanism that is *budget-balanced*, that is the sum of prices equals the cost of service for every input bid vector.

A central goal in mechanism design is to define mechanisms that are *strategyproof* in that no agent can gain by misreporting his value. This guarantees that the equilibrium bidding strategy of agents is robust and so the mechanism

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behaves as predicted. In cost-sharing problems, there is an inherent cooperative aspect: the cost of service changes drastically depending on which subset is serviced and so groups of agents may have aligned interests. In these problems, it makes sense to ask for an even more robust solution concept, *group-strategyproofness*. In a group-strategyproof mechanism, no group of agents can mutually gain by misreporting their values.

Group-strategyproofness is a very strong requirement. Nonetheless, there is a rich literature defining group-strategyproof mechanisms for various cost-sharing problems [5,8,7,12]. All these papers use the same general technique. They define a *cost-sharing scheme* which, given any subset of customers, defines the price each of them would have to pay if that subset was serviced. They then turn this scheme into a mechanism by applying a procedure of Moulin [10]. The resulting mechanism is group-strategyproof so long as the underlying cost-sharing scheme satisfies a property called *cross-monotonicity*. Intuitively, cross-monotonicity requires that as more agents are serviced, the price to each decreases. If the cross-monotonic cost-sharing scheme is (approximately) budget-balanced on every subset of customers, then the resulting group-strategyproof mechanism is also (approximately) budget-balanced.

Unfortunately, the use of this technique comes at a cost. While submodular cost functions always have fully budget-balanced cross-monotonic cost-sharing schemes [11], and many combinatorial optimization problems have approximately budget-balanced schemes [5,8,7,12], Immorlica et al. [6] showed that cross-monotonicity fundamentally limits achievable budget-balance factors for many combinatorial optimization problems. They also note that, while cross-monotonicity is a sufficient condition for giving rise to group-strategyproof mechanisms, it is not necessary. This left open the question of whether another approach might enable the design of group-strategyproof mechanisms with better budget-balance factors.

In recent work Pountourakis and Vidali [14] provided a complete characterization of group-strategyproof mechanisms. Their characterization is based on cost-sharing schemes that satisfy three technical properties. They then give a procedure that converts any such (approximately) budget-balanced cost-sharing scheme into an (approximately) budget-balanced group-strategyproof mechanism.[1]

In this work, we provide a natural cost-sharing scheme for the edge-cover problem and use the techniques of Pountourakis-Vidali to prove it gives rise to a group-strategyproof mechanism. In the edge-cover problem, the agents are the vertices of a graph, and the cost of a subset is the minimum number of edges that must be selected in order to cover every agent in the subset. The problem models, for example, assigning people to rooms either as a single occupant or with a compatible roommate (as defined by the edges of the graph).

It is shown that the best budget-balance factor of any cross-monotonic cost-sharing scheme for edge-cover is just 1/2 [6]. Using the complete characterization [14], we design a *fully* budget-balanced group-strategyproof mechanism for

[1] In general, this procedure is not known to be polynomial-time.

edge-cover. Thus our result improves upon *any* group-strategyproof mechanism designed using the standard cross-monotonic technique, thereby demonstrating the significance of the full characterization. Furthermore, the cost-sharing scheme that we define is very intuitive: for a given subset of agents, we compute a lexicographically first maximum matching of that subset, charge each matched agent a price of 1/2, and charge each remaining agent in the subset a price of 1. This natural scheme is not cross-monotonic. However, using a key lemma regarding alternating paths of certain matchings, we are able to prove that this scheme does satisfy the characterization of Pountourakis and Vidali [14].

We would like to stress out that we wanted to provide a simple mechanism for edge-cover, hence, we chose to restrict our attention to two possible prices. Even though, one could construct fully budget-balanced group-strategyproof mechanisms that use more than two prices, their analysis would be even more complicated. Our goal is to show the existence of such a mechanism rather characterize them, hence, we restricted to something intuitive and simple. Moreover, it allowed us to implement the allocation of the mechanism in polynomial time, whereas it is not certain that this would be possible if the cost-sharing scheme was more complicated. It is open whether natural and simple cost-sharing schemes for other interesting problems happen to satisfy the sufficient conditions [14] and in that case if we can find efficient implementation.

We also show that not all problems have fully budget-balanced group-strategyproof mechanisms. This is fairly obvious when one allows arbitrary (e.g., non-monotone) cost functions.[2] In this paper, we prove this result for the natural monotone cost function defined by the set cover problem, a generalization of the edge cover problem. For set cover, there is a bound of $n^{-1/2}$ (where n is the number of agents) on the budget-balance factor of cross-monotonic cost-sharing schemes [6], implying that the standard technique for designing group-strategyproof mechanisms is highly impractical. This negative result is particularly disturbing in light of the fact that there exists a trivial fully budget balanced strategyproof mechanism (see Example 4.1 [6]) for any non-decreasing cost-function if we don't take computational limits into consideration. Even imposing computational limits, we can obtain a $O(1/\log n)$-budget balanced mechanism that is strategyproof but not group-strategyproof [3]. In our work we are interested in bounding the power of group-strategyproofness without any computational assumption. We present a set-cover instance, where there is no cost-sharing scheme satisfying the characterization of Pountourakis and Vidali [14] with budget-balance factor better than 18/19. Since this characterization does not take computational constraints into consideration this implies a bound for every group-strategyproof mechanism independent of its running time.

Finally, we would like to note that while we try to deal with the limitations of cross-monotonic mechanisms by exploiting the full power of group-strategyproofness, another approach that has been followed so far was to relax group-strategyproofness. In particular, there is a general framework to design weak group-strategyproonf mechanisms [9]. This framework has been used

[2] See, for example [14].

to design mechanism with better budget balance guarantees for many combinatorial problems [9,1]. Specifically, cost-sharing schemes used by [9] were naturally derived by primal-dual schemes without any refinement that would ensure cross-monotonicity [12]. However, as we argue in Section 5 these cost-sharing schemes fail to satisfy the necessary condition to give rise to group-strategyproofness; hence, ideas from this literature cannot be directly applied to group-strategyproof mechanism design.

Related Work. In addition to the literature on cost-sharing mechanisms mentioned above, our work is related to the literature on combinatorial public projects. This problem was introduced in [13] who assume that a set of agents is interested in sharing a number of resources. Each agent has a private valuation for each subset of these resources. For some given k, a mechanism has to choose based on the valuations of the agent a set of k resources so as to maximize the social welfare.

There are communication and computational bounds for every strategyproof mechanism that solves this problem when the valuation functions satisfy submodularity [13]. There are similar bounds without the constraint of truthfulness, but slightly relaxing submodularity of valuation functions [15]. Finally, recent work [2] studies similar questions for sub-additive valuations and provide various upper and lower bound for specific valuation function classes.

This problem differs from cost-sharing in the sense that there are multiple resources the mechanism is called to choose upon, however, all the agents are going to share them. Moreover, they are interested in maximizing social welfare rather than budget balance. However, both of these problems have applications to resource sharing and particularly network formation.

2 Model

A set of agents $\mathcal{A} = \{1, 2, \ldots, n\}$ is interested in receiving a service. Each agent i has a private type v_i, which is her valuation for receiving the service. A *cost-sharing mechanism* inputs a bid b_i for each agent i and outputs the subset of agents $Q \subseteq \mathcal{A}$ that receive service and the price p_i that each agent i pays. Assuming quasi-linear utilities, each agent wants to maximize the quantity $v_i x_i - p_i$ where $x_i = 1$ if $i \in Q$ and $x_i = 0$ if $i \notin Q$. We concentrate on mechanisms that satisfy the following conditions [10,11]:

– *Voluntary Participation (VP)*: An agent that is not serviced is not charged ($i \notin Q \Rightarrow p_i = 0$) and a serviced agent is never charged more than her bid ($i \in Q \Rightarrow p_i \leq b_i$).
– *No Positive Transfer (NPT)*: The payment of each agent i is non-negative ($p_i \geq 0$ for all i).
– *Consumer Sovereignty (CS)*: For each agent i there exists a value $b_i^* \in \mathbb{R}$ such that if she bids b_i^*, then it is guaranteed that agent i will receive the service no matter what the other agents bid.

We also assume that the agents can bid in a way that they will definitely not receive the service. This can be done by allowing negative bids. Then VP implies that an agent that reports a negative amount has to be charged a negative amount if she is serviced, which is prohibited by NPT.

We are interested in mechanisms that are *group-strategyproof (GSP)*. A mechanism is GSP if for every two valuation vectors v, v' and every coalition of agents $S \subseteq \mathcal{A}$, satisfying $v_i = v'_i$ for all $i \notin S$, one of the following is true: *(a)* There is some $i \in S$, such that $v_i x'_i - p'_i < v_i x_i - p_i$ or *(b)* for all $i \in S$, it holds that $v_i x'_i - p'_i = v_i x_i - p_i$, where x'_i and p'_i is the allocation and payment of player i respectively when the agents report v'.

We also assume the existence of an underlying cost-function $C : 2^{\mathcal{A}} \to \mathbb{R}^+ \cup \{0\}$, where $C(S)$ specifies the cost of providing service to all agents in S. We say that a mechanism is α*-budget balanced* with respect to C if for all b, $\alpha C(Q) \leq \sum_{i \in \mathcal{A}} p_i \leq C(Q)$, where Q and $\{p_i\}$ are the prices and allocation output by the mechanism on input b.

2.1 Characterization

A cost-sharing scheme $\xi : \mathcal{A} \times 2^{\mathcal{A}} \to \mathbb{R}$ is a function that takes as input an agent and a set and outputs a real number. The amount $\xi(i, S)$ can be viewed as the payment of agent i when the set of agents S receives the service.[3] A cost-sharing scheme is α*-budget balanced* with respect to C if for all S, $\alpha C(S) \leq \sum_{i \in S} \xi(i, S) \leq C(S)$.

A property of the cost-sharing scheme, namely *cross-monotonicity*, has played a central role in the literature. Intuitively, cross-monotonicity requires that the cost-share of an agent can not increase as the serviced set grows. Moulin [10,11] showed that given a cross-monotone cost-sharing scheme we can construct a group-strategyproof mechanism with a budget-balance factor equal to that of the cost-sharing scheme. Moreover, if the underlying cost function is submodular then there exist a perfectly budget balanced cost-sharing scheme. However, when the cost function is given by the cost of the optimal solution of an optimization problem, the cost function is often not submodular. Subsequent work [6] proved bounds on the budget balance factor of cross-monotonic cost-sharing schemes. This gave rise to the question of whether there are group-strategyproof mechanisms for such problems with better budget-balance properties. A step towards answering this question was taken by [14], where they gave a complete characterization of the cost-sharing schemes that can give rise to group-strategyproof mechanisms. Let $\xi^*(i, L, U)$ be the minimum payment of player i for getting serviced when the serviced set is "between" sets L and U, i.e., $\xi^*(i, L, U) := \min_{\{L \subseteq S \subseteq U, i \ni S\}} \xi(i, S)$.

[3] This is a restrictive form of a payment policy as we exclude the possibility of charging different values given two different bid vectors where the mechanism provides the service to the same set of agents. Nevertheless, this is without loss of generality for the mechanisms of our setting [6].

Theorem 1 (Pountourakis and Vidali [14]). *A cost-sharing scheme ξ can give rise to a group-strategyproof mechanism if and only if for every $L \subseteq U \subseteq \mathcal{A}$ it satisfies the following three properties.*

(a) There exists a set S with $L \subseteq S \subseteq U$, such that for all $i \in S$, we have $\xi(i, S) = \xi^*(i, L, U)$.
(b) For each player $i \in U \setminus L$ there exists one set S_i with $i \in S_i$ and $L \subseteq S_i \subseteq U$, such that for all $j \in S_i \setminus L$, we have $\xi(j, S_i) = \xi^*(j, L, U)$. (Since $i \in S_i \setminus L$, it holds that $\xi(i, S_i) = \xi^*(i, L, U)$.)
(c) If for some $C \subset U$ there is a player $j \in C$ with $\xi(j, C) < \xi^*(j, L, U)$ (obviously $L \nsubseteq C$), then there exists a set $T \neq \emptyset$ with $T \subseteq L \setminus C$, such that for all $i \in T$, $\xi(i, C \cup T) = \xi^*(i, L, U)$.

2.2 Allocation

Additionally, there is a complete characterization of the allocation functions that can be coupled with a cost-sharing scheme satisfying the properties of Theorem 1 to yield a group-strategyproof mechanism [14] .

Theorem 2 (Pountourakis and Vidali [14]). *If ξ is a cost-sharing scheme that satisfies the properties of Theorem 1, then for every bid vector b there exist unique sets $L(b) \subseteq U(b) \subseteq \mathcal{A}$ such that*

1. *for all $i \in L(b)$, $b_i > \xi^*(i, L(b), U(b))$,*
2. *for all $i \in U(b) \setminus L(b)$, $b_i = \xi^*(i, L(b), U(b))$,*
3. *and for all $R \subseteq \mathcal{A} \setminus U(b)$, there exist $i \in R$ with $b_i < \xi^*(i, L(b), U(b) \cup R)$.*

Furthermore, the mechanism that on input b outputs allocation $Q = S$ and prices $p_i = \xi(i, S)$ where

1. *$L(b) \subseteq S \subseteq U(b)$, and*
2. *for all $i \in S$, $\xi(i, S) = \xi^*(i, L(b), U(b))$ (such a set must exist by Theorem 1(a)),*

is a group-strategyproof mechanism.

A way to implement the allocation is to exhaustively search for these sets $L(b)$ and $U(b)$ and a set S that satisfy the properties of the theorem. It is still not known if there is an algorithm that implements this procedure with asymptotically better running time in general. However, in Section 3, we provide a fully budget-balanced cost-sharing scheme and corresponding polynomial-time allocation when the cost function is given by the edge-cover problem.

3 Edge Cover

In this section, we give a fully budget-balanced group-strategyproof mechanism for the unweighted edge-cover cost-sharing game. To do so, we derive a cost-sharing scheme that satisfies the conditions presented in Theorem 1. Previous work [6] implies that group-strategyproof mechanisms for edge-cover designed via cross-monotonic cost-sharing schemes are at best 1/2-budget balanced. Thus, our work improves upon the previous results and demonstrates that the assumption of cross-monotonicity is not without loss for group-strategyproof mechanism design. We start with a definition of the edge cover game.

Definition 1. *In the* edge cover game *we are given a graph $G = (V, E)$ with no isolated vertices. The agents in the game are the vertices V of the graph G. Given a subset of vertices $S \subseteq V$, an edge-cover of S is a subset of edges $F \subseteq E$ such that for all vertices $v \in S$ there is some edge $e \in F$ such that $v \in e$. The cost of a set S is the minimum cardinality edge-cover of S.*

In the following subsections, we first present a cost-sharing scheme for the edge cover game that provably gives rise to a group-strategyproof mechanism. We then show how to use this scheme to define a computationally efficient group-strategyproof mechanism for our problem.

3.1 Cost-Sharing Scheme

Our cost-sharing scheme is based on the following well-known polynomial time algorithm [4] for finding the minimum edge cover F of a set S. Let F be the set of edges in the maximum matching on S, and then for each vertex $v \in S$ uncovered by F, add to F an edge e adjacent to v. Based on this algorithm, a natural cost-sharing scheme is to charge each agent $v \in S$ a price of 1/2 if v is in the matching found by the algorithm, and 1 otherwise. The problem that arises with this cost-sharing scheme is the existence of multiple maximum matchings. We demonstrate this in the full version of the paper using an example of bad tie-breaking rule among maximum matchings.

Definition 2. *Given $G = (V, E)$ on m edges, label edges E from 1 to m arbitrarily. For a subset of vertices $S \subseteq V$, let M_S denote the lexicographically first maximum matching according to the labeling. Moreover let $V(M_S) = \{v \mid \exists e \in M_S \text{ s.t. } v \in e\}$, i.e., $V(M_S)$ contains the vertices that are matched in M_S.*

Note that the lexicographically first maximum matching M_S of any set of vertices S can be found efficiently, for example by assigning a weight of $(1 + 2^i)/2^m$ to the i'th edge and then computing the maximum weight matching. We are now ready to formally define the cost-sharing scheme ξ. This cost-sharing scheme extends one introduced by [6] for an edge-cover instance on just three vertices as an example of a group-strategyproof mechanism without a cross-monotone cost-sharing scheme.

Definition 3. *Let $G = (V, E)$. For every $S \subseteq V$ and every $i \in V$ we define*

$$\xi(i, S) = \begin{cases} 0 & i \notin S \\ 1/2 & i \in V(M_S) \\ 1 & i \in S \setminus V(M_S) \end{cases}$$

By construction, the cost-sharing scheme of Definition 3 is 1-budget balanced (and therefore, by the results of [6], it is not cross-monotone). We show that it additionally satisfies all the necessary and sufficient conditions of group-strategyproofness.

Theorem 3. *The cost-sharing scheme ξ of Definition 3 satisfies all the necessary and sufficient conditions to give rise to a GSP mechanism. Consequently there is a 1-budget balanced GSP mechanism for the edge-cover problem.*

The proof uses the characterization presented in Theorem 1. The main challenge is to show that for any lower set L and upper set U, there is some intermediate set $S^*, L \subseteq S^* \subseteq U$, in which every agent in S^* achieves his minimum cost-share among all intermediate sets (i.e., property (a)). Since cost-shares are always either 1 or 1/2, this amounts to finding a set S^* in which each agent $i \in S^*$ either has cost-share 1/2, or has cost-share 1 for every intermediate set $S, L \subseteq S \subseteq U$.

The proof idea is as follows. We start with an arbitrary intermediate set S and work our way towards S^*. First, we prove in the following lemmas that for any set S, we can discard agents with cost-share equal to 1 without changing the solution for the other agents. Thus starting from an arbitrary intermediate set S, we can work our way towards S^* by discarding all agents in $S \setminus L$ with cost-share equal to 1. This leaves the question of whether agents in $S \cap L$ are receiving their minimum cost-share among the intermediate sets. Unfortunately, this is not necessarily the case: there may be some unhappy agent $i \in L \cap S$ with cost-share equal to 1 who has a cost-share equal to 1/2 in some other intermediate set S_i. In this case, we use the lexicographically first maximum matchings M_S and M_{S_i} to construct an alternating path starting from agent i. We prove that this alternating path ends in a node j that can either be added to or deleted from S in order to decrease the number of unhappy agents in $L \cap S$ (interestingly, agent i may still be unhappy after this fix, but at least one agent becomes happy). In this way, starting from an arbitrary intermediate set S, we can walk towards S^*. The complete proof can be found in the full version of the paper.

3.2 Polynomial-Time Allocation

We now argue that the group-strategyproof mechanism corresponding to the cost-sharing scheme in Definition 3 can be constructed in polynomial time. To do so, we must find, for any bid vector b, a set S satisfying the conditions of Theorem 2. Namely, we are looking for a set S that lies between some lower-bound set $L(b)$ and upper-bound set $U(b)$ such that:

1. for all $i \in L(b)$, $b_i > \xi^*(i, L(b), U(b))$,
2. for all $i \in U(b) \setminus L(b)$, $b_i = \xi^*(i, L(b), U(b))$,
3. and for all $R \subseteq \mathcal{A} \setminus U(b)$, there exist $i \in R$ with $b_i < \xi^*(i, L(b), U(b) \cup R)$,

and for all $i \in S$, $\xi(i, S) = \xi^*(i, L(b), U(b))$. In words, the elements in $L(b)$ should be bidding more than their minimum cost-share; the elements in $U(b) \setminus L(b)$

should be bidding equal to their minimum cost-share; and $U(b)$ is maximal in the sense that when we try to add elements to it, at least one of the newcomers can't afford his minimum cost-share. The set S allocated by the group-strategyproof mechanism is then any of the intermediate sets in which each agent is happy (gets its minimum cost-share), i.e., a set S with

$$L(b) \subseteq S \subseteq U(b), s.t.\ \forall i \in S, \xi(i, S) = \xi^*(i, L(b), U(b)).$$

For ease of notation, in the rest of this section we fix b and use L to denote $L(b)$ and U to denote $U(b)$.

The main difficulty in finding S is that we do not know L and U. However, using the structure of these sets and the fact that the only cost-shares in our scheme are 1 and 1/2, we can bound these two sets. Given a bid vector b let us define $P = \{i \mid b_i > \frac{1}{2}\}$ and $Q = \{i \mid b_i \geq \frac{1}{2}\}$. Then $L \subseteq P$ and $U \subseteq Q$. Hence we can search through intermediate sets of P and Q, looking for our S. Any such S will definitely contain L as $L \subseteq P$, but may not be contained in U; our algorithm must provide a separate guarantee for this containment.

Our algorithm for finding S is based on a local search procedure and corresponding potential function $\phi(.)$ which is strictly increasing with respect to the steps of this search. The search procedure iteratively adds an element to, or deletes an element from, the current set S while maintaining the invariant that $P \subseteq S \subseteq Q$. Our potential function $\phi(S)$ counts the number of happy elements in $L \subseteq S$, i.e.,

$$\phi(S) = |\{i \in L \mid \xi(i, S) = \xi^*(i, L, U)\}|.$$

We show that as long as $\phi(S) < |L|$, there is always a way to improve the potential. Since L is fixed and finite (given b), this procedure must terminate. Furthermore, by definition of the potential, when the procedure terminates, each agent in $L \subseteq S$ is happy. To guarantee that agents in $S \setminus L$ are happy and also that $S \subseteq U$, we need to prune S. As we show later, it is sufficient to simply remove agents from S whose bids are less than their cost-shares. The following procedure implements this local search.

1. $S \leftarrow P$.
2. Iterate as long as the set S changes:
   (a) Remove all players in $S \setminus P$ with $\xi(i, S) = 1$.
   (b) If there is some $i \in Q \setminus S$ such that the cardinality of the maximum matching in $S \cup \{i\}$ is increased, then set $S \leftarrow S \cup \{i\}$.
   (c) Else if there is some $i \in S \setminus P$ that was matched in $M(S)$ and the maximum matching in $S \setminus \{i\}$ does not decrease, then set $S \leftarrow S \setminus \{i\}$.
3. Set $S \leftarrow \{i \mid b_i \geq \xi(i, S)\}$.

We first observe that this algorithm runs in polynomial time. Specifically steps 2 (b) and 2 (c) reduce to finding whether the inclusion of some agent in $Q \setminus P$ forms an augmented path or whether an agent in $S \setminus P$ is not present in every maximum cardinality matching respectively. Both of these steps can be implemented in polynomial time. Since at each step the potential function increases, step 2 is performed at most as many times as the cardinality of L, which is bounded by the total number of agents.

Theorem 4. *This procedure outputs a set S, $L \subseteq S \subseteq U$, where for all $i \in S$, $\xi(i, S) = \xi^*(i, L, U)$.*

We refer the reader to the full version for the proof of this theorem.

4 Set Cover

In this section we show that it is impossible to construct a fully budget balanced group-strategyproof mechanism when the cost function is determined by the optimal objective function of the set-cover problem. It is known that no cross-monotonic cost-sharing scheme can have a budget-balance of better than $n^{-1/2}$ where n is the number of elements or the size of the largest subset in the set-cover instance [6]. Here we show that there are instances where no group-strategyproof mechanism can be (18/19)-budget-balanced. Thus, while group-strategyproof mechanisms may be able to improve upon the budget-balance factor of cross-monotonic ones, we show that they can not, in general, provide full budget-balance.

Definition 4. *In the* set cover game *we are given a ground set V and a collection of subsets $\mathcal{F} \subseteq 2^V$. The agents in the game are the elements V of the ground set. Given a subset of agents $S \subseteq V$, a set-cover of S is a collection of subsets $\mathcal{C} \subseteq \mathcal{F}$ such that $S \subseteq \bigcup_{C \in \mathcal{C}} C$, i.e., every element $i \in S$ belongs to some subset $C \in \mathcal{C}$. The cost of a set of agents S is the minimum cardinality set-cover of S.*

In the following subsection, in order to build intuition, we first prove that there is no fully budget-balanced group-strategyproof mechanism for the set-cover game. Our counter-example uses the following instance of a set-cover game. There are six elements $U = \{A_1, A_2, B_1, B_2, C_1, C_2\}$, and the collection of subsets is $\mathcal{F} = \{\{A_i, B_j, C_k\}_{i,j,k=1,2}\}$. In other words there are three groups of two elements and the available subsets are all those who contain exactly one element from each group. We then extend the proof to show the constant bound.

4.1 Impossibility of Full Budget-Balance

We will show by contradiction that there is no fully budget balanced group-strategyproof mechanism for this instance of the set-cover game. The first step to reach a contradiction is to show that the necessary properties together with full budget balance imply that at sets of the form $\{A_1, A_2, B_j, C_k\}$, there must be an unfair sharing of the cost in the sense that A_1 or A_2 must be responsible for their externality (their inclusion increases the cost of the optimal solution by one). Since one of the agents of the group A is responsible for her externality this puts an upper bound on the sum of the rest agents' payments. Then, since adding the missing agent from group B does not increase the cost, we show this does not change the upper bound. Finally, we exploit the symmetric form of this instance to derive the same bound with different agents in groups A and B. By summing, we deduce that the contribution of agent C_k is zero in the set U. Applying the same argument for every agent we deduce that no agent must be charged in U reaching a contradiction.

Theorem 5. *There is no fully budget-balanced group-strategyproof mechanism for the set-cover game.*

A slight different approach can be used to obtain a constant lower bound for this example.

Theorem 6. *There is no (18/19)-budget-balanced group-strategyproof mechanism for set cover.*

We refer the reader to the full version of the paper for the proofs of these theorems.

5 Conclusion

Our work is the first application of the complete characterization of group-strategyproof mechanisms [14]. We use the characterization to show bounds on the budget balance of group-strategyproof mechanisms for specific combinatorial problems. Particularly, we show that a very natural cost-sharing scheme for edge-cover satisfies the conditions of the characterization and is fully budget balanced. While the case of edge-cover is completely solved by this paper, it remains open to bound the optimal budget balance factor of group-strategyproof mechanisms for set-cover. Other problems of interest include facility location, vertex cover, Steiner tree, and Steiner forest. In the previous literature, these problems have only been solved using techniques involving cross-monotonic cost-sharing schemes, and it is known such an approach can not achieve perfect budget-balance for these problems.

Many constructions of cross-monotonic cost-sharing schemes rely on primal-dual schema. In these schemes, the natural linear-programming formulations of the combinatorial problems have constraints corresponding to the demand of the agents. The primal-dual scheme charges each agent her respective dual variable. In order to guarantee cross-monotonicity, these schemes introduce the notion of *ghost-shares*, i.e., the idea that variables contributing to a tight constraint are not frozen but rather keep growing and contributing to other constraints. Nevertheless, the payment of an agent is determined by the time that the dual variable was first involved in a tight constraint.

If we don't use the ghost-share technique the resulting cost-sharing scheme is not cross-monotone. However, in many cases it can be used to design weakly group-strategyproof mechanisms [9]. A natural question that arises is whether such a cost-sharing scheme satisfies the necessary conditions of group-strategyproofness despite the fact that it fails to satisfy cross-monotonicity. Unfortunately, the following observation indicates that this may not be true. Consider a cost-sharing scheme that is constructed by a primal-dual scheme and does not satisfy cross-monotonicity. Note that this implies the existence of a set S and two agents $i, j \in S$ such that $\xi(j, S \setminus \{i\}) < \xi(j, S)$. This means that the constraint that was responsible for freezing the dual variable of agent j becomes tight at a later time when i is present. This is only possible if there is another agent $k \in S$ that contributed to this constraint for subset $S \setminus \{i\}$; however, the inclusion of agent i caused the variable of k to freeze earlier, which means that $\xi(k, S \setminus \{i\}) > \xi(k, S)$. Such a cost-sharing scheme would not satisfy even the weaker necessary property of semi-cross monotonicity identified in [6].

The previous observation indicates that one should search beyond primal-dual schemes in order to design group-strategyproof mechanisms that perform strictly better than mechanisms captured by the cross-monotonic framework.

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On Coalitions and Stable Winners in Plurality

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Abstract. We consider elections under the Plurality rule, where all voters are assumed to act strategically. As there are typically many Nash equilibria for every preference profile, and strong equilibria do not always exist, we analyze the *most stable* outcomes according to their stability scores (the number of coalitions with an interest to deviate). We show a tight connection between the Maximin score of a candidate and the highest stability score of the outcomes where this candidate wins, and show that under mild conditions the Maximin winner will also be the winner in the most stable outcome under Plurality.

1 Introduction

Voting over potential possibilities is often used by societies to select an outcome that has a global effect. As the different individuals have different incentives, the way votes are aggregated to a final decision is important. Various voting systems have been suggested and analyzed, where a society with given preferences may often end up with radically different outcome, according to the voting system in use.[1] Thus, different voting systems offer different interpretations of consensus, or society's preferences. In political elections, the two prevalent systems are the single round *Plurality vote*, and the two round *Plurality with Runoff*, where a second round of Plurality occurs between the two leaders of the first round.

To complicate matters, even if we agree on a voting system that best implements the will of the society, voters may not reveal their true preferences. As Gibbard and Satterthwaite showed [4,13], it is unavoidable that in any non-trivial voting system there will be manipulations. One manipulation may trigger counter-manipulations by other voters, and so on. Therefore, the real question

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[1] There is an example with five candidates and five widely-used voting systems, where each candidate emerges as a winner under one of the systems, see here http://www.cs.cmu.edu/~arielpro/mfai_papers/lecture6.pdf

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that has to be asked when analyzing a voting system is not about the characterization of its *truthful outcome*, but rather what is its *likely outcome*. This is not a simple question, as the appropriate equilibrium concept is highly dependent on many parameters: whether voters vote simultaneously or sequentially, the information available to them, opportunities for collusion, etc.

The naïve approach of analyzing Nash equilibria typically fails due to the very large number of Nash equilibria, especially when there are many voters (and thus in almost all profiles no voter has real power to change the outcome alone). Several solutions have been proposed to the equilibria selection problem, typically by adding refinements that take into account voters' information or dynamics (see related work). Some of this work is driven by empirical findings on how people vote – either in lab experiments or in real elections. For example, a well-observed phenomenon in Plurality is that most votes are concentrated on a small number of candidates. A stronger form of this effect is the *Duverger's law*, which asserts that Plurality voting reduces to a match between only two candidates. A challenge to theorists is therefore to come up with models that replicate such effects and explain under what conditions they occur.

A very natural refinement, which directly addresses one of the main weaknesses of the Nash equilibrium, is to consider manipulations of *coalitions* rather than singletons. While a strong equilibrium (where no coalition can gain by deviating) does not always exist, some outcomes have fewer coalitional deviations than others. This counting measure of coalitional stability in games has been recently formalized under the name *stability scores* [3]. Stability scores can be used to compare various outcomes in a given game and highlight those outcomes that have the lowest number of deviating coalitions. They are therefore highly useful as an equilibrium selection criterion in situations where coalitional stability is important. In particular, a requirement for low stability scores can be thought of as a relaxation of strong equilibrium that is still stronger than Nash.

In this work we apply stability scores to identify the most stable outcomes under the Plurality rule. We then measure the stability of a *candidate* according to the most stable profile in which this candidate wins.

Our paper shows that the stability of each candidate strongly depends on her Maximin score. We study this dependency, and characterize the most stable winner under Plurality when voters are strategic.

1.1 Related Work

The equilibria outcomes of many voting rules have been studied. Due to its wide use and simplicity, Plurality was a natural focus for many of these studies. Myerson and Weber [10] tackled the multiple equilibria problem by assuming that voters have some level of uncertainty about the preferences of others. Thus every voter has some chance of being pivotal. They showed that every positional scoring rule has at least one equilibrium under these assumptions. Myerson and Weber, and later de Sinopoli [15,16] showed that under Plurality votes are typically more condensed in equilibrium, although they do not necessarily replicate Duverger's law.

A recent empirical study by Thompson et al. [17] studies a particular refinement of Nash equilibrium, that is achieved by adding a slight preference toward truthful reporting. They show that such equilibria tend to prefer winners that are ranked high on average (i.e. have high Borda score). While Thompson et al. did not measure the Maximin score of winners, we have no reason to believe that it is particularly high, as their equilibrium refinement is completely different from ours.

Other studies of the Plurality rule looked on equilibria that follow from a particular *dynamics*, e.g. when voters are voting in turns [2,7,20]. However, in all the papers we mentioned thus far, an outcome was considered to be an equilibrium if no *single* voter could gain by deviating.

Coalitional Stability. When considering manipulations by coalitions, an equilibrium is no longer guaranteed to exist. It has been shown by Kukushkin [6] that the only voting rule that guarantees the existence of a strong equilibrium (where no coalition of voters can gain) for any profile, is dictatorial. Messner and Polborn [8] considered *robust political equilibria* (RPE), which both assumes a level of uncertainty (also known as *trembling hand perfection*), and requires that deviating coalitions will themselves be stable. It turns out that trembling hand alone (even without considering coalitions) induces the Duverger's law under Plurality for any number of candidates. For three candidates, the authors characterize all RPEs and show conditions under which it exists and is unique. Interestingly, a sufficient condition is that there is *no* Condorcet winner.

Finally and closest to our model, Sertel and Sanver [14] characterized profiles in which strong equilibria exist in a wide variety of voting rules. They offered a criterion called *generalized Condorcet*, and proved that it is a necessary and sufficient condition to the existence of strong equilibrium. In particular, for the case of Plurality the generalized Condorcet criterion coincides with the standard definition. At the end of their paper, Sertel and Sanver leave as an open question whether there are relaxations of strong equilibrium for which their characterization could prove useful.

Counting Manipulations. Stability scores have been defined by Feldman et al. [3], and applied to congestion games and auctions. Our definitions follow their paper where possible. The proportion of coalitions of a given size that have a manipulation had been previously studied by Procaccia and Rosenschein and later by Xia and Conitzer [12,19]. However there are two significant differences between such counting methods and our stability scores approach. First, we do not assume truthful voting. Second, we compare the number of deviations assuming a fixed preference profile, rather than a distribution over profiles.

Another perspective on counting manipulations comes from the recent line of research on quantitative versions of robust social choice impossibility theorems, such as Arrow's and Gibbard-Satterthwaite's theorems. This line of research – having its roots in the study of the geometry of the Boolean cube, Analysis of Boolean functions, and discrete Fourier Analysis – started in [5]. The state-of-the-art in this direction is a theorem by Mossel and Racz [9] that bounds the *number of profitable manipulations* of any voting mechanism, in terms of its distance from a trivial mechanism, i.e. a dictatorship or a constant mechanism. An implication of this theorem is that in anonymous mechanisms, such as plurality, every profile has, on average, many profitable manipulations.

Such theorems study isoperimetric properties of the profile space, viewed as a graph whose edges are defined by profitable manipulations. However, like most work on voting, these results assume that there is only a single manipulator. The study we wish to initiate focuses on the extended definition of manipulation, where all voters may vote strategically, and deviations can be made by coalitions of any size. Of course, the scope of this paper is limited just to a single mechanism, namely Plurality.

1.2 Our Contribution

We relax the requirement for a strong equilibrium, and instead look for the *most stable* profiles and candidates in terms of their stability score. We show tight connection between the Maximin score of a candidate and its stability score, where the latter decreases roughly exponentially in the former. The Copeland score of a candidate also affects its coalitional stability, but has a secondary role.

As a corollary from the relation between the Maximin score and stability scores, we show that given mild conditions the Maximin winner always emerges as the most stable candidate under Plurality, i.e. as the winner in the most stable voting profile. This suggests that the Plurality rule *implements* the Maximin rule by selecting its winner under the stability scores solution concept. As a special case, we have that if and only if the Maximin score of a is at least $n/2$ (equivalently, when a is a Condorcet winner), it has a stability score of 0 - i.e. no coalition can deviate. For a lower Maximin score, the stability score increases gradually. Therefore our result generalizes the result of Sertel and Sanver on the Plurality rule to arbitrary profiles. We complement our results by showing that when our mild restrictions are relaxed, there are cases where the Maximin winner is not the winner of the most stable outcome.

To allow continuous reading, we moved most of the proofs to the appendix.

2 Preliminaries

Let A be a finite set of alternatives. $\boldsymbol{R}$ is a *preference profile*, i.e. a collection of n total orders $R_1, \ldots, R_n$ over A. $\boldsymbol{Q} = (a_1, \ldots, a_n)$ is a *voting profile* in Plurality, where voter i votes for candidate a_i. We assume that the preference profile $\boldsymbol{R}$ is fixed, whereas the voting profile $\boldsymbol{Q}$ may change due to strategic voting. We denote $a \succ_i b$ when a is preferred over b according to the preference order R_i.

$s(\boldsymbol{Q}) = (s_1(\boldsymbol{Q}), \ldots, s_m(\boldsymbol{Q}))$ is a scoring vector, where $s_i(\boldsymbol{Q})$ is the number of voters who voted for i in the profile $\boldsymbol{Q}$. The Plurality winner in $\boldsymbol{Q}$ is $f(\boldsymbol{Q}) = \text{argmax}_{a \in A}\, s_a(\boldsymbol{Q})$.

For every $a, a' \in A$, denote $W(a, a') = \{i \in N : a \succ_i a'\}$, and $w(a, a') = |W(a, a')|$. These pairwise matches induce a tournament graph over A, with an edge $a \rightarrow b$ whenever a beats b. We also denote $d(a) = |\{b \in A : w(a,b) < w(b,a)\}|$, i.e. the indegree of a in the tournament graph. Note that $m - d(a)$ is the Copeland score of a. The Maximin score of a in profile $\boldsymbol{R}$ is $ms(a, \boldsymbol{R}) = \min_{a' \neq a} w(a, a')$. For simplicity we assume throughout the paper that the number of voters n is odd (some minor adjustments are required if this assumption is relaxed). We denote the Maximin winner under profile $\boldsymbol{R}$ by $MX(\boldsymbol{R}) = \text{argmax}_{a \in A}\, ms(a, \boldsymbol{R})$ (only when it is unique).

Sertel and Sanver [14] define the set $C(\boldsymbol{R}; n, q)$ of (n, q)-Condorcet winners as follows: a is in $C(\boldsymbol{R}; n, q)$ if for all $b \in A$, $w(a, b) \geq q$. Thus this set coincides exactly with the set of candidates with Maximin score of at least q. They show that for a large collection of voting rules, the set of strong equilibrium winners can be characterized in terms of containment using $C(\boldsymbol{R}; n, q)$ for appropriate values of q. In particular, for the Plurality rule with odd n the appropriate threshold is $q = \lceil n/2 \rceil$, and thus $C(\boldsymbol{R}; n, q)$ consists of the Condorcet winner (if one exists).

Stability Scores. Given a preference profile $\boldsymbol{R}$ (which defines the game together with the voting rule), a voting profile $\boldsymbol{Q}$ and a candidate $a \in A$, the a-stability score of $\boldsymbol{Q}$ is defined as follows: For every $a \neq f(\boldsymbol{Q})$, $SC_a(\boldsymbol{Q}, t)$ is the number of coalitions $C \subseteq N$ s.t.

- $|C| = t$.
- $\forall i \in C$, $a \succ_i f(\boldsymbol{Q})$ (i.e. all voters in C prefer a over the current outcome).
- There is another voting profile $\boldsymbol{Q}'_C$ of C, s.t. $f(\boldsymbol{Q}_{-C}, \boldsymbol{Q}'_C) = a$.

The *stability score* of $\boldsymbol{Q}$, $SC(\boldsymbol{Q}, t)$, is simply the sum

$$SC(\boldsymbol{Q}, t) = \sum_{a \in A \setminus \{f(\boldsymbol{Q})\}} SC_a(\boldsymbol{Q}, t).$$

Since we get a different score for every size t, we need to aggregate these scores to a single number. We do so by treating all coalitions equally.[2] The *total stability score* is thus defined as $TSC(\boldsymbol{Q}) = \sum_{t \leq n} SC(\boldsymbol{Q}, t)$. Similarly, for all $a \in A$, $TSC_a(\boldsymbol{Q}) = \sum_{t \leq n} SC_a(\boldsymbol{Q}, t)$.

3 Stability of Outcomes under Plurality

Assume there is some given preference profile $\boldsymbol{R}$. Let $x(a) = n - ms(a, \boldsymbol{R})$, i.e. the largest support that any candidate has against a (note that lower $x(a)$ is better). For every candidate $a \in A$, let $\boldsymbol{Q}_a$ be the unanimity voting profile for this candidate, i.e. $\boldsymbol{Q}_a = (a, a, \ldots, a)$. Obviously, $f(\boldsymbol{Q}_a) = a$.

The next two lemmas show that $\boldsymbol{Q}_a$ is the most stable voting profile among all profiles where a wins.[3]

[2] We note that in the context of this work the exact way in which we aggregate scores over a and t does not really matter. See Discussion.

[3] While profiles where all voters vote the same are not very likely, we will later show that there are also other profiles with the same score.

Lemma 1. *For every candidate $a \in A$, it holds that*

1. *for all $b \neq a$, and any $t \geq \lceil n/2 \rceil$, $SC_b(\boldsymbol{Q}_a, t) = \binom{w(b,a)}{t}$. If $w(b,a) < n/2$ (i.e. a beats b) or $t < n/2$ then $SC_b(\boldsymbol{Q}_a, t) = 0$.*
2. $\max_{b \neq a} SC_b(\boldsymbol{Q}_a, t) = \binom{x(a)}{t}$.

Proof. In the voting profile $\boldsymbol{Q}_a$, a coalition C *can* change the outcome to b iff $|C| > n/2$. Also, C *wants* to change the outcome iff $C \subseteq W(b,a)$. To see why the last property holds, note that $x(a) = n - ms(a, \boldsymbol{R}) = n - \min_{b \neq a} w(a,b) = \max_{b \neq a} w(b,a)$.

Lemma 2. *Let $\boldsymbol{Q}$ be a voting profile s.t. $a = f(\boldsymbol{Q})$. For any $b \neq a$ and $t \geq \lceil n/2 \rceil$, $SC_b(\boldsymbol{Q}, t) \geq \binom{w(b,a)}{t}$.*

Proof. Let $C \subseteq W(b,a)$ be a coalition of size t. If all members of C will vote for b, then $s_b(\boldsymbol{Q}') \geq t > n/2$, and thus b will become the winner. Since b may already have some voters in C, it may be the case that many more coalitions of size t (or smaller coalitions) can change the outcome to b.

Let us now look on the "most stable" profile. That is, the voting profile $\boldsymbol{Q}$ that minimizes $TSC(\boldsymbol{Q})$. By the two lemmas above, it is sufficient to look on the m unanimous profiles $\{\boldsymbol{Q}_a\}_{a \in A}$. Since unanimous profiles are both most stable and simple, we will use them in our analysis. However the next lemma shows that there may still be *other* profiles where a wins, that have the same stability score.

Lemma 3. *Let $a, b \in A$ s.t. b is pair-wise losing to a. Let $\boldsymbol{Q}'$ be a voting profile where a wins s.t. (1) all voters vote for either a or b; and (2) all voters voting to b rank a last. Then $TSC(\boldsymbol{Q}') = TSC(\boldsymbol{Q}_a)$.*

We denote by $ST(\boldsymbol{R}) = \text{argmin}_{a \in A} TSC(\boldsymbol{Q}_a)$ the "most stable" winner under profile $\boldsymbol{R}$. In the remainder of this paper, we study the relations between $MX(\boldsymbol{R})$ and $ST(\boldsymbol{R})$. In particular, we seek conditions under which the two coincide.

Proposition 1. *Let a and a' be two candidates such that $x(a) \leq x(a') - k$ (i.e. a has a better Maximin score than a'), then*

$$TSC(\boldsymbol{Q}_a) \leq \max\left\{0, d(a)\left(\frac{n/2 - x(a')}{n - x(a')}\right)^k TSC(\boldsymbol{Q}_{a'})\right\},$$

and in particular $TSC(\boldsymbol{Q}_a) \leq d(a) 2^{-k} TSC(\boldsymbol{Q}_{a'})$.

This means that the stability score of candidates drops (i.e. improves) exponentially with their Maximin score (the complement of $x(a)$), and linearly with their Copeland score (the complement of $d(a)$). As a special case, we get that when $ms(a, \boldsymbol{R}) > n/2$ (i.e. when a is a Condorcet winner), then $\boldsymbol{Q}_a$ is a strong equilibrium – as has been showed by Sertel and Sanver. This is since for all $b \in A$, $w(b,a) < n/2$, and thus the right-hand side of the equation is 0.

3.1 Maximin and Stability

Our main result ties together high Maximin score and low stability score.

Theorem 2. *Let* $a = ST(\boldsymbol{R})$*, then* $ms(a, \boldsymbol{R}) \geq \max_b ms(b, \boldsymbol{R}) - \log d(b)$*. That is,* $ST(\boldsymbol{R})$ *is approximately the Maximin winner (under truthful voting).*

Proof. If a is the Maximin winner then we are done. Otherwise, let $b \in A$ be some candidate such that $ms(b, \boldsymbol{R}) > ms(a, \boldsymbol{R})$, and denote $k = ms(b, \boldsymbol{R}) - ms(a, \boldsymbol{R})$, thus $x(b) = x(a) - k$. By Proposition 1,

$$TSC(\boldsymbol{Q}_b) \geq d(b) 2^{-k} TSC(\boldsymbol{Q}_a) = 2^{d(b)-k} TSC(\boldsymbol{Q}_a).$$

However, since $TSC(\boldsymbol{Q}_a) \leq TSC(\boldsymbol{Q}_b)$, this implies that $2^{k - \log d(b)} \leq 1$, which means that $k \leq \log d(b)$. Thus $ms(a, \boldsymbol{R}) = ms(b, \boldsymbol{R}) - k \geq ms(b, \boldsymbol{R}) - \log d(b)$.

We next show that in most natural scenarios, the most stable winner is indeed the Maximin winner, and not just an approximation.

Let $MoV(\boldsymbol{R})$ deonte the *margin of victory* of the winner a under Maximin, in profile $\boldsymbol{R}$. That is, $MoV(\boldsymbol{R}) = \max_{b \in A}(ms(a, \boldsymbol{R}) - ms(b, \boldsymbol{R}))$, where a is the Maximin winner of $\boldsymbol{R}$.

From Theorem 2 it is easy to see that $MX(\boldsymbol{R}) = ST(\boldsymbol{R})$ whenever the $MoV(\boldsymbol{R})$ is higher than $\log m$. Our next result complements this observation by showing that even if the margin is low (say, $MoV(\boldsymbol{R}) = 1$), but the Maximin score of $MX(\boldsymbol{R})$ is sufficiently high, then it has to be the most stable winner.

Theorem 3. *Let* $a = MX(\boldsymbol{R})$*, and suppose that* $ms(a, \boldsymbol{R}) \geq n(\frac{1}{2} - \frac{1}{2(m-2)})$*. Then* $a = ST(\boldsymbol{R})$*.*

In particular, when there are 3 candidates, the condition always holds, and $ST(\boldsymbol{R}) = MX(\boldsymbol{R})$ for *any* profile $\boldsymbol{R}$. Using the fact that transitive votes pose certain constraints on the possible Maximin scores, we can strengthen this result even further.

Corollary 4. *For every profile* $\boldsymbol{R}$ *over (at most)* 4 *candidates,* $MX(\boldsymbol{R}) = ST(\boldsymbol{R})$*.*

For any number of candidates, this equality holds almost always, provided that there are enough voters. This is since the MoV in most voting rules including Maximin is typically of the order of $\sqrt{n}$ or more [18].[4] This is usually much larger than $\log m$, which means that the Maximin winner indeed minimizes the stability score.

However, it is not always the case that the Maximin winner is also the most stable. We construct such an example for 7 candidates, the details of which are given in the proof of the following proposition, in the appendix. Therefore, the additional requirements in Theorems 2 and 3 cannot be completely omitted.

[4] This holds whenever voters are drawn i.i.d. from some fixed distribution with full support over profiles.

Proposition 5. *There is a preference profile* $\boldsymbol{R}$ *over* 7 *candidates, s.t.* $MX(\boldsymbol{R}) \neq ST(\boldsymbol{R})$.

The main idea is to construct a voting profile with candidates a and b and 5 "dummy candidates", s.t. a beats b and also has a slightly higher Maximin score. However a loses to all dummy candidates, and by a larger margin than b does. We then show that it is in fact sufficient to count only coalitions whose size is close to $n/2$ (as these are almost all coalitions), and that for any such size there are more coalitions deviating from $\boldsymbol{Q}_a$ than from $\boldsymbol{Q}_b$. Thus a is not the most stable winner. Clearly, such example can also be constructed for any higher number of candidates.

4 Discussion and Implications

4.1 Implementation by Manipulation

Our analysis demonstrates that in the voting profile that is most stable against coalitional deviations, the winner is almost always the Maximin winner of the original profile. This result has two interpretations, or implications.

The first implication is a predictive one. In the presence of strategic voters who freely form coalitions, we may think of the most stable outcome (in terms of stability scores), as the *most likely outcome*. That is, we can predict that under Plurality, the Maximin winner will tend to be selected. Thus stability scores provide us with a solution concept, much like Nash equilibrium, strong equilibrium and other solution concepts are widely applied as predictions of the outcome. A drawback of this interpretation is that we did not compute the stability score of other profiles where the Maximin winner wins, whereas the unanimous profile is not a very likely outcome in itself.

We think the predictive interpretation is still valuable for two reasons. First, we conjecture that *on average*, other outcomes where the Maximin winner is selected will tend to be more stable than outcomes with a different winner. Second, even without proving such a conjecture, our prediction about the selection of the Maximin winner can still be tested empirically and experimentally.

The second implication of our result falls under mechanism design. A neutral arbitrator with access to the real preferences of the voters can *recommend* all voters to vote for a particular candidate (or any other recommendation). Voters may decide not to follow the recommendation, especially if they can find a sufficiently large group that by doing so will achieve a better outcome. Since low stability score indicates that there are few such coalitions, the most stable recommendation would be to vote for the Maximin winner. Note that in most cases voters who rank the Maximin winner *last* may avoid playing their dominated strategy (by ignoring the recommendation), without weakening the stablility score of the outcome.

This approach is very similar to the one taken by Peleg in his remarkable work on voting in committees [11]. In his paper, he describes a particular voting rule which is not truthful, but where an arbitrator can always recommend voters a particular voting profile which is itself a strong equilibrium - and selects the truthful winner. This is a rule that in a sense "implements itself" under strong equilibrium. Our results suggest that while Maximin may not implement itself, it can be implemented by Plurality under a somewhat weaker notion than strong equilibrium.

4.2 Variations of the Stability Score

Smaller coalitions are typically considered to be more likely to form than large ones. According to our definitions every deviating coalition is counted once, regardless of its size.

To the other extreme, one can choose to count only coalitions of size $t = \lceil \frac{n}{2} \rceil$, the smallest size of coalition that is guaranteed to exist in every profile (given that there is no Condorcet winner), and ignore larger coalitions. Even with this definition, our analysis remains correct. Notice that such coalitions contribute most to the stability score, as $SC(\boldsymbol{Q}_a, t)$ decreases exponentially in t. Therefore, such a choice is a good approximation of every definition of aggregation (in the unanimous profiles we analyzed).

Profitable coalitions of a given size can be counted with several resolutions:

- *By members*: In this model every combination of voters that can form a profitable coalition is counted once. This is the definition used by Feldman et al. [3] in the context of ad-auctions.
- *By members + target*: In this model every combination of voters that can form a profitable coalition is counted by the number of distinct profitable outcomes (i.e. winners) it can obtain.
- *By members + target + action*: In this model every profitable deviation by a set of voters is counted. This fits a model where the profitable deviation is chosen at random, since it is known to be computationally hard to obtain such a deviation in some mechanisms [1].

In this paper we chose to use the second resolution for two reasons. First, it somewhat simplifies the exposition, and second, it can be argued to be more appropriate in the context of voting, since there is a small defined set of alternative winners on which the coalition may try to agree. However, for the profiles that we analyzed and for coalitions of size $t = \lceil \frac{n}{2} \rceil$, the main results remain the same for all of the three models, subject to minor modifications of the analysis.

4.3 Future Work

Other than studying the connections between stability scores and coalitional dynamics, a natural direction is applying a similar stability analysis to voting rules other than Plurality. Such an analysis can reveal the properties that turn a candidate into a stable winner, and help to better predict the outcome in such settings.

A Proofs

Lemma 3. *Let $a, b \in A$ s.t. b is pair-wise losing to a. Let $\boldsymbol{Q}'$ be a voting profile where a wins s.t. (1) all voters vote for either a or b; and (2) all voters voting to b rank a last. Then $TSC(\boldsymbol{Q}') = TSC(\boldsymbol{Q}_a)$.*

Proof. We only need to show that every deviating coalition from $\boldsymbol{Q}'$ can still deviate in $\boldsymbol{Q}_a$. Note that even if all of $W(b,a)$ will vote for b, this will still not make b a winner, as $w(b,a) < w(a,b)$. Thus any coalition that can make b a winner must contain voters from $W(a,b)$, which do not want to deviate. Let C be a coalition deviating to some candidate b', and denote by B all voters voting for b in $\boldsymbol{Q}'$. Then $C \cup B$ is a deviating coalition in $\boldsymbol{Q}_a$.

Notice that by Lemma 1 and the definition of the total score,

$$TSC_b(\boldsymbol{Q}_a) = \sum_{t=\lceil \frac{n}{2} \rceil}^{w(b,a)} SC_b(\boldsymbol{Q}_a, t) = \sum_{t=\lceil \frac{n}{2} \rceil}^{w(b,a)} \binom{w(b,a)}{t}$$

We denote $\alpha(x) = \sum_{t=\lceil \frac{n}{2} \rceil}^{x} \binom{x}{t}$, so $TSC_b(\boldsymbol{Q}_a) = \alpha(w(b,a))$.

Lemma 4. $\alpha(x) \geq \alpha(x-1)\frac{x}{x-\lceil n/2 \rceil}$.

Proof. Assume $x \geq \lceil n/2 \rceil$, otherwise $\alpha(x) = 0$ and we get the inequality trivially. For all $t \geq \lceil n/2 \rceil$,

$$\binom{x}{t} = \binom{x-1}{t}\frac{x}{x-t} \geq \binom{x-1}{t}\frac{x}{x-\lceil n/2 \rceil}$$

Summing over all t,

$$\alpha(x) = \sum_{t=\lceil n/2 \rceil}^{x} \binom{x}{t} > \sum_{t=\lceil n/2 \rceil}^{x-1} \binom{x}{t}$$
$$\geq \sum_{t=\lceil n/2 \rceil}^{x-1} \binom{x-1}{t}\frac{x}{x-\lceil n/2 \rceil} = \frac{x}{x-\lceil n/2 \rceil}\alpha(x-1)$$

Proposition 1. *Let a and a' be two candidates such that $x(a) \leq x(a') - k$ (i.e. a has a better Maximin score than a'), then*

$$TSC(\boldsymbol{Q}_a) \leq d(a)\left(\frac{n/2 - x(a')}{n - x(a')}\right)^k TSC(\boldsymbol{Q}_{a'}) \leq d(a)2^{-k}TSC(\boldsymbol{Q}_{a'})$$

Proof. Let b^* be the candidate that beats a by the largest number of voters, thus $x(a) = w(b^*, a) = n - w(a, b^*)$. Similarly, there is a candidate $b' \in A$ s.t. $x(a') = w(b', a')$. Thus $w(b', a') - w(b^*, a) = x(a') - x(a) \geq k$.

By applying the inequality from Lemma 4 k times, we have that $\alpha(x) \geq \left(\frac{x}{x-\lceil n/2\rceil}\right)^k \alpha(x-k)$. Then

$$TSC_{b^*}(\boldsymbol{Q}_a) = \alpha(w(b^*,a)) \leq \alpha(w(b',a')-k) \leq \left(\frac{w(b',a')-\lceil n/2\rceil}{w(b',a')}\right)^k \alpha(w(b',a'))$$
$$= \left(\frac{\lfloor n/2\rfloor - w(a',b')}{n-w(a',b')}\right)^k \alpha(w(b',a')) = \left(\frac{\lfloor n/2\rfloor - w(a',b')}{n-w(a',b')}\right)^k TSC_{b'}(\boldsymbol{Q}_{a'}).$$

Also, we do not need to consider all $m-1$ candidates but only candidates that beat a (there are $d(a)$ such candidates). Thus

$$TSC(\boldsymbol{Q}_a) = \sum_{b:w(b,a)>n/2} TSC_b(\boldsymbol{Q}_a) \leq \sum_{b:w(b,a)>n/2} TSC_{b^*}(\boldsymbol{Q}_a) = d(a)TSC_{b^*}(\boldsymbol{Q}_a),$$

whereas $TSC(\boldsymbol{Q}_{a'}) \geq TSC_{b'}(\boldsymbol{Q}_a)$.

Summing over all possible deviations,

$$TSC(\boldsymbol{Q}_a) \leq d(a)TSC_{b^*}(\boldsymbol{Q}_a) \leq d(a)\left(\frac{n/2-w(a',b')}{n-w(a',b')}\right)^k TSC_{b'}(\boldsymbol{Q}_{a'})$$
$$\leq d(a)\left(\frac{n/2-x(a')}{n-x(a')}\right)^k TSC_{b'}(\boldsymbol{Q}_{a'})$$
$$\leq d(a)\left(\frac{n/2-x(a')}{n-x(a')}\right)^k TSC(\boldsymbol{Q}_{a'}) \leq d(a)2^{-k}TSC(\boldsymbol{Q}_{a'})$$

Theorem 3. *Let $a = MX(\boldsymbol{R})$, and suppose that $ms(a,\boldsymbol{R}) \geq n(\frac{1}{2} - \frac{1}{2(m-2)})$. Then $a = ST(\boldsymbol{R})$.*

Proof. Note that $ms(a,\boldsymbol{R}) \geq n(\frac{1}{2} - \frac{1}{2(m-2)})$ iff $\frac{\frac{n}{2}-x(a)}{n-x(a)} \leq \frac{1}{m-1}$. Also, $d(a) \leq m-1$ and $x(a) \leq x(a')$ for every $a' \in A$. Thus by Proposition 1, for every $a' \neq a$ s.t. $x(a') = x(a)+k$,

$$TSC(\boldsymbol{Q}_a) < d(a)\left(\frac{\frac{n}{2}-x(a')}{n-x(a')}\right)^k TSC(\boldsymbol{Q}_{a'}) \leq d(a)\frac{\frac{n}{2}-x(a')}{n-x(a')}TSC(\boldsymbol{Q}_{a'})$$
$$\leq (m-1)\frac{\frac{n}{2}-x(a)}{n-x(a)}TSC(\boldsymbol{Q}_{a'}) \leq (m-1)\frac{1}{m-1}TSC(\boldsymbol{Q}_{a'}) = TSC(\boldsymbol{Q}_{a'})$$

Lemma 5. *Let $C = (c_1,\ldots,c_k)$ be a tuple of k candidates. Then $\sum_{i=1}^k w(a_i, a_{(i \mod k)+1}) \geq n$.*

Proof. Otherwise, the union of all sets $W(a_i, a_{(i \mod k)+1})$ does not cover all voters, and thus there is a voter with cyclic preferences $c_k \succ c_{k+1} \succ \cdots \succ c_1 \succ c_k$.

Corollary 4. *For every profile* $\boldsymbol{R}$ *over (at most)* 4 *candidates,* $MX(\boldsymbol{R}) = ST(\boldsymbol{R})$*.*

Proof. Denote $a = MX(\boldsymbol{R}), b = ST(\boldsymbol{R})$. Assume, toward a contradiction, that $a \neq b$. Then from our previous corollaries we know that

- $d(a)\left(\frac{n/2-ms(b,\boldsymbol{R})}{n-ms(b,\boldsymbol{R})}\right)^t > 1$, where $t = ms(a,\boldsymbol{R}) - ms(b,\boldsymbol{R})$. Therefore:
- $d(a) \geq 3$. That is, a must be a beaten by at least (and exactly) 3 candidates.
- $t = 1$. That is, $ms(b,\boldsymbol{R}) = ms(a,\boldsymbol{R}) - 1$.

It thus follows, that

$$\begin{aligned}
&3\frac{n/2 - ms(b,\boldsymbol{R})}{n - ms(b,\boldsymbol{R})} > 1 && \Rightarrow \\
&3n/2 - 3ms(b,\boldsymbol{R}) > n - ms(b,\boldsymbol{R}) && \Rightarrow \\
&n/2 > 3ms(b,\boldsymbol{R}) - ms(b,\boldsymbol{R}) = 2ms(b,\boldsymbol{R}) && \Rightarrow \\
&ms(b,\boldsymbol{R}) < n/4.
\end{aligned}$$

Denote by c the candidate that beats b, then $w(b,c) = ms(b,\boldsymbol{R})$. c must also beat a, and since there is no Condorcet winner, the fourth candidate d beats c (and loses to b).

Since a is the Maximin winner, and the Maximin scores of a and b differ by 1, we have that $ms(c,\boldsymbol{R}), ms(d,\boldsymbol{R}) \leq ms(b,\boldsymbol{R}) < n/4$ as well. We also know that each of c, d is beaten by a single candidate, and thus $w(c,d) = ms(c,\boldsymbol{R}) < n/4$ (since d beats c), and $w(d,b) = ms(d,\boldsymbol{R}) < n/2$ (since b beats d). Therefore,

$$w(b,c) + w(c,d) + w(d,b) < n/4 + n/4 + n/2 < n,$$

in contradiction to Lemma 5.

Proposition 5. *There is a preference profile* $\boldsymbol{R}$ *over* 7 *candidates, s.t.* $MX(\boldsymbol{R}) \neq ST(\boldsymbol{R})$*.*

Proof. We will use $m = 7$ candidates denoted $A = \{a, b, c_1, \ldots, c_5\}$, and n voters, where $n = 4n' + 3$ for some integer n'. We note that as the number of voters grows, it becomes easier to construct such examples. Also, we can build similar constructions with more dummy candidates. The preference profile $\boldsymbol{R}$ is defined as follows.

- $n' + 1$ voters rank $a \succ b \succ C$.
- $2n' + 2$ voters rank $C \succ a \succ b$.
- n' voters rank $b \succ C \succ a$.

The preferences among the set C are set so that roughly $n/5$ voters rank c_i above $c_{(i \mod 5)+1}$ (this is easy to set via cyclic shifts). Note that $w(a,b) = n' + 1 + 2n' + 2 = 3n' + 3 = \lceil 3n/4 \rceil$, and that for all $c \in C$, $w(c,a) = 3n' + 2 = \lfloor 3n/4 \rfloor$, $w(c,b) = 2n' + 2 = \lceil n/2 \rceil$. It therefore holds that $ms(a,\boldsymbol{R}) = w(a,c) = n - w(c,a) = \lceil n/4 \rceil$, $ms(b,\boldsymbol{R}) = \lfloor n/4 \rfloor$, $\forall c \in C$ $ms(c,\boldsymbol{R}) \cong n/5$. In particular, a

is the Maximin winner. We next turn to compute the stability scores of $\boldsymbol{Q}_a$ and $\boldsymbol{Q}_b$, to show that the latter is more stable.

For $\boldsymbol{Q}_a$, we need to consider all winning coalitions that prefer c over a, for every $c \in C$. Note that $x(a) = w(c,a) = \lfloor 3n/4 \rfloor$, and $x(b) = w(a,b) = \lceil 3n/4 \rceil = x(a) + 1$.

$$\forall t \geq \lceil n/2 \rceil, SC(\boldsymbol{Q}_a, t) = |C| SC_c(\boldsymbol{Q}_a, t) = 5 \binom{x(a)}{t}.$$

As for $\boldsymbol{Q}_b$, there is exactly one coalition of size $\lceil n/2 \rceil$ that can deviate from $\boldsymbol{Q}_b$ to any c. For any higher t, the only possible deviations are to a. Thus $\forall t$ s.t. $\lceil n/2 + \log(n) \rceil \geq t > \lceil n/2 \rceil$

$$\begin{aligned}
SC(\boldsymbol{Q}_b, t) &= SC_a(\boldsymbol{Q}_b, t) = \binom{x(b)}{t} = \binom{x(a)+1}{t} = \binom{x(a)}{t} \frac{x(a)}{x(a)-t} \\
&= \frac{1}{5} SC(\boldsymbol{Q}_a, t) \frac{x(a)}{x(a)-t} \leq \frac{1}{5} SC(\boldsymbol{Q}_a, t) \frac{x(a)}{x(a) - \lceil n/2 + \log(n) \rceil} \\
&= \frac{1}{5} SC(\boldsymbol{Q}_a, t) \frac{x(a)}{x(a) - \lceil n/2 + \log(n) \rceil} = \frac{1}{5} SC(\boldsymbol{Q}_a, t) \frac{\lfloor 3n/4 \rfloor}{\lfloor 3n/4 \rfloor - \lceil n/2 + \log(n) \rceil} \\
&= \frac{1}{5} SC(\boldsymbol{Q}_a, t) \frac{\lfloor 3n/4 \rfloor}{\lfloor n/4 - \log(n) \rfloor} < \frac{1}{5} SC(\boldsymbol{Q}_a, t) \cdot 3.1 < \frac{7}{10} SC(\boldsymbol{Q}_a, t) \\
& \qquad \text{(for every } n > 100\text{)}
\end{aligned}$$

For $t = \lceil n/2 \rceil$, we need to add the 5 coalitions that deviate to C. We now turn to sum over al sizes of coalitions. Note that there are many deviating coalitions of size $t = \lceil n/2 \rceil$, but this size drops exponentially as t grows. In fact almost all deviations (say, 95%) are in the range $t \in [\lceil n/2 \rceil, \lceil n/2 + \log(n) \rceil]$. In other words

$$\sum_{t=\lceil n/2 \rceil}^{\lceil n/2 + \log(n) \rceil} SC_a(\boldsymbol{Q}_b, t) > 0.95 \sum_{t=\lceil n/2 \rceil}^{n - x(b)} SC_a(\boldsymbol{Q}_b, t) = 0.95 \cdot TSC_a(\boldsymbol{Q}_b, t), \tag{1}$$

whereas $TSC(\boldsymbol{Q}_b) = TSC_a(\boldsymbol{Q}_b) + 5$.

Adding all inequalities,

$$\begin{aligned}
TSC(\boldsymbol{Q}_b) = TSC_a(\boldsymbol{Q}_b) + 5 &< 5 + 1.06 \sum_{t=\lceil n/2 \rceil}^{\lceil n/2 + \log(n) \rceil} SC_a(\boldsymbol{Q}_b, t) \\
&< 5 + 1.06 \sum_{t=\lceil n/2 \rceil}^{\lceil n/2 + \log(n) \rceil} \frac{7}{10} SC(\boldsymbol{Q}_a, t) \\
&< 5 + \frac{8}{10} \sum_{t=\lceil n/2 \rceil}^{\lceil n/2 + \log(n) \rceil} SC(\boldsymbol{Q}_a, t) < 5 + \frac{8}{10} TSC(\boldsymbol{Q}_a) < TSC(\boldsymbol{Q}_a).
\end{aligned}$$

Thus for sufficiently large n, candidate b has a lower (better) stability score than a.

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On the Efficiency of Influence-and-Exploit Strategies for Revenue Maximization under Positive Externalities*

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Abstract. We consider the marketing model of (Hartline, Mirrokni, Sundararajan, WWW '08) for selling a digital product in a social network under positive externalities. The seller seeks for a marketing strategy, namely an ordering in which he approaches the buyers and the prices offered to them, that maximizes her revenue. We restrict our attention to the Uniform Additive Model of externalities, and mostly focus on Influence-and-Exploit (IE) marketing strategies. We show that in undirected social networks, revenue maximization is $\mathbf{NP}$-hard not only when we search for a general optimal marketing strategy, but also when we search for the best IE strategy. Rather surprisingly, we observe that allowing IE strategies to offer prices smaller than the myopic price in the exploit step leads to a significant improvement on their performance. Thus, we show that the best IE strategy approximates the maximum revenue within a factor of 0.911 for undirected and of roughly 0.553 for directed networks. Utilizing a connection between good IE strategies and large cuts in the underlying social network, we obtain polynomial-time algorithms that approximate the revenue of the best IE strategy within a factor of roughly 0.9. Hence, we significantly improve on the best known approximation ratio for the maximum revenue to 0.8229 for undirected and to 0.5011 for directed networks (from $2/3$ and $1/3$, respectively).

1 Introduction

Understanding the flow of information, influence, and epidemics through the social fabric has become increasingly important due to the high interconnectedness brought about by technological advances. The digitization of communications (e.g., cell phones, emails, text messages) and of the social interaction (e.g., Facebook, Twitter) not only has provided the researchers with a strong empirical footing upon which they can base their theories and test their predictions, but also has opened the frontier of algorithmic applications related to social networks. Particularly, there has been a shift from aggregate descriptive theories, in the spirit of *Diffusion of Innovations*, to models incorporating the structure of social networks, culminating with the algorithmic paradigm of *Influence Maximization*.

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Firms operating in such a reticular environment, where information about products and services diffuses rapidly between individuals, have acknowledged the importance of revisiting their approach. The availability of information about users and the mitigated effectiveness of traditional forms of marketing occasion the need for intelligent marketing strategies. Towards realizing this goal, there are three main challenges: mining individual preferences, quantifying the influence that buyers exert upon each other, and fusing these information along a marketing strategy. The ideal solution would be an algorithm that intelligently adjusts its actions (e.g., prices, individuals to approach) based on the current state of the network, and maximizes the seller's revenue.

In this work, we are interested in the latter challenge of designing efficient marketing strategies that exploit the positive influence between buyers. We focus on the setting where the utility of the product depends inherently on the scale of the product's adoption, e.g., the value of a social network depends on the fraction of the population using it on a regular basis. In fact, for many products, their value to a buyer depends on the set of her friends using them (e.g., cell phones, online gaming). In the presence of such positive externalities between the potential buyers, the seller seeks for a marketing strategy that guarantees a significant revenue through a wide adoption of the product, which leads to an increased value, and consequently, to a profitable pricing of it.

Marketing Model. More formally, we adopt the model of Hartline, Mirrokni, and Sundararajan [14], where a digital product is sold to a set of potential buyers under positive externalities. We assume an unlimited supply of the product and that there is no production cost for it. A (possibly directed) weighted social network $G(V, E, w)$ on the set V of potential buyers models how their value of the product is affected by other buyers who already own the product. Specifically, an edge $(j, i) \in E$ denotes that the event that j owns the product has a positive influence on i's value of the product. The strength of this influence is quantified by a non-negative weight w_{ji} associated with edge (j, i). Also, buyer i may have an intrinsic value of the product, quantified by a non-negative weight w_{ii}. The product's value to each buyer i is given by a non-decreasing function $v_i : 2^{N_i} \mapsto \mathbb{R}_+$, which depends on w_{ii} and on the set $S \subseteq N_i$ of i's neighbors who already own the product, where $N_i = \{j \in V \setminus \{i\} : (j, i) \in E\}$. The exact values $v_i(S)$ are unknown and are treated as random variables of which only the distributions $F_{i,S}$ are known to the seller. In particular, we assume that for each buyer i and each set $S \subseteq N_i$, the seller only knows the probability distribution $F_{i,S}(x) = \mathbb{P}\mathrm{r}[v_i(S) < x]$ that buyer i rejects an offer of price x for the product.

Regarding the distribution of $v_i(S)$'s, the most interesting cases outlined in [14] are: (i) the *Concave Graph Model*, where the weights w_{ji} are random variables, and the values $v_i(S)$ are determined by a concave function of the total influence $M_{i,S} = \sum_{j \in S \cup \{i\}} w_{ji}$ perceived by buyer i from the set S of her neighbors owning the product, and (ii) the *Uniform Additive Model*, where the weights w_{ji} are deterministic, and the values $v_i(S)$ are uniformly distributed in $[0, M_{i,S}]$. In this work, we restrict our attention to the Uniform Additive Model, which can be regarded as an extension of the widely accepted Linear Threshold Model of social influence [15]. Though technically simpler, the Uniform Additive Model incorporates all the main features of the marketing model of [14]. An important special case of the Uniform Additive Model is the undirected (or the symmetric) case, where $w_{ij} = w_{ji}$ for all edges $\{i, j\}$ of the social network.

In this setting, the seller approaches each potential buyer once and makes an offer to him. Thus, a *marketing strategy* $(\boldsymbol{\pi}, \boldsymbol{x})$ consists of a permutation $\boldsymbol{\pi}$ of the buyers and a pricing vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, where $\boldsymbol{\pi}$ determines the order in which the buyers are approached and $\boldsymbol{x}$ the prices offered to them. Given the set S of i's neighbors who own the product when the seller approaches her, buyer i accepts the offer with probability $1 - F_{i,S}(x_i)$, in which case she pays the price x_i, or rejects it, with probability $F_{i,S}(x_i)$, in which case she pays nothing and never receives an offer again. The seller's goal is to compute a marketing strategy $(\boldsymbol{\pi}, \boldsymbol{x})$ that maximizes her expected revenue, namely the total amount paid by the buyers who accept the offer.

Previous Work. Using a transformation from Maximum Acyclic Subgraph, Hartline et al. [14] proved that if we have complete knowledge of the buyers' valuations, computing a revenue-maximizing ordering is $\mathbf{NP}$-hard for directed social networks. Combined with the result of [12], this suggests an upper bound of 0.5 on the approximation ratio of revenue maximization for directed networks and deterministic additive valuations. On the positive side, they gave a polynomial-time dynamic programming algorithm for a fully symmetric special case, where the order of the buyers is insignificant.

An interesting contribution of [14] is a class of elegant marketing strategies called *Influence-and-Exploit* (IE). An IE strategy first offers the product for free to a selected subset of buyers, aiming to increase the value of the product to the remaining buyers (influence step). Then, in the exploit step, it approaches the remaining buyers, in a random order, and offers them the product at the so-called *myopic price*. The myopic price ignores the current buyer's influence on the subsequent buyers and maximizes the expected revenue extracted from her. In the Uniform Additive Model, each buyer accepts the myopic price with probability $1/2$. Hence, there is a notion of uniformity in the prices offered in the exploit step, in the sense that the buyers accept them with a fixed probability, and we can say that the IE strategy uses a *pricing probability* of $1/2$.

As for the revenue extracted by IE strategies compared against the maximum revenue extracted by general marketing strategies, Hartline et al. [14] proved that the best IE strategy approximates the maximum revenue within a factor of 0.25 for the Concave Graph Model, which improves to $\frac{e}{4e-2} \approx 0.306$ if the distributions $F_{i,S}$ satisfy the monotone hazard rate condition, and within a factor of 0.94 for the (polynomially solvable) fully symmetric case of the Uniform Additive Model. Combined with the recent algorithm of [16] for unconstrained submodular maximization, which can be used to approximate the revenue of the best IE strategy within a factor of 0.5, the results of [14] imply an approximation ratio of 0.125 for the maximum revenue in the Concave Graph Model, which improves to 0.153 if the distributions $F_{i,S}$ satisfy the monotone hazard rate condition. As for the Uniform Additive Model, Hartline et al. [14] proved that if each buyer is selected in the influence set randomly, with an appropriate probability, the expected revenue of IE is at least $2/3$ (resp. $1/3$) times the maximum revenue of undirected (resp. directed) networks. Since [14], the Influence-and-Exploit paradigm has been applied to a few other settings where one seeks to maximize revenue in the presence of positive externalities (see e.g. [4,5,13]).

Contribution and Techniques. Although IE strategies are simple, elegant, and promising in terms of efficiency, their performance against the maximum revenue and their approximability are not well understood. Moreover, the absence of any strong bounds on the fraction of the maximum revenue extracted by the best IE strategy and the poor approximation ratios for the maximum revenue in the Concave Graph Model suggest looking into simpler cases of the model. This is also suggested by previous work on Influence Maximization, where focusing on simpler cases provides insights, which, in turn, can enhance our understanding of more general settings. In this work, we focus on the important case of the Uniform Additive Model, and obtain a comprehensive collection of results on the efficiency and the approximability of IE strategies. Our results also imply a significant improvement on the best known approximation ratio for revenue maximization in the Uniform Additive Model.

We first show that in the Uniform Additive Model, revenue maximization is $\mathbf{NP}$-hard for undirected networks[1] not only when we search for a general optimal marketing strategy, but also when we search for the best IE strategy. Next, we embark on a systematic study of the algorithmic properties of IE strategies (Section 3). In [14], IE strategies are restricted, by definition, to the myopic pricing probability, which for the Uniform Additive Model is $1/2$. Rather surprisingly, we observe that we can achieve a significant improvement on the efficiency of IE strategies if we use smaller prices (equivalently, a larger pricing probability) in the exploit step. Thus, we let IE strategies use a carefully selected pricing probability $p \in [1/2, 1)$.

We prove the existence of an IE strategy with pricing probability 0.586 (resp. $2/3$) which approximates the maximum revenue, extracted by an unrestricted marketing strategy, within a factor of 0.911 for undirected (resp. 0.55289 for directed) networks. The proof assumes a revenue-maximizing pricing probability vector $\boldsymbol{p}$ and constructs an IE strategy with the desired expected revenue by applying randomized rounding to $\boldsymbol{p}$. An interesting consequence is that the upper bound of 0.5 on the approximation ratio of the maximum revenue for directed networks does not apply to the Uniform Additive Model. In Section 3, we discuss the technical reasons behind this and show a pair of upper bounds on the approximation ratio achievable for directed networks. Specifically, assuming the Unique Games conjecture, we show that it is $\mathbf{NP}$-hard to approximate the maximum revenue within a factor greater than $27/32$, if we use any marketing strategy, and greater than $3/4$, if we are restricted to IE strategies with pricing probability $2/3$.

The technical intuition behind most of our results comes from the apparent connection between good IE strategies and large cuts in the underlying social network. Following this intuition, we optimize the parameters of the random-partitioning IE strategy of [14] and slightly improve the approximation ratio to 0.686 (resp. 0.343) for undirected (resp. directed) networks. Building on the idea of generating revenue from large cuts in the network, we discuss, in Section 4, a natural generalization of IE strategies that use more than two pricing classes. We show that a simple random partitioning of the buyers in six pricing classes further improves the approximation ratio for the maximum revenue to 0.7032 for undirected networks and to 0.3516 for directed social networks.

The main hurdle in obtaining better approximation guarantess for the maximum revenue problem is the lack of any strong upper bounds on it. In Section 5, we introduce a strong Semidefinite Programming (SDP) relaxation for the problem of computing the

[1] If the seller has complete knowledge of the buyers' valuations, finding a revenue-maximizing ordering for undirected networks is polynomially solvable (Lemma 1). Therefore, the reduction of [14] does not imply that revenue maximization for undirected networks is $\mathbf{NP}$-hard.

best IE strategy with any given pricing probability. Our approach exploits the resemblance between computing the best IE strategy and the problems of MAX-CUT and MAX-DICUT, and builds on the elegant approach of Goemans and Williamson [11] and Feige and Goemans [8]. Solving the SDP relaxation and using randomized rounding, we obtain a 0.9032 (resp. 0.9064) approximation for the best IE strategy with a pricing probability of 0.586 for undirected networks (resp. of $2/3$ for directed networks). Combining these results with the bounds on the fraction of the maximum revenue extracted by the best IE strategy, we significantly improve on the best known approximation ratio for revenue maximization to 0.8229 for undirected networks and 0.5011 for directed networks (from $2/3$ and $1/3$, respectively, in [14]). To the best of our knowledge, this is the first time an (approximate) SDP relaxation for a pricing model under positive externalities is suggested and exploited to improve the approximation ratio for the corresponding revenue (or welfare) maximization problem. Actually, we believe that our SDP-based approach may find applications to other pricing models under externalities.

Other Related Work. Our work lies in the area of pricing and revenue maximization under positive externalities, and more generally, in the area of social contagion and influence maximization (see e.g., [7,15]). Recent research has studied the impact of externalities in a variety of settings (see e.g. [14,4,1,3,6,5,13,9]). Hartline et al. [14] were the first to consider social influence in the framework of revenue maximization. Since then, relevant research has focused either on posted price strategies, where there is no price discrimination, or on game theoretic considerations, where the buyers act strategically according to their value of the product. To the best of our knowledge, our work is the first that considers the approximability of the revenue extracted by an optimal strategy and by the best IE strategy, which were the central problems in [14].

Regarding posted pricing, Arthur et al. [4] considered a model where recommendations about the product cascade through the network from early adopters, and presented an IE-based $O(1)$-approximation algorithm for the maximum revenue. Akhlaghpour et al. [1] considered iterative posted pricing, where all interested buyers can buy the product at the same price at a given time. They studied revenue maximization under two different repricing models allowing for at most k prices. They proved that if frequent repricing is allowed, revenue maximization is $\mathbf{NP}$-hard to approximate, while if the repricing rate is limited, there is an FPTAS. Anari et al. [3] considered a posted price setting with historical externalities. Given a fixed price trajectory, the buyers decide when to buy the product. In this setting, they studied existence and uniqueness of equilibria, and presented an FPTAS for special cases of revenue maximization.

In a complementary direction, Chen et al. [6] investigated the (Bayesian-)Nash equilibria when each buyer's value of the product depends on the set of buyers who own the product. They focused on two classes of equilibria, pessimistic and optimistic ones, and showed how to compute these equilibria and how to find revenue-maximizing prices. Candogan et al. [5] investigated a scenario where a monopolist sells a divisible good to buyers under positive externalities. They considered a two-stage game where the seller first sets an individual price for each buyer, and then the buyers decide on their consumption level. They proved that the optimal price for each buyer is proportional to her Bonacich centrality, and that if the buyers are partitioned into two pricing classes (which is conceptually similar to IE), the problem is reducible to MAX-CUT.

2 The Model and Preliminaries

The Influence Model. The social network is a (possibly directed) weighted network $G(V, E, w)$ on the set V of potential buyers. There is a positive weight w_{ij} associated with each edge $(i, j) \in E$ (we assume that $w_{ij} = 0$ if $(i, j) \notin E$). A social network is undirected (or symmetric) if $w_{ij} = w_{ji}$ for all $i, j \in V$, and directed otherwise. There may exist a non-negative weight w_{ii} associated with each buyer i[2]. Each buyer i has a value $v_i : 2^{N_i} \mapsto \mathbb{R}_+$ of the product, which depends on w_{ii} and on the set $S \subseteq N_i$ of i's neighbors who already own the product, where $N_i = \{j \in V \setminus \{i\} : (j, i) \in E\}$. However, the exact values $v_i(S)$ are unknown to the seller, who, for each buyer i and each set $S \subseteq N_i$, only knows the probability distribution $F_{i,S}(x) = \mathbb{P}\text{r}[v_i(S) < x]$ that buyer i rejects an offer of price x for the product.

In the *Uniform Additive Model* [14, Section 2.1], the values $v_i(S)$ are drawn from the uniform distribution in $[0, M_{i,S}]$, where $M_{i,S} = \sum_{j \in S \cup \{i\}} w_{ji}$ is the total influence perceived by i by the set S of her neighbors owning the product. Then, the probability that buyer i rejects an offer of price x is $F_{i,S}(x) = x/M_{i,S}$.

Myopic Pricing. The *myopic price* disregards any externalities imposed by i on her neighbors, and simply maximizes the expected revenue extracted from buyer i, given that S is the current set of i's neighbors who own the product. For the Uniform Additive Model, the myopic price is $M_{i,S}/2$, the probability that buyer i accepts it is $1/2$, and the expected revenue extracted from her with the myopic price is $M_{i,S}/4$, which is the maximum revenue one can extract from buyer i alone.

Marketing Strategies and Revenue Maximization. We can usually extract more revenue from G by employing a marketing strategy that exploits the positive influence between the buyers. A *marketing strategy* $(\boldsymbol{\pi}, \boldsymbol{x})$ consists of a permutation $\boldsymbol{\pi}$ of the buyers and a pricing vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, where $\boldsymbol{\pi}$ determines the order in which the buyers are approached and $\boldsymbol{x}$ the prices offered to them.

We observe that for any buyer i and any probability p that i accepts an offer, there is an (essentially unique) price x_p such that i accepts an offer of x_p with probability p. For the Uniform Additive Model, $x_p = (1-p)M_{i,S}$ and the expected revenue extracted from buyer i with such an offer is $p(1-p)M_{i,S}$. Throughout this paper, we equivalently regard marketing strategies as consisting of a permutation $\boldsymbol{\pi}$ of the buyers and a vector $\boldsymbol{p} = (p_1, \ldots, p_n)$ of pricing probabilities. We note that if $p_i = 1$, i gets the product for free, while if $p_i = 1/2$, the price offered to i is (the myopic price of) $M_{i,S}/2$. We assume that $p_i \in [1/2, 1]$, since any expected revenue in $[0, M_{i,S}/4]$ can be achieved with such pricing probabilities. The expected revenue of a marketing strategy $(\boldsymbol{\pi}, \boldsymbol{p})$ is:

$$R(\boldsymbol{\pi}, \boldsymbol{p}) = \sum_{i \in V} p_i(1 - p_i) \left(w_{ii} + \sum_{j : \pi_j < \pi_i} p_j w_{ji} \right) \quad (1)$$

The problem of *revenue maximization* under the Uniform Additive Model is to find a marketing strategy $(\boldsymbol{\pi}^*, \boldsymbol{p}^*)$ that extracts a maximum revenue of $R(\boldsymbol{\pi}^*, \boldsymbol{p}^*)$ from a given social network $G(V, E, w)$.

[2] Wlog., we ignore w_{ii}'s for directed networks, since we can replace each w_{ii} by an edge (i', i) of weight w_{ii} from a new node i' with a single outgoing edge (i', i) and no incoming edges.

Bounds on the Maximum Revenue. Let $N = \sum_{i \in V} w_{ii}$ and $W = \sum_{i<j} w_{ij}$, if the social network G is undirected, and $W = \sum_{(i,j) \in E} w_{ij}$, if G is directed. Then an upper bound on the maximum revenue of G is $R^* = (W + N)/4$, and follows by summing up the myopic revenue over all edges of G [14, Fact 1]. A lower bound on the maximum revenue is $(W + 2N)/8$ (resp. $(W + 4N)/16$), if G is undirected (resp. directed), and follows by approaching the buyers in any order (resp. in a random order) and offering them the myopic price. Thus, myopic pricing achieves an approximation ratio of 0.5 for undirected networks and of 0.25 for directed networks.

Ordering and NP-Hardness. Revenue maximization exhibits a dual nature involving optimizing both the pricing probabilities and the sequence of offers. For directed networks, finding a good ordering $\boldsymbol{\pi}$ of the buyers bears a resemblance to the Maximum Acyclic Subgraph problem, where given $G(V, E, w)$, we seek for an acyclic subgraph of maximum total edge weight. In fact, any permutation $\boldsymbol{\pi}$ of V corresponds to an acyclic subgraph of G that includes all edges going forward in $\boldsymbol{\pi}$, i.e., all edges (i, j) with $\pi_i < \pi_j$. [14, Lemma 3.2] shows that given a directed network G and a pricing probability vector $\boldsymbol{p}$, computing an optimal ordering of the buyers (for the particular $\boldsymbol{p}$) is equivalent to computing a Maximum Acyclic Subgraph of G, with each edge (i, j) having a weight of $p_i p_j (1 - p_j) w_{ij}$. Consequently, computing an ordering $\boldsymbol{\pi}$ that maximizes $R(\boldsymbol{\pi}, \boldsymbol{p})$ is $\mathbf{NP}$-hard and Unique-Games-hard to approximate within a factor greater than 0.5 [12]. On the other hand, we can show that in the undirected case, if the pricing probabilities are given, we can easily compute the best ordering of the buyers.

Lemma 1. *Let $G(V, E, w)$ be an undirected social network, and let $\mathbf{p}$ be any pricing probability vector. Then, approaching the buyers in non-increasing order of their pricing probabilities maximizes the revenue extracted from G under $\mathbf{p}$.*

Therefore, [14, Lemma 3.2] does not imply the $\mathbf{NP}$-hardness of revenue maximization for undirected networks. The following lemma employs a reduction from monotone One-in-Three 3-SAT [10, LO4], and shows that revenue maximization is $\mathbf{NP}$-hard for undirected networks.

Lemma 2. *Computing a marketing strategy that extracts the maximum revenue from an undirected social network is $\mathbf{NP}$-hard.*

3 Influence-and-Exploit Strategies

An *Influence-and-Exploit* (IE) strategy $\mathrm{IE}(A, p)$ consists of a set of buyers A receiving the product for free and a pricing probability p offered to the remaining buyers in $V \setminus A$, approached in a random order. We slightly abuse the notation, and let $\mathrm{IE}(q, p)$ denote an IE strategy where each buyer is selected in A independently with probability q. For directed networks, $\mathrm{IE}(A, p)$ extracts an expected (wrt the random ordering of the exploit set) revenue of:

$$R_{\mathrm{IE}}(A, p) = p(1-p) \sum_{i \in V \setminus A} \left(w_{ii} + \sum_{j \in A} w_{ji} + \sum_{j \in V \setminus A,\ j \neq i} \frac{p\, w_{ji}}{2} \right) \qquad (2)$$

Specifically, $\mathrm{IE}(A, p)$ extracts a revenue of $p(1-p)w_{ji}$ from each edge (j, i) with $j \in A$ and $i \in V \setminus A$, and a revenue of $p^2(1-p)w_{ji}$ from each edge (j, i) with both $j, i \in V \setminus A$, if j is before i in the random order, which happens with probability $1/2$.

The problem of finding the best IE strategy is to compute a subset of buyers A^* and a pricing probability p^* that extract a maximum revenue of $R_{\mathrm{IE}}(A^*, p^*)$ from a given social network $G(V, E, w)$. The following lemma employs a reduction from monotone One-in-Three 3-SAT, and shows that computing the best IE strategy is $\mathbf{NP}$-hard.

Lemma 3. *The problems of computing the best IE strategy and of computing the best IE strategy with a given pricing probability p, for any fixed* $p \in [1/2, 1)$*, are* $\mathbf{NP}$*-hard, even for undirected networks.*

Simple IE strategies extract a significant fraction of the maximum revenue. E.g., for undirected networks, $R_{\mathrm{IE}}(\emptyset, 2/3) = (4W + 6N)/27$, and $\mathrm{IE}(\emptyset, 2/3)$ achieves an approximation ratio of $\frac{16}{27}$. Moreover, $\mathrm{IE}(X, 1/2)$ extracts the maximum revenue from any simple undirected bipartite network $G(X, Y, E)$. For directed networks, $R_{\mathrm{IE}}(\emptyset, 2/3) = (2W + 6N)/27$, and $\mathrm{IE}(\emptyset, 2/3)$ achieves an approximation ratio of $\frac{8}{27}$. We next show that carefully selected IE strategies extract a larger fraction of the maximum revenue.

Exploiting Large Cuts. A natural idea is to exploit the apparent connection between a large cut in the social network and a good IE strategy. For example, in the undirected case, an IE strategy $\mathrm{IE}(q, p)$ is conceptually similar to the randomized 0.5-approximation algorithm for MAX-CUT, which puts each node in set A with probability $1/2$. However, in addition to a revenue of $p(1-p)w_{ij}$ from each edge $\{i, j\}$ in the cut $(A, V \setminus A)$, $\mathrm{IE}(q, p)$ extracts a revenue of $p^2(1-p)w_{ij}$ from each edge $\{i, j\}$ between nodes in the exploit set $V \setminus A$. Thus, to optimize the performance of $\mathrm{IE}(q, p)$, we carefully adjust the probabilities q and p so that $\mathrm{IE}(q, p)$ balances between the two sources of revenue. The proof of Proposition 1 extends the proof of [14, Theorem 3.1].

Proposition 1. *Let* $G(V, E, w)$ *be an undirected (resp. directed) social network, let* $\lambda = N/W$*, and let* $q = \max\{1 - \frac{\sqrt{2}(2+\lambda)}{4}, 0\}$*, Then,* $\mathrm{IE}(q, 2-\sqrt{2})$ *approximates the maximum revenue of G within a factor of* 0.686 *(resp.* 0.343*).*

On the Efficiency of Influence-and-Exploit. IE makes a rough discretization of the pricing space, and exploits the fact that the combinatorial structure of partitioning the vertices into two sets is well understood. Nevertheless, we are left with the nontrivial task of correlating the maximum revenue with only two prices and the maximum revenue with any set of prices. We next show that the best IE strategy, which is $\mathbf{NP}$-hard to compute, manages to extract a significant fraction of the maximum revenue.

Theorem 1. *For any undirected social network, there exists an IE strategy with pricing probability* 0.586 *whose revenue is at least* 0.9111 *times the maximum revenue.*

Proof. We consider an undirected social network $G(V, E, w)$, start from the revenue-maximizing pricing probability vector $\boldsymbol{p}$, and obtain an IE strategy $\mathrm{IE}(A, \hat{p})$ by applying randomized rounding to $\boldsymbol{p}$. We show that for $\hat{p} = 0.586$, the expected (wrt the randomized rounding choices) revenue of $\mathrm{IE}(A, \hat{p})$ is at least 0.9111 times the revenue extracted from G by the best ordering for $\boldsymbol{p}$.

By Lemma 1, the best ordering is to approach the buyers in non-increasing order of pricing probabilities. Hence, we let $p_1 \geq \cdots \geq p_n$, and let $\boldsymbol{\pi}$ be the identity permutation. Then,

$$R(\boldsymbol{\pi}, \boldsymbol{p}) = \sum_{i \in V} p_i(1-p_i)w_{ii} + \sum_{i<j} p_i p_j (1-p_j) w_{ij}$$

For the IE strategy, we assign each buyer i to the influence set A independently with probability $I(p_i) = \alpha(p_i)(p_i - 0.5)$, and to the exploit set with probability $E(p_i) = 1 - I(p_i)$, where $\alpha(x) : [0.5, 1] \mapsto [0, 2]$ is a piecewise linear function with breakpoints at $(0.5, 0.7, 0.8, 0.9, 1.0)$ and values $(0.0, 1.0, 1.33, 1.63, 2.0)$ at these points. By linearity of expectation, the expected revenue of $\mathrm{IE}(A, \hat{p})$ is:

$$R_{\mathrm{IE}}(A, \hat{p}) = \sum_{i \in V} \hat{p}(1-\hat{p})E(p_i)w_{ii} + \sum_{i<j} \hat{p}(1-\hat{p})\left(I(p_i)E(p_j) + E(p_i)I(p_j) + \hat{p}\, E(p_i)E(p_j)\right) w_{ij}$$

Specifically, $\mathrm{IE}(A, \hat{p})$ extracts a revenue of $\hat{p}(1-\hat{p})w_{ii}$ from each loop $\{i, i\}$, if i is included in the exploit set. Moreover, $\mathrm{IE}(A, \hat{p})$ extracts a revenue of $\hat{p}(1-\hat{p})w_{ij}$ from each edge $\{i, j\}$, $i < j$, if one of i, j is included in the influence set A and the other is not, and a revenue of $\hat{p}^2(1-\hat{p})w_{ij}$ if both i and j are included in the exploit set $V \setminus A$.

The approximation ratio of $\mathrm{IE}(A, \hat{p})$ to the maximum revenue of G under $\boldsymbol{p}$ is derived as the minimum ratio between any pair of terms in $R(\boldsymbol{\pi}, \boldsymbol{p})$ and $R_{\mathrm{IE}}(A, \hat{p})$ corresponding to the same loop $\{i, i\}$ or to the same edge $\{i, j\}$. Therefore, the approximation ratio of $\mathrm{IE}(A, \hat{p})$ is no less than the minimum of:

$$\min_{0.5 \leq x \leq 1} \frac{\hat{p}\,(1-\hat{p})\,E(x)}{x\,(1-x)} \quad \text{and} \quad \min_{0.5 \leq y \leq x \leq 1} \frac{\hat{p}\,(1-\hat{p})(I(x)\,E(y) + E(x)\,I(y) + \hat{p}\,E(x)\,E(y))}{x\,y\,(1-y)}$$

Using calculus, we can show that for $\hat{p} = 0.586$, these ratios are at least 0.9111. □

For directed networks, we use the same approach, and obtain the following theorem.

Theorem 2. *For any directed social network, there is an IE strategy with pricing probability* $2/3$ *whose expected revenue is at least* 0.55289 *times the maximum revenue.*

Proof sketch. Working as in the proof of Theorem 1, we show that the approximation ratio of the IE strategy obtained by applying randomized rounding to the revenue-maximizing pricing probability vector is at least:

$$\min_{0.5 \leq x, y \leq 1} \frac{\hat{p}\,(1-\hat{p})(I(x)\,E(y) + 0.5\,\hat{p}\,E(x)\,E(y))}{x\,y\,(1-y)}$$

For $\hat{p} = 2/3$ and $\alpha(x) = 1.0$, for all x, this is simplified to $\min_{y \in [0.5,1]} \frac{2(3-2y)}{27y(1-y)}$, which attains its minimum value of ≈ 0.55289 at $y = \frac{3-\sqrt{3}}{2}$. □

Similarly, we can show that there is an IE strategy that uses the myopic pricing probability of $1/2$ and extracts a revenue of at least 0.8857 (resp. 0.4594) times the maximum revenue for undirected (resp. directed) social networks.

On the Approximability of the Maximum Revenue for Directed Networks. The results of [14, Lemma 3.2] and [12] suggest that given a pricing probability vector $\boldsymbol{p}$, it is Unique-Games-hard to compute a vertex ordering $\boldsymbol{\pi}$ of a directed network G for which $R(\boldsymbol{\pi}, \boldsymbol{p})$ is at least 0.5 times the maximum revenue of G under $\boldsymbol{p}$. An interesting consequence of Theorem 2 is that the inapproximability bound of 0.5 does not apply to revenue maximization in the Uniform Additive Model. In particular, given the prices $\boldsymbol{p}$, Theorem 2 computes, in linear time, an IE strategy with an expected revenue of at least 0.55289 times the maximum revenue of G under $\boldsymbol{p}$. This does not contradict the results of [14,12], because the pricing probabilities of the IE strategy are different from $\boldsymbol{p}$.

In the Uniform Additive Model, different acyclic (sub)graphs (equivalently, different vertex orderings) allow for a different fraction of their edge weight to be translated into revenue, while in the reduction of [14, Lemma 3.2], the weight of each edge in an acyclic subgraph is equal to its revenue. Thus, although the IE strategy of Theorem 2, with pricing probability $2/3$, gives a 0.55289-approximation to the maximum revenue of G under $\boldsymbol{p}$, its vertex ordering combined with $\boldsymbol{p}$ may generate a revenue of less than 0.5 times the maximum revenue of G under $\boldsymbol{p}$. Next, we obtain a pair of inapproximabity results for revenue maximization in the Uniform Additive Model.

Lemma 4. *Assuming the Unique Games conjecture, it is* $\mathbf{NP}$*-hard to approximate within a factor greater than* $27/32$ *(resp. to compute an IE strategy with pricing probability* $2/3$ *that approximates within a factor greater than* $3/4$*) the maximum revenue of a directed social network in the Uniform Additive Model.*

4 Generalized Influence-and-Exploit

Building on the idea of generating revenue from large cuts between pricing classes, we obtain a class of generalized IE strategies, which employ a partition of buyers in more than two pricing classes. A generalized IE strategy consists of $K \geq 3$ classes. Each class k, $k = 1, \ldots, K$, is associated with a pricing probability of $p_k = 1 - \frac{k-1}{2(K-1)}$, and each buyer is assigned to the class k independently with probability q_k, where $\sum_{k=1}^{K} q_k = 1$, and is offered a pricing probability of p_k. The buyers are considered in non-increasing order of pricing probability, i.e., the buyers in class k are considered before the buyers in class $k+1$, and the buyers in the same class are considered in a random order.

Let $\mathrm{IE}(\boldsymbol{q}, \boldsymbol{p})$ be such a generalized IE strategy, where $\boldsymbol{q} = (q_1, \ldots, q_K)$ is the assignment probability vector and $\boldsymbol{p} = (p_1, \ldots, p_K)$ is the pricing probability vector. We can show that the approximation ratio of $\mathrm{IE}(\boldsymbol{q}, \boldsymbol{p})$ for undirected networks is at least:

$$\min\left\{ 4\sum_{k=1}^{K} q_k p_k (1-p_k),\ \ 4\sum_{k=1}^{K} q_k p_k (1-p_k) \left(q_k p_k + 2\sum_{\ell=1}^{k-1} q_\ell p_\ell \right) \right\}, \qquad (3)$$

while for directed social networks, the approximation ratio of $\mathrm{IE}(\boldsymbol{q}, \boldsymbol{p})$ is at least half of the quantity in (3). We can now select the assignment probability vector $\boldsymbol{q}$ so that (3) is maximized. With the pricing probability vector $\boldsymbol{p}$ fixed, this involves maximizing a quadratic function of $\boldsymbol{q}$ over linear constraints. Thus, we obtain the following:

Theorem 3. *For any undirected (directed) network G, the generalized IE strategy with $K = 6$ classes and assignment probabilities* $\boldsymbol{q} = (0.183, 0.075, 0.075, 0.175, 0.261, 0.231)$ *approximates the maximum revenue of G within a factor of* 0.7032 *(*0.3516*).*

5 Influence-and-Exploit via Semidefinite Programming

The main hurdle in obtaining better approximation guarantees for the maximum revenue is the loose upper bound of $(N + W)/4$. We do not know how to obtain a stronger upper bound on the maximum revenue. However, in this section, we obtain a Semidefinite Programming (SDP) relaxation for the problem of computing the best IE strategy with any given pricing probability $p \in [1/2, 1)$. Our approach exploits the resemblance between computing the best IE strategy and the problems of MAX-CUT (for undirected networks) and MAX-DICUT (for directed networks), and builds on the approach of [11,8]. Solving the SDP relaxation and using randomized rounding, we obtain, in polynomial time, a good approximation to the best influence set for the given p. Then, employing the bounds of Theorems 1 and 2, we obtain strong approximation guarantees for the maximum revenue in both directed and undirected networks.

Directed Social Networks. The case of a directed network $G(V, E, w)$ is a bit simpler, because we can ignore loops (i, i) without loss of generality. We observe that for any given pricing probability $p \in [1/2, 1)$, the problem of computing the best IE strategy IE(A, p) is equivalent to solving the following Quadratic Integer Program:

$$\max \tfrac{p(1-p)}{4} \sum_{(i,j)\in E} w_{ij} \left(1 + \tfrac{p}{2} + (1 - \tfrac{p}{2})y_0 y_i - (1 + \tfrac{p}{2})y_0 y_j - (1 - \tfrac{p}{2})y_i y_j\right) \quad \text{(Q1)}$$

$$\text{s.t.} \qquad y_i \in \{-1, 1\} \qquad \forall i \in V \cup \{0\}$$

In (Q1), there is a variable y_i for each buyer i and an additional variable y_0 denoting the influence set A. A buyer i is assigned to A, if $y_i = y_0$, and to the exploit set, otherwise. For each edge (i, j), $1 + y_0 y_i - y_0 y_j - y_i y_j$ is 4, if $y_i = y_0 = -y_j$ (i.e., if i is assigned to the influence set and j is assigned to the exploit set), and 0, otherwise. Moreover, $\frac{p}{2}(1 - y_0 y_i - y_0 y_j + y_i y_j)$ is $2p$, if $y_i = y_j = -y_0$ (i.e., if both i and j are assigned to the exploit set), and 0, otherwise. Therefore, the contribution of each edge (i, j) to the objective function of (Q1) is equal to the revenue extracted from (i, j) by IE(A, p).

Following the approach of [11,8], we relax (Q1) to the following Semidefinite Program, where $v_i \cdot v_j$ denotes the inner product of vectors v_i and v_j:

$$\max \tfrac{p(1-p)}{4} \sum_{(i,j)\in E} w_{ij} \left(1 + \tfrac{p}{2} + (1 - \tfrac{p}{2})\, v_0 \cdot v_i - (1 + \tfrac{p}{2})\, v_0 \cdot v_j - (1 - \tfrac{p}{2})\, v_i \cdot v_j\right)$$

$$\begin{aligned}
\text{s.t.} \qquad & v_i \cdot v_j + v_0 \cdot v_i + v_0 \cdot v_j \geq -1 && \text{(S1)}\\
& v_i \cdot v_j - v_0 \cdot v_i - v_0 \cdot v_j \geq -1 \\
& -v_i \cdot v_j - v_0 \cdot v_i + v_0 \cdot v_j \geq -1 \\
& -v_i \cdot v_j + v_0 \cdot v_i - v_0 \cdot v_j \geq -1 \\
& v_i \cdot v_i = 1, \quad v_i \in \mathbb{R}^{n+1} && \forall i \in V \cup \{0\}
\end{aligned}$$

Any feasible solution to (Q1) can be translated into a feasible solution to (S1) by setting $v_i = v_0$, if $y_i = y_0$, and $v_i = -v_0$, otherwise. An optimal solution to (S1) can be computed within any precision ε in time polynomial in n and in $\ln \frac{1}{\varepsilon}$ (see e.g. [2]).

Given a directed social network $G(V, E, w)$, a pricing probability p, and a parameter $\gamma \in [0, 1]$, the algorithm SDP-IE(p, γ) first computes an optimal solution $v_0, v_1, \ldots, v_n$ to (S1). Then, following [8], the algorithm maps each vector v_i to a rotated vector v'_i which is coplanar with v_0 and v_i, lies on the same side of v_0 as v_i, and forms an angle with v_0 equal to $f_\gamma(\theta_i) = (1-\gamma)\theta_i + \gamma\pi(1-\cos\theta_i)/2$, where $\pi = 3.14\ldots$ and $\theta_i = \arccos(v_0 \cdot v_i)$ is the angle of v_0 and v_i. Finally, the algorithm computes a random vector r uniformly distributed on the unit $(n+1)$-sphere, and assigns each buyer i to the influence set A, if $\mathrm{sgn}(v'_i \cdot r) = \mathrm{sgn}(v_0 \cdot r)$, and to the exploit set $V \setminus A$, otherwise where $\mathrm{sgn}(x) = 1$, if $x \geq 0$, and -1, otherwise. We next show that:

Theorem 4. *For any directed social network G, SDP-IE*$(2/3, 0.722)$ *approximates the maximum revenue extracted from G by the best IE strategy with pricing probability* $2/3$ *within a factor of* 0.9064.

Proof. We let $v_0, v_1, \ldots, v_n$ be an optimal solution to (S1), let $\theta_{ij} = \arccos(v_i \cdot v_j)$ be the angle of any two vectors v_i and v_j, and let $\theta_i = \arccos(v_0 \cdot v_i)$ be the angle of v_0 and any vector v_i. Similarly, we let $\theta'_{ij} = \arccos(v'_i \cdot v'_j)$ be the angle of any two rotated vectors v'_i and v'_j, and let $\theta'_i = \arccos(v_0 \cdot v'_i)$ be the angle of v_0 and any rotated vector v'_i. Building on the proof of [11, Lemma 7.3.2], we can show that:

Lemma 5. *The IE strategy of SDP-IE*(p, γ) *extracts from each edge* (i, j) *an expected revenue of:*

$$w_{ij}\, p(1-p)\, \frac{(1-\frac{p}{2})\,\theta'_{ij} - (1-\frac{p}{2})\,\theta'_i + (1+\frac{p}{2})\,\theta'_j}{2\pi} \quad (4)$$

Since (S1) is a relaxation of the problem of computing the best IE strategy with pricing probability p, the revenue of an optimal IE(A, p) strategy is at most:

$$\tfrac{p(1-p)}{4} \sum_{(i,j)\in E} w_{ij} \left(1 + \tfrac{p}{2} + (1-\tfrac{p}{2})\cos\theta_i - (1+\tfrac{p}{2})\cos\theta_j - (1-\tfrac{p}{2})\cos\theta_{ij}\right) \quad (5)$$

On the other hand, by Lemma 5 and linearity of expectation, the IE strategy computed by SDP-IE(p, γ) generates an expected revenue of:

$$\tfrac{p(1-p)}{2\pi} \sum_{(i,j)\in E} w_{ij} \left((1-\tfrac{p}{2})\,\theta'_{ij} - (1-\tfrac{p}{2})\,\theta'_i + (1+\tfrac{p}{2})\,\theta'_j\right) \quad (6)$$

We recall that for each i, $\theta'_i = f_\gamma(\theta_i)$. In [8, Section 4], it is shown that for each i, j,

$$\begin{aligned}\theta'_{ij} &= \arccos\Big(\cos f_\gamma(\theta_i)\, \cos f_\gamma(\theta_j) + \tfrac{\cos\theta_{ij} - \cos\theta_i\, \cos\theta_j}{\sin\theta_i\, \sin\theta_j}\, \sin f_\gamma(\theta_i)\, \sin f_\gamma(\theta_j)\Big)\\ &\equiv g_\gamma(\theta_{ij}, \theta_i, \theta_j)\end{aligned}$$

The approximation ratio of SDP-IE(p, γ) is derived as the minimum ratio of any pair of terms in (6) and (5) corresponding to the same edge (i, j). Thus, the approximation ratio of SDP-IE(p, γ) is at least:

$$\rho(p,\gamma) = \frac{2}{\pi} \min_{0\le\theta_{ij},\theta_i,\theta_j\le\pi} \frac{(1-\frac{p}{2})\, g_\gamma(\theta_{ij},\theta_i,\theta_j) - (1-\frac{p}{2}) f_\gamma(\theta_i) + (1+\frac{p}{2}) f_\gamma(\theta_j)}{1+\frac{p}{2} + (1-\frac{p}{2})\cos\theta_i - (1+\frac{p}{2})\cos\theta_j - (1-\frac{p}{2})\cos\theta_{ij}},$$

where $\cos\theta_{ij} = v_i \cdot v_j$, $\cos\theta_i = v_0 \cdot v_i$, and $\cos\theta_j = v_0 \cdot v_j$ must satisfy the inequality constraints of (S1). It can be shown numerically that $\rho(2/3, 0.722) \ge 0.9064$. □

Combining Theorem 4 and Theorem 2, we conclude that:

Theorem 5. *For any directed social network G, the IE strategy of SDP-IE*$(2/3, 0.722)$ *approximates the maximum revenue of G within a factor of* 0.5011.

Undirected Social Networks. We apply the same approach to an undirected network $G(V, E, w)$. The important difference is that the objective function of the SDP relaxation now is:

$$\max \; \tfrac{p(1-p)}{2} \sum_{i\in V} w_{ii}\,(1 - v_0 \cdot v_i) + \tfrac{p(1-p)}{4} \sum_{i<j} w_{ij}\,(2 + p - p\,v_0 \cdot v_i - p\,v_0 \cdot v_j - (2-p)\,v_i \cdot v_j)$$

Apart from the SDP relaxation, the algorithm is the same as that for directed networks. Working as in the proof of Theorem 4, we can prove that:

Theorem 6. *For any undirected social network G, SDP-IE*$(0.586, 0.209)$ *approximates the maximum revenue extracted from G by the best IE strategy with pricing probability* 0.586 *within a factor of* 0.9032.

Combining Theorem 6 and Theorem 1, we conclude that:

Theorem 7. *For any undirected network G, the IE strategy of SDP-IE*$(0.586, 0.209)$ *approximates the maximum revenue of G within a factor of* 0.8229.

Remark. By the same approach, we compute the approximation ratio of SDP-IE(p, γ) against the best IE strategy, for any pricing probability $p \in [1/2, 1)$. Viewed as a function of p, both the best value of γ and the approximation ratio of SDP-IE(p, γ) against the best IE strategy increase slowly with p. For example, for directed networks, the approximation ratio of SDP-IE$(0.5, 0.653)$ (resp. SDP-IE$(0.52, 0.685)$ and SDP-IE$(0.52, 0.704)$) is 0.8942 (resp. 0.8955 and 0.9005). For undirected social networks, the approximation ratio of SDP-IE$(0.5, 0.176)$ (resp. SDP-IE$(0.52, 0.183)$ and SDP-IE$(2/3, 0.425)$) is 0.899 (resp. 0.9005 and 0.907). Then, for any $p \in [1/2, 1)$, we can multiply the approximation ratio of SDP-IE(p, γ) and the bound obtained by the approach of Theorems 1 and 2 on the fraction of the maximum revenue extracted by the best IE strategy with pricing probability p, and obtain the approximation ratio of SDP-IE(p, γ) against the (unrestricted) optimal marketing strategy.

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On the Efficiency of the Simplest Pricing Mechanisms in Two-Sided Markets

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Abstract. We study the price of anarchy of a trading mechanism for divisible goods in markets containing both producers and consumers (i.e. in two-sided markets). Each producer is asked to submit a linear pricing function (or, equivalently, a linear supply function) that specifies a per-unit price $p(d)$ as a function of the demand d that they face. Consumers then buy their preferred resource amounts at these prices.

We prove that having three producers for every resource guarantees the price of anarchy is bounded. In general, the price of anarchy depends heavily on the level of horizontal and vertical competition in the market, on the producers' cost functions, and on the elasticity of consumer demand. We show how these characteristics affect economic efficiency and in particular, we find that the price of anarchy equals 2/3 in a perfectly competitive market, 3/4 in a monopsony, and $2\epsilon(2-\epsilon)/(4-\epsilon)$ in a monopoly where consumer valuations have a fixed elasticity of ϵ. These results hold in markets with multiple goods, particularly in bandwidth markets over arbitrary graphs.

Pricing mechanisms are used in several real-world applications; our results suggest how to add formal efficiency guarantees to these mechanisms. On the theory side, we show that near-optimal efficiency can be achieved within two-sided markets by simple mechanisms in the spirit of Bertrand and Cournot. This result extends to the two-sided setting the analyses for fixed-supply and fixed-demand markets of Johari and Tsitsiklis (2005), Acemoglu and Ozdaglar (2007), and Correa et al. (2010).

1 Introduction

We consider the problem of designing a mechanism that enables the trading of divisible goods between producers and consumers. A natural way of achieving this is to set prices on goods according to some rule, and then simply let the users trade. Such pricing mechanisms are appealing due to their simplicity, and are often found in real-world applications, such as power engineering or networking.

In this paper, we analyze the following simple pricing rule for a market in which operate a set of consumers Q and a set of producers R. In the simplest

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setting, when there is only one good in the market, each producer $r \in R$ is asked to provide a linear pricing function (or, equivalently, a linear supply function) $p_r(f_r) = \gamma_r f_r$ with slope $\gamma_r > 0$, which specifies the per-unit price producer r will charge if its total demand is f_r. In other words, if a consumer buys x units from r, they will pay r a sum of $p_r(f_r)x = (\gamma_r f_r)x$. Although γ_r specifies a pricing function, for brevity we will often refer to γ_r as a "price". After seeing the producers' prices, each consumer $q \in Q$ chooses a resource amount d_{qr} to buy from producer r and pays for it $p_r(f_r)d_{qr}$. Because producers input scalar pricing information and consumers input resource quantities, we call this mechanism *Bertrand-Cournot* (somewhat stretching the usual terminology). Although the p_r can be also viewed as supply functions, we will only refer to them as pricing functions from now on.

When multiple goods are traded, we identify the market with a multigraph $G = (V, E)$. Each consumer q owns a source-sink pair $(s_q, t_q) \in V$, and each producer r operates on an edge $e_r \in E$. As in the single-good mechanism, producer r inputs a linear pricing function $p_r(f_r)$. Consumers now buy edge capacities from producers and derive utility from the size of the maximum (s_q, t_q)-flow they can send in the resulting capacitated graph G. Thus, capacities can be associated with goods. Specifically, each consumer q directly submits for each (s_q, t_q)-path p the size d_{qp} of the flow it wishes to send over p, and pays $\sum_{p \in P} \sum_{e \in p} p_e(f_e) d_{qp}$ where f_e is the total demand faced by the producer at edge e.

We choose to study this model because we seek a mechanism that is both efficient and conceptually simple. This puts our work within the research agenda of understanding the tradeoffs between economic efficiency and mechanism complexity, set by Johari and Tsitsklis in [7]. Although the simplest mechanism would consist in asking producers for fixed, scalar prices and consumers for resource amounts, this type of market has been shown to be very inefficient [3]. The mechanism we consider is slightly more complex, but it is provably efficient. To the best of our knowledge, it is the first to combine good economic efficiency (i.e. price of anarchy close to one), high scalability (i.e. scalar strategy spaces), and conceptual simplicity of the strategy space within two-sided markets.

Representing the market as a graph makes our mechanism directly applicable to real-world markets for goods such as transportation, bandwidth, and electricity. Perhaps more interestingly, this graphical structure allows us to study the effects of horizontal and vertical competition between producers. In the former, producers sell substitute goods that are graphically represented by parallel edges. In the latter, producers' goods are complements that are represented by consecutive edges on a path: capacity on one edge can be used only if it is also bought on all the others in the path.

2 Results

Our main result is to show that the price of anarchy of our mechanism is bounded by a constant, as long as there are at least three producers for every good. The precise value the price of anarchy takes depends heavily on the level of horizontal and vertical competition among producers, on their cost functions, and on the elasticity of demand. Within series-parallel graphs, our techniques yield closed-form expressions for market efficiency as a function of these characteristics.

More generally, we make the following contributions in this paper.

- *We show that near-optimal efficiency can be achieved within two-sided markets by simple mechanisms in the spirit of Bertrand and Cournot.* Almost all results for such mechanisms hold only in models in which either supply or demand is fixed [1,8], and the only result for two-sided markets that we know is negative [3]. Our mechanism is the first to possess formal guarantees in two-sided markets with atomic players.
- *Our results suggest how to improve pricing mechanisms that are used in practice.* Most mechanisms intended to be used in practice [5,12] tend to be inefficient [3]. On the other hand, mechanisms that are efficient are often quite unintuitive. The mechanism we propose is both efficient and conceptually simple.
- *We examine how market structure affects economic efficiency.* To the best of our knowledge, this has never been thoroughly studied within two-sided markets with atomic players. In our paper, we derive closed-form expressions that describe the effects of market structure on efficiency.

3 Related Work

Pricing mechanisms have been extensively studied within electrical engineering and computer science as a way of allocating bandwidth between users on a network. A seminal result in this field is the proportional allocation mechanism (PAM), which distributes a fixed supply of a resource among consumers [9]. In [7], Johari and Tsitsiklis show that the PAM has a price of anarchy of 3/4; in [11], Kuleshov and Vetta extend this fixed-supply result to two-sided markets.

In both settings, the PAM admits the best price of anarchy guarantee within a large class of mechanisms. Nonetheless, more natural mechanisms — especially ones that are Cournot or Bertrand — have also received significant attention, as they are easier to use in practice. In the Cournot setting, the price of anarchy varies between 0 and 2/3 when supply is fixed, depending on how resources are priced [6,8]. In the Bertrand setting, when demand is fixed, Acemoglu and Ozdaglar showed that the price of anarchy equals 5/6 in single-resource markets and 0 in multi-resource markets [1]. In [4], Correa et al. propose an alternative pricing scheme for fixed-demand markets that accepts from producers linear pricing functions instead of scalar prices. They establish constant price of anarchy bounds in the multi-resource setting and also discuss how market structure affects efficiency.

Many of the above models appear to be also studied in the economics literature on supply function equilibria [10] and their applications to electricity markets. Interestingly, our work appears to be among the first price of anarchy analyses for these models. We refer the reader to [2] for a survey of the literature on linear supply function bidding.

Pricing mechanisms for markets containing *both* consumers and producers have been studied by Chawla and Roughgarden in [3]. Their system operates as the Cournot mechanism of Johari and Tsitsiklis [8] on the demand side and as the Bertrand mechanism of Acemoglu and Ozdaglar [1] on the supply side. Although that mechanism is extremely intuitive, its price of anarchy is zero in most settings.

Here, we present a mechanism that is both easy to use and efficient. It combines the demand side of Johari and Tsitsiklis [8] and the supply side of Correa et al. [4].

4 Definitions and Assumptions

We first refer the reader to the introduction for a high-level definition of the mechanism. As we mentioned, in its most general form, the mechanism is defined over a multigraph $G = (V, E)$. We use P to denote the set of paths in G. We call a set of parallel edges between two vertices a *link*; the set of all links is denoted by L. Essentially, links correspond to goods. We call a path in the induced graph (V, L) a *route*; the set of all routes is denoted by T. Two sets of users operate on the multigraph: consumers Q and producers R. Consumer $q \in Q$ owns a source and a sink $s_q, t_q \in V$; producer $r \in R$ operates on some edge $e \in E$. The strategy of consumer q is a positive vector $\boldsymbol{d}_q = (d_{qp})_{p \in P_q}$, specifying a flow on each (s_q, t_q)-path in G; the strategy of producer r is a scalar $\gamma_r > 0$, specifying a linear pricing function $p_r(f) = \gamma_r f$. Several equivalent strategy spaces can be defined for this mechanism, which we will discuss later in the paper; here, we use one that is standard in the literature [7]. We also assume there is a one-to-one relationship between edges and producers and throughout the paper we may use both r and e to index providers.

We make the following assumptions on the utilities of the agents:

Assumption 1. *The utility of consumer q for sending a flow of size d_q is $U_q(d_q) = V_q(d_q) - \sum_{p \in P_q} d_{qp} \sum_{e \in p} p_e(f_e)$, where $V_q(d_q)$ is q's* valuation function*. The valuation functions $V_q(d_q) : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, increasing, concave, and differentiable for all $q \in Q$.*

Assumption 2. *The utility of producer r for supplying f_r units of capacity on its edge is $U_r(f_r) = p_r(f_r)f_r - C_r(f_r)$, where $C_r(f_r)$ is r's* cost function*. For all $r \in R$, the cost function $C_r(f) : \mathbb{R}_+ \to \mathbb{R}_+$ is of the form $C_r(f) = \int_0^f c_r(x)dx$ where $c_r(x) : \mathbb{R}_+ \to \mathbb{R}_+$ is the* marginal cost *function. It is continuous, strictly increasing, convex, and $c_r(0) = 0$.*

Both assumptions are standard in the literature, see for example [4,6]. Requiring convex marginal costs in a relatively strong assumption; fortunately, it holds in several important areas of application, such as in electricity markets, where generators tend to use their cheapest capacity first. We refer the reader to [11] for techniques for showing that the price of anarchy smoothly degrades to zero as the marginal cost functions become more concave.

From the above utilities we obtain the social welfare within the market:

Definition 1. *The* social welfare *within the mechanism equals* $\sum_{q \in Q} V_q(d_q) - \sum_{r \in R} C_r(f_r)$.

An important observation is that the demand f_r faced by producer r is a function of all the edge prices in the mechanism (which we denote by the vector $\boldsymbol{\gamma} = (\gamma_e)_{e \in E}$) when consumers are at a Nash equilibrium in the Cournot game with edge prices $\boldsymbol{\gamma}$. This function is well-defined because Nash equilibria in Cournot mechanisms always exist and are unique [8].

Definition 2. *A* Nash equilibrium *of the Bertrand-Cournot mechanism is a set of strategies* $\{\boldsymbol{d}_q, \gamma_r \mid q \in Q, r \in R\}$ *such that*

1. *The* $\boldsymbol{d}_q$ *form a* demand-side equilibrium *given prices* $\boldsymbol{\gamma}$ *and utilities* U_q. *That is, for all* $q \in Q$, $\boldsymbol{d}_q = \arg\max_{\boldsymbol{d}} U_q(\boldsymbol{d}, \boldsymbol{d}_{-q}, \boldsymbol{\gamma})$, *where* $\boldsymbol{d}_{-q}$ *are the strategies of all consumers except* q.
2. *The prices* $\boldsymbol{\gamma}$ *form a* supply-side equilibrium *given demand functions* f_r *and utilities* U_r. *That is, for all* $r \in R$, $\gamma_r = \arg\max_\gamma \left(\gamma f_r(\gamma, \boldsymbol{\gamma}_{-r}) - C_r(f_r(\gamma, \boldsymbol{\gamma}_{-r}))\right)$, *where* $\boldsymbol{\gamma}_{-r}$ *are the strategies of all providers except* r. *Thus* γ_r *is the best response to the other prices when* r *anticipates consumers' equilibrium demand.*

We measure economic efficiency using the concept of price of anarchy.

Definition 3. *The* price of anarchy *is defined as the smallest* welfare ratio

$$\left(\sum_{q \in Q} V_q(\boldsymbol{d}_q^{NE}) - \sum_{r \in R} C_r(f_r(\gamma^{NE}))\right) \Big/ \left(\sup_{\boldsymbol{d}_q, f_r} \sum_{q \in Q} V_q(\boldsymbol{d}_q) - \sum_{r \in R} C_r(f_r)\right),$$

where the $\boldsymbol{d}_q^{NE}, \gamma^{NE}$ *form Nash equilibrium.*

Our goal is to lower-bound the price of anarchy across all instances of the mechanism.

The producers' strategies in the mechanism are primarily determined by consumers' responses to price. In economics, the *elasticity* of demand with respect to price is the standard way of measuring these responses.

Definition 4. *Let* $f(x) : \mathbb{R} \to \mathbb{R}$ *be a differentiable function. The elasticity of* f *with respect to* x *is a function* $\epsilon_x f(y) : \mathbb{R} \to \mathbb{R}$ *defined as* $\epsilon_x f(y) = \frac{df}{dx}(y) \frac{y}{f(y)}$.

We mainly work with $\epsilon_{\gamma_r} f_r$, the elasticity of demand to producer r. When referring to $\epsilon_{\gamma_e} f(\gamma_e)$, we often drop the γ_e subscript and simply write $\epsilon_e f_e$ or ϵf_e. Throughout this paper, we extensively use some standrd properties of elasticity which are derived in the full version of the paper.

5 Markets with a Single Good

We begin our analysis with a market in which there is only a single resource. In this setting, the multigraph G consists of two nodes, s and t, and one link; every producer r offers to carry flow from s to t over an edge in the link.

We establish bounds on the price of anarchy within this market by analyzing the demand and supply sides of the market separately. On the demand side, welfare is usually lost because consumers that value the resource less end up receiving goods that should go to the consumers that value the resource the most. We call that *demand-side inefficiency.* On the supply side, welfare is lost because producers charge consumers at rates higher than at their marginal costs (marginal cost pricing can be shown to be optimal in terms of social welfare). We call that *supply-side inefficiency.* We adopt the following three-step procedure to measure these two inefficiencies:

1. **Defining a simplified version of the mechanism.** We define an equivalent mechanism in which we ask consumers for the size of the flow they want to send across the link, and have the mechanism split it across providers automatically. The per-unit flow price is set using a single linear pricing function $P(f) = \Gamma f$, whose slope Γ is defined as a function of the prices γ_e.
2. **Measuring inefficiency on the demand side.**
   (a) First, we show that the worst price of anarchy occurs in a game where valuations are linear and costs are quadratic.
   (b) We then formulate the price of anarchy as the minimum of an optimization problem that minimizes the welfare ratio over all possible linear valuations and marginal costs and over all relevant strategy profiles.
   (c) We analytically solve this problem and find that the price of anarchy equals $2\rho(2-\rho)/(4-\rho)$, where $0 \leq \rho \leq 1$ is a parameter measuring supply-side inefficiency.
3. **Measuring inefficiency on the supply side.** We derive bounds on ρ when there are at least three producers in the market, and we show how it varies with the number of producers and with the elasticity of consumer demand.

Interestingly, in later sections, we will use essentially the same approach to analyze the price of anarchy in markets with more complex structure.

5.1 Defining a Simplified Version of the Mechanism

Observe that from a consumer's perspective, there is only a single resource in the market: (s,t)-flow. It's therefore quite natural that we can define a single price for that flow and automatically split the resulting demand across providers.

Definition 5. *In the* simplified *single-link Bertrand-Cournot mechanism,*

1. *Producers submit linear pricing functions as in the regular mechanism. The aggregate pricing function is set to* $P(f) = \Gamma f$*, where* $\Gamma = \frac{1}{\sum_{e \in E} 1/\gamma_e}$.
2. *Consumer* q *submits an* (s,t)*-flow* d_q *and pays for it* $\Gamma f d_q$*, where* $f = \sum_{q \in Q} d_q$.
3. *The mechanism sends* $f_e = \frac{1/\gamma_e}{\sum_{e' \in E} 1/\gamma_{e'}} f$ *over edge* e *and pays the producer* $\gamma_e f_e^2$.

This new mechanism is easier to analyze, easier to use, and from the point of view of a consumer, its communication complexity no longer depends on the number of producers. It is also equivalent to the original mechanism.

Theorem 1. *The Nash equilibria of the standard and simplified mechanisms are identical, and at equilibrium, the utilities of each player are the same.* □

Interestingly, when the producers' costs are quadratic, we can also aggregate their cost functions $\frac{\beta_r}{2} f_r^2$ into a single cost function $C(f) = \frac{B}{2} f^2$ that specifies the smallest cost for sending a total flow of f across all the edges. We can then use this function to compute the size of the socially optimal flow as if there was only a single provider in the market.

Definition 6. *When the marginal costs at the edges are of the form $c_e(f) = \beta_e f$, the slope of the* aggregate cost *function of the link is defined to be $B = \frac{1}{\sum_{e \in E} 1/\beta_e}$.*

Theorem 2. *When producers' marginal costs are linear, a cost-minimizing allocation f has a total cost of $\frac{B}{2} f^2$.* □

Proofs of these theorems can be found in the full version of the paper.

5.2 Measuring Inefficiency on the Demand Side

Next, observe that the price of anarchy can be written out as the solution to the following optimization problem, taken over all possible functions $(V_q)_{q \in Q}$, $(C_r)_{r \in R}$ and over scalars $d_q, d_q^*, \gamma_r, f_r^*$ (the $f_r(\boldsymbol{\gamma})$ are implicitly defined by the V_q):

$$\min \quad \left(\sum_{q \in Q} V_q(d_q) - \sum_{r \in R} C_r(f_r(\boldsymbol{\gamma})) \right) \Big/ \left(\sum_{q \in Q} V_q(d_q^*) - \sum_{r \in R} C_r(f_r^*) \right) \tag{1}$$

s.t. The V_q are valuations and the C_r are costs satisfying Assumptions 1 and 2.
The d_q and the γ_r are equilibrium strategies given $(V_q)_{q \in Q}$, $(C_r)_{r \in R}$
The d_q^* and the f_r^* are optimum allocations given $(V_q)_{q \in Q}$, $(C_r)_{r \in R}$.

As it stands, the above formulation is not very useful. However, by our next lemma, we can restrict our attention to settings where the valuations and marginal costs are all linear (therefore the costs themselves are quadratic), in which case the above problem becomes finite-dimensional, and therefore much simpler.

Lemma 1. *Given any game instance $\mathcal{G}$, one can construct a new game instance $\mathcal{G}'$ where:*

1. *Consumers have linear valuations and producers have quadratic costs.*
2. *Producers set prices as if the demand functions f_r they were facing were the ones in $\mathcal{G}$.*

The price of anarchy of $\mathcal{G}'$ is a lower bound on that of $\mathcal{G}$. □

A more formal version of this lemma and a proof can be found in the full version of the paper.

Using the above lemma, we can show that for a fixed set of demand functions, the optimization problem (1) reduces to the following problem in only $2(Q+R)$ scalar variables d_q, γ_r, α_q and β_r and 3 "helper" variables f, Γ, B.

Lemma 2. *The price of anarchy is lower-bounded by the solution to the following system.*

$$\min \quad \frac{\sum_{q=1}^{Q} \alpha_q d_q - \frac{B}{2} f^2}{\max_{\bar{f}}(\max_{q \in Q} \alpha_q \bar{f} - \frac{B}{2}\bar{f}^2)}$$

$$\text{s.t.} \quad \begin{array}{ll} \Gamma f + \Gamma d_q \quad (\textit{for all } q \in Q) & \beta_r = \gamma_r \left(2 - \frac{1}{|\epsilon_{\gamma_r} f_r(\boldsymbol{\gamma})|}\right) \quad (\textit{for all } r \in R) \\ \Gamma = \frac{1}{\sum_{e \in E} 1/\gamma_e} & B = \frac{1}{\sum_{e \in E} 1/\beta_e} \\ \sum_{q \in Q} d_q = f & 0 \le \alpha_q, d_q, \Gamma, B \end{array}$$

When valuation functions are linear, this bound is tight. □

The variables α_q correspond to the slopes of the consumers' linear valuation functions; the β_r correspond to marginal cost slopes. The first two constraints are the necessary and sufficient conditions for an allocation to be a Nash equilibrium. The third constraint ensures that supply equals demand. Variables Γ, B correspond to the the aggregate prices that we defined in Section 5.1, and $\epsilon_{\gamma_r} f_r(\boldsymbol{\gamma})$ is the elasticity of the demand function f_r faced by provider r. When seeking the solution of the program, we take $\epsilon_{\gamma_r} f_r(\boldsymbol{\gamma})$ to be fixed; we will minimize over $\epsilon_{\gamma_r} f_r(\boldsymbol{\gamma})$ in the next section.

In the full version of the paper, we analytically solve the above problem using techniques developed in [1,7]. As a result, we obtain the following lemma.

Lemma 3. *The welfare ratio in a single-good market is bounded by* $2\rho(2-\rho)/(4-\rho)$*, where* $0 \le \rho \le 1$ *is an overcharging parameter. It equals* B/Γ*, where* Γ *is the equilibrium aggregate price. Thus the price of anarchy equals the minimal value of* ρ *over all* β_e, γ_e *that satisfy the supply-side Nash equilibrium condition*

$$\beta_r = \gamma_r \left(2 - \frac{1}{|\epsilon_{\gamma_r} f_r(\boldsymbol{\gamma})|}\right) \textit{ for all } r. \tag{2}$$

When valuations are linear, this bound is tight. □

Lemma 3 suggests that the price of anarchy has two distinct components: one arising from demand-side inefficiency and another from supply-side inefficiency. Demand-side inefficiency has been accounted for by the minimization problem in Lemma 2. All that remains is to combine that analysis with a measure of supply-side inefficiency ρ. Note this parameter corresponds to the ratio of true producer costs over the prices that they charge the users, which was how we defined supply-side inefficiency.

5.3 Measuring Inefficiency on the Supply Side

As usual, we start by looking at the simplest setting, in which there is only one monopolist producer in the market, so that $\varGamma = \gamma_e$.

Theorem 3. *Suppose the market is a monopoly. Suppose users have monomial valuation functions* $V_q(d_q) = \alpha_q d_q^x$*, where* $0 < x \leq 1$ *and* $\alpha_q > 0$*. The price of anarchy is bounded by* $2x(2-x)/(4-x)$*. When valuations are linear, the bound equals* $2/3$ *and is tight.* □

Notice that when $x \to 0$, the elasticity of demand decreases and the bound tends to zero. It can be shown that as $x \to 0$, this is actually tight. This observation is hardly surprising: if consumer demand changes very little with price, there is nothing to stop the monopolist from substantially overcharging its customers.

It can be shown this kind of overcharging can happen even when there are two producers, but with *three* competitors in the market, the price of anarchy can be bounded by a constant. This is our main result for the single-resource case.

Theorem 4. *Suppose there are at least 3 producers in the market. Suppose there is a* $0 < \varDelta \leq 1$ *such that* $\min_e \beta_e / \max_e \beta_e \geq \varDelta$*. Then the price of anarchy is bounded by a constant for any type of consumer demand.*

Proof. In the simplified mechanism, the flow f_e over edge e equals $S_e f$, where f is the total flow across the link, and S_e denotes the fraction that is routed through edge e. Using properties of elasticity, we obtain

$$\begin{aligned} \epsilon_{\gamma_e} f_e &= \epsilon_{\gamma_e} S_e + \epsilon_{\gamma_e} f = \epsilon_{\gamma_e} S_e + \epsilon_{\varGamma} f \, \epsilon_{\gamma_e} \varGamma \\ &= \epsilon_{\gamma_e} \frac{1/\gamma_e}{\sum_{e' \in E} 1/\gamma_{e'}} + \epsilon_{\varGamma} f \, \epsilon_{\gamma_e} \frac{1}{\sum_{e'} 1/\gamma_{e'}} \\ &= -\frac{\sum_{e' \neq e} 1/\gamma_{e'}}{\sum_{e'} 1/\gamma_{e'}} + \epsilon_{\varGamma} f \, \frac{1/\gamma_e}{\sum_{e'} 1/\gamma_{e'}} \end{aligned} \tag{3}$$

Now suppose for a contradiction that $\rho(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) \to 0$ for some sequence $(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n)_{n=1}^{\infty}$ (where $\boldsymbol{\beta}_n$ and $\boldsymbol{\gamma}_n$ are vectors of costs and equilibrium prices, indexed by edges). We claim that this implies $\beta_{en}/\gamma_{en} \to 0$ as $n \to \infty$ for all $e \in E$. If $\beta_{en}/\gamma_{en} \not\to 0$ for some e, then $\rho(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) = \frac{\sum_{e'} 1/\gamma_{e'n}}{\sum_{e'} 1/\beta_{e'n}} \geq (\sum_{e'} 1/\gamma_{e'n}) \varDelta \beta_{en} \geq \varDelta \frac{\beta_{en}}{\gamma_{en}} \not\to 0$ where n is arbitrary. We see that in this case ρ cannot go to zero.

Since $\beta_{en}/\gamma_{en} \to 0$ for all e, by equation (2) we have $\epsilon_{\gamma_e} f_e(\boldsymbol{\beta}_n, \boldsymbol{\gamma}_n) \to -1/2$ for all e as $n \to \infty$. In particular, for some $N \geq 0$ sufficiently high we must have $\epsilon_{\gamma_e} f_e(\boldsymbol{\beta}_N, \boldsymbol{\gamma}_N) \geq -1/2 - \epsilon/|E|$ for $\epsilon > 0$ and for all $e \in E$.

Inserting expression (3) into $\epsilon f_e \geq -1/2 - \epsilon/|E|$ and summing the result over all e, we obtain $\epsilon_{\varGamma} f - (|E| - 1) \geq -\frac{|E|}{2} - \epsilon$, which cannot be achieved for small values of ϵ when $|E| \geq 3$, because $\epsilon_{\varGamma} f$ cannot be positive. Thus we arrive at a contradiction. □

The requirement that $\min_e \beta_e / \max_e \beta_e \geq \Delta$ for some $0 < \Delta \leq 1$ ensures that producers are able to compete with each other. If one producer had significantly higher costs than the others, they could not never undercut their competitors.

The precise constant that bounds the price of anarchy depends on both Δ and the number of producers. For fixed values of these parameters, it can be computed numerically by formulating ρ as the minimum of an optimization problem. Interestingly, it tends to $2/3$ as $|R| \to \infty$, which is the same value it achieves when demand has an elasticity of 1. Thus market competition may entirely offset the effects of inelastic demand.

Theorem 5. *Consider a single-resource market over a link graph $G = ((s,t), E)$ Suppose there is a $0 < \Delta \leq 1$ such that $\min_{e\in E} \beta_e / \max_{e\in E} \beta_e \geq \Delta$ for any set of edges E. Then as $|E| \to \infty$, the price of anarchy goes to $2/3$.*

Proof. Consider a countably infinite set of producers R with quadratic cost functions $\{\frac{\beta_r}{2} f_r^2 \mid r \in R\}$, and let $R_m = \{r_1, ..., r_m\}$ denote the set of the first m producers. Let $\boldsymbol{\gamma}_m$ denote a vector of equilibrium prices in the game where the set of providers is R_m, and define $\rho_m = \frac{B_m}{\Gamma_m} = \frac{\sum_{r\in R_m} 1/\gamma_{mr}}{\sum_{r\in R_m} 1/\beta_r}$ to be the corresponding overcharging factor.

We have to show that $\rho_m \to 1$ as $m \to \infty$. For simplicity, assume that $\epsilon_\Gamma f = 0$; it is easy to show that more elastic demand functions always lead to less overcharging and a better supply-side efficiency measure ρ. Assuming $\epsilon_\Gamma f = 0$, one can easily derive from the equilibrium constraint (2) the identity

$$\beta_r = \gamma_{mr} \left(1 - \frac{1/\gamma_{mr}}{\sum_{r\in R_m} 1/\gamma_{mr}}\right), \tag{4}$$

which holds for all m, r.

First, we claim that for all $\epsilon > 0$, there is an N such that for all $m = 1, 2, ...,$ the number of players in R_m for which $\frac{1/\gamma_{mr}}{\sum_{r\in R_m} 1/\gamma_{mr}} > \epsilon$ is less than N. If not, then for some ϵ, we can find a set R_M such that the above holds for a set of at least $1/\epsilon$ players R'_M, and so we arrive at the following contradiction: $1 = \sum_{r\in R_M} \frac{1/\gamma_{Mr}}{\sum_{r\in R_M} 1/\gamma_{Mr}} \geq \sum_{r\in R'_M} \frac{1/\gamma_{Mr}}{\sum_{r\in R_M} 1/\gamma_{Mr}} > \frac{1}{\epsilon} \cdot \epsilon = 1$.

So fix an $\epsilon > 0$ and an m and let R_m^ϵ denote the set of producers in R_m for which $\frac{1/\gamma_{mr}}{\sum_{r\in R_m} 1/\gamma_{mr}} < \epsilon$. Note that by equation (4), we have for all these producers that $\beta_r > \gamma_{mr}(1-\epsilon)$.

We can now express the ratio ρ_m as $\rho_m = \frac{\sum_{r\in R_m} 1/\gamma_{mr}}{\sum_{r\in R_m} 1/\beta_r} \geq \frac{\sum_{r\in R_m^\epsilon} 1/\gamma_{mr}}{\sum_{r\in R_m} 1/\beta_r} > \frac{(1-\epsilon)\sum_{r\in R_m^\epsilon} 1/\beta_r}{\sum_{r\in R_m^\epsilon} 1/\beta_r + \sum_{r\in R_m \setminus R_m^e} 1/\beta_r}$. Since this holds for all m, since the set $R_m \setminus R_m^e$ is finite, and since there is a $0 < \Delta \leq 1$ such that $\min_{r\in R_m} \beta_r / \max_{r\in R_m} \beta_r \geq \Delta$ for any R_m, we can easily establish by picking m large enough that $\liminf_{m\to\infty} \rho_m \geq 1 - \epsilon$. But since ϵ was arbitrary, it must follow that $\liminf_{m\to\infty} \rho_m \geq 1$. This is precisely what we wanted to prove. □

Thus a perfectly competitive market has a price of anarchy of 2/3. The best possible price of anarchy guarantee, on the other hand, is achieved when there is no competition among consumers.

Corollary 6. *When there is only one user and an infinite number of producers, the price of anarchy equals 3/4.* □

The proof of this theorem can be found in the full version of our paper.

6 Multi-resource Markets over Series-Parallel Graphs

We now turn our attention to the more interesting setting where the market contains multiple resources. In this setting, the efficiency is highly dependent on whether producers compete horizontally or vertically with each other. Recall that in the former case, producers sell substitute goods that are graphically represented by parallel edges; in the latter, producers' goods are complements that are represented by consecutive edges on a path.

The effects of horizontal and vertical competition are most easily understood by looking at *series-parallel* graphs. Informally, a series-parallel graph is built recursively by connecting smaller series-parallel graphs in parallel or in series, starting from edges. See [4] for a full definition. For our purposes, it will be enough to look only at *two-level* series-parallel graphs, although our results also carry over to arbitrary series-parallel graphs (usually by an induction argument).

Definition 7. *A* two-level series-parallel graph G *consists of a set of* T *disjoint parallel routes that connect two special nodes: a source* s *and a target* t*.*

We also assume in this section that consumers have linear valuations $\{\alpha_q d_q \mid q \in Q\}$ and that providers have quadratic costs $\{\frac{\beta_r}{2} f_r^2 \mid r \in R\}$; in the next Section 7, we formally establish that this is the worst-case setting.

Our analysis follows the same plan as in the single-resource case. First, we show that the mechanism can be simplified on the consumer side like in Section 5.1. Then, using the same argument as in the single-resource setting, we derive a price of anarchy bound that is a function of supply-side inefficiency (an analogue of Lemma 3). Finally, we bound the supply-side inefficiency and obtain the full price of anarchy. See Section 5 for more details.

6.1 Defining a Simplified Version of the Mechanism

Let G be a two-level series-parallel graph with a a source and a target node shared by all consumers. We can define like in Section 5.1 a pricing function $P(f) = \Gamma f$ for (s,t)-flow f and show that charging consumers for their total flow using P results in a game that is equivalent to the original. More formally, given the graph G, we define Γ to be $\Gamma = \frac{1}{\sum_{t \in T} 1/\Gamma_t}$, where $\Gamma_t = \sum_{l \in t} \Gamma_l$ and $\Gamma_l = 1/\sum_{e \in l} 1/\gamma_e$. The intuition here is that the price of a route is a sum of the prices of its links, and parallel routes with prices Γ_t can be aggregated like edges in a link.

Definition 8. *In the* simplified *Bertrand-Cournot mechanism for a two-level series-parallel graph G,*

1. *Each producer r submits a linear pricing function with slope γ_r like in the regular mechanism, and the aggregate pricing function is set to $P(f) = \Gamma f$.*
2. *Each consumer q chooses to send d_q units of (s,t)-flow and pays $\Gamma f d_q$, where $f = \sum_{q \in Q} d_q$.*
3. *The mechanism divides payments and flow proportionally to the producers' contribution to Γ. The producer on edge e on link l on route t receives the following fraction of the flow and payments: $\frac{1/\Gamma_t}{\sum_{t' \in T} 1/\Gamma_{t'}} \frac{\Gamma_l}{\sum_{l' \in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}}$.*

Theorem 7. *The Nash equilibria of the standard and simplified mechanisms are identical, and at equilibrium, the utilities of each player are the same.* □

See the full paper for a proof In the same way as we did for Γ, we can also define a true cost $B = \frac{1}{\sum_{t \in T} 1/B_t}$, where $B_t = \sum_{l \in t} B_l$ and $B_l = 1/\sum_{e \in l} 1/\beta_e$. The lowest true cost of sending a flow of f can again be computed using this function, as if there was only a single producer in the market.

6.2 Measuring Inefficiency on the Demand Side

Since from a consumer's perspective, there is again only a single resource in the simplified mechanism, (s,t)-flow, the same argument as in the single-resource setting establishes the following bound on the price of anarchy (an analogue of Lemma 3), which is independent of graph structure.

Lemma 4. *The welfare ratio in a market over a two-level series-parallel graph is bounded by $2\rho(2-\rho)/(4-\rho)$, where $0 \le \rho \le 1$ is an overcharging parameter. It equals B/Γ, where Γ is the equilibrium aggregate price. Thus the price of anarchy is the minimum ρ over all β_e, γ_e that satisfy the supply-side Nash equilibrium condition*

$$\beta_r = \gamma_r \left(2 - \frac{1}{|\epsilon_{\gamma_r} f_r(\boldsymbol{\gamma})|}\right) \text{ for all } r. \tag{5}$$

When valuations are linear, this bound is tight. □

The parameter ρ can be seen as the ratio of the true cost of (s,t)-flow over the price that the users are charged. See the full paper for a proof.

6.3 Measuring Inefficiency on the Supply Side

Unlike consumer behavior, the behavior of producers depends heavily on the topology of the graph G. In particular, when producers are located on parallel edges, competition tends to drive down the price, whereas when producers are on edges connected in series, the opposite happens.

Although we don't have a closed-form expression for the price of anarchy as a function of graph structure, the following formula shows how horizontal and vertical competition affect the elasticity of the flow at an edge. The price of anarchy can then be obtained by plugging the expression for $\epsilon_{\gamma_e} f_e$ into ρ.

Theorem 8. *In a two-level series-parallel graph G, let e be an edge located on link l on route t. The elasticity of the (s,t)-flow f_e at e with respect to γ_e equals*

$$\epsilon_{\gamma_e} f_e = -\frac{\sum_{e'\in l; e'\neq e} 1/\gamma_{e'}}{\sum_{e'\in l} 1/\gamma_{e'}} - \frac{\sum_{t\in T; t'\neq t} 1/\Gamma_{t'}}{\sum_{t'\in T} 1/\Gamma_{t'}} \frac{\Gamma_l}{\sum_{l'\in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e'\in l} 1/\gamma_{e'}} + \frac{1/\Gamma_t}{\sum_{t'\in T} 1/\Gamma_{t'}} \frac{\Gamma_l}{\sum_{l'\in t} \Gamma_{l'}} \frac{1/\gamma_e}{\sum_{e'\in l} 1/\gamma_{e'}} \epsilon_\Gamma f. \qquad \square$$

This somewhat complicated-looking expression actually has three distinct terms. The first term approaches -1 (its best possible value) as horizontal competition at the link containing e increases. Similarly, the other two terms drive the elasticity up when the number of parallel routes increases. On the other hand, when the number of serial links increases, the last two terms tends to zero, and the elasticity worsens. Thus horizontal competition leads to higher efficiency, while vertical competition drives efficiency down.

The theorems below formalize this claim. Our first results pertain to *route graphs* — graphs containing exactly one route of L serial links — which turn out to admit the worst price of anarchy of all series-parallel graphs. We assume there are at least two edges in every link; otherwise there is no equilibrium in the market.

Theorem 9. *Let G be a route graph with m producers per link and suppose that there exists a $0 < \Delta \leq 1$ such that $\min_{e\in E} \beta_e / \max_{e\in E} \beta_e \geq \Delta$ for all m. Whenever $m \geq 3$, the price of anarchy is bounded by a constant. As m goes to infinity, ρ goes to one.* $\square$

In general series-parallel graphs, there is more competition among producers, since consumers are offered alternative routes. That turns out to improve the price of anarchy.

Theorem 10. *Let G be a two-level series-parallel graph with at least three providers on every link and suppose that and suppose that there exists a $0 < \Delta \leq 1$ such that $\min_{e\in E} \beta_e / \max_{e\in E} \beta_e \geq \Delta$. When the number of parallel routes of G goes to infinity, the elasticity of total demand in the graph $\epsilon_\Gamma f$ tends to one.* $\square$

Theorem 11. *Let G be a two-level series-parallel graph. The price of anarchy of G is lower-bounded by that of a route series-parallel graph.* $\square$

7 Multi-resource Markets over Arbitrary Graphs

Finally, we return to the general setting we described at the beginning of the paper, under Assumptions 1 and 2 and an arbitrary graph G. Although we can no longer describe how graph structure affects efficiency, our two most important results carry over to this general setting.

Theorem 12. *Let G be an arbitrary graph with m producers per link and suppose that $\min_{e\in E} \beta_e / \max_{e\in E} \beta_e \geq \Delta$ for some $0 < \Delta \leq 1$. Whenever $m \geq 3$, the price of anarchy is bounded. As $m \to \infty$, $\rho \to 1$ and the POA $\to 2/3$.*

These results are established using the same three-step process that was used in previous sections.

8 Existence of Nash Equilibria

We can also establish the following extension of the equilibrium result for fixed-demand mechanisms [4].

Theorem 13. *Let G be a series-parallel graph with at least two producers per link. When producers' costs are quadratic and that consumers' valuations are linear a Nash equilibrium exits and best-responses converge.* □

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Optimal Pricing Is Hard

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Abstract. We show that computing the revenue-optimal deterministic auction in unit-demand single-buyer Bayesian settings, i.e. the optimal item-pricing, is computationally hard even in single-item settings where the buyer's value distribution is a sum of independently distributed attributes, or multi-item settings where the buyer's values for the items are independent. We also show that it is intractable to optimally price the grand bundle of multiple items for an additive bidder whose values for the items are independent. These difficulties stem from implicit definitions of a value distribution. We provide three instances of how different properties of implicit distributions can lead to intractability: the first is a $\#P$-hardness proof, while the remaining two are reductions from the SQRT-SUM problem of Garey, Graham, and Johnson [14]. While simple pricing schemes can oftentimes approximate the best scheme in revenue, they can have drastically different underlying structure. We argue therefore that either the specification of the input distribution must be highly restricted in format, or it is necessary for the goal to be mere approximation to the optimal scheme's revenue instead of computing properties of the scheme itself.

1 Introduction

Designing auctions to maximize revenue in a Bayesian setting is a problem of high importance in both theoretical and applied economics [19–21]. While substantial progress has been made on designing mechanisms with revenue guarantees that are approximately optimal [4, 6, 9, 10], the question of determining the optimal mechanism exactly has been much more intricate [1, 2, 7, 8, 11, 16, 17, 22].

In this paper, we study the complexity of designing optimal deterministic auctions for single-bidder problems, i.e. optimal pricing mechanisms. Prior to our work, Briest showed that finding the optimal pricing mechanism for a unit-demand bidder is highly inapproximable when the bidder's values for different

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items are correlated [5]. Our work complements his by either considering single-item settings, or multi-item settings with product value distributions. We also investigate the complexity of optimally pricing the grand bundle of multiple items for an additive buyer whose values for the items are independent. For these problems we demonstrate that even when the optimal mechanism can only be one of two possibilities, it can be computationally difficult to determine which one achieves the highest expected revenue.

We note that all hard instances presented in this paper have fully polynomial-time approximation schemes, and thus our results preclude exact algorithms but not computationally efficient approximation schemes. From a practical perspective, a nearly optimal mechanism may be almost as desirable as an exact one. From a theoretical perspective, however, it is important to understand the structure of the exactly optimal mechanism [20], which may be drastically different than that of approximate ones. Computational barriers to determining the best mechanism, such as the ones presented here, reflect barriers to understanding its structure.

Our results suggest in particular that great care must be taken in how a bidder's value distributions are specified. Intricate distributions can be described succinctly, providing a simple outlet to encode computationally hard problems. We present three concrete scenarios where succinctly-represented distributions lead to computational hardness: Easy-to-describe discrete distributions may have exponential size support, may have mild irrationality in their support, or have mild irrationality in the probabilities they assign. Indeed, many (or all) of these features of discrete distributions can be present in simple continuous distributions. Thus, to obtain a robust theory of optimal Bayesian mechanism design, we must either aim for only approximate revenue guarantees or severely limit the types and specification format of allowable value distributions.

2 Preliminaries

In our model, there is a seller with n items and a buyer whose values for the items $v_1, ..., v_n$ are random variables drawn from known distributions $F_1, ..., F_n$. We will consider both *unit-demand* and *additive* buyer types:

- A (quasi-linear) unit-demand buyer is interested in buying at most one item; if the item prices are $p_1, ..., p_n$, the buyer buys the item maximizing his *utility*, $v_i - p_i$, as long as it is positive, breaking ties among the maximizers in some pre-determined way, e.g. lexicographic or in favor of the cheapest/most expensive item.
- A (quasi-linear) additive buyer values a subset S of items $\sum_{i \in S} v_i$. If subset S is priced P_S, his utility for buying that subset is $\sum_{i \in S} v_i - P_S$. The buyer buys the subset of items that maximizes his utility, as long as it is positive, breaking ties among subsets in some pre-determined way.

In the case of a unit-demand bidder, the seller's goal is to price the items to optimize the expected price paid by the buyer. Finding the optimal such prices is called the *unit-demand pricing problem*. In the case of an additive bidder, the seller's goal is to price all subsets of items to optimize the expected price paid by the buyer. Of course, the seller may not want to explicitly list the price of every subset but describe their prices in some succinct manner, or may want to offer only some subsets at a finite price. We are particularly interested in the *grand bundle pricing problem* where the seller wants to optimally price the set of all items (the grand bundle) and the buyer must take all items or nothing. As shown in [18], pricing just the grand bundle is optimal in several natural settings. Furthermore, it oftentimes achieves revenue close to the optimal mechanism [3, 15]. Optimally pricing the grand bundle is furthermore interesting in its own right [13].

Finally, the distributions $F_1, ..., F_n$ may be provided explicitly, by listing their support and the probabilities placed on each point in the support, or implicitly giving a closed-form formula for them. In this paper, we study how various ways to describe the distributions affect the complexity of the pricing problem.

3 Complexity of Sum-of-Attributes Distributions

We first consider the problem of optimally pricing a single item for a single buyer whose value for the item is a sum of independent random variables. The probability distribution of the item's value has an exponentially sized support, but has a succinct description in terms of each component variable's distribution. The seller must choose a price P for the item. The buyer will accept the offer (and pay P) if his value for it is at least P, and will reject the offer (giving the seller zero revenue) if his value is strictly less than P. The seller's goal is to choose P to maximize his expected revenue. In fact it follows from Myerson [20] that pricing the item at the optimal price is the optimal mechanism in this setting, even among randomized mechanisms.

This problem occurs fairly naturally. When selling a complex product (for example, a car), there are a number of attributes (color, size, etc) that a buyer may or may not value highly, and his value for the product may be the sum of his values for the individual attributes. If his values for the attributes are independent, the buyer's value for the product can be modeled as a sum of independent random variables.

Formally, the problem we study in this section is the following.

Definition 1 (The Sum-of-Attributes Pricing (SoAP) Problem). *Given n pairs of nonnegative integers $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)$ and rational probabilities $p_1, p_2, \ldots, p_n$, determine the price P^* which maximizes $P^* \cdot Pr[\sum_{i=1}^{n} X_i \geq P^*]$, where the X_i are independent random variables taking value u_i with probability p_i and v_i with probability $1 - p_i$.*

Notice that we can always view an instance of the sum-of-attributes pricing problem as an instance of the grand bundle pricing problem where we seek the optimal price to sell the "grand bundle" of a collection of n items that are independently distributed.

Theorem 1. *The Sum-of-Attributes Pricing problem and the Grand Bundle Pricing problem are #P-hard.*

Proof. We show how to use oracle access to the SoAP problem to solve the counting analog of the SUBSET-SUM problem, defined next, which is $\#P$-complete.[1]

#-SUBSET-SUM: Given as input a set of positive integers $\{a_1, a_2, \ldots, a_n\}$ and a positive integer $T \leq \sum_i a_i$, the goal is to determine the number of subsets of the a_i's which sum to at least T.

The idea of our reduction is to design an instance of the SoAP problem with n+1 attributes for which the optimal price is one of two possible prices. A single parameter (in particular, the probability p_{n+1} of the last attribute) determines which of these two prices is optimal. By repeatedly querying a SoAP oracle with varying values of p_{n+1}, we can determine the exact threshold value of p_{n+1}, which provides sufficient information to deduce the answer to the #-subset sum instance.

We proceed to provide the details of our reduction. Given an instance of the #-subset sum problem, we create an instance of SoAP with $n+1$ attributes, where for all $i \in \{1, \ldots, n\}$ we take $u_i = a_i$ and $v_i = 0$, while for the last attribute we take $u_{n+1} = T + 1$ and $v_{n+1} = 1$. Moreover, for all $i \in \{1, \ldots, n\}$, we set

$$p_i \triangleq \frac{1}{2^n n(n+1+\sum_{j=1}^n a_j)^2}.$$

Notice in particular that the first n attributes have the same probability of taking their highest value. Moreover, the probability that all the first n attributes have value 0 is:

$$\left(1 - \frac{1}{2^n n(n+1+\sum_{j=1}^n a_j)^2}\right)^n > 1 - \frac{1}{2^n (n+1+\sum_{j=1}^n a_j)^2}$$

i.e. very close to 1. We leave the probability p_{n+1} that the last attribute takes its highest value a free parameter, which we denote by p for convenience.

Now, suppose that we use price B for the SoAP instance. We claim the following:

1. If $B = 1$, the expected revenue is 1.
2. If $1 < B < T + 1$, then the expected revenue is at most
$$B\left(p + \frac{1-p}{2^n(n+1+\sum_{j=1}^n a_j)^2}\right).$$
3. If $B = T + 1$, then the expected revenue is at least $p(T + 1)$.
4. If $T + 1 < B \leq T + 1 + \sum_{j=1}^n a_j$, then the expected revenue is at most
$$\left(T + 1 + \sum_{i=1}^n a_i\right)\left(\frac{1}{2^n(n+1+\sum_{j=1}^n a_j)^2}\right) \leq \frac{1 + \sum_{j=1}^n a_i}{2^{n-1}(n+1+\sum_{j=1}^n a_j)^2} < 1.$$
5. If $B > T + 1 + \sum_{j=1}^n a_j$, then the expected revenue is 0.

[1] Indeed, the reduction from SAT to SUBSET-SUM as presented in [23] is parsimonious.

The fourth and fifth cases are never optimal, since they are both dominated by using $B = 1$. We claim that the second case is also never optimal. Suppose for the sake of contradiction that some integral price B strictly between 1 and $T+1$ were optimal. Then we would have the following two constraints:

- $B\left(p + \frac{1-p}{2^n(n+1+\sum_{j=1}^n a_j)^2}\right) \geq 1$
- $B\left(p + \frac{1-p}{2^n(n+1+\sum_{j=1}^n a_j)^2}\right) \geq (T+1)p.$

To show a contradiction, define for convenience

$$\epsilon \triangleq \frac{1}{2^n(n+1+\sum_{j=1}^n a_j)^2}.$$

We will show that no value of p exists for which both of the above constraints are simultaneously satisfied. From the first constraint, we deduce $p+\epsilon(1-p) \geq 1/B$ and thus

$$p \geq \frac{1/B - \epsilon}{1-\epsilon} \geq \frac{1/T - \epsilon}{1-\epsilon} > 1/T - \epsilon,$$

where for the last inequality we used that $T \leq \sum_{j=1}^n a_j$. Moreover,

$$1/T - \epsilon \geq \frac{1}{\sum_{j=1}^n a_i} - \frac{1}{2^n(n+1+\sum_{j=1}^n a_j)^2} \geq \frac{1}{\sum_{j=1}^n a_j} - \frac{1}{2^n \sum_{j=1}^n a_j} \geq \frac{1}{2\sum_{j=1}^n a_j}.$$

Therefore, the first constraint implies that $p > \frac{1}{2\sum a_j}$. From the second constraint, we deduce $B(p+\epsilon(1-p)) \geq (T+1)p$ and thus

$$p \leq \frac{B\epsilon}{T+1-B+B\epsilon},$$

where we used that $B \leq T$ so $T+1-B+B\epsilon > 1$. We further have

$$p < B\epsilon \leq T\epsilon \leq \sum_{j=1}^n a_j\epsilon = \frac{\sum_{j=1}^n a_j}{2^n(n+1+\sum_{j=1}^n a_j)^2} < \frac{1}{2\sum_{j=1}^n a_j}.$$

We get a contradiction as both constraints on p cannot be satisfied simultaneously. In summary, we have shown the following:

"For any p, the optimal price is either 1 or $T+1$."

We also note the following monotonicity property. If, for some p, the optimal price is $T+1$, then the optimal price is $T+1$ for any $p' > p$.[2] Therefore, there exists a unique p^* for which the expected revenue of selling at price $T+1$ is exactly the same as the expected revenue of selling at price 1.

[2] This follows from the fact that the expected revenue from selling at $T+1$ will only increase as p increases.

Suppose that we knew some p^* such that the expected revenue of selling at $T+1$ is exactly 1. Then, if we denote by V_n the total value of the first n attributes, p^* should satisfy:

$$1 = (T+1)\left(p^* + (1-p^*)P[V_n \geq T]\right);$$

so

$$P[V_n \geq T] = \frac{1/(T+1) - p^*}{1-p^*}.$$

Therefore, it is simple arithmetic to compute $P[V_n \geq T]$ from p^*. We also note that

$$P[V_n \geq T] = \sum_{k=0}^{n} p_1^k (1-p_1)^{n-k} \cdot S(k,T) = p_1^n \cdot \sum_{k=0}^{n} \left(\frac{1-p_1}{p_1}\right)^{n-k} \cdot S(k,t),$$

where $S(k,T)$ is the number of size k subsets of the a_i's which sum to at least T. By our choice of p_1 being sufficiently small, we know that $\frac{1-p_1}{p_1} = \frac{1}{p_1} - 1$ is an integer greater than 2^n. Therefore, the $S(k,t)$ are the unique integers in the base-$(\frac{1}{p_1} - 1)$ representation of $P[V_n \geq T]/p_1^n$, and can be found efficiently. So given p^* we can compute the total number of subsets of the a_i's that sum up to at least T, thereby solving the given instance of #-SUBSET SUM.

It remains to argue that we can compute p^* using oracle access to SoAP. We do binary search on p while maintaining all other parameters of the SoAP instance fixed, as described above. In every step of the binary search, we solve the corresponding SoAP instance, determining if the optimal price is 1 or $T+1$ and respectively increasing or decreasing the value of p for the next step, until we have pinned down p^* exactly. To argue that this takes polynomial time we notice that:

$$p^* = \frac{1/(T+1) - P[V_n \geq T]}{1 - P[V_n \geq T]}.$$

We also notice that $P[V_n \geq T]$ is a rational number that can be specified with a polynomial number of bits.[3] So p^* has polynomial accuracy and we need polynomially many calls to SoAP to determine it exactly. □

4 Complexity of Mildly Irrational Valuations

Issues of numerical precision may arise when analyzing value distributions which are implicitly described. Even very mild irrationality, such as the support of the distribution containing square roots of integers, can cause the resulting pricing problem to be computationally intricate. In particular, optimization may require deciding between two mechanisms whose expected utility differs only by an exponentially small amount. In this section, we present an example of how we can reduce a numerical problem whose status even in NP remains unknown to the pricing problem for a unit-demand buyer with mildly irrational valuations.

[3] In particular, each number of the form $p_1^i(1-p_1)^{n-i}$ has polynomial bit-length.

Definition 2 (The Square Root Sum Problem). *Given positive integers $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and K, the SQRT-SUM problem is to determine whether or not $\sum_{i=1}^{n} \sqrt{\alpha_i} > K$.*

While known to be in PSPACE, it remains an important open problem whether the square root sum problem is solvable in NP, let alone whether it is in P. [12, 14]

Remark 1. Checking whether $\sum_i \sqrt{a_i} = K$ for positive integers a_i, $i = 1, .., n$, and K can be done in polynomial time [14]. So the square root sum problem draws its computational difficulty from instances where equality between $\sum_i \sqrt{a_i}$ and K does not hold and we need to decide whether $\sum_i \sqrt{a_i}$ is $>$ or $<$ than K. In the hardness proofs of Theorems 2 and 3 we will implicitly assume that the given instance of the square root sum problem satisfies $\sum_i \sqrt{a_i} \neq K$. Given such instance we will construct an unit-demand pricing instance whose solution answers the question of whether $\sum_i \sqrt{a_i}$ is $>$ or $<$ than K.

Remark 2. The important computational difference between the square root of an integer and the sum of square roots of multiple integers is that the i-th bit of the former can be computed in time polynomial in i and the number's description complexity, while the same is not known to be true for the latter.

Theorem 2. *The unit-demand pricing problem is SQRT-SUM-hard when the item values are independent of support two with rational probabilities and each possible item value is the square root of an integer.*[4]

Proof. We will reduce SQRT-SUM to the pricing problem for a single unit-demand buyer whose values for the items are distributed independently, take one of two possible values with rational probabilities, and each of these possible values is the square root of an integer.

Given an input $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and K to the SQRT-SUM problem, we construct an input to the unit-demand pricing problem with $n + 1$ items. For $i = 1, \ldots, n$, item i has value $\sqrt{\alpha_i}$ with probability $1/i$, and value 0 with probability $1 - 1/i$. Finally, item $n + 1$ has value $T/2$ with probability $1/2 + \epsilon$ and value T with probability $1/2 - \epsilon$, where:

$$\epsilon \triangleq \frac{K}{4n \max(K, \alpha_n)} \leq \frac{1}{2}; \qquad T \triangleq \frac{(1/2 + \epsilon)K}{n\epsilon}.$$

Notice that $T/2 > \frac{K}{4n\epsilon} = \max(K, \alpha_n) \geq \alpha_n \geq \sqrt{\alpha_n}$.

We now claim that the optimal expected revenue for the unit-demand pricing instance we defined is the maximum of $T/2$ and

$$(1/2 - \epsilon)T + \frac{1/2 + \epsilon}{n} \left(\sqrt{\alpha_1} + \cdots + \sqrt{\alpha_n}\right).$$

[4] The item values are mildly irrational since the i-th bit of the square root of an integer can be computed exactly in time polynomial in i and the description complexity of the integer.

Indeed, it is clearly possible to achieve revenue $T/2$ by pricing item $n+1$ at $T/2$ and all other items at a price greater than $\sqrt{\alpha_n}$. Since $T/2 > \sqrt{\alpha_n}$, if item $n+1$ is priced less than or equal to $T/2$, the revenue cannot be higher than $T/2$.

Now what if item $n+1$ were priced at a price higher than $T/2$? Suppose, e.g., that we price item $n+1$ at T and all other items i at $\sqrt{\alpha_i}$. Then the expected revenue we would get is[5]

$$(1/2 - \epsilon)T + (1/2 + \epsilon)\left(\frac{1}{n}\sqrt{\alpha_n} + \frac{n-1}{n} \cdot \frac{1}{n-1}\sqrt{\alpha_{n-1}} + \cdots + \frac{1}{n}\sqrt{\alpha_1}\right) \quad (1)$$

We claim that this is the best revenue we could possibly achieve if item $n+1$ is priced at a price higher than $T/2$. Indeed, it is easy to see that the maximum of the values of items $1, \ldots, n$ is independent of the value of item $n+1$, it has expectation $\frac{1}{n}\sum_i \sqrt{\alpha_i}$ and, because $T/2 > \sqrt{\alpha_n}$, it is smaller than T with probability 1. So consider any pricing where the price of item $n+1$ is larger than $T/2$. In the event that the value of item $n+1$ is T (which happens with probability exactly $1/2-\epsilon$) the best revenue that the pricing could possibly get is at most T, while in the event that the value of item $n+1$ is $T/2$ (which happens with probability exactly $1/2+\epsilon$) the revenue cannot exceed the maximum of the values of items $1, \ldots, n$ which has expectation $\frac{1}{n}\sum_i \sqrt{\alpha_i}$ even after conditioning on the value of item $n+1$ as it is independent from the value of item $n+1$.

Observe that (1) is higher than $T/2$ if and only if

$$\epsilon T < \frac{(1/2+\epsilon)}{n}\left(\sqrt{\alpha_1} + \cdots + \sqrt{\alpha_n}\right),$$

which occurs precisely when $K < \sqrt{\alpha_1} + \cdots + \sqrt{\alpha_n}$. □

5 Complexity of Mildly Irrational Probabilities

The reduction of the previous section used distributions that were supported on irrational values. A possible critique of this in a discrete setting is that it may be unnatural for an individual to hold irrational values for an item. Contrastingly, it seems more natural to allow for a person's values to be rational but to depend on certain mildly irrational probabilities.

Perhaps the simplest form of an irrational probability is one for which we can efficiently compute arbitrary bits of its binary expansion correctly.[6] Notice that using a fair coin to sample exactly such probability, e.g. $\sqrt{1/3}$, is no more work than sampling exactly a rational probability, e.g. $1/3$: Imagine an infinite sequence of coin tosses. We reveal a prefix of that sequence until, viewed as a binary number, we can certify that the sequence lies above or below the target probability written in binary; if above, we output 1, otherwise we output 0.

[5] Suppose that ties are broken in favor of the most expensive item.

[6] This property is satisfied, for example, by a probability of the form $\sqrt{r}$, where r is a rational number; but, as remarked in section 4, it is unknown whether it is satisfied by a probability of the form $\sum_i \sqrt{r_i}$, for rational r_i's.

We now consider unit-demand pricing instances as in the previous section, except where the values are integral and the probabilities are irrational. As in the previous section, we will give a SQRT-SUM-hardness reduction.

Theorem 3. *The unit-demand pricing problem is SQRT-SUM-hard when the item values are independent of support two, have probabilities for which the i^{th} bit of their binary expansions can be computed in time polynomial in i, and each possible item value is integral.*

Proof. Let $a_1 \leq ... \leq a_n$ and K be an instance of the SQRT-SUM problem. Also let X be a large integer with $X > \max\{3K/n, a_n\}$. We define $a_{n+1} = X^2$ maintaining the monotonicity of the sequence a_i since $X > a_n$.

We reduce the given SQRT-SUM instance to an instance of the unit-demand pricing problem with $n+1$ items. For $i = 1, ..., n$, item i has value i with probability $p_i = 1 - \sqrt{a_i/a_{i+1}}$, and value 0 with probability $\sqrt{a_i/a_{i+1}}$. Finally, item $n+1$ has value $T/2$ with probability $3/4$ and value T with probability $1/4$, where:

$$T \triangleq 3\left(n - \frac{K}{X}\right).$$

Notice that by the choice of $X > 3K/n$ we have that $T/2 > n$, the highest possible value of any other item. Also, since the sequence of a_i's is non-decreasing, all probabilities p_i are well defined.

As in the proof of Theorem 2, we can argue that the optimal pricing either prices item $n+1$ at $T/2$ and the other items at infinity (call this "Scheme 1"), or prices all items at their high value (call this "Scheme 2"). In the former case the revenue is $T/2$. In the latter case the bidder will choose to buy the largest item he values high, i.e. will choose item $n+1$ if he values it high, otherwise item n if he values it high, and so on.[7] Therefore, Scheme 1 beats Scheme 2 if and only if:

$$\frac{T}{2} > \frac{T}{4} + \frac{3}{4}(p_n n + p_{n-1}(1-p_n)(n-1) + ... + p_1 \prod_{i=2}^{n}(1-p_i)),$$

which becomes, after substituting for the p_i's:

$$\frac{T}{2} > \frac{T}{4} + \frac{3}{4}\sum_{i=1}^{n}\left(i\left(\sqrt{\frac{a_{i+1}}{a_{n+1}}} - \sqrt{\frac{a_i}{a_{n+1}}}\right)\right).$$

Simplifying and using the fact that $\sqrt{a_{n+1}} = X$, our condition becomes

$$\frac{T}{2} > \frac{T}{4} + \frac{3}{4}\left(n - \frac{\sum_{i=1}^{n}\sqrt{a_i}}{X}\right).$$

[7] As in the proof of Theorem 2 we assume that ties are broken in favor of the most expensive item.

This occurs precisely when:

$$\sum_{i=1}^{n} \sqrt{a_i} > X(n - T/3) = K.$$

Therefore, Scheme 1 is strictly better than Scheme 2 precisely when $\sum_{i=1}^{n} \sqrt{\alpha_i} > K$, concluding our reduction from the SQRT-SUM problem. □

6 Future Work

Studying the complexity of optimal pricing in a Bayesian context is an important question, both theoretically and practically. However, to have a robust complexity model, great care must be taken in specifying the input distributions. Indeed, as shown in this paper, implicit distributions can easily embed hard problems into the distribution's parameters, and therefore any complexity theoretic model of pricing must take into account the complexity of the distributions themselves, and not just the length of a minimal specification.

A setting that avoids the computational barriers raised in this paper is that of several items, each distributed independently on some finite size support, with all values and probabilities rational and explicitly given. This problem is not yet resolved for either unit-demand or additive bidders. Moreover, while our paper has focused only on discrete distributions, issues of distributional specification are perhaps even more vital if one wishes to model the complexity of pricing with continuous distributions. It is of interest to propose a robust computational framework for studying the pricing problem with continuous distributions.

Finally, our results apply to computing the optimal deterministic mechanism, which in the case of a single buyer is tantamount to finding an optimal pricing scheme. It is an important open question to determine the complexity of the optimal mechanism design problem when randomized mechanisms are also allowed.

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Privacy Auctions for Recommender Systems

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Abstract. We study a market for private data in which a data analyst publicly releases a statistic over a database of private information. Individuals that own the data incur a cost for their loss of privacy proportional to the differential privacy guarantee given by the analyst at the time of the release. The analyst incentivizes individuals by compensating them, giving rise to a *privacy auction*. Motivated by recommender systems, the statistic we consider is a linear predictor function with publicly known weights. The statistic can be viewed as a prediction of the unknown data of a new individual, based on the data of individuals in the database. We formalize the trade-off between privacy and accuracy in this setting, and show that a simple class of estimates achieves an order-optimal trade-off. It thus suffices to focus on auction mechanisms that output such estimates. We use this observation to design a truthful, individually rational, proportional-purchase mechanism under a fixed budget constraint. We show that our mechanism is 5-approximate in terms of accuracy compared to the optimal mechanism, and that no truthful mechanism can achieve a $2-\varepsilon$ approximation, for any $\varepsilon > 0$.

1 Introduction

Recommender systems are ubiquitous on the Internet, lying at the heart of some of the most popular Internet services, including Netflix, Yahoo, and Amazon. These systems use algorithms to predict, *e.g.*, a user's rating for a movie, her propensity to click on an advertisement or to purchase a product online. By design, such prediction algorithms rely on access to large training datasets, typically comprising data from thousands (often millions) of individuals. This large-scale collection of user data has raised serious privacy concerns among researchers and consumer advocacy groups. Privacy researchers have shown that access to seemingly non-sensitive data (*e.g.*, movie ratings) can lead to the leakage of potentially sensitive information when combined with de-anonymization techniques [1]. Moreover, a spate of recent lawsuits [2, 3, 4] as well as behavioral studies [5] have demonstrated the increasing reluctance of the public to allow the unfettered collection and monetization of user data.

As a result, researchers and advocacy groups have argued in favor of legislation protecting individuals, by ensuring they can "opt-out" from data collection if they so desire [6]. However, a widespread restriction on data collection would

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be detrimental to profits of the above companies. One way to address this tension between the value of data and the users' need for privacy is through *incentivization*. In short, companies releasing an individual's data ought to appropriately compensate her for the violation of her privacy, thereby incentivizing her consent to the release.

We study the issue of user incentivization through *privacy auctions*, as introduced by Ghosh and Roth [7]. In a privacy auction, a data analyst has access to a database $\mathbf{d} \in \mathbb{R}^n$ of private data d_i, $i = 1, \ldots, n$, each corresponding to a different individual. This data may represent information that is to be protected, such as an individual's propensity to click on an ad or purchase a product, or the number of visits to a particular website. The analyst wishes to publicly release an estimate $\hat{s}(\mathbf{d})$ of a statistic $s(\mathbf{d})$ evaluated over the database. In addition, each individual incurs a privacy cost c_i upon the release of the estimate $\hat{s}(\mathbf{d})$, and must be appropriately compensated by the analyst for this loss of utility. The analyst has a budget, which limits the total compensation paid out. As such, given a budget and a statistic s, the analyst must (a) solicit the costs of individuals c_i and (b) determine the estimate $\hat{s}$ to release as well as the appropriate compensation to each individual.

Ghosh and Roth employ *differential privacy* [8] as a principled approach to quantifying the privacy cost c_i. Informally, ensuring that $\hat{s}(\mathbf{d})$ is ϵ-differentially private with respect to individual i provides a guarantee on the privacy of this individual; a small ϵ corresponds to better privacy since it guarantees that $\hat{s}(\mathbf{d})$ is essentially independent of the individual's data d_i. Privacy auctions incorporate this notion by assuming that each individual i incurs a cost $c_i = c_i(\epsilon)$, that is a function of the privacy guarantee ϵ provided by the analyst.

1.1 Our Contribution

Motivated by recommender systems, we focus in this paper on a scenario where the statistic s takes the form of a *linear predictor*:

$$s(\mathbf{d}) := \langle \mathbf{w}, \mathbf{d} \rangle = \textstyle\sum_{i=1}^{n} w_i d_i, \qquad (1)$$

where $\mathbf{w} \in \mathbb{R}^n$, is a publicly known vector of real (possibly negative) weights. Intuitively, the public weights w_i serve as measures of the similarity between each individual i and a new individual, outside the database. The function $s(\mathbf{d})$ can then be interpreted as a prediction of the value d for this new individual.

Linear predictors of the form (1) include many well-studied methods of statistical inference, such as the k-nearest-neighbor method, the Nadaranya-Watson weighted average, ridge regression, as well as support vector machines. We provide a brief review of such methods in Section 5. Functions of the form (1) are thus of particular interest in the context of recommender systems [9, 10], as well as other applications involving predictions (*e.g.*, polling/surveys, marketing). In the sequel, we ignore the provenance of the public weights $\mathbf{w}$, keeping in mind that any of these methods apply. Our contributions are as follows:

1. **Privacy-Accuracy Trade-off.** We characterize the accuracy of the estimate $\hat{s}$ in terms of the *distortion* between the linear predictor s and $\hat{s}$ defined

as $\delta(s,\hat{s}) := \max_{\mathbf{d}} \mathbb{E}\left[|s(\mathbf{d}) - \hat{s}(\mathbf{d})|^2\right]$, *i.e.*, the maximum mean square error between $s(\mathbf{d})$ and $\hat{s}(\mathbf{d})$ over all databases $\mathbf{d}$. We define a *privacy index* $\beta(\hat{s})$ that captures the amount of privacy an estimator $\hat{s}$ provides to individuals in the database. We show that any estimator $\hat{s}$ with low distortion must also have a low privacy index (Theorem 1).

2. **Laplace Estimators Suffice.** We show that a special class of *Laplace estimators* [8, 11] (*i.e.*, estimators that use noise drawn from a Laplace distribution), which we call Discrete Canonical Laplace Estimator Functions (DCLEFs), exhibits an order-optimal trade-off between privacy and distortion (Theorem 2). This allows us to restrict our focus on privacy auctions that output DCLEFs as estimators of the linear predictor s.
3. **Truthful, 5-Approximate Mechanism, and Lower bound.** We design a *truthful*, *individually rational*, and *budget feasible* mechanism that outputs a DCLEF as an estimator of the linear predictor (Theorem 3). Our estimator's accuracy is a 5-approximation with respect to the DCLEF output by an optimal, individually rational, budget feasible mechanism. We also prove a lower bound (Theorem 4): there is no truthful DCLEF mechanism that achieves an approximation ratio $2-\varepsilon$, for any $\varepsilon > 0$.

In our analysis, we exploit the fact that when $\hat{s}$ is a Laplace estimator minimizing distortion under a budget resembles the knapsack problem. As a result, the problem of designing a privacy auction that outputs a DCLEF $\hat{s}$ is similar in spirit to the knapsack auction mechanism [12]. However, our setting poses an additional challenge because the privacy costs exhibit *externalities*: the cost incurred by an individual is a function of which other individuals are being compensated. Despite the externalities in costs, we achieve the same approximation as the one known for the knapsack auction mechanism [12].

Due to space constraints we omit all proofs from this extended abstract, and refer the interested reader to the full version [13] of the paper.

1.2 Related Work

Privacy of Behavioral Data. Differentially-private algorithms have been developed for the release of several different kinds of online user behavioral data such as click-through rates and search-query frequencies [14], as well as movie ratings [15]. As pointed out by McSherry and Mironov [15], the reason why the release of such data constitutes a privacy violation is not necessarily that, *e.g.*, individuals perceive it as embarrassing, but that it renders them susceptible to *linkage* and *de-anonymization attacks* [1]. Such linkages could allow, for example, an attacker to piece together an individual's address stored in one database with his credit card number or social security number stored in another database. It is therefore natural to attribute a loss of utility to the disclosure of such data.

Privacy Auctions. Quantifying the cost of privacy loss allows one to study privacy in the context of an economic transaction. Ghosh and Roth initiate this study of privacy auctions in the setting where the data is binary and the statistic reported is the sum of bits, *i.e.*, $d_i \in \{0,1\}$ and $w_i = 1$ for all $i = 1,\ldots,n$ [7].

Unfortunately, the Ghosh-Roth auction mechanism cannot be readily generalized to asymmetric statistics such as (1), which, as discussed in Section 5, have numerous important applications including recommender systems. Our Theorems 1 and 2, which parallel the characterization of order-optimal estimators in [7], imply that to produce an accurate estimate of s, the estimator $\hat{s}$ *must provide different privacy guarantees to different individuals*. This is in contrast to the multi-unit procurement auction of [7]. In fact, as discussed the introduction, a privacy auction outputting a DCLEF $\hat{s}(\mathbf{d})$ has many similarities with a knapsack auction mechanism [12], with the additional challenge of externalities introduced by the Laplacian noise (see also Section 4).

Privacy and Truthfulness in Mechanism Design. A series of interesting results follow an orthogonal direction, namely, on the connection between privacy and truthfulness when individuals have the ability to misreport their data. Starting with the work of McSherry and Talwar [16] followed by Nissim *et al* [17], Xiao [18] and most recently Chen *et al* [19], these papers design mechanisms that are simultaneously truthful and privacy-preserving (using differential privacy or other closely related definitions of privacy). As pointed out by Xiao [18], all these papers consider an *unverified* database, *i.e.*, the mechanism designer cannot verify the data reported by individuals and therefore must incentivize them to report truthfully. Recent work on truthfully eliciting private data through a *survey* [20, 21] also fall under the unverified database setting [18]. In contrast, our setting, as well as that of Ghosh and Roth, is that of a *verified* database, in which individuals cannot lie about their data. This setting is particularly relevant to the context of online behavioral data: information on clicks, websites visited and products purchased is collected and stored in real-time and cannot be retracted after the fact.

Correlation between Privacy Costs and Data Values. An implicit assumption in privacy auctions as introduced in [7] is that the privacy costs c_i are *not* correlated with the data values d_i. This might not be true if, *e.g.*, the data represents the propensity of an individual to contract a disease. Ghosh and Roth [7] show that when the privacy costs are correlated to the data no individually rational direct revelation mechanism can simultaneously achieve non-trivial accuracy and differential privacy. As discussed in the beginning of this section, the privacy cost of the release of behavioral data is predominantly due to the risk of a linkage attack. It is reasonable in many cases to assume that this risk (and hence the cost of privacy loss) is not correlated to, *e.g.*, the user's movie ratings. Nevertheless, due to its importance in other settings such as medical data, more recent privacy auction models aim at handling such correlation [20, 21, 22]; we leave generalizing our results to such privacy auction models as future work.

2 Preliminaries

Let $[k] = \{1, \cdots, k\}$, for any integer $k > 0$, and define $\mathrm{I} := [R_{\min}, R_{\max}] \subset \mathbb{R}$ to be a bounded real interval. Consider a database containing the information of $n > 0$ individuals. In particular, the database comprises a vector $\mathbf{d}$, whose entries

$d_i \in \mathrm{I}$, $i \in [n]$, represent the private information of individual i. Each entry d_i is *a priori* known to the database administrator, and therefore individuals do not have the ability to lie about their private data. A data analyst with access to the database would like to publicly release an estimate of the statistic $s(\mathbf{d})$ of the form (1), *i.e.* $s(\mathbf{d}) = \sum_{i\in[n]} w_i d_i$, for some publicly known weight vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$. For any subset $H \subseteq [n]$, we define $w(H) := \sum_{i\in H} |w_i|$, and denote by $W := w([n]) = \sum_{i=1}^n |w_i|$ the ℓ_1 norm of vector $\mathbf{w}$. We denote the length of interval I by $\Delta := R_{\max} - R_{\min}$, and its midpoint by $\bar{R} := (R_{\min} + R_{\max})/2$. Without loss of generality, we assume that $w_i \neq 0$ for all $i \in [n]$; if not, since entries for which $w_i = 0$ do not contribute to the linear predictor, it suffices to consider the entries of $\mathbf{d}$ for which $w_i \neq 0$.

2.1 Differential Privacy and Distortion

Similar to [7], we use the following generalized definition of differential privacy:

Definition 1 (Differential Privacy). *A (randomized) function $f : \mathrm{I}^n \to \mathbb{R}^m$ is $(\epsilon_1, \ldots, \epsilon_n)$-differentially private if for each individual $i \in [n]$ and for any pair of data vectors $\mathbf{d}, \mathbf{d}^{(i)} \in \mathrm{I}^n$ differing in only their i-th entry, ϵ_i is the smallest value such that $\mathbb{P}[f(\mathbf{d}) \in S] \leq e^{\epsilon_i}\mathbb{P}[f(\mathbf{d}^{(i)}) \in S]$ for all $S \subset \mathbb{R}^m$.*

This definition differs slightly from the usual definition of ϵ-differential privacy [11], as the latter is stated in terms of the *worst case* privacy across all individuals. More specifically, according to the notation in [11], an $(\epsilon_1, \ldots, \epsilon_n)$-differentially private function is ϵ-differentially private, where $\epsilon = \max_i \epsilon_i$.

Given a deterministic function f, a well-known method to provide ϵ-differential privacy is to add random noise drawn from a Laplace distribution to this function [11]. This readily extends to $(\epsilon_1, \ldots, \epsilon_n)$-differential privacy.

Lemma 1 ([11]). *Consider a deterministic function $f : \mathrm{I}^n \to \mathbb{R}$. Define $\hat{f}(\mathbf{d}) := f(\mathbf{d}) + Lap(\sigma)$, where $Lap(\sigma)$ is a random variable sampled from the Laplace distribution with parameter σ. Then, $\hat{f}$ is $(\epsilon_1, \ldots, \epsilon_n)$-differentially private, where $\epsilon_i = S_i(f)/\sigma$, and $S_i(f) := \max_{\mathbf{d},\mathbf{d}^{(i)}\in\mathrm{I}^n} |f(\mathbf{d}) - f(\mathbf{d}^{(i)})|$, is the sensitivity of f to the i-th entry d_i, $i \in [n]$.*

Intuitively, the higher the variance σ of the Laplace noise added to f, the smaller ϵ_i, and hence, the better the privacy guarantee of $\hat{f}$. Moreover, for a fixed σ, entries i with higher sensitivity $S_i(f)$ receive a worse privacy guarantee (higher ϵ_i).

There is a natural tradeoff between the amount of noise added and the accuracy of the perturbed function $\hat{f}$. To capture this, we introduce the notion of *distortion* between two (possibly randomized) functions:

Definition 2. (Distortion). *Given two functions $f : \mathrm{I}^n \to \mathbb{R}$ and $\hat{f} : \mathrm{I}^n \to \mathbb{R}$, the* distortion*, $\delta(f, \hat{f})$, between f and $\hat{f}$ is given by*

$$\delta(f, \hat{f}) := \max_{\mathbf{d}\in\mathrm{I}^n} \mathbb{E}\left[|f(\mathbf{d}) - \hat{f}(\mathbf{d})|^2\right].$$

In our setup, the data analyst wishes to disclose an *estimator function* $\hat{s} : \mathrm{I}^n \to \mathbb{R}$ of the linear predictor s. Intuitively, a good estimator $\hat{s}$ should have a small distortion $\delta(s, \hat{s})$, while also providing good differential privacy guarantees.

2.2 Privacy Auction Mechanisms

Each individual $i \in [n]$ has an associated cost function $c_i : \mathbb{R}_+ \to \mathbb{R}_+$, which determines the cost $c_i(\epsilon_i)$ incurred by i when an $(\epsilon_1, \dots, \epsilon_n)$-differentially private estimate $\hat{s}$ is released by the analyst. As in [7], we consider linear cost functions, *i.e.*, $c_i(\epsilon) = v_i \epsilon$, for all $i \in [n]$. We refer to v_i as the *unit-cost* of individual i. The unit-costs v_i are not *a priori* known to the data analyst. Without loss of generality, we assume throughout the paper that $v_1 \leq \dots \leq v_n$.

Given a weight vector $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$, let M_s be a mechanism compensating individuals in $[n]$ for their loss of privacy from the release of an estimate $\hat{s}$ of the linear predictor $s(\mathbf{d})$. Formally, M_s takes as input a vector of reported unit-costs $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n_+$ and a budget B, and outputs

1. a payment $p_i \in \mathbb{R}_+$ for every $i \in [n]$, and
2. an estimator function $\hat{s} : I^n \to \mathbb{R}_+$.

Assume that the estimator $\hat{s}$ satisfies $(\epsilon_1, \dots, \epsilon_n)$-differential privacy. A mechanism is *budget feasible* if $\sum_{i \in [n]} p_i \leq B$, *i.e.*, the payments made by the mechanism are within the budget B. Moreover, a mechanism is *individually rational* if for all $i \in [n]$, $p_i \geq c_i(\epsilon_i) = v_i \epsilon_i$, *i.e.*, payments made by the mechanism exceed the cost incurred by individuals. Finally, a mechanism is *truthful* if for all $i \in [n]$, $p_i(v_i, v_{-i}) - v_i \epsilon_i(v_i, v_{-i}) \geq p_i(v'_i, v_{-i}) - v_i \epsilon_i(v'_i, v_{-i})$, *i.e.*, no individual can improve her utility by misreporting her private unit-cost.

2.3 Outline of Our Approach

We denote by $\delta_{M_s} := \delta(s, \hat{s})$ the distortion between s and the function output by the mechanism M_s. Ideally, a mechanism should output an estimator that has small distortion. However, the smaller the distortion, the higher the privacy violation and, hence, the more money the mechanism needs to spend. As such, the objective of this paper is to design a mechanism with minimal distortion, subject to the constraints of truthfulness, individual rationality, and budget feasibility.

To address this question, in Section 3, we first establish a privacy-distortion tradeoff for differentially-private estimators of the linear predictor. We then introduce a family of estimators, Discrete Canonical Laplace Estimator Functions (DCLEFs), and show that they achieve a near-optimal privacy-distortion tradeoff. This result allows us to limit our attention to DCLEF privacy auction mechanisms, *i.e.*, mechanisms that output a DCLEF $\hat{s}$. In Section 4, we present a mechanism that is truthful, individually rational, and budget feasible, while also being near-optimal in terms of distortion.

3 Privacy-Distortion Tradeoff and Laplace Estimators

Recall that a good estimator should exhibit low distortion and simultaneously give good privacy guarantees. In this section, we establish the privacy-distortion tradeoff for differentially-private estimators of the linear predictor. Moreover, we introduce a family of estimators that exhibits a near-optimal tradeoff between privacy and distortion. This will motivate our focus on privacy auction mechanisms that output estimators from this class in Section 4.

3.1 Privacy-Distortion Tradeoff

There exists a natural tension between privacy and distortion, as highlighted by the following two examples.

Example 1. Consider the estimator $\hat{s} := \bar{R}\sum_{i=1}^{n} w_i$, where recall that $\bar{R} = (R_{\min} + R_{\max})/2$. This estimator guarantees perfect privacy (*i.e.*, $\epsilon_i = 0$), for all individuals. However, $\delta(s, \hat{s}) = (W\Delta)^2/4$.

Example 2. Consider the estimator function $\hat{s} := \sum_{i=1}^{n} w_i d_i$. In this case, $\delta(s, \hat{s}) = 0$. However, $\epsilon_i = \infty$ for all $i \in [n]$.

In order to formalize this tension between privacy and distortion, we define the *privacy index* of an estimator as follows.

Definition 3. *Let $\hat{s} : \mathrm{I}^n \to \mathbb{R}$ be any $(\epsilon_1, \ldots, \epsilon_n)$-differentially private estimator function for the linear predictor. We define the* privacy index*, $\beta(\hat{s})$, of $\hat{s}$ as*

$$\beta(\hat{s}) := \max\left\{ w(H) : H \subseteq [n] \text{ and } \sum_{i \in H} \epsilon_i < 1/2 \right\}. \tag{2}$$

$\beta(\hat{s})$ captures the weight of the individuals that have been guaranteed good privacy by $\hat{s}$. Next we characterize the impossibility of having an estimator with a low distortion but a high privacy index. Note that for Example 1, $\beta(\hat{s}) = W$, *i.e.*, the largest value possible, while for Example 2, $\beta(\hat{s}) = 0$. We stress that the selection of $1/2$ as an upper bound in (2) is arbitrary; Theorems 1 and 2 still hold if another value is used, though the constants involved will differ.

Our first main result establishes a trade-off between the privacy index and the distortion of an estimator.

Theorem 1 (Trade-off between Privacy-index and Distortion). *Let $0 < \alpha < 1$. Let $\hat{s} : \mathrm{I}^n \to \mathbb{R}$ be an arbitrary estimator function for the linear predictor. If $\delta(s, \hat{s}) \leq (\alpha W\Delta)^2/48$ then $\beta(\hat{s}) \leq 2\alpha W$.*

In other words, if an estimator has low distortion, the weight of individuals with a good privacy guarantee (*i.e.*, a small ϵ_i) can be at most an α fraction of $2W$.

3.2 Laplace Estimator Functions

Consider the following family of estimators for the linear predictor $\hat{s} : \mathrm{I}^n \to \mathbb{R}$:

$$\hat{s}(\mathbf{d}; \mathbf{a}, \mathbf{x}, \sigma) := \sum_{i=1}^{n} w_i d_i x_i + \sum_{i=1}^{n} w_i a_i (1 - x_i) + \mathrm{Lap}(\sigma) \tag{3}$$

where $x_i \in [0,1]$, and each $a_i \in \mathbb{R}$ is a constant independent of the data vector $\mathbf{d}$. This function family is parameterized by $\mathbf{x}, \mathbf{a}$ and σ. The estimator $\hat{s}$ results from distorting s in two ways: (a) a randomized distortion by the addition of the Laplace noise, and (b) a deterministic distortion through a linear interpolation between each entry d_i and some constant a_i. Intuitively, the interpolation parameter x_i determines the extent to which the estimate $\hat{s}$ depends on entry d_i. Using Lemma 1 and the definition of distortion, it is easy to characterize the privacy and distortion properties of such estimators.

Lemma 2. *Given w_i, $i \in [n]$, let $s(\mathbf{d})$ be the linear predictor given by* (1)*, and $\hat{s}$ an estimator of s given by* (3)*. Then,*

1. *$\hat{s}$ is $(\epsilon_1, \ldots, \epsilon_n)$-differentially private, where $\epsilon_i = \frac{\Delta |w_i|\, x_i}{\sigma}$, $i \in [n]$.*
2. *The distortion satisfies $\delta(s,\hat{s}) \geq \left(\frac{\Delta}{2}\sum_{i=1}^n |w_i|(1-x_i)\right)^2 + 2\sigma^2$, with equality attained when $a_i = \bar{R}$, for all $i \in [n]$.*

Note that the constants a_i do not affect the differential privacy properties of $\hat{s}$. Moreover, among all estimators with given $\mathbf{x}$, the distortion $\delta(s,\hat{s})$ is minimized when $a_i = \bar{R}$ for all $i \in [n]$. In other words, to minimize the distortion without affecting privacy, it is always preferable to interpolate between d_i and $\bar{R}$. This motivates us to define the family of Laplace estimator functions as follows.

Definition 4. *Given w_i, $i \in [n]$, the* Laplace estimator function family *(LEF) for the linear predictor s is the set of functions $\hat{s} : \mathrm{I}^n \to \mathbb{R}$, parameterized by $\mathbf{x}$ and σ, such that*

$$\hat{s}(\mathbf{d};\mathbf{x},\sigma) = \sum_{i=1}^n w_i d_i x_i + \bar{R}\sum_{i=1}^n w_i(1-x_i) + Lap(\sigma) \tag{4}$$

We call a LEF *discrete* if $x_i \in \{0,1\}$. Furthermore, we call a LEF *canonical* if the Laplace noise added to the estimator has a parameter of the form

$$\sigma = \sigma(\mathbf{x}) := \Delta \sum_{i=1}^n |w_i|(1-x_i) \tag{5}$$

Recall that x_i controls the dependence of $\hat{s}$ on the entry d_i; thus, intuitively, the standard deviation of the noise added in a canonical Laplace estimator is proportional to the "residual weight" of data entries. Note that, by Lemma 2, the distortion of a canonical Laplace estimator $\hat{s}$ has the following simple form:

$$\delta(s,\hat{s}) = \frac{9}{4}\Delta^2\Big(\sum_{i=1}^n |w_i|(1-x_i)\Big)^2 = \frac{9}{4}\Delta^2\Big(W - \sum_{i=1}^n |w_i| x_i\Big)^2. \tag{6}$$

Our next result establishes that there exists a discrete canonical Laplace estimator function (DCLEF) with a small distortion and a high privacy index.

Theorem 2 (DCLEFs suffice). *Let* $0 < \alpha < 1$. *Let*

$$\hat{s}^* := \operatorname*{argmax}_{\hat{s}:\delta(s,\hat{s})\le(\alpha W \Delta)^2/48} \beta(\hat{s})$$

be an estimator with the highest privacy index among all $\hat{s}$ *for which* $\delta(s,\hat{s}) \le (\alpha W \Delta)^2/48$. *There exists a DCLEF* $\hat{s}^\circ : \mathrm{I}^n \to \mathbb{R}$ *such that* $\delta(s,\hat{s}^\circ) \le (9/4)(\alpha W \Delta)^2$, *and* $\beta(\hat{s}^\circ) \ge \frac{1}{2}\beta(\hat{s}^*)$.

In other words, there exists a DCLEF that is within a constant factor, in terms of both its distortion and its privacy index, from an optimal estimator $\hat{s}^*$. Theorem 2 has the following immediate corollary:

Corollary 1. *Consider an arbitrary estimator* $\hat{s}$ *with distortion* $\delta(s,\hat{s}) < (W\Delta)^2$ *48. Then, there exists a DCLEF* $\hat{s}^\circ$ *such that* $\delta(s,\hat{s}^\circ) \le 108\delta(s,\hat{s})$ *and* $\beta(\hat{s}^\circ) \ge \frac{1}{2}\beta(\hat{s})$.

Proof. Apply Theorem (2) with $\alpha = \sqrt{48\delta(s,\hat{s})}/(W\Delta)$. In particular, for this α and $\hat{s}$ as in the theorem statement, we have that $\hat{s}^* := \operatorname{argmax}_{\hat{s}':\delta(s,\hat{s}')\le\delta(s,\hat{s})} \beta(\hat{s}')$, hence $\beta(\hat{s}^*) \ge \beta(\hat{s})$. Therefore, there exists a DCLEF $\hat{s}^\circ$ such that $\delta(s,\hat{s}^\circ) \le (9/4)(\alpha W\Delta)^2 \le 108\delta(s,\hat{s})$, and $\beta(\hat{s}^\circ) \ge \frac{1}{2}\beta(\hat{s}^*) \ge \frac{1}{2}\beta(\hat{s})$.

Theorems 1 and 2 imply that, when searching for estimators with low distortion and high privacy index, it suffices (up to constant factors) to focus on DCLEFs. Similar results were derived in [7] for estimators of unweighted sums of bits.

4 Privacy Auction Mechanism

Motivated by Theorems 1 and 2, we design a truthful, individually rational, budget-feasible DCLEF mechanism (*i.e.*, a mechanism that outputs a DCLEF) and show that it is 5-approximate in terms of accuracy compared with the optimal, individually rational, budget-feasible DCLEF mechanism. Note that a DCLEF is fully determined by the vector $\mathbf{x} \in \{0,1\}^n$. Therefore, we will simply refer to the output of the DCLEF mechanisms described below as $(\mathbf{x},\mathbf{p})$, as the latter characterize the released estimator and the compensations to individuals.

4.1 An Optimal DCLEF Mechanism

Consider the problem of designing a DCLEF mechanism M that is individually rational and budget feasible (but not necessarily truthful), and minimizes δ_M. Given a DCLEF $\hat{s}$, define $H(\hat{s}) := \{i : x_i = 1\}$ to be the set of individuals that receive non-zero differential privacy guarantees. Eq. (6) implies that $\delta(s,\hat{s}) = \frac{9}{4}\Delta^2(W - w(H(\hat{s})))^2$. Thus, minimizing $\delta(s,\hat{s})$ is equivalent to maximizing $w(H(\hat{s}))$. Let $(\mathbf{x}_{opt},\mathbf{p}_{opt})$ be an optimal solution to the following problem:

$$\begin{aligned}
\text{maximize} \quad & S(\mathbf{x};\mathbf{w}) = \sum_{i=1}^{n} |w_i| x_i \\
\text{subject to:} \quad & p_i \geq v_i \epsilon_i(\mathbf{x}), \quad \forall i \in [n], \qquad \text{(individual rationality)} \\
& \sum_{i=1}^{n} p_i \leq B \qquad \text{(budget feasibility)} \\
& x_i \in \{0,1\}, \ \forall i \in [n] \qquad \text{(discrete estimator function)}
\end{aligned} \tag{7}$$

where, by Lemma 2 and (5),

$$\epsilon_i(\mathbf{x}) = \frac{\Delta |w_i| x_i}{\sigma(\mathbf{x})} = \frac{|w_i| x_i}{\sum_i |w_i|(1-x_i)} \text{ (canonical property).} \tag{8}$$

A mechanism M_{opt} that outputs $(\mathbf{x}_{opt}, \mathbf{p}_{opt})$ will be an optimal, individually rational, budget feasible (but not necessarily truthful) DCLEF mechanism. Let $OPT := S(\mathbf{x}_{opt}; \mathbf{w})$ be the optimal objective value of (7). We use OPT as the benchmark to which we compare the (truthful) mechanism we design below. Without loss of generality, we make the following assumption:

Assumption 5. *For all* $i \in [n]$, $|w_i| v_i / (W - |w_i|) \leq B$.

Observe that if an individual i violates this assumption, then $c_i(\epsilon_i(\mathbf{x})) > B$ for any $\mathbf{x}$ output by a DCLEF mechanism that sets $x_i = 1$. In other words, no DCLEF mechanism (including M_{opt}) can compensate this individual within the analyst's budget and, hence, will set $x_i = 0$. Therefore, it suffices to focus on the subset of individuals for whom the assumption holds.

4.2 A Truthful DCLEF Mechanism

To highlight the challenge behind designing a truthful DCLEF mechanism, observe that if the privacy guarantees were given by $\epsilon_i(\mathbf{x}) = x_i$ rather than (8), the optimization problem (7) would be identical to the budget-constrained mechanism design problem for knapsack studied by Singer [12]. In the reverse-auction setting of [12], an auctioneer purchases items valued at fixed costs v_i by the individuals that sell them. Each item i is worth $|w_i|$ to the auctioneer, while the auctioneer's budget is B. The goal of the auctioneer is to maximize the total worth of the purchased set of items, *i.e.*, $S(\mathbf{x}; \mathbf{w})$. Singer presents a truthful mechanism that is 6-approximate with respect to OPT. However, in our setting, the privacy guarantees $\epsilon_i(\mathbf{x})$ given by (8) introduce *externalities* into the auction. In contrast to [12], the ϵ_i's couple the cost incurred by an individual i to the weight of other individuals that are compensated by the auction, making the mechanism design problem harder. This difficulty is overcome by our mechanism, which we call FairInnerProduct, described in Algorithm 1.

The mechanism takes as input the budget B, the weight vector $\mathbf{w}$, and the vector of unit-costs $\mathbf{v}$, and outputs a set $O \subset [n]$, that receive $x_i = 1$ in the

Algorithm 1. FairInnerProduct($\mathbf{v}, \mathbf{w}, B$)

Let k be the largest integer such that $\frac{B}{w([k])} \geq \frac{v_k}{W-w([k])}$.
Let $i^* := \mathrm{argmax}_{i\in[n]} |w_i|$.
Let $\hat{p}$ be as defined in (9).
if $|w_{i^*}| > \sum_{i\in[k]\setminus\{i^*\}} |w_i|$ **then**
  Set $O = \{i^*\}$.
  Set $p_{i^*} = \hat{p}$ and $p_i = 0$ for all $i \neq i^*$.
else
  Set $O = [k]$.
  Pay each $i \in O,\ p_i = |w_i| \min\{\frac{B}{w([k])}, \frac{v_{k+1}}{W-w([k])}\}$, and for $i \notin O,\ p_i = 0$.
end if
Set $x_i = 1$ if $i \in O$ and $x_i = 0$ otherwise.

DCLEF, as well as a set of payments for each individual in O. Our construction uses a greedy approach similar to the Knapsack mechanism in [12]. In particular, it identifies users that are the "cheapest" to purchase. To ensure truthfulness, it compensates them within budget based on the unit-cost of the last individual that was not included in the set of compensated users. As in greedy solutions to knapsack, this construction does not necessarily yield a constant approximation w.r.t. OPT; for that, the mechanism needs to sometimes compensate only the user with the highest absolute weight $|w_i|$. In such cases, the payment of the user of the highest weight is selected so that she has no incentive to lie about here true unit cost.

Recall that $v_1 \leq \ldots \leq v_n$. The mechanism defines $i^* := \mathrm{argmax}_{i\in[n]} |w_i|$ as the individual with the largest $|w_i|$, and k as the largest integer such that $\frac{B}{w([k])} \geq \frac{v_k}{W-w([k])}$. Subsequently, the mechanism either sets $x_i = 1$ for the first k individuals, or, if $|w_{i^*}| > \sum_{i\in[k]\setminus\{i^*\}} |w_i|$, sets $x_{i^*} = 1$. In the former case, individuals $i \in [k]$ are compensated *in proportion to their absolute weights* $|w_i|$. If, on the other hand, only $x_{i^*} = 1$, the individual i^* receives a payment $\hat{p}$ defined as follows: Let

$$S_{-i^*} := \Big\{ t \in [n]\setminus\{i^*\} : \frac{B}{\sum_{i\in[t]\setminus\{i^*\}} |w_i|} \geq \frac{v_t}{W - \sum_{i\in[t]\setminus\{i^*\}} |w_i|} \text{ and} \sum_{i\in[t]\setminus\{i^*\}} |w_i| \geq |w_{i^*}| \Big\}.$$

If $S_{-i^*} \neq \emptyset$, then let $r := \min\{i : i \in S_{-i^*}\}$. Define

$$\hat{p} := \begin{cases} B, & \text{if } S_{-i^*} = \emptyset \\ \frac{|w_{i^*}| v_r}{W - |w_{i^*}|}, & \text{otherwise} \end{cases} \tag{9}$$

The next theorem states that FairInnerProduct has the properties we desire.

Theorem 3. *FairInnerProduct is truthful, individually rational and budget feasible. It is 5-approximate with respect to OPT. Further, it is 2-approximate when all weights are equal.*

We note that the truthfulness of the knapsack mechanism in [12] is established via Myerson's characterization of truthful single-parameter auctions (*i.e.*, by showing that the allocation is monotone and the payments are threshold). In contrast, because of the coupling of costs induced by the Laplace noise in DCLEFs, we are unable to use Myerson's characterization and, instead, give a direct argument about truthfulness.

We prove a 5-approximation by using the optimal solution of the fractional relaxation of (7). This technique can also be used to show that the knapsack mechanism in [12] is 5-approximate instead of 6-approximate. FairInnerProduct generalizes the Ghosh-Roth mechanism; in the special case when all weights are equal FairInnerProduct reduces to the Ghosh-Roth mechanism, which, by Theorem 3, is 2-approximate with respect to OPT. In fact, our next theorem states that the approximation ratio of a truthful mechanism is at least 2.

Theorem 4 (Hardness of Approximation). *For all $\varepsilon > 0$, there is no truthful, individually rational, budget feasible DCLEF mechanism that is also $2-\varepsilon$-approximate with respect to OPT.*

Our benchmark OPT is stricter than that used in [7]. In particular, Ghosh and Roth show that their mechanism is optimal among all truthful, individually rational, budget-feasible, and *envy-free* mechanisms. In fact, the example we use to show hardness of approximation is a uniform weight example, implying that the lower-bound also holds for uniform weight case. Indeed, the mechanism in [7] is 2-approximate with respect to OPT, although it is optimal among individually rational, budget feasible mechanisms that are also truthful *and* envy free.

5 Discussion on Linear Predictors

As discussed in the introduction, a statistic $s(\mathbf{d})$ of the form (1) can be viewed as a *linear predictor* and is thus of particular interest in the context of recommender systems. We elaborate on this interpretation in this section. Assume that each individual $i \in [n] = \{1, \dots, n\}$ is endowed with a public vector $\mathbf{y}_i \in \mathbb{R}^m$, which includes m publicly known features about this individual. These could be, for example, demographic information such as age, gender or zip code, that the individual discloses in a public online profile. Note that, though features $\mathbf{y}_i$ are public, the data d_i is perceived as private.

Let $\mathbf{Y} = [\mathbf{y}_i]_{i\in[n]} \in \mathbb{R}^{n\times m}$ be a matrix comprising public feature vectors. Consider a new individual, not belonging to the database, whose public feature profile is $\mathbf{y} \in \mathbb{R}^m$. Having access to $\mathbf{Y}$, $\mathbf{d}$, and $\mathbf{y}$, the data analyst wishes to release a prediction for the unknown value d for this new individual. Below, we give several examples where this prediction takes the form $s(\mathbf{d}) = \langle \mathbf{w}, \mathbf{d}\rangle$, for some $\mathbf{w} = \mathbf{w}(\mathbf{y}, \mathbf{Y})$. All examples are textbook inference examples; we refer the interested reader to, for example, [23] for details.

k-Nearest Neighbors. In k-Nearest Neighbors prediction, the feature space $\mathbb{R}^m$ is endowed with a distance metric (*e.g.*, the ℓ_2 norm), and the predicted value is given by an average among the k nearest neighbors of the feature vector $\mathbf{y}$ of the new individual. *I.e.*, $s(\mathbf{d}) = \frac{1}{k}\sum_{i\in\mathcal{N}_k(\mathbf{y})} d_i$ where $N_k(\mathbf{y}) \subset [n]$ comprises the k individuals whose feature vectors y_i are closest to $\mathbf{y}$.

Nadaranya-Watson Weighted Average. The Nadaranya-Watson weighted average leverages all data in the database, weighing more highly data closer to $\mathbf{y}$. The general form of the prediction is $5s(\mathbf{d}) = \sum_{i=1}^{n} K(\mathbf{y}, \mathbf{y}_i) d_i / \sum_{i=1}^{n} K(\mathbf{y}, \mathbf{y}_i)$ where the *kernel* $K : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_+$ is a function decreasing in the distance between its argument (*e.g.*, $K(\mathbf{y}, \mathbf{y}') = e^{-\|\mathbf{y}-\mathbf{y}'\|^2}$).

Ridge Regression. In ridge regression, the analyst first fits a linear model to the data, *i.e.*, solves the optimization problem

$$\min_{\mathbf{b} \in \mathbb{R}^m} \sum_{i=1}^{n} \left(d_i - \langle \mathbf{y}_i, \mathbf{b} \rangle\right)^2 + \lambda \|\mathbf{b}\|_2^2, \tag{10}$$

where $\lambda \geq 0$ is a regularization parameter, enforcing that the vector $\mathbf{b}$ takes small values. The prediction is then given by the inner product $\langle \mathbf{y}, \mathbf{b} \rangle$. The solution to (10) is given by $\mathbf{b} = (\mathbf{Y}^T \mathbf{Y} + \lambda \mathbf{I})^{-1} \mathbf{Y}^T \mathbf{d}$; as such, the predicted value for a new user with feature vector $\mathbf{y}$ is given by $s(\mathbf{d}) = \langle \mathbf{y}, \mathbf{b} \rangle = \mathbf{y}^T (\mathbf{Y}^T \mathbf{Y} + \lambda \mathbf{I})^{-1} \mathbf{Y}^T \mathbf{d}$.

In all three examples, the prediction $s(\mathbf{d})$ is indeed of the form (1). Note that the weights are non-negative in the first two examples, but may assume negative values in the last one.

6 Conclusion and Future Work

We considered the setting of an auction, where a data analyst wishes to buy, from a set of n individuals, the right to use their private data $d_i \in \mathbb{R}$, $i \in [n]$, in order to *cheaply* obtain an *accurate* estimate of a statistic. Motivated by recommender systems and, more generally, prediction problems, the statistic we consider is a linear predictor with publicly known weights. The statistic can be viewed as a prediction of the unknown data of a new individual based on the database entries. We formalized the trade-off between privacy and accuracy in this setting; we showed that obtaining an accurate estimate necessitates giving poor differential privacy guarantees to individuals whose cumulative weight is large. We showed that DCLEF estimators achieve an order-optimal trade-off between privacy and accuracy, and, consequently, it suffices to focus on DCLEF mechanisms. We use this observation to design a truthful, individually rational, budget feasible mechanism under the constraint that the analyst has a fixed budget. Our mechanism can be viewed as a proportional-purchase mechanism, *i.e.*, the privacy ϵ_i guaranteed by the mechanism to individual i is proportional to her weight $|w_i|$. We show that our mechanism is 5-approximate in terms of accuracy compared to an optimal (possibly non-truthful) mechanism, and that no truthful mechanism can achieve a $2 - \varepsilon$ approximation, for any $\varepsilon > 0$.

Our work is the first studying privacy auctions for asymmetric statistics, and can be extended in a number of directions. An interesting direction to investigate is characterizing the most general class of statistics for which truthful privacy auctions that achieve order-optimal accuracy can be designed. An orthogonal direction is to study the release of asymmetric statistics in other settings such as (a) using a different notion of privacy, (b) allowing costs to be correlated with the data values, and (c) survey-type settings where individuals first decide whether to participate and then reveal their private data.

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Redistribution of VCG Payments in Public Project Problems*

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Abstract. Redistribution of VCG payments has been mostly studied in the context of resource allocation. This paper focuses on another fundamental model—the public project problem. In this scenario, the VCG mechanism collects in payments up to $\frac{n-1}{n}$ of the total value of the agents. This collected revenue represents a loss of social welfare. Given this, we study how to redistribute most of the VCG revenue back to the agents. Our first result is a bound on the best possible efficiency ratio, which we conjecture to be tight based on numerical simulations. Furthermore, the upper bound is confirmed on the case with 3 agents, for which we derive an optimal redistribution function. For more than 3 agents, we turn to heuristic solutions and propose a new approach to designing redistribution mechanisms.

1 Introduction

Public good or public project problems refer to situations where a group of agents need to decide whether or not to undertake a project or to procure a good. The project is "public" in the sense that everyone will enjoy the benefits of it. A typical example is a community deciding to build a bridge. If the bridge is built, everyone will be able to cross it. The challenge in deciding whether or not the bridge should be built, lies in learning how much the people need the bridge. Each person has a value for the bridge, but this value is known to him alone. The *efficient* outcome is to build the bridge if and only if the total value exceeds the cost of the bridge. Public project problems have been studied extensively in both economics and computer science literature (see, e.g., [9–11, 7, 1]).

In this context, we are interested in mechanisms that satisfy dominant-strategy incentive compatibility (DSIC), and maximize social welfare. The social welfare

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is measured as the sum of the utilities of the agents. Specifically, any payments collected from the agents reduce social welfare. Some payments however are required by DSIC. The objective of social welfare is natural in public good provision problems: after all, the benefits are to be enjoyed by all non-exclusively, and public projects are normally undertaken in the interest of the participants. This is in contrast to private goods that are often sold to generate profit for the auctioneer (there is no auctioneer or residual claimant in the public good problems we consider).

Our focus here is on mechanisms that are efficient and weakly budget-balanced (i.e., do not require an external subsidy). The latter restriction is necessary, as otherwise one can achieve infinite social welfare by providing an infinite subsidy to the agents. To this end, mechanisms from the Groves class align the incentives of the agents with the objective of choosing the efficient outcome. Specifically, under a Groves mechanism, each agent prefers reporting her value truthfully regardless of the reports of the other agents. In fact, Groves mechanisms are the only mechanisms that are dominant-strategy incentive compatible (or, *truthful*) and efficient for public project problems [8]. The mechanisms within the Groves class differ in the amount of payment collected from the agents. An easy way to describe this class of mechanisms is through the most prominent Groves mechanism—the VCG mechanism: payment made by agent i under any Groves mechanism can be represented as the payment collected by the VCG mechanism minus a redistribution $h_i(v_{-i})$, which is a function of other agents' values. For efficient mechanisms without an auctioneer, the objective of maximizing social welfare is equivalent to the objective of minimizing the revenue collected. Under this objective, the VCG mechanism has a very poor performance (i.e., collects a lot of revenue) as we detail next. Therefore, the question we study in this paper is how to design the *redistribution* functions that maximize social welfare.

We do not assume any prior on agent valuations and we evaluate mechanisms based on the worst-case performance over all possible value profiles. Following previous work on redistribution in resource allocation settings (e.g., [12, 6]), we make the performance metric unit free by measuring the performance as a percentage of the value of the efficient outcome achieved. We will refer to this metric as the *ratio*. Since there are no external subsidies, the value of the efficient outcome is the highest welfare that can be achieved, had all values been publicly known. Thus, the highest possible ratio is one.

The ratio of the VCG mechanism is $\frac{1}{n}$, where n is the number of agents [7]. In this paper, we derive an upper bound on the optimal ratio. Unlike the ratio of VCG, which decreases with n, the upper bound increases with n. We conjecture the bound to be tight based on numerical simulations. Further, for the case of $n = 3$, we find an optimal mechanism which guarantees the upper bound ratio of $\frac{2}{3}$. Finally, we propose a general heuristic-based approach for deriving redistribution mechanisms. Using a simple sampling-based heuristic, we obtain a mechanism whose ratio is higher than that of VCG for $n = 4, 5, 6$.

Our work is related to, and builds upon, some recent research on redistribution mechanisms. The public good model and, in particular, the valuation function of the agents are the same as in [7]. There, non-efficient but strongly budget-balanced mechanisms are considered. The authors discuss a randomized allocation function that guarantees a high expected ratio, while restricting the payments to add up to zero. In contrast, here we study *deterministic* mechanisms optimizing only over the payment functions, while the allocation rule is fixed to choose the efficient allocation. Our upper bound results suggest that full social welfare may be achievable asymptotically without resorting to randomized mechanisms.

Other work in various allocation settings has studied the problem of finding payments for Groves mechanisms that are optimal in terms of social welfare. In particular, Moulin [12] and Guo and Conitzer [6] simultaneously derived the optimal redistribution for allocating identical items to agents with unit demand. The results were further extended to multi-unit demand in [6]. An optimal Groves mechanism for allocating heterogeneous items was derived in [5]. General techniques have also been proposed for optimizing payments according to the mechanism designer's objectives, for single-parameter and multi-parameter domains [13, 4]. In fact, we make use of a heuristic technique from [4] to derive an optimal solution for $n = 3$.

There are also other redistribution mechanisms aiming to minimize payments that can be applied to the public good setting. Bailey [2] proposed a redistribution mechanism for public good problems, but under the worst-case analysis it is not weakly budget-balanced. While the mechanism proposed by Cavallo [3] is efficient and weakly budget-balanced, it cannot redistribute any VCG revenue in public good problems [7]. In this paper, we propose weakly budget-balanced mechanisms that do redistribute some of the VCG revenue, which increases social welfare without requiring external subsidy.

The rest of the paper is organized as follows. We present the model in Section 2. A conjecture about the optimal ratio is derived analytically in Section 3. The optimal solution to the case with $n = 3$ is presented in Section 4. We then propose a heuristic-based approach for deriving redistribution mechanisms and analyze the resulting mechanism's performance in Section 5. Section 6 relaxes the assumption that allowed us to restrict the value space while deriving prior results. We conclude and discuss directions for future work in Section 7.

2 The Model

There are n agents deciding whether or not to undertake a project, such as building a bridge. The cost of the bridge is C, which is commonly known. Each agent has a private type $\theta_i \geq 0$ denoting how much he will benefit if the bridge is built. We will assume $\theta_i \in [0, C]$, and will demonstrate in Section 6 that it is without loss of generality to consider types that are bounded from above by C. Also, without loss of generality, we can assign labels to the agents so that agent 1 is the agent with the highest value, agent 2—with the second highest, etc. Thus, $C \geq \theta_1 \geq \theta_2 \geq \ldots \geq \theta_n \geq 0$, and we denote the space of agent values by $\Theta = \{\theta \in [0, C]^n \mid C \geq \theta_1 \geq \theta_2 \geq \ldots \geq \theta_n \geq 0\}$. A mechanism for this problem consists of the outcome and the payment functions. The outcome is denoted by $k(\theta) \in \{0,1\}$ with $k(\theta) = 1$ if the bridge is built, and $t_i(\theta) \in \mathbb{R}$ are payments made by each agent i. We fix k to be the efficient rule: $k(\theta) = 1$ iff $\sum_i \theta_i > C$.

The value of each agent depends on his type and whether or not the bridge is built. Following [7], we define the value of the efficient outcome as follows:

$$s(\theta) = \max(\sum_i \theta_i, C)$$

This definition corresponds to the interpretation that if the bridge is not built, the agents get to distribute C among themselves (or, equivalently, they do not spend C on the bridge). This is reflected in the valuation function, which lets each agent keep $\frac{C}{n}$ if the bridge is not built:

$$v_i(k(\theta), \theta_i) = \begin{cases} \theta_i & \text{if } k(\theta) = 1 \\ \frac{C}{n} & \text{otherwise} \end{cases}$$

Utility of agent i is quasi-linear in the payment $t_i \in \mathbb{R}$ collected from him:

$$u_i(\theta) = v_i(k(\theta), \theta_i) - t_i(\theta)$$

Without loss of generality, for efficient and dominant-strategy incentive compatible mechanisms, we focus on the Groves class. Furthermore, we focus on Groves mechanisms that are anonymous, which, for our objective of maximizing worst-case performance (see Equations 2 and 4), is without loss of generality [1]. These mechanisms implement the efficient outcome, $k(\theta) = 1$ iff $\sum_i \theta_i > C$. Note that $\sum_i v_i(k(\theta), \theta_i) = s(\theta)$ for the efficient $k(\theta)$. DSIC is achieved by selecting t_i that aligns an agent's utility with the goal of selecting the efficient outcome:

$$t_i(\theta) = v_i(k(\theta), \theta_i) - s(\theta) + h(\theta_{-i})$$

which yields

$$u_i(\theta) = s(\theta) - h(\theta_{-i}) \tag{1}$$

where $h : W \to \mathbb{R}$ is an arbitrary function of the values of the agents other than the agent whose redistribution (or rather, *rebate*) is computed.[1] Here, domain $W = \{w \in [0, C]^{n-1} \mid C \geq w_1 \geq w_2 \geq \ldots \geq w_{n-1} \geq 0\}$ of rebate function h (which we will also term the *rebate space*) refers to the space of values of $n-1$ agents (other than i). Importantly, the second term of utility, $h(\theta_{-i})$, characterizes all mechanisms within the Groves class. Our goal is to choose function h that maximizes social welfare subject to the constraint of weak budget balance.

Weak budget balance constraint means that the sum of payments made by the agents must be non-negative:

$$\sum_i t_i(\theta) = \sum_i (v_i(k(\theta), \theta_i) - s(\theta) + h(\theta_{-i})) = \sum_i h(\theta_{-i}) - (n-1)s(\theta) \geq 0 \quad \forall \theta$$

[1] Under Equation 1, h is the function that determines how much of the value of the efficient outcome the agent should pay back. This is hardly a redistribution/rebate, but we keep this terminology to be consistent with prior literature.

Next, we describe how the performance of a mechanism is measured. A mechanism guarantees the ratio r if the following holds:

$$\sum_i u_i(\theta) = ns(\theta) - \sum_i h(\theta_{-i}) \geq rs(\theta) \quad \forall\theta$$

Stated formally, we seek to solve the following optimization problem:

$$\max_{h:W\to\mathbb{R}, r\in\mathbb{R}} r \tag{2}$$

$$\sum_i h(\theta_{-i}) \geq (n-1)s(\theta) \quad \forall\ \theta \in \Theta \tag{3}$$

$$ns(\theta) - \sum_i h(\theta_{-i}) \geq rs(\theta) \quad \forall\ \theta \in \Theta \tag{4}$$

In words, we are looking for a mechanism with the highest ratio (Equations 2 and 4) that is weakly budget-balanced (Equation 3). Note that both constraints can be written in one line as

$$(n-r)s(\theta) \geq \sum_i h(\theta_{-i}) \geq (n-1)s(\theta) \quad \forall\theta \tag{5}$$

3 Optimal Ratio (Conjecture)

In this section, we describe an interesting structure of the optimization problem (2)-(4). The problem has an infinite number of constraints, but our numerical results showed that it is sufficient to consider only $n+1$ constraints to obtain an upper bound on the ratio, such that this ratio does not change when we add additional constraints (of course, we were only able to check finite sets of constraints). This provides numerical evidence that the upper bound is tight. Furthermore, we derive this upper bound in closed form, which we show in the rest of this section.

First, we discuss how the ratio can be upper bounded computationally using the technique `RestrictedProblem` from [4]. The idea is to solve the problem while only enforcing a finite subset of constraints. The solution may violate some of the excluded constraints, thus providing an upper bound on the objective value (we are considering a maximization problem). In more detail, the optimization problem (2)-(4) has an infinite number of constraints (one for each $\theta \in \Theta$) and optimizes over functions (equivalently, there is an infinite number of variables—a rebate $h(w)$ for each $w \in W$). To make the problem more manageable, we limit the space of value profiles to a finite subset $\hat{\Theta} \subset \Theta$. Notice that once the set of profiles is finite, the set of rebates that appear in the constraints is also finite. It can be obtained by "projecting" each value profile into n profiles by removing one of the elements while keeping the rest. For example, when we restrict the value space to the set of profiles $\hat{\Theta} = \{(a, b, c), (d, e, f)\}$, the relevant rebates are defined for each profile in $\hat{W} = \{(b, c), (a, c), (a, b), (e, f), (d, f), (d, e)\}$. The constraints (3) and (4) appear once for each value profile, and the number of variables is $|\hat{W}|$. With these restrictions, the optimization problem in (2)-(4) becomes a linear program, which we implemented and solved using CPLEX.

Clearly, the choice of the enforced constraints as governed by $\hat{\Theta}$ determines the quality of the upper bound. Adding more constraints can only improve the bound. Interestingly, we find that considering only $n+1$ "important" profiles gave the best upper bound we could find among all sets of $\hat{\Theta}$ that we tried. In more detail, for a given n, we obtained the profiles $\hat{\Theta}$ by discretizing the space of values an agent may have. For example, discretizing into $z+1$ possible values we get $\theta_i \in \{j\frac{C}{z}\}_{j=0}^{z}$. Without loss of generality we set $C=1$, and focus on $\theta_i \in \{\frac{j}{z}\}_{j=0}^{z}$. Looking deeper into the patterns, we observed an interesting structure, that let us characterize the upper bound analytically.

The best upper bound we observed numerically across n was obtained when solving the restricted problem with the following $n+1$ value profiles: the zero profile and the profiles $(\underbrace{\frac{1}{b},\ldots,\frac{1}{b}}_{k},\underbrace{0,\ldots,0}_{n-k})$ where b is the integer part of $\frac{n}{2}$ and $1 \le k \le n$. For example, for $n=5$ we have $b=2$, and the profiles $(0,0,0,0,0)$, $(\frac{1}{2},0,0,0,0)$, ..., $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$. We refer to these $n+1$ profiles as *important profiles*. Next we provide an optimal solution to the restricted problem analytically.

Theorem 1. *No mechanism can achieve a ratio above r.*

$$r = 1 - \left(2 + \frac{2(\frac{n}{2}!)^2}{n}\sum_{j=0}^{\frac{n-4}{4}}\frac{(3n-4j)}{(2j)!(n-2j)!}\right)^{-1} \qquad n = 4, 8, 12, \ldots \tag{6}$$

$$r = 1 - \left(2 + \frac{2(\frac{n}{2}!)^2}{n}\sum_{j=0}^{\frac{n-2}{4}}\frac{(3n-4j-2)}{(2j+1)!(n-2j-1)!}\right)^{-1} \qquad n = 6, 10, 14, \ldots \tag{7}$$

$$r = 1 - \left(\frac{n(\frac{n-1}{2}!)^2}{n-1}\sum_{j=0}^{\frac{n-1}{4}}\frac{(n+4j-1)}{(2j)!(n-2j)!}\right)^{-1} \qquad n = 5, 9, 13, \ldots \tag{8}$$

$$r = 1 - \left(\frac{n(\frac{n-1}{2}!)^2}{n-1}\sum_{j=0}^{\frac{n-3}{4}}\frac{(n+4j+1)}{(2j+1)!(n-2j-1)!}\right)^{-1} \qquad n = 3, 7, 11, \ldots \tag{9}$$

Proof. The proof is available in the full version of the paper. □

Considering much larger sets of value profiles never improved the bound. This leads us to believe that the bound is tight. Furthermore, performing sensitivity analysis revealed that only the constraints used to derive the bound were tight in optimal solutions to restricted problems that included supersets of the important profiles. If the ratio is indeed tight, then we also have optimal rebates for the n rebate profiles used in deriving the bound: these values are unique, and thus, they cannot change in a solution that achieves the bound.

Observing the behavior of this upper bound (see Figure 1), we see that it approaches 1 as the number of agents increases. Thus, if this bound on the ratio

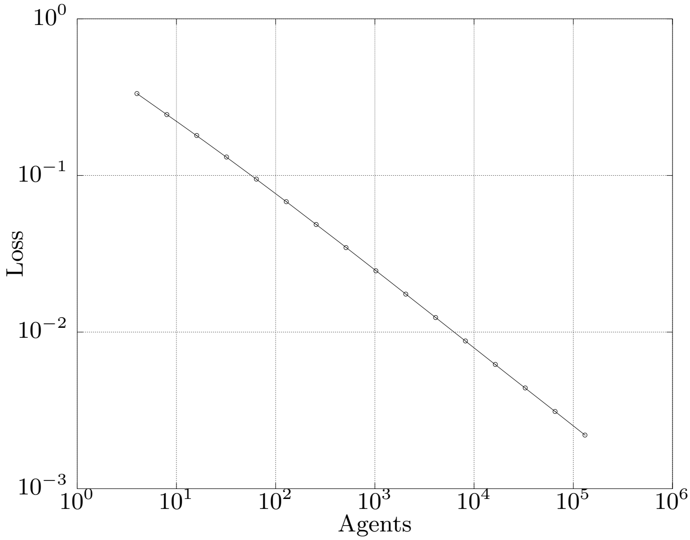

Fig. 1. The loss, $1 - r$, approaches zero as the number of agents increases

is tight, then an optimal mechanism for the public project problem will have a loss of social welfare approaching zero with additional agents. This is in contrast to the VCG mechanism, which has an overall social welfare of $\frac{1}{n}$ that approaches zero as the number of agents increase [7].

4 Optimal Redistribution for $n = 3$

For the case of $n = 3$, we obtain an optimal redistribution function. It was derived using techniques described in [4]. We provide the details next.

The upper bound linear program described in Section 3 can be modified to produce a heuristic redistribution function using another technique from [4]. The idea is to optimize over the space of rebate functions that are piecewise linear within a specified set of regions. The algorithm `LinearRebates` described in [4] takes a subdivision of the rebate space into regions and produces a redistribution function (and the ratio it achieves) that is optimal over all rebate functions that are linear within these regions. We use `LinearRebates` with the subdivision shown in Figure 2 to obtain a redistribution function. This piecewise linear function is composed of linear functions for each of the 4 regions

$$h(w) = \frac{2}{3}C + \begin{cases} 0 & \text{if } w \in \text{region } 0 \\ \frac{2}{3}w_1 + \frac{2}{3}w_2 - \frac{C}{3} & \text{if } w \in \text{region } 1 \\ \frac{1}{3}w_1 + \frac{2}{3}w_2 - \frac{C}{6} & \text{if } w \in \text{region } 2 \\ \frac{7}{6}w_1 + \frac{3}{2}w_2 - C & \text{if } w \in \text{region } 3 \end{cases}$$

This function can be represented more compactly. Let $s(\theta, C) = \max(\sum_i \theta_i, C)$, denote the value of the efficient outcome for agents defined by value profile θ and some total cost C. The optimal piecewise linear redistribution function is

$$h(\theta_{-i}) = \frac{5}{6}s(\theta_{-i}, C) + \frac{2}{3}s(\theta_{-i}, \frac{C}{2}) - \frac{1}{3}s(\theta^1_{-i}, \frac{C}{2}) - \frac{C}{3} \quad (10)$$

where θ^1_{-i} refers to the first element of the vector θ_{-i}.

The ratio obtained by this function is $\frac{2}{3}$. However, $\frac{2}{3}$ is also the upper bound on the ratio as computed in Equation 9. This means that the rebate function we found is optimal.

We next provide an interpretation of the rebate function, which may help generalize it to more than 3 agents. In the analytical form used in Equation 10 to express the function, each region boundary of the subdivision is encoded in a single $s(\cdot)$ term. Note that, without the coefficient, the first term is the rebate agent i would receive in a normal VCG mechanism. The second term is the VCG rebate for a project with cost $\frac{C}{2}$. The first two terms are piecewise linear, with boundaries at $\sum_{j \neq i} \theta_j = C$ and $\sum_{j \neq i} \theta_j = \frac{C}{2}$, respectively. In Figure 2, these are the region-2-3 boundary and region-0-1 boundary, respectively. Finally, since we assume agents are sorted, the max-valued agent in the third term is always agent w_1, and this third term is piecewise linear, with a boundary at $\max_{j \neq i} \theta_j = \frac{C}{2}$, i.e. the region-1-2 boundary.

The next step is to generalize the rebate function above to problems with more than 3 agents. One way to do this is through finding a subdivision of the rebate space such that an optimal mechanism for this subdivision improves over the VCG mechanism. However, generalizing the subdivision in Figure 2 to 3- or higher dimensional rebate spaces proved elusive, and the question remains open.

5 Heuristic-Based Redistribution

In the previous section, we have solved for an optimal mechanism for 3 agents. However, when there are more agents, we do not yet know how to solve for the optimal mechanisms. Given this, we propose a new heuristic-based approach for designing weakly budget-balanced mechanisms with high social welfare. By using a simple sampling-based heuristic, we derive the *sampling-based redistribution (SBR)* mechanism. We show that SBR's ratio is higher than that of VCG for $n = 4, 5, 6$, and conjecture that this is still the case for $n > 6$. Both the heuristic-based approach and the SBR mechanism are general enough that they may potentially be used in settings other than public project problems.

Our approach builds on the Cavallo mechanism [3], which works as follows: We first run VCG. Besides paying the VCG payment, agent i also receives

$$\frac{1}{n} \min_{\theta'_i} VCG(\theta'_i, \theta_{-i})$$

Here, $VCG(\theta'_i, \theta_{-i})$ represents the total VCG payment for the profile under which agent i reports θ'_i, and the other agents report θ_{-i}. In words, agent i

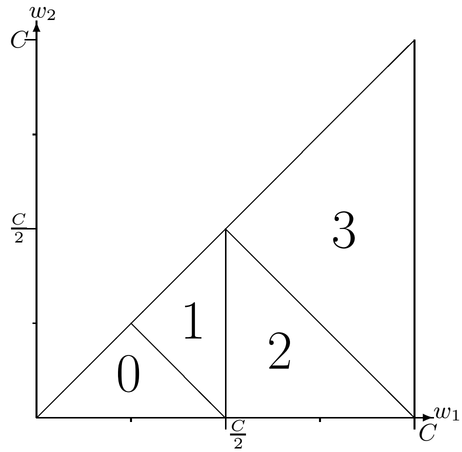

Fig. 2. Subdivision of the space θ_{-i} for 3 agents. The rebate function $h(\theta_{-i}) = h(w_1, w_2)$ is linear within each of the 4 regions.

receives $\frac{1}{n}$ times the *minimal possible* total VCG payment given that the other agents report θ_{-i}. Since the additional amount agent i receives is independent of her own type, the Cavallo mechanism is dominant-strategy incentive compatible. Then, since every agent at most receives $\frac{1}{n}$ times the actual total VCG payment, the Cavallo mechanism is weakly budget-balanced. In many settings (*e.g.*, resource allocation with free disposal and public good provision), VCG is pay-only. In these settings, the additional amount an agent receives is nonnegative. Unfortunately, for our model, the additional amount an agent receives is always 0.[2] That is, the Cavallo mechanism always coincides with VCG.

Our heuristic-based approach works as follows:

- We start with a dominant-strategy incentive-compatible mechanism (*e.g.*, VCG). Let $P(\theta)$ be the total payment under this mechanism for profile θ.
- Besides paying the payment under the initial mechanism, agent i also receives

$$\frac{1}{n} EM(\theta_{-i})$$

  Here, $EM(\theta_{-i})$ represents agent i's estimation of the total payment under the initial mechanism, given that the others report θ_{-i}. Agent i's estimation should not depend on her own report, which is to maintain dominant-strategy incentive compatibility. The estimation function EM can be based on any heuristic. (One naive choice would be $EM(\theta_{-i}) = P(0, \theta_{-i})$, which uses the total payment assuming $\theta_i = 0$ to be the estimation.) The goal of this step is to modify the initial mechanism, so that it becomes as close to strong budget

[2] $\min_{\theta'_i} VCG(\theta'_i, \theta_{-i})$ is always 0 [1]: if $\sum_{j \neq i} \theta_j \geq \frac{n-1}{n} C$, then set θ'_i to be C; otherwise, set θ'_i to be 0.

balance as possible. Generally, we cannot achieve perfect budget balance. That is, even if EM is based on a good heuristic, the mechanism at this point still incurs some small amount of waste or deficit, depending on the profile.
- To ensure weak budget balance, we finally collect from every agent $\frac{1}{n}$ times the *maximum possible* deficit, given the heuristic that we use (EM) and given the other agents' reports. Formally, we collect from agent i the following amount:

$$\frac{1}{n}\max_{\theta'_i}\{\sum_j \frac{1}{n}EM(\hat{\theta}_{-j}) - P(\hat{\theta})\}$$

Here, $\hat{\theta}$ represents the profile (θ'_i, θ_{-i}). It should be noted that this step is based on exactly the idea behind the Cavallo mechanism. Dominant-strategy incentive compatibility is maintained because the amount we charge from an agent does not depend on her own report. Furthermore, since the total amount we charge is never less than the actual deficit, the resulting mechanism is weakly budget-balanced.

We start with VCG, by using a simple sampling-based heuristic, we obtain a specific mechanism, which we call the *sampling-based redistribution (SBR)* mechanism. In detail, to estimate the total VCG payment given the others' report θ_{-i}, we just assume that agent i's type is drawn uniformly at random from θ_{-i}, and then use the expected total VCG payment as the estimation. Formally, EM is defined as follows:

$$EM(\theta_{-i}) = \frac{\sum_{j\neq i} VCG(\theta_j, \theta_{-i})}{n-1}$$

Next, we show how to derive a lower bound on the ratio of SBR. Without loss of generality, we let $C = 1$.

The social welfare under SBR is:

$$s(\theta) - VCG(\theta) + \sum_i \frac{1}{n}EM(\theta_{-i}) - \sum_i \frac{1}{n}\max_{\theta'_i}\{\sum_j \frac{1}{n}EM(\hat{\theta}_{-j}) - VCG(\hat{\theta})\}$$

We have:

$$-VCG(\theta) + \sum_i \frac{1}{n}EM(\theta_{-i}) \geq \min_{\theta}\{\sum_i \frac{1}{n}EM(\theta_{-i}) - VCG(\theta)\}$$

Also,

$$\sum_i \frac{1}{n}\max_{\theta'_i}\{\sum_j \frac{1}{n}EM(\hat{\theta}_{-j}) - VCG(\hat{\theta})\} \leq \sum_i \frac{1}{n}\max_{\theta}\{\sum_j \frac{1}{n}EM(\theta_{-j}) - VCG(\theta)\}$$

$$= \max_{\theta}\{\sum_i \frac{1}{n}EM(\theta_{-i}) - VCG(\theta)\}$$

We use $EMVCG(\theta)$ to denote $\sum_i \frac{1}{n} EM(\theta_{-i}) - VCG(\theta)$. The social welfare under SBR is then at least:

$$s(\theta) + \min_\theta EMVCG(\theta) - \max_\theta EMVCG(\theta)$$

The ratio of SBR is then:

$$\frac{s(\theta) + \min_\theta EMVCG(\theta) - \max_\theta EMVCG(\theta)}{s(\theta)}$$

$$\geq 1 + \min_\theta EMVCG(\theta) - \max_\theta EMVCG(\theta)$$

(We recall that $s(\theta)$ is at least $C = 1$.)

Given n, $\min_\theta EMVCG(\theta)$ and $\max_\theta EMVCG(\theta)$ are constants. For small n, we can numerically solve for their values. Specifically, instead of minimizing/maximizing over all possible profiles, we only consider profiles where every agent's report is an integer multiple of $1/N$. Larger values of N generally correspond to more accurate results. We notice that as long as N is a multiple of $2n$ (*e.g.*, $N = 2n$, $N = 4n$, $\ldots$, $N = 100n$), we always end up with the same maximizing/minimizing profiles. To double check, for every maximizing/minimizing profile obtained, we generate 10,000 random vectors, and perturb the profile along these 10,000 directions. At the end, no perturbation ever leads to higher maximum or lower minimum. The results are presented in the following table. We only considered $n \leq 6$ due to the exponential complexity of this approach.

	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$\max_\theta EMVCG(\theta)$	$2/9$	$1/4$	$6/25$	$2/9$
$\arg\max_\theta EMVCG(\theta)$	$(1,1,0)$	$(1,1,0,0)$	$(1,1,0,0,0)$	$(1,1,0,0,0,0)$
$\min_\theta EMVCG(\theta)$	$-4/9$	$-19/48$	$-2/5$	$-23/60$
$\arg\min_\theta EMVCG(\theta)$	$(1,0,0)$	$(5/8,3/8,0,0)$	$(3/5,2/5,0,0,0)$	$(7/12,5/12,0,0,0,0)$
$1 + \min_\theta EMVCG(\theta)$ $- \max_\theta EMVCG(\theta)$	$1/3$ ≈ 0.333333	$17/48$ ≈ 0.354167	$9/25$ ≈ 0.360000	$71/180$ ≈ 0.394444

There are two interesting observations. First, at least for $3 \leq n \leq 6$, the lower bound of the ratio of SBR increases as n increases. We conjecture that this trend remains when n is greater than 6. Second, when $n = 3$, the lower bound of the ratio of SBR is the same as VCG's ratio ($1/n$), and when $4 \leq n \leq 6$, the lower bound of the ratio of SBR is higher than VCG's ratio.

Finally, it should be noted that even though we do not know how to estimate the ratio of SBR when $n > 6$, we do know that SBR is always dominant-strategy incentive-compatible and weakly budget-balanced. Also, SBR's payments are computationally easy to calculate. Therefore, we can always apply it. It is just that for $n > 6$, we do not know how well it will perform. We tried to experimentally evaluate the ratio of SBR for larger values of n. For example, for $n = 10$, we randomly generated 1,000,000 profiles (every agent's type is drawn from *i.i.d.* uniform distribution from 0 to 1). For these profiles, the worst-case ratio of SBR is around 0.850. However, 1,000,000 is hardly a large enough sample size, because for these same set of profiles, the worst-case ratio of VCG is around 0.827, which we know is much higher than its actual ratio $1/n = 0.1$.

6 Extending the Solution for Values Below C to All Values

So far we have assumed that the agents' values are bounded from above by C. In this section, we show that this assumption is without loss of generality. Basically, if we can solve for a weakly budget-balanced mechanism with ratio r in the restricted setting where the agents' values are bounded from above by C, then we can extend this mechanism to cover all values, and achieve the same ratio. If a mechanism is optimal in the restricted setting where the agents' values are bounded from above by C, then the extended mechanism is also optimal in the more general setting where the agents' values are not bounded from above.

Let h be a feasible solution of the original model (the one with the assumption that the agents' values are bounded from above by C), and let r be the ratio achieved by h ($0 \leq r \leq 1$). Then, h together with r must satisfy the following constraints:

$$(n-r)s(\theta) \geq \sum_i h(\theta_{-i}) \geq (n-1)s(\theta) \quad \forall\, \theta \in \Theta$$

We introduce the following notation to convert values that are not bounded from above into values bounded from above by C:

$$\bar{\theta} = (\min\{\theta_1, C\}, \ldots, \max\{\theta_n, C\})$$

The values marked with the "bar" are capped at C. We construct h' as follows:

$$h'(\theta_{-i}) = \sum_{j \neq i} (\theta_j - \bar{\theta}_j) + h(\bar{\theta}_{-i})$$

It turns out that h' corresponds to a mechanism that is weakly budget-balanced and has ratio r even if we allow the agents' values to be greater than C. To show this, we need to prove that h' together with r satisfy the following:

$$(n-r)s(\theta) \geq \sum_i h'(\theta_{-i}) \geq (n-1)s(\theta) \quad \forall\, \theta \in \{\theta \in [0,\infty)^n \mid \theta_1 \geq \ldots \geq \theta_n \geq 0\}$$

Since h' coincides with h when θ_i are bounded from above by C, we only need to consider scenarios where $\theta_1 \geq C$. That is, we only need to prove:

$$(n-r)s(\theta) \geq \sum_i h'(\theta_{-i}) \geq (n-1)s(\theta) \quad \forall\, \theta \in \{\theta \in [0,\infty)^n \mid \theta_1 \geq C, \theta_1 \geq \ldots \geq \theta_n \geq 0\}$$

Again, since h' coincides with h when θ_i are bounded from above by C, we have:

$$\forall\, \theta \in \{\theta \in [0,\infty)^n \mid \theta_1 \geq C, \theta_1 \geq \ldots \geq \theta_n \geq 0\}$$

$$(n-r)s(\bar{\theta}) \geq \sum_i h'(\bar{\theta}_{-i}) = \sum_i h(\bar{\theta}_{-i}) \geq (n-1)s(\bar{\theta})$$

Now, if $\theta_1 \geq C$, then $s(\theta) = \sum_i \theta_i$ and $s(\bar{\theta}) = \sum_i \bar{\theta}_i$. That is, $s(\theta) = s(\bar{\theta}) + \sum_i(\theta_i - \bar{\theta}_i)$. Adding $(n-1)\sum_i(\theta_i - \bar{\theta}_i)$ to every term in the above inequality, after simplification, we get:

$$(1-r)s(\bar{\theta}) + (n-1)s(\theta) \geq \sum_i h'(\theta_{-i}) \geq (n-1)s(\theta)$$

Finally, since $s(\theta) \geq s(\bar{\theta})$, we obtain the required:

$$(n-r)s(\theta) \geq \sum_i h'(\theta_{-i}) \geq (n-1)s(\theta) \quad \forall\, \theta \in \{\theta \in [0,\infty)^n \mid \theta_1 \geq C, \theta_1 \geq \ldots \geq \theta_n \geq 0\}$$

7 Conclusions and Future Work

Public good provision is a fundamental problem in economic theory. However, unlike various allocation models, optimal Groves mechanisms (that is, optimal efficient and truthful mechanisms) for public good settings have not previously been considered. Against this background, we provided the first results for this problem. Specifically, we derived an upper bound on the best possible efficiency ratio, successfully characterized the optimal mechanism for 3 agents, and presented a new heuristic-based approach to designing weakly budget-balanced mechanisms with high social welfare.

The question of deriving an optimal mechanism for more than 3 agents remains open for future research. Another interesting direction is to consider public good problems where the choice involves multiple possible projects.

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Simultaneous Single-Item Auctions

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Abstract. In a combinatorial auction (CA) with item bidding, several goods are sold simultaneously via single-item auctions. We study how the equilibrium performance of such an auction depends on the choice of the underlying single-item auction. We provide a thorough understanding of the price of anarchy, as a function of the single-item auction payment rule.

When the payment rule depends on the winner's bid, as in a first-price auction, we characterize the worst-case price of anarchy in the corresponding CAs with item bidding in terms of a sensitivity measure of the payment rule. As a corollary, we show that equilibrium existence guarantees broader than that of the first-price rule can only be achieved by sacrificing its property of having only fully efficient (pure) Nash equilibria.

For payment rules that are independent of the winner's bid, we prove a strong optimality result for the canonical second-price auction. First, its set of pure Nash equilibria is always a superset of that of every other payment rule. Despite this, its worst-case POA is no worse than that of any other payment rule that is independent of the winner's bid.

1 Introduction

The problem of allocating multiple heterogeneous goods to a number of competing buyers is well motivated, notoriously difficult in practice, and, when buyers' preferences are private (i.e., unknown to the seller), central to the study of *algorithmic mechanism design*. More precisely, suppose there are m goods and each buyer i has a private *valuation* v_i that assigns a value $v_i(S)$ to each bundle (i.e., subset) S of goods. For example, each good could represent a license for exclusive use of a given frequency range in a given geographic area, buyers could correspond to mobile telecommunication companies, and valuations then describe a company's willingness to pay for a given collection of licenses [6]. One natural objective function, for example when the seller is the government, is *welfare maximization*: partition the goods into bundles $S_1, \ldots, S_n$, with S_i denoting the goods given to buyer i, to maximize the welfare $\sum_{i=1}^{n} v_i(S_i)$.

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A *combinatorial auction* is a protocol that elicits information from buyers about their private valuations, computes an allocation of the goods, and determines who pays what. There are at least three different types of obstacles to designing good combinatorial auctions. The first problem is information-theoretic: players' valuations have size exponential in m, so eliciting full valuations is not feasible unless m is small. The second problem is computational: the welfare-maximization problem is generally NP-hard, even to approximate, even when players' valuations have succinct representations. The third problem is game-theoretic: a player is happy to misreport its preferences to manipulate a poorly-designed auction to produce an outcome that favors the player. Thus designing combinatorial auctions requires compromises — on the welfare of the computed solution, the complexity of the mechanism, or the strength of the incentive-compatibility guarantee.

Most previous work on combinatorial auctions in the theoretical computer science literature focuses on *truthful approximation mechanisms* [3]. Such mechanisms run in time polynomial in n and m (with oracle access to players' valuations) and satisfy a very strong incentive-compatibility guarantee: for every player, reporting its true preferences in the auction is a dominant strategy (i.e., maximizes its utility, no matter what the other players do). The benefits of such mechanisms are clear: they require minimal work from and make minimal behavioral assumptions on the players, and are computationally tractable. They suffer from two major drawbacks, however. The first is that the strong requirement of a dominant-strategy implementation severely restricts what is possible: even for the relatively well-behaved class of submodular valuations,[1] no truthful approximation mechanism achieves a sub-polynomial approximation factor [7,9]. The second is that, even for settings where good truthful approximation mechanisms exist, these mechanisms are often quite complicated (see e.g. [8]).

The complexity and provable limitations of dominant-strategy implementations motivate the design of combinatorial auctions that have weaker incentive guarantees, in exchange for simpler formats or better approximation factors. One natural and practical auction format that has been studied recently is *combinatorial auctions (CA) with item bidding*. In a CA with item bidding, each player submits a single bid for each item, and each item is sold independently via a single-item auction. They were first studied in [5] and [4] with second-price single-item auctions. CAs with item bidding and first-price auctions were recently studied in [12].

Combinatorial auctions with item bidding are interesting for many reasons. First, they are one of the simplest auction formats that could conceivably admit performance guarantees for non-trivial combinatorial auction problems. By construction, they do not suffer from the informational problems of most combinatorial auctions — each player is forced to summarize its entire (exponential-size) valuation for the mechanism in the form of m bids — nor from the computational problems, since the auction outcome is as trivial to compute as in a

[1] A valuation v is *submodular* if, for every pair $S \subseteq T$ of goods and good $j \notin T$, $v(T \cup \{j\}) - v(T) \leq v(S \cup \{j\}) - v(S)$.

single-item auction. Of course, there is no hope for a "truthful" implementation — players are not even granted the vocabulary to express fully their preferences — and the incentive properties of CAs with item bidding will be weaker than in dominant-strategy implementations. Second, CAs with item bidding naturally arise "in the wild". They were first studied in the AI literature [4] because trading agents are often forced to participate in them — imagine, for example, an automated travel agent responsible for acquiring a vacation package by negotiating simultaneously with hotels, airlines, and tour guides. Similarly, single-item auction sites like eBay are presumably used by some buyers to acquire several goods in parallel, even when there are non-trivial substitutes or complements among the goods [5]. Third, the recent strong lower bounds on the performance of dominant-strategy CAs [7,9] imply that further progress in algorithmic mechanism design requires the systematic study of mechanisms with weaker incentive guarantees. CAs with item bidding are a natural and well-motivated starting point for this exploration. Fourth, as discussed in [12], equilibria in CAs with item bidding can be thought of as generalizations of price equilibria in settings with indivisible goods, where a conventional (i.e., Walrasian) price equilibrium need not exist.

The properties of a CA with item bidding depend on the format choice for the underlying single-item auctions. For example, CAs with item bidding and first-price auctions have Nash equilibria (in pure strategies) in strictly fewer settings than with second-price auctions; but Nash equilibria with first-price auctions are always welfare-maximizing, while those with second-price auctions are not [5,12].

The goal of this paper is to understand how the equilibrium set of a CA with item bidding depends on the format choice for its constituent single-item auctions.

(Q1) *How does the equilibrium performance of a combinatorial auction depend on the choice of the underlying single-item auction?*

(Q2) *Is there an "optimal" single-item auction for CAs with item bidding? Is there a single-item auction that shares the benefits of both the first- and second-price auctions?*

1.1 Our Results

We provide a thorough understanding of the price of anarchy of pure Nash equilibria, when such equilibria exist, in CAs with item bidding, as a function of the single-item auction payment rule. When the payment rule depends on the winner's bid (like in a first-price auction), we characterize the worst-case price of anarchy in the corresponding CAs with item bidding in terms of a "sensitivity measure" of the payment rule. As a corollary, we derive the following "undominated" property of the first-price payment rule: the *only way* to have broader equilibrium existence guarantees is to sacrifice the property of having only fully efficient equilibria.

For payment rules that are independent of the winner's bid, we prove a strong optimality result for the canonical second-price auction. First, its set of pure

Nash equilibria is always a superset of that of every other payment rule. Despite this, its worst-case POA is no worse than that of any other payment rule that is independent of the winner's bid.

1.2 Related Work

The literature on combinatorial auctions is too big to survey here; see the book [6] and book chapter [3] for general information on the topic. Related work on combinatorial auctions with item bidding, also mentioned above, are [2,5,21] for second-price auctions and [12] for first-price auctions. An alternative simple auction format is sequential (rather than simultaneous) single-item auctions; the price of anarchy in such auctions was studied recently in [16,20]. Most other work in theoretical computer science on combinatorial auctions has focused on truthful, dominant-strategy implementations (see [3]), with [1] being a notable exception.

A less obviously related paper is by Fu et al. [10]. This paper introduces the concept of a conditional equilibrium. Lavi (personal communication) showed that a conditional equilibrium exists for a valuation profile if and only if a "conservative" equilibrium (defined below) exists in the corresponding CA with item bidding with the second-price payment rule. The paper shows that, for every valuation profile, every conditional equilibrium has welfare at least 1/2 times that of an optimal allocation.

Finally, several previous works [13,19,18] consider the independent private values model and study how the Bayes-Nash equilibrium of a single-item auction varies with the choice of payment rule.

2 Preliminaries

Combinatorial Auctions. In a combinatorial auction (CA), there is a set of n players and a set M of m goods (or items). Each player i has a *valuation* $v_i : 2^M \to \mathbb{R}^+$ that describes its value for each subset of the goods. We always assume that $v_i(\emptyset) = 0$ and $v_i(S) \leq v_i(T)$ for all $S \subseteq T$. The *social welfare* $SW(\mathbf{X})$ of an allocation $\mathbf{X} := \{X_1, X_2, \ldots, X_n\}$ of the goods to the players is $\sum_{i=1}^{n} v_i(X_i)$.

For a valuation profile $\mathbf{v} = \{v_1, v_2, \ldots, v_n\}$, we denote the welfare-maximizing allocation by $OPT(\mathbf{v})$.

Item Bidding. In a CA with item bidding, each player i submits m bids, one for each good. Each good is allocated to the highest bidder at a price given by a payment rule p. We denote such a mechanism by $\mathcal{M}_p$.

For a fixed mechanism, we use $X_i(\mathbf{b})$ to denote the goods allocated to player i in the bid profile $\mathbf{b}$ and $SW(\mathbf{b}) = \sum_{i=1}^{n} v_i(X_i(\mathbf{b}))$ the social welfare of the resulting allocation. Player i's utility in a bid profile $\mathbf{b} = (b_1, b_2, \ldots b_n)$ is

$$u_i(\mathbf{b}) = v_i(X_i(\mathbf{b})) - \sum_{j \in X_i(\mathbf{b})} p_i(b_1(j), b_2(j), \ldots b_n(j)).$$

Payment Rules. We consider payment rules that meet the following natural conditions. We assume that the payment rule is anonymous. For such a payment rule p, the winner's payment when the bids are $x_1 \geq x_2 \geq \ldots \geq x_n$ is denoted by $p(x_1, x_2, \ldots, x_n)$. We further assume that the payment function is non-decreasing: raising bids can only increase the price charged to the winner. Finally, we assume that the payment function is continuous in every bid. For example, every payment rule given by a convex combination of the bids satisfies all of these assumptions.

For convenience, we also assume that the payment rule is not bounded or constant, and that the minimum price $p(0, 0, \ldots 0)$ is 0. As we show in the full version, payment rules that do not meet these assumptions are uninteresting — either there are never any equilibria, or such equilibria can be arbitrarily inefficient.

Auctions as Games. Players generally have no dominant strategies in a CA with item bidding, and we study the performance of an auction via the equilibria of the corresponding bidding game. In this paper, we focus on a full-information model, where players' valuations are publicly known, and on pure Nash equilibria. Recall that for a fixed valuation profile $\mathbf{v} = (v_1, v_2, \ldots v_n)$, a bid profile $\mathbf{b} = (b_1, b_2, \ldots b_n)$ is a *(pure) Nash equilibrium* if $u_i(\mathbf{b}) \geq u_i(b'_i, \mathbf{b}_{-i})$ for every player i and (feasible) deviation b'_i, where $(b'_i, \mathbf{b}_{-i})$ denotes the bid profile in which player i bids b'_i and all other players bid according to $\mathbf{b}$.

The *price of anarchy (POA)* is the ratio of the social welfare of an optimal allocation and that of the worst Nash equilibrium:

$$\text{POA} = \max_{\mathbf{b}\text{: a pure Nash eq.}} \frac{SW(OPT(\mathbf{v}))}{SW(\mathbf{b})}. \tag{1}$$

The POA is undefined when no equilibria exist.

3 Winner-Dependent Payment Rules

3.1 Overview

This section considers *winner-dependent* payment rules, such as the first-price rule, where the winner's payment is strictly increasing in its bid. The key property shared by such rules is that, in an equilibrium, the winner must bid the minimum amount required to win.

Are there winner-dependent payment rules that are "better" than the first-price rule? A drawback with CAs with item bidding and the first-price rule is that equilibria often fail to exist. Precisely, recall that a *Walrasian equilibrium* for a valuation profile is a set of prices $p_1, \ldots, p_m$ on the goods and a feasible allocation $(S_1, S_2, \ldots S_n)$ of the goods to the players so that each player obtains a bundle that maximizes its utility (i.e., value minus price). We say that a valuation profile is *Walrasian* if it admits a Walrasian equilibrium and *non-Walrasian* otherwise. Walrasian equilibria always exist when valuations meet the gross substitutes property, but not generally otherwise (see [11,14]). The pure Nash equilibria of a CA with item bidding and the first-price payment rule correspond to the Walrasian equilibria (if any) in a natural way, and are fully efficient when they exist [12].

Other winner-dependent payment rules can yield CAs with item bidding that possess equilibria *even in non-Walrasian instances*. We give an explicit example in the full version, for the payment rule that averages the highest and third-highest bids. This observation motivates the question: is there a payment rule that strictly dominates the first-price rule? That is, is there a payment rule that induces an equilibrium in at least one non-Walrasian instance and has worst-case POA equal to 1?

We answer this question negatively in the following theorem (proved in Section 3.3).

Theorem 1. *If the worst-case POA for the mechanism $\mathcal{M}_p$ is 1, then pure Nash equilibria exists under this mechanism only in Walrasian instances.*

Thus, for every winner-dependent payment rule p, either there is an instance in which some pure Nash equilibrium of the mechanism $\mathcal{M}_p$ is not efficient, or every instance in which a pure Nash equilibrium exists is a Walrasian instance.

The main step in our proof of Theorem 1 is a characterization of the worst-case POA in CAs with item bidding and winner-dependent payment rules. For a payment rule p, we define a sensitivity measure ζ by

$$\zeta(p) = \sup_{\boldsymbol{b}:b_1=b_2\geq\ldots\geq b_n} \frac{p(b_1, \boldsymbol{b_{-n}})}{p(\boldsymbol{b})}, \tag{2}$$

where we interpret 0/0 as 1.

The denominator in (2) is the winner's payment with the bid vector $\mathbf{b}$. The numerator is the payment of the lowest bidder in $\mathbf{b}$, after it switches to bidding the minimum amount necessary to win (namely, b_1). We restrict attention to bid vectors $\mathbf{b}$ with $b_1 = b_2$ because this property is satisfied in every equilibrium under a winner-dependent rule. Because p is monotone, $p(b_1, b_{-n}) \geq p(\boldsymbol{b})$ and hence $\zeta(p) \geq 1$. Similarly, if a bidder other the lowest in $\mathbf{b}$ changes its bid to b_1, then its payment is at most the numerator in (2).

For a concrete example, consider the payment rule (first-price +2·third-price)/3. The numerator is $(b_1 + 2b_2)/3 = (b_1 + 2b_1)/3 = b_1$, while the denominator is $(b_1 + 2b_3)/3 \geq b_1/3$. In the worst case this ratio is 3, and hence $\zeta(p) = 3$.

We show in Theorem 2 that the parameter $\zeta(p)$ is exactly the worst-case POA in CAs with item bidding and the payment rule p. It follows that the POA is exactly 1 only when $\zeta(p) = 1$. We use this fact to prove Theorem 1, that a pure Nash equilibrium exists for such a payment rule only in Walrasian instances.

3.2 Characterization of Worst-Case POA

We now prove that for every winner-dependent payment rule p, the worst case POA of CAs with item bidding and rule p is exactly $\zeta(p)$. The upper bound applies to every valuation profile for which an equilibrium exists. The lower bound already applies to bidders with submodular (or even "budgeted additive") valuations.

Theorem 2. *For every winner-dependent payment rule p with $\zeta(p)$ finite, the worst-case POA of CAs with item bidding and payment rule p is precisely $\zeta(p)$. For winner-dependent payment rules with $\zeta(p) = +\infty$, there are CAs with item bidding with arbitrarily high POA.*

Proof. We first prove an upper bound of $\zeta(p)$ on the POA. Fix a valuation profile $\mathbf{v} = (v_1, v_2, \ldots v_n)$. Let $\mathbf{b} = (\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_n})$ denote an equilibrium bid profile and $\mathbf{X}(\mathbf{b}) = \{X_1(\mathbf{b}), X_2(\mathbf{b}), \ldots, X_n(\mathbf{b})\}$ the corresponding allocation. For each good j, we use $b_{j1}, b_{j2}, \ldots, b_{jn}$ to denote the sorted set of bids on that good and use $\mathbf{b}_{j,-i}$ to denote the same set with the ith bid removed. Since the payment rule is winner dependent, the winner of each good bids the minimum amount required to win, and thus $b_{j1} = b_{j2}$ for each good j. We use p_j to denote the payment $p(b_{j1}, b_{j2}, \ldots, b_{jn})$ of the winner of good j.

We first relate equilibrium payments to equilibrium welfare. Since the utility of every player in an equilibrium is non-negative, $\sum_{j \in X_i(\mathbf{b})} p_j \leq v_i(X_i(\mathbf{b}))$ for every player i. Summing over the players gives $\sum_j p_j \leq SW(\mathbf{X}(\mathbf{b}))$.

Next we relate the optimal welfare to the equilibrium utilities. Let $\mathbf{O} = (O_1, O_2, \ldots O_n)$ denote an optimal allocation. For each player i, define the bid vector a'_i as equal to $b_{j1} + \epsilon$ on each good $j \in O_i$ and zero otherwise. If player i bids a'_i, it wins at least the set O_i and pays $p(b_{j1} + \epsilon, b_{j,-i})$ on each good $j \in O_i$. Since $\mathbf{b}$ is an equilibrium bid profile, $u_i(\mathbf{b}) \geq u_i(a'_i, b_{-i}) \geq v_i(O_i) - \sum_{j \in O_i} p(b_{j1} + \epsilon, \mathbf{b}_{j,-i})$. Since this inequality holds for every $\epsilon > 0$ and the payment rule is continuous, $u_i(\mathbf{b}) \geq v_i(O_i) - \sum_{j \in O_i} p(b_{j1}, \mathbf{b}_{j,-i})$. By the definition of ζ in (2), $p(b_{j1}, \mathbf{b}_{j,-i}) \leq \zeta(p) \cdot p_j$ for every $j \in O_i$. Thus

$$u_i(\mathbf{b}) \geq v_i(O_i) - \zeta(p) \cdot \sum_{j \in O_i} p_j.$$

Next, since $v_i(X_i(\mathbf{b})) - \sum_{j \in X_i} p_j = u_i(\mathbf{b})$ for every player i, we can derive

$$\begin{aligned} SW(\mathbf{X}(\mathbf{b})) - \sum_j p_j &= \sum_{i=1}^{n} u_i(\mathbf{b}) \\ &\geq \sum_i v_i(O_i) - \zeta(p) \sum_i \sum_{j \in O_i} p_j \\ &= SW(\mathbf{O}) - \zeta(p) \sum_j p_j. \end{aligned}$$

Since $\zeta(p) \geq 1$, and $\sum_j p_j \leq SW(\mathbf{X}(\mathbf{b}))$, rearranging terms gives $\zeta(p) \cdot SW(\mathbf{X}(\mathbf{b})) \geq SW(\mathbf{O})$. This shows that the POA is at most $\zeta(p)$.

To establish the lower bound, fix $\epsilon > 0$ and set $\zeta' = \zeta(p) - \epsilon$. If $\zeta(p) = +\infty$ we can set ζ' to an arbitrarily large number. There must exist a bid vector $\boldsymbol{b}$ with $b_1 = b_2 \geq \ldots \geq b_n$ such that $\zeta' \leq p(b_1, \boldsymbol{b_{-n}})/p(\boldsymbol{b})$. Let $p_1 = p(b_1, \boldsymbol{b_{-n}})$ and let

$p_2 = p(\boldsymbol{b})$. Clearly $p_1 \geq p_2$. We construct an instance with n players where the equilibrium welfare is at most $p_2/p_1 \leq 1/\zeta'$ times that of the optimal allocation.

Consider an instance with n players and 2 goods denoted A, B. Player 1 values good A for p_1, good B for p_2 and both goods for p_1. Player 2 values good A for p_2, good B for p_1, and the two together for p_1. All other players value every subset of goods at 0. We show that the following bid profile is an equilibrium: player 1 bids (b_n, b_1), player 2 bids (b_1, b_n), and player i for $3 \leq i \leq n$ bids (b_{i-1}, b_{i-1}).

Fix a tie-breaking rule to favor player 2 over player 3 on good A and player 1 over player 3 on good B. (Note that the upper bound above is independent of the tie-breaking rule). In this bid profile, player 2 wins good A and player 1 wins good B. They both pay p_2 for the goods they win. If either of them tries to deviate to win the other good they have to pay p_1. Since their values for the good they currently win is p_2 and their value for the other good is p_1, these deviations are not profitable. No other player has an incentive to deviate.

The optimal allocation in this instance is to allocate good A to player 1 and good B to player 2. This allocation has welfare $2p_1$ while the equilibrium allocation has welfare $2p_2$. Thus the POA is at least $p_1/p_2 \geq \zeta'$. □

3.3 Proof of Theorem 1

Consider a winner-dependent payment rule p with worst-case POA equal to 1. We show that every instance for which the mechanism $\mathcal{M}_p$ has an equilibrium is a Walrasian instance.

Fix a valuation profile and an equilibrium bid profile $\mathbf{b}$ for the mechanism $\mathcal{M}_p$ with some deterministic tie-breaking rule. Let $(S_1, S_2, \ldots S_n)$ denote an equilibrium allocation and $p_1, p_2, \ldots, p_m$ the prices paid by the winner on each good. We argue by contradiction that the S_i's and p_i's form a Walrasian equilibrium.

Suppose the equilibrium allocation with prices $p_1, p_2, \ldots p_m$ is not a Walrasian equilibrium. Then there must exist a player i and a set X of goods such that $u_i(S_i, p) < u_i(X, p)$, where $u_i(S, p)$ denotes the utility $v_i(S) - \sum_{j \in S} p_j$ of player i when receiving bundle S at prices p. Let δ satisfy $0 < \delta < u_i(X, p) - u_i(S_i, p)$.

Let $b_{j1} \geq b_{j2} \geq b_{j3} \ldots \geq b_{jn}$ denote the nondecreasing set of equilibrium bids on a good j. Since the payment rule is winner-dependent, $b_{j1} = b_{j2}$ for every good j. Since the payment rule p is assumed to induce only CAs with item bidding with fully efficient equilibria, Theorem 2 implies that $\zeta(p) = 1$. This fact and the monotonicity of p imply that $p(b_{j1}, \mathbf{b}_{j,-i}) = p_j$ for every j. By the continuity of p, we can identify an ϵ such that $\sum_{j \in X} p(b_{j1} + \epsilon, \mathbf{b}_{j,-i}) - p_j \leq \delta$. Then,

$$v_i(S_i) - \sum_{j \in S_i} p_j < v_i(X) - \sum_{j \in X} p(b_{j1} + \epsilon, \mathbf{b}_{j,-i}).$$

Player i can win set X by bidding $b_{j1} + \epsilon$ on each element $j \in X$ and bidding zero on the rest, and this deviation increases its utility. This contradicts the assumption that $\mathbf{b}$ is an equilibrium bid profile and completes the proof. □

We can sharpen Theorem 1 when there are only two players. Every winner-dependent payment rule p that depends only on the two highest bids satisfies $\zeta(p) = 1$. This holds, in particular, for every winner-dependent rule in a two-player setting. From the proof of Theorem 1, we conclude the following corollary.

Corollary 1. *For every winner-dependent payment rule p and two-player instance, $\mathcal{M}_p$ has an equilibrium only if it is a Walrasian instance.*

It is easy to construct non-Walrasian two-player instances. We conclude that no winner-dependent payment rule guarantees existence in all two-player instances.

4 Winner-Independent Payment Rules

This section focuses on *winner-independent* payment rules, for which the winner's payment does not depend on its bid. We prove that among all payment rules in this class, the second-price rule has the best worst-case POA while guaranteeing equilibrium existence most often.

First, we prove that there are more pure Nash equilibria under the second-price payment rule than under any other rule. This "maximal existence: guarantee has a possible drawback, however, in the form of a larger worst-case POA bound. We show that this drawback does not materialize: the second-price rule, despite the relative profusion of equilibria, leads to a worst-case POA that is as good as with any other winner-independent rule.

4.1 γ-Conservative Equilibria

To make meaningful statements about equilibrium efficiency in CAs with item bidding and winner-independent payment rules, we need to parameterize the equilibria in some way. The reason is that every winner-independent payment rule suffers from arbitrarily bad equilibria.[2]

We consider equilibria where the players' bids satisfy a certain "conservativeness" condition. This assumption is fairly standard in the POA of auctions literature [2,5,15,17]. The conservativeness condition assumes that the equilibrium bids guarantee each player positive utility on the set it wins, even when all other players bid the same as this player. More generally, we relax this idea in two ways: parameterizing it with a parameter $\gamma \geq 1$, and applying it only to the bundles that players win in the equilibrium (rather than to all bundles they might hypothetically win). Players have the freedom to bid as high as they want on the goods they lose and can contemplate arbitrary deviations.

[2] Let $y > 0$ satisfy $p(y, y) > 0$ and consider an instance with two players and one good. Player 1 values the good at 0 and player 2 values it at $p(y, y)$. Then player 1 bidding y and player 2 bidding zero is an equilibrium and this equilibrium is clearly very inefficient.

Definition 1 (γ-conservative). *Suppose a player bidding $(b_1, b_2, \dots b_m)$ wins a set S in the equilibrium. We say that the bid is* γ-conservative *if it satisfies*

$$\sum_{j \in S} p(b_j, b_j, \dots, b_j) \leq \gamma \cdot v(S).$$

An equilibrium allocation is γ-conservative *if every player uses a γ-conservative bid in the equilibrium.*

4.2 The Second-Price Rule Has the Most Equilibria

Next we show that every γ-conservative equilibrium allocation for a payment rule p can also be realized as a γ-conservative equilibrium for the second-price rule. This transformation does not change the prices that the winners pay on the goods that they win. We use Σ_p^γ to denote the set of γ-conservative equilibrium allocations of the mechanism $\mathcal{M}_p$. In particular, $\Sigma_{s.p.}^\gamma$ denotes the set of γ-conservative equilibrium allocations of the item bidding mechanism with the second-price payment rule.

Theorem 3. *For every payment rule p, $\Sigma_p^\gamma \subseteq \Sigma_{s.p.}^\gamma$.*

Proof. We start with an equilibrium of the mechanism $\mathcal{M}_p$. Let $(S_1, S_2, \dots S_n)$ denote the allocation. Focus on a good j, and let $b_{j1} \geq b_{j2} \geq \dots \geq b_{jn}$ be the ordered bids on the good. While reasoning about individual goods we refer to the players by their rank in this ordering. The payment the winner (player 1) makes in this case is $p(b_{j1}, b_{j2}, \dots b_{jn})$. Denote this as p_{j1}.

Let $p_{j2} = p(b_{j1}, b_{j1}, \dots b_{j1})$. If any player i deviates, it will have to bid at least b_{j1} and pay at least $p(b_{j1}, b_{j,-i})$. Here $b_{j,-i}$ denotes the bids on good j by all players other than player i. Since the payment rule is monotone, this payment is at most that $p(b_{j1}, b_{j1}, \dots b_{j1}) = p_{j2}$ when all players bid b_{j1}. By monotonicity, $p_{j2} \geq p_{j1}$.

Construct an equilibrium under the second-price rule as follows. Fix a player i. On good $j \in S_i$, player i bids p_{j2}, one other player bids p_{j1}, and all other players bid zero. Note that bidding p_{j2} is feasible for player i. This is because in the given equilibrium instance for payment rule p, the players' bids on the sets they win are γ-conservative. Hence for every player i, $\sum_{j \in S_i} p_{j2} \leq \gamma \cdot v_i(S_i)$. This is the same as the γ-conservativeness condition for the second price rule, as for the second price rule when all players bid p_{j2} the payment is p_{j2} as well.

In this construction the winner's payment on a good is the same as that in the equilibrium for payment rule p. Any player currently not winning a good has to pay at least p_{j2} if it deviates to win that good. Deviations are then not profitable, as in the equilibrium for payment rule p players do not find them profitable at even lower prices. The constructed bid profile is an equilibrium for the second-price rule. The equilibrium allocation and the prices paid by the winners remain the the same. □

Theorem 3 shows that the second-price payment rule has at least as large a set of γ-conservative equilibrium allocations as any other payment rule p. We include in the full version an example showing that this inclusion can be strict.

Theorem 3 has immediate implications, both positive and negative, for all winner-independent payment rules. On the negative side, it allows us to port equilibrium non-existence results for CAs with item bidding and the second-price rule — like the fact that with subadditive valuations (where $v_i(S \cup T) \leq v_i(S) + v_i(T)$ for every player i and bundles S, T), γ-conservative equilibria need not exist (see [2] and the full version) — to those with an arbitrary winner-independent rule. On the positive side, Theorem 3 implies that POA bounds for CAs with item bidding and the second-price rule carry over to all winner-independent rules. For example, we show in the full version, by modifying a result in [2], that the POA of γ-conservative equilibria with the second-price rule is at most $\gamma + 1$ (in instances where such an equilibrium exists). Using Theorem 3, this bound holds more generally for all winner-independent rules.

4.3 POA Lower Bounds

The results of the previous section imply that, for every $\gamma \geq 1$, the POA of γ-conservative equilibria of CAs with item bidding is as bad with the second-price rule as with any other winner-independent rule. This section proves the converse, for every $\gamma \geq 1$.

Theorem 4. *For every winner-independent payment rule p, the worst-case POA of γ-conservative equilibria of $\mathcal{M}_p$ is at least $\gamma + 1$.*

We prove this theorem by establishing a stronger result: when there are only two players, the set of γ-conservative equilibrium allocations is the same for all winner-independent payment rules. The POA lower bound then follows from a lower bound construction for the second-price rule that uses only two players.

Lemma 1. *In a two-player CA with item bidding, every equilibrium of the second-price payment rule is an equilibrium of every winner-independent payment rule p.*

Proof. Consider an equilibrium under the second-price payment rule. Let S_1, S_2 denote the equilibrium allocation. Fix a player i, and suppose that on good $j \in S_i$ the player i bids b_j and pays p_j. Clearly $b_j \geq p_j$. The other player would have to bid at least b_j to win this good and would then pay b_j. The conservativeness condition for the second-price payment rule implies that for each player i, $\sum_{j \in S_i} b_j \leq \gamma \cdot v_i(S_i)$.

Since the given payment rule p is winner-independent and there are only two players, the payment only depends on the non-winning player's bid. To mimic the second-price equilibrium allocation with the mechanism $\mathcal{M}_p$, we first identify for each good a bid vector such that $p(b_{1j}, b_{1j}) = p_j$. This exists because the payment rule p is continuous and has full range. Similarly, we can identify a bid x_j such that $p(x_j, x_j) = b_j$. Since $b_j \geq p_j$, $x_j \geq b_{1j}$ Since the payment is independent of the highest bid it doesn't change if we raise the winner's bid to x_j. Hence, $p_j(x_j, b_{2j}) = p_j$.

Focus on a player i and set S_i. Set player i's bid on good j in S_i to x_j. Since x_j satisfies $p(x_j, x_j) = b_j$ and $\sum_{j \in S_i} b_j \leq \gamma \cdot v_i(S_i)$, these bids form a γ-conservative strategy for player i. The other player bids b_{1j} on each good $j \in S_i$. In case of a tie, we employ the same tie-breaking rule used in the second-price equilibrium, resulting in the tie being broken in favor of player i.

If the other player wishes to deviate to win good j it must bid at least x_j. By the choice of x_j, it would have to pay at least b_j. Since in the second-price equilibrium neither player wants to deviate when faced with the price b_j, no player wants to deviate in this constructed bid profile either. This bid profile is an equilibrium with the same allocation and payments as the given equilibrium under the second-price rule.

To complete the proof that the second-price rule has the best-possible worst-case POA of γ-conservative equilibria (for every fixed $\gamma \geq 1$), we give a two-player example with POA equal to $\gamma + 1$.

Example 1. There are two goods denoted A, B and two players. Player 1 values A for 1, B at $\gamma + 1$, and both for $\gamma + 1$. Player 2 values A for $\gamma + 1$, B for 1, and both for $\gamma + 1$.

The bid profile where player 1 bids $(\gamma, 0)$ and player 2 bids $(0, \gamma)$ is an equilibrium of the the CA with item bidding and the second-price payment rule. These bids are γ-conservative. The welfare of this equilibrium allocation is 2 while the optimal welfare is $2(\gamma + 1)$.

5 Conclusions

There are a number of opportunities for interesting further work. One important direction is to extend our study of CAs with item bidding to mixed-strategy Nash equilibria of the full-information model and to Bayes-Nash equilibria in incomplete information models. These more general equilibrium concepts are not well understood even for the second- and first-price payment rules [2,12]. A second topic is allocation rules different from the one studied here, where the highest bidder always wins. For example, can reserve prices improve the performance of CAs with item bidding in any sense? A third direction is to study systematically different single-item payment rules in sequential auctions, thereby extending the recent work in [16,20]. Finally, it would be very interesting to analyze restricted auction formats that extend simultaneous or sequential single-item auctions, such as combinatorial auctions with restricted package bidding.

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Smooth Inequalities and Equilibrium Inefficiency in Scheduling Games

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Abstract. We study coordination mechanisms for Scheduling Games (with unrelated machines). In these games, each job represents a player, who needs to choose a machine for its execution, and intends to complete earliest possible. In the paper, we focus on a general class of ℓ_k-norm (for parameter k) on job completion times as social cost, that permits to balance overall quality of service and fairness. Our goal is to design scheduling policies that always admit a pure Nash equilibrium and guarantee a small price of anarchy for the ℓ_k-norm social cost. We consider strongly-local and local policies (the policies with different amount of knowledge about jobs). First, we study the inefficiency in ℓ_k-norm social costs of a strongly-local policy SPT that schedules the jobs non-preemptively in order of increasing processing times. We show that the price of anarchy of policy SPT is $O(k^{\frac{k+1}{k}})$ and this bound is optimal (up to a constant) for all deterministic, non-preemptive, strongly-local and non-waiting policies (non-waiting policies produce schedules without idle times). Second, we consider the makespan (ℓ_∞-norm) social cost by making connection within the ℓ_k-norm functions. We present a local policy Balance. This policy guarantees a price of anarchy of $O(\log m)$, which makes it the currently best known policy among the anonymous local policies that always admit a pure Nash equilibrium.

1 Introduction

With the development of the Internet, large-scale systems consisting of autonomous decision-makers (players) become more and more important. The rational behavior of players who compete for the usage of shared resources generally leads to an unstable and inefficient outcome. This creates a need for *resource usage policies* that guarantee stable and near-optimal outcomes.

From a game theoretical point of view, stable outcomes are captured by the concept of *Nash equilibria*. Formally, in a game with n players, each player j chooses a strategy x_j from a set S_j and this induces a cost $c_j(\mathbf{x})$ for player j depending all chosen strategies $\mathbf{x}$. A strategy profile $\mathbf{x} = (x_1, \ldots, x_n)$ is a *pure Nash equilibrium* if no player can decrease its cost by a unilateral deviation,

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i.e., $c_j(x'_j, x_{-j}) \geq c_j(\mathbf{x})$ for every player j and $x'_j \in S_j$, where x_{-j} denotes the strategies selected by players different from j.

The *better-response dynamic* is the process of repeatedly choosing an arbitrary player that can improve its cost and let it take a better strategy while other player strategies remain unchanged. It is desirable that in a game the better-response dynamic converges to a Nash equilibrium as it is a natural way that selfish behavior leads the game to a stable outcome. A *potential game* is a game in which for any instance, the better-response dynamic always converges [10].

A standard measure of inefficiency is the *price of anarchy (PoA)*. Given a game with an objective function and a notion of equilibrium (e.g pure Nash equilibrium), the PoA of the game is defined as the ratio between the largest cost of an equilibrium and the cost of an optimal profile, which is not necessarily an equilibrium. The PoA captures the worst-case paradigm and it guarantees the efficiency of every equilibrium.

The social cost of a game is an objective function measuring the quality of strategy profiles. In the literature there are two main extensively-studied objective functions: (i) the *utilitarian social cost* is the total individual costs; while (ii) the *egalitarian social cost* is the maximum individual cost. The two objective functions are included in a general class of social costs: the class of ℓ_k norms of the individual costs, with utilitarian and the egalitarian social costs corresponding to the cases $k = 1$ and $k = \infty$, respectively. There is a need to design policies that guarantee the efficiency (e.g the PoA) of games under some specific objective function. Moreover, it would be interesting to come up with a policy, that would be efficient for every social costs from this class. Note that the optimum is defined as the strategy profile minimizing the social cost. As such it depends on the fixed norm but not on the scheduling policy.

1.1 Coordination Mechanisms in Scheduling Games

In a scheduling game, there are n jobs and m unrelated machines. Each job needs to be scheduled on exactly one machine. We consider the unrelated parallel machine model, where each machine could be specialized for a different type of jobs. In this general setting, the processing time of job j on machine i is some given arbitrary value $p_{ij} > 0$. A strategy profile $\mathbf{x} = (x_1, \ldots, x_n)$ is an assignment of jobs to machines, where x_j denotes the machine (strategy) of job j in the profile. The *cost* c_j of a job j is its completion time and every job strategically chooses a machine to minimize the cost. In the game, we consider the social cost as the ℓ_k-norm of the individual costs. The *social cost* of profile $\mathbf{x}$ is $C(\mathbf{x}) = \left(\sum_j c_j^k\right)^{1/k}$.

The traditional ℓ_1, ℓ_∞-norms represent the total completion time and the makespan, respectively. Both objectives are natural. Minimizing the total completion time guarantees a quality of service while minimizing the makespan ensures the fairness of schedule. Unfortunately, in practice schedules which optimize the total completion time are not implemented due to a lack of fairness. Implementing a fair schedule is one of the highest priorities in most systems [16].

A popular and practical method to enforce the fairness of a schedule is to optimize the ℓ_k-norm of completion times for some fixed k. By optimizing the ℓ_k-norm of completion time, one balances overall quality of service and fairness, which is generally desirable. So the system takes into account a trade-off between quality of service and fairness by optimizing the ℓ_k-norm of completion time [14,16].

A *coordination mechanism* is a set of *scheduling policies*, one for each machine, that determine how to schedule the jobs assigned to a machine. The idea is to connect the individual cost to the social cost, in such a way that the selfishness of the agents will lead to equilibria with small social cost. We distinguish between *local* and *strongly-local* policies. These policies are classified in the decreasing order of the amount of information that ones could use for their decisions. Formally, let $\mathbf{x} = (x_1, \ldots, x_n)$ be a profile.

- A policy is *local* if the scheduling of jobs on machine i depends only on the processing times of jobs assigned to the machine, i.e., $\{p_{i'j} : x_j = i, 1 \leq i' \leq m\}$.
- A policy is *strongly-local* if the policy of machine i depends only on the processing times for this machine i for all jobs assigned to i, i.e., $\{p_{ij} : x_j = i\}$.

In addition, a policy is *anonymous* if it does not use any *global* ordering of jobs or any *global* job identities. Note that for any deterministic policy, *local* job identities are necessary as a machine may need such information in order to break ties (a job may have different identities on different machines). Moreover, we call a policy *non-waiting* if the schedule contains no idle time between job executions.

Instead of specifying the actual schedule, we rather describe a scheduling policy as a function, mapping every job j to some completion time $c_j(\mathbf{x})$. Such a policy is said *feasible* if for any profile $\mathbf{x}$, there exists a schedule where job j completes at time $c_j(\mathbf{x})$. Formally, for any job j, we must have $c_j(\mathbf{x}) \geq \sum_{j'} p_{ij'}$ where the sum is taken over all jobs j' with $x_j = x_{j'}$ and $c_{j'}(\mathbf{x}) \leq c_j(\mathbf{x})$. Certainly, any designed deterministic policy needs to be feasible.

1.2 Overview and Contributions

Recently, Roughgarden [12] developed the *smoothness argument*, a unifying method to show upper bounds of the PoA for utilitarian games. This canonical method is elegant in its simplicity and its power. Here we give a brief description of this argument.

A cost-minimization game with the total cost objective $C(\mathbf{x}) = \sum_j c_j(\mathbf{x})$ is (λ, μ)-*smooth* if for every profile $\mathbf{x}$ and $\mathbf{x}^*$,

$$\sum_j c_j(x_j^*, x_{-j}) \leq \mu \sum_j c_j(\mathbf{x}) + \lambda \sum_j c_j(\mathbf{x}^*)$$

The smooth argument [12] states that the robust price of anarchy (including the PoA of pure, mixed, correlated equilibria, etc) of a cost-minimization game is bounded by

$$\inf\left\{\frac{\lambda}{1-\mu} : \lambda \geq 0, \mu < 1, \text{ the game is } (\lambda,\mu)\text{-smooth}\right\}.$$

We will make use of this argument to settle the equilibrium inefficiency in scheduling games. We will prove the robust PoA by applying the smooth argument to the game with $C^k(\mathbf{x}) = \sum_j c_j^k(\mathbf{x})$ where $C(\mathbf{x})$ is the ℓ_k-norm social cost of Scheduling Games. The main difficulty in applying the smooth argument to Scheduling Games has arisen from the fact that jobs on the same machine have different costs, which is in contrast to Congestion Games [11] where players incur the same cost at the same resource. The key technique in this paper is a system of inequalities, called *smooth inequalities*, that are useful to prove the smoothness of the game.

Our contributions are the following:

1. We study the equilibrium inefficiency for the ℓ_k-norm objective function. We consider a strongly-local policy SPT that schedules the jobs non-preemptively in order of increasing processing times (with a deterministic tie-breaking rule for each machine)[1]. We prove that the PoA of the game under the deterministic strongly-local policy SPT is at most $O(k^{\frac{k+1}{k}})$. Moreover, we show that any deterministic non-preemptive, non-waiting and strongly-local policy has a PoA at least $\Omega(k^{\frac{k+1}{k}})$, which matches to the performance of SPT policy. Hence, for any ℓ_k-norm social cost, SPT is optimal among deterministic non-preemptive, non-waiting, strongly-local policy. (The cases $k = 1$ and $k = \infty$ are confirmed in [6] and [2,9], respectively.) If one considers theoretical evidence to classify algorithms for practical use then SPT is a good candidate due to its simplicity and theoretically guaranteed performance on any combination of the quality and the fairness of schedules.
2. We study the equilibrium inefficiency for the makespan objective function (e.g., ℓ_∞-norm) for local policies by making connection between ℓ_k-norm functions. We present a policy Balance (definition is given is Section 4). The game under that policy always admits Nash equilibrium and induces the PoA of $O(\log m)$ — the currently best performance among anonymous local policies that always possess pure Nash equilibria.

Our results naturally extend to the case when jobs have weights and the objective is the ℓ_k-norm of weighted completion times, i.e., $(\sum_j (w_j c_j(\mathbf{x}))^k)^{1/k}$.

1.3 Related Results

The smooth argument has been formalized in [12]. It has been used to establish tight PoA of congestion games [11], a fundamental class of games. The argument is also applied to prove bounds on the PoA of weighted congestion games [3]. Subsequently, Roughgarden and Schoppman [13] have extended the argument to prove tight bounds on the PoA of atomic splittable congestion games for a large class of latencies.

[1] Formal definition of SPT is given in Section 3

Coordination mechanisms for scheduling games were introduced in [5] where the makespan (ℓ_∞-norm) objective was considered. For strongly-local policies, Immorlica et al. [9] gave a survey on the existence and inefficiency of different policies such as SPT, LPT, RANDOM. Some tight bounds on the PoA under different policies were given. Azar et al. [2] initiated the study on local policies. They designed a non-preemptive policy with PoA of $O(\log m)$. However, the game under that policy does not necessarily guarantee a Nash equilibrium. The authors modified the policy and gave a preemptive one that always admits an equilibrium with a larger PoA as $O(\log^2 m)$. Subsequently, Caragiannis [4] derived a non-anonymous local policy ACOORD and anonymous local policies BCOORD and CCOORD with PoA of $O(\log m)$, $O(\log m/\log\log m)$ and $O(\log^2 m)$, respectively where the first and the last ones always admit a Nash equilibrium. Fleischer and Svitkina [7] showed a lower bound of $\Omega(\log m)$ for all deterministic non-preemptive, non-waiting local policies. Recently, Abed and Huang [1] proved that every deterministic (even preemptive) local policy, that satisfies natural properties, has price of anarchy at least $\Omega(\log m/\log\log m)$.

Cole et al. [6] studied the game with total completion time (ℓ_1-norm) objective. They considered strongly-local policies with weighted jobs, and derived a non-preemptive policy inspired by the Smith's rule which has PoA = 4. This bound is tight for deterministic non-preemptive non-waiting strongly-local policies. Moreover, some preemptive policies are also designed with better performance guarantee.

1.4 Organization

In Section 2, we state some smooth inequalities that will be used in settling the PoA for different policies. In Section 3, we study the scheduling game with the ℓ_k-norm social cost. We define and prove the inefficiency of the policiy SPT. We also provide a lower bound on the PoA for any deterministic non-preemptive non-waiting strongly-local policy. In Section 4, we consider the makespan (ℓ_∞-norm) social cost for local policies. We define and analyze the performance of policy Balance. Due to the space constraint, some proofs are given in the appendix.

2 Smooth Inequalities

In the section we show various inequalities that are useful for the analysis.

Lemma 1. *Let k be a positive integer. Let $0 < a(k) \leq 1$ be a function on k. Then, for any $x, y > 0$, it holds that*

$$y(x+y)^k \leq \frac{k}{k+1} a(k) x^{k+1} + b(k) y^{k+1}$$

where α is some constant and

$$b(k) = \begin{cases} \Theta\left(\alpha^k \cdot \left(\frac{k}{\log ka(k)}\right)^{k-1}\right) & \text{if } \lim_{k\to\infty}(k-1)a(k) = \infty, \quad (1a) \\ \Theta\left(\alpha^k \cdot k^{k-1}\right) & \text{if } (k-1)a(k) \text{ are bounded } \forall k, \quad (1b) \\ \Theta\left(\alpha^k \cdot \frac{1}{ka(k)^k}\right) & \text{if } \lim_{k\to\infty}(k-1)a(k) = 0. \quad (1c) \end{cases}$$

Note that the case (1a) of Lemma 1 could be used to settle the tight bound on the PoA of Congestion Games in which delay functions are polynomials with positive coefficients. [15] proved this case for $a(k) = 1$ and $b(k) = \Theta(\frac{1}{k}(k/\log k)^k)$ in order to upper bound of the PoA in Selfish Load Balancing Games.

Lemma 2. *It holds that $(k+1)z \geq 1-(1-z)^{k+1}$ for all $0 \leq z \leq 1$ and for all $k \geq 0$.*

Proof. Consider $f(z) = (k+1)z - 1 + (1-z)^{k+1}$ for $0 \leq z \leq 1$. We have $f'(z) = (k+1)-(k+1)(1-z)^k \geq 0\ \forall 0 \leq z \leq 1$. So f is non-decreasing function, thus $f(z) \geq f(0) = 0$. Therefore, $(k+1)z \geq 1-(1-z)^{k+1}$ for all $0 \leq z \leq 1$. □

In the following, we prove inequalities to bound the PoA of the scheduling game. Remark that until the end of the section, we use i, j as the indices. The following is the main lemma to show the upper bound $O(k^{(k+1)/k})$ of the PoA under policy SPT in the next section.

Lemma 3. *For any non-negative sequences $(n_i)_{i=1}^P$, $(m_i)_{i=1}^P$, and for any positive increasing sequence $(q_i)_{i=1}^P$, define $A_{i,j} := n_1q_1 + \ldots + n_{i-1}q_{i-1} + j \cdot q_i$ for $1 \leq i \leq P, 1 \leq j \leq n_i$ and $B_{i,j} := m_1q_1 + \ldots + m_{i-1}q_{i-1} + j \cdot q_i$ for $1 \leq i \leq P, 1 \leq j \leq m_i$. Then, it holds that*

$$\sum_{i=1}^{P}\sum_{j=1}^{m_i}(A_{i,n_i} + j \cdot q_i)^k \leq \mu_k \sum_{i=1}^{P}\sum_{j=1}^{n_i} A_{i,j}^k + \lambda_k \sum_{i=1}^{P}\sum_{j=1}^{m_i} B_{i,j}^k,$$

where $\mu_k = \frac{k+1}{k+2}$ and $\lambda_k = \Theta(\alpha^k(k+1)^k)$ for some constant α.

3 ℓ_k-norms of Completion Times under Strongly-Local Policies

We consider the coordination mechanism under the strongly-local policy SPT that schedules jobs in the order of non-decreasing processing times. The formal definition of that policy is the following.

Policy SPT Let $\mathbf{x}$ be a strategy profile. Let $\prec_i$ be an order of jobs on machine i, where $j' \prec_i j$ iff $p_{ij'} < p_{ij}$ or $p_{ij'} = p_{ij}$ and j is priority over j' (machine i

chooses a local preference over jobs based on their local identities to break ties). The cost of job j under the SPT [9] policy is

$$c_j(\mathbf{x}) = \sum_{\substack{j':\ x_{j'}=i \\ j' \preceq j}} p_{ij'}.$$

Note that the policy SPT is feasible. Since all p_{ij} could be written as a multiple of ϵ (a small precision) without loss of generality, assume that all jobs processing times (scaling by ϵ^{-1}) are integers and upper-bounded by P.

Lemma 4. *Let* $\mathbf{x}$ *be an assignment of jobs to machines. Then, among all feasible schedules, SPT policy minimizes the* ℓ_k*-norm of job completion times with respect to this assignment.*

Theorem 1. *The PoA of SPT with respect to the* ℓ_k*-norm of job completion times is* $O(k^{\frac{k+1}{k}})$.

Proof. Let $\mathbf{x}$ and $\mathbf{x}^*$ be two arbitrary profiles. We focus on a machine i. Let $n_1, \ldots, n_P$ be the numbers of jobs in $\mathbf{x}$ which are assigned to machine i and have processing times $1, \ldots, P$, respectively. Similarly, $m_1, \ldots, m_P$ are defined for profile $\mathbf{x}^*$. Note that n_a and m_a are non-negative for $1 \leq a \leq P$. Applying Lemma 3 for non-negative sequences $(n_a)_{a=1}^P$, $(m_a)_{a=1}^P$ and the positive increasing sequence $(a)_{a=1}^P$, we have:

$$\sum_{a=1}^{P} \left[\left(\sum_{b=1}^{a} bn_b + a \right)^k + \left(\sum_{b=1}^{a} bn_b + 2a \right)^k + \ldots + \left(\sum_{b=1}^{a} bn_b + m_a \cdot a \right)^k \right]$$

$$\leq \frac{k+1}{k+2} \cdot \sum_{a=1}^{P} \left[\left(\sum_{b=1}^{a-1} bn_b + a \right)^k + \left(\sum_{b=1}^{a-1} bn_b + 2a \right)^k + \ldots + \left(\sum_{b=1}^{a-1} bn_b + n_a \cdot a \right)^k \right]$$

$$+ \Theta\left(\alpha^k (k+1)^k\right) \cdot \sum_{a=1}^{P} \left[\left(\sum_{b=1}^{a-1} bm_b + a \right)^k + \left(\sum_{b=1}^{a-1} bm_b + 2a \right)^k + \ldots + \left(\sum_{b=1}^{a-1} bm_b + m_a \cdot a \right)^k \right]$$

where α is a constant.

Observe that, by definition of the cost under the SPT policy, the left-hand side (of the inequality above) is an upper bound for $\sum_{j:x_j^*=i} c_j^k(x_{-j}, x_j^*)$, while the right-hand side is exactly $\frac{k+1}{k+2} \cdot \sum_{j:x_j=i} c_j^k(\mathbf{x}) + \Theta\left(\alpha^k (k+1)^k\right) \cdot \sum_{j:x_j^*=i} c_j^k(\mathbf{x}^*)$. Thus,

$$\sum_{j:x_j^*=i} c_j^k(x_{-j}, x_j^*) \leq \frac{k+1}{k+2} \cdot \sum_{j:x_j=i} c_j^k(\mathbf{x}) + \Theta\left(\alpha^k (k+1)^k\right) \cdot \sum_{j:x_j^*=i} c_j^k(\mathbf{x}^*)$$

As the inequality above holds for every machine i, summing over all machines we have:

$$\sum_j c_j^k(x_{-j}, x_j^*) \le \frac{k+1}{k+2} \cdot \sum_j c_j^k(\mathbf{x}) + \Theta\left(\alpha^k (k+1)^k\right) \cdot \sum_j c_j^k(\mathbf{x}^*)$$

By the smooth argument, $C^k(\mathbf{x}) \le \left(\alpha^k (k+1)^{k+1}\right) C^k(\mathbf{x}^*)$. Therefore, we have $C(\mathbf{x}) \le O(k^{\frac{k+1}{k}}) C(\mathbf{x}^*)$.

Choosing $\mathbf{x}^*$ as an optimal assignment. By Lemma 4, the optimal schedule for this assignment could be done using the SPT policy, i.e., the optimal social cost is $C(\mathbf{x}^*)$. Therefore, the PoA is $O(k^{\frac{k+1}{k}})$. □

The following theorem proves that the bound on the PoA is tight. The construction is a generalization of the one in [6] where the authors showed a tight bound for the ℓ_1-norm.

Theorem 2. *The PoA of any deterministic non-preemptive non-waiting strongly-local policy is $\Omega(k^{\frac{k+1}{k}})$ with respect to the ℓ_k-norm of job completion times.*

Proof. Using the technique described in [6], it is sufficient to prove that the PoA of SPT is $\Omega(k^{\frac{k+1}{k}})$.

Let t and m be integers such that $m = \prod_{u=1}^t u^k$. (In fact, for the proof it is enough to choose m such that m/u^k is integer for every $1 \le u \le t$.) Consider an instance in which there are m machines and the jobs are $\{j_{u,v} : 1 \le u \le t, 1 \le v \le m/u^k\}$. A job $j_{u,v}$ has unit processing time on every machine $1 \le i \le v$ and has processing time infinity on other machines. In other words, job $j_{u,v}$ is allowed to be scheduled only on machine with index at most v. We say that a job $j_{u,v}$ has more *priority* than job $j_{u',v'}$ if $v > v'$; or if $v = v'$ and $u < u'$. If two jobs $j_{u,v}$ and $j_{u',v'}$ are both assigned to the same (allowed) machine then the job with higher priority will be scheduled before the other (note that those jobs have the same unit processing times in the machine).

We first give an assignment of jobs to machines with a small social cost. Consider an assignment $\mathbf{x}^*$ in which job $j_{u,v}$ for $1 \le u \le t, 1 \le v \le m/u^k$ is scheduled in machine v. An illustration is given in the left of Figure 1. By the priority order, the completion time of job $j_{u,v}$ is u. By the construction, the number of jobs with completion time u for $1 \le u \le t$ is m/u^k. Hence, the social cost of the assignment satisfies $C^k(\mathbf{x}^*) = \sum_{u=1}^t u^k m/u^k = mt$.

Now we construct a Nash equilibrium with high social cost. Roughly speaking, for each $1 \le s \le t$, we will assign the set of jobs $\mathcal{J}_s = \{j_{u,v} : 1 \le u \le s, m/(s+1)^k < v \le m/s^k\}$ to a subset of machines i for $1 \le i \le m/s^{k+1}$ in such a way that their completion times are between $k(s-1)+1$ and ks. Moreover, in the assignment apart of those jobs, no other has completion time in $[k(s-1)+1, ks]$. As there are t such sets $\mathcal{J}_s$ and each set gives rise to k units in the completion times, the desired lower bound follows.

Formally, fix $1 \leq s \leq t$ and consider the set $\mathcal{J}_s = \{j_{u,v} : 1 \leq u \leq s, m/(s+1)^k < v \leq m/s^k\}$. Partition $\mathcal{J}_s = \mathcal{J}_{s,1} \cup \ldots \cup \mathcal{J}_{s,k}$ where

$$\mathcal{J}_{s,a} := \left\{ j_{u,v} : 1 \leq u \leq s, \frac{m}{s^{k-a}(s+1)^a} < v \leq \frac{m}{s^{k+1-a}(s+1)^{a-1}} \right\}.$$

for $1 \leq a \leq k$. The cardinal of $\mathcal{J}_{s,a}$ is

$$|\mathcal{J}_{s,a}| = s\left(\frac{m}{s^{k+1-a}(s+1)^{a-1}} - \frac{m}{s^{k-a}(s+1)^a} \right) = \frac{m}{s^{k-a}(s+1)^a}.$$

Note that by definition, jobs in $\mathcal{J}_{s',a'}$ have higher priority then the ones in $\mathcal{J}_{s,a}$ in case $s > s'$ or in case $s = s'$ and $a > a'$. In total, there are $k \cdot t$ sets $\mathcal{J}_{s,a}$ since $1 \leq s \leq t$ and $1 \leq a \leq k$.

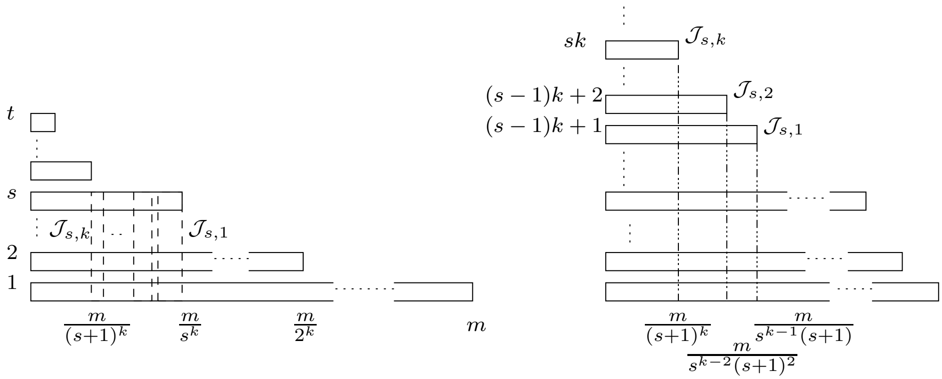

Fig. 1. Illustration of profiles $\mathbf{x}$ (in the left) and $\mathbf{x}^*$ (in the right). The horizontal and vertical axes represent machines and completion times, respectively.

Consider a profile $\mathbf{x}$ in which jobs in $\mathcal{J}_{s,a}$ for $1 \leq s \leq t$ and $1 \leq a \leq k$ are assigned arbitrarily one-to-one to machines $1, 2, \ldots, |\mathcal{J}_{s,a}|$. It is feasible since a job $j_{u,v} \in \mathcal{J}_{s,a}$ has index $v > \frac{m}{s^{k-a}(s+1)^a} = |\mathcal{J}_{s,a}|$, meaning that the job could be scheduled on any machine in $1, 2, \ldots, |\mathcal{J}_{s,a}|$. In this assignment, jobs in the same set $\mathcal{J}_{s,a}$ have the same cost, which is $(s-1)k + a$. An illustration is given in the right of Figure 1. We show that profile $\mathbf{x}$ is indeed a Nash equilibrium. Let $j_{u,v}$ be a job in $\mathcal{J}_{s,a}$. This job has cost $(s-1)k + a$ and cannot be scheduled on any machine with index larger then $\frac{m}{s^{k+1-a}(s+1)^{a-1}}$. Recall that if $a > 1$, $\frac{m}{s^{k+1-a}(s+1)^{a-1}} = |\mathcal{J}_{s,a-1}|$; and if $a = 1$ and $s > 1$, $\frac{m}{s^{k+1-a}(s+1)^{a-1}} = |\mathcal{J}_{s-1,k}|$. In profile $\mathbf{x}$, the jobs assigned to machines $1, 2, \ldots, \frac{m}{s^{k+1-a}(s+1)^{a-1}}$ with cost strictly smaller then $(s-1)k + a$ are jobs in $\mathcal{J}_{s',a'}$ where either $s' < s$ or $s' = s$ and $a' < a$. The jobs have higher priority then $j_{u,v}$. Therefore, job $j_{u,v} \in \mathcal{J}_{s,a}$ for $(s, a) \neq (1, 1)$ cannot unilaterally change machine to improve its cost. Besides, jobs in $\mathcal{J}_{1,1}$ have no incentive to change their machines as their cost are 1 and they cannot strictly decrease by doing so. Thus, $\mathbf{x}$ is a Nash equilibrium.

In profile $\mathbf{x}$, there are exactly $|\mathcal{J}_{s,a}|$ jobs with cost $(s-1)k+a$. Therefore, the social cost $C(\mathbf{x})$ satisfies:

$$C^k(\mathbf{x}) = \sum_{s=1}^{t}\sum_{a=1}^{k} \frac{m}{s^{k-a}(s+1)^a}[(s-1)k+a]^k \geq k^k m \sum_{s=1}^{t}\sum_{a=1}^{k} \frac{(s-1)^k}{s^{k-a}(s+1)^a}$$
$$\geq k^{k+1} m \sum_{s=1}^{t} \frac{(s-1)^k}{(s+1)^k} \geq k^{k+1} m(t-1)\frac{1}{3^k}$$

Hence, we deduce that $C(\mathbf{x})/C(\mathbf{x}^*) \geq \frac{1}{4}k^{\frac{k+1}{k}}$. □

4 ℓ_∞-norms of Completion Times under Local Policies

For any profile $\mathbf{x}$, the social cost $C(\mathbf{x}) = \max_j c_j$. Let $\mathbf{x}(i) = \{j : x_j = i\}$ be the set of jobs assigned to machine i. Define $L(\mathbf{x}(i)) := \sum_{j:x_j=i} p_{ij}$ as the *load* of of machine i for $1 \leq i \leq m$ in profile $\mathbf{x}$. The makespan of the profile is $L(\mathbf{x}) := \max_i L(\mathbf{x}(i))$. Observe that in an optimal assignment $\mathbf{x}^*$, $C(\mathbf{x}^*) = L(\mathbf{x}^*)$ since there is no idle-time in an optimal schedule. For each job j, denote $q_j := \min\{p_{ij} : 1 \leq i \leq m\}$ and define $\rho_{ij} := p_{ij}/q_j$ for all i, j. Note that a local policy can compute q_j for every job j while a strongly-local one cannot.

A profile $\mathbf{x}$ is *m-efficient* if $\rho_{x_j,j} \leq m$ for every job j. The following lemma guarantees that the restriction to the m-efficient profiles worsens the optimal social cost only by a constant factor.

Lemma 5 ([4]). *Let $\mathbf{y}^*$ be an optimal assignment. Then, there exits a m-efficient assignment $\mathbf{x}^*$ such that $L(\mathbf{x}^*) \leq 2L(\mathbf{y}^*)$.*

Policy Balance Let $\mathbf{x}$ be a strategy profile. Let $\prec_i$ be a total order on the jobs assigned to machine i, which is a SPT-like order. Formally, $j \prec_i j'$ if $p_{ij} < p_{ij'}$, or $p_{ij} = p_{ij'}$ and j is priority over j' (machine i chooses a local preference over jobs based on their local identities to break ties). Note that the policy does not need global job identities (there is no communication cost between machines about job identities) and a job may have different priority on different machines. The policy is clearly anonymous.

The cost c_j of job j assigned to machine i is defined as follows where h is a positive integer constant to be chosen later.

$$c_j^h(\mathbf{x}) = \begin{cases} \frac{1}{q_j}\left[\left(p_{ij} + \sum_{\substack{j':j'\prec_i j\\ x_{j'}=i}} p_{ij'}\right)^{h+1} - \left(\sum_{\substack{j':j'\prec_i j\\ x_{j'}=i}} p_{ij'}\right)^{h+1}\right] & \text{if } \rho_{ij} \leq m, \\ \infty & \text{otherwise.} \end{cases}$$

Intuitively, the cost of a job scheduled on a machine is proportional to its *marginal contribution* to the load of the machine (up to some power). Moreover, by the definition, jobs are allowed to be scheduled only on machines with inefficiency smaller than m.

Observe that the cost $c_j(\mathbf{x})$ of job j satisfies

$$c_j^h(\mathbf{x}) \geq \frac{1}{q_j}\left[\Big(p_{ij} + \sum_{j':j'\prec_i j,\ x_{j'}=i} p_{ij'}\Big)^{h+1} - \Big(\sum_{j':j'\prec_i j,\ x_{j'}=i} p_{ij'}\Big)^{h+1}\right]$$
$$\geq \frac{p_{ij}}{q_j}\Big(p_{ij} + \sum_{j':j'\prec_i j,\ x_{j'}=i} p_{ij'}\Big)^h \geq \Big(p_{ij} + \sum_{j':j'\prec_i j,\ x_{j'}=i} p_{ij'}\Big)^h$$

since $p_{ij}/q_j \geq 1$. As that holds for every job j assigned to machine i, policy Balance is feasible.

Lemma 6. *The best-response dynamic under the* Balance *policy converges to a Nash equilibrium.*

Proof. By the definition of the policy, any job j will choose a machine i such that $\rho_{ij} \leq m$. Moreover, since q_j is fixed for each job j, the behavior of jobs is similar to that in the following game. In the latter, the set of strategies of a player j is the same as in the former except the machines i with $\rho_{ij} > m$. Moreover, in the new game, player j in profile $\mathbf{x}$ has cost $c'_j(\mathbf{x})$ such that

$$\Big(c'_j(\mathbf{x})\Big)^h = \Big(p_{ij} + \sum_{j'\prec_i j} p_{ij'}\Big)^{h+1} - \Big(\sum_{j'\prec_i j} p_{ij'}\Big)^{h+1}$$

Hence, it is sufficient to prove that the better-response dynamic in the new game always converges. The argument is the same as the one to prove the existence of Nash equilibrium for policy SPT [9]. Here we present a proof based on a geometrical approach [8].

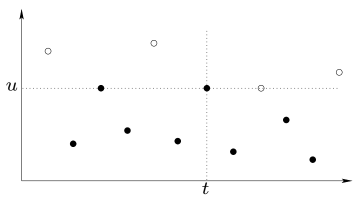

Fig. 2. A geometrical illustration of $|\mathbf{x}|_{u,t}$, every dot is a $(j, c_j(\mathbf{x}))$ pair, colored black if counted in $|\mathbf{x}|_{u,t}$

First, define $\mathtt{pos}_i(j) := 1 + |\{j' : j' \prec_i j, 1 \leq j' \neq j \leq n\}|$ which represents the priority of job j on machine i. For a value $u \in \mathbb{R}^+$ and a job index $1 \leq t \leq n$, we associate to every profile $\mathbf{x}$ the quantity

$$|\mathbf{x}|_{u,t} := |\{j : c'_j(\mathbf{x}) < u \text{ or } c'_j(\mathbf{x}) = u, \mathtt{pos}_{x_j}(j) \leq t\}|.$$

We use it to define a partial order $\prec$ on profiles. Formally $\mathbf{x} \prec \mathbf{y}$ if for the lexicographically smallest pair (u, t) such that $|\mathbf{x}|_{u,t} \neq |\mathbf{y}|_{u,t}$ we have $|\mathbf{x}|_{u,t} < |\mathbf{y}|_{u,t}$.

We show that the profile strictly increases according to this order, whenever a job changes to another machine while decreasing its cost. Let j be such a job changing from machine a in profile $\mathbf{x}$ to machine b, resulting in a profile $\mathbf{y}$. We know that $c'_j(\mathbf{y}) < c'_j(\mathbf{x})$. Remark that only jobs j' with $x_{j'} = b$ might have the cost in $\mathbf{y}$ larger than that in $\mathbf{x}$ (by definition of the cost c'). Moreover, such job j' with $x_{j'} = b$ and j' has a different costs in $\mathbf{x}$ and $\mathbf{y}$, it must be $j \prec_b j'$, which also implies $c'_{j'}(\mathbf{x}) \geq c'_j(\mathbf{y})$. In the same spirit, some jobs j' with $x_{j'} = a$ might decrease their cost, but not below $c'_j(\mathbf{x})$.

Consider $u = c'_j(\mathbf{y})$ and $t = \mathtt{pos}_b(j)$. We have that $|\mathbf{x}|_{u',t'} = |\mathbf{y}|_{u',t'}$ for all $u' < u$ and all t'. If job j is the only job with processing time p_{bj} among the ones $\{j' : x_{j'} = b\}$, then $|\mathbf{y}|_{u,t} = |\mathbf{x}|_{u,t} + 1$. Otherwise, $|\mathbf{y}|_{u,t'} = |\mathbf{x}|_{u,t'}$ for $t' < t$ and $|\mathbf{y}|_{u,t} = |\mathbf{x}|_{u,t} + 1$.

Therefore (u, t) is the first lexicographical pair where $|\mathbf{x}|_{u,t} \neq |\mathbf{y}|_{u,t}$ and $|\mathbf{y}|_{u,t} > |\mathbf{x}|_{u,t}$. Hence, since the set of strategy profiles is finite, the better-response dynamic must converge to a pure Nash equilibrium. This completes the proof. □

Remark that the game under Balance convergences fast to Nash equilibria in the best-response dynamic (the argument is the same as [9, Theorem 12]).

Lemma 7. *Let* $\mathbf{x}$ *and* $\mathbf{x}^*$ *be an equilibrium and an* m*-efficient arbitrary profile, respectively. Then,* $\sum_{i=1}^{m} L^{h+1}(\mathbf{x}(i)) \leq O(\alpha^h h^{h+1}) \sum_{i=1}^{m} L^{h+1}(\mathbf{x}^*(i))$ *where* α *is some constant.*

Proof. We focus on an arbitrary job j. Denote $i = x_j$ and $i^* = x^*_j$. As $\mathbf{x}$ is an equilibrium, we have $c^h_j(\mathbf{x}) \leq c^h_j(x_{-j}, x^*_j)$, i.e,

$$\begin{aligned}
&\Big(p_{ij} + \sum_{\substack{j' : j' \prec_i j \\ x_{j'} = i}} p_{ij'}\Big)^{h+1} - \Big(\sum_{\substack{j' : j' \prec_i j \\ x_{j'} = i}} p_{ij'}\Big)^{h+1} \\
&\quad \leq \Big(p_{i^*j} + \sum_{\substack{j' : j' \prec_{i^*} j \\ x_{j'} = i^*}} p_{i^*j'}\Big)^{h+1} - \Big(\sum_{\substack{j' : j' \prec_{i^*} j \\ x_{j'} = i^*}} p_{i^*j'}\Big)^{h+1} \\
&\quad \leq \Big(p_{i^*j} + L(\mathbf{x}(i^*))\Big)^{h+1} - \Big(L(\mathbf{x}(i^*))\Big)^{h+1} \\
&\quad \leq (h+1) p_{i^*j} \Big(p_{i^*j} + L(\mathbf{x}(i^*))\Big)^{h} \qquad (2)
\end{aligned}$$

where the second inequality is due to the fact that $(z + a)^{h+1} - z^{h+1}$ is increasing in z (for $a > 0$) and $\sum_{\substack{j' : j' \prec_i j \\ x_{j'} = i^*}} p_{i^*j'} \leq L(\mathbf{x}(i^*))$; the third inequality is due to

Lemma 2 (by dividing both sides by $(p_{i^*j} + L(\mathbf{x}(i^*)))^{h+1}$ and applying $z = \frac{p_{i^*j}}{p_{i^*j}+L(\mathbf{x}(i^*))}$ in the statement of Lemma 2). Therefore,

$$\begin{aligned}\sum_{i=1}^{m} L^{h+1}(\mathbf{x}(i)) &= \sum_{i=1}^{m} \sum_{j:x_j=i} q_j c_j^h(\mathbf{x}) \leq \sum_{i=1}^{m} \sum_{j:x_j=i} q_j c_j^h(x_{-j}, x_j^*) \\ &\leq \sum_{i=1}^{m} \sum_{j:x_j^*=i} (h+1) p_{ij} \Big(p_{ij} + L(\mathbf{x}(i)) \Big)^h \\ &\leq (h+1) \sum_{i=1}^{m} L(\mathbf{x}^*(i)) \Big(L(\mathbf{x}(i)) + L(\mathbf{x}^*(i)) \Big)^h \\ &\leq (h+1) \sum_{i=1}^{m} \frac{h}{(h+1)^2} L^{h+1}(\mathbf{x}(i)) + O\left(\alpha^h h^{h-1}\right) L^{h+1}(\mathbf{x}^*(i))\end{aligned}$$

where the first inequality is because $\mathbf{x}$ is an equilibrium; the second inequality is due to the sum of Inequality (2) taken over all jobs j; and the fourth inequality follows by applying case (1b) of Lemma 1 for $a(h) = 1/(h+1)$. Arranging the terms, the lemma follows. □

Theorem 3. *The PoA of the game under policy* Balance *is at most* $O(\log m)$ *by choosing* $h = \lfloor \log m \rfloor$.

Proof. Let $\mathbf{y}^*$ be an optimal assignment and $\mathbf{x}^*$ be an m-efficient assignment with property of Lemma 5. Let $\mathbf{x}$ be an equilibrium. Remark that $\mathbf{x}$ is a m-efficient assignment since every job can always get a bounded cost. Consider a job j assigned to machine i in profile $\mathbf{x}$. As $\mathbf{x}$ is a m-efficient assignment, by the definition of policy Balance

$$\begin{aligned}c_j^h(\mathbf{x}) &= \frac{1}{q_j} \left[\Big(p_{ij} + \sum_{\substack{j':j' \prec_i j \\ x_{j'}=i}} p_{ij'} \Big)^{h+1} - \Big(\sum_{\substack{j':j' \prec_i j \\ x_{j'}=i}} p_{ij'} \Big)^{h+1} \right] \\ &\leq \frac{1}{q_j} \left[\Big(L(\mathbf{x}(i)) \Big)^{h+1} - \Big(L(\mathbf{x}(i)) - p_{ij} \Big)^{h+1} \right] \leq (h+1) \rho_{ij} L^h(\mathbf{x}(i))\end{aligned}$$

where the first inequality is because function $(a+x)^{h+1} - x^{h+1}$ is increasing; and the last inequality is due to Lemma 2 (by dividing both sides by $L^{h+1}(\mathbf{x}(i))$ and applying $z = \frac{p_{ij}}{L(\mathbf{x}(i))}$ in the statement of Lemma 2). Moreover, by Lemma 7, we have

$$L^{h+1}(\mathbf{x}) \leq \sum_{i=1}^{m} L^{h+1}(\mathbf{x}(i)) \leq O(\alpha^h h^{h+1}) \sum_{i=1}^{m} L^{h+1}(\mathbf{x}^*(i)) \leq O(\alpha^h h^{h+1} m) L^{h+1}(\mathbf{x}^*)$$

for some constant α. Therefore,

$$\begin{aligned}C(\mathbf{x}) = \max_j c_j(\mathbf{x}) &\leq \max_{i,j} \Big((h+1)\rho_{ij} \Big)^{1/h} L(\mathbf{x}(i)) \leq \Big((h+1)m \Big)^{1/h} L(\mathbf{x}) \\ &\leq O\left(\Big(h^{h+2} m^2 \Big)^{1/h} \right) L(\mathbf{x}^*) \leq O\left(\Big(h^{h+2} m^2 \Big)^{1/h} \right) L(\mathbf{y}^*)\end{aligned}$$

where the last inequality is due to Lemma 5. Choosing $h = \lfloor \log m \rfloor$, the theorem follows. □

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Social Context in Potential Games

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Abstract. A prevalent assumption in game theory is that all players act in a purely selfish manner, but this assumption has been repeatedly questioned by economists and social scientists. In this paper, we study a model that allows to incorporate the social context of players into their decision making. We consider the impact of other-regarding preferences in potential games, one of the most popular and central classes of games in algorithmic game theory. Our results concern the existence of pure Nash equilibria and potential functions in games with social context. The main finding is a tight characterization of the class of potential games that admit exact potential functions for any social context. In addition, we prove complexity results on deciding existence of pure Nash equilibria in numerous popular classes of potential games, such as different classes of load balancing, congestion, cost and market sharing games.

1 Introduction

Game theory deals with the mathematical study of the interaction of rational agents. A prevalent assumption in many game-theoretic works is that agents are selfish, they consider only their own well-being and act upon their own interest. The assumption that players are purely selfish disregards complicated externalities or correlations in agent interests and has been repeatedly questioned by economists and social scientists [19,12,13]. In many applications, agents are embedded in a social context resulting in other-regarding preferences that are not captured by standard game-theoretic models. There are numerous examples, such as bidding frenzies in auctions [22] or altruistic contribution behavior on the Internet, in which players are spiteful or altruistic and (partially) disregard their own well-being to influence the well-being of others. Despite some recent efforts, the impact of such other-regarding preferences on fundamental results in game theory is not well-understood.

In this paper, we study a general appoach to incorporate externalities in the form of other-regarding preferences into strategic games. Our model is in line with a number of recent approaches on altruistic and spiteful incentives in games. We transform a base game into another strategic game, in which players aggregate dyadic influence values combined with personal utility of other players. Relying on dyadic relations is also a popular approach in social network

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analysis. Consequently, we refer to the set of dyadic influence values as *social context* [3]. Our results concentrate on (exact) potential games, a prominent class of games with many applications that has received much attention in algorithmic game theory. Most notably, potential games always possess pure Nash equilibria, and a potential function argument shows that arbitrary better-response dynamics converge. Our interest is to understand how these conditions change when social context comes into play.

Not surprisingly, potential functions and pure Nash equilibria might not exist with social contexts, even in very simple load balancing games [3]. For a variety of prominent classes of simple potential games, such as load balancing, congestion, or fair cost-sharing games, we even show hardness of deciding existence of pure Nash equilibria. On the positive side, our main finding is a tight characterization of all games that remain exact potential games under social context. We prove that every such game is isomorphic to a congestion game with affine delays. In this sense, our characterization is similar to the celebrated result by Monderer and Shapley [21] that shows isomorphism between exact potential games and congestion games. The main difference is that our result also allows to specify the delays as affine functions. In turn, our hardness results imply that in general the isomorphism result of [21] must use non-affine delays.

Model. We consider strategic games $\Gamma = (K, (\mathcal{S}_i)_{i\in K}, (c_i)_{i\in K})$ with a set K of k *players*. Each player $i \in K$ picks a *strategy* $S_i \in \mathcal{S}_i$. A *state* or *strategy profile* S is a collection of strategies, one for each player. The *(personal) cost* for player i in state S is $c_i(S)$. Each player tries to unilaterally improve his cost by optimizing his strategy choices against the choices of the other players. A state S has a unilateral *improvement move* for player $i \in K$ if there is $S_i' \in \mathcal{S}_i$ with $c_i(S_i', S_{-i}) < c_i(S)$. A state without improvement move for any player is a *pure Nash equilibrium (PNE)*.

In an *(exact) potential game*, we have a *potential function* $\Phi(S)$ such that $c_i(S) - c_i(S_i', S_{-i}) = \Phi(S) - \Phi(S_i', S_{-i})$ for every state S, player $i \in K$ and strategy $S_i' \in \mathcal{S}_i$. Φ simultaneously encodes the cost changes for all players in the game. The local optima of Φ are exactly the PNE, and every sequence of improvement moves is guaranteed to converge to such a PNE. It is well-known that every exact potential game is isomorphic to a congestion game [21]. In a *congestion game* [23] we have a set R of *resources* and for each $i \in K$ the strategy space $\mathcal{S}_i \subseteq 2^R$. For state S, we define the *load* $n_r(S)$ of resource r to be the number of players i with $r \in S_i$. Each resource $r \in R$ has a *delay* $d_r(S) = d_r(n_r(S))$, and the personal cost of player $i \in K$ is $c_i(S) = \sum_{r\in S_i} d_r(S)$.

We consider the effects of social context on the existence of potential functions. We extend a strategic game Γ by a *social context* defined by a set of weights F that contains a numerical influence value $f_{ij} \in \mathbb{R}$ for each pair of players $i, j \in K$, $i \neq j$. In particular, the *perceived cost* of player $i \in K$ is given by his personal cost and a weighted sum of cost of other players

$$c_i(S, F) = c_i(S) + \sum_{j\in K, j\neq i} f_{ij} c_j(S) \ .$$

We will assume throughout that the context is *symmetric*, i.e., $f_{ij} = f_{ji}$. In games with social context a state has a unilateral improvement move for player i if i can decrease his perceived cost by switching to another strategy. A PNE in a game with social context is a state without improvement moves for perceived costs. For our lower bounds, we will restrict to *binary contexts* F with $f_{ij} = f_{ji} \in \{0, 1\}$. Our existence results, however, do also allow non-binary and negative values. In the following, we say two players i and j are *friends* if $f_{ij} = f_{ji} = 1$.

We will consider social contexts in a variety of well-studied classes of potential games which we define more formally in the respective sections.

Results. In Section 2 we provide the following tight characterization of the existence of potential functions in strategic games with social context. Every strategic game that admits an exact potential function for every binary context is isomorphic to a congestion game with affine delay functions. In turn, every congestion game with affine delays has an exact potential function for every social context. Hence, the class of games that allows exact potential functions for all social contexts is exactly given by congestion games with affine delays.

In the following sections we consider many popular classes of potential games and examine deciding existence of a PNE for a given game and a given binary context. In most of these games, however, a PNE might not exist and deciding existence is NP-hard. In Section 3.1 we show that this holds even for simple classes of congestion games with increasing delays, e.g., for singleton congestion games with concave delays, general congestion games with convex delays, or weighted load balancing games on identical machines. For decreasing delays, we show in Section 3.2 NP-hardness of deciding PNE existence in Shapley cost-sharing games, even in broadcast games on undirected networks where every node is a player. If we consider cost sharing with priority-based sharing rules such as the Prim rule [6], it turns out that PNE exist in undirected broadcast games, but not necessarily in directed broadcast games. While PNE exist in undirected networks, convergence of improvement dynamics is not guaranteed. In fact, we show that such games might not even be weakly acyclic. Finally, in Section 3.3 we briefly consider hardness of PNE existence in market sharing games. All proofs missing from this extended abstract are deferred to the full version of this paper.

Related Work. The study of social contexts and other-regarding preferences has prompted increased interest in recent years, especially in well-studied classes of potential games such as load balancing [24] or congestion games [23]. Existence of equilibrium with binary contexts and different aggregation functions in simple congestion and load balancing games was studied in [3]. Binary contexts with sum aggregation were also considered in inoculation games [20]. More recently, social cost of worst-case equilibria with and without context were quantified for general non-negative contexts in load balancing games [4]. Coalitional stability concepts in a model with social context and aggregation via minimum cost change were studied for load balancing games in [16].

Several works examined the impact of altruism on the price of anarchy [8,5,7] and equilibrium existence [17,18] in congestion and load balancing games, and in fair cost-sharing games [11]. Altruism in these works is also modelled via a weighted sum of personal and social cost. For a recent characterization of stability of social optima in several classes of games with altruism see [2].

The impact of social context with sum aggregation was also studied in other game-theoretic scenarios, for instance in auctions (see, e.g., [22] or [9] and the references therein), market equilibria [10], stable matching [1], and others.

Characterizing the existence of potential functions and pure Nash equilibria was recently discussed in weighted potential games [14,15]. The results imply existence only for the classes of linear and exponential delay functions. This characterization refers to existence of potential functions and pure Nash equilibria for *all games* with the same class of delay functions. For example, for every set of non-linear or non-exponential delays *there is at least one game* that has no pure Nash equilibrium, but there might be others with such delays that have one.

In contrast, we provide a stronger result similar to [21] in the form of a one-to-one correspondence for each individual game under consideration.

2 Characterization

We start by characterizing the prevalence of potential functions under social contexts. We say a potential game has a *context-potential* Φ if there exists a function $\Phi(S,F)$ with $c_i(S,F) - c_i(S'_i, S_{-i}, F) = \Phi(S,F) - \Phi(S'_i, S_{-i}, F)$ for all states S, social contexts F, players $i \in K$, and strategies $S'_i \in \mathcal{S}_i$. Thus, if a potential game has a context-potential, it remains a potential game under every given social context F. We show the following theorem.

Theorem 1. *A strategic game has a context-potential if and only if it is isomorphic to a congestion game with affine delay functions.*

We prove the theorem in two steps. We first show that a game Γ that has a context-potential for every binary context must be isomorphic to a congestion game with affine delays by constructing an isomorphic game. Afterwards, we show that these games admit a potential also for every non-binary social context by providing a context-potential.

Lemma 1. *If a strategic game has a context-potential for every binary context, then it is isomorphic to a congestion game with affine delay functions.*

Proof. It is insightful to consider an arbitrary 4-tuple of states involving the deviations of 2 players, say players i and j. Here we denote $S^1 = (S_i, S_j, S_{-\{i,j\}})$, $S^2 = (S'_i, S_j, S_{-\{i,j\}})$, $S^3 = (S'_i, S'_j, S_{-\{i,j\}})$, and $S^4 = (S_i, S'_j, S_{-\{i,j\}})$. For the cycle $(S_1, S_2, S_3, S_4, S_1)$ consider the difference in personal cost of the moving players $\Delta_i^{12} = c_i(S^2) - c_i(S^1)$, $\Delta_j^{23} = c_j(S^3) - c_j(S^2)$, $\Delta_i^{34} = c_i(S^4) - c_i(S^3)$, $\Delta_j^{41} = c_j(S^1) - c_j(S^4)$. Note that existence of an exact potential function is equivalent to assuming that this difference is 0, i.e.,

$$\Delta_i^{12} + \Delta_j^{23} + \Delta_i^{34} + \Delta_j^{41} = 0 \ , \tag{1}$$

for every pair of players i and j and every 4-tuple of states as detailed above [21].

Now suppose Γ is an exact potential game for every binary context F. Note that for 2 players, every exact potential game is isomorphic to a congestion game with affine delays, because each resource is used by at most 2 players. Hence, consider a game with at least three players. The main idea of the proof is to characterize the impact on the personal cost of player h when a different player i makes a strategy switch. Using this characterization, we then construct resources and affine delay functions.

Consider three different players $i, j, h \in K$ and F with $f_{ih} = f_{hi} = 1$ and 0 for all other pairs of players in the game. We assume that the resulting game has an exact potential, we have

$$\Delta_i^{12} + c_h(S^2) - c_h(S^1) + \Delta_j^{23} + \Delta_i^{34} + c_h(S^4) - c_h(S^3) + \Delta_j^{41} = 0 \ ,$$

and by using Eqn. (1) above and the definition of $S^1, \ldots, S^4$, we see that

$$c_h(S_i', S_j, S_{-\{i,j\}}) - c_h(S_i, S_j, S_{-\{i,j\}}) = c_h(S_i', S_j', S_{-\{i,j\}}) - c_h(S_i, S_j', S_{-\{i,j\}}) \ .$$

The sides of this equation describe the change of personal cost of h when i switches from S_i to S_i', once with j playing S_j (left) and once with j playing S_j' (right). We can derive this identity for all strategies of each player $j \neq i, h$. This shows that when i changes his strategy from S_i to S_i', then the change in personal cost of h is independent of the strategy of any other player j. Hence, there is

$$\Delta_h(S_i', S_i, S_h) = c_h(S_i', S_h, S_{-\{i,h\}}) - c_h(S_i, S_h, S_{-\{i,h\}}) \ .$$

To show that these values are pairwise consistent, we again consider F with $f_{ih} = f_{hi} = 1$ and 0 for all other pairs of players. However, this time i and h do the strategy switches. By considering a 4-cycle as above and using Eqn. (1), we obtain

$$\begin{aligned} & c_h(S_i', S_h, S_{-\{i,h\}}) - c_h(S_i, S_h, S_{-\{i,h\}}) \\ + \ & c_i(S_i', S_h', S_{-\{i,h\}}) - c_i(S_i', S_h, S_{-\{i,h\}}) \\ + \ & c_h(S_i, S_h', S_{-\{i,h\}}) - c_h(S_i', S_h', S_{-\{i,h\}}) \\ + \ & c_i(S_i, S_h, S_{-\{i,h\}}) - c_i(S_i, S_h', S_{-\{i,h\}}) = 0 \ , \end{aligned}$$

or, equivalently,

$$\Delta_h(S_i', S_i, S_h) + \Delta_i(S_h', S_h, S_i') + \Delta_h(S_i, S_i', S_h') + \Delta_i(S_i, S_i', S_h) = 0 \ . \qquad (2)$$

We now construct an equivalent congestion game Γ' with affine delay functions. We consider each pair of players $i \neq h$ and introduce a single resource $r_{\{S_i, S_h\}}$ for every unordered pair of strategies in $\mathcal{S}_i \times \mathcal{S}_h$. For every player $j \in K$ and each strategy $S_j \in \mathcal{S}_j$, we assume that S_j contains all resources for which it appears in the index. Let us first restrict our attention to one pair of players i and h. Due to the fact that the values Δ_i and Δ_h can be given separately for each pair $\{i, h\}$ and do not depend on other player strategies, we can effectively reduce the game to a set of 2-player games played simultaneously.

For each resource r associated with strategies of both i and h, we set all delays $d_r(1) = 0$. The delay $d_r(2)$ is set to 1 for one arbitrarily chosen resource $r_{\{S_i,S_h\}}$. The other delays $d_r(2)$ simply are derived via the differences Δ_h and Δ_i. In particular, with $r' = r_{\{S'_i,S_h\}}$ and $r = r_{\{S_i,S_h\}}$ we have $d_{r'}(2) = d_r(2) + \Delta_h(S'_i, S_i, S_h)$. Similarly, with $r' = r_{\{S_i,S'_h\}}$ and $r = r_{\{S_i,S_h\}}$ we have $d_{r'}(2) = d_r(2) + \Delta_i(S'_h, S_h, S_i)$. The set of values $d_r(2)$ defined in this way is consistent, because Eqn (2) essentially proves existence of an exact potential function when differences are given by Δ_h and Δ_i values, as the sum of changes in all 4-cycles of the state graph is 0. By our assignment, we essentially use this potential function for the $d_r(2)$ values.

In our construction so far, we guarantee that in Γ' player i suffers from the same cost change as in Γ when *the other player moves*. So far, however, it does not necessarily implement the correct personal cost or cost change for *the moving player*. For this we introduce a single resource r_{S_i} for strategy $S_i \in \mathcal{S}_i$ of every player $i \in K$. This resource is used only by player i and only if he plays strategy S_i. We again set the delay $d_r(1) = 1$ for some arbitrary resource r_{S_i}. Then consider a state (S_i, S_{-i}) and the deviation to (S'_i, S_{-i}). The difference in cost for player i is denoted by $\Delta_i(S'_i, S_i, S_{-i})$, and with $r = r_{S_i}$, $r' = r_{S'_i}$, $R_{ij} = \{r_{\{S_i,S_j\}} \mid S_j \in \mathcal{S}_j\}$ and $R'_{ij} = \{r_{\{S'_i,S_j\}} \mid S_j \in \mathcal{S}_j\}$ we get

$$d_{r'}(1) = d_r(1) + \Delta_i(S'_i, S_i, S_{-i}) + \sum_{\substack{j\in K\\ j\neq i}} \left(\sum_{s\in R_{ij}} d_s(S_i, S_{-i}) - \sum_{s\in R'_{ij}} d_s(S'_i, S_{-i}) \right).$$

Thus, we simply account for all delay changes from the sets of resources R_{ij} and R'_{ij} and correct the cost to implement the correct delay change of $\Delta_i(S'_i, S_i, S_{-i})$ via our resource r_{S_i}. Note that this gives a consistent set of values for $d_r(1)$. For a fixed S_{-i}, this implies the same cost changes for i as in Γ. To show that this correctly implements all cost changes for player i as in Γ, consider the switch from S_i to S'_i for a different set of strategies S'_{-i} and the cost change $\Delta_i(S'_i, S_i, S'_{-i})$. To see that the correct cost change is present also in Γ', we implement the deviation via the following shift. We first let all players other then i change to S_{-i}. By construction this changes i's personal cost as in Γ. Then we let i deviate to S_i in state (S'_i, S_{-i}). This yields a change in personal cost as in Γ by definition. Afterwards, we let other players switch back to S'_{-i}. Again, the cost changes of player i are implemented as in Γ. Hence, in conclusion, by implementing the correct cost change $\Delta_i(S'_i, S_i, S_{-i})$ for a single strategy switch of S_i to S'_i, all other cost changes for switches among these strategies are uniquely and correctly determined.

This shows that we can turn Γ into a congestion game Γ' with the same potential function, in which every resource is accessed by at most two players. Trivially, for every such resource we can generate the required delays $d_r(1)$ and $d_r(2)$ via an affine delay function $d_r(x) = a_r \cdot x + b_r$. □

Lemma 2. *A congestion game with affine delay functions has a context-potential for every social context.*

Proof. The context-potential function is given by

$$\Phi(S,F) = \sum_{r\in R}\sum_{j=1}^{n_r(S)} d_r(j) + \sum_{\substack{i\neq j\in K,\\ r\in S_i\cap S_j}} f_{ij}a_r$$

For simplicity of presentation, we consider affine delays $d_r(x) = a_r \cdot x + b_r$ in the form of linear delays $d_r(x) = a_r \cdot x$ by appropriate introduction of player-specific resources with linear delays that account for the offsets b_r. Then, if i changes from S_i to S_i', the change of cost for player j is given by 0 for the resources of S_j that are used in neither or both S_i and S_i'. The change is a_r or $-a_r$ for each resource r that is joined or left by i, respectively. Hence, when we examine the potential, we see that

$$\begin{aligned}
&\Phi(S_i,S_{-i}) - \Phi(S_i',S_{-i})\\
&= \Delta_i(S_i',S_i,S_{-i}) + \sum_{\substack{i\neq j\in K,\\ r\in S_i'\cap S_j}} f_{ij}a_r - \sum_{\substack{i\neq j\in K,\\ r\in S_i\cap S_j}} f_{ij}a_r\\
&= \Delta_i(S_i',S_i,S_{-i}) + \sum_{\substack{i\neq j\in K,\\ r\in (S_i'-S_i)\cap S_j}} f_{ij}a_r - \sum_{\substack{i\neq j\in K,\\ r\in (S_i-S_i')\cap S_j}} f_{ij}a_r\\
&= \Delta_i(S_i',S_i,S_{-i}) + \sum_{i\neq j\in K} f_{ij}\cdot \Delta_j(S_i',S_i,S_{-i})\ ,
\end{aligned}$$

as desired. This proves the lemma. □

3 Computational Results

In this section, we study the computational complexity of deciding existence of PNE for a given potential game with a given social context. Throughout this section, we focus on binary contexts. We will say that player i *is friends with* player j if $f_{ij} = f_{ji} = 1$.

3.1 Congestion Games

We first focus on congestion games as introduced above. For this central class of games we can prove a NP-completeness result even for singleton games, in which $|S_i| = 1$ for all players $i \in K$ and all strategies $S_i \in \mathcal{S}_i$. We start with a game that does not have a PNE. This game is then used below in our construction to show NP-completeness of the decision problem.

Example 1. Consider a congestion game Γ consisting of the set of players $K = \{1, 2, 3, 4\}$ and the set of resources $R = \{r_1, r_2\}$. Players 1 and 2 have only one strategy each, with $\mathcal{S}_1 = \{\{r_1\}\}$ and $\mathcal{S}_2 = \{\{r_2\}\}$. Players 3 and 4 both have two strategies, $\mathcal{S}_3 = \mathcal{S}_4 = \{\{r_1\}, \{r_2\}\}$. Both resources have the same delay function d_r with $d_r(1) = 4$, $d_r(2) = 8$ and $d_r(3) = d_r(4) = 9$. The binary context is such that player 4 is friends with all other players. Every other player is only friends with player 4.

It is easy to verify that this game has no PNE: In a state in which both resources are used by two players, player 4 has an improvement move by moving to the other resources. In a strategy profile in which player 3 and 4 are both on the same resource, player 3 has an improvement move by moving to the other resource.

Theorem 2. *It is* NP*-complete to decide if a singleton congestion game with binary context has a pure Nash equilibrium.*

The previous result uses concave delay functions to construct a game without PNE. It is an open problem if PNE always exist in singleton congestion games with binary context and convex delays. For more general structures of strategy spaces, however, convex delay functions are not sufficient. Again, we use the example below to prove NP-completeness of deciding existence.

Example 2. We consider a congestion game with six players denoted by $K = \{1, \ldots, 6\}$. Player 1 is friends with 3 and 4. Player 2 is friends with 5 and 6. The set of resources is $R = \{r_1, r_2, r_3, r_4, r_5\}$. Players 1 and 2 have two strategies. The strategies of player 1 are $\mathcal{S}_1 = \{\{r_1\}, \{r_2, r_3\}\}$. The strategies of player 2 are $\mathcal{S}_2 = \{\{r_2, r_4\}, \{r_3, r_5\}\}$. The remaining players have one strategy each, $\mathcal{S}_3 = \{\{r_1\}\}$, $\mathcal{S}_4 = \{\{r_3\}\}$, $\mathcal{S}_5 = \{\{r_2\}\}$ and $\mathcal{S}_6 = \{\{r_5\}\}$.

Note that r_4 is used by at most 1 player, r_1, r_5 by at most 2 players each, r_2, r_3 by at most 3 players. We define the convex delays only for the required number of players. For r_1 we have $d_{r_1}(1) = 15$ and $d_{r_2}(2) = 16$. Resources r_2 and r_3 have the same delay function with $d_r(1) = 5.5$, $d_r(2) = 6$ and $d_r(3) = 10$. Resource r_4 has delay $d_{r_4}(1) = 1$. Finally, r_5 has delay $d_{r_5}(1) = 0$ and $d_{r_5}(2) = 1$.

Note that only players 1 and 2 have more than one strategy. Thus, to verify that this game does not have a PNE, we have to check the four possible states represented by the strategies of players 1 and 2. In state $(\{r_1\}, \{r_2, r_4\})$ the perceived cost of player 1 is $16 + 16 + 6 = 38$ and he would improve by changing to strategy $\{r_2, r_3\}$ resulting in perceived cost of $15 + 10 + 6 + 6 = 37$. In state $(\{r_2, r_3\}, \{r_2, r_4\})$, the perceived cost of player 2 is $10 + 10 + 1 + 0 = 21$ and he would improve by changing to strategy $\{r_3, r_5\}$ resulting in perceived cost of $10 + 1 + 6 + 1 = 18$. In state $(\{r_2, r_3\}, \{r_3, r_5\})$, the perceived cost of player 1 is $6 + 10 + 15 + 10 = 41$ and he would improve by changing to strategy $\{r_1\}$ resulting in perceived cost of $16 + 16 + 6 = 38$. In state $(\{r_1\}, \{r_3, r_5\})$, the perceived cost of player 2 is $6 + 1 + 5.5 + 1 = 13.5$ and he would improve by changing to strategy $\{r_2, r_4\}$ resulting in perceived cost of $6 + 1 + 6 + 0 = 13$.

Theorem 3. *It is* NP*-complete to decide if a general congestion game with binary context has a pure Nash equilibrium even if the delay functions are convex.*

As an extension to ordinary congestion games, we also consider weighted congestion games. In this case, each player $i \in K$ has a weight $w_i \in \mathbb{N}$. Instead of the *number* of players using resource r, the delay function d_r now takes the *sum of weights* of players using r as input and maps it to a delay value. The personal cost of a player is the sum of delays of chosen resources. Weighted congestion games are known to possess PNE for linear and exponential delay functions, see [15]. Here we show that with a binary context, even singleton weighted congestion games with identical linear delays might not have a PNE. This example can be used in a reduction to show NP-completeness of deciding PNE existence.

Example 3. Consider the following game on two identical resources. Each resource r has the delay function $d_r(x) = x$. The game consists of four players with weights 1,1,4, and 9, respectively. The binary context is such that the three players with weights 1 and 4 are all friends with each other, but the player with weight 9 is not friends with anyone. It is easy to verify that this game does not have a PNE.

Theorem 4. *It is NP-complete to decide if a weighted singleton congestion game with binary context has a pure Nash equilibrium even if all delay functions are linear.*

3.2 Cost Sharing

In this section, we consider several classes of cost sharing games. We first study *Shapley* or *fair cost-sharing games.* These games are congestion games with delay functions $d_r(x) = c_r/x$, where $c_r \in \mathbb{N}$ is the cost of the resource. The cost of a resource is assigned in equal shares to all players using the resource. As a subclass, we consider *broadcast games* with Shapley sharing in which there is a directed or undirected graph $G = (V, E)$ with a single sink node $t \in V$. Every edge $e \in E$ is a resource. Every node $v_i \in V$, $v_i \neq t$ is associated to a different player i. The strategy set $\mathcal{S}_i$ consists of all v_i-t-paths in G.

A different cost sharing scheme proposed in [6] yields *Prim cost-sharing games.* In this case, resources are edges of a directed or undirected graph $G = (V, E)$ and players are situated at a subset of the nodes in this graph. There is a single sink node t, and the set of strategies for a player i in node v_i is the set of v_i-t-paths in G. There is a global ranking of players and the cost of an edge is assigned fully to the highest ranked player using it. The ranking of players derives from the ordering, in which Prim's algorithm would add players to construct a minimum spanning tree (MST). In particular, the first player i is the one which has the cheapest path to t in G. The second player is the one, which has the cheapest path to $\{t, v_i\}$, and so on. Again, in a *broadcast game* with Prim sharing every node $v \neq t$ is associated with a different player.

We first show that Shapley cost-sharing games with binary context might not possess a PNE. Remarkably, this even holds for broadcast games with undirected edges as the following example shows. We then use this example game as a building block in our NP-completeness result for broadcast games with Shapley sharing and binary context.

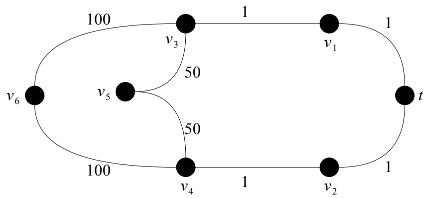

Fig. 1. A Shapley cost-sharing game that does not have a pure Nash equilibrium. The players v_3, v_4, and v_5 are friends.

Example 4. Consider a broadcast game with Shapley sharing in the network depicted in Figure 1. The edges are labeled with their costs. The players that belong to the vertices v_3, v_4, and v_5 are mutual friends. If player v_5 chooses the path via v_3 (or v_4), the best response of player v_6 is to choose his path via v_3 (or v_4, respectively). However, the best response of player v_5 is inverted. If player v_6 chooses the path via v_3 (or v_4), the best response of player v_5 is to choose his path via v_4 (or v_3, respectively). Thus, no PNE exists.

Theorem 5. *It is* NP*-hard to decide if a broadcast game with Shapley sharing and binary context has a pure Nash equilibrium.*

For Prim cost-sharing games existence and convergence results become more interesting. In particular, for undirected broadcast games with Prim sharing and arbitrary binary context there always exists a PNE. However, we first show that a similar result does not hold for directed broadcast games. The following example shows that such games with binary contexts do not have PNE in general. The main idea to prove existence of PNE and convergence without social context is that the player priorities induce a lexicographic potential function for the game. If we allow additive social context, the lexicographic improvement property breaks. This is then used to prove NP-hardness of deciding existence of PNE below.

Example 5. Figure 2 shows an example of a Prim cost-sharing game that does not have a PNE. In this game player d is friends with all other players and the players b and c are friends. Observe that in every state, d uses the edge of cost 1000. Hence, this cost is part of the perceived cost of every player in every state. Therefore, the players never have an incentive to use one of dashed edges. On the other hand, these are the edges that define the priorities of the players. Given their priorities, it is straightforward to verify that the players never agree on a subset of the edges of small cost to buy. Hence, no state of the game qualifies as a PNE. To turn this game into a broadcast game, note that we can safely add another player to every intermediate (non-filled) node. These players have only one strategy each, they will end up with lowest priority, and thus they do not change the cyclic incentives of players a, b, c and d described above.

Theorem 6. *It is* NP*-hard to decide if a directed broadcast game with Prim sharing has a pure Nash equilibrium.*

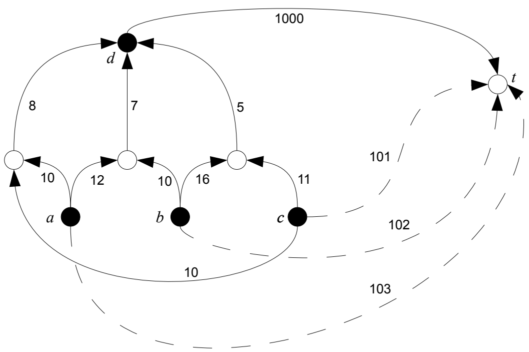

Fig. 2. An example of a Prim cost-sharing game with a binary context that does not have a pure Nash equilibrium. Here, player d is friends with a, b, and c and the players b and c are friends.

In contrast, if we consider undirected broadcast games with Prim sharing and binary contexts, we can construct a PNE using an efficient centralized assignment algorithm. While this shows existence of a PNE, convergence of improvement moves might still be absent. In fact, our theorem below shows the slightly stronger statement that these games are not even weakly acyclic.

Theorem 7. *For every undirected broadcast game with Prim cost sharing there is a pure Nash equilibrium if the social context F satisfies $f_{ij} = f_{ji} \in [0,1]$ for all $i, j \in K$. The pure Nash equilibrium can be computed in polynomial time.*

Proof. The proof of the theorem is mainly a consequence of classic arguments showing non-emptiness of the core in cooperative minimum spanning tree games. We here use Prim's MST algorithm not only to define the priority ordering of players but also to construct a PNE. We first consider the cheapest incident edge to t and assign the incident player v to play strategy $\{v, t\}$. Subsequently, consider the set V' of players connected to t. Consider the cheapest edge connecting a player of V' to a player in $V - V'$. We denote the players incident to this edge by $v' \in V'$ and $v \in V - V'$. Now we expand V' by assigning v to play the strategy composed of edge (v, v') and the path that v' uses to connect to t. This inductively constructs a state, in which the cost of a MST is shared. Note that the players are added in order of their priority, and hence every player pays exactly for the first edge on his path to t. We will argue that this state is a PNE for every social context with $f_{ij} = f_{ji} \in [0,1]$ for all $i, j \in K$.

Assume that a player i has a profitable strategy switch that decreases his perceived cost. This switch does not change the personal cost of any higher ranked player, these players will stay connected to t by sharing the cost of their subtree. In addition, the set of all players shares the cost of a MST, i.e., a minimum cost network connecting all players to t. Hence, the sum of all personal costs cannot decrease in a strategy switch. First, suppose the personal cost of player i strcitly decreases in the strategy switch. Note that all players connecting to t via his node v_i have lower priority. Hence, we could construct a cheaper network by letting all these players imitate i's strategy switch, because this would not change the personal cost of the imitating players. In this way, we would obtain a strictly cheaper network connecting all players to t, a contradiction.

Thus, the only way to improve the perceived cost is to strictly decrease the cost of other players that he is friends with. However, player i can only decrease the cost of lower ranked players by paying some of the edges currently assigned to them. As $f_{ij} = f_{ji} \leq 1$, he obtains no benefit from paying these edges himself. As $f_{ij} = f_{ji} \geq 0$, he obtains no benefit from forcing lower ranked players to pay the edges he vacates. Hence, if he strictly lowers his perceived cost in this way, then he must also strcitly decrease his personal cost, which is impossible as noted above. □

Theorem 8. *There is an undirected broadcast game with Prim sharing and binary context with the property that there exists a starting state from which there is no sequence of improvement moves to a PNE.*

Proof. We construct an example game and an appropriate starting state. Our game is an adaptation of the game in Fig. 2. We simply turn every directed edge into an undirected edge. The social context is as before, but here we also assume that the three auxiliary players in non-filled nodes are all friends with d. In our starting state, player d uses the edge of cost 1000, and all other players use some cycle-free path to t that goes over node d. The main invariant is that players c and b always remain on the edge of cost 1000. Given this condition, player c has no incentive to switch to a path containing an edge of cost 101, because otherwise b would be assigned to pay a cost of 1000. If c is assigned to pay the cost of 1000, all players have an incentive to join c on this edge as the corresponding paths become cheaper. Thus, no player will have an improvement move purchasing some of the edges of cost 101, 102 or 103. However, it is straightforward to verify that without these edges, no PNE can be obtained, and hence no sequence of improvement moves leads to a PNE. □

3.3 Market Sharing Games

Market sharing games are a class of congestion games that model content distribution in ad-hoc networks. There is a set of players and a set of markets. Each player i has a budget B_i, each market has a cost C_i. In addition, a market has a query rate q_i. There is a bipartite network specifying which player can participate in which market. From the set of markets a player is connected to, he can choose as strategy any subset for which the sum of costs is at most his budget. The reward from a market is the query rate, and it is shared equally by the players that pick the market. Every player gets as utility the sum of rewards of markets chosen in his strategy. More generally, market games are congestion games with utility-maximizing players and reward functions $d_r(x) = q_r/x$. Market costs and budgets determine the structure of the strategy spaces.

In market sharing games with binary context, we again observe absence of PNE and NP-completeness of deciding PNE existence.

Example 6. Consider the following market sharing game with two identical markets. Each market has cost of 1 and its query rate (revenue) is 1. There are four players 1, 2, 3, and 4 in this game. Each player is interested in both markets and each player has a budget of 1. The players 1, 2, and 3 are mutual friends. It is easy to see that this game does not have an equilibrium. The players 1, 2, and 3 prefer an outcome in which one of them is in a market by himself.

Theorem 9. *It is* NP*-hard to decide if a market sharing game with a binary context has a pure Nash equilibrium.*

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Take It or Leave It: Running a Survey When Privacy Comes at a Cost

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Abstract. In this paper, we consider the problem of estimating a potentially sensitive (individually stigmatizing) statistic on a population. In our model, individuals are concerned about their privacy, and experience some *cost* as a function of their privacy loss. Nevertheless, they would be willing to participate in the survey if they were compensated for their privacy cost. These cost functions are not publicly known, however, nor do we make Bayesian assumptions about their form or distribution. Individuals are rational and will misreport their costs for privacy if doing so is in their best interest. Ghosh and Roth recently showed in this setting, when costs for privacy loss may be correlated with private types, if individuals value *differential privacy*, no individually rational direct revelation mechanism can compute any non-trivial estimate of the population statistic. In this paper, we circumvent this impossibility result by proposing a modified notion of how individuals experience cost as a function of their privacy loss, and by giving a mechanism which does not operate by direct revelation. Instead, our mechanism has the ability to randomly approach individuals from a population and offer them a take-it-or-leave-it offer. This is intended to model the abilities of a surveyor who may stand on a street corner and approach passers-by.

1 Introduction

Voluntarily provided data is a cornerstone of medical studies, opinion polls, human subjects research, and marketing studies. Suppose you are a researcher and you would like to collect data from a population and perform an analysis on it. Presumably, you would like your sample, or at least your analysis, to be representative of the underlying population. Unfortunately, individuals' decisions of whether to participate in your study may skew your data: perhaps people with an embarrassing medical condition are less likely to respond to a survey whose results might reveal their condition. Some data collectors, such as the US Census, can get around the issue of voluntary participation by legal mandate, but this is rare. How might we still get analyses that represent the underlying population?

Statisticians and econometricians have of course attempted to address selection and non-response bias issues. One approach is to assume that the effect of

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unobserved variables has mean zero. The Nobel-prize-winning Heckman correction method [1] and the related literature instead attempt to correct for non-random samples by formulating a theory for the probabilities of the unobserved variables and using the theorized distribution to extrapolate a corrected sample. The limitations of these approaches is precisely in the assumptions they make on the structure of the data. Is it possible to address these issues without needing to "correct" the observed sample, while simultaneously minimizing the cost of running the survey?

One could try to incentivize participation by offering a reward for participation, but this only serves to skew the survey in favor of those who value the reward over the costs of participating (e.g., hassle, time, detrimental effects of what the study might reveal), which again may not result in a representative sample. Ideally, you would like to be able to find out exactly how much you would have to pay each individual to participate in your survey (her "value", akin to a reservation price), and offer her exactly that much. Unfortunately, traditional mechanisms for eliciting player values truthfully are not a good match for this setting because a player's value may be correlated with her private information (for example, individuals with an embarrassing medical condition might want to be paid extra in order to reveal it). Standard mechanisms based on the revelation principle are therefore no longer truthful. In fact, Ghosh and Roth [2] showed that when participation costs can be arbitrarily correlated with private data, no direct revelation mechanism can simultaneously offer non-trivial accuracy guarantees and be individually rational for agents who value their privacy.

The present paper tackles this problem of conducting a survey on sensitive information when the costs of participating might be correlated with the information itself. In order to allow us to focus on the problem of incentivizing participation, we set aside the problem of *truthfulness*, and assume that once someone has decided to voluntarily participate in our survey, she must respond truthfully. This can most simply be justified by assuming that survey responses are verifiable or cannot easily be fabricated (e.g., the surveyor requires documentation of answers, or, more invasively, actually collects a blood sample from the participant). While the approach we present in this paper works well with such verifiable responses, in addition, our framework provides a formal "almost-truthfulness" guarantee, that the expected utility a participant could gain by lying in the survey is at most very small.

Motivated by the negative result of Ghosh and Roth [2], we move away from direct revelation mechanisms, to a framework where the surveyor is allowed to make "take-it-or-leave-it" offers to randomly sampled members of the underlying population. The simplest "take-it-or-leave-it" mechanism one might construct is simply to offer all sampled individuals the same low price in return for their participation in the survey (where participation might come with, e.g., a guarantee of differential privacy on their private data). If it turns out that this price is not high enough to induce sufficient rates of participation, one would double the price and restart the mechanism with a fresh sample of individuals, repeating until a target participation rate is reached (or the survey budget is exhausted).

The statistics released from the survey would then be based (perhaps in a differentially private manner) on the private information of the participants at the final (highest) price.

One might hope that such a simple doubling scheme would suffice to get "reasonable" participation rates at "reasonably" low cost. In order to deduce when take-it-or-leave-it offers will be accepted, though, we need a concrete model for how individuals value their privacy. Ghosh and Roth [2] provide such a model—essentially, they interpret the differential privacy parameter as the parameter governing individuals' costs. However, as we argue, this model can be problematic.

Our Results. Our first contribution is to document the "paradox of differential privacy"—in Section 2, we observe that the manner in which Ghosh and Roth propose to model privacy costs results in clearly nonsensical behavioral predictions, even in a quite simple take-it-or-leave-it setting. In Section 5, we offer an alternative model for the value of privacy in multi-stage protocols, using the tools and language of differential privacy. We then, in Section 6, present a privacy-preserving variant of the simple "double your offer" algorithm above, and examine its ability to incentivize participation in data analyses when the subjects' value for their private information may be correlated with the sensitive information itself. We show that our simple mechanism allows us to compute accurate statistical estimates, addressing the survey problem described above, and we present an analysis of the costs of running the mechanism relative to a fixed-price benchmark.

2 The Paradox of Differential Privacy

Over the past decade, differential privacy has emerged as a compelling privacy definition, and has received considerable attention. While we provide formal definitions in Section 4, differential privacy essentially bounds the sensitivity of an algorithm's output to arbitrary changes in individual's data. In particular, it requires that the probability of *any* possible outcome of a computation be insensitive to the addition or removal of one person's data from the input. Among differential privacy's many strengths are (1) that differentially private computations are approximately truthful [3] (which gives the almost-truthfulness guarantee mentioned above), and (2) that differential privacy is a property of the *mechanism* and is independent of the input to the mechanism.

A natural approach taken by past work (e.g., [2]) in attempting to model the cost incurred by participants in a computation on their private data is to model individuals as experiencing cost as a function of the *differential privacy* parameter ε associated with the mechanism using their data. We argue here, however, that modeling an individual's cost for privacy loss solely as any function $f(\varepsilon)$ of the privacy parameter ε predicts unnatural agent behavior and incentives.

Consider an individual who is approached and offered a deal: she can participate in a survey in exchange for \$100, or she can decline to participate and walk away. She is given the guarantee that both her participation decision and her input to the survey (if she opts to participate) will be treated in an ε-differentially private manner. In the usual language of differential privacy, what does this mean? Formally, her input to the mechanism will be the tuple containing her participation decision and her private type. If she decides not to participate, the mechanism output is not allowed to depend on her private type, and switching her participation decision to "yes" cannot change the probability of any outcome by more than a small multiplicative factor. Similarly, fixing her participation decision as "yes", any change in her stated type can only change the probability of any outcome by a small multiplicative factor.

How should she respond to this offer? A natural conjecture is that she would experience a higher privacy cost for participating in the survey than not (after all, if she does not participate, her private type has *no* effect on the output of the mechanism – she need not even have provided it), and that she should weigh that privacy cost against the payment offered, and make her decision accordingly.

However, if her privacy cost is solely some function $f(\varepsilon)$ of the privacy parameter of the mechanism, she is actually incentivized to behave quite differently. Since the privacy parameter ε is *independent* of her input, her cost $f(\varepsilon)$ will be identical *whether she participates or not*. Indeed, her participation decision does not affect her privacy cost, and only affects whether she receives payment or not, and so she will always opt to participate in exchange for any positive payment.

We view this as problematic and as not modeling the true decision-making process of individuals: real people are unlikely to accept arbitrarily low offers for their private data. One potential route to addressing this "paradox" would be to move away from modeling the value of privacy solely in terms of an input-independent privacy guarantee. This is the approach taken by [4]. Instead, we retain the framework of differential privacy, but introduce a new model for how individuals reason about the cost of privacy loss. Roughly, we model individuals' costs as a function of the differential privacy parameter only of the portion of the mechanism they participate in, and assume they do not experience cost from the parts of the mechanism that process data that they have not provided (or that have no dependence on their data).

3 Related Work

In recent years, differential privacy [5] has emerged as the standard solution concept for privacy in the theoretical computer science literature. There is by now a very large literature on this fascinating topic, which we do not attempt to survey here, instead referring the interested reader to a survey by Dwork [6].

McSherry and Talwar [3] propose that differential privacy could itself be used as a *solution concept* in mechanism design (an approach later used by Gupta et al. [7] and others). They observe that any differentially private mechanism is approximately truthful, while simultaneously having some resilience to collusion. Using differential privacy as a solution concept (as opposed to dominant strategy truthfulness) they give improved results in a variety of auction settings.

This literature was extended by a series of elegant papers by Nissim, Smorodinsky, and Tennenholtz [8], Xiao [9], Nissim, Orlandi, and Smorodinsky [10], and Chen et al. [4]. This line of work observes ([8,9]) that differential privacy does not lead to exactly truthful mechanisms, and indeed that manipulations might be easy to find, and then seeks to design mechanisms that are exactly truthful even when agents explicitly value privacy ([9,10,4]). Recently, Huang and Kannan show that the mechanism used by McSherry and Talwar is *maximal in range*, and so can be made exactly truthful through the use of payments [11].

Feigenbaum, Jaggard, and Schapira consider (using a different notion of privacy) how the implementation of an auction can affect how many bits of information are leaked about individuals' bids [12].

Most related to this paper is an orthogonal direction initiated by Ghosh and Roth [2], who consider the problem of a data analyst who wishes to buy data from a population for the purposes of computing an accurate estimate of some population statistic. Individuals experience cost as a function of their privacy loss (as measured by differential privacy), and must be incentivized by a truthful mechanism to report their true costs. In particular, [2] show that if individuals experience disutility as a function of differential privacy, and if costs for privacy can be arbitrarily correlated with private types, then no individually rational direct revelation mechanism can achieve any nontrivial accuracy. Fleischer and Lyu [13] overcome this impossibility result by moving to a Bayesian setting, in which costs are drawn from known prior distributions which depend on the individual's private data, and by proposing a relaxation of how individuals experience privacy cost. In this paper, we also overcome this impossibility result, but by an abandoning the direct revelation model in favor of a model in which a surveyor can approach random individuals from the population and offer them take-it-or-leave-it offers, and by introducing a slightly different model for how individuals experience cost as a function of privacy. In contrast to [13], our results allow for *worst-case* correlations between private data and costs for privacy, and do not require any Bayesian assumptions. Also in this line of work, Roth and Schoenebeck [14] consider the problem of deriving Bayesian optimal survey mechanisms for computing minimum variance unbiased estimators of a population statistic from individuals who have costs for participating in the survey. Although the motivation of this work is similar, the results are orthogonal. In the present paper, we take a prior-free approach and model costs for private access to data using the formalism of differential privacy. In contrast, [14] takes a Bayesian approach, assuming a known prior over agent costs, and does not attempt to provide any privacy guarantee, and instead only seeks to pay individuals for their participation.

4 Preliminaries

We model databases as an ordered multiset of elements from some universe X: $D \in X^*$ in which each element corresponds to the data of a different individual. We call two databases *neighbors* if they differ in the data of only a single individual.

Definition 1. *Two databases of size n $D, D' \in X^n$ are* neighbors *with respect to individual i if for all $j \neq i \in [n]$, $D_j = D'_j$.*

We can now define *differential privacy*. Intuitively, differential privacy promises that the output of a mechanism does not depend too much on any single individual's data.

Definition 2 ([5]). *A randomized algorithm A which takes as input a database $D \in X^*$ and outputs an element of some arbitrary range R is ε_i-differentially private with respect to individual i if for all databases $D, D' \in X^*$ that are neighbors with respect to individual i, and for all subsets of the range $S \subseteq R$, we have:*

$$\mathbb{P}\mathrm{r}[A(D) \in S] \leq \exp(\varepsilon_i) Pr[A(D') \in S]$$

A is ε_i-minimally differentially private with respect to individual i if $\varepsilon_i = \inf(\varepsilon \geq 0)$ such that A is ε-differentially private with respect to individual i. When it is clear from context, we will simply write ε_i-differentially private to mean ε_i-minimally differentially private.

A simple and useful fact is that *post-processing* does not affect differential privacy guarantees.

Fact 41. *Let $A : X^* \to R$ be a randomized algorithm which is ε_i-differentially private with respect to individual i, and let $f : R \to T$ be an arbitrary (possibly randomized) function mapping the range of A to some abstract range T. Then the composition $g \circ f : X^* \to T$ is ε_i-differentially private with respect to individual i.*

A useful distribution is the *Laplace* distribution.

Definition 3 (The Laplace Distribution). *The Laplace Distribution with mean 0 and scale b is the distribution with probability density function: $Lap(x|b) = \frac{1}{2b}\exp(-\frac{|x|}{b})$. We will sometimes write $Lap(b)$ to denote the Laplace distribution with scale b, and will sometimes abuse notation and write $Lap(b)$ simply to denote a random variable $X \sim Lap(b)$.*

A fundamental result in data privacy is that perturbing low sensitivity queries with Laplace noise preserves ε-differential privacy.

Theorem 1 ([5]). *Suppose $f : X^* \to \mathbb{R}^k$ is a function such that for all adjacent databases D and D', $||f(D) - f(D')||_1 \leq 1$. Then the procedure which on input D releases $f(D) + (X_1, \ldots, X_k)$, where each X_i is an independent draw from a $Lap(1/\varepsilon)$ distribution, preserves ε-differential privacy.*

We consider a (possibly infinite) collection of individuals drawn from some distribution over types $\mathcal{D}$. There exists a finite collection of private types T. Each individual is described by a private type $t_i \in T$, as well as a nondecreasing cost function $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ that measures her disutility $c_i(\varepsilon_i)$ for having her private type used in a computation with a guarantee of ε_i-differential privacy.

Agents interact with the mechanism as follows. The mechanism will be endowed with the ability to select an individual i uniformly at random (without replacement) from $\mathcal{D}$, by making a call to a *population oracle* $\mathcal{O}_{\mathcal{D}}$. Once an individual i has been sampled, the mechanism can present i with a *take-it-or-leave-it offer*, which is a tuple $(p_i, \varepsilon_i^1, \varepsilon_i^2) \in \mathbb{R}_+^3$. p_i represents an offered payment, and ε_i^1 and ε_i^2 represent two privacy parameters. The agent then makes her participation decision, which consists of one of two actions: she can *accept* the offer, or she can *reject* the offer. If she accepts the offer, she communicates her (verifiable) private type t_i to the auctioneer, who may use it in a computation which is ε_i^2-differentially private with respect to agent i. In exchange she receives payment p_i. If she rejects the offer, she need not communicate her type, and receives no payment. Moreover, the mechanism guarantees that the bit representing whether or not agent i accepts the offer is used only in an ε_i^1-differentially private way, regardless of her participation decision.

5 An Alternate Model of Privacy Costs

We model agents as caring only about the privacy of their private type t_i, but because of possible correlations between costs and types they may also experience a cost when information about their cost function $c_i(\varepsilon_i)$ is revealed. To capture this while avoiding Bayesian assumptions, we take the following approach.

Implicitly, there is a (possibly randomized) process f_i which maps a user's private type t to her cost function c_i, but we make no assumption about the form of this map. This takes a worst case view—i.e., we have no prior over individuals' cost functions. For a point of reference, in a Bayesian model, the function f_i would represent user i's marginal distribution over costs conditioned on her type.

When individual i is faced with a take-it-or-leave-it offer, her type may affect two computations: first, her participation decision (which may be a function of her type) is used in some computation A_1 which will be ε_i^1-differentially private. Then, if she accepts the offer, she allows her type to be used in some ε_i^2-differentially private computation, A_2.

We model individuals as *caring about the privacy of their cost function only insofar as it reveals information about their private type.* Because their cost function is determined as a function of their private type, if P is some predicate over cost functions, if $P(c_i) = P(f_i(t_i))$ is used in a way that guarantees ε_i-differential privacy, then the agent experiences a privacy loss of some $\varepsilon_i' \leq \varepsilon_i$ (which corresponds to a disutility of some $c_i(\varepsilon_i') \leq c_i(\varepsilon_i)$). We write $g_i(\varepsilon_i) = \varepsilon_i'$ to denote this correspondence between a given privacy level and the effective privacy loss due to use of the cost function at that level of privacy. For example, if f_i is a deterministic injective mapping, then $f_i(t_i)$ is as disclosive as t_i and so $g_i(\varepsilon_i) = \varepsilon_i$. On the other hand, if f_i produces a distribution independent of the user's type, then $g_i(\varepsilon_i) = 0$ for all ε_i. Note that by assumption, $0 \leq g_i(\varepsilon_i) \leq \varepsilon_i$ for all ε_i and g_i.

5.1 Cost Experienced from a Take-It-or-Leave-It Mechanism

Definition 4. *A Private Take-It-Or-Leave-It Mechanism is composed of two algorithms,* A_1 *and* A_2*.* A_1 *makes offer* $(p_i, \varepsilon_i^1, \varepsilon_i^2)$ *to individual* i *and receives a binary* participation decision*. If player* i participates*, she receives a payment of* p_i *in exchange for her private type* t_i*.* A_1 *performs no computation on* t_i*. The privacy parameter* ε_i^1 *for* A_1 *is computed by viewing the input to* A_1 *to be the vector of participation decisions, and the output to be the number of individuals to whom offers were made, the offers* $(p_i, \varepsilon_i^1, \varepsilon_i^2)$*, and an* ε_i^1*-differentially private count of the number of players who chose to participate at the highest price we offer.*

Following the termination of A_1*, a separate algorithm* A_2 *computes on the reported private types of these participating individuals and outputs a real number* $\widehat{s}$*. The privacy parameter* ε_i^2 *of* A_2 *is computed by viewing the input to be the private types of the participating agents, and the output as* $\widehat{s}$*.*

We assume that agents have quasilinear utility (cost) functions: given a payment p_i, an agent i who declines a take-it-or-leave-it offer (and thus receives no payment) and whose participation decision is used in an ε_i^1-differentially private way experiences utility $u_i = -c_i(g_i(\varepsilon_i^1)) \geq -c_i(\varepsilon_i^1)$. An agent who accepts a take-it-or-leave-it offer and receives payment p, whose participation decision is used in an ε_i^1-differentially private way, and whose private type is subsequently used in an ε_i^2-differentially private way experiences utility $u_i = p_i - c_i(\varepsilon_i^2 + g_i(\varepsilon_i^1)) \geq p_i - c_i(\varepsilon_i^2 + \varepsilon_i^1)$, by a composition property of differential privacy.

Remark 1. This model captures the correct cost model in a number of situations. Suppose, for example, that costs have correlation 1 with types, and $c_i(\varepsilon) = \infty$ if and only if $t_i = 1$, otherwise $c_i(\varepsilon) \ll p_i$. Then, asking whether an agent wishes to accept an offer $(p_i, \varepsilon_i, \varepsilon_i)$ is equivalent to asking whether $t_i = 1$ or not, and those accepting the offer are in effect answering this question twice. In this case, we have $g_i(\varepsilon) = \varepsilon$. On the other hand, if types and costs are completely uncorrelated, then there is no privacy loss associated with responding to a take-it-or-leave-it offer. This is captured by setting $g_i(\varepsilon) = 0$.

Agents wish to maximize their utility, and so the following lemma is immediate:

Lemma 1. *A utility-maximizing agent* i *will accept a take-it-or-leave-it offer* $(p_i, \varepsilon_i^1, \varepsilon_i^2)$ *when* $p_i \geq c_i(\varepsilon_i^1 + \varepsilon_i^2)$

Proof. We simply compare the lower bound on an agent's utility when accepting an offer with an upper bound on an agent's utility when rejecting an offer to find that agent i will always accept when

$$p_i - c_i(\varepsilon_i^1 + \varepsilon_i^2) \geq 0.$$

Remark 2. Note that this lemma is tight exactly when agent types are uncorrelated with agent costs, i.e., when $g_i(\varepsilon) = 0$. When agent types are highly correlated with costs, then rejecting an offer becomes more costly, and agents may accept take-it-or-leave-it offers at lower prices.

We make no claims about how agents respond to offers $(p_i, \varepsilon_i^1, \varepsilon_i^2)$ for which $p_i < c_i(\varepsilon_i^1 + \varepsilon_i^2)$. Note that since agents can suffer negative utility even by rejecting offers, it is possible that they will accept offers that lead to experiencing negative utility. Thus, in our setting, take-it-or-leave-it offers do not necessarily result in participation decisions that *truthfully* reflect costs in the standard sense. Nevertheless, Lemma 1 will provide a strong enough guarantee for us of *one-sided truthfulness*: we can guarantee that rational agents will accept all offers that guarantee them non-negative utility.

Note that our mechanisms will satisfy only a relaxed notion of *individual rationality*: we have not endowed agents with the ability to avoid having been given a take-it-or-leave it offer, even if both options (taking or rejecting) would leave her with negative utility. Agents who reject take-it-or-leave-it offers can experience negative utility in our mechanism because their rejection decision is observed and used in a computation; we limit this negative utility and the corresponding deviation from individual rationality by treating their rejection decision in a differentially private manner. Once the take-it-or-leave-it offer has been presented, agents are free to behave selfishly. We feel that both of these relaxations (of truthfulness and individual rationality) are well motivated by real world mechanisms in which surveyors may approach individuals in public, and crucially, they are necessary in overcoming the impossibility result in [2].

Most of our analysis holds for arbitrary cost functions c_i, but we do a benchmark cost comparison assuming *linear* utility functions of the form $c_i(\varepsilon) = v_i\varepsilon$, for some quantity v_i.

5.2 Accuracy

Our mechanism is designed to be used by a data analyst who wishes to compute some statistic about the private type distribution of the population. Specifically, the analyst gives the mechanism some function $Q : T \to [0, 1]$, and wishes to compute $a = \mathbb{E}_{t_i \sim \mathcal{D}}[Q(t_i)]$, the average value that Q takes among the population of agents $\mathcal{D}$. The analyst wishes to obtain an *accurate* answer, defined as follows:

Definition 5. *A randomized algorithm, given as input access to a population oracle $\mathcal{O}_{\mathcal{D}}$ which outputs an estimate $M(\mathcal{O}_{\mathcal{D}}) = \widehat{a}$ of a statistic $a = \mathbb{E}_{t_i \sim \mathcal{D}}[Q(t_i)]$ is α-accurate if:*

$$\Pr[|\widehat{a} - a| > \alpha] < \frac{1}{3}$$

where the probability is taken over the internal randomness of the algorithm and the randomness of the population oracle.

The constant $\frac{1}{3}$ is arbitrary, and is fixed only for convenience. It can be replaced with any other constant value without qualitatively affecting our results.

5.3 Cost

We will evaluate the cost incurred by our mechanism using a bi-criteria benchmark: For a parametrization of our mechanism which gives accuracy α, we will compare our mechanism's cost to a benchmark algorithm that has perfect knowledge of each individual's cost function, but is constrained to make every individual the same take-it-or-leave-it offer (the same fixed price is offered to each person in exchange for some fixed ε'-differentially private computation on her private type) while obtaining $\alpha/32$ accuracy.[1] That is, the benchmark mechanism must be "envy-free", and may obtain better accuracy than we do, but only by a constant factor. On the other hand, the benchmark mechanism has several advantages: it has full knowledge of each player's cost, and need not be concerned about sample error. For simplicity, we will state our benchmark results in terms of individuals with linear cost functions.

6 Mechanism and Analysis

Due to space constraints, proofs can be found in the full version.

6.1 The Take-It-or-Leave-It Mechanism

In this section we describe our mechanism. It is *not* a direct revelation mechanism, and instead is based on the ability to present take-it-or-leave-it offers to uniformly randomly selected individuals from some population. This is intended to model the scenario in which a surveyor is able to stand in a public location and ask questions or present offers to passers by (who are assumed to arrive randomly). Those passing the surveyor have the freedom to accept or reject the offer, but they cannot avoid having heard it.

Our mechanism consists of two algorithms. Algorithm 1, the Harassment Mechanism, is run on samples from the population with privacy guarantee ε_0, until it terminates at some final epoch $\widehat{j}$; and then Algorithm 2, the Estimation Mechanism, is run on $(\text{AcceptedSet}_{\widehat{j}}, \text{EpochSize}(\widehat{j}), \varepsilon_0)$. The Harassment Mechanism operates in epochs, wherin a large number of individuals are each offered the same price. The price we offer increases by a multiplicative $(1+\eta)$ in each epoch, for some fixed η . If a differentially private count of the number of players accepting the offer in a given epoch is high enough, we call this the final epoch, and hand the participating individuals over to the Estimation Mechanism. The Estimation Mechanism then computes a differentially private (noisy) version of the desired statistic over this set of individuals who chose to participate at the highest price.

6.2 Privacy

Note that our mechanism offers the same ε_0 in every take-it-or-leave-it offer.

Theorem 2. *The Harassment Mechanism is ε_0-differentially private with respect to the participation decision of each individual approached.*

[1] Note that we have made no attempt to optimize the constant.

Algorithm 1. Algorithm A_1, the "Harassment Mechanism". It is parametrized by an accuracy level α, and we view its *input* to be the participation decision of each individual approached with a take-it-or-leave-it offer, and its *observable output* to be the number of individuals approached, the payments offered, and the noisy count of the number of players who accepted the offer in the final epoch.

Let $\text{EpochSize}(j) \leftarrow \frac{100(\log j+1)}{\alpha^2}$.
Let $j \leftarrow 1$.
Let $\varepsilon_0 = \alpha$
while TRUE do
  Let $\text{AcceptedSet}_j \leftarrow \emptyset$ and $\text{NumberAccepted}_j \leftarrow 0$ and $\text{Epoch}_j \leftarrow \emptyset$
  for $i = 1$ to $\text{EpochSize}(j)$ **do**
    Sample a new individual x_i from $\mathcal{D}$.
    Let $\text{Epoch}_j \leftarrow \text{Epoch}_j \cup \{x_i\}$.
    Offer x_i the take-it-or-leave it offer $(p_j, \varepsilon_0, \varepsilon_0)$ with $p_j = (1+\eta)^j$
    if i accepts **then**
      Let $\text{AcceptedSet}_j \leftarrow \text{AcceptedSet}_j \cup \{x_i\}$ and
        $\text{NumberAccepted}_j \leftarrow \text{NumberAccepted}_j + 1$.
  Let $\nu_j \sim \text{Lap}(1/\varepsilon_0)$ and $\text{NoisyCount}_j = \text{NumberAccepted}_j + \nu_j$
  if $\text{NoisyCount}_j \geq (1-\alpha/8)\text{EpochSize}(j)$ **then**
    Call Estimate$(\text{AcceptedSet}_j, \text{EpochSize}(j), \varepsilon_0)$.
  else
    Let $j \leftarrow j+1$

Algorithm 2. The Estimation Mechanism. We view its *inputs* to be the private types of each participating individual from the final epoch, and its *output* is a single numeric estimate.

Estimate$(\text{AcceptedSet}, \text{EpochSize}, \varepsilon)$:
  Let $\widehat{a} = \sum_{x_i \in \text{AcceptedSet}} Q(x_i) + \text{Lap}(1/\varepsilon)$
  Output $\widehat{a}/\text{EpochSize}$.

Theorem 3. *The Estimation Mechanism is ε_0-differentially private with respect to the participation decision and private type of each individual approached.*

Note that these two theorems, together with Lemma 1, imply that each agent will accept her take-it-or-leave-it offer of $(p_j, \varepsilon_0, \varepsilon_0)$ whenever $p_j \geq c_i(2\varepsilon_0)$.

6.3 Accuracy

Theorem 4. *Our overall mechanism, which first runs the Harassment Mechanism and then hands the types of the accepting players from the final epoch to the Estimation Mechanism, is α-accurate.*

6.4 Benchmark Comparison

In this section we compare the cost of our mechanism to the cost of an omniscient mechanism that is constrained to make envy-free offers and achieve $\Theta(\alpha)$-accuracy. For the purposes of the cost comparison, in this section we assume that the individuals our algorithm approaches have linear cost functions: $c_i(\varepsilon) = v_i\varepsilon$ for some $v_i \in \mathbb{R}^+$.

Let $v(\alpha)$ be the smallest value v such that $\Pr_{x_i \sim \mathcal{D}}[v_i \leq v] \geq 1 - \alpha$. In other words, $(v(\alpha) \cdot 2\varepsilon, \varepsilon, \varepsilon)$ is the cheapest take-it-or-leave-it offer for ε-units of privacy that in the underlying population distribution would be accepted with probability at least $1 - \alpha$, It follows that:

Lemma 2. *Any $(\alpha/32)$-accurate mechanism that makes the same take-it-or-leave-it offer to every individual $x_i \sim \mathcal{D}$ must in expectation pay in total at least $\Theta(\frac{v(\frac{\alpha}{8})}{\alpha})$. Note that because here we assume cost functions are linear, this quantity is fixed independent of the number of agents the mechanism draws from $\mathcal{D}$.*

We now wish to bound the expected cost of our mechanism, and compare it to our benchmark cost, $\text{BenchMarkCost} = \Theta(\frac{v(\frac{\alpha}{8})}{\alpha})$.

Theorem 5. *The total expected cost of our mechanism is at most*

$$\begin{aligned}\mathbb{E}[\mathit{MechanismCost}] &= O\left(\log\log\left(\alpha \cdot v(\alpha/8)\right) \cdot \mathit{BenchMarkCost} + \frac{1}{\alpha^2}\right) \\ &= O\left(\log\log\left(\alpha^2 \cdot \mathit{BenchMarkCost}\right) \cdot \mathit{BenchMarkCost} + \frac{1}{\alpha^2}\right).\end{aligned}$$

Remark 3. Note that the additive $1/\alpha^2$ term is necessary only in the case in which $v(\alpha/8) \leq (1+\eta)/\alpha$: i.e., only in the case in which the *very first* offer will be accepted by a $1 - \alpha/8$ fraction of players with high probability. In this case, we have started off offering too much money, right off the bat. An additive term is necessary, intuitively, because we cannot compete with the benchmark cost in the case in which the benchmark cost is arbitrarily small.[2]

7 Discussion

In this paper, we have proposed a method for accurately estimating a statistic from a population that experiences cost as a function of their privacy loss. The statistics we consider here take the form of the expectation of some predicate over the population. We leave to future work the consideration of other, nonlinear, statistics. We have circumvented the impossibility result of [2] by using a mechanism empowered with the ability to approach individuals and make them take-it-or-leave-it offers (instead of relying on a direct revelation mechanism),

[2] We thank Lisa Fleischer and Yu-Han Lyu for pointing out the need for the additive term.

and by relaxing the measure by which individuals experience privacy loss. Moving away from direct revelation mechanisms seems to us to be inevitable: if costs for privacy can be correlated with private data, then merely asking for individuals to report their costs is inevitably disclosive, for any reasonable measure of privacy. On the other hand, we do not claim that the model we use for how individuals experience cost as a function of privacy is “the” right one. Nevertheless, we have argued that some relaxation away from individuals experiencing privacy cost entirely as a function of the differential privacy parameter of the entire mechanism is inevitable (as made particularly clear in the setting of take-it-or-leave-it offers, in which individuals in this model would accept arbitrarily low offers). In particular, we believe that the style of survey mechanism presented in this paper, in which the mechanism may approach individuals with take-it-or-leave-it offers, is realistic, and any reasonable model for how individuals value their privacy should predict reasonable behavior in the face of such a mechanism.

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The Max-Distance Network Creation Game on General Host Graphs*

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Abstract. In this paper we study a generalization of the classic *network creation game* to the scenario in which the n players sit on a given arbitrary *host graph*, which constrains the set of edges a player can activate at a cost of $\alpha \geq 0$ each. This finds its motivations in the physical limitations one can have in constructing links in practice, and it has been studied in the past only when the routing cost component of a player is given by the sum of distances to all the other nodes. Here, we focus on another popular routing cost, namely that which takes into account for each player its *maximum* distance to any other player. For this version of the game, we first analyze some of its computational and dynamic aspects, and then we address the problem of understanding the structure of associated pure Nash equilibria. In this respect, we show that the corresponding price of anarchy (PoA) is fairly bad, even for several basic classes of host graphs. More precisely, we first exhibit a lower bound of $\Omega(\sqrt{n/(1+\alpha)})$ for any $\alpha = o(n)$. Notice that this implies a counter-intuitive lower bound of $\Omega(\sqrt{n})$ for the case $\alpha = 0$ (i.e., edges can be activated for free). Then, we show that when the host graph is restricted to be either k-regular (for any constant $k \geq 3$), or a 2-dimensional grid, the PoA is still $\Omega(1+\min\{\alpha, \frac{n}{\alpha}\})$, which is proven to be tight for $\alpha = \Omega(\sqrt{n})$. On the positive side, if $\alpha \geq n$, we show the PoA is $O(1)$. Finally, in the case in which the host graph is very sparse (i.e., $|E(H)| = n-1+k$, with $k = O(1)$), we prove that the PoA is $O(1)$, for any α.

1 Introduction

In a *network creation game* (NCG), we are given n players identified as the nodes of a graph, and each player attempts to connect itself to all the other players. In such a decentralized process, each player aims to selfishly optimize a certain

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routing cost towards the other players. Thus, its action consists of choosing a suitable subset of players, which are then made adjacent through the activation of the corresponding set of incident edges. Unavoidably, activating a link incurs a cost to the player, and so the overall *building* cost should be strategically balanced with the aforementioned routing cost.

Due to their generality, it is in clear evidence that NCGs can model very different practical situations, depending on how all the build-up ingredients are mixed. In the very classic formulation of the game [7], each player has no limitations in choosing a subset of adjacent players, its routing cost is a function of the sum of distances to all the other players (i.e., the so called *sum cost*), and activating a link has a fixed cost $\alpha \geq 0$. Not surprisingly, this model was devised by the economists, which were mainly interested in understanding whether the attainment of an equilibrium status (i.e., a status in which players are not willing to move from) for a mutual-relationships social system is compatible with the behavior of the players, which tend to establish selfishly their personal contacts.

With the recent advent of the algorithmic game theory, the interest on NCGs has been reawakened. This is especially due to the fact that NCGs are fit to model the decentralized construction of *communication* networks, in which the constituting components (e.g., routers and links) are activated and maintained by different owners, as in the Internet. According to its performance measurement philosophy, computer scientists put a new special emphasis on the challenge of understanding how the social utility for the (very large) system as a whole is affected by the selfish behavior of the players. This trend originated from the paper of Fabrikant *et al.* [6], and was then followed by a sequel of papers, as detailed in the following.

Previous Work. As said before, the canonical form of a NCG, also known as SumNCG, is as follows: We are given a set of n players, say V, where the strategy space of player $v \in V$ is the power set $2^{V \setminus \{v\}}$. Given a combination of strategies $\sigma = (\sigma_v)_{v \in V}$, let $G(\sigma)$ denote the underlying undirected graph whose node set is V, and whose edge set is $E(\sigma) = \{(v,u) : v \in V \wedge u \in \sigma_v\}$. Then, the *cost* incurred by player v under σ is

$$C_v(\sigma) = \alpha \cdot |\sigma_v| + \sum_{u \in V} d_{G(\sigma)}(u,v) \tag{1}$$

where $d_{G(\sigma)}(u,v)$ is the distance between nodes u and v in $G(\sigma)$. Thus, the cost function implements the inherently antagonistic goals of a player, which on the one hand attempts to buy as little edges as possible, and on the other hand aims to be as close as possible to the other nodes in the resulting network. These two criteria are suitably balanced in (1) by making use of the parameter $\alpha \geq 0$. Consequently, the *Nash Equilibria*[1] (NE) space of the game is a function of it. Actually, if we characterize such a space in terms of the *Price of Anarchy* (PoA), then this has been shown to be constant for all values of α except for $n^{1-\varepsilon} \leq \alpha \leq 273\,n$, for any $\varepsilon \geq 1/\log n$ (see [9]).

[1] In this paper, we only focus on *pure* strategies Nash equilibria.

A first natural variant of SumNCG was introduced in [3], where the authors redefined the player cost function as follows

$$C_v(\sigma) = \alpha \cdot |\sigma_v| + \max\{d_{G(\sigma)}(u, v) : u \in V\}. \tag{2}$$

This variant, named MaxNCG, received further attention in [9], where the authors improved the PoA of the game on the whole range of values of α, obtaining in this case that the PoA is constant for all values of α except for $129 > \alpha = \omega(1/\sqrt{n})$.

Besides these two basic models, many variations on the theme have been defined. In an effort of defining α-free models, in [8] the authors proposed a variant in which a player, when forming the network, has a limited budget to establish links to other players. This way, the player cost function will only describe the goal of the player, namely either the maximum distance or the total distance to other nodes. Since in [8] links and hence the resulting graph are seen as directed, a natural variant of the model was given in [5], where the undirected case was considered. Afterwards, in [2] the authors proposed a model complementing the one given in [5]. More precisely, they assumed that the cost function of each player now only consists of the number of bought edges (without any budget on them), and a player needs to connect to the network by satisfying the additional constraint of staying within a given either maximum or total distance to the rest of players. Then, in [1] the authors proposed a further variant, called BasicNCG, in which given some existing network, the only improving transformations allowed are *edge swap*, i.e., a player can only modify a *single* incident edge, by either replacing it with a new incident edge, or by removing it. Recently, this model has been extended to the case in which edges are oriented and players can swap only outleading edges [12]. Notice that, differently from the previous models, in BasicNCG we have the positive news that for a player it is *not* NP-hard to find a best response to the strategies of other players.

Generally speaking, in all the above models the obtained results on the PoA are asymptotically worse than those we get in the two basic models, and we refer the reader to the cited papers for the actual bounds.

Our Results. In this paper we concentrate on a seemingly underplayed generalization of NCGs, namely that in which for each player the set of possible adjacent nodes is constrained by a given connected, undirected graph H, which in the end will host the created network. This finds its practical motivations in the physical limitations of constructing links, and was originally studied in [4] for SumNCG, where it is shown that the PoA is upper bounded by $O(1 + \sqrt{\alpha})$ and $\min\{O(\sqrt{n}), 1 + n^2/\alpha\}$ for $\alpha < n$ and $\alpha \geq n$, respectively, and lower bounded by $\Omega(1 + \min\{\alpha/n, n^2/\alpha\})$. Here, we focus on the max-distance version, that we call MaxNCG(H), and we show that also in this case the PoA is fairly bad,[2] even when the host graph is restricted to some basic standard layout patterns. More precisely, we show that for k-regular (with any constant $k \geq 3$) and 2-dimensional grid host graphs, the PoA is $\Omega(1 + \min\{\alpha, n/\alpha\})$, and this is asymptotically tight for $\alpha = \Omega(\sqrt{n})$ and $\alpha \leq n$, since we can prove a general upper bound of $O\left(1+\frac{n}{\alpha+\rho_H}\right)$, where ρ_H is the radius of H. Moreover, on general host graphs, we exhibit a lower bound of $\Omega\left(\sqrt{\frac{n}{1+\alpha}}\right)$ for $0 \leq \alpha = o(n)$. Quite surprisingly, this implies that the PoA is $\Omega(\sqrt{n})$ even when the players can build edges for free. On the positive side, if $\alpha \geq n$, we show the PoA is at most 2 (this is a direct consequence of the fact that in this case any equilibrium is a tree). Finally, in the meaningful practical case in which the host graph is sparse (i.e., $|E(H)| = n - 1 + k$, with $k = O(n)$), we prove that the PoA is $O(1 + k)$, and so for very sparse graphs, i.e. $k = O(1)$, we obtain that the PoA is constant.

Preliminarily to the above study, we also provide some results concerning the computational and dynamic aspects of the game. First, we prove that computing a best response for a player is NP-hard for any $0 < \alpha = O(n^{1-\epsilon})$, thus extending a similar result given in [9] for MaxNCG when $\alpha = 2/n$. Then, we prove that MaxNCG(H) is not a potential game, by showing that an improving response dynamic does not guarantee to converge to an equilibrium, even if we assume a minimal *liveness* property that no player is prevented from moving for arbitrarily many steps. This implies that an improving response dynamic may not converge for the MaxNCG game as well (after relaxing such a liveness property). To the best of our knowledge, a similar negative result is only known for the sum-distance version of BasicNCG [11].

The paper is organized as follows: in Section 2 we analyze the computational/convergence aspects of the game, while Sections 3 and 4 discuss the upper and lower bounds to the PoA, respectively.

2 Preliminary Results

First of all, we observe that, as for the sum-distance version of the problem studied in [4], it is open to establish whether MaxNCG(H) always admits an equilibrium. This problem is particularly intriguing, since the topology of H could play a discriminating role on that. We conjecture an affirmative answer to this question for any $\alpha > 0$ (for $\alpha = 0$ it is trivially true). As a first step towards this direction, observe that given any H, a breadth-first search tree rooted at a center of H, and in which each node owns the edges towards its children, is an equilibrium whenever $\alpha \geq 2\rho_H - 1$, where ρ_H is the radius of H.

[2] According to the spirit of the game, we concentrate on connected equilibria only. In fact, to avoid pathological disconnected equilibria, we can slightly modify the player's cost function (2) as it was done in [5], in order to incentive the players to converge to connected equilibria. Alternatively, this can be obtained by assuming that initially the players sit on a connected network (embedded in the hosted graph), and they move (non-simultaneously) with a myopic best/improving response dynamics.

Besides that, and similarly to other NCGs, we also have the bad news that it is hard for a player computing a best response, as stated in the following theorem (whose proof will be given in the full version of the paper). Notice that this is true for any $0 < \alpha = O(n^{1-\epsilon})$, and so this extends the NP-hardness proof given in [9] and holding for complete host graphs and $\alpha = 2/n$.

Theorem 1. *For every constant $\epsilon > 0$ and for every $0 < \alpha = O(n^{1-\epsilon})$, the problem of computing a best response strategy of a player is* NP*-hard.*

On the other hand, it is interesting to notice that the problem of finding a best response is fixed-parameter tractable w.r.t. to the maximum degree of H. Hence, for virtually all practical purposes, it is reasonable to assume that players can actually adhere to the best response dynamics. Incidentally, this could also help in circumventing a possible convergence problem which may arise by following instead an improving response dynamics, as suggested by the next. □

Theorem 2. *For every value of $\alpha < \frac{n}{2} - 6$,* MaxNCG(H) *is not a potential game. Moreover, if $\alpha > 0$, an improving response dynamics may not converge to an equilibrium.*

Proof. To prove that no potential function can exist we will show a cyclic sequence of strategy profiles where, at the end of each cycle, the total cost of the moving players will decrease by a positive constant amount.

Let $l > \alpha + 6$ be an integer satisfying $l \equiv 1 \pmod 3$ and consider a host graph H similar to the one shown in Figure 1. H is composed by a cycle of l nodes labelled from 1 to l, by a path of length $l-1$ with endpoints x and z, and by all the edges between x and the nodes of the cycle. The strategy profile σ_a being played is shown by using a graphical notation explained in the caption.

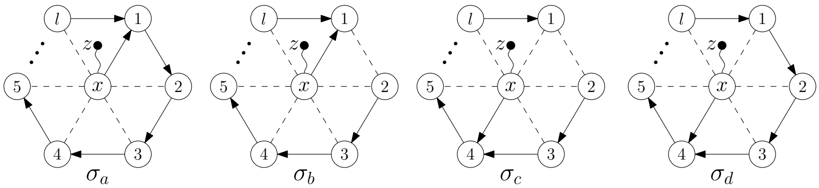

Fig. 1. Representation of the strategy changes used in the proof of Theorem 2. On the left side, the initial configuration, where directed edges exit from their respective owner node, dashed edges are non-bought edges of H, and the spline denotes a path between z and x, whose edges are arbitrarily owned.

In such a status, player 1 is paying $\alpha + l - 1$, whilst changing the strategy to σ_b by removing the edge $(1, 2)$ yields a cost of $l-1$, thus saving α. Observe that now $C_x(\sigma_b)$ is $\alpha + l$, and so x has interest in swapping the edge $(x, 1)$ with the edge $(x, 4)$, thus obtaining the strategy σ_c and saving 1. In such configuration $C_1(\sigma_c)$ has increased to $2l - 4$, therefore player 1 can buy back the edge $(1, 2)$, as shown in strategy σ_d, thus reducing its cost to $\alpha + l + 2$, i.e., saving $l - (\alpha + 6) > 0$.

Notice how configuration σ_d is similar to σ_a, with the only difference being the edge bought by x. Since $l \equiv 1 \pmod 3$, by repeating l times these strategy changes, every node in the cycle $\langle 1, \ldots, l \rangle$ will play a move at least once, and the resulting configuration is identical to σ_a, hence the players will cycle.

To prove the latter part of the claim it suffices to note that after each cycle: (i) for $\alpha > 0$ each strategy change is an improving response, and (ii) that the nodes in the path from x to z other than x, can never change their strategy. □

Actually, the above proof shows that the improving response dynamics may not converge even if the minimal liveness property that each player takes a chance to make an improving move every fixed number of steps is guaranteed. Indeed, as observed in the proof, the players sitting on the path appended to x do not move just because they cannot. However, if we modify the above proof by letting H be complete and by preventing such nodes to move, then the same arguments continue to hold. This shows that the improving response dynamics may not converge on complete host graphs as well, i.e., for the classic NCG.

3 Upper Bounds

In this section we prove some upper bounds to the PoA for the game. In what follows, for a generic graph G, we denote by ρ_G and δ_G its radius and its diameter, respectively, and by $\varepsilon_G(v)$ the eccentricity of node v in G. Moreover, we denote by $SC(\sigma)$ the *social cost* of a generic strategy profile σ (i.e., the sum of players' individual costs), and by Opt a strategy profile minimizing the social cost. Then

Lemma 1. *Let $G = G(\sigma)$ be a NE, and let $\alpha = O(n)$. Then, we have that $SC(\sigma)/SC(\text{Opt}) = O\big(\frac{\rho_G}{\alpha+\rho_H}\big)$.*

Proof. Let u be a center of G, and let T be a shortest path tree of G rooted at u. Clearly, the diameter of T is at most $2\,\rho_G$. Now, for every node v, let us denote by k_v the number of edges of T bought by v in σ. The key argument is that if a node v bought only the k_v edges of T, its eccentricity would be at most $\varepsilon_T(v) \leq 2\,\rho_G$. Hence, since σ is a NE, we have that $C_v(\sigma) \leq \alpha k_v + 2\,\rho_G$. By summing up the inequalities over all nodes, we obtain

$$SC(\sigma) = \sum_v C_v(\sigma) \leq \alpha \sum_v k_v + 2n\,\rho_G = \alpha(n-1) + 2n\,\rho_G.$$

Now, since $SC(\text{Opt}) \geq \alpha(n-1) + n\,\rho_H$, we have

$$\frac{SC(\sigma)}{SC(\text{Opt})} \leq \frac{\alpha(n-1) + 2n\,\rho_G}{\alpha(n-1) + n\,\rho_H} \leq 1 + \frac{2n\,\rho_G}{\alpha(n-1) + n\,\rho_H} = O\Big(\frac{\rho_G}{\alpha + \rho_H}\Big). \qquad \square$$

As an immediate consequence, we obtain the following:

Theorem 3. *For $\alpha = O(n)$, the PoA is $O(\frac{n}{\alpha+\rho_H})$.* □

Another interesting consequence of the Lemma 1 concerns sparse host graphs:

Theorem 4. *If the host graph H has $n-1+k$ edges, and $k=O(n)$, then the PoA is $O(k+1)$.*

Proof. Let $G=G(\sigma)$ be an equilibrium network. Since G must be connected, we have that $|E(H)\setminus E(G)|\leq k$. This is sufficient to provide an upper bound to the diameter of G. Indeed, in [10] it is shown that the diameter of a connected graph obtained from a supergraph of diameter δ by deleting h edges is at most $(h+1)\delta$. This implies that in our case $\delta_G \leq (1+k)\delta_H$. Now, the claim follows from Lemma 1. □

Therefore, for very sparse host graphs, i.e., $|E(H)|=n-1+k$, with $k=O(1)$, we have that the PoA is $O(1)$, for any α.

The next theorem shows that the PoA is low when α is at least n. The result has been proved in [3] for the case in which H is a complete graph. In fact, it turns out that the same proof can be used to extend the result to any H.

Theorem 5 ([3]). *For $\alpha \geq n$, the PoA is at most 2.*

We end this section by showing that when either α is small, or the host graph has small diameter, every stable tree (if any) is a good equilibrium. This generalizes a result in [9] given for complete host graphs, which states that the social cost of every acyclic equilibrium is $O(1)$ times the optimum. □

Lemma 2. *Let $(u,v)\in E(H)$ be an edge of the host graph. Then, for every stable graph G, we have $|\varepsilon_G(u)-\varepsilon_G(v)|\leq 1+\alpha$.*

Proof. W.l.o.g. assume $\varepsilon_G(u)\geq\varepsilon_G(v)$. If $(u,v)\in E(G)$, then the claim trivially holds. Otherwise, if u buys the edge (u,v) then its eccentricity will decrease at least by $\varepsilon_G(u)-\varepsilon_G(v)-1$, while its building cost will increase by α. Since G is stable, we have $\varepsilon_G(u)-\varepsilon_G(v)-1\leq\alpha$, and the claim follows. □

Corollary 1. *For every $u,v\in V$ and for every stable graph G, it holds that $|\varepsilon_G(u)-\varepsilon_G(v)|\leq(1+\alpha)\,d_H(u,v)$.* □

Lemma 3. *Let $G=G(\sigma)$ be a stable graph. If there are two nodes $u,v\in V$ such that $\varepsilon_G(v)\geq c\cdot\varepsilon_G(u)+k$ with $c>1$ and $k\in\mathbb{R}$, then $\frac{\delta_G}{\delta_H}\leq 2\cdot\frac{1+\alpha-\frac{k}{\delta_H}}{c-1}$.*

Proof. We have

$$\begin{aligned}\varepsilon_G(v)-\varepsilon_G(u) &\geq c\cdot\varepsilon_G(u)+k-\varepsilon_G(u)\\ &\geq (c-1)\varepsilon_G(u)+k\geq(c-1)\rho_G+k\geq(c-1)\frac{1}{2}\delta_G+k.\end{aligned}$$

Moreover, from Corollary 1, we have

$$\varepsilon_G(v)-\varepsilon_G(u)\leq(1+\alpha)d_H(u,v)\leq(1+\alpha)\delta_H,$$

from which, we obtain

$$(c-1)\frac{1}{2}\delta_G+k\leq(1+\alpha)\delta_H$$

and hence the claim. □

We are now ready to give the following

Theorem 6. *Let σ be a NE such that $G = G(\sigma)$ is a tree. Then,* $\frac{SC(\sigma)}{SC(\text{OPT})} \leq \min\{O(1+\alpha), O(\rho_H)\}$.

Proof. Let us consider a center u of G, and let v be a node in the periphery of G, namely $\varepsilon_G(v) = \delta_G$. Since G is a tree, we have $\varepsilon_G(v) = \delta_G \geq 2\,\rho_G - 1 = 2\,\varepsilon_G(u) - 1$. Now, using Lemma 3 and Lemma 1, the claim follows. □

4 Lower Bounds

In this section we prove some lower bounds to the PoA of the game, as summarized in Table 1.

Table 1. Obtained lower bounds to the PoA

α	$O(\sqrt[3]{n})$	$O(\sqrt{n})$	$\Omega(\sqrt{n})$
PoA	$\Omega\left(\sqrt{\frac{n}{1+\alpha}}\right)$	$\Omega(\alpha)$	$\Omega\left(1+\frac{n}{\alpha}\right)$

Before getting to the technical details, let us discuss the significance of the above bounds. First of all, we notice that the lower bound for $\alpha = \Omega(\sqrt{n})$ is tight, due the upper bound given in the previous section. Moreover, observe that we can obtain such a lower bound for two prominent classes of host graphs, namely for *k-regular graphs* (for any constant $k \geq 3$) and for *2-dimensional grids*.[3] We view this as a meaningful result, due to the practical relevance of such host topologies.[4] Concerning the case $\alpha \in \Omega(\sqrt[3]{n}) \cap O(\sqrt{n})$, we notice that the lower bound holds for the same classes of host graphs, but now it is not tight. Finally, for $\alpha = O(\sqrt[3]{n})$, to prove the lower bound we make use of a specific host graph, but the surprising fact here is that we are able to show a quite large lower bound (i.e., $\Omega(\sqrt{n})$) even for $\alpha = 0$. Summarizing, we point out that we get a polynomial lower bound for any $\alpha = O(n^{1-\varepsilon})$, for any $\varepsilon > 0$, in strong contrast with the almost everywhere constant upper bound to the PoA of MaxNCG.

Theorem 7. *The PoA is* $\Omega\big(1 + \min\{\alpha, \frac{n}{\alpha}\}\big)$*, even when the host graph is a 2-dimensional grid.*

Proof. Let $k = 2p$ where p is an odd number, and let H be a 2-dimensional square grid of $n = k \times k$ vertices. In the rest of the proof, we assume that the vertex in the i-th row and j-th column of the grid is labeled with $\langle i, j\rangle$, where $1 \leq i, j \leq k$.

[3] Notice that a 2-dimensional grid is also planar and bipartite.

[4] Actually, using different constructions, we are also able to prove the same lower bound for outerplanar and series-parallel graphs, but we postpone such result to the full paper version.

For every $1 \leq j \leq k$, let P_j be the path in H which spans all the vertices of the j-th column of H. Let $k^* = \min\{1 + \lfloor \frac{\alpha}{2} \rfloor, k\}$. Let F be the set of edges linking vertex $\langle 1, j \rangle$ with vertex $\langle 1, j+1 \rangle$ iff j is even and let F' be the set of edges linking vertex $\langle k^*, j \rangle$ with $\langle k^*, j+1 \rangle$ iff j is odd.

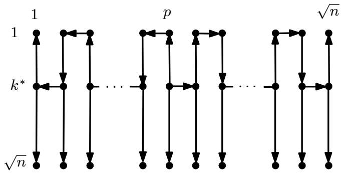

Fig. 2. The stable graph G when the host graph H is a square grid of n vertices

Let G be the subgraph of H whose edge set is $E(G) = F \cup F' \cup \bigcup_{j=1}^{k} E(P_j)$ (see also Figure 2). Observe that G is a tree of radius greater than or equal to $\frac{1}{2}kk^* = \Omega\big(\sqrt{n} \cdot \min\{1+\alpha, \sqrt{n}\}\big)$. Observe also that $\langle k^*, p \rangle$ and $\langle k^*, p+1 \rangle$ are the two centers of G. Let $\langle k^*, p \rangle$ be the root of G and let $\bar{G}$ be the directed version of G where all root-to-leaf paths are directed towards the leaves. Finally, let σ be the strategy profile induced by $\bar{G}$, i.e., each player v is buying exactly the edges in $\bar{G}$ outgoing from v. Clearly, $G(\sigma) = G$.

To prove that σ is a NE, it is enough to show that every vertex $\langle i, j \rangle$, with $1 \leq i \leq k$ and $1 \leq j \leq p$, is playing a best response strategy. Indeed, if we show that $\langle i, j \rangle$ is playing a best response strategy, then, by symmetry, also $\langle i, k-j+1 \rangle$ is playing a best response strategy.

Let i and j be two fixed integers such that $1 \leq i \leq p$ and $1 \leq j \leq k$ and let t be the number of edges bought by $\langle i, j \rangle$ in σ. Since G is a tree and since $\langle k^*, p \rangle$ and $\langle k^*, p+1 \rangle$ are the two centers of G, there exists a vertex $\langle i', j' \rangle$, with $1 + p \leq i' \leq k$ and $1 \leq j' \leq k$, such that the distance in G from $\langle i, j \rangle$ to $\langle i', j' \rangle$ is exactly equal to the eccentricity of $\langle i, j \rangle$ in G. Observe also that the (unique) path in G from $\langle i, j \rangle$ to $\langle i', j' \rangle$ traverses the root as well as the vertex $\langle k^*, p+1 \rangle$. Let $\langle i', j' \rangle$ be any vertex such that $1 + p \leq i' \leq k$ and $1 \leq j' \leq k$. First of all, observe that if we add to G all the edges adjacent to $\langle i, j \rangle$ in H, then the distance from $\langle i, j \rangle$ to $\langle i', j' \rangle$ decreases by at most α. Since the cost of activating new links is at least α, $\langle i, j \rangle$ cannot improve its cost function by buying more than t edges. Now we prove that $\langle i, j \rangle$ cannot improve its cost function by buying at most t edges. First of all, observe that t is the minimum number of edges $\langle i, j \rangle$ has to buy to guarantee connectivity. Moreover, to guarantee connectivity, $\langle i, j \rangle$ has to buy an edge towards some vertex of every subtree of G rooted at any of its t children. Since the subtree of G rooted at $\langle i, j \rangle$ does not contain $\langle i', j' \rangle$ when $\langle i, j \rangle$ is not the root, $\langle i, j \rangle$ cannot improve its eccentricity, and thus its cost function, by buying an edge towards some vertex of every subtree of G rooted at any of its t children. Furthermore, if $\langle i, j \rangle$ is the root of G, then $\langle i, j \rangle$ cannot improve its eccentricity, and thus its cost function, by buying an edge towards some vertex of every subtree of G rooted at any of its t children as $\langle i, j \rangle$ is already buying the unique edge of H linking it to the subtree of G rooted at $\langle k^*, 1+p \rangle$.

To complete the proof, observe that $SC(\text{Opt})$ is upper bounded by the social cost of building H, i.e., $SC(\text{Opt}) = O\big(n(\alpha + \sqrt{n})\big)$. Since $SC(\sigma) \geq \alpha(n-1) + \frac{1}{2}kk^*n = \Omega\big(n^{3/2}\min\{1+\alpha, \sqrt{n}\}\big)$, we have that

$$\frac{SC(\sigma)}{SC(\text{Opt})} = \frac{\Omega\big(n^{3/2}\min\{1+\alpha,\sqrt{n}\}\big)}{O\big(n(\alpha+\sqrt{n})\big)} = \Omega\big(1+\min\{\alpha, n/\alpha\}\big). \qquad \square$$

We now show that a similar lower bound holds also when the host graph is k-regular.

Theorem 8. *If the host graph is k-regular, with $k \geq 3$, the PoA is $\Omega\big(1+\min\{\alpha, \frac{n}{\alpha k}\}\big)$.*

Proof. We will consider $\alpha = \omega(1)$ and $\alpha = o(n)$, since otherwise the claim trivially holds. The proof will be given only for even values of k, while for odd values it will be just sketched since the construction is very similar.

Let l be the greatest integer such that $l \leq \lfloor \alpha + 1 \rfloor$, and let η be a large value such that $\eta \equiv 1 \pmod{l}$. Notice that if the number of players n is sufficiently large, then $l \geq 3$. We will use a host graph H composed by: (i) a path P of η nodes, numbered from 0 to $\eta - 1$, (ii) a set of shortcut edges on P, and (iii) a set of gadgets appended to P and used to increase to k the degree of its vertices. In the following we describe formally how H is built.

Concerning the shortcut edges, let u_i be the node on P numbered $i \cdot l$ for $i = 0, \dots, g$, where $g = \frac{\eta-1}{l}$. Then, a shortcut edge connects u_i to u_{i+1}, for $0 \leq i < g$. Notice that any node on P has now degree 2, but for $u_1, \dots, u_{g-1}$ which have degree 4.

Concerning the gadgets, for each node u on P that has degree $d < k$, augment H in the following way:

- consider a complete, loop-free, graph K on $k+1$ vertices;
- remove $\frac{k-d}{2}$ vertex-disjoint edges from K, so that every vertex in K has degree k except $k-d$ vertices that have degree $k-1$;
- add the resulting graph to H, by connecting u to the nodes with degree $k-1$.

At the end of this process the resulting host graph H is k-regular. Consider now a strategy profile σ such that:

- all the edges of the path P are bought (arbitrarily) by vertices other than u_i, $i = 0, \dots, g$;
- the vertices of the gadgets that have an edge towards a node on P, buy it;
- the remaining vertices of the gadgets buy a single edge towards a vertex adjacent to a node of P.

An example of the resulting configuration for $k = 4$ along with the edges of the host graph is shown in Figure 3(a).

This configuration is stable. Indeed, every node u_i can only change its strategy by buying either one or two edges, but this can decrease its eccentricity by at most $l-1$, while increasing its building cost of at least $\alpha \geq l-1$. Moreover, the

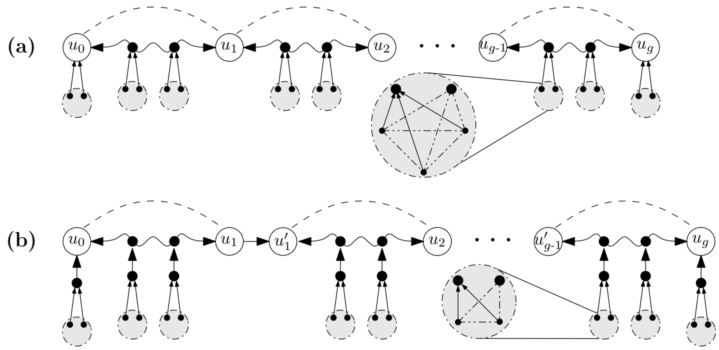

Fig. 3. Representation of the host graph and the equilibrium used in the proof of Theorem 8 for (a) $k = 4$, and (b) $k = 3$

remaining nodes in P cannot change their strategy, as doing so will cause the disconnection of the graph. Finally, the nodes of the gadgets buy just a single edge, and no other choice can decrease their eccentricity.

Clearly $SC(\sigma) = \Omega(\alpha n + n\eta)$, as $G(\sigma)$ is a tree with radius $\Theta(\eta)$. Let now $\widehat{G}$ be the graph obtained by adding to $G(\sigma)$ the shortcut edges of H. The number of edges of $\widehat{G}$ is $n-1+g \leq 2n$, and its diameter is bounded by $2 \cdot \varepsilon_{\widehat{G}}(u_0) \leq 2 \cdot (g+l+2)$, as u_0 can take advantage of the shortcut edges. As a consequence, with a small abuse of notation, we have $SC(\widehat{G}) = O(\alpha n + n(g + l))$.

Using the relations $l = \Theta(\alpha)$, $\eta = \Theta(lg)$, and $n = \Theta(\eta k)$, we have that

$$\text{PoA} \geq \frac{SC(\sigma)}{SC(\widehat{G})} = \frac{\Omega(\alpha n + n\eta)}{O(\alpha n + n(g + l))} = \Omega\left(\frac{\eta}{\alpha + \frac{\eta}{\alpha}}\right) = \Omega\left(\frac{n}{\alpha k + \frac{n}{\alpha}}\right)$$

from which the claim easily follows.

If k is odd, then a host graph similar to that shown in Figure 3(b) (for the case $k = 3$) is considered. Notice that the shortcut edges are now vertex-disjoint, and each node incident to them has degree 3, but for u_0 and u_g that have degree 2. Then, by appending the appropriate gadget to each node of the path, one can increase the degree of each node to k. □

We end this section by proving a non-constant lower bound to the PoA when $\alpha = o(n)$. Remarkably, our lower bound implies a non-constant lower bound to the PoA for the case $\alpha = 0$, i.e., players buy edges for free. Our lower bounding construction is a non-trivial modification of the 2D-torus-rotated-45° construction used in [1] to prove a lower bound for BasicNCG.

Theorem 9. *For $\alpha = o(n)$, the PoA is $\Omega\left(\sqrt{\frac{n}{1+\alpha}}\right)$.*

Proof. Let $k \in \mathbb{N}$ and let $\bar{H}$ be an edge-weighted 2D-torus-rotated-45° consisting of $2k^2$ vertices that we call junction vertices. For every pair of integers $0 \leq i, j < 2k$, with $i + j$ even, there is exactly one vertex of $\bar{H}$ labeled with $\langle i, j \rangle$. We treat the two integers of a vertex label as modulo $2k$. Each vertex $\langle i, j\rangle$ has exactly four neighbors in $\bar{H}$: $\langle i \pm 1, j \pm 1\rangle$. All edge weights are equal to $\ell = 2(1 + \lceil\alpha\rceil)$. For every pair of integers $0 \leq i, j < 2k$, let $X_{i,j} = \{\langle i', j'\rangle \mid i' = i \text{ or } j' = j\}$. The properties satisfied by $\bar{H}$ are the following:

(i) $\bar{H}$ is vertex transitive, i.e., any vertex can be mapped to any other by a vertex automorphism, i.e., a relabeling of vertices that preserves edges;
(ii) the distance between two vertices $\langle i, j\rangle$ and $\langle i', j'\rangle$ in $\bar{H}$ is equal to $\ell \cdot \max\left\{\bar{d}(i, i'), \bar{d}(j, j')\right\}$, where $\bar{d}(h, h') = \min\left\{|h - h'|, 2k - |h - h'|\right\}$;
(iii) the eccentricity of each vertex in $\bar{H}$ is equal to ℓk;
(iv) for every $0 \leq i, j < 2k$, the distance from every vertex $v \in X_{i,j}$ to vertex $\langle |i - k|, |j - k|\rangle$ is equal to ℓk;
(v) for every edge e of $\bar{H}$, the eccentricity of both endpoints of e in $\bar{H} - e$ is greater than or equal to $\ell(k + 1)$;
(vi) for every edge e of $\bar{H}$ and for every vertex $\langle i, j\rangle$, the distance from $\langle i, j\rangle$ and the closest endpoint of e is less than or equal to $\ell(k - 1)$.

It is easy to see that (i) holds and it is also easy to see that (iv) holds once (ii) has been proved. To prove (ii), it is enough to observe that each label can change by ± 1 each time we move from one vertex to any of its neighbors. To prove (iii), we use (i) and the fact that the distance from vertex $\langle i', j'\rangle$ to $\langle k, k\rangle$, which is equal to $\max\{|k - i'|, |k - j'|\}$, is maximized for $i' = j' = 0$. To prove (v), we first use (i) to assume that, w.l.o.g, e is the edge linking $\langle k, k\rangle$ with $\langle k-1, k-1\rangle$. Next, we observe that any path in $\bar{H} - e$ going from $\langle k, k\rangle$ to $\langle 1, 1\rangle$ must traverse a neighbor v of $\langle k, k\rangle$ in $\bar{H} - e$ and the distance between v and $\langle 1, 1\rangle$ in H is equal to ℓk because one of the two integers in the label of v is equal to $k + 1$. Finally, to prove (vi), we first use (i) to assume that, w.l.o.g., $i = j = k$, i.e., $\langle i, j\rangle$ is $\langle k, k\rangle$, and the two endpoints of e are, respectively, $\langle i', j'\rangle$ and $\langle i' + 1, j' + 1\rangle$, where $0 \leq i', j' < k$. Using (ii), it is easy to see that the distance from $\langle k, k\rangle$ to $\langle i' + 1, j' + 1\rangle$ is less than or equal to $\ell(k - 1)$.

Let G be an unweighted graph obtained from $\bar{H}$ by replacing each edge of $\bar{H}$ with a path of length ℓ via the addition of $\ell - 1$ new vertices per edge of $\bar{H}$. Let H be the host graph obtained from G by adding an edge between $\langle i, j\rangle$ and every vertex in $X_{i,j}$, for every junction vertex $\langle i, j\rangle$ (see also Figure 4). Notice that the number of vertices of H is $n = 2k^2 + 4k^2(\ell - 1) = \Theta\big((1 + \alpha)k^2\big)$. In what follows, we call the vertices in H which are not in $\bar{H}$ path vertices.

Let σ be any strategy profile such that $G(\sigma) = G$ and all edges of $G(\sigma)$ are bought by players sitting on path vertices, i.e., no edge of $G(\sigma)$ is bought by some player sitting on junction vertices. We prove that σ is a NE.

We start proving that players sitting on junction vertices are playing a best response strategy. Let $\langle i, j\rangle$ be a junction vertex. Observe that $\langle i, j\rangle$ is not buying any edge, therefore it suffices to show that $\langle i, j\rangle$ cannot improve its cost function by buying edges. First of all, observe that by (ii) and (vi), the eccentricity of $\langle i, j\rangle$ in G is equal to ℓk. Indeed, if v is a path vertex of a path P corresponding to edge e of $\bar{H}$, then the distance from $\langle i, j\rangle$ to the closest endpoint of P (which corresponds to the closest endpoint of e) is less than or equal to $\ell(k - 1)$. Therefore, the distance from $\langle i, j\rangle$ to v is less than or equal to ℓk. To prove that $\langle i, j\rangle$

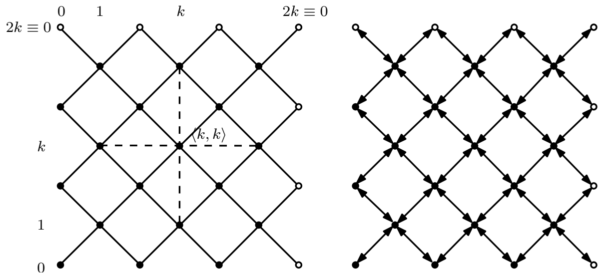

Fig. 4. The lower bound construction of Theorem 9. On the left side, the host graph H is depicted. For the sake of readability, only junction vertices are visible and not all the edges are shown. The white vertices of row $2k \equiv 0$ are copies of the vertices of row 0 while the white vertices of column $2k \equiv 0$ are copies of the vertices of column 0. The solid edges are paths of length ℓ, while the dashed edges are all the other edges adjacent to vertex $\langle k, k\rangle$. On the right side, the stable graph G is depicted.

is in equilibrium, simply observe that if we add to G all edges of H incident to $\langle i, j\rangle$, i.e., all edges linking $\langle i, j\rangle$ to vertices in $X_{i,j}$, then by (iv) the distance from $\langle i, j\rangle$ to $\langle |i-k|, |j-k|\rangle$ is still ℓk.

Now, we prove that players sitting on path vertices are playing a best response strategy. Let v be a path vertex. First of all, the eccentricity of v in G is less than or equal to $\ell k + \frac{1}{2}\ell$ by (vi). Indeed, if v is a vertex of a path P corresponding to edge e of $\bar{H}$, then the distance from any junction vertex to the closest endpoint of P (which corresponds to the closest endpoint of e) is less than or equal to $\ell(k-1)$. Therefore, the distance from v to every junction vertex is less than or equal to ℓk and the distance from v to every other path vertex is less than or equal to $\ell k + \frac{1}{2}\ell$. Now, observe that G already contains all edges of H incident to v and, moreover, the degree of v in G is equal to 2. Therefore, v might improve its cost function by removing exactly one edge incident to it, i.e., by buying fewer edges than those it is buying in σ. However, if v removes any of its incident edges in G, thus saving an α factor from its building cost, then by (v) the eccentricity of the unique junction vertex closest to v becomes greater than or equal to $\ell(k+1)$ and thus, the eccentricity of v also becomes greater than or equal to $\ell(k+1)$. Since $\ell(k+1) - \alpha \geq \ell(k+1) - \frac{\ell-2}{2} > \ell k + \frac{1}{2}\ell$, v does not improve its cost function by buying fewer edges than those it is buying in σ.

To complete the proof, we have to show that PoA is $\Omega\left(\sqrt{\frac{n}{1+\alpha}}\right)$. First of all, observe that the radius of H is $\Theta(\ell) = \Theta(1+\alpha)$. Let T be a breadth-first-search tree rooted at $\langle k, k\rangle$. Clearly the radius of T is also $\Theta(1+\alpha)$. Furthermore, the social cost of Opt is upper bounded by the social cost of building T, i.e., $SC(\text{Opt}) = \alpha(n-1) + n \cdot O(1+\alpha) = O\big((1+\alpha)n\big)$. As $SC(\sigma) \geq \alpha(4\ell k^2) + n\ell k = \Omega(n\ell k) = \Omega\big((1+\alpha)nk\big)$, we have that

$$\frac{SC(\sigma)}{SC(\text{Opt})} = \frac{\Omega\big((1+\alpha)nk\big)}{O\big((1+\alpha)n\big)} = \Omega(k) = \Omega\left(\sqrt{\frac{n}{1+\alpha}}\right). \qquad \square$$

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The Power of Local Information in Social Networks

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Abstract. We study the power of *local information algorithms* for optimization problems on social and technological networks. We focus on sequential algorithms where the network topology is initially unknown and is revealed only within a local neighborhood of vertices that have been irrevocably added to the output set. This framework models the behavior of an external agent that does not have direct access to the network data, such as a user interacting with an online social network.

We study a range of problems under this model of algorithms with local information. When the underlying graph is a preferential attachment network, we show that one can find the root (i.e. initial node) in a polylogarithmic number of steps, using a local algorithm that repeatedly queries the visible node of maximum degree. This addresses an open question of Bollobás and Riordan. This result is motivated by its implications: we obtain polylogarithmic approximations to problems such as finding the smallest subgraph that connects a subset of nodes, finding the highest-degree nodes, and finding a subgraph that maximizes vertex coverage per subgraph size.

Motivated by problems faced by recruiters in online networks, we also consider network coverage problems on arbitrary graphs. We demonstrate a sharp threshold on the level of visibility required: at a certain visibility level it is possible to design algorithms that nearly match the best approximation possible even with full access to the graph structure, but with any less information it is impossible to achieve a non-trivial approximation. We conclude that a network provider's decision of how much structure to make visible to its users can have a significant effect on a user's ability to interact strategically with the network.

1 Introduction

In the past decade there has been a surge of interest in the nature of complex networks that arise in social and technological contexts; see [9] for a recent survey of the topic. In the computer science community, this attention has been directed largely towards algorithmic issues, such as the extent to which network structure

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can be leveraged into efficient methods for solving complex tasks. Common problems include finding influential individuals, detecting communities, constructing subgraphs with desirable connectivity properties, and so on.

The standard paradigm in these settings is that an algorithm has full access to the network graph structure. More recently there has been growing interest in *local* algorithms, in which decisions are based upon local rather than global network structure. This locality of computation has been motivated by applications to distributed algorithms [17,11], improved runtime efficiency [10,20], and property testing [15,18]. In this work we consider a different motivation: in some circumstances, an optimization is performed by an external user who has inherently restricted visibility of the network topology. For such a user, the graph structure is revealed incrementally within a local neighborhood of nodes for which a connection cost has been paid. The use of local algorithms in this setting is necessitated by constraints on network visibility, rather than being a means toward an end goal of efficiency or parallelizability.

As a motivating example, consider an agent in a social network who wishes to find (and link to) a highly connected individual. For instance, this agent may be a newcomer to a community (such as an online gaming or niche-based community) wanting to interact with influential or popular individuals, or a recruiter attempting to form strategic connections in a social network application. Finding a high-degree node is a straightforward algorithmic problem without information constraints, but many online and real-world social networks reveal graph structure only within one or two hops from a user's existing connections.

Is it possible for an agent to solve such a problem using only the local information available on an online networking site? This question is relevant not only for individual users, but also to the designer of a social networking service who must decide how much information to reveal. For example, LinkedIn allows each user to see the degree of nodes two hops away in the network, whereas Facebook does not reveal this information by default. We ask: what impact do such design decisions have on an individual's ability to interact with the network?

More generally, we consider graph algorithms in a setting of restricted network visibility. We focus on optimization problems for which the goal is to return a subset of the nodes in the network; this includes coverage, connectivity, and search problems. An algorithm in our framework proceeds by incrementally and adaptively building an output set of nodes, corresponding to those vertices of the graph that have been queried (or connected to) so far. When the algorithm has queried a set S of nodes, the structure of the graph within a small radius of S is revealed, guiding future queries. The principle challenge in designing such an algorithm is that decisions must be based solely on local information, whereas the problem to be solved may depend on the global structure of the graph. In addition to these restrictions, we ask for algorithms that run in polynomial time.

For many problems we derive strong lower bounds on the performance of local algorithms in general networks. We therefore turn to the class of preferential attachment (PA) graphs, which model properties of many real-world social and technological networks. For PA networks, we prove that local information algorithms do well at many optimization problems, including shortest path routing and finding the k vertices of highest degree (up to polylogarithmic factors).

We also consider node coverage problems on general graphs, where the goal is to find a small set of nodes whose neighborhood covers all (or much) of the network. Such coverage problems are especially motivated in our context by applications to employment-focused social networking platforms such as LinkedIn, where there is benefit in having as many nodes as possible within a few hops of one's direct connections[1]. We design local information algorithms whose performances approximately match the best possible even when information about network structure is unrestricted. We also demonstrate that the amount of local information available is of critical importance: strong positive results are possible at a certain range of visibility (made explicit below), but non-trivial algorithms become impossible when less information is made available. This observation has implications for the design of online networks, such as the amount of information to provide a user about the local topology: seemingly arbitrary design decisions may have a significant impact on a user's ability to interact with the network.

Results and Techniques. Our first set of results concerns algorithms for preferential attachment (PA) networks. Such networks are defined by a process by which nodes are added sequentially and form random connections to existing nodes, where the probability of connecting to a node is proportional to its degree.

We first consider the problem of finding the root (first) node in a PA network. A random walk would encounter the root in $\tilde{O}(\sqrt{n})$ steps (where n is the number of nodes in the network). The question of whether a better local information algorithm exists for this problem was posed by Bollobas and Riordan [5]. They conjecture that such short paths can be found locally in $\Theta(\log n)$ steps. We make the first progress towards this conjecture by showing that polylogarithmic time is sufficient: there is an algorithm that finds the root of a PA network in $O(\log^4(n))$ time, with high probability. We show how to use this algorithm to obtain polylogarithmic approximations for finding the smallest subgraph that connects a subset of nodes (including shortest path), finding the highest-degree nodes, and finding a subgraph that maximizes vertex coverage per subgraph size.

The local information algorithm we propose uses a natural greedy approach: at each step, it queries the visible node with highest degree. Demonstrating that such an algorithm reaches the root in $O(\log^4(n))$ steps requires a probabilistic analysis of the PA process. A natural intuition is that the greedy algorithm will find nodes of ever higher degrees over time. However, such progress is impeded by the presence of high-degree nodes with only low-degree neighbors. We show that these bottlenecks are infrequent enough that they do not significantly hamper the algorithm's progress. To this end, we derive a connection between node degree correlations and supercritical branching processes to prove that a path of high-degree vertices leading to the root is always available to the algorithm.

[1] For example, LinkedIn allows recruiters to execute searches for potential job candidates among all nodes within distance 3 from the recruiter, additionally displaying resume information for those within distance 2.

We then consider general graphs, where we explore local information algorithms for dominating set and coverage problems. A dominating set is a set S such that each node in the network is either in S or the neighborhood of S. We design a randomized local information algorithm for the minimum dominating set problem that achieves an approximation ratio that nearly matches the lower bound on polytime algorithms with no information restriction. As has been noted in [14], the greedy algorithm that repeatedly selects the visible node that maximizes the size of the dominated set can achieve a very bad approximation factor. We consider a modification of the greedy algorithm: after each greedy addition of a new node v, the algorithm will also add a random neighbor of v. We show that this randomized algorithm obtains an approximation factor that matches the known lower bound of $\Omega(\log \Delta)$ (where Δ is the maximum degree in the network) up to a constant factor. We also show that having enough local information to choose the node that maximizes the incremental benefit to the dominating set size is crucial: no algorithm that can see only the degrees of the neighbors of S can achieve a sublinear approximation factor.

Finally, we extend these results to related coverage problems. For the partial dominating set problem (where the goal is to cover a given constant fraction of the network with as few nodes as possible) we give an impossibility result: no local information algorithm can obtain an approximation better than $O(\sqrt{n})$ on networks with n nodes. However, a slight modification to the local information algorithm for minimum dominating set yields a bicriteria result (in which we compare performance against an adversary who must cover an additional ϵ fraction of the network). We also consider the "neighbor-collecting" problem, in which the goal is to minimize $c|S|$ plus the number of nodes left undominated by S, for a given parameter c. For this problem we show that the minimum dominating set algorithm yields an $O(c \log \Delta)$ approximation (where Δ is the maximum degree in the network), and that the dependence on c is unavoidable.

Related Work. Over the last decade there has been a substantial body of work on understanding the power of sublinear-time approximations. In the context of graphs, the goal is to understand how well one can approximate graph properties in a sublinear number of queries. See [18] and [13] for recent surveys. Motivated by distributed computation, a notion of local computation was formalized by [19] and further developed in [1]. They define a local computation algorithm as computing only certain specified bits of a global solution. In contrast, our notion of locality is motivated by information constraints imposed upon a sequential algorithm. h Local algorithms motivated by efficient computation, rather than informational constraints, were explored by [2,20]. These works explore local approximation of graph partitions to efficiently find a global solution.

Preferential attachment (PA) networks were suggested by [3] as a model for large social networks. There has been much work studying the properties of such networks, such as degree distribution [6] and diameter [5]; see [4] for a short survey. The problem of finding high degree nodes, using only uniform sampling and local neighbor queries, is explored in [7]. The low diameter of PA graphs can be used to implement distributed algorithms in which nodes repeatedly broadcast information to their neighbors [11,8]. A recent work [8] showed that such algorithms can be used for fast rumor spreading. Our results on the ability to find short paths in such graphs differs in that our algorithms are sequential, with a small number of queries, rather than applying broadcast techniques.

The ability to quickly find short paths in social networks has been the focus of much study, especially in the context of small-world graphs [16,12]. It is known that local routing using short paths is possible in such models, given some awareness of global network structure (such as coordinates in an underlying grid). In contrast, our shortest-path algorithm for PA graphs does not require an individual know the graph structure beyond the degrees of his neighbors. However, our result requires that routing can be done from both endpoints; in other words, both nodes are trying to find each other.

For the minimum dominating set problem, Guha and Khuller [14] designed a local $O(\log \Delta)$ approximation algorithm. As a local information algorithm, their method requires that the network structure is revealed up to distance two from the current dominating set. By contrast, our local information algorithm requires less information to be revealed on each step. Our focus, and the motivation behind this distinction, is to determine sharp bounds on the amount of local information required to approximate this problem (and others) effectively.

2 Model and Preliminaries

Graph Notation. We write $G = (V, E)$ for an undirected graph with node and edge sets V and E, respectively. We write n_G for the number of nodes in G, $d_G(v)$ for the degree of a vertex v in G, and $N_G(v)$ for the set of neighbors of v. Given a subset of vertices $S \subseteq V$, $N_G(S)$ is the set of nodes adjacent to at least one node in S. We also write $D_G(S)$ for the set of nodes *dominated* by S: $D_G(S) = N_G(S) \cup S$. We say S is a *dominating set* if $D_G(S) = V$. Given nodes u and v, the distance between u and v is the number of edges in the shortest path between u and v. The distance between vertex sets S and T is the minimum distance between a node in S and a node in T. Given a subset S of nodes of G, the subgraph induced by S is the subgraph consisting of S and every edge with both endpoints in S. Finally, Δ_G is the maximum degree in G. In all of the above notation we often suppress the dependency on G when clear from context.

Algorithmic Framework. We focus on graph optimization problems in which the goal is to return a minimal-cost[2] set of vertices S satisfying a feasibility constraint. We will consider a class of algorithms that build S incrementally under local information constraints. We begin with a definition of local neighborhoods.

Definition 1 (Local Neighborhood). *Given a set of nodes S in the graph G, the* r-closed neighborhood around S *is the induced subgraph of G containing all nodes at distance less than or equal to r from S, plus the degree of each node at distance r from S. the* r-open neighborhood around S *is the r-closed neighborhood around S, after the removal of all edges between nodes at distance exactly r from S.*

[2] In most of the problems we consider, the cost of set S will simply be $|S|$.

Definition 2 (Local Information Algorithm). *Let G be an undirected graph unknown to the algorithm, where each vertex is assigned a unique identifier. For integer $r \geq 1$, a (possibly randomized) algorithm is an r^+-local algorithm if:*

1. *The algorithm proceeds sequentially, growing step-by-step a set S of nodes, where S is initialized either to $\emptyset$ or to some seed node.*
2. *Given that the algorithm has queried a set S of nodes so far, it can only observe the r-closed neighborhood around S.*
3. *On each step, the algorithm can add a node to S either by selecting a specified vertex from the r-closed neighborhood around S (a* crawl*) or by selecting a vertex chosen uniformly at random from all graph nodes (a* jump*).*
4. *In its last step the algorithm returns the set S as its output.*

Similarly, for $r \geq 1$, we call an algorithm a *r-local algorithm* if its local information (i.e. in item 2) is made from the r-open neighborhood around S.

We focus on computationally efficient (i.e. polytime) local algorithms. Our framework applies most naturally to coverage, search, and connectivity problems, where the family of valid solutions is upward-closed. More generally, it is suitable for measuring the complexity, using only local information, for finding a subset of nodes having a desirable property. In this case the size of S measures the number of queries made by the algorithm; we think of the graph structure revealed to the algorithm as having been paid for by the cost of S.

For our lower bound results, we will sometimes compare the performance of an r-local algorithm with that of a (possibly randomized) algorithm that is also limited to using Jump and Crawl queries, but may use full knowledge of the network topology to guide its query decisions. The purpose of such comparisons is to emphasize instances where it is the lack of information about the network structure, rather than the necessity of building the output in a local manner, that impedes an algorithm's ability to perform an optimization task.

3 Preferential Attachment Graphs

We consider graphs generated by the preferential attachment (PA) process, conceived by Barabási and Albert [3]. The process is defined sequentially with nodes added one by one. When a node is added it sends m links to previously created nodes, connecting to a node with probability proportional to its current degree.

We will use the following, now standard, formal definition of the process, due to [5]. Given $m \geq 1$, we inductively define random graphs G_m^t, $1 \leq t \leq n$. The vertex set for G_m^t is $[t]$. G_m^1 is the graph with node 1 and m self-loops. Given $G_m^{(t-1)}$, form G_m^t by adding node t and then forming m edges from t to nodes in $[t]$, say $p_1(t), \ldots, p_m(t)$. The nodes $p_k(t)$ are referred to as the *parents* of t. The edges are formed sequentially. For each k, node s is chosen with probability $\deg(s)/z$ if $s < t$, or $(\deg(s)+1)/z$ if $s = t$, where z is a normalization factor. Note that $\deg(s)$ denotes degree in G_m^{t-1}, counting previously-placed edges.

We first present a 1-local approximation algorithm for the following simple problem on PA graphs: given an arbitrary node u, return a minimal connected subgraph containing nodes u and 1 (i.e. the root of G_m^n).

Algorithm 1. TraverseToTheRoot

1: Initialize a list L to contain an arbitrary node $\{u\}$ in the graph.
2: **while** L does not contain node 1 **do**
3: Add a node of maximum degree in $N(L)\backslash L$ to L.
4: return L.

Algorithm 2. s-t-Connect

1: $P_1 \leftarrow$ TraverseToTheRoot(G, s)
2: $P_2 \leftarrow$ TraverseToTheRoot(G, t)
3: Return $P_1 \cup P_2$

Our algorithm, TraverseToTheRoot, is listed as Algorithm 1. The algorithm grows a set S of nodes by starting with $S = \{u\}$ and then repeatedly adding the node in $N(S)\backslash S$ with highest degree. We will show that, with high probability, this algorithm traverses the root node within $O(\log^4(n))$ steps.

Theorem 1. *With probability* $1 - o(1)$ *over the preferential attachment process on* n *nodes, TraverseToTheRoot returns a set of size* $O(\log^4(n))$.

Remark: For convenience, we have defined TraverseToTheRoot assuming that the algorithm can determine when it has successfully traversed the root. This is not necessary in general; our algorithm will have the guarantee that, after $O(\log^4(n))$ steps, it has traversed node 1 with high probability.

Before proving Theorem 1, we discuss its algorithmic implications below.

3.1 Applications of Fast Traversal to the Root

We now describe how to use TraverseToTheRoot to implement local algorithms for other problems on PA networks. Proofs are omitted due to space constraints.

s-t Connectivity. The s-t connectivity (shortest path) problem is to find a small connected subgraph containing two given nodes s and t in an undirected graph.

Corollary 1. *Let* G *be a PA graph on* n *nodes. Then, with probability* $1 - o(1)$ *over the PA process, Algorithm 2 (listed above), a* 1*-local algorithm, returns a connected subgraph of size* $O(\log^4(n))$ *containing vertices* s *and* t.

This result implies that a subset of k nodes can be connected by a local algorithm in $O(k\log^4(n))$ steps, using a subset of size $O(k\log^4(n))$. Also, in the full version of the paper we show that Corollary 1 does not extend to general graphs: local algorithms cannot achieve sublinear approximations.

Finding High Degree Nodes. A natural problem on graphs is to find a node with maximal degree. The algorithm TraverseToTheRoot gives a polylogarithmic approximation to this problem with high probability. This follows because, with high probability, the root of a PA network has approximately maximal degree.

Corollary 2. *Let G be a preferential attachment graph on n nodes. Then, with probability $1 - o(1)$, algorithm TraverseToTheRoot will return a node of degree at least $\frac{1}{\log^2(n)}$ of the maximum degree in the graph, in time $O(\log^4(n))$.*

In the full version we show that Corollary 2 does not extend to general graphs.

Maximizing Coverage versus Cost. In the full version of the paper we consider the optimization problem of finding set S such that $|D(S)|/|S|$ is maximized. For this problem the TraverseToTheRoot algorithm obtains a polylogarithmic approximation in $O(\log^4(n))$ queries, and we prove no such result is possible for general graphs.

3.2 Analysis of TraverseToTheRoot

We now turn to the proof of Theorem 1. Let us provide some intuition. We would like to show that TraverseToTheRoot queries nodes of progressively higher degrees over time. However, if we query a node i of degree d, there is no guarantee that subsequent nodes will have degree greater than d; the algorithm may encounter local maxima. Suppose, however, that there were a path from i to the root consisting entirely of nodes with degree at least d. In this case, the algorithm will only ever traverse nodes of degree at least d from that point onward. One might therefore hope that the algorithm finds nodes that lie on such "good" paths for ever higher values of d, representing progress toward the root.

Motivated by this intuition, we will study the probability that any given node i lies on a path to the root consisting of only high-degree nodes (i.e. not much less than the degree of i). We will argue that many nodes in the network lie on such paths. We prove this in two steps. First, we show that for any given node i and parent $p_k(i)$, $p_k(i)$ will have high degree relative to i with probability greater than 1/2 (Lemma 2). Second, since each node i has at least two parents, we use the theory of supercritical branching processes to argue that, with constant probability for each node i, there exists a path to a node close to the root following links to such "good" parents (Lemma 3).

This approach is complicated by the fact that existence of such good paths is highly correlated between nodes; this makes it difficult to argue that such paths occur "often" in the network. To address this issue, we show that good paths are likely to exist even after a large set of nodes (Γ in our argument below) is adversarially removed from the network. We can then argue that each node is likely to have a good path independently of many other nodes, as we can remove all nodes from one path before testing the presence of another.

We now provide an outline of the proof. The proofs of technical lemmas appear in the full version. Set $s_0 = 160\log(n)(\log\log(n))^2$ and $s_1 = \frac{n}{2^{25}\log^2 n}$. We think of vertices in $[1, s_0]$ as close to the root, and vertices in $[s_1, n]$ as very far from the root. Let $I_t = [2^t + 1, 2^{t+1}]$ be a partition of $[n]$ into intervals. Define constants $\beta = 1/4$ and $\zeta = 30$.

Definition 3 (Typical node). *A node i has* typical degree *if either* $\deg(i) \geq \frac{m}{2\zeta}\sqrt{\frac{n}{i}}$ *or* $i \leq s_0$.

Lemma 1. *The following are true with probability* $1 - o(1)$*:*

- $\forall i \geq s_0 : \ \deg(i) \leq 6m \log(n) \sqrt{\frac{n}{i}}$.
- $\forall i \leq s_0 : \ \deg(i) \geq \frac{m\sqrt{n}}{5 \log^2(n)}$.
- $\forall i \geq s_0 : \ \mathbb{P}[i \text{ is connected to } 1] \geq \frac{3.9}{\log(n)\sqrt{i}}$.
- $\forall j \geq i \geq s_0, k \leq m : \ \mathbb{P}[p_k(i) < j] \geq \frac{0.9\sqrt{i}}{\sqrt{j}}$.

Our next lemma states that, for any set Γ that contains sufficiently few nodes from each interval I_t, and any given parent of a node i, with probability greater than 1/2 the parent will be typical, not in Γ, and not in the same interval as i.

Definition 4 (Sparse set). *A subset of nodes* $\Gamma \subseteq [n]$ *is* sparse *if* $|\Gamma \cap I_t| \leq |I_t|/\log\log(n)$ *for all* $\log s_0 \leq t \leq \log s_1$.

Lemma 2. *Fix sparse set* Γ*. Then for each* $i \in [s_0, s_1]$ *and* $k \in [m]$*, the following are true with probability* $\geq 8/15$ *:* $p_k(i) \notin \Gamma$*,* $p_k(i) \leq i/2$*, and* $p_k(i)$ *is typical.*

We now claim that, for any given node i and sparse set Γ, there is likely a short path from i to vertex 1 consisting entirely of typical nodes that do not lie in Γ. Our argument is via a coupling with a supercritical branching process. Consider growing a subtree, starting at node i, by adding to the subtree any parent of i that satisfies the conditions of Lemma 2, and then recursively growing the tree in the same way from any parents that were added. Since each node has $m \geq 2$ parents, and each satisfies the conditions of Lemma 2 with probability $> 1/2$, this growth process is supercritical and should survive with constant probability (within the range of nodes $[s_0, s_1]$). We should therefore expect that, with constant probability, such a subtree would contain some node $j < s_0$.

The argument above leads to the following lemma, which we will use in our analysis of the algorithm TraverseToTheRoot. First a definition.

Definition 5 (Good paths). *For any* $i \in [s_0, s_1]$*, we say* i has a good path *if there is a path from* i *to a node* $j \leq s_0$ *consisting of nodes with typical degree.*

Lemma 3. *Choose any set* T *of at most* $16 \log n$ *nodes from* $[s_0, s_1]$*. Then each* $i \in T$ *has a good path with probability at least* $1/5$*, independently for each* i*.*

We will apply Lemma 3 to the set of nodes queried by TraverseToTheRoot to argue that progress toward the root is made after every sequence of polylogarithmically many steps. We can now complete the proof of Theorem 1, which we sketch below; a full proof appears in the full version of the paper.

Our analysis of Algorithm 1 consists of three steps, corresponding to three phases of the algorithm. The first phase consists of all steps until the first time we traverse a node $i < s_1$ with a good path. The second phase then lasts until the algorithm queries a node $i < s_0$. The third phase ends when the algorithm traverses node 1. We will show that each phase lasts at most $O(\log^4(n))$ steps.

We will make use of Lemma 3 in our analysis whenever we consider whether a node has a good path. We will check at most $16 \log n$ nodes in this manner, and hence the conditions of Lemma 3 will be satisfied throughout the analysis.

Analysis of Phase 1. Phase 1 ends when the algorithm traverses a node $i < s_1$ with a good path. The value of s_1 is set large enough so that every node queried by the algorithm has index $\leq s_1$ with probability at least $\frac{1}{O(\log n)}$, regardless of previous nodes traversed. By Lemma 3, each such node has a good path with probability at least 1/5. Multiplicative Chernoff bounds therefore imply that the phase will end after at most $O(\log^2(n))$ steps, with high probability.

Analysis of Phase 2. We split phase 2 into a number of epochs. For each $t \in [\log s_0, \log s_1]$, epoch t begins when some node $i \in I_t$ with a good path has been traversed (and ends when epoch $t+1$ begins). Define random variable Y_t to be the length of epoch t. The total number of steps in phase 2 is $\sum_{t=\log s_0}^{\log s_1} Y_t$.

Suppose the algorithm is in epoch t, having traversed node $i \in I_t$ with a good path. Then i has a parent $j \in I_u$ with $deg(j) \geq \frac{m}{2\zeta}\sqrt{\frac{n}{i}}$ and $u < t$. This node j could be traversed by the algorithm, so any node queried before j must have at least this degree. Moreover, traversing node j would end epoch t, so every step in epoch t traverses a node with degree at least $\frac{m}{2\zeta}\sqrt{\frac{n}{i}}$. By Lemma 1, any such node ℓ satisfies $\ell < zi\log^2(n)$ where z is a constant. But now, by Lemma 1, each node ℓ traversed in epoch t has a parent $r < i/\log^2(n)$ with probability at least $\frac{1}{O(\log^2(n))}$. Any such node r has degree greater than any node in I_t, again by Lemma 1, so if a queried node had such a parent then the subsequent step must query a node of index at most 2^t. Any such node is on a good path with probability at least 1/5, by Lemma 3, in which case epoch t would end.

To summarize, each step of epoch t causes an end to the epoch with probability at least $\frac{1}{O(\log^2(n))}$. We conclude that $\sum_{t=\log s_0}^{\log s_1} Y_t$ is dominated by the sum of at most $\log n$ geometric random variables, each with mean $O(\log^2(n))$. Concentration bounds for geometric random variables then imply that, with high probability, epoch 2 ends in $O(\log^3(n))$ steps.

Analysis of Phase 3. We first note that the induced graph on the first s_0 nodes is connected with high probability (see [8], corollary 5.15). By Lemma 1 every node $i \leq s_0$ has degree at least $d = \frac{m\sqrt{n}}{5\log^{1.9}(n)}$, so the algorithm will only traverse nodes of degree at least d in phase 3. By Lemma 1, any node j with degree at least d must satisfy $j < (60\zeta)^2\log^{5.8}(n)$. Also by Lemma 1, for each such j, the probability that j is adjacent to the root is at least $\frac{1}{2^{11}\log^{3.9}(n)}$. Chernoff bounds then imply that such an event will occur with high probability after at most $O(\log^4(n))$ steps. Thus, with high probability, phase 3 will end after at most $s_0 + O(\log^4(n)) = O(\log^4(n))$ steps. This completes the proof of Theorem 1.

4 Minimum Dominating Set on Arbitrary Networks

We now consider the problem of finding a dominating set S of minimal size for an arbitrary graph G. Even with full (non-local) access to the network structure, it is known to be hard to approximate the Minimum Dominating Set Problem to within a factor of $H(\Delta)$ in polynomial time, where $H(n) \approx \ln(n)$ is the nth

Algorithm 3. AlternateRandom

Select an arbitrary node u from the graph and initialize $S = \{u\}$.
while $D(S) \neq V$ **do**
  Choose $x \in \arg\max_{v \in N(S)}\{|N(v) \setminus D(S)|\}$ and add x to S.
  if $N(x) \setminus S \neq \emptyset$ **then**
    Choose $y \in N(x) \setminus S$ uniformly at random and add y to S.
return S.

harmonic number. In this section we explore how much local network structure must be made visible in order for it to be possible to match this lower bound.

Guha and Khuller [14] design an $O(H(\Delta))$-approximate algorithm for the minimum dominating set problem, which can be interpreted in our framework as a 2^+-local algorithm. As we show, the ability to observe network structure up to distance 2 is unnecessary if we allow the use of randomness: we will construct a randomized $O(H(\Delta))$ approximation algorithm that is 1^+-local. We then show that this level of local information is crucial: no algorithm with less local information can return a non-trivial approximation. Proofs in this section are omitted due to space constraints, but appear in the full version of the paper.

4.1 A 1^+-Local Algorithm

We now present a 1^+-local randomized $O(H(\Delta))$-approximation algorithm for the min dominating set problem. Our algorithm obtains this approximation factor both in expectation and with high probability in the optimal solution size[3].

Roughly speaking, our approach is to greedily grow a subtree of the network, repeatedly adding vertices that maximize the number of dominated nodes. Such a greedy algorithm is 1^+-local, as this is the amount of visibility required to determine how much a given node will add to the number of dominated vertices. Unfortunately, this greedy approach does not yield a good approximation; it is possible for the algorithm to waste significant effort covering a large set of nodes that are all connected to a single vertex just beyond the algorithm's visibility. To address this issue, we introduce randomness into the algorithm: after each greedy addition of a node x, we will also query a random neighbor of x. The algorithm is listed above as Algorithm 3 (AlternateRandom).

We now show that AlternateRandom obtains an $O(H(\Delta))$ approximation, both in expectation and with high probability. In what follows, $\mathcal{OPT}$ will denote the size of the optimal dominating set in an inplicit input graph. The proof follows by bounding, for each node v in the optimal solution, the expected number of neighbors of v that are queried before v is queried.

Theorem 2. *AlternateRandom is 1^+-local and returns a dominating set S where $\mathbb{E}[|S|] \leq 2(1 + H(\Delta))\mathcal{OPT} + 1$ and $\mathbb{P}[|S| > 2(2 + H(\Delta))\mathcal{OPT}] < e^{-\mathcal{OPT}}$.*

[3] Our algorithm actually generates a connected dominating set, so it can also be seen as an $O(H(\Delta))$ approximation to the connected dominating set problem.

We end this section by showing that 1^+-locality is necessary for constructing good local approximation algorithms. The example we consider is a clique with one edge (u, v) removed, plus two additional nodes u' and v' that are adjacent to nodes u and v respectively.

Theorem 3. *For any randomized* 1*-local algorithm* A *for the min dominating set problem, there exists an input instance* G *for which* $\mathbb{E}[|S|] = \Omega(n)\mathcal{OPT}$*, where* S *denotes the output generated by* A *on input* G*.*

4.2 Partial Coverage Problems

We next study problems in which the goal is not necessarily to cover all nodes in the network, but rather dominate only sections of the network that can be covered efficiently. We consider two central problems in this domain: the partial dominating set problem and the neighbor collecting problem.

Partial Dominating Set. In the partial dominating set problem we are given a parameter $\rho \in (0, 1]$. The goal is to find the smallest set S such that $|D(S)| \geq \rho n$.

We begin with a negative result: for any constant k and any k-local algorithm, there are graphs for which the optimal solution has constant size, but with high probability $\Omega(\sqrt{n})$ queries are required to find any ρ-partial dominating set. Our example will apply to $\rho = 1/2$, but can be extended to any constant $\rho \in (0, 1)$. The example is a graph with two embedded stars, one with $n/2 - \sqrt{n}$ leaves and one with only $\sqrt{n}$ leaves; the optimal solution contains the center of each star, but the smaller star requires many queries to locate.

Theorem 4. *For any randomized* k*-local algorithm* A *for the partial dominating set problem with* $\rho = 1/2$*, there exists an input* G *with optimal partial dominating set* $\mathcal{OPT}$ *for which* $\mathbb{E}[|S|] = \Omega(\sqrt{n}) \cdot |\mathcal{OPT}|$*, where* S *denotes the output generated by* A *on input* G*.*

Motivated by this lower bound, we consider a bicriterion result: given $\epsilon > 0$, we compare the performance of an algorithm that covers ρn nodes with the optimal solution that covers $\rho(1 + \epsilon)n$ nodes (assuming $\rho(1 + \epsilon) \leq 1$). We show that a modification to Algorithm 3, in which jumps to uniformly random nodes are interspersed with greedy selections, yields an $O((\rho\epsilon)^{-1}H(\Delta))$ approximation. The proof is similar in spirit to Theorem 2.

Theorem 5. *Given any* $\rho \in (0, 1)$*,* $\epsilon \in (0, \rho^{-1} - 1)$*, and set of nodes* $\mathcal{OPT}$ *with* $|D(\mathcal{OPT})| \geq \rho(1 + \epsilon)n$*, Algorithm 4 (AlternateRandomAndJump) returns a set* S *of nodes with* $|D(S)| \geq \rho n$ *and* $\mathbb{E}[|S|] \leq 3|\mathcal{OPT}|(\rho\epsilon)^{-1}H(\Delta)$*.*

The Neighbor Collecting Problem. We next consider the objective of minimizing the total cost of the selected nodes plus the number of nodes left uncovered: choose a set S of G that minimizes $f(S) = c|S| + |V \backslash D(S)|$ for a given parameter $c > 0$. This problem is motivated by the Prize-Collecting Steiner Tree problem. The proof is similar in spirit to Theorem 2, noting that the optimal dominating set is no worse than a c-approximation to the optimal solution.

Algorithm 4. AlternateRandomAndJump

Initialize $S = \emptyset$.
while $|D(S)| < \rho n$ **do**
  Choose a node u uniformly at random from the graph and add u to S.
  Choose $x \in \arg\max_{v \in N(S)} \{|N(v) \backslash D(S)|\}$ and add x to S.
  if $N(x) \backslash S \neq \emptyset$ **then**
    Choose $y \in N(x) \backslash S$ uniformly at random and add y to S.
return S.

Theorem 6. *For any $c \geq 1$ and set $\mathcal{OPT}$ minimizing $f(\mathcal{OPT})$, algorithm AlternateRandom returns a set S for which $\mathbb{E}[f(S)] \leq 2c(1 + H(\Delta))f(\mathcal{OPT})$.*

We show in the full version that the dependency on c is unavoidable and that Theorem 6 cannot be extended to 1-local algorithms without significant loss.

5 Conclusions

We presented a model of computation in which algorithms are constrained in the information they have about the input structure, which is revealed over time as expensive exploration decisions are made. Our motivation lies in determining whether and how an external user in a network, who cannot make arbitrary queries of the graph structure, can efficiently solve optimization problems in a local manner. Our results suggest that inherent structural properties of social networks may be crucial in obtaining strong performance bounds.

Another implication is that the designer of a network interface, such as an online social network platform, may gain from considering the power and limitations that come with the design choice of how much network topology to reveal to individual users. On one hand, revealing too little information may restrict natural social processes that users expect to be able to perform, such as searching for potential new connections. On the other hand, revealing too much information may raise privacy concerns, or enable unwanted behavior such as automated advertising systems searching to target certain individuals. Our results suggest that even minor changes to the structural information made available to a user may have a large impact on the class of optimization problems that can be reasonably solved by the user.

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The Price of Anarchy for Selfish Ring Routing Is Two

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Abstract. We analyze the network congestion game with atomic players, asymmetric strategies, and the maximum latency among all players as social cost. This important social cost function is much less understood than the average latency. We show that the price of anarchy is at most two, when the network is a ring and the link latencies are linear. Our bound is tight. This is the first sharp bound for the maximum latency objective.

1 Introduction

Selfish routing is a fundamental problem in algorithmic game theory, and was one of the first problems which were intensively studied in this field [1–4]. A main question in this field concerns the cost of selfishness: how much performance is lost because agents behave selfishly, without regard for the other agents or for any global objective function?

The established measure for this performance loss is the price of anarchy (PoA) [1]. This is the worst-case ratio between the value of a Nash equilibrium, where no player can deviate unilaterally to improve, and the value of the optimal routing.

Of particular interest to computer science are network congestion games, where agents choose routing paths and experience delays (latencies) depending on how much other players also use the edges on their paths. Such games are guaranteed to admit at least one Nash equilibrium [5]. Generally, the price of anarchy for a selfish routing problem may depend on the network topology, the number of players (including the *non-atomic* case where an infinite number of players each controls a negligible fraction of the solution), the type of latency functions on the links, and the objective functions of the players and of the system (the latter is often called the *social cost function*).

Most of the existing research has focused on the price of anarchy for minimizing the *total* latency of all the players [6, 7]. Indeed, this measure is so standard that it is often not even mentioned in titles or abstracts. In most cases, a symmetric setting was considered where all players have the same source node and

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the same destination node, and hence the same strategy set. [8] and [9] independently proved that the PoA of the atomic congestion game (symmetric or asymmetric) with linear latency is at most 2.5. This bound is tight. The bound grows to 2.618 for weighted demands [9], which is again a tight bound. In non-atomic congestion games with linear latencies, the PoA is at most 4/3 [3]. This is witnessed already by two parallel links. The same paper also extended this result to polynomial latencies.

In this work, we regard as social cost function the *maximum* latency a player experiences. While this cost function was suggested already in [1], it seems much less understood. For general topologies, the maximum PoA of atomic congestion games with linear latency is 2.5 in single-commodity networks (symmetric case, all player choose paths between the same pair of nodes), but it grows to $\Theta(\sqrt{k})$ in k-commodity networks (asymmetric case, k players have different nodes to connect via a path) [8]. The PoA further increases with additional restrictions to the strategy sets. [10] showed that when the graph consists of n parallel links and each player's choice can be restricted to a particular subset of these links, the maximum PoA lies in the interval $[n-1, n)$.

For non-atomic selfish routing, [11] showed that the PoA of symmetric games on n-node networks with arbitrary continuous and non-decreasing latency functions is $n-1$, and exhibited an infinite family of asymmetric games whose PoA grows exponentially with the network size.

Our Setting: In this work, we analyze the price of anarchy of a maximum latency network congestion game for a concrete and useful network topology, namely rings. Rings are frequently encountered in communication networks. Seven self-healing rings form the EuroRings network, the largest, fastest, best-connected high-speed network in Europe, spanning 25,000 km and connecting 60 cities in 18 countries. As its name suggests, the Global Ring Network for Advanced Applications Development (GLORIAD) [12] is an advanced science internet network constructed as an optical ring around the Northern Hemisphere. The global ring topology of the network provides scientists, educators and students with advanced networking tools, and enables active, daily collaboration on common problems. It is therefore worthwhile to study this topology in particular. Indeed, considerable research has already gone into studying rings, in particular in the context of designing approximation algorithms for combinatorial optimization problems [13–17].

As in most previous work, we assume that traffic may not be split, because this causes the problem of packet reassembly at the receiver and is therefore generally avoided. Furthermore, we assume that the edges ("links") have linear latency functions. That is, each link e has a latency function $\ell_e(x) = a_e x + b_e$, where x is the number of players using link e and a_e and b_e are nonnegative constants.

For the problem of minimizing the maximum latency, even assuming a central authority, the question of how to route communication requests optimally is nontrivial; it is not known whether this problem is in P. It is known for general (directed or undirected) network topologies that already the price of stability

(PoS), which is the ratio of the value of the *best* Nash equilibrium to that of the optimal solution [18], is unbounded for this goal function even for linear latency functions [19, 20]. However, this is not the case for rings. It has been shown that for any instance on a ring, either its PoS equals 1, or its PoA is at most 6.83, giving a universal upper bound 6.83 on PoS for the selfish ring routing [19]. The same paper also gave a lower bound of 2 on the PoA. Recently, an upper bound of 16 on the PoA was obtained [20].

Our Results: In this paper, we show that the PoA for minimizing the maximum latency on rings is exactly 2. This improves upon the previous best known upper bounds on both the PoA and the PoS [19, 20]. Achieving the tight bound required us to upper bound a high-dimensional nonlinear optimization problem. Our result implies that the performance loss due to selfishness is relatively low for this problem. Thus, for ring routing, simply allowing each agent to choose its own path will always result in reasonable performance. The lower bound example (see Figure 1) can be modified to give a lower bound of 2^d for latency functions that are polynomials of degree at most d.

Proof Overview: Our proof consists of two main parts: first, we analyze for Nash equilibria the maximum ratio of the latency of any player to the latency of the entire ring, and then we analyze the ratio of the latency of the entire ring in a Nash equilibrium to the maximum player latency in an optimal routing. In the first part we show that this ratio is at most roughly 2/3; the precise value depends on whether or not every link of the ring is used by at least one player in the Nash equilibrium.

For the second ratio, we begin by showing the very helpful fact that it is sufficient to consider only instances where no player uses the same path in the Nash routing as in the optimal routing. For such instances, we need to distinguish two cases. The first case deals with instances for which there exists a link that in the Nash equilibrium is not used by any player. For such instances we use a structural analysis to bound the second ratio from above by $2 + 2/k$, where k is the number of agents in the system.

For the main case in which the paths of the players in the Nash equilibrium cover the ring, we show that the second ratio is at most 3. We begin by using the standard technique of adding up the Nash inequalities which state that no player can improve by deviating to its alternative path. This gives us a constraint which must be satisfied for any Nash equilibrium, but this does not immediately give us an upper bound for the second ratio. Instead, we end up with a nonlinear optimization problem: maximize the ratio under consideration subject to the Nash constraint. The analysis of this problem was the main technical challenge of this paper. We use a series of modifications to reach an optimization problem with only five variables, which, however, is still nonlinear. It can be solved by Maple, but we also provide a formal solution.

2 The Selfish Ring Routing Model

Let $\mathcal{I} = (R, \ell, (s_i, t_i)_{i\in[k]})$ be a selfish ring routing (SRR) instance, where $R = (V, E)$ is a ring and where for each agent $i \in [k]$ the pair (s_i, t_i) denotes the source and the destination nodes of agent i. We sometimes refer to the agents as *players*. For every *link* $e \in E$ we denote the *latency function* by $\ell_e(x) = a_e x + b_e$, where a_e and b_e are nonnegative constants; without loss of generality we assume that a_e, b_e are nonnegative integers. This is feasible since real-valued inputs can be approximated arbitrarily well by integers by scaling the input appropriately.

For any subgraph P of R (written as $P \subseteq R$), we slightly abuse the notation and identify P with its link set $E(P)$. If Q is a path on R with end nodes s and t, we use $P\backslash Q$ to denote the graph obtained from P by removing all nodes in $V(P) \cap V(Q) \setminus \{s, t\}$ (all internal nodes of Q which are contained in P), and all links in $P \cap Q$ (all links of Q which are contained in P).

For any feasible routing $\pi = \{P_1, \ldots, P_k\}$, where P_i is a path on R between s_i and t_i, $i = 1, \ldots, k$, we denote by $M(\pi) := \max_{i\in[k]} \ell(P_i, \pi)$ the maximum latency of any of the k agents. Here we abbreviate by $\ell(P, \pi)$ the latency

$$\ell(P, \pi) := \sum_{e\in P} (a_e|\{i \in [k] \mid e \in P_i\}| + b_e)$$

of a subgraph $P \subseteq R$ in π. We say that π is a *Nash equilibrium* (*routing*) if no agent $i \in [k]$ can reduce its latency $\ell(P_i, \pi)$ by switching P_i to the alternative path $R\backslash P_i$, provided other agents do not change their paths.

Sometimes we are only interested in the latency caused by one additional agent and we write $||P||_a := \sum_{e\in P} a_e$. Similarly we abbreviate $||P||_b := \sum_{e\in P} b_e$.

Let $\pi^N = \{N_1, \ldots, N_k\}$ be some fixed worst Nash routing (i.e., a Nash equilibrium with maximum system latency $M(\pi^N)$), and let Π^* be the set of optimal routings of $\mathcal{I}$.

For any $\pi = \{Q_1, \ldots, Q_k\} \in \Pi^*$, let

$$h(\pi) := |\{i \in [k] : N_i \neq Q_i\}|.$$

I.e., $h(\pi)$ is the number of agents for which their Nash routings are not the same as their optimal routings. We choose $\pi^* = \{Q_1, \ldots, Q_k\} \in \Pi^*$ to be an optimal routing that minimizes $h = h(\pi^*)$. Without loss of generality, we assume that $\{i \in [k] : N_i \neq Q_i\} = [h] := \{1, \ldots, h\}$. We call the agents $1, \ldots, h$ *switching* agents and we refer to the agents in $[k]\backslash[h]$ as *non-switching* ones.

For brevity, we write $\ell^*(P) := \ell(P, \pi^*)$ and $\ell^N(P) := \ell(P, \pi^N)$. Abusing notation, for any link $e \in R$, we set

$$\pi^*(e) := |\{i \in [h] \mid e \in Q_i\}|,$$

the number of *switching* (!) players whose optimal paths traverse e. Analogously, $\pi^N(e) := |\{i \in [h] \mid e \in N_i\}|$.

3 Main Result and Outline of the Proof

The purpose of this paper is the proof of the following statement.

Theorem 1. *The price of anarchy for selfish ring routing with linear latencies is* 2*.*

As mentioned in the introduction, a simple example for which the price of anarchy is two has been given already in [19]. This is the example given in Figure 1. As is easy to verify, $M(\pi^*) = 1$ and $M(\pi^N) = 2$.

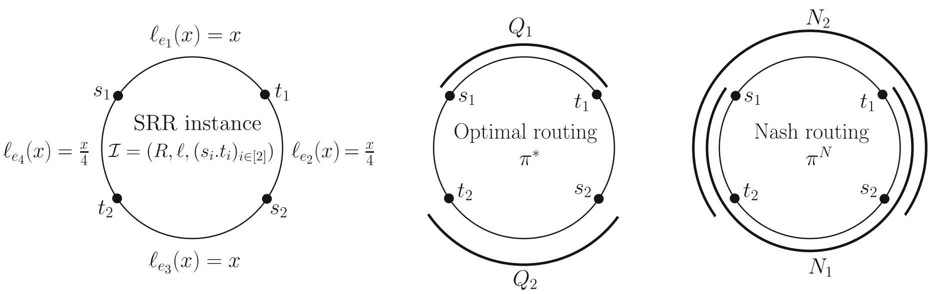

Fig. 1. A 2-player SRR instance with PoA = 2

Hence, our result is tight. We can resort to proving the upper bound in Theorem 1. That is, we need to show that for all SRR instances $\mathcal{I}$ the ratio $M(\pi^N)/M(\pi^*)$ is at most two. The main steps are as follows.

1. We begin by restricting the set of Nash routings we need to consider. We show that we can assume without loss of generality that in π^N there is at most *one* player that uses the same path as in π^*, i.e., $h \geq k-1$ (Section 3.1). We call the case where there is such a player the *singular* case; if there is no such a player, we are in the *nonsingular* case.
2. We say that the Nash equilibrium π^N is a *covering* equilibrium if the Nash paths of the switching agents $1, \ldots, h$ cover the ring, i.e., if $\cup_{i \in [h]} N_i = R$. For any non-covering equilibrium, we use a structural analysis of π^N to show (Section 4) that the PoA is less than two for $h \geq 3$.
3. We proceed by showing (Lemma 5) that for every covering equilibrium, the ratio $M(\pi^N)/\ell^N(R)$ is at most $2/3$.
4. Finally, in the remainder of Section 5, we show that $\ell^N(R)/M(\pi^*, I) \leq 3$ for any covering equilibrium π^N. This is the main part of the proof. We give a computer assisted proof here in this extended abstract, and we refer to the full paper [21] for a formal mathematical proof. Combining this with the third statement concludes the proof of Theorem 1 for covering equilibria.

Some specific cases with small values of h need to be handled separately, and are omitted in the extended abstract. Our proof needs the following technical lemma which is true for both covering and non-covering equilibria. It shows that any two Nash paths of agents that use different paths in π^N and in π^* share at least one common link.

Lemma 1. *For all $i, j \in [h]$, N_i and N_j are not link-disjoint.*

Proof. Assume there exist two agents $i, j \in [h]$ such that N_i and N_j have no link in common. Hence their complements, the optimal paths Q_i and Q_j jointly cover the entire ring, that is, $Q_i \cup Q_j = R$.

Consider the routing π' which is exactly the same as π^*, except for these two agents who use their Nash paths N_i, N_j instead. For any link $e \in Q_i \cap Q_j$ we have $\pi'(e) = \pi^*(e) - 2$, and for every link $e \in (Q_i \backslash Q_j) \cup (Q_j \backslash Q_i)$ the number of agents on this link does not change, i.e., $\pi'(e) = \pi^*(e)$. Since $a_e \geq 0$ for all $e \in E$, this yields $M(\pi') \leq M(\pi^*)$. Hence, $\pi' \in \Pi^*$. But we also have $h(\pi') < h(\pi^*)$, contradicting the choice of π^* given in Section 2. □

3.1 Reduction to Singular and Nonsingular Instances

Lemma 2. *Consider any selfish ring routing instance $\mathcal{I} = (R, \ell, (s_i, t_i)_{i \in [k]})$ with linear latencies. Let π^* be an optimal routing and let π^N be a Nash routing. Suppose there is an agent $q \in [k]$ that uses the same path in π^N as in π^*. Then there exists a selfish routing instance $\mathcal{I}' = (R, \ell', (s_i, t_i)_{i \in [k] \backslash \{q\}})$ with linear latency functions $\ell'_e(x)$ such that*

- *the non-switching agent q is removed from $\mathcal{I}$ to get $\mathcal{I}'$,*
- *the routing π^N restricted to the remaining agents, denoted as $\pi^{N'}$, is a Nash equilibrium for $\mathcal{I}'$,*
- *the total ring latencies satisfy $\ell'^N(R) := \ell'(R, \pi^{N'}) = \ell^N(R)$, and*
- *we have $M'(\mathrm{opt}') \leq M(\pi^*)$ for the maximum latencies of individual agents. Here, opt' denotes an optimal routing for $\mathcal{I}'$ and $M'(\cdot)$ denotes the maximum latency of a routing in $\mathcal{I}'$.*

Proof. By definition, player q uses path Q_q in both $\pi^N = \{N_i : i \in [k]\}$ and $\pi^* = \{Q_i : i \in [k]\}$. Remove player q from $\mathcal{I}$. For every link $e \in Q_q$ set $\ell'_e(x) := \ell_e(x) + a_e = a_e x + b_e + a_e$. The latency functions of all other links are unchanged. Denote the resulting instance $(R, \ell', (s_i, t_i)_{i \in [k] \backslash \{q\}})$ by $\mathcal{I}'$.

Every routing π for $\mathcal{I}$ induces a routing π' for $\mathcal{I}'$ in the natural way, by omitting the routing for player q. From the modified latency defined in the proof, we see that the latency of every edge in an induced routing is the same as the original latency in $\mathcal{I}$. It follows immediately that

- a routing which is a Nash equilibrium in $\mathcal{I}$ induces a Nash equilibrium routing in $\mathcal{I}'$,
- the latency of the entire ring of an induced routing is also the same as the ring latency of the original routing in $\mathcal{I}$, and
- the maximum latency of the induced routing $\pi^{*\prime}$ of the optimal routing π^* is not larger than the maximum latency of the optimal routing itself, i.e., $M'(\pi^{*\prime}) \leq M(\pi^*)$.

By definition, the *optimal* routing opt′ for instance $\mathcal{I}'$ cannot be worse than the feasible routing $\pi^{*\prime}$, and we conclude $M'(\text{opt}') \leq M'(\pi^{*\prime}) \leq M(\pi^*)$. □

We call the Nash routing π^N *singular* if $M(\pi^N) > \max_{i\in[h]} \ell^N(N_i)$, i.e., if the maximum latency in π^N is obtained only by an agent which uses the same routing in π^N as it uses in π^*. We call π^N *nonsingular* otherwise. That is, π^N is nonsingular if $M(\pi^N) = \max_{i\in[h]} \ell^N(N_i)$. Since we are interested in upper bounding the ratio $M(\pi^N)/M(\pi^*)$, applying Lemma 2 repeatedly enables us to make the following assumption.

Assumption 1. *$h \leq k \leq h+1$ and $h = k+1$ if and only if π^N is singular.*

Under Assumption 1, for any singular case $(\pi^N, \mathcal{I})$, Lemma 2 produces a nonsingular case $(\pi^{N'}, \mathcal{I}')$ with $\ell'^N(R, \mathcal{I}')/M'(\text{opt}', \mathcal{I}') \geq \ell^N(R, \mathcal{I})/M(\pi^*, \mathcal{I})$. Therefore we can upper bound the price of anarchy for the SRR problem as follows:

- analyze the ratio $\ell^N(R, \mathcal{I})/M(\pi^*, \mathcal{I})$ only for nonsingular instances $\mathcal{I}$ where no player uses the same path in π^N and π^*, and
- analyze the ratio $M(\pi^N, \mathcal{I})/\ell^N(R, \mathcal{I})$ for general instances $\mathcal{I}$.

This is what we will do in the remainder of the paper. We refer to the full version of the paper [21] for the details and proofs omitted due to the space limit.

4 Non-covering Equilibria

Theorem 2. *The ratio $M(\pi^N)/M(\pi^*)$ is at most $\frac{4}{3} + \frac{5}{3h}$ for instances for which $\cup_{i\in[h]} N_i \neq R$.*

The proof of Theorem 2 consists of the following two steps. First we show that the ratio $\ell^N(R)/M(\pi^*)$ is at most $2 + \frac{2}{h}$. This is Lemma 3. Next we show (Lemma 4) that for any uncovered instance, if $\ell^N(R)/M(\pi^*) \leq \alpha$ for some constant α, then $M(\pi^N)/M(\pi^*)$ is at most $(2\alpha + \frac{1}{h})/3$. This proves Theorem 2, which itself proves Theorem 1 for the non-covered case with $h \geq 3$. The remaining case of non-covering equilibria with $h = 2$ is handled in [21], where we show $M(\pi^N)/M(\pi^*) \leq 2$ directly by utilizing the structural properties of rings.

Lemma 3. *Let $\mathcal{I}$ be an SRR instance with $\cup_{i\in[h]} N_i \neq R$. Then $\ell^N(R)/M(\pi^*) \leq 2 + \frac{2}{h}$.*

Proof. By Lemma 2 it suffices to consider the nonsingular case. That is, we assume without loss of generality that $k = h$, i.e., we assume that all agents change their paths. There exist two agents $p, q \in [h]$ such that $\cup_{i\in[h]} N_i \subseteq N_p \cup N_q \subsetneq R$, and all h paths in $N_1, N_2, \ldots, N_h$ share a common link in $N_p \cap N_q$. This holds because if there were three agents that do not all share a same link, then two of them would not share a link at all. This is due to the assumption $\cup_{i\in[h]} N_i \neq R$. However, this contradicts Lemma 1. Therefore we can take P to be

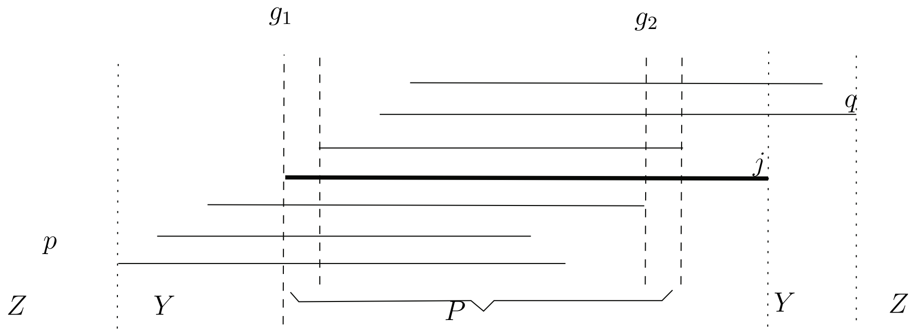

Fig. 2. Proof for non-covering equilibria. For this figure, we have mapped the ring to the real line.

the longest path in $N_p \cup N_q$ with end link g_1 and g_2 (possibly $\{g_1\} = \{g_2\} = P$) such that $\pi^N(g_i) > h/2$ for $i = 1, 2$ and

$$\pi^N(g) \leq h/2 \text{ for any link } g \in N_p \cup N_q \setminus P. \tag{1}$$

See Figure 2. Since we have $g_1 = g_2$ or $\pi^N(g_1) + \pi^N(g_2) > h$, there exists an agent $j \in [h]$ such that $\{g_1, g_2\} \subseteq N_j$ and thus $P \subseteq N_j$. Let $Y \subseteq Q_j$ consist of links e with $\pi^N(e) \geq 1$ and $Z = Q_j \backslash Y$. It can be seen from (1) that $\ell^N(Q_j) \leq \frac{h}{2}||Y||_a + ||Y||_b + ||Z||_b$ and therefore

$$\begin{aligned} \ell^N(R) = \ell^N(Q_j) + \ell^N(N_j) &\leq 2\ell^N(Q_j) + ||Y||_a + ||Z||_a \\ &\leq (h+1)||Y||_a + 2||Y||_b + ||Z||_a + 2||Z||_b. \end{aligned} \tag{2}$$

Since

$$\ell^*(Q_j) \geq \frac{h}{2}||Y||_a + ||Y||_b + h||Z||_a + ||Z||_b, \tag{3}$$

the ratio of the upper bound (2) for $\ell^N(R)$ to the lower bound (3) for $\ell^*(Q_j)$ is maximized for $||Z||_a = ||Z||_b = ||Y||_b = 0$ and is $(h+1)/(h/2) = 2 + 2/h$. $\square$

To conclude the proof of Theorem 2, we finally show the following.

Lemma 4. *The ratio $M(\pi^N)/M(\pi^*)$ is at most $(2\alpha + \frac{1}{h})/3$ for instances for which $\cup_{i \in [h]} N_i \neq R$ and $\ell^N(R)/M(\pi^*) \leq \alpha$.*

Proof. It suffices to show that for any agent $i \in [k]$ the inequality $\ell^N(N_i) \leq \frac{1}{3}(2\alpha + \frac{1}{h})M(\pi^*)$ holds. Consider an arbitrary agent $i \in [k]$. Let $C_i := R \backslash N_i$, the complement of player i's path N_i. We partition the link set of C_i into the set of links $Y := \{e \in C_i \mid \pi^N(e) \geq 1\}$ which, in routing π^N, have at least one agent on it and the set of links $Z := C_i \backslash Y$ with no players on it in routing π^N.

Since h is the number of players whose paths in π^N deviate from the one in π^*, the links e in Z satisfy $\pi^*(e) \geq h$, that is, there are at least h players using these links in the routing π^*. Hence $M(\pi^*) \geq h||Z||_a$. In the routing π^N, if player i would switch from path N_i to C_i, it would have a latency of at most $\ell^N(C_i) + ||Y||_a + ||Z||_a$. Since π^N is a Nash equilibrium, we have

$$\ell^N(N_i) \leq \ell^N(C_i) + ||Y||_a + ||Z||_a \leq 2\ell^N(C_i) + \frac{1}{h}M(\pi^*). \tag{4}$$

By assumption we also have $\ell^N(N_i)+\ell^N(C_i) = \ell^N(R) \leq \alpha M(\pi^*)$. Adding twice this inequality to (4) gives $3\ell^N(N_i) \leq (2\alpha + \frac{1}{h})M(\pi^*)$, as required. □

5 Covering Equilibria

For covering equilibria, we show that the price of anarchy is at most 2. This is again a two-step approach. First, the covering property implies an upper bound 2/3 on $M(\pi^N)/\ell^N(R)$ as follows.

Lemma 5. *If* $\cup_{i\in[h]}N_i = R$, *then* $M(\pi^N)/\ell^N(R) \leq 2/3$.

Proof. Take $Q \in \pi^N$ with $\ell^N(Q) = M(\pi^N)$. Then $\ell^N(Q) \leq \ell^N(R\backslash Q) + ||R\backslash Q||_a$ as π^N is covering. From $\ell^N(R) = \ell^N(Q) + \ell^N(R\backslash Q) \geq 2\ell^N(Q) - ||R\backslash Q||_a \geq 2\ell^N(Q) - \ell^N(R\backslash Q) = 3\ell^N(Q) - \ell^N(R)$, we deduce that $M(\pi^N) = \ell^N(Q) \leq \frac{2}{3}\ell^N(R)$. □

Second, we prove $\ell^N(R)/M(\pi^*) \leq 3$ by distinguishing between the case $h \leq 2$ and $h > 2$.

Theorem 3. *If* $\cup_{i\in[h]}N_i = R$, *then* $\ell^N(R)/M(\pi^*) \leq 3$.

The former case $h \leq 2$ is proved in [21], which along with Lemma 8 in this section establishes Theorem 3.

By Lemma 2, we only need to bound ratio $\ell^N(R)/M(\pi^*)$ for nonsingular case where $h = k$. In this section we consider the $k = h \geq 3$ switching players. For each switching player $i \in [h]$, we can formulate an inequality $\ell^N(N_i) \leq \ell^N(Q_i)+||Q_i||_a$ saying that its Nash path may not have a longer latency than its alternative path, if one unit load is added on every link of the latter. We obtain a constraint by adding up all of these inequalities.

We can assume that every link has a latency function of x or 1. This can be achieved by replacing a link e with latency function $a_e x + b_e$ by a_e links with latency function x followed by b_e links with latency function 1. Now there are only two types of links left, the ones with latency function x and the ones with latency 1. We introduce variables which count the number of links of both types which are used by a certain number of players, and write the constraint that we constructed above in terms of these variables. We then give an upper bound for $\ell^N(R)/M(\pi^*)$ in terms of these variables as well.

We end up with a nonlinear optimization problem: maximize the ratio under consideration subject to the Nash constraint. For this problem, we first show that, for the links with latency function 1, only the *total* number of players on all these links affects the upper bound. For any fixed number of players h that do not use the same path in the Nash routing as in the optimal routing, this still leaves us with $h+3$ variables, since we have one variable for each possible number of players on the links with latency function x. We now use a centering argument to show that only at most two of these h variables are nonzero in an optimal solution of this optimization problem.

Using normalization, this finally gives us an optimization problem with five variables. This problem unfortunately is still not linear. It can be solved by Maple, but we also provide a formal solution. To do this, we fix h and another variable, and solve the remaining problem; we then determine the optimal overall values of the fixed h and that variable.

Summing the Nash Inequalities. For a given path $P \subseteq R$, let P^a be the subset of links with latency function x and let P^b be the subset of links with latency function 1.

Consider a link $e \in R^a$ (resp. R^b). By definition and our assumption that $k = h$, this link occurs in $\pi^N(e)$ Nash paths. That is, this link occurs $\pi^N(e)$ times on the left-hand side of the h Nash inequalities given above—each time with coefficient $\pi^N(e)$ (resp. 1). On the other hand, it occurs $h - \pi^N(e)$ times on the right-hand side of the inequalities, each time with coefficient $\pi^N(e) + 1$ (resp. 1).

Formally, we have for $i = 1, \ldots, h$

$$\sum_{e \in N_i^a} \pi^N(e) + \sum_{e \in N_i^b} 1 = \ell^N(N_i) \leq \ell^N(Q_i) + ||Q_i||_a = \sum_{e \in Q_i^a} (\pi^N(e) + 1) + \sum_{e \in Q_i^b} 1$$

and, by summation,

$$\sum_{e \in R^a} (\pi^N(e))^2 + \sum_{e \in R^b} \pi^N(e) \leq \sum_{e \in R^a} (h - \pi^N(e))(\pi^N(e) + 1) + \sum_{e \in R^b} (h - \pi^N(e)),$$

or $\sum_{e \in R^a} \left(2(\pi^N(e))^2 - h\right) + \sum_{e \in R^b} 2\pi^N(e) \leq \sum_{e \in R^a} (h-1)\pi^N(e) + \sum_{e \in R^b} h.$

Writing A_i (resp. B_i) as the number of links with i players on it and a latency function of x (resp. 1), we can group links with the same numbers of players and write the above as

$$\sum_{i=1}^{h} ((2i^2 - h)A_i + 2iB_i) \leq \sum_{i=1}^{h} ((h-1)iA_i + hB_i) \tag{5}$$

$$\Rightarrow \sum_{i=1}^{h} \left(\left(\frac{2i}{h} - \frac{1}{i} \right) C_i + \frac{2i}{h^2} B_i \right) \leq \sum_{i=1}^{h} \left(\frac{h-1}{h} C_i + \frac{1}{h} B_i \right) \tag{6}$$

where we have written $C_i = \frac{i}{h} A_i$ and divided by h^2.

Bounding the Optimal Latency. For the optimal routing we also have, by definition and the fact that we are in the nonsingular case, h inequalities of the form $M(\pi^*) \geq \ell^*(Q_i)$, $i \in [h]$. Summing all the inequalities and dividing by h implies a lower bound on $M(\pi^*)$, namely

$$M(\pi^*) \geq \frac{1}{h}\sum_{i=1}^{h} \ell^*(Q_i) = \frac{1}{h}\sum_{i=1}^{h} \left((h-i)^2 A_i + (h-i)B_i\right).$$

Thus we have

$$\frac{\ell^N(R)}{M(\pi^*)} \leq \frac{\sum_{i=1}^h (iA_i + B_i)}{\sum_{i=1}^h \left(\frac{(h-i)^2}{h} A_i + \frac{h-i}{h} B_i\right)} = \frac{\sum_{i=1}^h \left(C_i + \frac{1}{h} B_i\right)}{\sum_{i=1}^h \left(\frac{(h-i)^2}{ih} C_i + \frac{h-i}{h^2} B_i\right)} \tag{7}$$

and we want to find an upper bound for this expression under the restriction (6).

Lemma 6. *If* $\sum_{i=1}^h C_i = 0$*, then* $\ell^N(R)/M(\pi^*) \leq 2$.

Proof. Since $C_i \geq 0$ by definition, we have $C_i = 0$ for all $i \in [h]$. Condition (6) implies that $\sum_{i=1}^h \frac{i}{h} B_i \leq \frac{1}{2}\sum_{i=1}^h B_i$. Therefore, by (7), the ratio $\ell^N(R)/M(\pi^*)$ is at most $(\sum_{i=1}^h B_i)/(\sum_{i=1}^h B_i - \sum_{i=1}^h \frac{i}{h} B_i) \leq (\sum_{i=1}^h B_i)/(\frac{1}{2}\sum_{i=1}^h B_i) = 2$. □

Rewriting the Problem. Henceforth we assume $\sum_{i=1}^h C_i > 0$. Using $\frac{(h-i)^2}{ih} = \frac{h}{i} + \frac{i}{h} - 2$, from (7) we arrive at the following inequality after dividing numerator and denominator by $\sum_{j=1}^h C_j > 0$.

$$\frac{\ell^N(R)}{M(\pi^*)} \leq \frac{1 + \sum_{i=1}^h \frac{B_i}{h\sum_{j=1}^h C_j}}{\sum_{i=1}^h \left(\left(\frac{h}{i} + \frac{i}{h}\right)\frac{C_i}{\sum_{j=1}^h C_j} + \frac{h-i}{h^2}\frac{B_i}{\sum_{j=1}^h C_j}\right) - 2}$$
$$\leq \frac{1+\beta}{\sum_{i=1}^h \left(\frac{h}{i} + \frac{i}{h}\right) D_i - 2 + \beta - z}$$

where $\beta := \frac{\sum_{i=1}^h B_i}{h\sum_{j=1}^h C_j} \geq 0$, $z := \sum_{i=1}^h \frac{iB_i}{h^2 \sum_{j=1}^h C_j} \in [\frac{\beta}{h}, \beta]$, and $D_i := \frac{C_i}{\sum_{j=1}^h C_j}$ for every $i \in [h]$. Notice that $\sum_{i=1}^h D_i = 1$. We divide both sides of (6) by $\sum_{j=1}^h C_j$ and obtain the constraint $\sum_{i=1}^h \left(\frac{2i}{h} - \frac{1}{i}\right) D_i + 2z \leq \frac{h-1}{h} + \beta$. Our problem now looks as follows.

$$\frac{\ell^N(R)}{M(\pi^*)} \leq \max \frac{1+\beta}{\sum_{i=1}^h \left(\frac{h}{i} + \frac{i}{h}\right) D_i - 2 + \beta - z} \tag{8}$$

$$\text{s.t.} \sum_{i=1}^h \left(\frac{2i}{h} - \frac{1}{i}\right) D_i + 2z \leq \frac{h-1}{h} + \beta \tag{9}$$

$$\sum_{i=1}^h D_i = 1, \qquad D_i \geq 0 \ \forall i \in [h] \tag{10}$$

$$\beta \geq z \geq \beta/h \tag{11}$$

To bound the ratio $\ell^N(R)/M(\pi^*)$ from above we will solve the general problem (8)-(11), where we ignore our definitions of β and z above and thus allow β and z to take any nonnegative real values (subject to (11)).

Since $\frac{h}{i} + \frac{i}{h} \geq 2$ for all $i \geq 1$ and $h \geq 1$, we see that for any $\beta \geq 0$ and $h \geq 1$, the denominator in (8) is positive for every feasible solution $(\{D_i\}_{i=1}^h, z)$ of (9)–(11). We can therefore also consider the following equivalent *minimization* problem:

$$\min \left\{ \sum_{i=1}^{h} \left(\frac{h}{i} + \frac{i}{h} \right) D_i - z \,\middle|\, (9)\text{–}(11) \right\} . \tag{12}$$

In what follows, we solve (12) for any fixed h and β, and then determine which values of h and β give the highest overall value for (8). For fixed h and β, any solution $(\{D_i\}_{i=1}^h, z)$ of (9) – (11) is either an optimal solution to both problem (8)–(11) and problem (12) or to neither of them. The next lemma helps to simplify our problem (12), and hence problem (8)–(11).

Lemma 7. *There is an optimal solution* $(\{D_i\}_{i=1}^h, z)$ *of (12), which is also an optimal solution of (8) – (11), such that* $D_i > 0$ *for at most two values of* i*. If there are two such values, they are consecutive.*

For an optimal solution $(\{D_i\}_{i=1}^h, z)$ to (8)–(11) as given in Lemma 7, let $x \in [h-1]$ be the minimum index such that $D_i = 0$ for all $i \in [h]\backslash\{x, x+1\}$. That is, x is the minimum index such that $D_x > 0$, or $x = h-1$. Writing y for D_{x+1}, we have $D_x = 1 - y$, and problem (8) – (11) transforms to the following relaxation which drops the upper bound β on z in (11).

$$\frac{\ell^N(R)}{M(\pi^*)} \leq \max \frac{1+\beta}{\frac{h}{x} + \frac{x}{h} - \left(\frac{h}{x(x+1)} - \frac{1}{h} \right) y - 2 + \beta - z} \tag{13}$$

$$\text{s.t.} \frac{2x}{h} - \frac{1}{x} + \left(\frac{2}{h} + \frac{1}{x(x+1)} \right) y + 2z \leq \frac{h-1}{h} + \beta \tag{14}$$

$$1 \leq x \leq h-1, x \in \mathbb{N} \tag{15}$$

$$0 \leq y \leq 1 \tag{16}$$

$$\beta/h \leq z \tag{17}$$

Upper Bounding Problem (13)–(17). We can theoretically prove that the optimal objective value of problem (13)–(17) is at most 3 for any instances $\mathcal{I}$ with $h \geq 3$ except for the case $h = 5$. For this case we have to resort to a different approach. Here in this extended abstract, we provide a computer verification using Maple. In the Maple program we note that for $h \geq 7$ we allow x to take arbitrary real values. That is, we relax the integer constraint in (15). Still we obtain an upper bound of less than 3. For a smaller number of players h we check the several options for integer values $x \in [h-1]$, again using Maple. The results are presented below, where the last five columns give the values of constants and variables at which the bound is attained.

Case	Bound	h	β	x	y	z
$h = 3, x = 1$	3	3	0.5	1	1	0.167
$h = 3, x = 2$	3	3	0.5	2	0	0.167
$h = 4, x \leq 2$	2.91	4	0	2	0.375	0
$h = 4, x = 3$	2.59	4	0.833	3	0	0.208
$h = 6, x \leq 2$	2.4	6	0	2	1	0.0833
$h = 6, x = 3$	2.87	6	0.375	3	1	0.0625
$h = 6, x \geq 4$	2.87	6	0.375	4	0	0.0625
$h \geq 7, x \geq 1$	2.93	7	0.167	3.24	1	0.0238

To sum up, we have shown the following result.

Lemma 8. *If* $\cup_{i\in[h]} N_i = R$ *and* $h > 2$*, then* $\ell^N(R)/M(\pi^*) \leq 3$.

6 Concluding Remarks

We have shown that the PoA of network congestion game is two, when the network is a ring and the link latencies are linear. It is left open whether the PoA is exactly 2^d for polynomial latency functions of degree d. Another challenging open question is what happens in more complicated network topologies. It is interesting to see if our proof technique can be extended to the more general class of games where each player can choose between a set of resources and its complement.

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Triadic Consensus

A Randomized Algorithm for Voting in a Crowd

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Abstract. Typical voting rules do not work well in settings with many candidates. If there are even several hundred candidates, then a simple task such as evaluating and choosing a top candidate becomes impractical. Motivated by the hope of developing group consensus mechanisms over the internet, where the numbers of candidates could easily number in the thousands, we study an urn-based voting rule where each participant acts as a voter and a candidate. We prove that when participants lie in a one-dimensional space, this voting protocol finds a $(1 - \epsilon/\sqrt{n})$ approximation of the Condorcet winner with high probability while only requiring an expected $O(\frac{1}{\epsilon^2} \log^2 \frac{n}{\epsilon^2})$ comparisons on average per voter. Moreover, this voting protocol is shown to have a quasi-truthful Nash equilibrium: namely, a Nash equilibrium exists which may not be truthful, but produces a winner with the same probability distribution as that of the truthful strategy.

1 Introduction

Voting is often used as a method for achieving consensus among a group of individuals. This may happen, for example, when a committee chooses a representative or friends go out to watch a movie. When the group is small, this process is relatively easy; however, for larger groups, the typical requirement of ranking all candidates becomes impractical and heuristics are often applied to narrow down opinions to a few representative ones before a vote is taken.

This problem of large-scale preference aggregation is even more interesting in light of the rising potential of crowdsourcing. Suppose that a city government wanted to ask its constituencies to contribute solutions for an "ideal budget that cuts 50 percent of the deficit".[1] Soliciting such proposals may be relatively straightforward; however, it is not clear how these proposals should be aggregated. In particular, a participant cannot even look through each proposal, making seemingly simple tasks such as choosing top ranked proposals, difficult. A solution to this problem would enable a new level of collaboration, a key step towards unleashing the full potential of crowdsourcing.

In this paper, we propose a randomized voting rule designed for scenarios like the above. In our problem setting, each participant submits exactly one proposal,

[1] See, for example, widescope.stanford.edu, aimed at tackling the federal budget deficit.

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representing his or her stance on the question of interest. A random triad of participants is then selected and each selected member is made to vote between the other two. Roughly speaking (details are elaborated in Sect. 2), if there is a three-way tie, the participants are thrown out from the election; otherwise, the losers are replaced by 'copies' of the winner. This is then repeated until there is a single participant remaining, who is declared the winner.

We show that for single peaked preferences, Triadic Consensus converges approximately to the Condorcet winner[2] with high probability, while only requiring an average of $\sim \log^2 n$ (conjectured to be $\sim \log n$) comparisons per individual. We also show that Triadic Consensus has nice properties for protecting against manipulation. Suppose that the rankings of candidates are induced from an underlying distance metric and suppose that each candidate has a concave utility in that distance. Then Triadic Consensus has a *quasi-truthful* Nash equilibrium. Specifically, (see Sect. 2.1) a Nash equilibrium exists which may not be truthful, but still chooses a winner *with the same probability distribution* as if every participant voted truthfully. Surprisingly, we achieve this result by counterintuitively allowing voters to express cyclical preferences (e.g. $a > b$, $b > c$, and $c > a$). Finally, we show simulations that indicate the practicality of Triadic Consensus outside of the single-peaked domain and make comparisons to a couple other algorithms. Because of space constraints, some results and proofs are expanded on in the longer version[1].

1.1 Related Work and Our Contributions

Given the long history of work on voting theory, it is not surprising that the problems we tackle have been, for the most part, thought about before. Here, we give a brief overview of related work, followed by a summary of our contributions. For in-depth reading, we refer the reader to Brandt et al. [2].

Voting Rule Criteria. One of the earliest criteria introduced for evaluating voting rules is known as the Condorcet criteria, introduced by Marquis de Condorcet[3]. It states that if a candidate exists who would win against every other candidate in a majority election, then this candidate should be elected. Unfortunately, such a candidate does not always exist. Since then, many other criteria have been introduced as ways to evaluate voting rules. However, in the surprising result known as Arrow's Impossibility Theorem, Arrow [4] proved that there were three desirable criterion that no deterministic voting rule could satisfy. This was expanded by Pattanaik and Peleg [5] to show that a similar result holds for probabilistic voting rules.

Strategic Manipulation. This sparked a wave of impossibility results, including the classical Gibbard-Satterthwaite Impossibility Theorem. Define a voting

[2] The candidate who would beat any other candidate in a pairwise majority election. In single dimensional spaces, this happens to be the median participant.

[3] See Young [3] for a fascinating historical description of the early work of Condorcet.

rule to be *strategy-proof* if it is always in a voter's interest to submit his true preference, regardless of the other voter rankings. Gibbard [6] and Satterthwaite [7] independently showed that all deterministic, strategy-proof voting rules must either be dictatorships or never allow certain candidates to win. This was extended to show that only very simple probabilistic voting rules were strategy-proof[8].

Numerous attempts at circumventing these impossibility result have been made. Bartholdi et al. [9] first proposed using computational hardness as a barrier against manipulation in elections. However, despite many NP-hardness results on manipulation of voting rules[10], it was shown that there do not exist any voting rules that are *usually* hard to manipulate[11].

Procaccia [12] used the simple probabilistic voting rules of Gibbard [8] to approximate common voting rules in a strategy-proof way, but the approximations are weak and they show that, for many of these voting rules, no strategy-proof approximations can be much stronger. Birrell and Pass [13] extended this idea to approximately strategy-proof voting, proving that there exist tight approximations of any voting rule that are close to strategy-proof.

Communication Complexity. When the number of candidates is large, it is important to study voting rules from the perspective of the burden on voters. Conitzer and Sandholm [14] studied the worst case number of bits that voters need to communicate (e.g. pairwise comparisons) in order to determine the ranking or winner of common voting rules; for many of these voting rules, it was shown that the number of bits required is essentially the same as what is required for reporting the entire ranking. In addition, they showed [15] that for many common voting rules, determining how to elicit preferences efficiently is NP-complete, even when perfect knowledge about voter preferences is assumed. Lu and Boutilier [16] proposed the idea of reducing communication complexity under approximate winner determination. Though they do not present theoretical guarantees, they propose a regret minimizing algorithm and show significant reductions in communication when run on experimental data sets.

Single-Peaked Preferences. One special case that avoids the many discouraging results above is that of single-peaked preferences[17] (or other domain restrictions). Single-peaked preferences are those for which candidates can be described as lying on a line. Every voter's utility function is peaked at one candidate and drops off on either side. For such preferences, a Condorcet winner always exists and is the candidate who is the median of all voter peaks. This winner can be found by the classical median voting rule, which has each voter state their peak and returns the median of these peaks. It turns out that the median voting rule is both strategy-proof[18] and has a low communication complexity of $O(n \log m)$[19], where n is the number of voters and m is the number of candidates. Conitzer [20] also studies the problem of eliciting voter preferences or the aggregate ranking using comparison queries.

The median voting rule has one weakness: it requires knowledge of an axis, which can make it impractical in practice. First, the *algorithm* requires knowledge of the axis in order to pick the median of peaks. When an axis isn't known, Escoffier et al. [19] provides an $O(mn)$ algorithm for finding such an axis with additional queries, but with no strategic guarantees. Second, the *voter* also requires knowledge of the axis. In situations where proposals have multiple criterion, but are still single peaked (for example, in a linear combination of the criterion), it may not be obvious to the voter where the axis is. Third, and more subtle, even if an axis is known, it may not be practical to *express* a voter's position on this axis. Take, for example, the canonical liberal-conservative axis used to support the single-peaked setting. It is obvious that one extreme of the axis is an absolute liberal and that the other is an absolute conservative. But how would a voter express any position in between? It would not make sense for a voter to express his or her peak as "seventy percent liberal".[4]

Our Contributions. Triadic Consensus solves the previous problems by eliminating the need for an axis. The only task voters are required to perform is a series of comparisons between two candidates. Likewise, the central algorithm does not require any knowledge about proposal positions. With these properties, we prove the following guarantees (as made precise in Sects. 3 and 4):

1. For single-peaked preferences, Triadic Consensus finds a $(1-\epsilon/\sqrt{n})$ approximation of the Condorcet winner with high probability with a communication complexity of $O(\frac{n}{\epsilon^2}\log^2\frac{n}{\epsilon^2})$, i.e. $\sim n\log^2 n$ (conjectured to be $\sim n\log n$) for a $1-\frac{1}{\sqrt{n}}$ approximation and $\sim\frac{1}{\epsilon^2}\log^2\frac{1}{\epsilon^2}$ for a $1-\epsilon$ approximation.
2. For a single-dimensional setting, Triadic Consensus has a quasi-truthful Nash equilibrium when participants have concave utility functions.

These results are especially interesting given that they are coupled with the following novel concepts:

1. *A localized consensus mechanism for large groups.* We propose Triadic Consensus as an approach for large groups to make decisions using small decentralized decisions among groups of three.
2. *Quasi-truthful voting rules and cyclical preferences.* When each participant is a voter and a candidate, we demonstrate that allowing participants to express cyclical preferences ($a > b$, $b > c$, and $c > a$) can introduce strategies that detect and protect against strategic manipulation.

Outline of the Paper. Before continuing, we briefly describe the structure of the remaining sections. In Sect. 2, we detail Triadic Consensus and introduce the notion of quasi-truthfulness. This is followed by Sect. 3, which presents the approximation and communication complexity results, and Sect. 4, which describes the quasi-truthfulness results. Finally, Sect. 5 concludes by describing topics elaborated on in the longer version[1] and a discussion on future directions.

[4] Note that he cannot just state his favorite candidate as his peak because this would require looking through all n candidates.

ALGORITHM 1: Triadic Consensus

```
Input: An urn with k labeled balls for each participant 1, 2, ..., n
Output: A winning candidate i.
while there is more than one label do
    Sample three balls (with labels x, y, z) uniformly at random with replacement;
    w = TriadicVote(x, y, z);
    if w ≠ ∅ then
        Relabel all the sampled balls with the winning label w;
    else
        /* For example, remove the three sampled balls from the urn     */
        TriadicMechanism(x, y, z);
if at least one ball remains then
    return the id of any remaining ball;
else
    return the id of a random ball from the last removed;
```

ALGORITHM 2: TriadicVote

```
Input: Candidates x, y, z
Output: One of {x, y, z} if there is a winner, ∅ otherwise
if two of more of x, y, z have the same id then
    return the majority candidate;
x votes between y and z; y votes between x and z; z votes between x and y;
if each received exactly one vote then
    return ∅;
else
    return the candidate with two votes;
```

2 Triadic Consensus and Quasi-truthfulness

Triadic Consensus applies to scenarios where the set of candidates and voters coincide. We use x to refer to both the participant x and the candidate solution that he or she proposes. For $x, y, z \in \{1, 2, \ldots, n\}$, we use $\succ_x$ to denote the ranking of participant x and $y \succ_x z$ to denote that x prefers y over z.[5]

The best way to understand Triadic Consensus (Algorithm 1) is to imagine an urn with balls, each of which is labeled by a participant id. The urn starts with k balls for each of the n participants.[6] At each step, the algorithm samples three balls uniformly at random (with replacement) and performs a TriadicVote (Algorithm 2) on the three corresponding participants.

If the three participants x, y, and z are unique, the TriadicVote subroutine consists of a single comparison for each of the selected participants: x votes

[5] We assume a strict ordering, but it is not hard to generalize the algorithm to ties.

[6] The intuition for k is that it is a tradeoff between approximation and time. Increasing k makes the approximation tighter, but requires more comparisons to converge.

ALGORITHM 3: The Remove mechanism

Input: Balls x, y, and z

Remove the three sampled balls from the urn;

ALGORITHM 4: The RepeatThenRemove mechanism

Input: Balls x, y, and z

$w = \text{TriadicVote}(x, y, z)$;

if $w \neq \emptyset$ **then**

  Relabel all the sampled balls with the winning label w;

else

  Remove(x, y, z);

between y and z, y between x and z, and z between x and y. These votes can be distributed in some permutation of $2, 1, 0$ or split $1, 1, 1$. In the first case, the participant who received two votes is returned as the winner. In the second case, a tie (represented as $\emptyset$) is returned. If two or more of the selected ids are the same, i.e. are the same person, then he is automatically returned as the winner.

If a winner was returned from the TriadicVote, then the three balls are relabeled with the winning id and placed back into the urn; otherwise, one of several mechanisms can be applied to resolve the tie. This process is repeated until there is only one participant id remaining, which is declared the winner.

In our paper, we propose two possible mechanisms, each of which has a quasi-truthful Nash equilibrium. The simplest is Remove (Algorithm 3), in which the three balls are simply removed. In RepeatThenRemove (Algorithm 4), the three balls are made to vote again; if there is another three way tie, then they are removed. Surprisingly, repeating the TriadicVote before elimination results in a simpler (and more practical) strategy that is a quasi-truthful Nash equilibrium.

2.1 Truthfulness and Quasi-truthfulness

For our analysis of strategic behavior, we will assume that each individual is represented as a point x in some space X and that his or her preference ranking is induced by a distance metric $d(x, \cdot)$ on X. If $d(x, y) \leq d(x, z)$, then $y \succ_x z$; that is, x prefers proposals that are closer to him. Since the individuals voting in a TriadicVote are also the candidates being voted for, there can never be a three-way tie in a truthful vote. Otherwise, all three of $d(x, y) < d(x, z)$, $d(y, z) < d(y, x)$, and $d(z, x) < d(z, y)$ must be simultaneously true, which is impossible so long as $d(\cdot, \cdot)$ satisfies the natural property that $d(x, y) = d(y, x)$.

Consider a TriadicVote between participants x, y, and z. If they vote truthfully, then there are situations when players may be incentivized to deviate.

Example 1. Four participants lie in space $X = \mathbb{R}$ at positions 0, 5, 6, and 7. Suppose participants 0, 5, and 7 are selected for a TriadicVote. Since they are

ALGORITHM 5: Quasi-truthful Nash for the Remove mechanism

Input: Voter x, candidates y, z
Output: One of $\{y, z\}$
if *x thinks he should win* **then**
  if *y would prefer a win for x rather than a three-way tie in a truthful world* **then**
    return y;
  else
    return z;
else
  return a truthful comparison between y and z;

voting truthfully, 0 votes for 5, 5 votes for 7, and 7 votes for 5. As a result, 5 wins and the resulting urn consists of three balls for 5 and one for 6.

Now suppose participant 7 were to vote strategically for participant 0. This would result in a tie and all the selected balls would be eliminated, leaving only participant 6. Clearly, participant 7 would prefer this second situation.

At this point, we might note that the truthful winner's vote (e.g. 5) does not change the result and that he can use his vote to disincentivize others from manipulating the TriadicVote. We define any such behavior to be quasi-truthful when it results in the same outcome as that of truthful voting.

Example 2. Suppose that the participants of Ex. 1 are trying to minimize the expected distance of the winning proposal to their position. Then a quasi-truthful strategy would be for 0 to vote for 5, 5 to vote for 0 and 7 to vote for 5. As in Ex. 1, 5 wins and the resulting urn consists of three balls for 5 and one for 6.

Now suppose participant 7 deviates from this strategy and votes for 0. Then participant 0 gets two votes and he wins. The resulting urn consists of three balls for 0 and one for 6, which is clearly worse for participant 7. Likewise, suppose participant 0 deviates from this strategy and votes for 7. Then there is a three-way tie and all selected balls get eliminated. The resulting urn consists of a single ball for 6, which is clearly worse for participant 0.

From this example, we get the intuition for Algorithm 5, a quasi-truthful Nash for the Remove mechanism. If a participant (x WLOG) is the truthful winner, then he should look for the participant who would prefer a win for him over a removal of all three balls. Such a participant will be shown to exist when all players have concave utilities. The same idea gives us Algorithm 6, a quasi-truthful Nash for the RepeatThenRemove mechanism and a more practical strategy to implement. The participant that would prefer a win for x over a removal of all three balls (y WLOG) will not deviate from voting for x. If x votes for y in the first round, z cannot cause a tie. If x votes for z in the first round, z can only cause a second round, during which x will then vote for y. We will show that these intuitions do indeed translate to rigorous proofs in Sect. 4.

ALGORITHM 6: Quasi-truthful Nash for the RepeatThenRemove mechanism

```
Input: Voter x, candidates y, z
Output: One of {y, z}
if x thinks he should win then
    if it is the first TriadicVote then
        /* For example, a truthful comparison                                  */
        return either of y or z;
    else
        return the candidate that he didn't vote for in the first round;
else
    return a truthful comparison between y and z;
```

3 Triadic Consensus Approximates the Condorcet Winner with Low Communication Complexity

3.1 Background: Fixed Size Urns and Urn Functions

The primary idea in proving the results in this section is to reduce the Triadic Consensus urn to previously known results for fixed size urns with urn functions. A fixed size urn contains some number of balls, which are each colored either red or blue. Let R_t and B_t be the number of red and blue balls respectively at time t, where $R_t + B_t = n$. Also, let $p_t = \frac{R_t}{n}$ denote the fraction of red balls. At every discrete time t, either a red ball is sampled with probability $f(p_t)$, a blue ball is sampled with probability $f(1 - p_t)$, or nothing happens with the remaining probability. The function $f : [0, 1] \to [0, 1]$ is called an urn function and satisfies $0 \leq f(x) + f(1-x) \leq 1$ for $0 \leq x \leq 1$. If a ball was sampled, it is then recolored to the opposite color and placed back into the urn. This process repeats until some time T when all the balls are the same color, i.e. $R_T = n$ or $R_T = 0$.

We will show in the following section that Triadic Consensus is closely related to fixed size urns with urn function $f(p) = 3p(1-p)^2$. We will then use the following theorems derived from those in Lee and Bruck [21]

Theorem 1. *Let a fixed size urn start with R_0 red balls out of n total balls and have an urn function $f(p) = 3p(1-p)^2$. Let T denote the first time when either $R_T = n$ or $R_T = 0$. Then,*

$$\mathbf{Pr}[R_T = n] = \left(\frac{1}{2}\right)^{n-1} \sum_{j=1}^{R_0} \binom{n-1}{j-1}$$

Theorem 2. *Let a fixed size urn start with R_0 red balls out of n total balls and have an urn function $f(p) = 3p(1-p)^2$. Let T denote the first time when either $R_T = n$ or $R_T = 0$. Then,*

$$\mathbb{E}[T] \leq n \ln n + O(n)$$

3.2 Reduction from Triadic Consensus to Fixed Size Urns

Recall that our results are for the case of single-peaked preferences, for which the candidates can be said to lie on some axis. Every voter's utility is described by a peak on that axis which falls off on either direction. Without loss of generality, we let the participant ids be labeled from one end of the axis to the other, i.e. $1 < 2 < \ldots < n$.

Lemma 1. *Let x, y, and z be three unique participants whose peaks lie on an axis such that $x < y < z$. Then the winner of a quasi-truthful TriadicVote(x, y, z) must be the median participant y.*

Proof. Since y would win in a truthful vote, this follows from the definition of quasi-truthfulness. □

Lemma 2. *For single-peaked preferences, let the participant ids be labeled from one end of the axis to the other, i.e. $1 < 2 < \ldots < n$. Color balls with ids $1, 2, \ldots, i$ red and balls with ids $i+1, i+2, \ldots, n$ blue. Then if participants vote quasi-truthfully, Triadic Consensus (for $k = 1$) will produce a red winner with the same probability as that of a fixed size urn with urn function $f(p) = 3p(1-p)^2$.*

Proof. Let p_r and p_b denote the fraction of red and blue balls respectively. Each time three balls are sampled, the median ball must win by Lemma 1, which implies that the majority color must win. Then we have the following four cases:

Three red With probability p_r^3, there is no change in colors.
Two red, one blue With probability $3p_r^2 p_b$, one blue ball is recolored red.
One red, two blue With probability $3p_r p_b^2$, one red ball is recolored blue.
Three blue With probability p_b^3, there is no change in colors.

These are the transition probabilities for a fixed size urn with i red balls, $n-i$ blue balls, and urn function $f(p) = 3p(1-p)^2$. Since every transition probability is identical, the final probability of a red winner must be identical. □

3.3 Main Results

Theorem 3. *For single-peaked preferences, let the participant ids be labeled from one end of the axis to the other, i.e. $1 < 2 < \ldots < n$. Then if participants vote quasi-truthfully, Triadic Consensus (for $k = 1$) will produce a winner w with probability*

$$\mathbf{Pr}[w = i] = \left(\frac{1}{2}\right)^{n-1} \binom{n-1}{i-1}$$

Proof. If balls $1, 2, \ldots, i$ are colored red, then $w \leq i$ iff the winning ball is red. Then applying Theorem 1 and Lemma 2, we get $\mathbf{Pr}[w \leq i]$. By subtracting $\mathbf{Pr}[w \leq i-1]$ from $\mathbf{Pr}[w \leq i]$, we get our final expression. □

A similar argument extends the above theorem for general k. Using standard probabilistic arguments[22], we get the following corollary.

Corollary 1. *Let there be n single-peaked participants and let w denote the winning id after running Triadic Consensus with $k = O(\frac{1}{\epsilon^2}\log\frac{1}{\delta})$. Then assuming that participants vote quasi-truthfully, w will be a $(1-\epsilon/\sqrt{n})$ approximation of the Condorcet winner with probability at least $1-\delta$.*

Theorem 4. *For single-peaked preferences and quasi-truthful voting, Triadic Consensus has a total communication complexity of $O(kn\log^2(kn))$.*

Proof. Theorem 2 is an upper bound on the expected time it takes to halve the number of remaining participants (since we can color half the participants red and half blue). For kn balls, this gives us $\leq kn\ln(kn)+O(kn)$ time to halve the participants. Each time the urn converges to a single color, we can recolor half the remaining participants and repeat. After $\log n$ rounds, we will be done. □

The above theorem is an upper bound on the communication complexity. In reality, at each recoloring, the balls will not be split evenly between the two colors. Based on this intuition and simulations, we conjecture that the communication complexity is only $O(kn\log kn)$.

4 Triadic Consensus Has a Quasi-truthful Nash Equilibrium for Concave Utilities

To discuss strategic behavior, we need to define the utilities for each participant. Let $U_x(y)$ denote the utility that x gets from a proposal y. The utility x derives from y depends on the distance from x to y, i.e. $U_x(y) = f_x(d(x,y))$, where $f(\cdot)$ must be decreasing in distance so that $U_x(y) > U_x(z)$ whenever $y \succ_x z$. We say that a participant x has a concave utility function if $f_x(\cdot)$ is a concave function.

Theorem 5. *If all participants have concave utility functions, then Algorithms 5 and 6 are quasi-truthful Nash equilibria for Triadic Consensus when using the Remove and RepeatThenRemove mechanisms respectively.*

Proof. We prove our main result with the following proof by induction. Since the proofs for the Remove mechanism and the RepeatThenRemove mechanism are almost identical, we will refer solely to the Remove mechanism for simplicity.

Base Case: Algorithm 5 is a Nash equilibrium for $n = 1, 2, 3$ balls (Lemma 3).

Inductive Step: Assume that Algorithm 5 is a Nash equilibrium for $n-3$ balls. Now consider a participant x who is considering deviating from Algorithm 5 in an urn with n balls:

1. For any TriadicVote with participants $x < y < z$ in an urn with n balls, if y votes for x, then by the definition of the strategy and the fact that one of x and z must prefer y to a three-way tie (Lemma 4), we know that x must prefer y to win over a three-way tie, which means x should not deviate.

2. For any TriadicVote with participants $x < y < z$ in an urn with n balls, if y votes for z, then given that the previous statement is true, we show that x should prefer a win for y over a win for z (Lemma 5). This is done by defining a comparison relation between urns that formalizes this intuition that participants should prefer closer balls. With this definition, we can define a coupling of two urns: one in which x plays an optimal strategy, and one in which x always plays according to Algorithm 5. We show that for every coupled history, the urn from Algorithm 5 does at least as well as the optimal urn in expected utility. This means that Algorithm 5 is also an optimal strategy for x in this case.

By carrying out the Inductive Hypothesis, we get our result for all n. □

4.1 Supporting Lemmas

Lemma 3. *Algorithm 5 is a Nash equilibrium for Triadic Consensus with the Remove mechanism when $n = 1$, 2, or 3 balls.*

Proof. This is trivially true for $n = 1$ and 2 since no votes take place. For $n = 3$, suppose that the three participants are $x < y < z$. In this case, the only situation when participants cast votes is when TriadicVote is performed with all three unique participants. After such a situation occurs, there will either be a winner or all balls will be eliminated and no further votes take place. Therefore, our analysis can be constrained to this single TriadicVote.

If participants vote according to Algorithm 5, we know that y will be the winner since x and z both vote for him. Suppose y votes for x WLOG. Then if z deviates, x will win, which is clearly suboptimal. If x deviates, then there is a three-way tie and all are eliminated, resulting in a uniformly random winner.

The difference in utility lost for x by deviating is $\Delta U_x = U_x(y) - \frac{1}{3}(U_x(x) + U_x(y) + U_x(z))$. Letting d_1 be the distance between x and y and d_2 the distance between y and z, we have $\Delta U_x = \frac{1}{3}(f_x(d_1) - f_x(0)) - \frac{1}{3}(f_x(d_1 + d_2) - f_x(d_1))$ and

$$\Delta U_x \geq 0 \iff f_x(d_1) - f_x(0) \geq f_x(d_1 + d_2) - f_x(d_1) \iff \frac{f_x(d_1) - f_x(0)}{f_x(d_1 + d_2) - f_x(d_1)} \leq 1$$

Similarly,

$$\Delta U_z \geq 0 \iff \frac{f_z(d_2) - f_z(0)}{f_z(d_1 + d_2) - f_z(d_2)} \leq 1$$

For concave, monotonically non-increasing f_x and f_z, we know that (detailed in the long version[1]):

$$\frac{f_x(d_1) - f_x(0)}{f_x(d_1 + d_2) - f_x(d_1)} \leq \frac{d_1}{d_2} \qquad \text{and} \qquad \frac{f_z(d_2) - f_z(0)}{f_z(d_1 + d_2) - f_z(d_2)} \leq \frac{d_2}{d_1}$$

But then, at least one of $\frac{d_1}{d_2}$ or $\frac{d_2}{d_1}$ is less than or equal to 1, which means that at least one of ΔU_x and ΔU_z is greater than or equal to 0 and prefers a win for y over a three-way tie. By the definition of Algorithm 5, y will vote for this person when he exists. Therefore, since y voted for x, we know $\Delta U_x \geq 0$, which concludes the proof. □

Lemma 4. *Assume that Algorithm 5 is a Nash equilibrium for any configuration of $n-3$ balls. Then for a TriadicVote among participants $x < y < z$ in an urn with n balls, at least one of x or z prefers a win for y over a three-way tie, so long as they both have concave utilities.*

Proof. Because of space constraints, we will only outline the proof here, leaving the notation and algebra for the longer version. The proof has two parts:

Part A. Suppose all balls are positioned somewhere between x and z, i.e. in the interval $[x, z]$. Then, if x and z have concave utility functions, at least one of x and z prefers a win for y over a three-way tie. The proof for this statement is similar to the one in Lemma 3, albeit more complex.

Part B. For any configuration of n balls, moving any ball at position x leftwards and moving any ball at position z rightwards can only increase both ΔU_x and ΔU_z. Put another way, given any configuration, we can move all balls left of x to x and all balls right of z to z, while only decreasing ΔU_x and ΔU_z. Once moved in this way, the configuration of balls falls under the jurisdiction of Part 1, which states that at least one of ΔU_x and ΔU_z is greater than or equal to 0. Therefore, the same participant in the original configuration must also have a positive ΔU, which means he prefers a win for y over a three-way tie. □

For the final lemma, we require the following definition.

Definition 1. *Given two urns R and S, each with n balls, number the balls in R from left to right as $r_1, r_2, \ldots, r_n$ and number the balls in S from left to right as $l_1, l_2, \ldots, l_n$. Then R x-dominates S if*

$$s_i \leq r_i \text{ for } r_i < x$$
$$s_i = r_i \text{ for } r_i = x$$
$$s_i \geq r_i \text{ for } r_i > x$$

Lemma 5. *Assume that Algorithm 5 is a Nash equilibrium for any configuration of $n-3$ balls. Then for a TriadicVote among participants $x < y < z$ in an urn with n balls, if y votes for z (WLOG), x does not benefit by voting strategically for z.*

Proof. Our proof strategy will be to use a coupling argument. Let OPT denote the optimal strategy for x. We consider two urns R and S. In urn R, x plays according to Algorithm 5. In urn S, x plays according to OPT, the strategy that maximizes his expected utility. We couple the TriadicVote's of these urns in the following way:

1. Let $r_1, r_2, \ldots, r_n$ denote the balls in urn R as indexed from leftmost position to rightmost position. Let $s_1, s_2, \ldots, s_n$ denote the balls in urn S as indexed from leftmost position to rightmost position.

2. Then for every TriadicVote, when balls r_i, r_j, r_k are randomly drawn from urn R, balls s_i, s_j, s_k will be drawn from urn S.

Suppose R x-dominates S and then each undergoes a coupled TriadicVote where balls $r_i < r_j < r_k$ are selected from R and $s_i < s_j < s_k$ are selected from S. After they vote, we show that the resulting urns R' and S' must still satisfy R' x-dominates S'. By the coupling rule, this is trivially true when 1) x is not selected, 2) x is represented in two or more balls, and 3) x is the middle participant. This is because x either does not vote or cannot affect the result in these cases (remember that all other participants are voting according to Algorithm 5). The only remaining case is when x is one of the side participants (s_i WLOG). In this case, r_j wins in urn R since x plays according to Algorithm 5 in this urn. Suppose s_j voted for s_k. Then regardless of who x votes for, one of s_j or s_k must win, both of which will still satisfy R' x-dominates S'. Now suppose s_j voted for s_i. Then x could eliminate all three participants by voting for s_k. However, by Lemma 4 and the definition of Algorithm 5, this would be suboptimal, which means that x cannot play this strategy in urn S. Therefore, R' x-dominates S'.

Finally, we note that before any TriadicVote's take place, R and S are identical, i.e. R x-dominates S. Then, the winner of R must also x-dominate the winner of S, which means that urn R is better for x in every coupled history. □

5 Other Results and Future Directions

A couple results have been left to [1] due to space constraints. One of these is a series of simulations demonstrating that Triadic Consensus works well for preferences that are not single-peaked. We also make several comparisons to other algorithms, which yield an intuition that Triadic Consensus is good at eliminating outliers. Finally, we point out that the approximation factor produced by Triadic Consensus cannot be improved significantly given natural assumptions.

There are many future directions for this work. One clear step is to analyze higher dimensional spaces and attempt to find voting rules that can achieve low communication complexity for any set of rankings. Along with this comes the question of whether triads, quasi-truthfulness, and cyclical preferences can be extended to general settings. For example, one could imagine the following variant of the Borda count: for each of the $\binom{n}{3}$ triads, add one point to the score of the winner. Finally, it would be interesting to study more collaborative dynamics of group consensus mechanisms as opposed to only voting.

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Truthful Mechanism Design for Multidimensional Covering Problems*

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Abstract. We investigate *multidimensional covering mechanism-design* problems, wherein there are m items that need to be covered and n agents who provide covering objects, with each agent i having a private cost for the covering objects he provides. The goal is to select a set of covering objects of minimum total cost that together cover all the items.

We focus on two representative covering problems: uncapacitated facility location (UFL) and vertex cover (VC). For multidimensional UFL, we give a black-box method to transform any *Lagrangian-multiplier-preserving* ρ-approximation algorithm for UFL to a truthful-in-expectation, ρ-approx. mechanism. This yields the first result for multidimensional UFL, namely a truthful-in-expectation 2-approximation mechanism.

For multidimensional VC (Multi-VC), we develop a *decomposition method* that reduces the mechanism-design problem into the simpler task of constructing *threshold mechanisms*, which are a restricted class of truthful mechanisms, for simpler (in terms of graph structure or problem dimension) instances of Multi-VC. By suitably designing the decomposition and the threshold mechanisms it uses as building blocks, we obtain truthful mechanisms with approximation ratios (n is the number of nodes): (1) $O(\log n)$ for Multi-VC on any minor-closed family of graphs; and (2) $O(r^2 \log n)$ for r-dimensional VC on any graph. These are the first truthful mechanisms for Multi-VC with non-trivial approximation guarantees.

1 Introduction

Algorithmic mechanism design (AMD) deals with efficiently-computable algorithmic constructions in the presence of strategic players who hold the inputs to the problem, and may misreport their input if doing so benefits them. The challenge is to design algorithms that work well with the true (privately-known) input. In order to achieve this task, a *mechanism* specifies both an algorithm and a pricing or payment scheme that can be used to incentivize players to reveal their true inputs. A mechanism is said to be *truthful*, if each player maximizes his utility by revealing his true input regardless of the other players' declarations.

In this paper, we initiate a study of *multidimensional covering mechanism-design* problems, often called *reverse auctions* or *procurement auctions* in the mechanism-design literature. These can be abstractly stated as follows. There are m items that

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need to be covered and n agents who provide covering objects, with each agent i having a private cost for the covering objects he provides. The goal is to select (or buy) a suitable set of covering objects from each player so that their union covers all the items, and the total covering cost incurred is minimized. This *cost-minimization* (CM) problem is equivalent to the *social-welfare maximization* (SWM) (where the social welfare is − (total cost incurred by the players and the mechanism designer)), so ignoring computational efficiency, the classical VCG mechanism [26,4,15] yields a truthful mechanism that always returns an optimal solution. However, the CM problem is often *NP*-hard, so we seek to design a *polytime* truthful mechanism where the underlying algorithm returns a near-optimal solution to the CM problem.

Although multidimensional packing mechanism-design problems have received much attention in the AMD literature, multidimensional covering CM problems are conspicuous by their absence in the literature. For example, the packing SWM problem of combinatorial auctions has been studied (in various flavors) in numerous works both from the viewpoint of designing polytime truthful, approximation mechanisms [10,21,9,13], and from the perspective of proving lower bounds on the capabilities of computationally- (or query-) efficient truthful mechanisms [20,14,11]. In contrast, the lack of study of multidimensional covering CM problems is aptly summarized by the blank table entry for results on truthful approximations for procurement auctions in Fig. 11.2 in [25] (a recent result of [12] is an exception; see "Related work"). In fact, to our knowledge, the only multidimensional problem with a covering flavor that has been studied in the AMD literature is the makespan-minimization problem on unrelated machines [22,2], which is not an SWM problem.

Our Results and Techniques. We study two representative multidimensional covering problems, namely (metric) *uncapacitated facility location (UFL)*, and *vertex cover* (VC), and develop various techniques to devise polytime, truthful, approximation mechanisms for these problems.

For multidimensional UFL (Section 3), wherein players own (known) different facility sets and the assignment costs are public, we present a *black-box reduction from truthful mechanism design to algorithm design*. We show that any ρ-approximation algorithm for UFL satisfying an additional *Lagrangian-multiplier-preserving* (LMP) property (that indeed holds for various algorithms) can be converted in a black-box fashion to a truthful-in-expectation ρ-approximation mechanism (Theorem 3). This is the *first* such black-box reduction for a multidimensional covering problem, and it leads to the first result for multidimensional UFL, namely, a truthful-in-expectation, 2-approximation mechanism. Our result builds upon the convex-decomposition technique in [21]. Lavi and Swamy [21] primarily focus on packing problems, but remark that their convex-decomposition idea also yields results for *single-dimensional* covering problems, and leave open the problem of obtaining results for multidimensional covering problems. Our result for UFL identifies an interesting property under which a ρ-approximation algorithm for a covering problem can be transformed into a truthful, ρ-approximation mechanism in the multidimensional setting.

In Section 4, we consider multidimensional VC, where each player owns a (known) set of nodes. Although, algorithmically, VC is one of the simplest covering problems, it becomes a surprisingly challenging mechanism-design problem in the *multidimensional* mechanism-design setting, and, in fact, seems significantly more difficult than multidimensional UFL. This is in stark contrast with the single-dimensional setting, where each player owns a single node. Before detailing our results and techniques, we mention some of the difficulties encountered. We use Multi-VC to distinguish the multidimensional mechanism-design problem from the algorithmic problem.

For *single-dimensional* problems, a simple monotonicity condition characterizes the *implementability* of an algorithm, that is, whether it can be combined with suitable payments to obtain a truthful mechanism. This condition allows for ample flexibility and various algorithm-design techniques can be leveraged to design monotone algorithms for both covering and packing problems (see, e.g., [3,21]). For single-dimensional VC, many of the known 2-approximation algorithms for the algorithmic problem (based on LP-rounding, primal-dual methods, or combinatorial methods) are either already monotone, or can be modified in simple ways so that they become monotone, and thereby yield truthful 2-approximation mechanisms [7]. However, the underlying algorithm-design techniques fail to yield algorithms satisfying *weak monotonicity* (WMON)—a necessary condition for implementability (see Theorem 2)—even for the simplest multidimensional setting, namely, 2-dimensional VC, where *every player owns at most two nodes*. In the full version of the paper, we give examples that show this for various LP-rounding methods and primal-dual algorithms.

Furthermore, various techniques that have been devised for designing polytime truthful mechanisms for multidimensional packing problems (such as combinatorial auctions) do not seem to be helpful for Multi-VC. For instance, the well-known technique of constructing a *maximal-in-range*, or more generally, a *maximal-in-distributional-range* (MIDR) mechanism—fix some subset of outcomes and return the best outcome in this set—does not work for Multi-VC [12] (and more generally, for multidimensional covering problems). (More precisely, any algorithm for Multi-VC whose range is a proper subset of the collection of minimal vertex covers, cannot have bounded approximation ratio.) This also rules out the convex-decomposition technique of [21], which we exploit for multidimensional UFL, because, as noted in [21], this yields an MIDR mechanism.

Thus, we need to develop new techniques to attack Multi-VC (and multidimensional covering problems in general). We devise two main techniques for Multi-VC. We introduce a simple class of truthful mechanisms called *threshold mechanisms* (Section 4.1), and show that despite their restrictions, threshold mechanisms can achieve non-trivial approximation guarantees. We next develop a *decomposition method* for Multi-VC (Section 4.2) that provides a general way of reducing the mechanism-design problem for Multi-VC into simpler—either in terms of graph structure, or problem dimension—mechanism-design problems by using threshold mechanisms as building blocks. We believe that these techniques will also find use in other mechanism-design problems.

By leveraging the decomposition method along with threshold mechanisms, we obtain various truthful, approximation mechanisms for Multi-VC, which yield the *first* truthful mechanisms for multidimensional vertex cover with non-trivial approximation guarantees. We obtain a truthful, $O(\log n)$-approximation mechanism (Theorem 13) for any proper minor-closed family of graphs (such as planar graphs). Our decomposition method shows that any instance of r-dimensional VC can be broken up into $O(r^2 \log n)$ instances of *single-dimensional VC*; this in turn leads to a truthful, $O(r^2 \log n)$-approximation mechanism for r-dimensional VC (Theorem 14). In particular, for any fixed r, we obtain an $O(\log n)$-approximation for any graph. Here n is the number of nodes.

It is worthwhile to note that in addition to their usefulness in the design of truthful, approximation mechanisms for Multi-VC, some of the mechanisms we design also enjoy good frugality properties. We obtain (Theorem 16) the *first* mechanisms for Multi-VC that are polytime, truthful and *simultaneously* achieve bounded approximation ratio *and* bounded frugality ratio with respect to the benchmarks in [5,19]. This nicely complements a result of [5], who devise such a mechanism for single-dimensional VC.

Related Work. As mentioned earlier, there is little prior work on the CM problem for multidimensional covering problems. Dughmi and Roughgarden [12] give a general technique to convert an FPTAS for an SWM problem to a truthful-in-expectation FPTAS. However, for covering problems, they obtain an additive approximation, which does not translate to a (worst-case) multiplicative approximation. In fact, as they observe, a multiplicative approximation ratio is impossible (in polytime) using their technique, or any other technique that constructs a MIDR mechanism whose range is a proper subset of all outcomes.

For single-dimensional covering problems, various other results, including black-box results, are known. Briest et al. [3] consider a closely-related generalization, which one may call the “single-value setting”; although this is a multidimensional setting, it admits a simple monotonicity condition sufficient for implementability, which makes this setting easier to deal with than our multidimensional settings. They show that a pseudopolynomial time algorithm (for covering and packing problems) can be converted into a truthful FPTAS.

Single-dimensional covering problems have been well studied from the perspective of *frugality*. Here the goal is to design mechanisms that have bounded (over-)payment with respect to some benchmark, but one does not (typically) care about the cost of the solution returned. Starting with the work of Archer and Tardos [1], various benchmarks for frugality have been proposed and investigated for various problems including VC, k-edge-disjoint paths, spanning tree, s-t cut; see [18,6,19,5] and the references therein. Some of our mechanisms for Multi-VC are inspired by the constructions in [19,5], and simultaneously achieve bounded approximation ratio and bounded frugality ratio.

Our decomposition method, where we combine mechanisms for simpler problems into a mechanism for the given problem, is somewhat in the same spirit as the construction in [24]. They give a toolkit for combining truthful mechanisms, identifying sufficient conditions under which this combination preserves truthfulness. But they work only with the single-dimensional setting, which is much more tractable to deal with.

Finally, as noted earlier, there are a wide variety of results on truthful mechanism-design for packing SWM problems, such as combinatorial auctions [10,21,9,13,20,14,11].

2 Preliminaries

In a *multidimensional covering mechanism-design problem*, we have m items that need to be covered, and n agents/players who provide covering objects. Each agent i provides a set $\mathcal{T}_i$ of covering objects. All this information is public knowledge. We use $[k]$ to denote the set $\{1,\ldots,k\}$. Each agent i has a *private cost* (or type) vector $c_i = \{c_{i,v}\}_{v\in\mathcal{T}_i}$, where $c_{i,v}$ is the cost he incurs for providing object $v \in \mathcal{T}_i$; for $T \subseteq \mathcal{T}_i$, we use $c_i(T)$ to denote $\sum_{v\in T} c_{i,v}$. A feasible solution or allocation selects a subset $T_i \subseteq \mathcal{T}_i$ for each agent i, denoting that i provides the objects in T_i. Given this solution, each agent i incurs the private cost $c_i(T_i)$. Also, the mechanism designer incurs a publicly-known cost $pub(T_1,\ldots,T_n)$. The goal is to minimize the total cost $\sum_i c_i(T_i) + pub(T_1,\ldots,T_n)$ incurred. We call this the *cost minimization* (CM) problem. Note that we can encode any feasibility constraints in the covering problem by simply setting $pub(a) = \infty$ if a is not a feasible allocation. Observe that if we view the mechanism designer also as a player, then the CM problem is equivalent to maximizing the social welfare, which is given by $\sum_i -c_i(T_i) - pub(T_1,\ldots,T_n)$.

Various covering problems can be cast in the above framework. For example, in the mechanism-design version of *vertex cover* (Section 4), the items are edges of a graph. Each agent i provides a subset $\mathcal{T}_i$ of the nodes of the graph and incurs a private cost $c_{i,v}$ if node $v \in T_i$ is used to cover an edge. We can set $pub(T_1,\ldots,T_n) = 0$ if $\bigcup_i T_i$ is a vertex cover, and ∞ otherwise, to encode that the solution must be a vertex cover. It is also easy to see that the mechanism-design version of *uncapacitated facility location* (UFL; Section 3), where each agent provides some facilities and has private facility-opening costs, and the client-assignment costs are public, can be modeled by letting $pub(T_1,\ldots,T_n)$ be the total client-assignment cost given the set $\bigcup_i T_i$ of open facilities.

Let C_i denote the set of all possible cost functions of agent i, and $\mathcal{O}$ be the (finite) set of all possible allocations. Let $C = \prod_{i=1}^n C_i$. For a tuple $x = (x_1,\ldots,x_n)$, we use x_{-i} to denote $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$. Similarly, let $C_{-i} = \prod_{j\neq i} C_j$. For an allocation $a = (T_1,\ldots,T_n)$, we sometimes use a_i to denote T_i, $c_i(a)$ to denote $c_i(a_i) = c_i(T_i)$. A (direct revelation) *mechanism* $M = (\mathcal{A}, p_1,\ldots,p_n)$ for a covering problem consists of an allocation algorithm $\mathcal{A} : C \mapsto \mathcal{O}$ and a payment function $p_i : C \mapsto \mathbb{R}$ for each agent i, and works as follows. Each agent i reports a cost function c_i (that might be different from his true cost function). The mechanism computes the allocation $\mathcal{A}(c) = (T_1,\ldots,T_n)$, and pays $p_i(c)$ to each agent i. Throughout, we use $\overline{c}_i$ to denote the true cost function of i. The *utility* $u_i(c_i, c_{-i}; \overline{c}_i)$ that player i derives when he reports c_i and the others report c_{-i} is $p_i(c) - \overline{c}_i(T_i)$, and each agent i aims to maximize his own utility (rather than the social welfare).

A desirable property for a mechanism to satisfy is *truthfulness*, wherein every agent i maximizes his utility by reporting his true cost function. All our mechanisms will also satisfy the natural property of *individual rationality* (IR), which means that every agent has nonnegative utility if he reports his true cost.

Definition 1. *A mechanism $M = \big(\mathcal{A}, \{p_i\}\big)$ is truthful if for every agent i, every $c_{-i} \in C_{-i}$, and every $\overline{c}_i, c_i \in C_i$, we have $u_i(\overline{c}_i, c_{-i}; \overline{c}_i) \geq u_i(c_i, c_{-i}; \overline{c}_i)$. M is IR if for every i, every $\overline{c}_i \in C_i$ and every $c_{-i} \in C_{-i}$, we have $u_i(\overline{c}_i, c_{-i}; \overline{c}_i) \geq 0$.*

To ensure that truthfulness and IR are compatible, we consider *monopoly-free* settings: for every player i, there is a feasible allocation a (i.e., $pub(a) < \infty$) with $a_i = \emptyset$. (Otherwise, if there is no such allocation, then i needs to be paid at least $\min_{v \in \mathcal{T}_i} c_{i,v}$ for IR, so he can lie and increase his utility arbitrarily.)

For a *randomized mechanism M*, where $\mathcal{A}$ or the p_i's are randomized, we say that M is *truthful in expectation* if each agent i maximizes his expected utility by reporting his true cost. We now say that M is IR if for *every* coin toss of the mechanism, the utility of each agent is nonnegative upon bidding truthfully.

Since the CM problem is often *NP*-hard, our goal is to design a mechanism $M = \big(\mathcal{A}, \{p_i\}\big)$ that is truthful (or truthful in expectation), and where $\mathcal{A}$ is a ρ-approximation algorithm; that is, for every input c, the solution $a = \mathcal{A}(c)$ satisfies $\sum_i c_i(a) + pub(a) \leq \rho \cdot \min_{b \in \mathcal{O}} \big(\sum_i c_i(b) + pub(b)\big)$. We call such a mechanism a *truthful, ρ-approximation mechanism.*

The following theorem gives a necessary and sometimes sufficient condition for when an algorithm $\mathcal{A}$ is *implementable*, that is, admits suitable payment functions $\{p_i\}$ such that $\big(\mathcal{A}, \{p_i\}\big)$ is a truthful mechanism. Say that $\mathcal{A}$ satisfies *weak monotonicity* (WMON) if for all i, all $c_i, c'_i \in C_i$, and all $c_{-i} \in C_{-i}$, if $\mathcal{A}(c_i, c_{-i}) = a$, $\mathcal{A}(c'_i, c_{-i}) = b$, then $c_i(a) - c_i(b) \leq c'_i(a) - c'_i(b)$. Define the dimension of a covering problem to be $\max_i |\mathcal{T}_i|$. It is easy to see that for a single-dimensional covering problem—so $C_i \subseteq \mathbb{R}$ for all i—WMON is equivalent to the following simpler condition: say that $\mathcal{A}$ is *monotone* if for all i, all $c_i, c'_i \in C_i$, $c_i \leq c'_i$, and all $c_{-i} \in C_{-i}$, if $\mathcal{A}(c_i, c_{-i}) = a$, $\mathcal{A}(c'_i, c_{-i}) = b$ then $b_i \subseteq a_i$.

Theorem 2 (Theorems 9.29 and 9.36 in [25]). *If a mechanism $\big(\mathcal{A}, \{p_i\}\big)$ is truthful, then $\mathcal{A}$ satisfies WMON. Conversely, if the problem is single-dimensional, or if C_i is convex for all i, then every WMON algorithm $\mathcal{A}$ is implementable.*

3 A Black-Box Reduction for Multidimensional Metric UFL

In this section, we consider the multidimensional metric *uncapacitated facility location* (UFL) problem and present a *black-box reduction* from truthful mechanism design to algorithm design. We show that any ρ-approximation algorithm for UFL satisfying an additional property can be converted in a black-box fashion to a truthful-in-expectation ρ-approximation mechanism (Theorem 3). This is the first such result for a multidimensional covering problem. As a corollary, we obtain a truthful-in-expectation, 2-approximation mechanism (Corollary 5).

In the mechanism-design version of UFL, we have a set $\mathcal{D}$ of clients that need to be serviced by facilities, and a set $\mathcal{F}$ of locations where facilities may be opened. Each agent i may provide facilities at the locations in $\mathcal{T}_i \subseteq \mathcal{F}$. By making multiple copies of a location if necessary, we may assume that the $\mathcal{T}_i$s are disjoint. Hence, we will simply say "facility ℓ" to refer to the facility at location $\ell \in \mathcal{F}$. For each facility $\ell \in \mathcal{T}_i$ that is opened, i incurs a private opening cost of $\overline{f}_{i,\ell}$, and assigning client j to an open facility ℓ incurs a publicly known assignment/connection cost $c_{\ell j}$. To simplify notation, given a tuple $\{f_{i,\ell}\}_{i\in[n],\ell\in\mathcal{T}_i}$ of facility costs, we use f_ℓ to denote $f_{i,\ell}$ for $\ell \in \mathcal{T}_i$. The goal is to open a subset $F \subseteq \mathcal{F}$ of facilities, so as to minimize $\sum_{\ell\in F} \overline{f}_\ell + \sum_{j\in\mathcal{D}} \min_{\ell\in F} c_{\ell j}$. We will assume throughout that the $c_{\ell j}$s form a metric. It will be notationally convenient to allow our algorithms to have the flexibility of choosing the open facility $\sigma(j)$ to which a client j is assigned (instead of $\text{argmin}_{\ell\in F} c_{\ell j}$); since assignment costs are public, this does not affect truthfulness, and any approximation guarantee achieved also clearly holds when we drop this flexibility.

We can formulate (metric) UFL as an integer program, and relax the integrality constraints to obtain the following LP. Throughout, we use ℓ to index facilities in $\mathcal{F}$ and j to index clients in $\mathcal{D}$.

$$\min \sum_{\ell} f_\ell y_\ell + \sum_{j,\ell} c_{\ell j} x_{\ell j} \quad \text{s.t.} \quad \sum_{\ell} x_{\ell j} \geq 1 \quad \forall j, \quad 0 \leq x_{\ell j} \leq y_\ell \leq 1 \quad \forall \ell, j. \tag{FL-P}$$

Here, $\{f_\ell\}_\ell = \{f_{i,\ell}\}_{i\in[n],\ell\in\mathcal{T}_i}$ is the vector of reported facility costs. Variable y_ℓ denotes if facility ℓ is opened, and $x_{\ell j}$ denotes if client j is assigned to facility ℓ; the constraints encode that each client is assigned to a facility, and that this facility must be open.

Say that an algorithm $\mathcal{A}$ is a *Lagrangian multiplier preserving* (LMP) ρ-approximation algorithm for UFL if for every instance, it returns a solution $\big(F, \{\sigma(j)\}_{j\in\mathcal{D}}\big)$ such that $\rho \sum_{\ell\in F} f_\ell + \sum_j c_{\sigma(j)j} \leq \rho \cdot OPT_{(\text{FL-P})}$. The main result of this section is the following black-box reduction.

Theorem 3. *Given a polytime, LMP ρ-approximation algorithm $\mathcal{A}$ for UFL, one can construct a polytime, truthful-in-expectation, individually rational, ρ-approximation mechanism M for multidimensional UFL.*

Proof. We build upon the convex-decomposition idea used in [21]. The randomized mechanism M works as follows. Let $f = \{f_\ell\}$ be the vector of reported facility-opening costs, and c be the public connection-cost metric.

1. Compute the optimal solution (y^*, x^*) to (FL-P) (for the input (f, c)). Let $\{p_i^* = p_i^*(f)\}$ be the payments made by the *fractional* VCG mechanism that outputs the optimal LP solution for every input. That is, $p_i^* = \big(\sum_\ell f_\ell y'_\ell + \sum_{\ell,j} c_{\ell j} x'_{\ell j}\big) - \big(\sum_{\ell\notin\mathcal{T}_i} f_\ell y^*_\ell + \sum_{\ell,j} c_{\ell j} x^*_{\ell j}\big)$, where (y', x') is the optimal solution to (FL-P) with the additional constraints $y_\ell = 0$ for all $\ell \in \mathcal{T}_i$.

2. Let $\mathbb{Z}(P) = \{(y^{(q)}, x^{(q)})\}_{q\in\mathcal{I}}$ be the set of all integral solutions to (FL-P). In Lemma 4, we prove the key technical result that using $\mathcal{A}$, one can compute, in polynomial time, nonnegative multipliers $\{\lambda^{(q)}\}_{q\in\mathcal{I}}$ such that $\sum_q \lambda^{(q)} = 1$, $\sum_q \lambda^{(q)} y_\ell^{(q)} = y^*_\ell$ for all ℓ, and $\sum_{q,\ell,j} \lambda^{(q)} c_{\ell j} x_{\ell j}^{(q)} \leq \rho \sum_{\ell,j} c_{\ell j} x^*_{\ell j}$.

3. With probability $\lambda^{(q)}$: (a) output the solution $\big(y^{(q)}, x^{(q)}\big)$; (b) pay $p_i^{(q)}$ to agent i, where $p_i^{(q)} = 0$ if $\sum_{\ell \in \mathcal{T}_i} f_\ell y_\ell^* = 0$, and $\sum_{\ell \in \mathcal{T}_i} f_\ell y_\ell^{(q)} \cdot \frac{p_i^*}{\sum_{\ell \in \mathcal{T}_i} f_\ell y_\ell^*}$ otherwise.

Clearly, M runs in polynomial time. Fix a player i. Let $\overline{f}_i$ and f_i be the true and reported cost vector of i. Let f_{-i} be the reported cost vectors of the other players. Let (y^*, x^*) be an optimal solution to (FL-P) for (f, c). Note that $\mathrm{E}\big[p_i(f)\big] = p_i^*(f)$ since $\sum_q \lambda^{(q)} y^{(q)} = y_\ell^*$ for all ℓ. (If $\sum_{\ell \in \mathcal{T}_i} f_\ell y_\ell^* = 0$ then $p_i^*(f) = 0$.) So $\mathrm{E}\big[u_i(f_i, f_i; \overline{f}_i)\big] = \mathrm{E}\big[p_i\big] - \sum_q \lambda^{(q)} \sum_{\ell \in \mathcal{T}_i} \overline{f}_\ell y_\ell^{(q)} = p_i^*(f) - \sum_{\ell \in \mathcal{T}_i} \overline{f}_\ell y_\ell^*$. Since p_i^* and y^* are respectively the payment to i and the assignment computed for input (f_i, f_{-i}) by the fractional VCG mechanism, which is truthful, it follows that player i maximizes his utility in the VCG mechanism, and hence, his expected utility under mechanism M, by reporting his true opening costs.

Thus, M is truthful in expectation. This also implies the ρ-approximation guarantee because the convex decomposition obtained in Step 2 shows that the expected cost of the solution computed by M for input (f, c) (where we may assume that f is the true cost vector) is at most $\rho \cdot OPT_{\text{(FL-P)}}(f, c)$. Finally, since the fractional VCG mechanism is IR, for any agent i, the VCG payment $p_i^*(f)$ satisfies $p_i^*(f) \geq \sum_{\ell \in \mathcal{T}_i} f_\ell y_\ell^*$, and therefore $p_i^{(q)} \geq \sum_{\ell \in \mathcal{T}_i} f_\ell y_\ell^{(q)}$. So M is IR. □

Lemma 4. *The convex decomposition in Step 2 can be computed in polytime.*

Proof Sketch. It suffices to show that the LP (P) can be solved in polynomial time and its optimal value is 1. Recall that $\{(y^{(q)}, x^{(q)})\}_{q \in \mathcal{I}}$ is the set of all integral solutions to (FL-P). The LP (D) is the dual of (P).

$$
\begin{aligned}
\max \quad & \sum_q \lambda^{(q)} && \text{(P)}\\
\text{s.t.} \quad & \sum_q \lambda^{(q)} y_\ell^{(q)} = y_\ell^* \quad \forall \ell \\
& \sum_{j,\ell,q} \lambda^{(q)} c_{\ell j} x_{\ell j}^{(q)} \leq \rho \sum_{j,\ell} c_{\ell j} x_{\ell j}^* \\
& \sum_q \lambda^{(q)} \leq 1, \quad \lambda \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \sum_\ell y_\ell^* \alpha_\ell + \Big(\rho \sum_{j,\ell} c_{\ell j} x_{\ell j}^*\Big)\beta + z && \text{(D)}\\
\text{s.t.} \quad & \sum_\ell y_\ell^{(q)} \alpha_\ell + \Big(\sum_{j,\ell} c_{\ell j} x_{\ell j}^{(q)}\Big)\beta + z \geq 1 \quad \forall q && (1)\\
& z, \beta \geq 0.
\end{aligned}
$$

Clearly, $OPT_{\text{(D)}} \leq 1$ since $z = 1$, $\alpha_\ell = 0 = \beta$ for all ℓ is a feasible dual solution. If there is a feasible dual solution (α', β', z') of value smaller than 1, then the rough idea is that by running $\mathcal{A}$ on the UFL instance with facility costs $\{\frac{\alpha'_\ell}{\rho}\}$ and connection costs $\{\beta' c_{\ell j}\}$, we can obtain an integral solution whose constraint (1) is violated. (This idea needs be modified a bit since α'_ℓ could be negative.) Hence, we can solve (D) efficiently via the ellipsoid method using $\mathcal{A}$ to provide the separation oracle. This also yields an equivalent dual LP consisting of only the polynomially many violated inequalities found during the ellipsoid method. The dual of this compact LP gives an LP equivalent to (P) with polynomially many $\lambda^{(q)}$ variables whose solution yields the desired convex decomposition. □

By using the polytime LMP 2-approximation algorithm for UFL devised by Jain et al. [17], we obtain the following corollary of Theorem 3.

Theorem 5. *There is a polytime, IR, truthful-in-expectation, 2-approximation mechanism for multidimensional* UFL.

4 Truthful Mechanisms for Multidimensional VC

We now consider the multidimensional vertex-cover problem (VC), and devise various polytime, truthful, approximation mechanisms for it. We often use Multi-VC to distinguish multidimensional VC from its algorithmic counterpart.

Recall that in Multi-VC, we have a graph $G = (V, E)$ with n nodes. Each agent i provides a subset $\mathcal{T}_i$ of nodes. For simplicity, we first assume that the $\mathcal{T}_i$s are disjoint, and given a cost-vector $\{c_{i,u}\}_{i\in[n],u\in\mathcal{T}_i}$, we use c_u to denote $c_{i,u}$ for $u \in \mathcal{T}_i$. Monopoly-free then means that each $\mathcal{T}_i$ is an independent set. In Remark 11 we argue that many of the results obtained in this disjoint-$\mathcal{T}_i$s setting (in particular, Theorems 13 and 14) also hold when the $\mathcal{T}_i$s are not disjoint (each $\mathcal{T}_i$ is still an independent set). The goal is to choose a minimum-cost *vertex cover*, i.e., a min-cost set $S \subseteq V$ such that every edge is incident to a node in S.

As mentioned earlier, VC becomes a rather challenging mechanism-design problem in the *multidimensional* mechanism-design setting. Whereas for *single-dimensional VC*, many of the known 2-approximation algorithms for VC are implementable, none of these underlying techniques yield implementable algorithms even for the simplest multidimensional setting, 2-dimensional VC, where *every player owns at most two nodes* (see the full version for examples). Moreover, no maximal-in-distributional-range (MIDR) mechanism whose range is a proper subset of all outcomes can achieve a bounded multiplicative approximation guarantee [12]. This also rules out the convex-decomposition technique of [21], which yields MIDR mechanisms.

We develop two main techniques for Multi-VC in this section. In Section 4.1, we introduce a simple class of truthful mechanisms called *threshold mechanisms*, and show that although seemingly restricted, threshold mechanisms can achieve non-trivial approximation guarantees. In Section 4.2, we develop a *decomposition method* for Multi-VC that uses threshold mechanisms as building blocks and gives a general way of reducing the mechanism-design problem for Multi-VC into simpler mechanism-design problems.

By leveraging the decomposition method along with threshold mechanisms, we obtain various truthful, approximation mechanisms for Multi-VC, which yield the *first* truthful mechanisms for multidimensional vertex cover with non-trivial approximation guarantees. (1) We obtain a truthful, $O(\log n)$-approximation mechanism (Theorem 13) for any proper minor-closed family of graphs (such as planar graphs). (2) We show that any instance of r-dimensional VC can be decomposed into $O(r^2 \log n)$ *single-dimensional VC* instances; this leads to a truthful, $O(r^2 \log n)$-approximation mechanism for r-dimensional VC (Theorem 14). In particular, for any fixed r, we obtain an $O(\log n)$-approximation.

Theorem 16 shows that our mechanisms also enjoy good frugality properties. We obtain the first mechanisms for Multi-VC that are polytime, truthful, and achieve bounded approximation ratio *and* bounded frugality ratio. This complements a result of [5], who devise such mechanisms for single-dimensional VC.

4.1 Threshold Mechanisms

Definition 6. *A* threshold mechanism *M for Multi-VC works as follows. On input c, for every i and every node $u \in \mathcal{T}_i$, M computes a threshold $t_u = t_u(c_{-i})$ (i.e., t_u does not depend on i's reported costs). M then returns the solution $S = \{v \in V : c_v \le t_v\}$ as the output, and pays $p_i = \sum_{u \in S \cap \mathcal{T}_i} t_u$ to agent i.*

If t_u only depends on the costs in the neighbor-set $N(u)$ of u, for all $u \in V$ (note that $N(u) \cap \mathcal{T}_i = \emptyset$ if $u \in \mathcal{T}_i$), we call M a *neighbor-threshold mechanism.* A special case of a neighbor-threshold mechanism is an *edge-threshold mechanism*: for every edge $uv \in E$ we have edge thresholds $t_u^{(uv)} = t_u^{(uv)}(c_v)$, $t_v^{(uv)} = t_v^{(uv)}(c_u)$, and the threshold of a node u is given by $t_u = \max_{v \in N(u)}(t_u^{(uv)})$.

In general, threshold mechanisms may not output a vertex cover, however it is easy to argue that threshold mechanisms are always truthful and IR.

Lemma 7. *Every threshold mechanism for Multi-VC is IR and truthful.*

Proof. IR is immediate from the definition of payments. To see truthfulness, fix an agent i. For every $\overline{c}_i, c_i \in C_i, c_{-i} \in C_{-i}$ we have $u_i(c_i, c_{-i}; \overline{c}_i) = \sum_{v \in \mathcal{T}_i : c_v \le t_v} (t_v - \overline{c}_v)$. It follows that i's utility is maximized by reporting $c_i = \overline{c}_i$. □

Inspired by [19,5], we define an x*-scaled* edge-threshold mechanism as follows: fix a vector $(x_u)_{u \in V}$, where $x_u > 0$ for all u, and set $t_u^{(uv)} := x_u c_v / x_v$ for every edge (u, v). We abuse notation and use $\mathcal{A}_x$ to denote both the resulting edge-threshold mechanism and its allocation algorithm. Also, define $\mathcal{B}_x$ to be the neighbor-threshold mechanism where we set $t_u := \sum_{v \in N(u)} x_u c_v / x_v$. Define $\alpha(G; x) := \max_{u \in V} \big(\max_{S \subseteq N(u) : S \text{ independent}} \frac{x(S)}{x_u} \big)$.

Lemma 8. *$\mathcal{A}_x$ and $\mathcal{B}_x$ output feasible solutions and have approximation ratio $\alpha(G; x) + 1$.*

Proof. Clearly, every node selected by $\mathcal{A}_x$ is also selected by $\mathcal{B}_x$. So it suffices to show that $\mathcal{A}_x$ is feasible, and to show the approximation ratio for $\mathcal{B}_x$. For any edge (u, v), either $c_u \le x_u c_v / x_v$ and u is output, or $c_v \le x_v c_u / x_u$ and v is output. So $\mathcal{A}_x$ returns a vertex cover.

Let S be the output of $\mathcal{B}_x$ on input c, and let S^* be a min-cost vertex cover. We have $c(S) = c(S \cap S^*) + c(S \setminus S^*) \le c(S^*) + \sum_{u \in S \setminus S^*} t_u = c(S^*) + \sum_{u \in S \setminus S^*} \sum_{v \in N(u)} x_u c_v / x_v$. Note that $S \setminus S^*$ is an independent set since S^* is a vertex cover, so $\sum_{u \in S \setminus S^*} \sum_{v \in N(u)} x_u c_v / x_v \le \sum_{v \in S^*} \frac{c_v}{x_v} \sum_{u \in N(v) cap S^*} x_u \le \sum_{v \in S^*} c_v \cdot \alpha(G; x)$. Hence $c(S) \le (\alpha(G; x) + 1) c(S^*)$. It is not hard to construct examples showing that this approximation guarantee is tight. □

Corollary 9. *(i) Setting $x = \mathbf{1}$ gives $\alpha(G; x) \le \Delta(G)$, which is the maximum degree of a node in G, so $\mathcal{A}_\mathbf{1}$ has approximation ratio at most $\Delta(G) + 1$.*
(ii) Taking x to be the eigenvector corresponding to the largest eigenvalue $\lambda_{\max}$ of the adjacency matrix of G ($x > 0$ by the Perron-Frobenius theorem) gives $\alpha(G; x) \le \lambda_{max}$ (see [5]), so $\mathcal{A}_x$ has approximation ratio $\lambda_{max} + 1$.

Although neighbor-threshold mechanisms are more general than edge-threshold mechanisms, Lemma 10 shows that this yields limited dividends in the approximation ratio. Define $\alpha'(G) = \min_{\text{orientations of } G}\big(\max_{u\in V, S\subseteq N^{\text{in}}(u): S \text{ independent}} |S|\big)$, where $N^{\text{in}}(u) = \{v \in N(u) : (u,v) \text{ is directed into } u\}$. Note that $\alpha'(G) \leq \alpha(G;\mathbf{1}) \leq \Delta(G)$. If $G = (V,E)$ is *everywhere γ-sparse*, i.e., $|\{(u,v) \in E : u,v \in S\}| \leq \gamma|S|$ for all $S \subseteq V$, then $\alpha'(G) \leq \gamma$; this follows from Hakimi's theorem [16]. A well-known result in graph theory states that for every proper family $\mathcal{G}$ of graphs that is closed under taking minors (e.g., planar graphs), there is a constant γ, such that every $G \in \mathcal{G}$ is has at most $\gamma|V(G)|$ edges [23] (see also [8], Chapter 7, Ex. 20); since $\mathcal{G}$ is minor-closed, this also implies that G is *everywhere* γ-sparse, and hence $\alpha'(G) \leq \gamma$ for all $G \in \mathcal{G}$.

Lemma 10. *A (feasible) neighbor-threshold mechanism M for graph G with approximation ratio ρ, yields an $O\big(\rho\log(\alpha'(G))\big)$-approximation edge-threshold mechanism for G. This implies an approximation ratio of (i) $O(\rho\log\gamma)$ if G is an everywhere γ-sparse graph; (ii) $O(\rho)$ if G belongs to a proper minor-closed family of graphs (where the constant in the $O(.)$ depends on the graph family).*

Remark 11. Any neighbor-threshold mechanism M with approximation ratio ρ that works under the disjoint-$\mathcal{T}_i$s assumption can be modified to yield a truthful, ρ-approximation mechanism when we drop this assumption. Let $A_u = \{i : u \in \mathcal{T}_i\}$. Set $\hat{c}_u = \min_{i\in A_u} c_{i,u}$ for each $u \in V$ and let $\hat{t}_u$ be the neighbor-threshold of u for the input $\hat{c}$. Note that $\hat{t}_u$ depends only on c_{-i} for every $i \in A_u$. Set $t^i_u := \min\{\hat{t}_u, \min_{j\neq i: u\in\mathcal{T}_j} c_{j,u}\}$ for all $i, u \in \mathcal{T}_i$. Consider the threshold mechanism M' with $\{t^i_u\}$ thresholds, where we use a fixed tie-breaking rule to ensure that we pick u for at most one agent $i \in A_u$ with $c_{i,u} = t^i_u$. Then the outputs of M on c, and of M' on input $\hat{c}$ coincide. Thus, M' is a truthful, ρ-approximation mechanism.

4.2 A Decomposition Method

We now propose a general reduction method for Multi-VC that uses threshold mechanisms as building blocks to reduce the task of designing truthful mechanisms for Multi-VC to the task of designing threshold mechanisms for simpler (in terms of graph structure or the dimensionality of the problem) Multi-VC problems. This reduction is useful because designing good threshold mechanisms appears to be a much more tractable task for Multi-VC. By utilizing the threshold mechanisms designed in Section 4.1 in our decomposition method, we obtain an $O(\log n)$-approximation mechanism for any proper minor-closed family of graphs, and an $O(r^2\log n)$-approximation mechanism for r-dimensional VC.

A *decomposition mechanism* M for $G = (V,E)$ is constructed as follows.

- Let $G_1,\ldots,G_k$ be subgraphs of G such that $\bigcup_{q=1}^k E(G_q) = E$,
- Let $M_1,\ldots,M_k$ be threshold mechanisms for $G_1,\ldots,G_k$ respectively. For any $v \in V$, let t^q_v be v's threshold in M_q if $v \in V(G_i)$, and 0 otherwise.
- Define M to be the threshold mechanism obtained by setting the threshold for each node v to $t_v := \max_{q=1,\ldots,k}(t^q_v)$ for any $v \in V$. The payments of M are then as specified in Definition 6. Notice that if all the M_is are neighbor threshold mechanisms, then so is M.

Lemma 12. *The decomposition mechanism M described above is IR and truthful. If $\rho_1, \ldots, \rho_k$ are the approximation ratios of $M_1, \ldots, M_k$ respectively, then M has approximation ratio $\left(\sum_q \rho_q\right)$.*

Proof. Since M is a threshold mechanism, it is IR and truthful by Lemma 7. The optimal vertex cover for G induces a vertex cover for each subgraph G_q. So M_q outputs a vertex cover S_q of cost at most $\rho_q \cdot OPT$, where OPT is the optimal vertex-cover cost for G. It is clear that M outputs $\bigcup_q S_q$, which has cost at most $\left(\sum_q \rho_q\right) \cdot OPT$. □

Theorem 13. *If $G = (V, E)$ is everywhere γ-sparse, then one can devise a polytime, $O(\gamma \log |V|)$-approximation decomposition mechanism for G. Hence, there is a polytime, truthful, $O(\log n)$-approximation mechanism for* Multi-VC *on any proper minor-closed family of graphs. These guarantees also hold when the $\mathcal{T}_i$s are not disjoint.*

Proof. Let $n = |V|$. Since $|E| \leq \gamma n$, there are at most $n/2$ nodes with degree larger than 4γ. Let H_1 be the subgraph of G consisting of the edges incident to the vertices of G with degree at most 4γ. Now, $G_1 = G \setminus H_1$ (i.e., we delete the nodes and edges of H_1 to obtain G_1) is also γ-sparse. So, we can similarly find a subgraph H_2 that contains at least half of the nodes of G_1. Continuing this process, we obtain subgraphs $H_1, \ldots, H_k$ that partition G, where each subgraph H_q has maximum degree at most 4γ and $|V(H_q)| \geq |V(G \setminus (H_1 \cup \ldots H_{q-1})|/2$. Hence, $k \leq \log n$. Using the (edge-threshold) mechanism $\mathcal{A}_\mathbf{1}$ defined in Corollary 9, for each subgraph gives a $(4\gamma+1)$-approximation for each H_q, and hence a $(4\gamma + 1)\log n$-approximation neighbor-threshold mechanism for G. By Remark 11, this also holds when the $\mathcal{T}_i$s are not disjoint.

As noted in Section 4.1, every proper minor-closed family of graphs is everywhere γ-sparse for some $\gamma > 0$. Thus, the above result implies a truthful, $O(\log n)$-approximation for any proper minor-closed family (where the constant in the $O(.)$ depends on the graph family; e.g., for planar graphs $\gamma \leq 4$). □

We next present a decomposition mechanism whose guarantee depends only on the dimensionality of the problem, and not on the underlying graph structure.

Theorem 14. *For any r-dimensional instance of* Multi-VC *on $G = (V, E)$, one can obtain a polytime, $O(r^2 \log |V|)$-approximation, decomposition mechanism, even when the $\mathcal{T}_i$s are not disjoint.*

Proof. We decompose G into single-dimensional subgraphs, by which we mean subgraphs that contain at most one node from each $\mathcal{T}_i$. Initialize $j = 1$, $V_j = \emptyset$. While, $\bigcup_{q=1}^{j-1} E(G_q) \neq E$, we do the following: for every agent i, we pick one of the nodes of $\mathcal{T}_i$ uniformly at random and add it to V_j. We also add all the nodes in $V \setminus \left(\bigcup_{i=1}^{n} \mathcal{T}_i\right)$ to V_j. Let G_j be the induced subgraph on V_j; set $j \leftarrow j + 1$.

For any edge $e \in E$, the probability that both of its ends appear in some subgraph G_j, for any $i = 1, \ldots, l$, is at least $1/r^2$. So, the expected value of $|E \setminus \bigcup_{q=1}^{j-1} E(G_q)|$ decreases by a factor of at least $(1 - 1/r^2)$ with j. Hence, the expected number of subgraphs produced above is $O\left(\frac{\log |E|}{\log(r^2/(r^2-1))}\right) = O(r^2 \log |V|)$

(this also holds with high probability). Each G_j yields a single-dimensional VC instance (where a node may be owned by multiple players). Any truthful mechanism for a 1D-problem is a threshold mechanism. So we can use any truthful, 2-approximation mechanism for single-dimensional VC for the G_js and obtain an $O(r^2 \log n)$-approximation for r-dimensional Multi-VC. □

The following lemma shows that the decomposition obtained above into single-dimensional subgraphs is essentially the best that can hope for, for $r = 2$.

Lemma 15. *There are instances of 2-dimensional VCP that require $\Omega(\log |V(G)|)$ single-dimensional subgraphs in any decomposition of G.*

Proof. Define G^n to be the bipartite graph with vertices $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ and edges $\{(u_i, v_j) : i \neq j\}$. Each agent $i = 1, \dots, n$ owns vertices u_i and v_i.

For $n = 2$ the claim is obvious. Let q_n be the minimum number of single-dimensional subgraphs needed to decompose G^n. Suppose the claim is true for all $j < n$ and we have decomposed G^n into single-dimensional subgraphs $D = \{G_1, \dots, G_{q_n}\}$. We may assume that $V(G_1) = \{u_1, \dots, u_k, v_{k+1}, \dots, v_n\}$ (if G_1 has less than n nodes, pad it with extra nodes). Let H_1 and H_2 be the subgraphs of G induced by $\{u_1, \dots, u_k, v_1, \dots, v_k\}$ and $\{u_{k+1}, \dots, u_n, v_{k+1}, \dots, v_n\}$, respectively. The graphs in $D \setminus \{G_1\}$ must contain a decomposition of H_1 and a decomposition of H_2. So $q_n \geq 1 + \max(q_k, q_{n-k})$, and hence, by induction, we obtain that $q_n \geq 1 + (1 + \log_2(n/2)) = 1 + \log_2 n$. □

Frugality Considerations. Karlin et al. [18] and Elkind et al. [6] propose various benchmarks for measuring the *frugality ratio* of a mechanism, which is a measure of the (over-)payment of a mechanism. The mechanisms that we devise above also enjoy good frugality ratios with respect to the benchmark introduced by [6], which is denoted by $\nu(G, c)$ in [19] (and $\text{NTU}_{\max}$ in [6]).

The *frugality ratio* of a mechanism $M = \big(\mathcal{A}, \{p_i\}\big)$ on G is defined as $\phi_M(G) := \sup_c \frac{\sum_i p_i(c)}{\nu(G,c)}$. The proof of Lemma 8 is easily modified to show that the x-scaled mechanism $\mathcal{A}_x$ satisfies $\sum_i p_i(c) \leq \sum_u t_u \leq \beta(G; x) c(V)$, where $\beta(G; x) = \max_{u \in V} \frac{x(N(u))}{x_u}$. Since [6] show that $\nu(G, c) \geq c(V)/2$, this implies that $\phi_{\mathcal{A}_x}(G) \leq 2\beta(G; x)$. Also, if M is a decomposition mechanism constructed from threshold mechanisms $M_1, \dots, M_k$, where each M_q satisfies $\sum_u t_u^q \leq \phi_q \cdot c(V(G_q))$, then it is easy to see that $\phi_M(G) \leq 2 \sum_{q=1}^k \phi_q$. Thus, we obtain the following results.

Theorem 16. *Let $G = (V, E)$ be a graph with n nodes. We can obtain a polytime, truthful, IR mechanism M with the following approximation $\rho = \rho_M(G)$ and frugality $\phi = \phi_M(G)$ ratios.*

(i) $\rho = (\beta(G; x) + 1)$, $\phi \leq 2\beta(G; x)$ for Multi-VC on G;

(ii) $\rho, \phi = O(\gamma \log n)$ for Multi-VC on G when G is everywhere γ-sparse; hence, we achieve $\rho, \phi = O(\log n)$ for Multi-VC on any minor-closed family;

(iii) $\rho = O(r^2 \log n)$, $\phi = O\big(r^2 \log n \cdot \Delta(G)\big)$ for r-dimensional Multi-VC on G (using a 2-approximation mechanism with frugality ratio $2\Delta(G)$ [6] for single-dimensional VC in the construction of Theorem 14).

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What I Tell You Three Times Is True: Bootstrap Percolation in Small Worlds*

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Abstract. A bootstrap percolation process on a graph G is an "infection" process which evolves in rounds. Initially, there is a subset of infected nodes and in each subsequent round each uninfected node which has at least r infected neighbours becomes infected and remains so forever. The parameter $r \geqslant 2$ is fixed.

We analyse this process in the case where the underlying graph is an inhomogeneous random graph, which exhibits a power-law degree distribution, and initially there are $a(n)$ randomly infected nodes. The main focus of this paper is the number of vertices that will have been infected by the end of the process. The main result of this work is that if the degree sequence of the random graph follows a power law with exponent β, where $2 < \beta < 3$, then a sublinear number of initially infected vertices is enough to spread the infection over a linear fraction of the nodes of the random graph, with high probability.

More specifically, we determine explicitly a critical function $a_c(n)$ such that $a_c(n) = o(n)$ with the following property. Assuming that n is the number of vertices of the underlying random graph, if $a(n) \ll a_c(n)$, then the process does not evolve at all, with high probability as n grows, whereas if $a(n) \gg a_c(n)$, then there is a constant $\varepsilon > 0$ such that, with high probability, the final set of infected vertices has size at least εn. This behaviour is in sharp contrast with the case where the underlying graph is a $G(n,p)$ random graph with $p = d/n$. Recent results of Janson, Łuczak, Turova and Vallier have shown that if the number of initially infected vertices is sublinear, then with high probability the size of the final set of infected vertices is approximately equal to $a(n)$. That is, essentially there is lack of evolution of the process.

It turns out that when the maximum degree is $o(n^{1/(\beta-1)})$, then $a_c(n)$ depends also on r. But when the maximum degree is $\Theta(n^{1/(\beta-1)})$, then $a_c(n) = n^{\frac{\beta-2}{\beta-1}}$.

Keywords: bootstrap percolation, contagion, power-law random graphs.

* L. Carroll *The Hunting of the Snark.*

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1 Introduction

Models for the processes by which new ideas and new behaviors propagate through a population have been studied in a number of domains, including the epidemiology, political science, agriculture, finance and the effects of word of mouth (also known as viral marketing) in the promotion of new products. An idea or innovation appears (for example, the use of a new technology among college students) and it can either die out quickly or make significant advances into the population. The hypothesis of viral marketing is that by initially targeting a few influential members of the network (e.g., by giving them free samples of the product), we can trigger a cascade of influence by which friends will recommend the product to other friends, and many individuals will ultimately try it. But how should we choose the few key individuals to use for seeding this process? This problem is known as "the influence maximization problem"; hardness results have been obtained in [29], [30] and there is a large literature on this topic (see for example [31] and the references therein). However, in most practical cases, the structure of the underlying network is not known and then one has to initially target the popular and attractive individuals with many connections.

In this paper, we consider a simple model of diffusion, known as "bootstrap percolation model". Bootstrap percolation was introduced by Chalupa, Leath and Reich [13] in 1979 in the context of magnetic disordered systems and has been re-discovered since then by several authors mainly due to its connections with various physical models. A *bootstrap percolation process* with *activation threshold* an integer $r \geqslant 2$ on a graph $G = G(V, E)$ is a deterministic process which evolves in rounds. Every vertex has two states: it is either *infected* or *uninfected*. Initially, there is a subset $\mathcal{A}_0 \subseteq V$ which consists of infected vertices, whereas every other vertex is uninfected. This set can be selected either deterministically or randomly. Subsequently, in each round, if an uninfected vertex has at least r of its neighbours infected, then it also becomes infected and remains so forever. This is repeated until no more vertices become infected. We denote the final infected set by $\mathcal{A}_f$.

Bootstrap percolation processes (and extensions) have been used as models to describe several complex phenomena in diverse areas, from jamming transitions [27] and magnetic systems [24] to neuronal activity [3], [26] and spread of defaults in banking systems (see e.g. [4] with a more refined model). A short survey regarding applications of bootstrap percolation processes can be found in [1].

In the context of real-world networks and in particular in social networks, a bootstrap percolation process can be thought of as a primitive model for the spread of ideas or new trends within a set of individuals which form a network. Each of them has a threshold r and $\mathcal{A}_0$ corresponds to the set of individuals who initially are "infected" with a new belief. If for an "uninfected" individual at least r of its acquaintances have adopted the new belief, then this individual adopts it as well. Bootstrap percolation processes have also been studied on a variety of graphs, such as trees [8], [18], grids [12], [20], [7], [6], hypercubes [5], as well as on several distributions of random graphs [9], [22], [2].

More than a decade ago, Faloutsos et al. [17] observed that the Internet exhibits a *power-law* degree distribution, meaning that the proportion of vertices of degree k scales like $k^{-\beta}$, for all sufficiently large k, and some $\beta > 2$. In particular, the work of Faloutsos et al. [17] suggested that the degree distribution of the Internet at the router level follows a power law with $\beta \approx 2.6$. Kumar et al. [23] also provided evidence on the degree distribution of the World Wide Web viewed as a directed graph on the set of web pages, where a web page "points" to another web page if the former contains a link to the latter. They found that the indegree distribution follows a power law with exponent approximately 2.1, whereas the outdegree distribution follows also a power law with exponent close to 2.7. Other empirical evidence on real-world networks has provided examples of power law degree distributions with exponents between 2 and 3.

Thus, in the present work, we focus on the case where $2 < \beta < 3$. More specifically, the underlying random graph distribution we consider was introduced by Chung and Lu [14], who invented it as a general purpose model for generating graphs with a power-law degree sequence. Consider the vertex set $[n] := \{1, \ldots, n\}$. Every vertex $i \in [n]$ is assigned a positive weight w_i, and the pair $\{i, j\}$, for $i \neq j \in [n]$, is included in the graph as an edge with probability proportional to $w_i w_j$, independently of every other pair. Note that the expected degree of i is close to w_i. With high probability the degree sequence of the resulting graph follows a power law, provided that the sequence of weights follows a power law (see [28] for a detailed discussion). Such random graphs are also characterized as *ultra-small worlds*, due to the fact that the typical distance of two vertices that belong to the same component is $O(\log \log n)$ – see [15] or [28].

Regarding the initial conditions of the bootstrap percolation process, our general assumption will be that the initial set of infected vertices $\mathcal{A}_0$ is chosen randomly among all subsets of vertices of a certain size.

The aim of this paper is to analyse the evolution of the bootstrap percolation process on such random graphs and, in particular, the typical value of the ratio $|\mathcal{A}_f|/|\mathcal{A}_0|$. The main finding of the present work is the existence of a critical function $a_c(n)$, which is sublinear, such that when $|\mathcal{A}_0|$ "crosses" $a_c(n)$ we have a sharp change on the evolution of the bootstrap percolation process. When $|\mathcal{A}_0| \ll a_c(n)$, then typically the process does not evolve, but when $|\mathcal{A}_0| \gg a_c(n)$, then a linear fraction of vertices is eventually infected. Of course the non-trivial case here is when $|\mathcal{A}_0|$ is sublinear. What turns out to be the key to such a dissemination of the infection is the vertices of high weight. These are typically the vertices that have high degree in the random graph and, moreover, they form a fairly dense graph. We exploit this fact and show how this causes the spread of the infection to a linear fraction of the vertices (see Theorem 2 below). Interpreting this from the point of view of a social network, these vertices correspond to popular and attractive individuals with many connections – these are the *hubs* of the network. Our analysis sheds light to the role of these individuals in the infection process.

These results are in sharp contrast with the behaviour of the bootstrap percolation process in $G(n, p)$ random graphs, where every edge on a set of n vertices is included independently with probability p. Recently, Janson, Łuczak, Turova and Vallier [22] came up with a complete analysis of the bootstrap percolation process for various ranges of the probability p. Since the random graphs we consider have constant average degree, we focus on their findings regarding the range where $p = d/n$ and $d > 0$ is fixed. Among the findings of Janson et al. [22] (see Theorem 5.2 there) is that when $|\mathcal{A}_0| = o(n)$, then typically the process essentially does not evolve. More precisely, the ratio $|\mathcal{A}_f|/|\mathcal{A}_0|$ converges to 1 in probability – see below for the definition of this notion. In other words, the density of the initially infected vertices must be positive in order for the density of infected vertices to grow. We note that similar behavior to the case of $G(n, p)$ has been observed in the case of random regular graphs [9], and in random graphs with given vertex degrees constructed through the configuration model, studied by the first author in [2], when the sum of the square of degrees scales linearly with n, the size of the graph. The later case includes random graphs with power-law degree sequence with exponent $\beta > 3$. Our results imply that the two regimes $2 < \beta < 3$ and $\beta > 3$ have completely different behaviors.

Basic Notations. Let $\mathbb{R}^+$ be the set of positive real numbers. For non-negative sequences x_n and y_n, we describe their relative order of magnitude using Landau's $o(.)$ and $O(.)$ notation. We write $x_n = O(y_n)$ if there exist $N \in \mathbb{N}$ and $C > 0$ such that $x_n \leqslant C y_n$ for all $n \geqslant N$, and $x_n = o(y_n)$, if $x_n/y_n \to 0$, as $n \to \infty$. We also write $x_n \ll y_n$ when $x_n = o(y_n)$ and $x_n \gg y_n$ when $y_n = o(x_n)$.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued random variables on a sequence of probability spaces $\{(\Omega_n, \mathbb{P}_n)\}_{n\in\mathbb{N}}$. If $c \in \mathbb{R}$ is a constant, we write $X_n \xrightarrow{p} c$ to denote that X_n *converges in probability to* c. That is, for any $\varepsilon > 0$, we have $\mathbb{P}_n(|X_n - c| > \varepsilon) \to 0$ as $n \to \infty$.
Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers that tends to infinity as $n \to \infty$. We write $X_n = o_p(a_n)$, if $|X_n|/a_n$ *converges to 0 in probability.* Additionally, we write $X_n = O_p(a_n)$, to denote that for any positive-valued function $\omega(n) \to \infty$, as $n \to \infty$, we have $\mathbb{P}(|X_n|/a_n \geqslant \omega(n)) = o(1)$. If $\mathcal{E}_n$ is a measurable subset of Ω_n, for any $n \in \mathbb{N}$, we say that the sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ occurs *asymptotically almost surely (a.a.s.)* if $\mathbb{P}(\mathcal{E}_n) = 1 - o(1)$, as $n \to \infty$.

Also, we denote by $\mathsf{Be}(p)$ a Bernoulli distributed random variable whose probability of being equal to 1 is p. The notation $\mathsf{Bin}(k, p)$ denotes a binomially distributed random variable corresponding to the number of successes of a sequence of k independent Bernoulli trials each having probability of success equal to p.

2 Models and Results

The random graph model that we consider is asymptotically equivalent to a model considered by Chung and Lu [15], and is a special case of the so-called *inhomogeneous random graph*, which was introduced by Söderberg [25] and was generalised and studied in great detail by Bollobás, Janson and Riordan in [11].

2.1 Inhomogeneous Random Graphs – The Chung-Lu Model

In order to define the model we consider for any $n \in \mathbb{N}$ the vertex set $[n] := \{1, \ldots, n\}$. Each vertex i is assigned a positive weight $w_i(n)$, and we will write $\mathbf{w} = \mathbf{w}(n) = (w_1(n), \ldots, w_n(n))$. We assume in the remainder that the weights are deterministic, and we will suppress the dependence on n, whenever this is obvious from the context. However, note that the weights could themselves be random variables; we will not treat this case here, although it is very likely that under suitable technical assumptions our results generalize to this case as well. For any $S \subseteq [n]$, set

$$W_S(\mathbf{w}) := \sum_{i \in S} w_i.$$

In our random graph model, the event of including the edge $\{i, j\}$ in the resulting graph is independent of the events of including all other edges, and equals

$$p_{ij}(\mathbf{w}) = \min \left\{ \frac{w_i w_j}{W_{[n]}(\mathbf{w})}, 1 \right\}. \quad (1)$$

This model was considered by Chung et al., for fairly general choices of $\mathbf{w}$, who studied in a series of papers [14–16] several typical properties of the resulting graphs, such as the average path length or the component distribution. We will refer to this model as the *Chung-Lu* model, and we shall write $CL(\mathbf{w})$ for a random graph in which each possible edge $\{i, j\}$ is included independently with probability as in (1). Moreover, we will suppress the dependence on $\mathbf{w}$, if it is clear from the context which sequence of weights we refer to.

Note that in a Chung-Lu random graph, the weights essentially control the *expected* degrees of the vertices. Indeed, if we ignore the minimization in (1), and also allow a loop at vertex i, then the expected degree of that vertex is $\sum_{j=1}^{n} w_i w_j / W_{[n]} = w_i$. In the general case, a similar asymptotic statement is true, unless the weights fluctuate too much. Consequently, the choice of $\mathbf{w}$ has a significant effect on the degree sequence of the resulting graph. For example, the authors of [15] choose $w_i = d\frac{\beta-2}{\beta-1}(\frac{n}{i+i_0})^{1/(\beta-1)}$, which typically results in a graph with a power-law degree sequence with exponent β, average degree d, and maximum degree proportional to $(n/i_0)^{1/(\beta-1)}$, where i_0 was chosen such that this expression is $O(n^{1/2})$. Our results will hold in a more general setting, where larger fluctuations around a "strict" power law are allowed, and also larger maximum degrees are possible, thus allowing a greater flexibility in the choice of the parameters.

2.2 Power-Law Degree Distributions

Following van der Hofstad [28], let us write for any $n \in \mathbb{N}$ and any sequence of weights $\mathbf{w} = (w_1(n), \ldots, w_n(n))$

$$F_n(x) = n^{-1} \sum_{i=1}^{n} \mathbf{1}[w_i(n) < x], \quad \forall x \in [0, \infty)$$

for the empirical distribution function of the weight of a vertex chosen uniformly at random. We will assume that F_n satisfies the following two conditions.

Definition 1. *We say that* $(F_n)_{n \geqslant 1}$ *is* regular, *if it has the following two properties.*

- **[Weak convergence of weight]** *There is a distribution function* $F : [0, \infty) \to [0, 1]$ *such that for all* x *at which* F *is continuous* $\lim_{n \to \infty} F_n(x) = F(x)$*;*
- **[Convergence of average weight]** *Let* W_n *be a random variable with distribution function* F_n*, and let* W_F *be a random variable with distribution function* F*. Then we have* $\lim_{n \to \infty} \mathbb{E}[W_n] = \mathbb{E}[W_F]$.

The regularity of $(F_n)_{n \geqslant 1}$ guarantees two important properties. Firstly, the weight of a random vertex is approximately distributed as a random variable that follows a certain distribution. Secondly, this variable has finite mean and therefore the resulting graph has bounded average degree. Apart from regularity, our focus will be on weight sequences that give rise to power-law degree distributions.

Definition 2. *We say that a regular sequence* $(F_n)_{n \geqslant 1}$ *is* of power law with exponent β, *if there are* $0 < \gamma_1 < \gamma_2$, $x_0 > 0$ *and* $0 < \zeta \leqslant 1/(\beta - 1)$ *such that for all* $x_0 \leqslant x \leqslant n^{\zeta}$

$$\gamma_1 x^{-\beta+1} \leqslant 1 - F_n(x) \leqslant \gamma_2 x^{-\beta+1},$$

and $F_n(x) = 0$ *for* $x < x_0$*, but* $F_n(x) = 1$ *for* $x > n^{\zeta}$.

Thus, we may assume that for $1 \leqslant i \leqslant n(1 - F_n(n^{\zeta}))$ we have $w_i = n^{\zeta}$, whereas for $(1 - F_n(n^{\zeta}))n < i \leqslant n$ we have $w_i = [1 - F_n]^{-1}(i/n)$, where $[1 - F_n]^{-1}$ is the generalized inverse of $1 - F_n$, that is, for $x \in [0, 1]$ we define $[1 - F_n]^{-1}(x) = \inf\{s \ : \ 1 - F_n(s) < x\}$. Note that according to the above definition, for $\zeta > 1/(\beta - 1)$, we have $n(1 - F_n(n^{\zeta})) = 0$, since $1 - F_n(n^{\zeta}) \leqslant \gamma_2 n^{-\zeta(\beta-1)} = o(n^{-1})$. So it is natural to assume that $\zeta \leqslant 1/(\beta - 1)$. Recall finally that in the Chung-Lu model [15] the maximum weight is $O(n^{1/2})$.

2.3 Results

The main theorem of this paper regards the random infection of the whole of $[n]$. We determine explicitly a critical function which we denote by $a_c(n)$ such that when we infect randomly $a(n)$ vertices in $[n]$, then the following threshold phenomenon occurs. If $a(n) \ll a_c(n)$, then a.a.s. the infection spreads no further than $\mathcal{A}_0$, but when $a(n) \gg a_c(n)$, then at least εn vertices become eventually infected, for some $\varepsilon > 0$. We remark that $a_c(n) = o(n)$.

Theorem 1. *For any* $\beta \in (2, 3)$ *and any integer* $r \geqslant 2$*, we let*

$$a_c(n) = n^{\frac{r(1-\zeta)+\zeta(\beta-1)-1}{r}} \tag{2}$$

for all $n \in \mathbb{N}$*. Let* $a : \mathbb{N} \to \mathbb{N}$ *be a function such that* $a(n) \to \infty$*, as* $n \to \infty$*, but* $a(n) = o(n)$*. Let also* $\frac{r-1}{2r-\beta+1} < \zeta \leqslant \frac{1}{\beta-1}$*. If we initially infect randomly* $a(n)$ *vertices in* $[n]$*, then the following holds:*

– *if* $a(n) \ll a_c(n)$*, then a.a.s.* $\mathcal{A}_f = \mathcal{A}_0$*;*
– *if* $a(n) \gg a_c(n)$*, then there exists* $\varepsilon > 0$ *such that a.a.s.* $|\mathcal{A}_f| > \varepsilon n$*.*

Note that the above theorem implies that when the maximum weight of the sequence is $n^{1/(\beta-1)}$, then the threshold function becomes equal to $n^{\frac{\beta-2}{\beta-1}}$ and does not depend on r.

The second theorem has to do with the targeted infection of $a(n)$ vertices where $a(n) \to \infty$, as $n \to \infty$. Let $f : \mathbb{N} \to \mathbb{R}^+$ be a function. We define the f-*kernel* to be

$$\mathcal{K}_f := \{i \in [n] \ : \ w_i \geqslant f(n)\}.$$

We will denote by $CL[\mathcal{K}_f]$ the subgraph of $CL(\mathbf{w})$ that is induced by the vertices of $\mathcal{K}_f$. We show that there exists a function f such that if we infect randomly $a(n)$ vertices of $\mathcal{K}_f$, then this is sufficient to infect almost the whole of the C-kernel, for some constant $C > 0$, with high probability. In other words, the gist of this theorem is that there is a specific part of the random graph of size $o(n)$ such that if the initially infected vertices belong to it, then this is enough to spread the infection to a positive fraction of the vertices.

Theorem 2. *Let* $a : \mathbb{N} \to \mathbb{N}$ *be a function such that* $a(n) \to \infty$*, as* $n \to \infty$*, but* $a(n) = o(n)$*. Assume also* $\frac{r-1}{2r-\beta+1} < \zeta \leqslant \frac{1}{\beta-1}$*. If* $\beta \in (2,3)$*, then there exists an* $\varepsilon_0 = \varepsilon_0(\beta, \gamma_1, \gamma_2)$ *such that for any positive* $\varepsilon < \varepsilon_0$ *there exists a constant* $C = C(\gamma_1, \gamma_2, \beta, \varepsilon, r) > 0$ *and a function* $f : \mathbb{N} \to \mathbb{R}^+$ *such that* $f(n) \to \infty$ *as* $n \to \infty$ *but* $f(n) \ll n^{\zeta}$ *satisfying the following. If we infect randomly* $a(n)$ *vertices in* $\mathcal{K}_f$*, then at least* $(1-\varepsilon)|\mathcal{K}_C|$ *vertices in* $\mathcal{K}_C$ *become infected a.a.s.*

In both theorems, the sequence of probability spaces we consider are the product spaces of the random graph together with the random choice of $\mathcal{A}_0$.

We finish this section, by stating the result of [2] concerning bootstrap percolation in the case of power-law random graphs with exponent $\beta > 3$. (Note that the result in [2] is stated for random graphs with given vertex degrees constructed through the configuration model.) We assume that at time zero each node becomes infected with probability α independently of all the other vertices. Then if p_k denotes the fraction of nodes with degree k and $p_k \propto k^{-\beta}$ for $\beta > 3$, the final fraction of infected nodes satisfies

$$\frac{|\mathcal{A}_f|}{n} \xrightarrow{p} 1 - (1-\alpha) \sum_k p_k \mathbb{P}(\mathsf{Bin}(k, 1-y^*) < r),$$

where y^* is the largest solution in $[0,1]$ to the following fixed point equation

$$y^2 \sum_k k p_k = (1-\alpha) y \sum_k k p_k \mathbb{P}(\mathsf{Bin}(k-1, 1-y) < r).$$

Our results imply that the two regimes $2 < \beta < 3$ and $\beta > 3$ have completely different behaviors.

3 Proof of Theorem 1

In this section we present a sketch of the proof of Theorem 1.

3.1 Subcritical Case

We will use a first moment argument to show that if $a(n) = o(a_c(n))$, then a.a.s. there are no vertices outside $\mathcal{A}_0$ that have at least r neighbours in $\mathcal{A}_0$ and, therefore, the bootstrap percolation process does not actually evolve. Here we assume that initially each vertex becomes infected with probability $a(n)/n$, independently of every other vertex.

For every vertex $i \in [n]$, we define an indicator random variable X_i which is 1 precisely when vertex i has at least r neighbours in $\mathcal{A}_0$. Let $X = \sum_{i\in[n]} X_i$. Our aim is to show that $\mathbb{E}[X] = o(1)$, thus implying that a.a.s. $X = 0$.

For $i \in [n]$ let $p_i = \mathbb{E}[X_i] = \mathbb{P}[X_i = 1]$. We will first give an upper bound on p_i and, thereafter, the linearity of the expected value will conclude our statement.

Lemma 1. *For all integers $r \geqslant 2$ and all $i \in [n]$, we have*

$$p_i \leqslant \left(\frac{ew_i a(n)}{rn}\right)^r .$$

From this, we can use the linearity of the expected value to deduce an upper bound on $\mathbb{E}[X]$. We have

$$\mathbb{E}[X] = \sum_{i\in[n]} p_i \leqslant \sum_{i\in[n]} \left(\frac{ew_i a(n)}{rn}\right)^r = o\left(\left(\frac{a_c(n)}{n}\right)^r\right) \sum_{i\in[n]} w_i^r . \qquad (3)$$

We now need to give an estimate on $\sum_{i\in[n]} w_i^r$.

Claim. For all integers $r \geqslant 2$ and for $\beta \in (2,3)$ we have

$$\sum_{i\in[n]} w_i^r = \Theta\left(n^{1+\zeta(r-\beta+1)}\right).$$

Substituting this bound into the right-hand side of (3), we obtain:

$$\mathbb{E}[X] = o\left(\frac{n^{r(1-\zeta)+\zeta(\beta-1)-1}}{n^r}\, n^{1+\zeta(r-\beta+1)}\right).$$

But

$$r(1-\zeta) + \zeta(\beta-1) - 1 - r + 1 + \zeta(r-\beta+1) = 0,$$

thus implying that $\mathbb{E}[X] = o(1)$.

3.2 Supercritical Case

We begin with stating a recent result due to Janson, Łuczak, Turova and Vallier [22] regarding the evolution of bootstrap percolation processes on Erdős-Rényi random graphs, as these will be needed in our proofs. These results regard the binomial model $G(N, p)$ introduced by Gilbert [19] and subsequently became a major part of the theory of random graphs (see [10] or [21]). Here N is a natural number and p is a real number that belongs to $[0,1]$. We consider the set $[N] =: \{1,\ldots,N\}$ and create a random graph on the set $[N]$, including each pair $\{i,j\}$, where $i \neq j \in [N]$, independently with probability p. The following theorem from [22] considers the bootstrap percolation process on $G(N,p)$, when p as a function of N does not decay too quickly.

Theorem 3 (Theorem 5.8 [22]). *Let $r \geqslant 2$ and assume that initially a uniformly random subset of $[N]$ that has size $a(N)$ becomes infected. If $p \gg N^{-1/r}$ and $a(N) \geqslant r$, then a.a.s. $|\mathcal{A}_f| = N$.*

Now we proceed with the proof of Theorem 1. In this part of the proof, we shall be assuming that $a_c(n) = o(a(n))$. Additionally, we shall assume that the initially infected set is the set of the $a(n)$ vertices of smallest weight.

We will show first that there exists a function $f : \mathbb{N} \to \mathbb{R}^+$ such that $f(n) \to \infty$ as $n \to \infty$ but $f(n) = o(n^\zeta)$ for which a.a.s. $\mathcal{K}_f$ will become completely infected. This is where we use Theorem 3. More precisely, the subgraph of $CL(\mathbf{w})$ that is induced by the vertices of $\mathcal{K}_f$, which we denote by $CL[K_f]$, stochastically contains $G(N_f, p_f)$, where $N_f = |\mathcal{K}_f|$ and p_f is a lower bound on the probability that two vertices in $\mathcal{K}_f$ are adjacent – essentially p_f is equal to $\min\{f^2(n)/W_{[n]}, 1\}$. That is, one can construct a probability space that accommodates both $CL(\mathcal{K}_f)$ and $G(N_f, p_f)$, on the same vertex set and with the correct distributions, in such a way that always the latter is a subgraph of the former.

We then show that any given vertex in $\mathcal{K}_f$ has at least r neighbours in $\mathcal{A}_0$ with some probability p_{Inf} which we determine later in (4). In other words, each vertex in $\mathcal{K}_f$ becomes infected in one round with probability p_{Inf} independently of every other vertex. Hence, as we may consider $G(N_f, p_f)$ as a subgraph of $CL[\mathcal{K}_f]$ on the same vertex set, we deduce that the final set of infected vertices in $\mathcal{K}_f$ is bounded from below by the size of the final set of infected vertices in a bootstrap percolation process on $G(N_f, p_f)$, assuming that the set of initially infected vertices is the set of vertices which have at least r neighbours in $\mathcal{A}_0$. We will show that p_{Inf}, N_f and p_f satisfy the premises of Theorem 3, whereby we will deduce that in fact $\mathcal{K}_f$ becomes completely infected a.a.s. Thereafter, we use the following proposition, whose proof is rather lengthy and technical and, for this reason, we omit it. We consider a bootstrap percolation process on $CL(\mathbf{w})$ where the initially infected set is a large subset of $\mathcal{K}_f$.

Proposition 1. *Let $r \geqslant 2$ and let $f : \mathbb{N} \to \mathbb{R}^+$ be a function such that $f(n) \to \infty$ as $n \to \infty$ but $f(n) = o(n^\zeta)$. Then there exists an $\varepsilon_0 = \varepsilon_0(\beta, \gamma_1, \gamma_2) > 0$ such that for any positive $\varepsilon < \varepsilon_0$ there exists $C = C(\gamma_1, \gamma_2, \beta, \varepsilon, r) > 0$ for which the following holds. If $(1-\varepsilon)|\mathcal{K}_f|$ vertices of $\mathcal{K}_f$ have been infected, then a.a.s. at least $(1-\varepsilon)|\mathcal{K}_C|$ vertices of $\mathcal{K}_C$ become infected.*

We deduce by above proposition that there exists a real number $C > 0$ such that with high probability $\mathcal{K}_C$ will be almost completely infected. This and Definition 2 imply that there exists an $\varepsilon > 0$ such that a.a.s. at least εn vertices become infected.

Spreading the Infection to a Positive Fraction of the Vertices. We begin with determining the function f. To this end, we need to bound from below the probability that an arbitrary vertex in $\mathcal{K}_f$ becomes infected. In fact, we shall bound from below the probability that an arbitrary vertex in $\mathcal{K}_f$ will become infected already in the first round. Note that this amounts to bounding the probability that such a vertex has at least r neighbours in $\mathcal{A}_0$. Therefore, this forms a collection of independent events which is equivalent to the random independent infection of the vertices of $\mathcal{K}_f$ with probability equal to the derived lower bound. Recall that the random graph induced on $\mathcal{K}_f$ stochastically contains an Erdős-Rényi random graph with the appropriate parameters. This observation allows us to determine f. To be more specific, if the probability that any given vertex in $\mathcal{K}_f$ exceeds the complete infection threshold of this Erdős-Rényi random graph and the premises of Theorem 3 is satisfied, then a.a.s. $\mathcal{K}_f$ eventually becomes completely infected.

Under the assumption that $\mathcal{A}_0$ consists of the $a(n)$ vertices of smallest weight, we will bound from below the probability a vertex $v \in \mathcal{K}_f$ has at least r neighbours in $\mathcal{A}_0$. We denote the degree of v in $\mathcal{A}_0$ by $d_{\mathcal{A}_0}(v)$ and note that this random variable is equal to $\sum_{i\in\mathcal{A}_0} \mathsf{Be}\left(\frac{w_v w_i}{W_{[n]}}\right)$, where the summands are independent Bernoulli distributed random variables. Note also that for all n and for all $i \in [n]$ we have $w_i \geqslant x_0$. Thus, we can deduce the following (parts of it hold for n sufficiently large)

$$\begin{aligned}
\mathbb{P}\left[\sum_{i\in\mathcal{A}_0} \mathsf{Be}\left(\frac{w_v w_i}{W_{[n]}}\right) \geqslant r\right] &\geqslant \mathbb{P}\left[\sum_{i\in\mathcal{A}_0} \mathsf{Be}\left(\frac{w_v x_0}{W_{[n]}}\right) \geqslant r\right] \\
&= \mathbb{P}\left[\mathsf{Bin}\left(a(n), \frac{w_v x_0}{W_{[n]}}\right) \geqslant r\right] \\
&\geqslant \binom{a(n)}{r} \left(\frac{w_v x_0}{W_{[n]}}\right)^r \left(1 - \frac{w_v x_0}{W_{[n]}}\right)^{a(n)-r} \\
&\geqslant \frac{a(n)^r}{1.5\ r!} \left(\frac{f(n) x_0}{W_{[n]}}\right)^r \left(1 - \frac{f(n) x_0}{W_{[n]}}\right)^{a(n)-r}.
\end{aligned}$$

Thus, assuming that $a(n)f(n) = o(n)$ we have

$$\left(1 - \frac{f(n)x_0}{W_{[n]}}\right)^{a(n)-r} = 1 - o(1).$$

Therefore, for n sufficiently large

$$\mathbb{P}\left[\sum_{i\in\mathcal{A}_0} \mathsf{Be}\left(\frac{w_v w_i}{W_{[n]}}\right) \geqslant r\right] \geqslant \frac{1}{2r!} \left(\frac{a(n)f(n)x_0}{W_{[n]}}\right)^r =: p_{Inf}. \tag{4}$$

Thus, every vertex of $\mathcal{K}_f$ becomes infected during the first round with probability at least p_{Inf}, independently of every other vertex in $\mathcal{K}_f$.

Recall that $\frac{2r-\beta+1}{r-1} \leqslant \zeta \leqslant \frac{1}{\beta-1}$ and $a_c(n) = n^{\frac{r(1-\zeta)+\zeta(\beta-1)-1}{r}}$. Let us assume that $a(n) = \omega(n) n^{\frac{r(1-\zeta)+\zeta(\beta-1)-1}{r}}$, where $\omega : \mathbb{N} \to \mathbb{R}^+$ is some increasing function that grows slower than any polynomial. Setting $f = f(n) = \frac{n^\zeta}{\omega^{1+1/r}(n)}$, we will consider $CL[\mathcal{K}_f]$. Before doing so, we will verify the assumption that $a(n)f(n) = o(n)$. Indeed, we have

$$a(n)f(n) = \frac{1}{\omega^{1/r}(n)}\; n^{\frac{r(1-\zeta)+\zeta(\beta-1)-1}{r}+\zeta}.$$

But

$$\begin{aligned}\frac{r(1-\zeta)+\zeta(\beta-1)-1}{r}+\zeta &= \frac{r(1-\zeta)+\zeta(\beta-1)-1+r\zeta}{r} \\ &= 1+\frac{\zeta(\beta-1)-1}{r} \leqslant 1,\end{aligned}$$

since $\zeta \leqslant 1/(\beta-1)$, whereby $a(n)f(n) \leqslant \frac{n}{\omega^{1/r}(n)} = o(n)$.

Now, note that if $\zeta > \frac{1}{2}$, then $CL[\mathcal{K}_f]$ is the complete graph on $|\mathcal{K}_f|$ vertices. However, when $\zeta \leqslant \frac{1}{2}$, then $CL[\mathcal{K}_f]$ stochastically contains $G(N_f, p_f)$, where $N_f = |\mathcal{K}_f|$ and $p_f = \frac{f^2(n)}{W_{[n]}}$. We will treat these two cases separately.

Case I: $\frac{1}{2} < \zeta \leqslant \frac{1}{\beta-1}$.

In this case, as $CL[\mathcal{K}_f]$ is the complete graph, it suffices to show that with high probability at least r vertices of $\mathcal{K}_f$ become infected already at the first round. In fact, we will show that the expected number of vertices of $\mathcal{K}_f$ that become infected during the first round tends to infinity as n grows. Note that this number is equal to $N_f p_{Inf}$. Thus, once we show that $N_f p_{Inf} \to \infty$, as $n \to \infty$, then Chebyschev's inequality or a standard Chernoff bound can show that with probability $1-o(1)$, there are at least r infected vertices in $\mathcal{K}_f$ and, thereafter, the whole of $\mathcal{K}_f$ becomes infected in one round.

By Definition 2 we have

$$N_f = |\mathcal{K}_f| = \Omega\left(n\left(\frac{\omega(n)}{n^\zeta}\right)^{\beta-1}\right),$$

and by (4) we have

$$p_{Inf} = \Theta\left(\frac{1}{\omega(n)}\;\left(\frac{n^{\frac{r(1-\zeta)+\zeta(\beta-1)-1}{r}}\cdot n^\zeta}{n}\right)^r\right) = \Theta\left(\frac{n^{\zeta(\beta-1)-1}}{\omega(n)}\right).$$

Hence

$$N_f p_{Inf} = \Omega\left(\omega^{\beta-2}(n)\right).$$

Case II: $\frac{r-1}{2r-\beta+1} < \zeta \leqslant \frac{1}{2}$.

As we mentioned above, $CL[\mathcal{K}_f]$ stochastically contains $G(N_f, p_f)$, where $p_f = \frac{f^2(n)}{W_{[n]}}$, as $\zeta \leqslant \frac{1}{2}$. We will show that here $N_f p_f^r \to \infty$ as $n \to \infty$ and by Theorem 3 we deduce that $\mathcal{K}_f$ becomes completely infected with probability $1-o(1)$. We have

$$N_f p_f^r = \Theta \left(\omega^{\beta-1}(n) n^{1-\zeta(\beta-1)} \frac{n^{2\zeta r}}{\omega^{2r+2}(n) n^r} \right). \tag{5}$$

and the expression on the right-hand side is

$$\omega^{-(2r-\beta+3)}(n) n^{-(r-1)+\zeta(2r-\beta+1)} \to \infty,$$

by our assumption on ζ.

For each one of the above cases, Proposition 1 implies that for any real $\varepsilon > 0$ that is small enough there exists a real number $C = C(\gamma_1, \gamma_2, \beta, \varepsilon) > 0$ such that a.a.s. at least $(1 - \varepsilon)|\mathcal{K}_C|$ vertices of $\mathcal{K}_C$ become infected. But we have $|\mathcal{K}_C| = \Theta(n)$ and the second part of Theorem 1 follows.

4 Conclusion

In this paper, we analyse the evolution of a bootstrap percolation process in a class of inhomogeneous random graphs which exhibits a power law degree distribution with exponent β between 2 and 3. The main result of this work is that a sublinear initially infected set is enough to spread the infection to a linear fraction of vertices of the random graph. We further explore the role of hub vertices of the random graph and demonstrate their function in the evolution of the process.

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Ad Allocation for Browse Sessions

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Abstract. A user's session of information need often goes well beyond his search query and first click on the search result page and therefore is characterized by both search and browse activities on the web. In such settings, the effectiveness of an ad (measured as CtoC ratio, as well as #(conversions) per unit payment) could change based on what pages the user visits and the ads he encounters earlier in the session. *We assume that an advertiser's welfare is solely derived from conversions.*

Our first contribution is to show that the effectiveness of an ad depends upon the past events in the session, namely past exposure to self as well as to competitors. To this end, we analyze logs of user activity over a period of one month from Microsoft AdCenter Delivery Engine. We then propose a new bidding language that allows the advertiser to specify his valuation of a user's click as a function of these externalities, and study the improvement in prediction of conversion events with the new bidding language. We also study theoretical aspects of the allocation problem under new bidding language and conduct an extensive empirical analysis to measure effectiveness of our proposed allocation schemes.

1 Introduction

The increasing amount of time a user spends online conducting e-commerce transactions has led to a widespread use of online advertising by merchants to attract the potential customer to their sites and/or products. Often, this shopping experience of a user extends beyond his query to a search engine and includes visiting multiple web sites learning more about the product. A *browse session* is a contiguous sequence of webpages visited by a user; and two consecutive browse sessions are separated by a period of user's inactivity.

There has been work on understanding externalities in context of interplay between advertisements on the same page [1,9,6,8,7], however they neglect an important aspect that the user is not an independent entity on each page, and events in a browse session affect effectiveness of ads shown later in the session. Thus an ad allocation scheme needs to consider the session as a whole, rather than running independent auctions on each page. We initiate the study of understanding externalities and ad allocation for a browse session.

Now we describe the problem in detail: the most prevalent model of payment in search and contextual advertisements is *pay-per-click*, where advertisers bid for an ad position on a the page and they pay their bid value on a user's click. The advertiser's real welfare is derived from the sale of the good or service (i.e.

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a conversion), and the additional traffic (or the awareness about the product) generated by a click can contribute to its increase. *The advertiser would want to bid for a <user, page> based on his perceived probability that the given user's click on that page would lead to a conversion.* In other words, the advertiser's welfare and payment are in different "currencies", and if events in the browse session affect his *CtoC ratio* or his *welfare (measured in #(conversions)) per unit payment*, then he would want to change his bid accordingly.

Contributions of This Study: We analyzed the entire set of logs of user activity over a period of one month obtained from Microsoft AdCenter Delivery Engine to study the effect of the following two events in a user's browse session on an advertiser's CtoC ratio as well as his welfare per unit payment, namely (a) how many times the ad has been repeated already in the session, and (b) how many competing ads have been shown earlier in the session. We observed that these events affect the CtoC ratio (and the welfare per unit payment) negatively by up to 50%. *While our findings about the externalities from competing ads agree with previous studies in other contexts such as TV advertising which show that competitive advertising has a negative effect on the focal brand [5], our observations for the repeated exposure of ads are contrary to perceptions in other media (such as TV) in which it is considered beneficial to the advertiser [11].*

We model the prior on a user's browse session by a browse graph, and propose a natural language that allows advertisers to express their values of a click as a function of two main externality events. We perform an exhaustive set of experiments to show that *the model can be used to predict the conversion events in the session with better accuracy.* We study theoretical aspects of the allocation problem under new bidding language, and perform an empirical analysis of some natural heuristics on data. Our bidding language is simple, and can be considered as each advertiser specifying his *discount factors* for each externality event. E.g. an advertiser can ask to reduce his bid by a factor of $\texttt{disc\_self}(j)+\texttt{disc\_comp}(k)$ if he is already shown in the session j times and k competing ads have been in the past. Further, *our techniques can also be used internally by the ad allocation engines without exposing the details to advertiser, where the discount factors are computed by the engine, and it scales advertisers' bids with the discount factors.*

Related Work: There has been work on understanding externalities in online advertising [1,9,6,8,7]. One model of externality that has been studied is the effect of cascade models of user's browsing on the click through rates of ads [1,9]. Gomes *et al* [8] consider the role of information and position externalities in a similar cascade model. Ghosh *et al* [7] consider a special case where the each advertiser expresses a two bids for a user, for exclusive and non-exclusive display. Modeling externalities into an auction has also been studied using richer bidding languages [3,10,4].

Organization: We illustrate the experiments performed to establish the sources of externality in Section 2. We give a brief overview of our bidding language, and empirical analysis of some heuristics for ad allocation in Section 3. The details of the bidding language, its relevance to the conversion events in the data (in terms of accuracy of prediction) and theoretical aspects of the allocation problem with the new bidding language are deferred to the full version of the paper [2].

2 Existence of Externalities

We begin by performing a set of experiments to establish the existence of externalities in a user's browse session.

Data Sets: We used the entire set of user activity logs over a period of one month (June 2011) obtained from Microsoft AdCenter Delivery Engine. These logs consisted information about the user query, the set of competing ads, their bids, ads shown, and the click as well as the conversion information. The pages in consideration were essentially the sponsored search properties as well those sites enrolled in the Microsoft publisher network. We associated all requests coming from same IP (anonymized) to a single user. From this data, we extracted the session information of a user. We defined a *session* as the set of contiguous requests by a user such that two consecutive requests are no more than 10 minutes apart. We labeled an advertiser as a *valid advertiser* if his ad impressions got at least 1000 clicks in the month. *Our experiments are restricted to the set of valid advertisers for whom the conversion data is available.* We further note that, the data is for search and contextual ads, where the payment model in use is *pay-per-click.* We define *two advertisers as competitors* if there are at least 10000 sessions in the month in which both are assigned an impression. In our experiments, *a valid advertiser without conversion information is allowed to play the role of a competitor.*

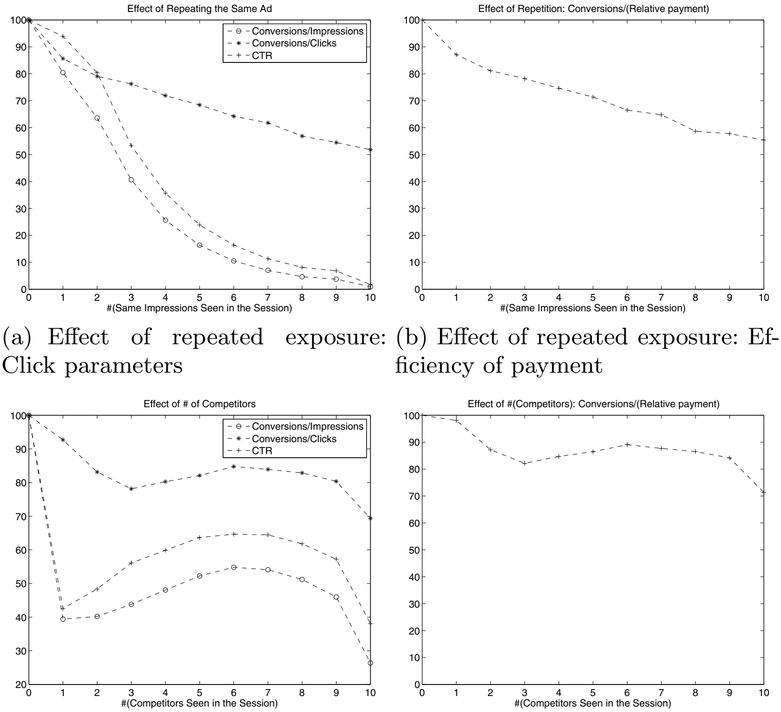
Advertiser's Welfare Model: We assume that an advertiser's welfare is solely derived from a conversion even as the payment model is pay-per-click; and for any fixed advertiser, the welfare derived from a conversion remains same. As an advertiser's welfare is derived solely from conversions, he would want to pay the same (or similar) amount per unit conversion, and we would expect his relative bid on a given page to be directly proportional to the "anticipated" CtoC ratio from a click on the page. Given advertiser $\mathbb{A}_i$ and a page p, let $b_{i,p}$ be his bid for a click on that page, and let $b_{i,avg}$ be his average bid over all pages across all sessions. We define his *relative bid* for page p as $\frac{b_{i,p}}{b_{i,avg}}$; if there is a click on p, his *relative payment* for the click is the value of his relative bid for p.

Existence of Externality: We performed two sets of experiments to establish each type of externality. We considered two types of externality events: (a) the current ad is the advertiser's j^{th} ad in the browse session, and (b) the ad is shown after showing i competitors' ads in the session, where i and j are parameters. Let E be the externality event in consideration.

Effect on CtoC Ratio: We measure the change in the CtoC ratio over all advertisers as a result of externality event E. Its value is computed as follows: let conversions(E) and clicks(E) be the number of conversions and clicks summed over all advertisers when event E is true. Then the value of CtoC ratio under event E, denote by CtoC(E), is defined to be

$$\texttt{CtoC}(E) = \frac{\texttt{conversions}(E)}{\texttt{clicks}(E)}$$

We note that the experiment ignores changes in advertisers' bids from their corresponding average bid values.



(a) Effect of repeated exposure: Click parameters

(b) Effect of repeated exposure: Efficiency of payment

(c) The effect of preceding competitors: Click parameters

(d) The effect of preceding competitors: Efficiency of payment

Fig. 1. Different externalities that are present in a user's browse session

Effect on Advertiser's Welfare per Unit Payment: We measure the *efficiency-of-payment* for advertisers when the event E is true. It is defined as follows: let `conversions`(E) and `payment`(E) be the number of conversions and the total relative payment summed over all advertisers when event E is true, then the efficiency-of-payment under event E, denote by `EoP`(E), is defined to be

$$\texttt{EoP}(E) = \frac{\texttt{conversions}(E)}{\texttt{payment}(E)}$$

As this experiment scales the clicks by advertisers' bid values, it removes the *effects of external parameters and events* on CtoC ratio such as bad quality of impressions (or less relevant users), as *the relative-bid value of an advertiser is a good indicator of importance* of a (user on a) page to the advertiser. If both experiments show a similar quantitative behavior for the externality event in consideration, then it establishes the externality for the event. Now we illustrate our experimental findings.

Effect of Repeating the Same Ad in the Browse Session – Figure 1(a) plots the values of click parameters over all advertisers based on the prior exposure to the same ad in the current browse session. We measure the prior exposure in terms of the number of times the same ad is shown previously in the session. All values in the plot are *relative* to the maximum possible value of the corresponding click parameter, which happens when the ad is shown for the first time. We observe that the CtoC ratio decreases almost linearly as the ad is repeated multiple times; for instance, it drops to 52% of its maximum value when the same ad is repeated 10 times in the session. Figure 1(b) plots the *efficiency-of-payment* for this externality; we observe that its value also drops linearly and it is around 55% of its maximum value when the ad is repeated 10 times. Thus these two quantities show a similar quantitative behavior.

Effect of Competitors – Next we analyze how important it is for an advertiser that his ad is shown before his competitors in a browse session. Toward this end, we analyzed the effect on an advertiser's click parameters when competing ads are shown on earlier pages in the the current browse session. Figures 1(c) and 1(d) measure the CtoC ratio and the efficiency-of-payment for advertisers as a function of number of preceding competitors in the session. We note that both parameters show a similar quantitative behavior and their values decrease as more competing ads are shown previously in the session. The values of the CtoC ratio and the efficiency-of-payment drop to 70% and 80% respectively of their maximum values as the ad is shown after 10 competing ads in the session. In other experiments, we observed that most advertisers are affected negatively by prior competing ads. In fact, 30% advertisers have their CtoC ratio dropped by more than 50% by preceding competitors, and overall, around 80% advertisers are affected negatively.

3 The Bidding Language and Ad Allocation

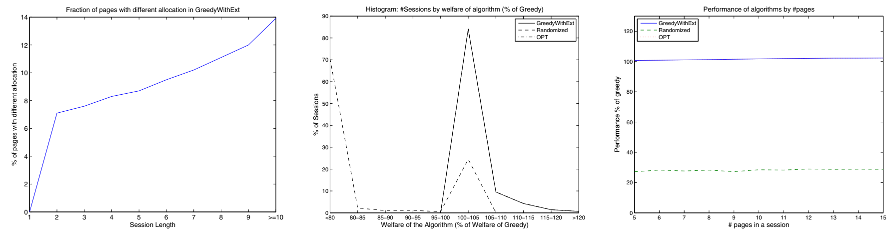
In this section, we give a brief overview of a richer bidding language which enables advertisers to adjust their bids based on the events in the browse session, and study the effectiveness of some natural heuristics for the ad allocation problem. The details of the prior on the user's browse session, the improvement in accuracy of prediction of conversion events with the bidding language and theoretical aspects of the ad allocation problem are deferred to the full version of the paper.

Bidding Language: Advertiser $\mathbb{A}_i$ specifies the *set of his competitors*, and two *discount factors*: $\texttt{disc\_self}_i : N \rightarrow R$ and $\texttt{disc\_comp}_i : N \rightarrow R$. Let b be the valuation of a click for $\mathbb{A}_i$ on page u. Let j be the number of times advertiser $\mathbb{A}_i$'s ad is shown in the session so far and k be the number of competing ads shown in the session so far, then we have

$$\texttt{ext\_self}(i,j) = \texttt{disc\_self}_i(j) \quad \text{and} \quad \texttt{ext\_comp}(i,k) = \texttt{disc\_comp}_i(k)$$

The total externality is the sum of both externalities. $\mathbb{A}_i$'s *effective valuation* of node u with externality effect is $b(1 + \texttt{ext\_self}(i,j) + \texttt{ext\_comp}(i,k))$.

Expressive Power: We discuss some examples to illustrate the expressive power of the simplified bidding language.



(a) Fraction of pages with different allocation in GreedyWithExt and Greedy a function of the length of the session

(b) Histogram of sessions for various algorithms (% of Greedy).

(c) Performance of various algorithms as a function of number of pages (% ofGreedy)

Fig. 2. Performance of various allocation algorithms

- If advertiser $\mathbb{A}_i$ wants his jth repetition in the session to be discounted by $10 \times j\%$, then it can be expressed by setting $\texttt{disc\_self}_i(j) = -0.1 \times j$.
- If advertiser $\mathbb{A}_i$ has two different bids for node u, his bid is b_1 when he is the first among his competitors in the session, and b_2 otherwise, then we can set $w(i,u) = b_1$, $\texttt{disc\_comp}_i(0) = 0$ and $\texttt{disc\_comp}_i(>0) = \frac{b_2 - b_1}{b_1}$. This example is similar to the setting considered in Ghosh *et al* [7], where as advertiser specifies two bids, one for the exclusive display on a page and the other for the non-exclusive display.

The simplified bidding language can also be used as a tool by the ad serving engine, where it computes the discount factors for advertisers to scale their bids, so that the advertisers have better value for their money.

Heuristics for Ad Allocation: As the externalities observed in data are (mostly) negative, the allocation problem is hard to approximate. Hence, we study the performance of some natural heuristics for the allocation problem on real data, using the experimental setup described in Section 2.

a) Greedy allocation with past externality (GreedyWithExt:) Assign the page to the advertiser with maximum (`effectivebid`) × `CTR` value, where `effective bid` is his bid considering the externality in the session.
b) Randomized allocation (Random:) Chosen advertiser randomly with probability proportional to his (`effectivebid`) × `CTR` value.
c) Greedy Allocation (Greedy) Assign each page to the advertiser with the maximum (`bid` × `CTR`) value. This is the optimal allocation in absence of externality effects.
d) Optimal allocation (OPT) (Computed using a dynamic program.)

Observations: The experimental results are given in Figure 2. We note some salient observations:

(1) The first plot (Figure 2(b)) – We measure performances of three algorithms with respect to the Greedy algorithm for sessions with at least 5 pages. For

each algorithm, we classify sessions into ten buckets based on the relative welfare compared to the Greedy algorithm. We observe that for every session, the performance of GreedyWithExt is at least as good as Greedy with at least 5% better for 5% sessions, and the performance of GreedyWithExt is indistinguishable from OPT.

(2) The second plot (Figure 2(c)) – We classify sessions based on their length. For a session type, we measure the ratio of the total welfare of the algorithm in consideration summed over these sessions compared to the total welfare of the greedy algorithm for these sessions. As we can see, when the number of pages in a session is small, OPT and GreedyWithExt are not substantially better than Greedy. This is because, there is less externality in a small session, and the allocation remains almost same even by ignoring it. As session-length increases to 15 pages, the externality effect becomes significant and GreedyWithExt performs 3% better than Greedy. Furthermore, we note that there is no noticeable difference between the performance of GreedyWithExt and OPT. This suggests that GreedyWithExt works well in practice.

(3) The third plot (Figure 2(a))– We classify sessions based on the number of pages in the session, and measure the fraction of pages that have different allocation in GreedyWithExt and Greedy algorithms for a given session-type. The difference in allocation increases with an increase in the session length, with 15% difference for sessions with length $\geq$ 10 pages.

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Computing a Profit-Maximizing Sequence of Offers to Agents in a Social Network

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Abstract. Firms have ever-increasing amounts of information about possible customers available to them; furthermore, they are increasingly able to push offers to them rather than having to passively wait for a consumer to initiate contact. This opens up enormous new opportunities for intelligent marketing. In this paper, we consider the limit case in which the firm can predict consumers' preferences and relationships to each other perfectly, and has perfect control over when it makes offers to consumers. We focus on how to optimally introduce a new product into a social network of agents, when that product has significant externalities. We propose a general model to capture this problem, and prove that there is no polynomial-time approximation unless P=NP. However, in the special case where agents' relationships are symmetric and externalities are positive, we show that the problem can be solved in polynomial time.

1 Introduction

Often the utility that a person derives from a technology depends on whether her neighbors are using the same technology. Examples include various kinds of office software (calendar management, word processing, spreadsheets), mobile phones, etc. In such a context, the technology-provider may need to charge early adopters lower prices (or even give them compensations). Moreover, as firms obtain increasing amounts of data on consumers, they are able to individualize offers to them, in terms of both the timing of the offer and price quoted. This results in a challenging optimization problem for the provider: choose intelligently to which agents to make offers, and in which order.

We assume that a new provider is introducing a single new technology. There may be competing technologies in the market, but in any case the existing situation is static. This rules out possibilities such as existing providers modifying their own prices or otherwise acting in response to the new provider's actions. We also assume that the agents are myopically rational: when made an offer, an agent decides on the offer based on the technologies currently used by her neighbors. The agent does not attempt to predict whether her neighbors will later switch technologies themselves. Finally, we restrict ourselves to situations where the new provider can perfectly predict how much an agent is be willing to pay.

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We show that the general problem is hard to approximate unless $P = NP$ (Section 3). However, in an interesting special case where the agents have symmetric utilities and positive externalities, the problem can be formulated as an integer program whose constraint matrix is totally unimodular. Hence, we get a polynomial time algorithm (Section 4).

Previous Work. There is an extensive literature on marketing policies over a social network [7]. The generic setting is as follows. Initially, the firm convinces a certain subset of agents to use the new technology and those agents, in turn, influence their neighbors. The process continues, and more agents adopt the new technology due to a cascading effect. A standard objective [11] is to select an initial subset of at most k agents so as to maximize the *influence*, which is defined as the total number of agents who adopt the new technology at the end of the cascading process.

In contrast to the *influence* maximization, we optimize the *profit* over a social network [3,9]. The two papers [10,4] are particularly relevant to our setting. They consider a Bayesian model. Here, an agent's valuation for the new technology is private knowledge, but it is drawn from a publicly known distribution. This distribution depends on the subset of her neighbors who have already switched to the new technology. The new firm visits the agents one by one, and while visiting an agent, it offers her the new technology at some price. The agents behave myopically, and the objective is to maximize the expected sum of total payments collected from all the agents. The authors give simple *influence and exploit* policies that are constant factor approximations to optimal profit: In the first stage, a select subset of agents gets the new technology for free. In the next stage, the remaining agents are visited in a sequence chosen uniformly at random, and each of those agents is offered the new technology at the myopically optimal price. Our work is different from these results in three crucial aspects: 1) Unlike these previous papers, we consider a perfect-information (non-Bayesian) setting. 2) In our model, the firm incurs a nonnegative cost for producing each unit of the product, and the objective is to maximize the total payments made by the agents *minus* the total production cost. Hence, marketing policies that make offers to a large subset of agents at low prices can be extremely suboptimal. 3) We allow the agents to have positive utilities for being in the initial state, which captures settings where an existing technology is already in use, and our firm wants to enter the market and compete with an incumbent.

2 The Problem: OPTIMAL-OFFER-SEQUENCE

Consider a simple undirected graph $G = (V, E)$. Every node $i \in V$ denotes an agent, and there is an edge $\{i, j\} \in E$ iff $i \neq j$ and i and j are neighbors. Initially, every agent $i \in V$ is in state $\mathcal{A}$. A new firm (say $\mathcal{B}$) now wants to enter the market, and its objective is to maximize profit by exploiting the network structure. If some agent $i \in V$ decides to be a customer of firm $\mathcal{B}$, then we say that agent i *switches* (or *converts*) to state $\mathcal{B}$.

The vector $\mathbf{S}$ captures the states of all the agents at any particular instant. Component $i \in V$ of vector $\mathbf{S}$ is denoted by $\mathbf{S}_i$, and the notation $\mathbf{S}_{-i}$ denotes all the components *except* component i. Specifically, we set $\mathbf{S}_i = \mathcal{A}$ (resp. $\mathbf{S}_i = \mathcal{B}$) iff agent i is in state $\mathcal{A}$ (resp. state $\mathcal{B}$). Let $U_i(\mathbf{S})$ be the utility of agent $i \in V$. It is a function of the state vector, and can be expressed as the sum of two terms:

$$U_i(\mathbf{S}) = In_i(\mathbf{S}_i) + \Gamma_i(\mathbf{S}_i, \mathbf{S}_{-\mathbf{i}}) \quad (1)$$

In the above equation, the term $In_i(\mathbf{S_i})$ denotes the *intrinsic* utility agent $i \in V$ derives from being in state $\mathbf{S_i}$; whereas her *extrinsic* utility is captured by the term $\Gamma_i(\mathbf{S_i}, \mathbf{S_{-i}})$ and it is determined in the following manner. Let $\Phi_{t,t'}(i,j)$ be the (nonnegative) utility agent i derives from her friend j, when i is in state $t \in \{\mathcal{A}, \mathcal{B}\}$ and j is in state $t' \in \{\mathcal{A}, \mathcal{B}\}$. In general, these utilities may be *asymmetric*, that is, we may have $\Phi_{t,t'}(i,j) \neq \Phi_{t',t}(j,i)$. For all $t, t' \in \{\mathcal{A}, \mathcal{B}\}$ and $i, j \in V$, we set $\Phi_{t,t'}(i,j) = 0$ if the agents i, j are not friends with each other. Now:

$$\Gamma_i(\mathbf{S_i}, \mathbf{S_{-i}}) = \sum_{\mathbf{j} \in \mathbf{V}} \mathbf{\Phi}_{\mathbf{S_i}, \mathbf{S_j}}(\mathbf{i}, \mathbf{j}) \tag{2}$$

Initially, every agent is in state $\mathcal{A}$. Next, firm $\mathcal{B}$ selects a subset $V^* \subseteq V$, and computes a *ranking* $\pi : V^* \to \{1, \ldots, |V^*|\}$ of the agents in V^*. The rank of agent $i \in V^*$ is given by $\pi(i)$. Firm $\mathcal{B}$ now *visits* the agents in V^* in increasing order of their ranks. While visiting an agent i, firm $\mathcal{B}$ offers her the new technology at a price p_i.

Without any loss of generality, we can assume that every agent $i \in V^*$ accepts her offer.[1] Let $\mathbf{S}$ be the state vector just before firm $\mathcal{B}$ makes an offer to agent i. Agent i behaves myopically and utilities are quasilinear. Hence, if she is to switch her state, then we must have: $In_i(\mathcal{B}) + \Gamma_i(\mathcal{B}, \mathbf{S}_{-i}) - p_i \geq In_i(\mathcal{A}) + \Gamma_i(\mathcal{A}, \mathbf{S_{-i}})$. Since firm $\mathcal{B}$ wants to maximize its profit, it sets p_i to the highest possible value. Thus, we have:

$$p_i = In_i(\mathcal{B}) + \Gamma_i(\mathcal{B}, \mathbf{S}_{-i}) - In_i(\mathcal{A}) - \Gamma_i(\mathcal{A}, \mathbf{S_{-i}}) \tag{3}$$

The price p_i can be negative, which implies a subsidy. The idea is that firm $\mathcal{B}$ may have to subsidize some agents in the beginning, when few agents are in state $\mathcal{B}$ and they may incur a loss for switching to the new technology. As more and more agents convert to state $\mathcal{B}$, the firm will be able to exploit the resulting positive externalities and generate a large profit, due to the customers who switch in later stages. Firm $\mathcal{B}$ also incurs a manufacturing cost of c per unit of the product. We want to maximize its net profit, given by the expression $\sum_{i \in V^*}(p_i - c)$. Throughout the rest of the paper, we refer to this optimization problem as OPTIMAL-OFFER-SEQUENCE.

Lemma 1. *Let* PROFIT(j) *be the profit from agent* j. *For all* $i \in V^*$, *let* $\pi_-(i)$ *be the set of agents switching to state* $\mathcal{B}$ before *agent* i, *i.e.,* $\pi_-(i) = \{j \in V^* : \pi(j) < \pi(i)\}$.

$$\text{PROFIT}(i) = \begin{cases} 0, & \text{if } i \in V \setminus V^*. \\ \big(In_i(\mathcal{B}) - In_i(\mathcal{A}) - c\big) + \sum\limits_{j \in \pi_-(i)} \big(\Phi_{\mathcal{B},\mathcal{B}}(i,j) - \Phi_{\mathcal{A},\mathcal{B}}(i,j)\big) & \\ + \sum\limits_{j \in V \setminus \pi_-(i)} \big(\Phi_{\mathcal{B},\mathcal{A}}(i,j) - \Phi_{\mathcal{A},\mathcal{A}}(i,j)\big), & \\ & \text{if } i \in V^*. \end{cases}$$

The total profit of firm $\mathcal{B}$ *is given by:* $\sum_{i \in V}$ PROFIT$(i) = \sum_{i \in V^*}$ PROFIT(i).

[1] Otherwise, we could delete agent i from the set V^*.

Proof. Fix any agent $i \in V^*$. Note that $\textsc{Profit}(i) = p_i - c$. Let $\mathbf{S}$ be the state vector just before i switches to state $\mathcal{B}$. By Equation 3, $\textsc{Profit}(i)$ is equal to:

$$In_i(\mathcal{B}) - In_i(\mathcal{A}) - c + \Gamma_i(\mathcal{B}, \mathbf{S}_{-i}) - \Gamma_i(\mathcal{A}, \mathbf{S}_{-i}) \quad (4)$$

Expanding the right hand side of Equation 2, we can show:

$$\Gamma_i(\mathcal{B}, \mathbf{S}_{-i}) = \sum_{j \in \pi_-(i)} \Phi_{\mathcal{B},\mathcal{B}}(i,j) + \sum_{j \in V \setminus \pi_-(i)} \Phi_{\mathcal{B},\mathcal{A}}(i,j)$$
$$\Gamma_i(\mathcal{A}, \mathbf{S}_{-i}) = \sum_{j \in \pi_-(i)} \Phi_{\mathcal{A},\mathcal{B}}(i,j) + \sum_{j \in V \setminus \pi_-(i)} \Phi_{\mathcal{A},\mathcal{A}}(i,j)$$

Finally, we substitute the above expressions back in Eq. 4. □

3 A Hardness Result

In this section, we show that (see Lemma 3) it is NP-hard to decide whether firm $\mathcal{B}$ can make positive profit, by a reduction from the Maximum Arc Set on Tournaments (MAST) problem. This rules out the existence of any polynomial-time approximation algorithm for OPTIMAL-OFFER-SEQUENCE, unless $P = NP$ (see Theorem 1).

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed tournament graph; that is, for any two distinct nodes $i, j \in \mathcal{V}$, we have $|\mathcal{E} \cap \{(i,j),(j,i)\}| = 1$. Let $\pi : \mathcal{V} \to \{1, \ldots, |\mathcal{V}|\}$ be a ranking of the set of nodes $\mathcal{V}$, where $\pi(i)$ denotes the rank of node $i \in \mathcal{V}$, and $\pi(i) \neq \pi(j)$ if $i \neq j$. We say that an edge $(i,j) \in \mathcal{E}$ is a *forward edge* (resp. *backward edge*) w.r.t. ranking π if $\pi(i) < \pi(j)$ (resp. $\pi(i) > \pi(j)$).

Maximum Acyclic Subgraph on Tournaments (MAST): An instance $\mathcal{F}$ of the problem consists of an ordered pair $(\mathcal{G}, \theta)$, where $\theta \geq 1$ is a positive integer, and $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a directed tournament graph. The objective is to decide if there exists a ranking of $\mathcal{V}$ where the number of backward edges is at least θ. This problem is NP-hard [6,2,5,1].

The Reduction. Given an instance $\mathcal{F}$ of the MAST problem $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), \theta)$, we construct the following instance $\mathcal{I}_{\mathcal{F}}$ of OPTIMAL-OFFER-SEQUENCE. It is easy to see that the reduction can be implemented in polynomial time.

- $G = (V, E)$ is a complete undirected graph, defined on the same node set as that of $\mathcal{G}$; that is, $V = \mathcal{V}$ and $E = \{\{i,j\} : i, j \in V, i \neq j\}$.
- For all $i, j \in V$: if $(i,j) \in \mathcal{E}$ then $\Phi_{\mathcal{B},\mathcal{B}}(i,j) = 1$, else $\Phi_{\mathcal{B},\mathcal{B}}(i,j) = 0$.
- For all $i, j \in V$: we have $\Phi_{\mathcal{A},\mathcal{B}}(i,j) = \Phi_{\mathcal{B},\mathcal{A}}(i,j) = \Phi_{\mathcal{A},\mathcal{A}}(i,j) = 0$.
- For all $i \in V$: we set $In_i(\mathcal{A}) = In_i(\mathcal{B}) = 0$.
- The cost per unit c is set in such a way that

$$-c \times |V| + \theta = 1 \quad (5)$$

According to the above reduction, the profit (Lemma 1) from the instance $\mathcal{I}_{\mathcal{F}}$ equals:

$$\textsc{Profit} = -c|V^*| + \sum_{i \in V^*} \sum_{j \in \pi_-(i)} \Phi_{\mathcal{B},\mathcal{B}}(i,j) \quad (6)$$

Let $\mathcal{G}[V^*] = (V^*, \mathcal{E}^*)$ be the subgraph of $\mathcal{G}$ induced by the node set $V^* \subseteq V$, so that:

$$\mathcal{E}^* = \{(i,j) \in \mathcal{E} : i,j \in V^*, i \neq j\} \tag{7}$$

Let $\mathcal{E}^*_\pi$ be the set of backward edges in $\mathcal{G}[V^*]$ w.r.t. π. Since $\Phi_{\mathcal{B},\mathcal{B}}(i,j) = 1$ when $(i,j) \in \mathcal{E}$, and $\Phi_{\mathcal{B},\mathcal{B}}(i,j) = 0$ when $(i,j) \notin \mathcal{E}$, Equation 6 implies that

$$\text{PROFIT} = -c|V^*| + \sum_{i \in V^*} \sum_{j \in \pi_-(i)} \Phi_{\mathcal{B},\mathcal{B}}(i,j) = -c|V^*| + |\mathcal{E}^*_\pi| \tag{8}$$

Lemma 2. *In the instance $\mathcal{I}_\mathcal{F}$ of* OPTIMAL-OFFER-SEQUENCE*, the profit-maximizing solution either converts* all *the agents to state $\mathcal{B}$, or it does not convert* any *agent to state $\mathcal{B}$; that is, it sets either $V^* = \emptyset$ or $V^* = V$.*

Proof. In the profit-maximizing solution, suppose that the agents in V^* switch to state $\mathcal{B}$ according to the ranking $\pi : V^* \to \{1, \ldots, |V^*|\}$. For the sake of contradiction, suppose that the lemma is false, and the profit-maximizing solution sets $\emptyset \subset V^* \subset V$. Since the profit is nonnegative, Equation 8 implies that $-c|V^*| + |\mathcal{E}^*_\pi| \geq 0$, or equivalently, $c \leq |\mathcal{E}^*_\pi|/|V^*|$. Since $|\mathcal{E}^*_\pi| \leq \binom{|V^*|}{2}$, we derive $c < |V^*|/2$.

Fix any $k \in V \setminus V^*$. Let $\delta^+(k, V^*)$ (resp. $\delta^-(k, V^*)$) be the number of outgoing (resp. incoming) edges of k whose other endpoints lie in V^*. Since the graph $\mathcal{G}$ is a tournament, either $\delta^-(k, V^*) \geq |V^*|/2$ or $\delta^+(k, V^*) \geq |V^*|/2$.

Case 1. $\delta^-(k, V^*) \geq |V^*|/2$.
In this case, we construct a new solution that converts all the nodes in $V^* \cup \{k\}$ to state $\mathcal{B}$ in the following order: First, it converts node k. Next, it converts the nodes in V^* according to ranking π. Let the new profit be P′. Clearly, we have:

$$\text{P}' = -c(|V^*| + 1) + \delta^-(k, V^*) + |\mathcal{E}^*_\pi| > -c|V^*| + |\mathcal{E}^*_\pi|$$

The inequality holds since $c < |V^*|/2$ and $\delta^-(k, V^*) \geq |V^*|/2$. Thus, the new profit is strictly greater than the maximum profit, which is a contradiction.

Case 2. $\delta^+(k, V^*) \geq |V^*|/2$.
In this case, we construct another solution that converts all the nodes in $V^* \cup \{k\}$ to state $\mathcal{B}$ in the following order: First, it converts the nodes in V^* according to ranking π. Next, it converts node k. Applying an argument similar to Case 1, we show that the new profit is strictly greater than the maximum profit, which is a contradiction. □

Lemma 3. *Firm $\mathcal{B}$ can get positive profit from the instance $\mathcal{I}_\mathcal{F}$ of the* OPTIMAL-OFFER-SEQUENCE *problem* if and only if *the instance $\mathcal{F}$ of the MAST problem admits a ranking where the number of backward edges is at least θ.*

Proof. Suppose that the optimal solution to the instance $\mathcal{I}_\mathcal{F}$ converts the agents in $V^* \subseteq V$ to state $\mathcal{B}$ according to the ranking π. Lemma 2 implies that it is possible to get positive profit from the instance $\mathcal{I}_\mathcal{F}$ iff $V^* = V$, and in that case, applying Equation 8:

$$\text{PROFIT} = -c|V| + |\mathcal{E}^*_\pi| = 1 - \theta + |\mathcal{E}^*_\pi| > 0.$$

The second equality holds because of Equation 5. Since θ is an integer, $1 - \theta + |\mathcal{E}^*_\pi| > 0$ iff $|\mathcal{E}^*_\pi| \geq \theta$. Since π is also a ranking for the MAST instance $\mathcal{F}$, the lemma follows.

Lemma 3 implies Theorem 1.

Theorem 1. *The* OPTIMAL-OFFER-SEQUENCE *problem does not admit any polynomial-time approximation algorithm, unless* $P = NP$.

Next, we describe a family of instances that admit a 2-approximation in poly-time. Theorem 2 follows from a result by Guruswami et al. [8].

Theorem 2. *Consider a family of instances of the* OPTIMAL-OFFER-SEQUENCE *problem where* $c = 0$, $In_i(\mathcal{A}) = In_i(\mathcal{B}) = 0$ *for all* $i \in V$, *and* $\Phi_{\mathcal{A},\mathcal{B}}(i,j) = \Phi_{\mathcal{B},\mathcal{A}}(i,j) = \Phi_{\mathcal{A},\mathcal{A}}(i,j) = 0$ *and* $\Phi_{\mathcal{B},\mathcal{B}}(i,j) \geq 0$ *for all* $i, j \in V$. *Under such settings, there exists a poly-time* 2 *approximation algorithm for the* OPTIMAL-OFFER-SEQUENCE *problem, and it is Unique Games hard to get better than* 2 *approximation.*

4 Symmetric Utility Functions: Polynomial Time Algorithm

In this section, for all $\{i,j\} \in E$, we require that $\Phi_{\mathcal{A},\mathcal{A}}(i,j) = \Phi_{\mathcal{A},\mathcal{A}}(j,i)$, $\Phi_{\mathcal{B},\mathcal{B}}(i,j) = \Phi_{\mathcal{B},\mathcal{B}}(j,i)$, and $\Phi_{\mathcal{A},\mathcal{B}}(i,j) = \Phi_{\mathcal{B},\mathcal{A}}(j,i) = 0$. Such utility functions are *symmetric*, and we write $\Phi_{\mathcal{A},\mathcal{A}}(\{i,j\})$ and $\Phi_{\mathcal{B},\mathcal{B}}(\{i,j\})$ instead of $\Phi_{\mathcal{A},\mathcal{A}}(i,j)$ and $\Phi_{\mathcal{B},\mathcal{B}}(i,j)$. Under symmetric utilities, the problem can be solved in polynomial time (see Theorem 3).

Lemma 4. *If the utility functions are symmetric, then the profit of firm* $\mathcal{B}$ *is given by:*

$$\sum_{i \in V^*} \big(In_i(\mathcal{B}) - In_i(\mathcal{A}) - c\big) + \sum_{\{i,j\} \subseteq V^*} \Phi_{\mathcal{B},\mathcal{B}}(\{i,j\}) - \sum_{\{i,j\} \cap V^* \neq \emptyset} \Phi_{\mathcal{A},\mathcal{A}}(\{i,j\})$$

Proof. Since the utility functions are symmetric, we have:

$$\sum_{i \in V^*} \sum_{j \in \pi_-(i)} \big(\Phi_{\mathcal{B},\mathcal{B}}(i,j) - \Phi_{\mathcal{A},\mathcal{B}}(i,j)\big) = \sum_{\{i,j\} \subseteq V^*} \Phi_{\mathcal{B},\mathcal{B}}(\{i,j\}) \tag{9}$$

$$\sum_{i \in V^*} \sum_{j \in V \setminus \pi_-(i)} \big(\Phi_{\mathcal{B},\mathcal{A}}(i,j) - \Phi_{\mathcal{A},\mathcal{A}}(i,j)\big) = - \sum_{\{i,j\} \cap V^* \neq \emptyset} \Phi_{\mathcal{A},\mathcal{A}}(\{i,j\}) \tag{10}$$

The lemma follows from Equations 9, 10 and Lemma 1. □

Lemma 4 implies that the profit of firm $\mathcal{B}$, under symmetric utility functions, is uniquely determined by the set of agents who switch to state $\mathcal{B}$, and is independent of the order in which those agents are offered the new technology. We now give an integer programming formulation (IP-1) for our problem. Note that in IP-1, the variables $\gamma_{\{i,j\}}, \lambda_{\{i,j\}}$ are defined over *unordered* pairs of nodes $\{i,j\} \in E$.

IP-1

$$\text{Max.} \sum_{i \in V} \big(In_i(\mathcal{B}) - In_i(\mathcal{A}) - c\big) x_i + \sum_{\{i,j\}} \big(\Phi_{\mathcal{B}\mathcal{B}}(\{i,j\}) \gamma_{\{i,j\}} - \Phi_{\mathcal{A}\mathcal{A}}(\{i,j\}) \lambda_{\{i,j\}}\big)$$

$$\text{s.t.} \quad \gamma_{\{i,j\}} - x_i \leq 0 \quad \forall i \in V, \{i,j\} \in E \tag{11}$$

$$x_i - \lambda_{\{i,j\}} \leq 0 \quad \forall i \in V, \{i,j\} \in E \tag{12}$$

$$x_i \in \{0,1\} \quad \forall i \in V \tag{13}$$

$$\gamma_{\{i,j\}}, \lambda_{\{i,j\}} \in \{0,1\} \quad \forall \{i,j\} \in E \tag{14}$$

Lemma 5. *The constraints of IP-1 ensure that in an optimal solution:*

- *The variable* $x_i = 1$ *iff node* $i \in V$ *switches to state* $\mathcal{B}$*, that is, when* $i \in V^*$.
- *The variable* $\gamma_{\{i,j\}} = 1$ *iff both the endpoints of edge* $\{i, j\}$ *switch to state* $\mathcal{B}$.
- *The variable* $\lambda_{\{i,j\}} = 1$ *iff at least one endpoint of edge* $\{i, j\}$ *switches to state* $\mathcal{B}$.

Hence, Lemma 4 implies that IP-1 gives an integer programming formulation of the OPTIMAL-OFFER-SEQUENCE *problem in the special case of symmetric utilities.*

Proof. We show that the interpretation of $\gamma_{\{i,j\}}$ is consistent with the interpretation of x_i. Each $\gamma_{\{i,j\}}$ has a nonnegative coefficient in the objective. Hence, in an optimal solution, $\gamma_{\{i,j\}}$ is set to the largest possible value. Constraint 11 establishes an upper bound of $\min(x_i, x_j)$ on the variable $\gamma_{\{i,j\}}$. It follows that $\gamma_{\{i,j\}} = 1$ iff $x_i = x_j = 1$.

Each $\lambda_{\{i,j\}}$ has a nonpositive coefficient in the objective. Thus, in an optimal solution, $\lambda_{\{i,j\}}$ is set to the smallest possible value. Constraint 12 establishes a lower bound of $\max(x_i, x_j)$ on the variable $\gamma_{\{i,j\}}$. Hence, $\lambda_{\{i,j\}}$ is set to 0 iff $x_i = x_j = 0$. □

Theorem 3. *The constraint matrix of IP-1 is totally unimodular. Hence, we can find an optimal solution of IP-1 in polynomial time. Thus, the* OPTIMAL-OFFER-SEQUENCE *problem can be solved efficiently when the utility functions are symmetric.*

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Convergence Analysis for Weighted Joint Strategy Fictitious Play in Generalized Second Price Auction

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Abstract. Generalized Second Price (GSP) auction is one of the most commonly used auction mechanisms in sponsored search. As compared to conventional equilibrium analyses on GSP auctions, the convergence analysis on the dynamic behaviors of the bidders can better describe real-world sponsored search systems, and give them a more useful guideline for making improvement. However, most existing works on convergence analysis assume the bidders to be greedy in taking actions, i.e., they only utilize the bid information in the current round of auction when determining the best strategy for the next round. We argue that real-world professional advertisers are more capable and can utilize the information in a longer history to optimize their strategies. Accordingly, we propose modeling their behaviors by a weighted joint strategy fictitious play (wJSFP). In the proposed model, bidders determine their optimal strategies based on their beliefs on other bidders' bid prices, and the beliefs are updated by considering all the information they have received so far in an iterative manner. We have obtained the following theoretical results regarding the proposed model: 1) when there are only two ad slots, the bid profile of the bidders will definitely converge; when there are multiple slots, there is a sufficient condition that guarantees the convergence of the bid profile; 2) as long as the bid profile can converge, it converges to a Nash equilibrium of GSP. To the best of our knowledge, this is the first time that the joint strategy fictitious play is adopted in such a complex game as sponsored search auctions.

1 Introduction

Sponsored search has become an increasingly important advertising channel nowadays. When a web user submits a query keyword to a search engine, besides the organic search results, he/she will also see a ranked list of paid ads. If the user clicks on one of these ads, the corresponding advertiser will be charged by a certain amount of money. In most search engines today, the ranking and pricing of the ads are determined by a keyword auction mechanism. Generalized Second

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Price (GSP) auction is one of the most widely used keyword auction mechanisms. With GSP, the ads are ranked according to the products of their quality scores and bid prices, and the payment for a clicked ad equals the minimal bid price for the ad to maintain its current rank position.

The theoretical properties of GSP have been well studied in the literature [1][3][4][5][10]. In particular, Edelman et al.[5] and Varian[10] discussed a type of Nash equilibrium of GSP named locally envy free equilibrium in the full information setting; Christodoulou et al.[3] studied the Bayesian-Nash equilibrium in the partial information setting; and Bhawalkar et al.[1] provided a guarantee on the social welfare and revenue in equilibrium for GSP. However, almost all these works suffer from the following problems: (i) they assume that every bidder knows the true values of the other bidders (either completely or in a Bayesian manner), however, the private values are inaccessible in keyword auctions, even to the auctioneer; (ii) they only investigate the fixed point of the bid profile (i.e., equilibrium) and cannot explain how the equilibrium is (progressively) achieved.

To tackle these problems, a number of bidder behavior models were proposed [2][9][11]. For example, Edelman et al. [2] modeled greedy bidding strategies and discussed their resultant revenue, convergence, and robustness. Zhou and Lukose [11] modeled vindictive bidding strategies and showed that most Nash equilibria are vulnerable to vindictive bidding. Noam Nisan et al. [9] analyzed the best response models and showed that the simple and myopic best-response dynamics converge to the VCG outcome in several well studied auction environments including GSP. These models have a couple of advantages. First, they do not assume bidders to know competitors' true values (but instead only their bid prices). Second, they can be used to explain how an equilibrium is achieved in a dynamic environment. However, these models also have their limitations. Specifically, these models assume the bidders to be greedy in taking actions, i.e., they only utilize the bid information in the current round of auction when determining the best strategy for the next round. It is clear that many real-world professional advertisers are more capable than assumed in these works, and can utilize the information in a longer history to optimize their strategies.

To better reflect the capability of professional advertisers, we propose modeling their behaviors in GSP auctions using a weighted joint strategy fictitious play (wJSFP). In this proposed model, every bidder forms a belief (i.e., a distribution) on other bidders' future bids and iteratively updates it by considering all the information he/she has received in the history. The bidder then chooses his/her best bid strategy by maximizing the expected utility according to the beliefs. The parameter in the proposed model can be estimated from real data, and can be used to predict the future behaviors of the bidder.

Although the fictitious play model has been proposed for over sixty years, previous studies on its convergence property are all for relatively simple games, such as the two-player zero-sum games[7], and the potential games [8]. Its convergence property in such a complex game as GSP auctions is unclear due to the existence of multiple players and multiple strategies per player. To perform meaningful convergence analysis, we consider the structure of bidders' utilities in GSP and obtain the following results. 1) When there are only two ad slots, the bid profile of the bidders will definitely converge. 2) When there are multiple slots, we obtain a sufficient condition that can guarantee the convergence of the bid profile. 3) As long as the bid profile converges, it will converge to a Nash equilibrium of GSP.

To the best of our knowledge, it is the first time that JSFP is adopted in sponsored search auctions, and it is also the first time that the convergence properties of JSFP in such a complex setting is comprehensively investigated.

The rest of this paper is organized as follows. In Section 2, we give a brief introduction to the GSP mechanism and the proposed wJSFP behavior model in GSP. Our theoretical results are presented in Section 3. The conclusion and future work are given in the last section.

2 Weighted Joint Strategy Fictitious Play in GSP

In this section, we first give a brief introduction to the keyword auctions in sponsored search and the GSP mechanism. Then we describe our proposed wJSFP model.

Consider the application of sponsored search. Suppose there are m bidders $\{1, 2, \cdots, m\}$ who bid for n ad slots, where $m \geq n$. Each bidder i has a private value v_i and bid set X_i. Without loss of generality, we assume $v_1 \geq v_2 \geq \cdots \geq v_m$. The click-through rate for ad slot k is denoted as β_k, which satisfies $\beta_k > 0$ if $k \leq n$, $\beta_k = 0$ if $k > n$, and $\beta_1 > \beta_2 > ... > \beta_n$. At time period t, the bid profile of all the bidders is denoted as $b^t = (b_1^t, b_2^t, \ \cdots b_m^t) \in X_1 \times X_2 \times \cdots \times X_m \triangleq X$, and the bid profile of other bidders except bidder i is denoted as $b_{-i}^t = \{b_1^t, \cdots b_{i-1}^t, b_{i+1}^t, \cdots b_m^t\}$. Then a keyword auction (A, p) is performed based on the bid profile b^t, where the allocation rule A allocates ads to ad slots and the pricing rule p charges the bidders for the clicks on their ads. Consequently, bidder i receives his/her utility $u_i(b_i^t, b_{-i}^t) = (v_i - p_i(b_i^t, b_{-i}^t))\beta_{A_i(b_i^t, b_{-i}^t)}$.

When the GSP mechanism is adopted, the allocation rule ranks the ads according to the products of their quality scores $\{q_i; i = 1, \ \cdots, m\}$ and bid prices[1], i.e., $A_i(b) = 1 + \sum_{j=1}^{m} I_{[q_j b_j > q_i b_i]}$; the pricing rule charges a bidder for his/her clicked ad by the minimum bid price that can maintain the current rank position of his/her ad, i.e., $p_i(b) = q_{A^{-1}(A_i(b)+1)} b_{A^{-1}(A_i(b)+1)} / q_i$. For simplicity and without loss of generality, we assume the quality scores to be identical in the following discussion.[2]

Next we describe our proposed wJSFP model, which contains the following three steps.

1. *Belief Update*: Each bidder has a belief on other bidders' future bids and updates it in an iterative manner by considering all the information he/she

[1] For simplicity, we break ties (if there exists) by allocating the ad slot to the bidder with a smaller index.

[2] Please note that this assumption does not affect any of our analyses, since one can absorb the original quality scores into the bid prices.

has received in the history. Specifically, we denote the belief of bidder i as $\pi_{-i}(t)$, which is a distribution on X_{-i} and is updated as follows:

$$\pi_{-i}(t) = (1-\gamma_i^{t-1})\pi_{-i}(t-1) + \gamma_i^{t-1} P_{\delta(b_{-i}^{t-1})}, \tag{1}$$

where $0 \leq \gamma_i^t \leq 1$ is the *belief update* parameter for bidder i at time period t, $P_{\delta(\cdot)}$ is the delta distribution, and $\pi_{-i}(1)$ is the prior belief for bidder i at time period 1. The formula indicates that each bidder updates his/her belief to a weighted average of his/her current belief and other bidders' current bid prices. In some sense, the parameter γ reflects the capability of a bidder in information collection, processing, and analysis.

2. *Utility Maximization*: Each bidder computes his/her expected utility based on his/her belief, i.e.,

$$u_i(b_i,\ \pi_{-i}(t)) = E_{b_{-i}\sim\pi_{-i}(t)} u_i(b_i,\ b_{-i}), where\ b_i \in X_i. \tag{2}$$

Then, bidder i computes his/her best response set with respect to belief $\pi_{-i}(t)$,

$$BR(\pi_{-i}(t)) \triangleq \arg\max_{b_i \in X_i} u_i(b_i,\ \pi_{-i}(t)). \tag{3}$$

3. *Bid Update*: Each bidder selects a bid from the best response set $BR(\pi_{-i}(t))$ in the following ways: if the current bid belongs to the best response set, the bidder will not change his/her bid; otherwise, he/she will randomly select a bid from $BR(\pi_{-i}(t))$. The selected bid will then serve as the bid price for the next round of auction.[3]

Based on the steps described above, we can rewrite each bidder i's belief by using the weight parameter $\tau_i^{s,t}$ as below,

$$\pi_{-i}(t) = \tau_i^{0,t-1}\pi_{-i}(1) + \sum_{s=1}^{t-1} \tau_i^{s,t-1} P_{\delta(b_{-i}^s)}, \tag{4}$$

where $\tau_i^{s,t} = \gamma_i^s \prod_{k=s+1}^t (1-\gamma_i^k)$ for $s > 0$ and $\tau_i^{0,t} = \prod_{k=1}^t (1-\gamma_i^k)$. The bids profile for other bidders before time period t is $b_{-i}^s (s < t)$. It is not difficult to verify that, if $0 < \gamma_i^t < 1$ and $\gamma_i^{t+1} \geq \frac{\gamma_i^t}{1+\gamma_i^t}$, we have $\tau_i^{t,t} \geq \tau_i^{t-1,t} \geq \cdots \geq \tau_i^{1,t} > 0$. That is, in this case, bidders will put higher weights to the bid vectors closer to the current time in their belief and will take into consideration all the information they have received in the history. According to the linearity property of expectation, we reformulate bidder's expected utility as below, which will be used in the next section.

$$u_i(b_i, \pi_{-i}(t)) = \tau_i^{0,t-1} u_i(b_i, \pi_{-i}(1)) + \sum_{s=1}^{t-1} \tau_i^{s,t-1} u_i(b_i, b_{-i}^s). \tag{5}$$

Please note that the proposed wJSFP behavior model is a generalization of the best response model [2] and the classical JSFP model [6]. If $\gamma_i^t \equiv 1$ for $t \geq 1$, the

[3] In this paper, we mainly focus on this particular bid update rule for sake of simplicity. However, please note that one can also apply other bid update rules. For example, with the random bid update rule, the bidder will randomly choose a bid from $BR(\pi_{-i}(t))$; with the competitor busting bid update rule, the bidder will bid the maximum of $BR(\pi_{-i}(t))$.

wJSFP model will be reduced to the best-response model; If $\gamma_i^t = \frac{1}{t}$ for $t \geq 1$, the wJSFP model will become the classic JSFP model. As compared to these two reduced versions of our proposed model, our model can better describe the behaviors of real-world advertisers due to the removal of some strict and unreasonable assumptions.

3 Convergence Analysis

In this section we prove the convergence properties of the proposed wJSFP behavior model.

Without loss of generality, we assume that no bidders will over bid, and we quantize the bids by δ. That is, $X_i \subseteq \{0, \delta, 2\delta, ...\}$ for all $i \in \{1,\ 2,\ \cdots,\ m\}$, where δ is the smallest bid that a bidder could choose (in real applications, the bid price is always quantized since it is money, and δ is one cent).

Lemma 1. Denote S_t as the set of bidders who can get a slot at time t. If all bidders behave according to the wJSFP model, $\lim_{t\to\infty} S_t = \{1,\ 2,\ \cdots,\ n\}$.

Proof. Let $b_{(n)}^t$ be the bid of the bidder in slot n at time t. It is clear that $b_{(n)}^t \leq v_n$, since bidders will not over bid. Thus, $B \triangleq \liminf_{t\to\infty} b_{(n)}^t \leq v_n$. Since $b_{(n)}^t \in \{0, \delta, 2\delta, ...\}$, $\exists\ T$ and $\{t_k\}_{k\geq 1}$ ($T < t_1 < t_2 < \cdots$) s.t. $b_{(n)}^t \geq B$ ($\forall t > T$) and $b_{(n)}^{t_k} = B$ ($\forall k \geq 1$). Therefore, for $\forall$ bidder i $\forall$ sufficiently large t, if $b_i^t < B$, $u_i(b^t) = 0$; if $b_i^t > B$, $u_i(b^t) \geq \delta\beta_n$. If $b_i < B$,

$$\begin{aligned} u_i(b_i, \pi_{-i}(t)) &= \tau_i^{0,t-1} u_i(b_i, \pi_{-i}(1)) + \sum_{s=1}^{t-1} \tau_i^{s,t-1} u_i(b_i, b_{-i}^s) \\ &= \tau_i^{T,t-1} \frac{1}{\gamma_i^T} (\tau_i^{0,T} u_i(b_i, \pi_{-i}(1)) + \sum_{s=1}^{T} \tau_i^{s,T} u_i(b_i, b_{-i}^s)) \\ &\leq \tau_i^{T,t-1} \frac{1}{\gamma_i^T} \max_{b_i<B} (\tau_i^{0,T} u_i(b_i, \pi_{-i}(1)) + \sum_{s=1}^{T} \tau_i^{s,T} u_i(b_i, b_{-i}^s)) \\ &\triangleq \tau_i^{T,t-1} A(T). \end{aligned} \tag{6}$$

If $v_i \geq b_i > B$,

$$u_i(b_i, \pi_{-i}(t)) \geq \sum_{T<t_k<t} \tau_i^{t_k,t-1} u_i(b_i, b_{-i}^s) \geq \tau_i^{T,t-1} \sum_{T<t_k<t} \delta\beta_n > \tau_i^{T,t-1} A(T). \tag{7}$$

Therefore, for bidder i s.t. $v_i \geq B$, $b_i^t \geq B$.

If $B < v_{n+1}$, for $i \leq n+1$, $v_i \geq v_{n+1} \geq B$. Thus $b_i^t \geq B$. Since we allocate slot to the bidder with a smaller index when a tie appears, if $b_{n+1}^t \leq B$, $u_{n+1}(b^t) = 0$. Thus, when $b_{n+1} \leq B$ inequality (6) still holds. Further considering (7), we have $b_{n+1}^t > B$ when t is sufficiently large. So, if $b_n^t \leq B$, $u_n(b^t) = 0$. Similar to the analysis for bidder $n+1$, $b_n^t > B$. With the same logic, we have $b_i^t > B$ ($i = 1, 2, \cdots, n+1$) which is contradicted with the definition of B. Thus, $B \geq v_{n+1}$. For $i \leq n$, $v_i \geq v_n \geq B$, thus $b_i^t \geq B$. For $i > n$, $b_i^t \leq v_i \leq v_{n+1} \leq B$. Therefore, $\{1,\ 2,\ \cdots,\ n\}$ will win. ■

The above lemma shows that in the wJSFP model, only those bidders whose private values are ranked in the top n positions can get a slot after a long-term update. Next, we discuss the convergence property of their bid profile.

Theorem 1. Consider a GSP auction with m bidders and 2 slots. If all the bidders behave according to the wJSFP model, their bid profile will converge.

Proof. According to Lemma 1, for bidder i ($i > 2$), $\exists\, T > 0$, if $t > T$, $u_i(b^t) = 0$ and thus

$$u_i(b_i, \pi_{-i}(t)) = \tau_i^{0,t-1} u_i(b_i, \pi_{-i}(1)) + \sum_{s=1}^{T} \tau_i^{s,t-1} u_i(b_i, b_{-i}^s) = \tau_i^{T,t-1} C(T) \quad (8)$$

Thus, $b_i^t \in BR(\pi_{-i}(t+1))$. According to the bid update rule, b_i^t will remain unchanged. Then, we consider bidder 1 and bidder 2. Without loss of generality, we assume $\beta_1 = 1$, $\beta_2 = \beta$. Since the other bidder' bids remain unchanged, so does the price for slot 2, which is denoted as p_2. Thus, $u_1(b_1, b_2) = v_1 - b_2, if\ b_1 \geq b_2$, and $u_1(b_1, b_2) = (v_1 - p_2)\beta, if\ b_1 < b_2$. Now, we define the following two sequences based on $\{b_2^t\}$,

$$U_2^t = \begin{cases} b_2^t, & if\ b_2^t > (1-\beta)v_1 + \beta p_2 \\ v_1, & if\ b_2^t \leq (1-\beta)v_1 + \beta p_2 \end{cases} \qquad L_2^t = \begin{cases} b_2^t, & if\ b_2^t < (1-\beta)v_1 + \beta p_2 \\ p_2, & if\ b_2^t \geq (1-\beta)v_1 + \beta p_2. \end{cases} \quad (9)$$

Let $U = \liminf_{t\to\infty} U_2^t$ and $L = \limsup_{t\to\infty} L_2^t$. Thus $\exists\, T'$, if $t > T'$, $U_2^t \geq U > (1-\beta)v_1 + \beta p_2 > L \geq L_2^t$. Similar to the proof of Lemma 1, we have, $\max_{L \leq b_1 < U} u_1(b_1, \pi_{-1}(t)) > \max_{b_1 < L, or\ b_1 \geq U} u_1(b_1, \pi_{-1}(t))$, when t is sufficiently large. So $b_1^t \in [L, U)$ and remain unchanged according to the bid update rule, so does b_2^t. In this way, we have proven this theorem. ∎

The discussions on the case with multiple ad slots are more complicated. We give a sufficient condition that can guarantee the convergence of the bid profile in this case. Under this condition there is a unique class of Nash equilibrium, and each bidder i will get slot i for $i \in \{1, 2, \cdots, n\}$.

Theorem 2. In GSP, if $(v_i - v_{i+1})\beta_i > v_i\beta_{i+1}$, $\forall i \in \{1, 2, \cdots, n-1\}$, and all bidders behave according to the wJSFP model, their bid profile will converge.

Proof. We consider bidder 1. Since $(v_i - v_{i+1})\beta_i > v_i\beta_{i+1}$, if he/she wins slot 1, his/her utility $u_1 = (v_1 - p_1)\beta_1 \geq (v_1 - v_2)\beta_1 > v_1\beta_2 \geq (v_1 - p_k)\beta_k$. That is, slot 1 can bring the largest utility to bidder 1. Thus $b_1^t \geq \max_{i>1, s<t} b_1^s$ and is increasing when t is sufficiently large. So, $\lim_{t\to\infty} b_1^t$ exists, and remain unchanged since the bid is quantized. We could conduct similar analysis to other bidders, and come to the conclusion that the bid profile will converge. ∎

In the next theorem, we show that as long as the bid profile converges, it will converge to a Nash equilibrium of GSP.

Theorem 3. If the bid profile converges to $b \in X$ in GSP, with the wJSFP behaviors, b must be a Nash equilibrium of GSP.

Proof. Since the bids are quantized and the bid profile converges to b, $\exists\, T > 0$, s.t. all bidders' bids remain for any $t > T$. If b is not a Nash equilibrium of GSP, we could find a bidder i and bid $b_i' \neq b_i$ s.t. $u_i(b_i', b_{-i}) > u_i(b_i, b_{-i})$. We define $A(T) = max_{b_i \in X_i} \frac{1}{\gamma_i^T}(\tau_i^{0,T} u_i(b_i, \pi_{-i}(1)) + \sum_{s=1}^{T} \tau_i^{s,T} u_i(b_i, b_{-i}^s)) < \infty$ and $\varepsilon = u_i(b_i', b_{-i}) - u_i(b_i, b_{-i}) > 0$. When $t > T + \frac{A(T)}{\varepsilon} + 1$ we have:

$$u_i(b_i, \pi_{-i}(t)) \leq \tau_i^{T,t-1} A(T) + \sum_{s=T+1}^{t-1} \tau_i^{s,t-1} u_i(b_i, b_{-i}^s). \quad (10)$$

So $u_i(b_i, \pi_{-i}(t)) < \sum_{s=T+1}^{t-1} \tau_i^{s,t-1}(u_i(b_i, b_{-i}^s) + \varepsilon) \leq u_i(b_i^{'}, \pi_{-i}(t))$ for any $t > T + \frac{A(T)}{\varepsilon} + 1$. Since the bid of bidder i remains b_i for all $t > T$, we have $b_i \in BR(\pi_{-i}(t))$. This contradicts with $u_i(b_i, \pi_{-i}(t)) < u_i(b_i^{'}, \pi_{-i}(t))$. So b must be a Nash equilibrium of GSP. ■

4 Conclusions

In this paper, we have proposed a weighted joint strategy fictitious play model to describe bidders' behaviors in GSP auctions. In this model, bidders update their beliefs on other bidders' bid strategies by iteratively involving the information they receive. We have proven that with the proposed model, when there are only two ad slots, the bid profile of the bidders will definitely converge; when there are multiple ad slots, we give a sufficient condition that can guarantee the convergence of the bid profile. Furthermore, as long as the bid profile can converge, it will converge to a Nash equilibrium of GSP.

As for the future work, we plan to work on the following aspects. 1) We will conduct experiments on real data to verify the effectiveness of our proposed model. 2) We will investigate the necessary and sufficient condition for the convergence of the proposed model in the setting of multiple ad slots.

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Convergence of Best-Response Dynamics in Games with Conflicting Congestion Effects

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Abstract. We study the model of resource allocation games with conflicting congestion effects introduced by Feldman and Tamir (2012). In this model, an agent's cost consists of its resource's load (which increases with congestion) and its share in the resource's activation cost (which decreases with congestion). The current work studies the convergence rate of best-response dynamics (BRD) in the case of homogeneous agents. Even within this simple setting, interesting phenomena arise. We show that, in contrast to standard congestion games with identical jobs and resources, the convergence rate of BRD under conflicting congestion effects might be super-linear in the number of jobs. Nevertheless, a specific form of BRD is proposed, which is guaranteed to converge in linear time.

1 Introduction

Resource allocation is considered to be a fundamental problem in algorithmic game theory, and has naturally been the subject of intensive research within this field. Most of the game-theoretic literature on resource allocation settings emphasizes either the negative or the positive congestion effects on the individual cost of an agent. The former approach assumes that the cost of a resource is some non-decreasing function of its load. This literature includes job scheduling and routing models [11,20]. In these cases an individual user will attempt to avoid sharing its resource with others as much as possible. The second approach, in stark contrast, assumes that a resource's cost is a decreasing function of its load. This is the case, for example, in network design and cost sharing connection games, in which each resource has some *activation cost*, which should be covered by its users [2,6]. In these cases, an individual user wishes to share its resource with as many other users as possible in attempt to decrease its share in the cost.

In reality, most applications have cost functions that exhibit both negative and positive congestion effects. Accordingly, more practical models that integrate the two congestion effects into a unified cost function have been considered [1,9,15]. The present paper studies the resource allocation setting that is introduced by Feldman and Tamir [9], in which the individual cost of an agent is composed of two components, one that exhibits positive externalities, and the other that exhibits negative externalities. More specifically, every resource has some activation cost, that is shared among all the agents using it. The individual cost of

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an agent is the sum of its chosen resource's load (reflecting the negative externalities) and its share in the resource's activation cost (reflecting the positive externalities). This model is applicable to a large set of applications, including job scheduling, network routing, and network design settings.

The induced game, unlike its two "parent games," is not a potential game[1] when played by heterogeneous agents. Indeed, it has been shown in [9] that best-response dynamics (BRD) do not necessarily converge in this setting. Yet, in the special case where agents are identical, the induced game is a potential game; consequently, any BRD is guaranteed to converge to a Nash equilibrium [9]. The rate of the convergence, however, has been overlooked thus far. It is argued that the convergence rate is crucial for the Nash equilibrium hypothesis to hold; that is, it is more plausible that a Nash equilibrium will be reached if natural dynamics lead to such an outcome within a small number of moves.

In this paper, we study the convergence rate of BRD in a job scheduling game with conflicting congestion effects and identical agents.

1.1 Our Results

It is fairly easy to see that for unit-size jobs, convergence to a Nash equilibrium is linear in the number of jobs in both of the "parent" models; namely, if the the cost function equals the resource's load or if it equals the job's share in the resource's activation cost. We find that if the cost function takes both components into consideration, the convergence rate might be super-linear. We then introduce a specific form of BRD, referred as *max-cost*, where the job that incurs the highest cost is the one to perform its best move. The motivation behind this BRD is clear: the job that incurs the highest cost has the strongest incentive to improve its state. For *max-cost*-BRD, linear convergence rate is guaranteed. Due to space constraints, we defer some proofs to the full version [10].

1.2 Related Work

A lot of research has been conducted in the analysis of job-scheduling applications using a game-theoretic approach, where the jobs are owned by players who choose the machine to run on. The questions that are commonly analyzed under this approach are Nash equilibrium existence, the convergence of best-response dynamics to a Nash equilibrium, and the inefficiency of Nash equilibria (quantified mainly by the price of anarchy [16,18] and price of stability [2] measures).

It is well known that every congestion game is a potential game [19,17], and therefore admits a pure Nash equilibrium, and every best-response dynamics converges to a pure Nash equilibrium. However, the convergence time may, in general, be exponentially long [1,8,21]. This observation has led to a large amount of work that identified special classes of congestion games, where best-response dynamics converge to a Nash equilibrium in polynomial time or even linear time. This agenda has been the focus of [7,12] in a setting with negative congestion

[1] Potential games have been introduced by [17].

effects, and was also studied in a setting of positive congestion effects [2]. In particular, it has been shown that it takes at most n steps (where n is the number of users) to converge to a Nash equilibrium if the network is composed of parallel links [7], and this result has been later extended to extension-parallel networks [12]. For resource selection games (i.e., where feasible strategies are composed of singletons), it has been shown in [14] that better-response dynamics converge within at most mn^2 steps for general cost functions (where m and n are the number of resources and users, respectively). In addition to standard better- and best-response dynamics, a few variants have been explored. One example is the study of the convergence rate of α-Nash dynamics to an approximate Nash equilibrium [5] and to an approximate optimal solutions [3]. Also, the robustness of best-response convergence to altruistic agents has been studied in [13], where it has been shown that BRD may cycle as a result of altruism.

In this paper we study the congestion models with conflicting congestion effects introduced in [9] and studied also in [4]. This model can also be seen as a special case of the model introduced in [2], where the network is composed of parallel links and the setup cost is determined through the cost-sharing rule.

2 Model and Preliminaries

We consider a *job-scheduling* setting with identical machines and identical (unit-size) jobs. There is a set of machines $M = \{M_1, M_2, \dots\}$ of unlimited size,[2] each associated with an *activation cost*, B. An instance of our problem is given as a tuple (n, B), where n denotes the number of jobs. An assignment method produces an assignment $s = (s_1, \dots, s_n)$ of jobs into machines, where $s_j \in M$ denotes the machine to which job j is assigned. We use the terms assignment, schedule, and profile interchangeably. The load of a machine M_i in a schedule s, denoted $L_i(s)$, is the number of jobs assigned to M_i in s.

Given a job-scheduling setting and an activation cost B, a *job-scheduling game* is induced where the set of players is the set of jobs, and the action space of each player is the set of machines. The cost function of job j in a given schedule is the sum of two components: the load on j's machine and j's share in the machine's activation cost. It is assumed that the activation cost B is shared equally between all the jobs that use a particular machine. That is, given a profile s in which $s_j = M_i$, the cost of job j is $c_j(s) = L_i(s) + \frac{B}{L_i(s)}$. We denote the cost of a job that is assigned to a machine with load x by $c(x)$, where $c(x) = x + \frac{B}{x}$. It can be easily verified that the cost function exhibits the following structure.

Observation 1. *The function $c(x) = x + B/x$ for $x > 0$ attains its minimum at $x = \sqrt{B}$, is decreasing for $x \in (0, \sqrt{B})$, and increasing for $x > \sqrt{B}$.*

Practically, the input to the cost function is an integral value. If B is a perfect square, then the integral load achieving the minimal cost is exactly $\sqrt{B}$. For example, if $B = 100$, then being assigned to a machine with load 10 is optimal.

[2] In any instance, though, the number of machines will clearly be less than n.

In general, however, the optimal integral load (i.e., the load that minimizes the cost function) may be either $\left\lfloor\sqrt{B}\right\rfloor$ or $\left\lceil\sqrt{B}\right\rceil$, and for some values of B it may be both. For example, if $B = 12$ then both 3 and 4 are optimal loads, as $c(3) = c(4) = 12$. We denote an optimal load by $\ell^* = \ell^*(B)$. Assuming a unique integral optimal load, it is easy to verify that the cost function is decreasing for $x \in [1, \ell^*]$ and increasing for $x \geq \ell^*$. For two optimal integral loads, $\ell^* - 1$ and ℓ^*, the cost function is decreasing for $x \in [1, \ell^* - 1]$ and increasing for $x \geq \ell^*$.

An assignment $s \in S$ is a *pure Nash equilibrium* (NE) if no job $j \in N$ can benefit from unilaterally deviating from its machine to another machine (possibly a new machine). In our game, this implies that for every job j assigned to M_i and every $i' \neq i$, it holds that $c(L_i(s)) \leq \min(c(1), c(L_{i'}(s) + 1))$.

3 Convergence of Best-Response Dynamics

Best-Response Dynamics (BRD) is a local search method where in each step some player plays its best-response, given the strategies of the others. In systems where the agents always reach a Nash equilibrium after repeatedly performing improvement steps, the notion of a pure Nash equilibrium is well justified. This section explores the convergence rate of best-response dynamics into a pure NE.

In the general case, in which jobs have arbitrary lengths and the activation cost of a machine is shared by the jobs proportionally to their length, BRD is not guaranteed to converge to a NE [9]. In contrast, if the jobs are identical, then the induced game is equivalent to a congestion game with n resources [19]. One can easily verify that the function $\Phi(s) = \sum_i \left(B \cdot H_{\ell_i} + \frac{1}{2}\ell_i^2\right)$, where ℓ_i denotes the number of jobs on machine i, $H_0 = 0$, and $H_k = 1 + 1/2 + \ldots + 1/k$, is a potential function for the game. Convergence to a NE is guaranteed in potential games, but the convergence time might be exponential.

Here, we study the convergence time of BRD of unit-length jobs. We show that the convergence in general might take $\Omega(n \log \frac{n}{B})$ moves, and propose a specific BRD that ensures convergence within a linear number of moves. Specifically,

> **Max-cost BRD:** At every time step, a job that incurs the highest cost among those who can benefit from migration, is chosen to perform its best-response move (where ties are broken arbitrarily).

The analysis of the convergence rate of BRD and max-cost BRD (MC-BRD hereafter) is quite complicated and requires several preparations and terminology. Recall that all jobs assigned to a machine with load x incur the same cost $c(x) = x + B/x$. We denote by ℓ^* a load achieving minimal cost. By Observation 1, ℓ^* may be either $\left\lfloor\sqrt{B}\right\rfloor$ or $\left\lceil\sqrt{B}\right\rceil$, and for some values of B it may be both. For simplicity, throughout this section we assume a unique optimal load. All the results hold also for the case of two optimal loads, where minor straightforward modifications are required in the proofs.

We denote by ℓ_i^t the load of machine M_i at time t, i.e., *before* the migration of iteration t takes place. A machine that has load at least (respectively, smaller than) ℓ^* is said to be a *high* (*low*) machine.

We observe that if at some iteration a job migrates to a low machine, then in subsequent iterations that machine will attract more jobs up to load at least ℓ^*. Indeed, since $c(\ell+1) < c(\ell)$ for $\ell < \ell^*$, a low best-response machine continues to be a best response until it is filled up to load at least ℓ^*. Formally,

Observation 2. *If at some iteration t there is a migration to a low machine M_i such that $\ell_i^t = \ell^* - x$ for some $x > 0$, then the following $x-1$ iterations will involve migrations to M_i.*[3]

Properties of MC-BRD: By the design of the MC-BRD process and as a direct corollary of Observation 1, every migration in the MC-BRD process is from either the lowest or the highest machine into either the lowest-high or the highest-low machine (see Figure 3).

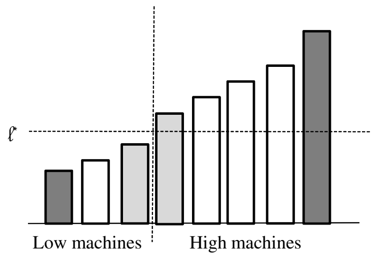

Fig. 1. MC-BRD process. Every migration is from one of the extreme machines into one of the middle grey machines.

Since all jobs on a particular machine share the same cost, the MC-BRD process can be described as if it acts on machines rather than on jobs. Specifically, in every iteration t, one job migrates from machine M_i to machine $M_k, k \neq i$, where (i) $c(\ell_k^t + 1)$ is minimal, (ii) $c(\ell_k^t + 1) < c(\ell_i^t)$, and (iii) $c(\ell_i^t)$ is maximal among all the machines from which a beneficial migration exists. While the MC-BRD process does not specify which job is migrating from M_i, for simplicity we will assume a LIFO (last in first out) job selection rule. Specifically, the job that entered M_i last is the one to migrate. If all jobs on M_i were assigned to it in the initial configuration, then an arbitrary job is selected. Since the BRD-process can be characterized by the load-vector of the machines in every time step, the number of iterations is independent of the job-selection rule. Consequently, our analysis of the convergence rate of MC-BRD applied with a LIFO job-selection rule is valid for any MC-BRD process.

Note that M_i, the machine from which a job is selected to migrate in iteration t, is not necessarily the machine for which $c(\ell_i^t)$ is maximal. For example, suppose that $B = 100$ and there are two active machines, a low one with load 3, and a high one with load 33. It is easy to verify that $c(4) < c(33) < c(3) < c(34)$. In this case, $c(3)$ is the maximal cost, but jobs on the low machine have no beneficial move (since $c(34) > c(3)$). On the other hand, jobs on the high machine wish to

[3] It is possible that the system reaches a NE and the BRD process terminates before $x-1$ iterations are performed.

migrate to the low one (since $c(4) < c(33)$). Thus, the high machine is the one selected by MC-BRD to perform a migration, although the low machine is the one incurring max-cost. Clearly, such a case can only occur if the machine that incurs the max-cost is itself the best-response machines, as summarized in the following observation.

Observation 3. *If at time t the machine M_i that incurs max-cost is not the one from which a job is selected to migrate in MC-BRD, then $c(\ell_i^t + 1)$ is the best-response, in particular, this implies that M_i is low.*

We next observe that in MC-BRD, if at some iteration a job leaves some low machine, then in the following iterations all the jobs assigned to that machine leave it one by one until the machine empties out. To see this, note that $c(\ell - 1) > c(\ell)$ for $\ell < \ell^*$; thus, if a low machine incurs the highest cost, it continues to incur the highest cost after its load decreases. It remains to show that if a beneficial move out of M_i exists when it has load $\ell < \ell^*$, then it is also beneficial to leave M_i when it has load $\ell - 1$. This is ensured by Observation 3. Specifically, if it is not beneficial, then $c(\ell)$ is the cost of the best-response machine. But this is impossible since $c(\ell)$ was the max-cost in the previous iteration.

Observation 4. *If at some iteration t there is a migration from a low machine M_i such that $\ell_i^t = \ell^* - x$ for some $x > 0$, then the following $\ell^* - x - 1$ iterations will involve migrations from M_i.*

We are now ready to state the bound on the convergence rate of MC-BRD. As shown in the full version [10], the following bound is almost tight.

Theorem 1. *For every job scheduling game with identical jobs, every MC-BRD process converges to a NE within at most $\max\{\frac{3n}{2} - 3, n - 1\}$ steps.*

In contrast to MC-BRD, the convergence time of arbitrary BRD, might not be linear in n.

Theorem 2. *There exists a job scheduling game with identical jobs and a BRD process such that the convergence time to a NE is $\Omega(n \log \frac{n}{B})$.*

While the convergence rate of general BRD is super-linear, the following theorem establishes an upper bound of n^2. Closing the gap remains open.

Theorem 3. *For every job scheduling game with identical jobs, every BRD process converges to a NE within at most n^2 steps.*

It is interesting to compare our results to those established for the standard model that considers only the negative congestion effects (i.e., where a job's cost is simply the load of its chosen machine). It has been shown by [7] that if the order of the jobs performing their best-response moves is determined according to their lengths (i.e., longer job first), then best-response dynamics reaches a pure Nash equilibrium within at most n improvement steps. In contrast, if the jobs move in an arbitrary order, then convergence to a Nash equilibrium might take an exponential number of steps. These results imply that for the special case of equal-length jobs, convergence occurs within at most n steps. Our results provide evidence that when there are conflicting congestion effects, it might take longer to reach a Nash equilibrium. Nevertheless, for the special case of max-cost BRD, the consideration of positive congestion effects (through activation costs) does not lead to a longer convergence time.

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Forming Networks of Strategic Agents with Desired Topologies

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Abstract. Many networks such as social networks and organizational networks in global companies consist of self-interested agents. The topology of these networks often plays a crucial role in important tasks such as information diffusion and information extraction. Consequently, growing a stable network having a certain topology is of interest. Motivated by this, we study the following important problem: given a certain desired network topology, under what conditions would best response (link addition/deletion) strategies played by self-interested agents lead to formation of a stable network having that topology. We study this interesting reverse engineering problem by proposing a natural model of recursive network formation and a utility model that captures many key features. Based on this model, we analyze relevant network topologies and derive a set of sufficient conditions under which these topologies emerge as pairwise stable networks, wherein no node wants to delete any of its links and no two nodes would want to create a link between them.

Keywords: Social Networks, Network Formation, Pairwise Stability, Network Topology, Strategic Agents.

1 Introduction

In a social network, individuals gain certain benefits from other individuals and at the same time, pay a certain cost for maintaining links with their friends. Owing to the tension between benefits and costs, self-interested or rational nodes think strategically while choosing their immediate neighbors. A stable network that forms out of this process will have a topological structure as dictated by the individual utilities and best response strategies of the nodes.

Often, stakeholders such as a social network owner or a social planner, who work with the networks so formed, would like the network to have a certain desirable topology to accomplish certain tasks. Typical examples of these tasks include enabling optimal communication among nodes for maximum efficiency (knowledge management), extracting certain critical information from the nodes (information extraction), broadcasting some information to the nodes (information diffusion), etc. If a particular topology is the most appropriate for the set of tasks to be handled, it would be useful to orchestrate network formation in a way that the required topology emerges as a stable configuration as a culmination of the network formation process.

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1.1 Motivation

One of the key problems addressed in the literature on social network formation is: given a set of self-interested nodes and a model of social network formation, which topologies would be stable and which would be efficient (maximizing sum of utilities of all nodes). In this paper, our focus is on the inverse problem, namely, given a certain desired topology, under what conditions would best response strategies played by self-interested agents lead to formation of a stable network with that topology. We motivate this problem with some relevant topologies.

Consider a network where there is a need to rapidly spread some crucial information received by any of the nodes, requiring precautions against link failures. In such cases, a complete network is ideal. Consider a different scenario where the information needs to be spread rapidly, however there needs to be a moderator to verify the authenticity of the information before spreading it to the other nodes in the network (for example, it could be a rumor). Here a star network is desirable. Consider a generalization of the star network where there is a need for decentralization for efficiently controlling information in the network. It has multiple centers, each linked to every other, and the leaf nodes are divided among the centers as evenly as possible. We call it, k-star network. Consider a necessity of having two sections where some or all members of a section receive certain information simultaneously and there is a need to forward it to the other section, taking care of link failures. Moreover, it is desirable to not have intra-section links to save on resources. A bipartite Turán network is ideal in this case as both communities are practically desirable to be of nearly equal size.

It is clear that depending on the tasks for which the network is used, a certain topology might be better than others. This provides the motivation for our work.

1.2 Relevant Work

Jackson [5] reviews several models of network formation in the literature. Watts [10] provides a sequential move game model where nodes are myopic; however, the resulting network is based on the ordering in which links are altered and so it is unclear which networks emerge [6]. Hummon [4] uses simulations to explore the dynamics of network evolution. Doreian [2] analytically arrives at specific networks that are pairwise stable; but its complexity increases exponentially with the number of nodes and so the analysis is limited to only five nodes.

There have been a few approaches to design incentives for nodes so that the resulting network is efficient [8,11]. Though it is often assumed that the welfare of a network is based only on its efficiency, there are many situations where this may not be true. A particular network may not be efficient in itself, but it may be desirable for reasons external to the network, as explained in Section 1.1.

The models of social network formation in literature assume that all nodes are present throughout the evolution of a network, which allows nodes to form links that may not be consistent with the desired network. Furthermore, with all nodes present in an unorganized network, a random ordering over them in sequential network formation models adds to the complexity of analysis. However, in most

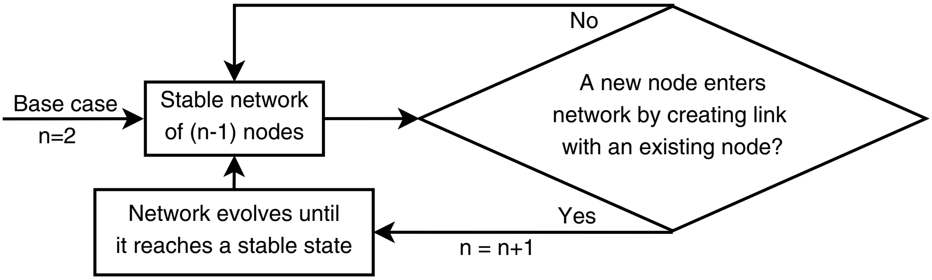

Fig. 1. Proposed model of network formation

social networks, not all nodes are present from beginning itself. A network starts building up from a few nodes and gradually grows to its capacity.

1.3 Contributions of the Paper

- We propose a recursive model of network formation using which it is possible to guarantee that the network retains its topology in each of its stable states; also the analysis can be carried out independent of the current number of nodes in the network. We also propose a utility model that captures many key features, including an *entry fee* for entering the network.
- We derive sufficient conditions under which star network, complete network, bipartite Turán network, and k-star network, emerge as pairwise stable.

2 A Recursive Model of Network Formation

The game is played amongst self-interested nodes, which we consider to be all homogeneous and having global knowledge of the network[1]. Each node, which gets to make a move, has a set of strategies at any given time and it chooses its myopic best response strategy[2]. A strategy can be of one of the three types, namely (a) creating a link with a node that is not its immediate neighbor with its consent, (b) deleting a link with an immediate neighbor without its consent, or (c) maintaining status quo. Moreover, consistent with the notion of pairwise stability, if a node gets to make a move, and proposing or deleting a link does not strictly increase its utility, then it prefers not to do so. But a node will accept a link proposed by some other node provided its utility does not decrease.

The game starts with one node and the process goes on as depicted in Figure 1. Now given that a stable network of $n-1$ nodes is formed, the n^{th} node considers entering the network. We make an intuitive assumption that in order to be a part of the network, the n^{th} node has to propose a link with one of the existing nodes and not vice versa. For successful link creation, utility of the latter should not decrease. After the new node enters the network, nodes who get to make

[1] As assumed in most of the literature on social network formation [6].
[2] The assumption of nodes behaving myopically has experimental justifications [9].

Table 1. Notation for the proposed utility model

N	set of nodes present in the network
u_j	net utility that node j gets from the network
d_j	degree of node j
b_i	benefits obtained from a node at distance i (where $b_{i+1} < b_i$)
c	cost incurred in maintaining link with an immediate neighbor
$l(j, w)$	shortest path distance between nodes j and w
$E(j, w)$	set of nodes essential to connect j and w
γ	fraction of indirect benefits paid to the corresponding set of essential nodes
c_0	network entry factor (see Section 2.1)
$\mathrm{T}(j)$	existing node in the network to which node j connects to enter the network
$\mathbf{I}_{\{j=\mathrm{NE}\}}$	1 when j is the newly entering node about to create its first link, else 0

their move are chosen at random at all time and the network evolves until it becomes a stable network consisting of n nodes. Following this, a new $(n+1)^{th}$ node considers entering the network and the process goes on recursively[3].

2.1 Utility Model

Our utility model takes the idea of essential nodes proposed by Goyal and Vega-Redondo [3]. A node j is said to be essential for y and z if j lies on every path that joins y and z in the network. Any two nodes pay a fraction of the benefits obtained from each other, as intermediation rents in the form of additional favors or monetary transfers to the corresponding set of essential nodes[4]. This fraction is assumed to be equally divided among the essential nodes connecting that pair. Thus, nodes get bridging benefits for being an essential node for each such pair.

We introduce a notion of network entry fee which corresponds to some cost a node has to bear in order to be a part of the network. If a newly entering node wants its first connection to be with an existing node of high importance or degree, then it has to spend more time or effort. So we assume the entry fee that the former pays to be an increasing function of the degree of the latter, say d_T. For simplicity of analysis, we assume the fee to be directly proportional to d_T and call the proportionality constant, network entry factor c_0.

Table 1 enlists the notation used in the paper. For a node j, the utility function is a function of the network, that is $u_j : g \to \mathbb{R}$ and is given by

[3] The assumption that a node considers entering the network only when it is pairwise stable might seem artificial in general social networks, but can be justified in organizational networks where entry of nodes can be controlled by an administrator.

[4] In order to avoid discrete constraints on rents, such as summation of the fractions paid to be less than one, we assume that irrespective of the number of essential nodes (provided positive) connecting y and z, they lose the same fraction $\gamma \in [0, 1)$. As a result, the real producers of benefits are guaranteed at least $(1 - \gamma)$ fraction of it.

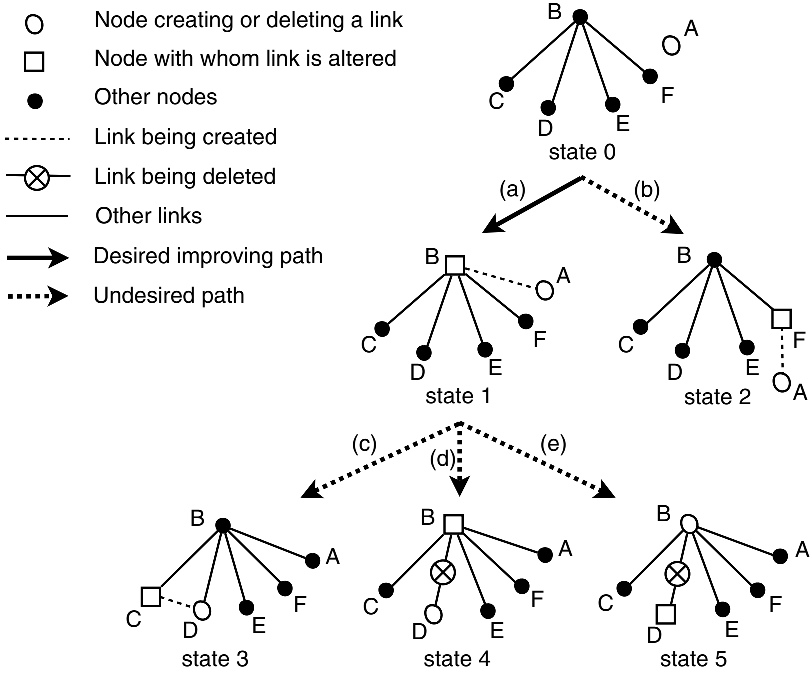

Fig. 2. Directing Network Evolution for the Formation of Star Topology

$$u_j = -c_0 d_{\mathrm{T}(j)} \mathbf{I}_{\{j=\mathrm{NE}\}} + d_j b_1 - d_j c + \sum_{\substack{w \in N \\ l(j,w)>1}} b_{l(j,w)} - \sum_{\substack{w \in N \\ E(j,w) \neq \phi}} \gamma b_{l(j,w)} + \sum_{\substack{y,z \in N \\ j \in E(y,z)}} \left(\frac{\gamma}{|E(y,z)|} \right) 2 b_{l(y,z)} \quad (1)$$

The individual terms of Equation (1) represent (a) network entry fee, (b) benefits from immediate neighbors, (c) costs of maintaining links with immediate neighbors, (d) benefits from indirect neighbors, (e) intermediation rents paid, and (f) bridging benefits, respectively.

2.2 Directing Network Evolution

We consider the sequential move game model and so the process of network evolution can be represented as a game tree. The entry of each node in the network results in a game tree. An *improving path* is a sequence of networks, where each transition is obtained by either two nodes choosing to add a link or one node choosing to delete a link [7]. Thus, a pairwise stable network is one from which there is no improving path leaving it. Hence, our objective is to direct the network evolution along a desired improving path in the game tree.

The procedure for deriving sufficient conditions for the formation of a given topology is similar to *mathematical induction*. Consider a base case network with very few nodes (two in our analysis). We derive conditions so that the network formed with these few nodes has the desired topology. Then using induction, we assume that a network with $n-1$ nodes has the desired topology, and derive conditions so that, the network with n nodes, also has that topology.

In Figure 2, we direct the network evolution by imposing a set of conditions ensuring that the resulting pairwise stable network is a star. Let $u_j(s)$ be the utility of node j when the network is in state s and let $leaf \in \{C, D, E, F\}$. As all leaf nodes are equivalent up to relabeling, considering utility of one such node is sufficient. The conditions sufficient to direct the network evolution along the desired improving path and avoid any undesired paths (be they improving or not) are (a) $u_A(1) > u_A(0)$ and $u_B(1) \geq u_B(0)$, (b) $u_A(1) > u_A(2)$ or $u_{leaf}(2) < u_{leaf}(0)$, (c) $u_{leaf}(1) \geq u_{leaf}(3)$, (d) $u_{leaf}(1) \geq u_{leaf}(4)$, and (e) $u_B(1) \geq u_B(5)$.

3 Sufficient Conditions for Relevant Topologies

In this section, we provide sufficient conditions for the formation of relevant topologies. We use Equation (1) for mathematically deriving these conditions. For the proofs, the reader is referred to the full version of this paper [1]. It also shows that with the derived sufficient conditions, star network and complete network are efficient, and for sufficiently large number of nodes, efficiencies of bipartite Turán network and k-star network are respectively, half and $\frac{1}{k}$ of that of the efficient network in the worst case and the networks are close to being efficient in the best case.

Theorem 1. *For a network, if* $b_1 - b_2 + \gamma b_2 \leq c < b_1$ *and* $c_0 < (1-\gamma)(b_2 - b_3)$*, the resulting topology is a star graph.*

Theorem 2. *For a network, if* $c < b_1 - b_2$ *and* $c_0 \leq (1-\gamma) b_2$*, the resulting topology is a complete graph.*

Theorem 3. *For a network with* $\gamma < \frac{b_2 - b_3}{3b_2 - b_3}$*, if* $b_1 - b_2 + \gamma(3b_2 - b_3) < c < b_1 - b_3$ *and* $(1-\gamma)(b_2 - b_3) < c_0 \leq (1-\gamma) b_2$*, the resulting topology is a bipartite Turán graph.*

In the case of certain topologies, under a given utility model, the conditions required for its formation on discretely small number of nodes, are inconsistent with that required on arbitrarily large number of nodes. Under the proposed utility model, k-star ($k \geq 3$) is one such topology [1]. A possible and reasonable solution to overcome this problem is to analyze the network formation process, starting with a graph that overcomes the conditions required for discretely small number of nodes. This graph can be obtained by some other method, one of which could be providing additional incentives to the nodes of this graph.

Theorem 4. *For a network starting with complete network on k centers ($k \geq 3$) with the centers connecting to one leaf node each, and $\gamma = 0$, if $c = b_1 - b_3$ and $b_2 - b_3 < c_0 < b_2 - b_4$, the resulting topology is a k-star graph.*

The value of c_0 lays the foundation for the degree distribution in a network as it dictates the first connection of a newly entering node. For instance, the values of c_0 in Theorems 1, 3 and 4 ensure that the degree of the first connection of a newly entering node is high, low, and intermediate, respectively. Furthermore, the constraints on γ arise owing to contrasting natures of connectivity in a network. For instance, in a bipartite Turán network, nodes from different partitions are densely connected with each other, while that from the same partition are not connected at all. Similarly, in a k-star network, there is an extreme contrast in the densities of connections (dense amongst centers and sparse for leaf nodes).

4 Discussion and Future Work

We proposed a model of recursive network formation where nodes enter a network sequentially, thus triggering evolution of the network each time a new node enters. Though we have assumed a sequential move game model with myopic nodes and pairwise stability as the solution concept, the model, as depicted in Figure 1, is independent of the model of network evolution, the solution concept used for equilibrium state, and also the utility model. The recursive nature of our model enabled us to directly analyze the network formation game using an elegant induction based technique. We derived sufficient conditions for relevant topologies by directing the network evolution along a desired improving path in the sequential move game tree.

It would be interesting to design incentives such that agents in a network comply with the derived sufficient conditions. Our analysis ensures that irrespective of the chosen node at any point in time, the network evolution is directed as desired. A possible solution for simplifying the analysis for more involved topologies is to carry out probabilistic analysis for deriving conditions so that a network has the desired topology with high probability. Another interesting direction, from a practical viewpoint, is to study the problem of forming networks where the topology need not be exactly the one which is ideally desirable, for example, a near-k-star network instead of a precise k-star.

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Homophily in Online Social Networks

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Abstract. We develop a parsimonious and tractable dynamic social network formation model in which agents interact in overlapping social groups. The model allows us to analyse network properties and homophily patterns simultaneously. We derive analytical expressions for the distributions of degree and, importantly, of homophily indices, using mean-field approximations. We test our model using a large dataset from Facebook covering student friendship networks in 10 American colleges in 2005. We find that our analytical expressions and simulations fit the homophily patterns, degree distribution, and individual clustering coefficients well with the data.

1 Introduction

Friendships are an essential part of economic life and social networks affect many areas of public policy. In many social network formation models in the economics literature agents are anonymous and the network structure depends entirely on the formation process. Yet we can think of numerous examples, such as information transmission, peer-to-peer lending, or sexual contacts, which suggest that the network topology is not only explained by the network formation process, but also by node characteristics.

We develop a dynamic network formation model that uses information on node characteristics to explain friendship patterns in online social networks and we test it against the data on Facebook networks in American colleges. In our model, agents spend time interacting with others across various social categories, such as attending lectures and spending time in their dorm. Naturally, the time allocation could be established institutionally by timetables or geographical proximity. The time allocation determines who agents are likely to meet and with whom they document their resulting friendship on Facebook. Our parsimonious model has only three parameters and is simple enough to allow us to derive analytic solutions for structural properties of the network. Conceptually, the model is related to affiliation networks introduced by [1]. However, these models typically contain a large number of parameters and most, such as [2,3,4] rely entirely on simulations.

A particular focus of this paper is homophily – the tendency of individuals to associate with those similar to themselves – which has been well documented in sociology [5]. [6] make it clear that the observed racial homophily patterns in American high schools do not necessarily arise from an exogenous bias in preferences towards people of the same race. In our model, we do not assume that agents have any preference bias. Rather the entire process is governed by the allocation of time and by the relative size

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of the social groups in which agents interact. Homophily therefore emerges purely from the correlations in agents' likelihood of interaction in similar social groups.

The empirical part of this paper provides striking support for our model. Using the analytical expressions, we find the best-fitting parameter values, which determine the allocation of time across social categories, for ten separate Facebook networks. Students' friendships reveal that they spend more time socialising in class than in their dorms. Despite its parsimony, the model closely matches the empirical degree and homophily distributions in gender and year at the best-fitting parameter values. Remarkably, the simulations run at these values show that the individual clustering distributions also match the empirical clustering patterns.

2 Model

2.1 Characteristics of Agents

Let $\mathscr{K} = [K^0, \dots, K^R]$ be a finite ordered list of social categories. An element K^r is the r^{th} category and $k \in K^r$ is a characteristic within that category. Let $\mathscr{R} = \{0, 1, \dots, R\}$. Every agent $i \in N$ is represented by a vector $\mathbf{k}_i = (k_i^0, \dots, k_i^R)$ of characteristics, where for each $r \in \mathscr{R}$, $k_i^r \in K^r$. For any pair $i, j \in N$, let $k_i^0 = k_j^0$.[1] For each $r \in \mathscr{R}$, define a social group $\gamma_i^r = \{j \in N | k_i^r = k_j^r\} \setminus \{i\}$, which is the set of all agents (other than i) that share the characteristic k_i^r within the social category r with i. Note that $\gamma_i^0 = N \setminus \{i\}$. Finally, for each non-empty subset of social category indices $S \subseteq \mathscr{R}$, define

$$\pi_i(S) = \bigcap_{r \in S} \gamma_i^r \setminus \bigcup_{r \in R \setminus (S \cup \{0\})} \gamma_i^r \ , \tag{1}$$

which induces a partition $\Pi_i = \{\pi_i(S) | S \subseteq \mathscr{R}, S \neq \emptyset\}$ on $N \setminus \{i\}$.[2] Therefore, $\pi_i(S)$ is the set of agents (other than i) that share *only* the characteristics within the set of categories indexed by S with i.

Example 1. In a university context, we could have

$$\mathscr{K} = [K^0, K^1, K^2, K^3, K^4] = [student, class, dorm, gender, year\ of\ graduation] \ .$$

All agents are students ($k_i^0 = k_j^0$ for all $i, j \in N$). $K^1 \in \mathscr{K}$, which represents class, can include $k \in \{maths, literature, biology\}$. Suppose, that agent i is represented by a vector $\mathbf{k}_i = (student, maths, campus, female, 2006)$. Let us consider $S = \{1, 3\}$. γ_i^1 is the set of all maths students other than i and γ_i^3 is the set of all female students other than i. Therefore, $\pi_i(S)$ is the set of female maths students, who do not live on campus and are of a different graduating year than i. $\pi_i(\{0\})$ would be the set of all male non-mathematicians, who do not live on campus and are of a different graduating year than i. Π_i represents the partition into disjoint sets of students, who share exactly 1, 2, 3, 4 or 5 social categories with i.

[1] This does not restrict the characteristics space in any way. The zeroth category, which greatly simplifies notation, is one in which all agents share the same characteristic.

[2] Note that $\pi_i(S) = \pi_i(S \cup \{0\})$ for all non-empty $S \subseteq \mathscr{R}$. Furthermore, since $\gamma_i^r = \cup_{\pi \in \{\pi_i(S) | r \in S\}} \pi$, a social group is a union of disjoint partition elements.

2.2 Network Formation Process

We model our network as a simple undirected graph with a finite set of nodes N (which represent agents), a finite set of edges (which represent friendships), and no self-loops. The degree of an agent is the number of the agent's friends. At time period $t = 0$ all agents are active and have no friends. Let $\mathbf{q} = (q^0, ..., q^R)$ and $\sum_{r \in \mathscr{R}} q^r = 1$. In each period $t \in \{1, 2, 3 \ldots\}$, an active agent interacts with agents in the social group γ_i^r with probability $q^r \geq 0$. We can thus interpret q^r as the proportion of time in period t that agent i spends with agents in the social group γ_i^r (one can think of $\gamma_i^0 = N \backslash \{i\}$ as the social group that i interacts with during i's "free time"). During the interaction in a social group, the agent is linked uniformly at random to another active agent in that group with whom the agent is not yet a friend. If the agent is already linked to every other active agent in that social group, the agent makes no friends in that period. Friendships are always reciprocal, so all links are undirected. Finally, in every period, an agent remains active with a given probability $p \in (0, 1)$ until the following period and becomes inactive with probability $1 - p$. If the agent i becomes inactive, i retains all friendships, but can no longer form any links with other agents in all subsequent periods.

There must be reasons, *other than having linked with every user in the network*, for why people stop adding new friends online: losing interest, finding an alternative online social network, reaching a cognitive capacity for social interaction, and so on. Including all these explanations would require a much richer model, so we simply capture them as a random process with the inactivity probability $1 - p$.

We are interested in how the agents' degrees change over time. Let us call $d_i(t)$ the expected degree of agent i in period t. We analyse a mean-field approximation to this dynamic system. This technique is commonly used in statistical mechanics in order to simplify many-body systems. Essentially, it assumes that the realisation of any random variable in any time period is its expected value. Hence, we chose to approximate our model by a discrete-time system, which changes deterministically at the rate proportional to the expected change (see [7,8]).

The probability with which agent i interacts with an agent from $\pi_i(S)$ is given by

$$q^{\pi_i(S)} = |\pi_i(S)| \left[\sum_{r \in S \cup \{0\}} \frac{q^r}{|\gamma_i^r|} \right] . \tag{2}$$

Indeed, with probability q^r, an agent is assigned to social group γ_i^r, and the probability that he meets an agent in $\pi_i(S) \subseteq \gamma_i^r$ is given by $\frac{|\pi_i(S)|}{|\gamma_i^r|}$. Note that $\sum_{\pi \in \Pi_i} q^\pi = 1$.

For every $\pi \in \Pi_i$, let $R^\pi(t)$ be the number of remaining active agents in π at t (other than i) with whom i is not yet linked. Furthermore, recall that an agent makes a link in every period and on average receives a link with probability $\frac{1}{R^\pi(t)}$ from each of the $R^\pi(t)$ agents (in each π weighted by q^π). Since i interacts with agents in π with probability q^π, i makes $2q^\pi$ links with agents in π in every period until T^π – the expected number of periods for i to form links with *every* agent in π. We find T^π by solving

$$R^\pi(t+1) = p\left[R^\pi(t) - 2q^\pi\right] . \tag{3}$$

This difference equation states that $R^{\pi}(t+1)$ is the number of agents who remain active in π out of $R^{\pi}(t)$ less the number of agents that i links with in π at t. Solving for $R^{\pi}(t)$ with initial condition $R^{\pi}(0) = |\pi|$ and setting $R^{\pi}(T^{\pi}) = 0$ gives us

$$T^{\pi} = \frac{\ln\left(\frac{2q^{\pi}p}{2q^{\pi}p+(1-p)|\pi|}\right)}{\ln(p)} \text{ (except if } q^{\pi} = 0 \text{ then } T^{\pi} = 0) \ . \quad (4)$$

This allows us to obtain the expected degree of agent i at time t

$$d_i(t) = \sum_{\pi\in\Pi_i} d_i^{\pi}(t) = \sum_{\pi\in\Pi_i} 2q^{\pi}\left[t\mathbf{1}(t \le T^{\pi}) + T^{\pi}\mathbf{1}(t > T^{\pi})\right] \ , \quad (5)$$

where $d_i^{\pi}(t)$ is the expected number of link i has with agents in $\pi \in \Pi_i$ in period t. Note that $d_i(t)$ is concave, piecewise linear, and strictly increasing in the range $[0, \max_{\pi\in\Pi_i}\{T^{\pi}\}]$. Hence, active agents make friends at a decreasing rate over time. Since an agent remains active exactly x periods with probability $p^x(1-p)$, we have that $\Pr(t \le x) = \sum_{t=0}^{t=x} p^t(1-p) = 1 - p^{x+1}$. Therefore, the probability that node i has degree at most d is given by $G_i(d) \equiv \Pr(d_i(t) \le d) = \Pr(t \le t_i(d)) = 1 - p^{t_i(d)+1}$, where

$$t_i(d) \equiv d_i^{-1}(d) = \frac{d - \sum_{\pi\in\Pi_i} 2q^{\pi}T^{\pi}\mathbf{1}(d > d_i(T^{\pi}))}{\sum_{\pi\in\Pi_i} 2q^{\pi}\mathbf{1}(d \le d_i(T^{\pi}))} \ . \quad (6)$$

Finally, the overall average degree distribution is $G(d) = \frac{1}{|N|}\sum_{i\in N} G_i(d)$.

2.3 Homophily

Homophily captures the tendency of agents to form links with those similar to themselves. Let $\Pi_i^r = \{\pi_i(S) \in \Pi_i | r \in S\}$ be the set of partition elements containing agents that share the characteristic k_i^r in category r with i. The individual homophily index in social category r of agent i in period t is defined as

$$H_i^r(t) = \frac{\text{number of friends of } i \text{ at } t \text{ that share } k_i^r}{\text{number of friends of } i \text{ at } t} = \frac{\sum_{\pi\in\Pi_i^r} d_i^{\pi}(t)}{d_i(t)} \ . \quad (7)$$

This is a standard definition from which we can easily recover various other definitions of homophily given in [6]. Finally, it will be useful to define a composition function $h_i^r(d) \equiv (H_i^r \circ t_i)(d)$, which expresses individual homophily as a function of degree rather than as a function of time.

2.4 Test of the Mean-Field Approximation

Since we used a mean-field method to derive the analytical expressions, we must test the accuracy of its approximations against simulations [9]. We did this for degree distributions and the individual homophily distribution against an average of 100 runs of the simulation for multiple parameter values. In general, the fits were good. An example is illustrated in Fig. 2.[3]

[3] There is some loss of accuracy at extreme values of the cumulative distribution of the individual homophily index: (7) makes it clear that the individual homophily index is unlikely to be near 0 or 1. Yet the mean-field approximation of the *average* is good.

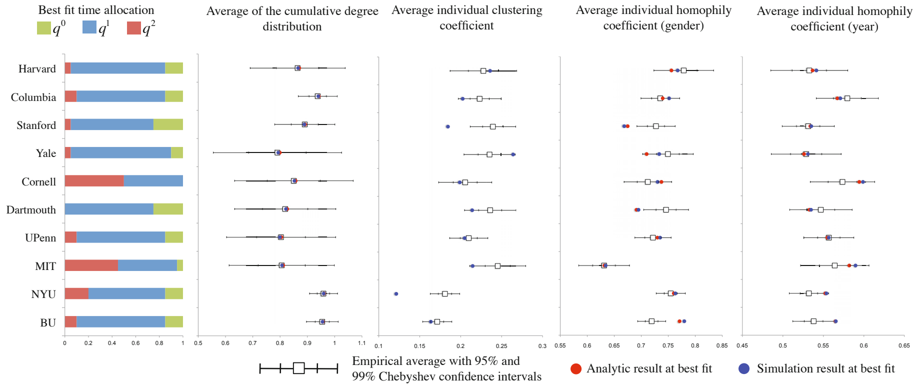

Fig. 1. Results for all colleges

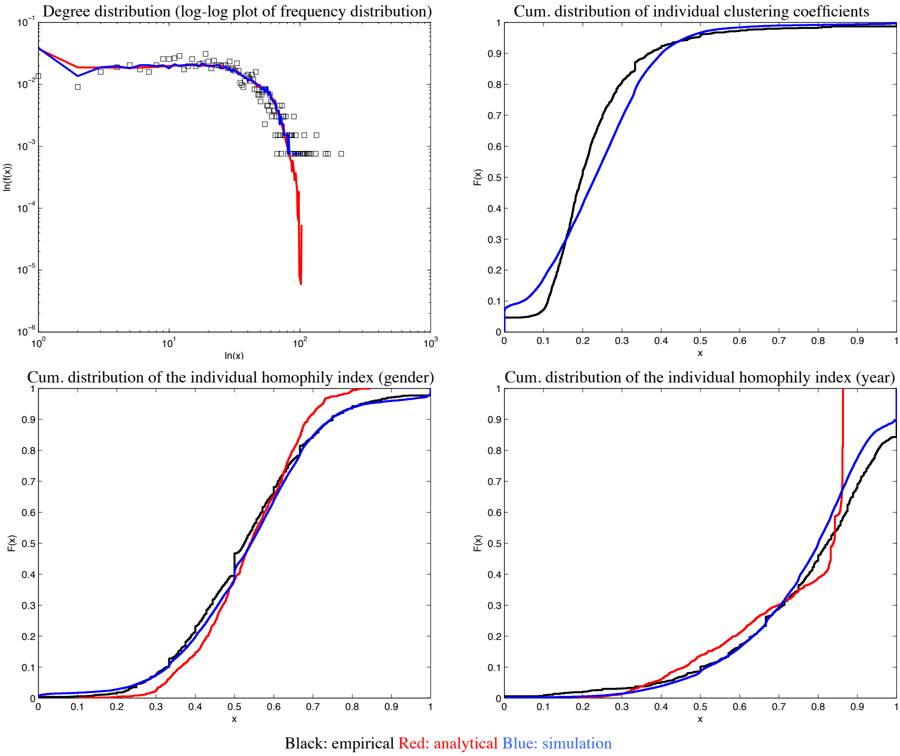

Fig. 2. Detailed results for Harvard University

3 Data

We use the September 2005 cross-section of the complete structures of social connections on www.facebook.com within (but not across) the first ten American colleges that joined Facebook (see [10]). We observe six social categories for each user: gender, year of graduation, major, minor, dorm, and high school. Since all personal data were provided voluntarily, some users did not submit all their information. We dropped any user (and their links), who has not provided all the personal characteristics other than high school. We therefore look only at students graduating between 2006 and 2009, who have supplied all the relevant personal characteristics (except high school).

4 Empirical Strategy

We test our model against the data using the social categories identified in the Example 1. Using the available information in our dataset, we define agents i and j to be in the same *class* if they are in the same year and major or in the same year and minor. We assume that every agent i interacts in i's class and dorm with respective probabilities q^1 and q^2. The probabilities of interacting with the gender and year social categories are set to zero ($q^3 = q^4 = 0$) since it is unreasonable to suppose that agents allocate time *specifically* to interacting with agents in these categories. Meeting agents of the same gender or year happens only through the interactions in the other social groups. Finally, $q^0 = 1 - q^1 - q^2$ is the proportion of time spent interacting with all other agents (their "free" time). Hence, the model has 4 parameters and 3 degrees of freedom.

We focus on explaining empirical homophily patterns in gender and year of graduation. Measuring homophily in these social categories is appropriate because gender and year of graduation are entirely immutable agent categories: unlike class and dorm, there is no feedback loop between social category membership and homophily.

4.1 Fitting the Model to Data

In order to fit the model to the data (degree distribution and homophily), we used a grid search on parameters q^0, q^1, q^2, and p.[4] For the degree distribution, we computed the analytical degree distribution, and, for homophily, we found the analytical homophily index in gender and year as a function of i's empirical degree at each point in the grid. We then found the values q^0, q^1, q^2, and p that minimise an intuitive loss function, which measures the "overall error" of the fit by taking the product of the normalised sums of squared distances between the analytical and the empirical distributions for degree and homophily in gender and year at each point in the grid.

4.2 Results

For each college, we ran 100 simulations at its best-fitting values of q^0, q^1, q^2, and p.[5] Figure 1 presents results for all colleges showing that our model closely matches

[4] For q^0, q^1 and q^2 we took values from 0 to 1 in steps of 0.05. For p, we took values from 0.9 to 0.9975 in steps of 0.0025.

[5] The results shown are averages over the 100 runs.

average degree, average homophily, and the average individual clustering coefficient (see [9, p. 35] for a standard definition).[6] Unsurprisingly, students spend most of their time interacting with others in their class. Interestingly, q^0 is small, which suggests that friendship patterns are far from random. Figure 2 shows the empirical, analytical, and simulated degree, homophily (in gender and year), and individual clustering distributions for Harvard University. These fits are representative of the other colleges.

5 Conclusions

We presented a network formation model, which provides rich microfoundations for the macroscopic properties of online social networks. The friendship and homophily patterns generated by the model find good support in data. We were also able to estimate how much time agents spend in particular social groups. There is still scope for further theoretical work, including finding accurate analytical approximations to the clustering measures and diameter.

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[6] In order to avoid making any assumptions about the distributions, we estimated standard errors around the empirical averages non-parametrically. Figure 1 therefore represents the Chebyshev confidence intervals at the 95% and 99% levels. Note that clustering appears to fit relatively well even though it did not appear in our loss function.

Limited Supply Online Auctions for Revenue Maximization*

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Abstract. We study online truthful auctions for prior-free revenue maximization, to sell copies of a good in limited supply to unit-demand bidders. The model is reminiscent of the secretary problem, in that the order of the bidders' arrival is chosen uniformly at random. For two variants of limited supply, a hard constraint of k available copies and a production cost per copy given by a convex curve, we reduce the problem algorithmically to the unlimited supply case. For the case of k available copies we obtain a $26e$-competitive auction, which improves upon a previously known ratio from [Hajiaghayi, Kleinberg, Parkes, ACM EC 2004].

1 Introduction

We study truthful online auctions for prior-free revenue maximization in the context of limited supply of a single good, in a model studied previously in [5,7]. This model is a blend of the *Secretary Problem* and the framework of *Competitive Auctions* proposed by Goldberg *et al.* [4], for prior-free revenue maximization.

Mechanism design for revenue maximization – referred to as *optimal mechanism design* – has seen extensive study in the economics community, following the seminal work of Myerson [8]. Most models assume a prior probability distribution on the values of bidders and describe maximization of the expected revenue over this distribution. Goldberg *et al.* introduced in [4] *Competitive Auctions*, along with a model for *prior-free* revenue maximization, in allocating identical copies of a single good to unit-demand bidders. The revenue of a truthful auction is compared against the *optimum single price revenue* $\mathcal{F}^{(2)}$, which allocates at least two units of the good. Auctions that approximate this *benchmark* within a constant, are called *competitive*. The rationale behind $\mathcal{F}^{(2)}$ is that the (unconstrained) optimum single price revenue approximates the optimum social welfare – an absolute upper bound on the achievable revenue by any mechanism – within factor $\Theta(\log n)$ [4]. As shown in [4], no truthful mechanism can be competitive

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against the unconstrained optimum single price revenue. This justifies the minimal constraint of allocating to at least two bidders in determining the optimum single price. For this model the authors give two competitive auctions based on random sampling, the *Random Sampling Optimal Price* auction (RSOP) and the *Sampling Cost Sharing* auction (SCS). SCS was shown in [4] to be 4-competitive; RSOP has been the subject of much study though [4,3,1], as its conjectured 4-competitiveness is open. The best known ratio is 4.68 [1]. The best-performing auction is 3.243-competitive [6]; a lower bound of 2.42 was proven in [4].

Competitive auctions were taken *online* in [5,7], in a way reminiscent of the *Secretary Problem* [2]. The bids are decided by an adversary and supplied in an online fashion, in uniformly random order. Hajiaghayi, Kleinberg and Parkes in [5] studied limited supply of k available copies; they designed a constant-competitive auction with respect to both, social welfare and revenue – against an appropriate adaptation of $\mathcal{F}^{(2)}$ that we also use here. Koutsoupias and Pierrakos studied in [7] the online *unlimited* supply case, using $\mathcal{F}^{(2)}$ as a benchmark. They showed that any *offline* ρ-competitive auction yields a 2ρ-competitive *online* auction. This implies a 6.486-competitive online auction, by the result of [6]. They proved a lower bound of 4 on the competitive ratio of any online auction (also valid in the settings considered here) and proposed an online auction termed Best-Price-So-Far (BPSF), that is conjectured to be 4-competitive.

We study further the online model of [5,7], under two limited supply settings: **(i)** a constraint of k available copies, as in [5], and **(ii)** a production cost for each allocated copy, given by a positive non-decreasing convex function. We design algorithmic reductions of both settings to the unlimited supply case of [7]; using a ρ-competitive online auction for unlimited supply as a black box in our algorithms, we obtain truthful $4e\rho$- and $64e\rho$-competitive auctions for each of **(i)** and **(ii)** respectively, against appropriate adaptations of $\mathcal{F}^{(2)}$. Our result, along with the reduction of [7] and the best known offline auction of [6] yields a $26e$-competitive auction for **(i)**, that improves upon the 6338 ratio of [5].

Definitions. We consider n potential buyers/bidders, with a positive private value v_i each, $i = 1, \ldots, n$, for a single unit of a single good. The values v_i are determined by an adversary. The bidders are revealed online to a mechanism, in order chosen uniformly at random beforehand. We study *truthful* mechanisms, that make a price offer p_t to the t-th arriving bidder i_t, before the bidder reports his private value v_{i_t}. Bidder i_t accepts the offer if $p_{i_t} \leq v_{i_t}$ and buys a unit of the good, or rejects it, if $p_{i_t} > v_{i_t}$. After the haggling is over, the mechanism learns the private value v_{i_t}. Since we study truthful mechanisms, we refer to the bidders' values as *bids* which we denote by $b_i \equiv v_i$, $i = 1, \ldots, n$. We assume $b_1 \geq \cdots \geq b_n$. The vector of all bids is denoted by $\mathbf{b}$ and we define $\mathbf{b}_{[i]} \equiv (b_1, \ldots, b_i)$. We assume that n is known to the mechanism and that it may refuse to sell a unit to any bidder. This option can be justified by assumption of knowledge of a very high upper bound on the bids which, if offered to a bidder as a price, it will be definitely rejected. This assumption facilitates the observation of bids (i.e., sampling) and cease of selling after some item copies have been sold. We consider two settings, *Limited Supply* and *Supply with Production Cost*.

Limited Supply. We assume that the auctioneer has a limited supply of $k \geq 2$ identical units (copies) of a single good to sell to the bidders. The revenue of the auction after at most k units have been sold, will be compared against the revenue of $\mathcal{F}_k^{(2)}(\mathbf{b}) = \max_{2 \leq i \leq k}(i \cdot b_i)$, the optimal single-price auction that *does* sell at least 2 item copies. Our aim is to devise and study limited supply online auctions that are constant-competitive against $\mathcal{F}_k^{(2)}$. The bid index corresponding to the optimum price that gives $\mathcal{F}_k^{(2)}(\mathbf{b})$ is denoted by $i^* = \arg\max_{k \geq i \geq 2}(i \cdot b_i)$.

Supply with Production Cost. This setting generalizes both the unlimited and limited supply models, in that the j-th copy of the good is available to the auctioneer at a cost $c(j)$, where c is a non-negative non-decreasing function, with $c(0) = 0$. We consider a slight generalization of convex cost functions, satisfying:

$$\frac{1}{\alpha}\int_0^{\alpha} c(x)dx \leq \frac{c(\alpha)}{\beta}, \quad \text{for some } \beta > 1 \tag{1}$$

This class includes all *convex* functions, since they all satisfy (1) for $\beta = 2$. The actual *profit* made in this setting from sales of item copies equals the raised revenue *minus* the cumulative production cost of the sold item copies. We extend $\mathcal{F}^{(2)}$ for the optimum single price profit in this case as follows:

$$\mathcal{F}_c^{(2)}(\mathbf{b}) = \max_{i \geq 2}\left(i \cdot b_i - \sum_{j \leq i}^{i} c(j)\right) \tag{2}$$

$\mathcal{F}_c^{(2)}$ is motivated similarly to $\mathcal{F}^{(2)}$ in [4]; $\mathcal{F}_c(\mathbf{b}) = \max_{i \geq 1}\left(i \cdot b_i - \sum_{j \leq i} c(j)\right)$ can be shown to approximate the optimum social welfare within factor $O(\log k)$.

2 Revenue Maximization under Limited Supply

We present an algorithmic reduction of the online problem for *limited supply* k, to the online problem for *unlimited supply*. We only consider online unlimited supply auctions $\mathcal{A}$ that offer each bidder i_t arriving at time t a price independent of b_{i_t}, that may depend on the set of bids $\{b_{i_r} | r \leq t\}$ observed so far. Let ρ denote the competitive ratio of $\mathcal{A}$ against $\mathcal{F}^{(2)}$. At time t, the auction $\mathcal{A}$ processes the bids seen so far and makes a price offer to the t-th arriving bidder, i_t. We use $\mathcal{A}$ in designing a limited supply auction $\texttt{LSOA}_{\mathcal{A}}$, as described by algorithm 1.

The mechanism has 2 phases. In the initial sampling phase, for an appropriately determined value of $t_0 = \lceil \frac{n}{k} \rceil$, t_0 bids are *only* observed and the highest is inserted in a reservoir R_{t_0}. From time $t \geq t_0 + 1$ on, the mechanism offers each bidder i_t the price $\mathcal{A}(R_{t-1})$ computed by $\mathcal{A}$ for the subset of bidders held in R_{t-1}. The *online* auction $\mathcal{A}$ is assumed to process a subset of the bids and determine a price for the bid arriving next. If bidder i_t accepts the offered price

Sampling Phase;

begin
  Initialize $R_{t_0} \leftarrow \emptyset$;
  if $k \leq n/2$ **then** $t_0 \leftarrow \lceil n/k \rceil$;
  else $t_0 \leftarrow 1$;
  Sample $B_0 \leftarrow \{i_t | t \leq t_0\}$;
  Set $R_{t_0} \leftarrow \arg\max_{i \in B_0} b_i$;
end

Price Offering Phase;
repeat for bidder i_t, $t \geq t_0 + 1$
  $R_t \leftarrow R_{t-1}$;
  Offer i_t price $p = \mathcal{A}(R_t)$;
  if $p \leq b_{i_t}$ **then** $k \leftarrow k - 1$;
  if $b_{i_t} \geq \min R_{t-1}$ **then**
    $R_t \leftarrow R_t \cup \{i_t\}$;
  end
until $k = 0$;

Algorithm 1: ($\mathtt{LSOA}_{\mathcal{A}}$) A scheme for online limited supply auctions

the number of available units is decreased. If his bid is higher than the lowest bid held in R_{t-1}, then R_t is updated to $R_{t-1} \cup \{i_t\}$ (otherwise, R_t is set equal to R_{t-1}). The following lemma provides a simple fact for the sampling phase:

Lemma 1. *In a uniformly distributed random permutation π of n numbers, let $\mathcal{E}_i$ be the event that: the i-th highest number out of the $k \leq \frac{n}{2}$ largest numbers, for any $i \in [k]$, is observed as the maximum among the first $\lceil n/k \rceil$ numbers in π. Then:*

$$\Pr[\mathcal{E}_i] = \binom{n-i}{\lceil n/k \rceil - 1} \cdot \binom{n}{\lceil n/k \rceil}^{-1} \geq (2ek)^{-1}$$

Theorem 1. *A ρ-competitive online auction for unlimited supply can be transformed into a $4e\rho$-competitive online auction for limited supply.*

Proof. First we examine the case of $k \leq \frac{n}{2}$. Consider the sampling phase of B_0, and the bid $i_0 = \arg\max_{i \in B_0} b_i$ chosen from B_0 and inserted in R_{t_0}. For any index value of i_0, the execution of the online auction $\mathcal{A}$ within $\mathtt{LSOA}_{\mathcal{A}}$ is equivalent to execution of $\mathcal{A}$ over any random permutation of the subset of $i_0 - 1$ bids $\mathbf{b}_{[i_0-1]}$. This holds because no bid with index $i > i_0$ – i.e., smaller than b_{i_0} – is ever inserted in R_t for any value of $t \geq t_0$.

Let $\mathbb{E}\Big[\mathtt{LSOA}_{\mathcal{A}}(\mathbf{b})\Big]$ denote the expected profit of the auction. We make two observations, in order to lower bound $\mathbb{E}\Big[\mathtt{LSOA}_{\mathcal{A}}(\mathbf{b})\Big]$. First, that if $\arg\max_{j \in B_0} b_j = 1$, the profit extracted is zero in the worst case; this is because b_1 may be the strictly largest bid of all, and it is the only bid that $\mathcal{A}$ will observe in R_t, for $t = 1, \ldots, n$. Then offering it as a price to any of the rest of the bids will not result in a purchase. The second observation has to do with the performance of $\mathcal{A}$ on $\mathbf{b}_{[i_0-1]}$, where $i_0 = \arg\max_{i \in B_0} b_i$. Because $b_{i_0} < b_i$ for all $b_i \in \mathbf{b}_{[i_0-1]}$, we may assume that no online auction will actually make a profit by selling to bidder i_0, because it must first observe at least b_{i_0}. Thus, for any $b_i \in \mathbf{b}$ we have that: $\mathbb{E}\left[\mathcal{A}(\mathbf{b}_{[i-1]}) \,\middle|\, \mathcal{E}_i\right] \geq \mathbb{E}\Big[\mathcal{A}(\mathbf{b}_{[i]})\Big]$ Then, by lemma 1, we have for $\mathbb{E}\Big[\mathtt{LSOA}_{\mathcal{A}}(\mathbf{b})\Big]$:

$$\mathbb{E}\Big[\mathtt{LSOA}_{\mathcal{A}}(\mathbf{b})\Big] \geq \sum_{i=2}^{k} \Pr\left[\mathcal{E}_i\right] \times \mathbb{E}\Big[\mathcal{A}(\mathbf{b}_{[i]})\Big] \geq \frac{1}{2ek} \sum_{i=2}^{k} \mathbb{E}\Big[\mathcal{A}(\mathbf{b}_{[i]})\Big] \tag{3}$$

$$\geq \sum_{i=2}^{i^*-1} \frac{i \cdot b_i}{2ek\rho} + \sum_{i=i^*}^{k} \frac{i^* \cdot b_{i^*}}{2ek\rho} \geq \sum_{i=2}^{i^*-1} \frac{i \cdot b_{i^*}}{2ek\rho} + \sum_{i=i^*}^{k} \frac{i^* \cdot b_{i^*}}{2ek\rho}$$
$$= \frac{1}{2ek\rho} \cdot \left(\sum_{i=1}^{i^*-2} \frac{i+1}{i^*} + \sum_{i=i^*}^{k} 1 \right) \cdot \mathcal{F}_k^{(2)}(\mathbf{b})$$
$$= \frac{1}{2ek\rho} \cdot \left(\frac{(i^*-2)(i^*+1)}{2i^*} + k - i^* + 1 \right) \cdot \mathcal{F}_k^{(2)}(\mathbf{b})$$
$$= \frac{1}{2ek\rho} \cdot \left(\frac{2k - i^* + 1}{2} - \frac{1}{i^*} \right) \cdot \mathcal{F}_k^{(2)}(\mathbf{b})$$
$$\geq \frac{2k - i^*}{4ek\rho} \cdot \mathcal{F}_k^{(2)}(\mathbf{b}) \geq \frac{1}{4e\rho} \cdot \mathcal{F}_k^{(2)}(\mathbf{b}) \tag{4}$$

The 2nd line is by the competitive ratio ρ of $\mathcal{A}$ and, given that $\mathcal{F}_k^{(2)}(\mathbf{b}) = i^* \cdot b_{i^*}$, we have that $\mathcal{F}_k^{(2)}(\mathbf{b}_{[i]}) = \mathcal{F}_k^{(2)}(\mathbf{b})$ for $i \geq i^*$ and $\mathcal{F}_k^{(2)}(\mathbf{b}_{[i]}) = i \cdot b_i$ for $i < i^*$, because $b_i \geq b_{i^*}$. The last inequality is due to $i^* \leq k$. Now (4) gives the result for $k \leq \frac{n}{2}$. For the case of $k > \frac{n}{2}$, the first (in the random permutation) encountered bid b_{i_0} is any particular bid $i = 1, \ldots, k$ of the k highest with probability at least $\frac{1}{2k}$. Then, we obtain competitive ratio 4ρ, from (3):

$$\mathbb{E}\Big[\texttt{LSOA}_{\mathcal{A}}(\mathbf{b})\Big] \geq \frac{1}{2k} \sum_{i=2}^{k} \mathbb{E}\Big[\mathcal{A}(\mathbf{b}_{[i]})\Big] \geq \frac{1}{4\rho} \cdot \mathcal{F}_k^{(2)}(\mathbf{b}) \tag{5}$$

where the second inequality follows by the same analysis that led to (4). □

3 Profit Maximization with Production Cost

We examine next the case of supply under a non-negative non-decreasing production cost function c; the cost of the i-th item copy is $c(i)$, with $c(0) = 0$. The developed reduction yields competitive online auctions for all such cost functions c satisfying (1). We use an alternative sampling phase for this case, described as algorithm 2. The price offering phase is identical to the one given in algorithm 1, with the exception that the offered price is the maximum of the current production cost and the computed price by an online competitive auction $\mathcal{A}$ for unlimited supply. The benchmark we compare the algorithm's performance against is $\mathcal{F}_c^{(2)}(\mathbf{b})$ as given by (2). In our analysis we use the following lemma:

Lemma 2. *In a single-good online prior-free profit maximization problem* $(\mathbf{b}, c)$*, with non-negative non-decreasing production cost function* c*, let* $k = max\{i | b_i \geq c(i+1)\}$*. If* $i_0 = \arg\max_{2 \leq i \leq k}(i \cdot b_i)$ *and there exists* $\beta = O(1)$ *such that* c *satisfies* $\int_0^{\alpha} c(x)dx \leq \alpha\beta^{-1} \cdot c(\alpha)$ *for any* $\alpha \geq 0$*, then:*

$$\mathcal{F}_c^{(2)}(\mathbf{b}) \leq \frac{2\beta}{2\beta - 3} \cdot \left(\mathcal{F}_k^{(2)}(\mathbf{b}) - \sum_{i \leq i_0} c(i) \right) \tag{6}$$

1 **Sampling Phase**;
2 **begin**
3 Initialize $t \leftarrow 0$; $B_0 \leftarrow \emptyset$;
4 **repeat**
5 $t \leftarrow t + 1$;
6 $B_t \leftarrow B_{t-1} \cup \{i_t\}$;
7 **until** $\max_{j \in B_t} b_j > c\left(\lceil \frac{n}{t} \rceil\right)$ `OR` $t \geq \frac{n}{2}$;
8 $t_0 \leftarrow t$;
9 Set $R_{t_0} \leftarrow \{\arg\max_{i \in B_t} b_i\}$;
10 **end**

1 **Price Offering Phase**;
2 $\chi^{t_0} \leftarrow 0$; `// # copies sold`
3 For bidder i_t on time $t \geq t_0 + 1$;
4 **begin**
5 $\chi^t \leftarrow \chi^{t-1}$; $R_t \leftarrow R_{t-1}$;
6 Offer $p = \max\{\mathcal{A}(R_t), c(1 + \chi^t)\}$;
7 **if** $p \leq b_{i_t}$ **then** $\chi^t \leftarrow \chi^t + 1$;
8 **if** $b_{i_t} \geq \min R_{t-1}$ **then**
9 $R_t \leftarrow R_t \cup \{i_t\}$;
10 **end**
11 **end**

Algorithm 2: ($\mathtt{LSOA}^c_{\mathcal{A}}$) A scheme for online auctions with production cost

Theorem 2. *A ρ-competitive online auction for unlimited supply can be transformed into a $\frac{32e\beta\rho}{2\beta-3}$-competitive online auction for profit maximization against $\mathcal{F}_c^{(2)}$, for any non-negative non-decreasing cost function c satisfying (1).*

Proof. Define $k = \max\{i | b_i \geq c(i+1)\}$. This is the point where the k-th highest bid almost reaches the production cost of the $(k+1)$-th copy. In effect, no profit can be made by sale of any copy from the $(k+2)$-nd onwards. First we examine the case of $n \geq 6$ and $k \leq \frac{n}{2}$. Consider the bid picked by the sampling phase of algorithm 2. This is the first bid that passes the test mentioned in line 7 of the sampling phase and it causes termination of the phase. As we have argued previously in the proof of theorem 1, if a bid b_j causes termination of the sampling phase, then the price offering phase is essentially equivalent to executing the online algorithm $\mathcal{A}$ on a random permutation of the subset of bids $\mathbf{b}_{[j]}$, conditionally on the fact that b_j is observed first in the permutation. Denote by $\mathcal{E}'_i$ the event that bid b_i, $i = 1, \ldots, k$ causes termination of the sampling phase at any sampling step. $\Pr[\mathcal{E}'_i]$ is at least equal to the probability that b_i is the maximum bid among the first $\lceil \frac{n}{k} \rceil$ bids in the random permutation, and the rest $\lceil \frac{n}{k} \rceil - 1$ bids are in $\{b_{k+2}, \ldots, b_n\}$. Indeed, by definition of k, for every bid b_r, $r = k+2, \ldots, n$, it is $b_r \leq b_{k+1} \leq c(k+2) \leq c(r)$. Then:

$$\Pr[\mathcal{E}'_i] \geq \binom{n-1-k}{\lceil n/k \rceil - 1} \cdot \binom{n}{\lceil n/k \rceil}^{-1} \tag{7}$$

$$= \frac{n - \lceil n/k \rceil - k + 1}{n-k} \cdot \binom{n-k}{\lceil n/k \rceil - 1} \cdot \binom{n}{\lceil n/k \rceil}^{-1}$$

$$\geq \frac{n - n/k - k}{n-k} \binom{n-k}{\lceil n/k \rceil - 1} \cdot \binom{n}{\lceil n/k \rceil}^{-1}$$

$$\geq \frac{n - 2 - \frac{n}{2}}{n-2} \binom{n-k}{\lceil n/k \rceil - 1} \cdot \binom{n}{\lceil n/k \rceil}^{-1} \geq \frac{1}{4} \cdot \Pr\left[\mathcal{E}_k\right] \tag{8}$$

where $\mathcal{E}_k$ is the instantiation of the event defined in lemma 1, for $i = k$. The last two inequalities occur by $\frac{n-n/k-k}{n-k}$ being increasing in k for $2 \leq k \leq n/2$, thus minimized for $k = 2$ to at least $\frac{1}{4}$ for $n \geq 6$. Then:

$$\mathbb{E}\left[\mathtt{LSOA}^c_{\mathcal{A}}(\mathbf{b})\right] \geq \sum_{i=2}^{k} \Pr\left[\mathcal{E}'_i\right] \mathbb{E}\left[\mathcal{A}(\mathbf{b}_{[i]})\right] \geq \frac{1}{4} \sum_{i=2}^{k} \Pr\left[\mathcal{E}_k\right] \mathbb{E}\left[\mathcal{A}(\mathbf{b}_{[i]})\right]$$
$$\geq \frac{1}{8ek} \sum_{i=2}^{k} \mathbb{E}\left[\mathcal{A}(\mathbf{b}_{[i]})\right] \quad \geq \frac{1}{16e\rho} \mathcal{F}^{(2)}_k(\mathbf{b}) \geq \frac{2\beta - 3}{32e\rho\beta} \mathcal{F}^{(2)}_c(\mathbf{b})$$

The second inequality follows by (8), the third by lemma 1 applied for $i = k$, the fourth by (4) in the proof of theorem 1 and the final one by lemma 2.

For the case of $k > \frac{n}{2}$, $\Pr[\mathcal{E}'_i]$ can be easily lower bounded by $\frac{1}{2k}$ and a calculation similar to (5) yields a better ratio. The only remaining case is $n = 5$ and $k = 2$; we find directly from (7) $\Pr[\mathcal{E}'_i] \geq \frac{1}{5}$ and proceeding with similar calculations as right above, we derive $\mathbb{E}\left[\mathtt{LSOA}^c_{\mathcal{A}}(\mathbf{b})\right] \geq \frac{2\beta-3}{10\beta\rho} \mathcal{F}^{(2)}_c(\mathbf{b})$. □

To obtain the best competitive ratios for the problems considered, we combine theorems 1 and 2 with the reduction given in [7](Theorem 1) of online unlimited suply to offline unlimited supply, and with the best known (offline) competitive auction from [6], with competitive ratio $\rho = 3.243 < 3.25$.

Corollary 1. *There exist truthful* $26e$*- and* $416e$*-competitive online auctions for limited supply and for supply under non-negative convex cost, respectively.*

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Lower Bounds on Revenue of Approximately Optimal Auctions

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Abstract. We obtain revenue guarantees for the simple pricing mechanism of a single posted price, in terms of a natural parameter of the distribution of buyers' valuations. Our revenue guarantee applies to the single item n buyers setting, with values drawn from an arbitrary joint distribution. Specifically, we show that a single price drawn from the distribution of the maximum valuation $V_{\max} = \max\{V_1, V_2, \ldots, V_n\}$ achieves a revenue of at least a $\frac{1}{e}$ fraction of the geometric expectation of $V_{\max}$. This generic bound is a measure of how revenue improves/degrades as a function of the concentration/spread of $V_{\max}$.

We further show that in absence of buyers' valuation distributions, recruiting an additional set of identical bidders will yield a similar guarantee on revenue. Finally, our bound also gives a measure of the extent to which one can simultaneously approximate welfare and revenue in terms of the concentration/spread of $V_{\max}$.

Keywords: Revenue, Auction, Geometric expectation, Single posted price.

1 Introduction

Here is a natural pricing problem: A single item is to be sold to one among n buyers. Buyers' valuations are drawn from some known joint distribution. How good a revenue can be achieved by posting a single price for all the buyers, and giving the item to the first buyer whose value exceeds the price? Can we lower bound the revenue in terms of some properties of the distribution? Such a single pricing scheme is often the only option available, for several natural reasons. In

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many situations, it is illegal or not in good taste to price discriminate between buyers; furthermore often it is not possible to implement a pricing scheme with multiple prices.

We define the geometric expectation of a random variable before describing our result: the geometric expectation of a random variable X is given by $e^{\mathbb{E}[\log(X)]}$ (see, e.g., [5]). The geometric expectation is always lower than the expectation, and the more concentrated the distribution, the closer they are; indeed, the ratio between the geometric expectation and the expectation is a natural measure of concentration around the mean. We illustrate how the ratio of geometric and actual expectations captures the spread of a random variable through an example in Section 2.

Constant fraction of geometric expectation. We show that a single price obtains a $\frac{1}{e}$ fraction of the geometric expectation of the maximum among the n valuations $(V_1, \ldots, V_n)$, i.e. geometric expectation of $V_{\max} = \max\{V_1, \ldots, V_n\}$. Thus for distributions that are concentrated enough to have a geometric expectation of $V_{\max}$ that is close to the expectation of $V_{\max}$, a single pricing scheme extracts a good fraction of the social surplus. In particular, when the ratio of geometric and actual expectations is larger than $e/4$, our revenue guarantee is larger than a $1/4$ fraction of the welfare (and hence the optimal revenue), thus beating the currently best known bound of $1/4$ by Hartline and Roughgarden [4]. In the special case when the distribution of $V_{\max}$ satisfies the monotone hazard rate (MHR) property, a single price can extract a $\frac{1}{e}$ fraction of the expected value of $V_{\max}$ ([3]). However, since several natural distributions fail to satisfy the MHR property, establishing a generic revenue guarantee in terms of the geometric expectation, and then bounding the ratio of the geometric and actual expectation is a useful route. For instance, in Section 2 we compute this ratio for power law distributions (which do not satisfy the MHR property) and show that for all exponents $m \geq 1.56$ this ratio is larger than $e/4$ thus beating the currently known bound.

Why geometric expectation?

1. Since the concentration of a distribution is a crucial property in determining what fraction of welfare (expectation of $V_{\max}$) can be extracted as revenue, it is natural to develop revenue guarantees expressed in terms of some measure of concentration.
2. While there are several useful measures of concentration for different contexts, in this work we suggest that for revenue in auctions the ratio of the geometric and actual expectations is both a generic and a useful measure — as explained in the previous paragraph, for some distributions our revenue guarantees are the best known so far.
3. The ratio of the two expectations is a dimensionless quantity (i.e., scale free).

Second price auction with an anonymous reserve price. A natural corollary of the lower bound on single pricing scheme's revenue is that the second price auction (or the Vickrey auction) with a single anonymous reserve obtains a fraction $\frac{1}{e}$ of the geometric expectation of $V_{\max}$. When buyers' distributions are independent and satisfy a technical regularity condition, Hartline and Roughgarden [4] show that the second price auction with a single anonymous reserve price obtains a four approximation to the optimal revenue obtainable. Here again, our result shows that for more general settings, where bidders values could be arbitrarily correlated, Vickrey auction with a single anonymous reserve price guarantees a $\frac{1}{e}$ fraction of geometric expectation of $V_{\max}$.

Second price auction with additional bidders. When estimating the distribution is not feasible (and hence computing the reserve price is not feasible), a natural substitute is to recruit extra bidders to participate in the auction to increase competition. We show that if we recruit another set of bidders distributed identically to the first set of n bidders, and run the second price auction on the $2n$ bidders, the expected revenue is at least a $\frac{2}{e}$ fraction of the geometric expectation of $V_{\max}$. As in the previous result, for the special case of independent distributions that satisfy the regularity condition, Hartline and Roughgarden [4] show that recruiting another set of n bidders identical to the given n bidders obtains at least half of the optimal revenue; our result gives a generic lower bound for arbitrary joint distributions.

In the course of proving this result we also prove the following result: in the single pricing scheme result, the optimal single price to choose is clearly the monopoly price of the distribution of $V_{\max}$. However we show that a random price drawn from the distribution of $V_{\max}$ also achieves a $\frac{1}{e}$ fraction of geometric expectation of $V_{\max}$.

Related Work. For the special single buyer case, Tamuz [6] showed that the monopoly price obtains a constant fraction of the geometric expectation of the buyer's value. We primarily extend this result by showing that for the n buyer setting, apart from the monopoly reserve price of $V_{\max}$, a random price drawn from the distribution of $V_{\max}$ also gives a $\frac{1}{e}$ fraction of geometric expectation of $V_{\max}$. This is important for showing our result by recruiting extra bidders. Daskalakis and Pierrakos [2] study simultaneous approximations to welfare and revenue for settings with independent distributions that satisfy the technical regularity condition. They show that Vickrey auction with non-anonymous reserve prices achieves a $\frac{1}{5}$ of the optimal revenue and welfare in such settings. Here again, for more general settings with arbitrarily correlated values, our result gives a measure how the quality of such simultaneous approximations degrades with the spread of $V_{\max}$. The work of Hartline and Roughgarden [4] on second price auction with anonymous reserve price / extra bidders has been discussed already.

2 Definitions and Main Theorem

Consider the standard auction-theoretic problem of selling a single item among n buyers. Each buyer i has a private (non-negative) valuation V_i for receiving the item. Buyers are risk neutral with utility $u_i = V_i x_i - p_i$, where x_i is the probability of buyer i getting the item and p_i is the price he pays. The valuation profile $(V_1, V_2, \ldots, V_n)$ of the buyers is drawn from some arbitrary joint distribution that is known to the auctioneer. Let $V_{\max} = \max_i V_i$ be the random variable that denotes the maximum value among the n bidders. We denote with $F_{\max}$ the cumulative density function of the distribution of $V_{\max}$.

Definition 1. *For a positive random variable X, the geometric expectation $\mathbb{G}[X]$ is defined as:*

$$\mathbb{G}[X] = \exp(\mathbb{E}[\log X])$$

We note that by Jensen's inequality $\mathbb{G}[X] \leq \mathbb{E}[X]$ and that equality is achieved only when X is a deterministic random variable. Further, as noted in the introduction, the ratio of geometric and actual expectations of a random variable is a useful measure of concentration around the mean. We illustrate this point through an example.

Example 1. Consider the family $F_m(x) = 1 - 1/x^m$ of power-law distributions for $m \geq 1$. As m increases the tail of the distribution decays faster, and thus we expect the geometric expectation to be closer to the actual expectation. Indeed, the geometric expectation of such a random variable can be computed to be $e^{1/m}$ and the actual expectation to be $\frac{m}{m-1}$. The ratio $e^{1/m}(1 - 1/m)$ is an increasing function of m. It reaches 1 at $m = \infty$, i.e., when the distribution becomes a point-mass fully concentrated at 1. The special case of $m = 1$ gives the equal-revenue distribution, where the geometric expectation equals e and the actual expectation is infinity. However this infinite gap (or the zero ratio) quickly vanishes as m grows; at $m = 1.56$, the ratio already crosses $e/4$ thus making our revenue guarantee better than the current best $1/4$ of optimal revenue; at $m = 4$, the ratio already equals 0.963.

For a random variable X drawn from distribution F, define $\mathbb{R}_p[X]$ as:

$$\mathbb{R}_p[X] = p\mathbb{P}[X \geq p] \geq p\mathbb{P}[X > p] = p(1 - F(p))$$

If X is the valuation of a buyer, $\mathbb{R}_p[X]$ is the expected revenue obtained by posting a price of p for this buyer. Therefore $\mathbb{R}_p[V_{\max}]$ is the revenue of a pricing scheme that posts a single price p for n buyers with values $V_1, \ldots, V_n$ and $V_{\max} = \max\{V_1, \ldots, V_n\}$.

We show that the revenue of a posted price mechanism with a single price drawn randomly from the distribution of $V_{\max}$, achieves a revenue that is at least a $\frac{1}{e}$ fraction of the geometric expectation of $V_{\max}$, or equivalently a $\frac{1}{e}$ fraction of the geometric expectation of the social surplus.

Theorem 1 (Main Theorem). *Let r be a random price drawn from the distribution of $V_{\max}$. Then:*

$$\mathbb{E}_r[\mathbb{R}_r[V_{\max}]] \geq \frac{1}{e}\mathbb{G}[V_{\max}]. \tag{1}$$

Proof. By the definition of $\mathbb{R}_r[V]$ we have:

$$\mathbb{E}_r[\mathbb{R}_r[V_{\max}]] \geq \mathbb{E}_r[r\ (1 - F_{\max}(r))]. \tag{2}$$

By taking logs on both the of the above equation, and using Jensen's inequality we get:

$$
\begin{aligned}
\log(\mathbb{E}_r\left[\mathbb{R}_r\left[V_{\max}\right]\right]) \geq\ & \log\left(\mathbb{E}_r\left[r(1-F_{\max}(r))\right]\right) \\
\geq\ & \mathbb{E}_r\left[\log(r(1-F_{\max}(r)))\right] \\
=\ & \mathbb{E}_r\left[\log(r)\right] + \mathbb{E}_r\left[\log(1-F_{\max}(r))\right].
\end{aligned}
$$

For any positive random variable X drawn from a distribution F we have:

$$\mathbb{E}\left[\log(1-F(X))\right] = \int_{-\infty}^{\infty} \log(1-F(x))dF(x) = \int_0^1 \log(1-y)dy = -1. \quad (3)$$

So we have:

$$
\begin{aligned}
\log(\mathbb{E}_r\left[\mathbb{R}_r\left[V_{\max}\right]\right]) &\geq \mathbb{E}_r\left[\log(r)\right] - 1 \\
\mathbb{E}_r\left[\mathbb{R}_r\left[V_{\max}\right]\right] &\geq \frac{1}{e}\exp(\mathbb{E}_r\left[\log(r)\right] = \frac{1}{e}\mathbb{G}\left[V_{\max}\right].
\end{aligned}
$$

where the last equality follows from the fact that the random reserve r is drawn from $F_{\max}$. □

Since a random price drawn from $F_{\max}$ achieves this revenue, it follows that there exists a deterministic price that achieves this revenue and hence the best deterministic price will achieve the same.

We define the monopoly price η_F of a distribution F to be the optimal posted price in a single buyer setting when the buyer's valuation is drawn from distribution F, i.e.:

$$\eta_F = \arg\sup_r r(1-F(r))$$

So a direct corollary of our main theorem is the following:

Corollary 1. *Let $\eta_{\max}$ be the monopoly price of distribution $F_{\max}$. Then:*

$$\mathbb{R}_{\eta_{\max}}\left[V_{\max}\right] \geq \frac{1}{e}\mathbb{G}\left[V_{\max}\right]$$

3 Applications to Approximations in Mechanisms Design

Single Reserve Mechanisms for Non-iid Irregular Settings. A corollary of our main theorem is that in a second price auction with a single anonymous reserve, namely a reserve drawn randomly from the distribution of $F_{\max}$ or a deterministic reserve of the monopoly price of $F_{\max}$, will achieve revenue that is a constant approximation to the geometric expectation of the maximum value. When the maximum value distribution is concentrated enough to have the geometric expectation is close to expectation it immediately follows that an anonymous reserve mechanism's revenue is close to that of the expected social surplus and hence the expected optimal revenue.

Corollary 2. *The second price auction with a single anonymous reserve achieves a revenue of at least $\frac{1}{e}\mathbb{G}\left[V_{\max}\right]$ for arbitrarily correlated bidder valuations.*

Approximation via replicating buyers in Irregular Settings. When the auctioneer is unable to estimate the distribution of $V_{\max}$, and therefore unable to compute the reserve price, a well known alternative [1] to achieve good revenue is to recruit additional bidders to participate in the auction to increase competition. In our setting, recruiting a set of n bidders distributed identically as the initial set of n bidders (i.e. following joint distribution F) will simulate having a reserve drawn randomly from $F_{\max}$. In fact it performs even better than having a reserve — one among the additionally recruited agents could be the winner and he pays the auctioneer, as against the reserve price setting. More formally, observe that in the setting with $2n$ bidders, half of the revenue is achieved from the original n bidders, and half from the new bidders (by symmetry). But the revenue from each of these parts is exactly that of the second price auction with a random reserve drawn from the distribution of $V_{\max}$. Hence, the revenue of this extended second price mechanism will be twice the revenue of a second price mechanism with a single random reserve drawn from the distribution of $V_{\max}$. This fact, coupled with our main theorem gives us the following corollary.

Corollary 3. *The revenue of a second price auction with an additional set of bidders drawn from joint distribution F is at least $\frac{2}{e}\mathbb{G}[V_{\max}]$.*

Approximately Optimal and Efficient Mechanisms. Finally, we note that when the geometric expectation of $V_{\max}$ is close to its expectation, all our mechanisms (both the single pricing scheme, and Vickrey with a single reserve) are also approximately efficient.

Corollary 4. *If $\mathbb{G}[V_{\max}] = c\mathbb{E}[V_{\max}]$, a single price drawn randomly from the distribution of $F_{\max}$ is simultaneously $\frac{c}{e}$ approximately revenue-optimal and $\frac{c}{e}$ approximately efficient.*

Proof. Since expected social welfare of a pricing scheme is at least its expected revenue, we have:

$$\mathbb{E}[\text{Social Welfare}] \geq \mathbb{E}[\text{Revenue}] \geq \frac{1}{e}\mathbb{G}[V_{\max}] \geq \frac{c}{e}\mathbb{E}[V_{\max}]$$

□

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On Fixed-Price Marketing for Goods with Positive Network Externalities

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Abstract. In this paper we discuss marketing strategies for goods that have positive network externalities, i.e., when a buyer's value for an item is positively influenced by others owning the item. We investigate revenue-optimal strategies of a specific form where the seller gives the item for free to a set of users, and then sets a fixed price for the rest. We present a $\frac{1}{2}$-approximation for this problem under assumptions about the form of the externality. To do so, we apply ideas from the influence maximization literature [13] and also use a recent result on non-negative submodular maximization as a black-box [3,7].

1 Introduction

Consumer goods and services often exhibit positive network externalities—a buyer's value for the good or service is influenced positively by other buyers owning the good or using the service. Such positive network externalities arise in various ways. For instance, XBox Live is an online gaming service that allows users to play with each other. Thus, the value of an XBox to a user increases as more of her friends also own an XBox. Popular smartphone platforms (such as Android, iOS, or Windows Mobile) actively support developer networks, because developers add 'Applications' that make the phone more useful to other users. Thus, the value of a smartphone to a user increases with the size of the developer network. Many consumer goods, especially those that have been newly introduced, benefit from word-of-mouth effects. Prospective buyers use this word-of-mouth to judge the quality of the item while making a purchase decision. If the good or service is of good quality, the word-of-mouth will cause a positive externality.

Irrespective of how positive network externalities arise, it is clear that they are worth paying attention to in designing a good marketing/pricing strategy. Companies that own smartphone platforms often hand out upcoming devices to developers. Manufacturers send out a new version of a device to technology review websites. Detergent companies, and manufacturers of health foods, hand out free samples of new products. The hope is that giving out the item for free drives up demand for the good/service and increases the revenue generated from future sales.

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In this paper we attempt to identify a revenue maximizing marketing strategy of the following form: *The seller selects a set S of buyers and gives them the good for free, and then sets a fixed per-unit price p at which other consumers can buy the item.* The strategy is consistent with practice as the examples above illustrate and is easy to implement. However, optimizing revenue poses two challenges. First, the choice of the set S and the price p are coupled and must be traded-off optimally: expanding the set S loses potential revenue from the set S, but may increase the positive externality on buyers not in S and may allow the seller to extract more revenue from them. A second, more subtle, issue is that it is important to have a handle on the dynamics of adoption. For a fixed set S and a price p, a buyer $j \notin S$ who is initially unwilling to buy the item at a price p, may later do so as other buyers (who are not in S and are willing to buy the item at a price p) go first. This may result in a 'cascade' of sales and it is important to have a handle on this revenue when optimizing for S and p.

Our Results. The related problem of *influence maximization* (as opposed to our revenue maximization problem) is well-studied (e.g., Chapter 23 in [13]). The canonical question in this literature, first posed by Domingos and Richardson [5], is: Which set I of influential nodes of cardinality k in a social network should be convinced to use a service, so that subsequent adoption of the service is maximized? This literature has made substantial progress in understanding the cascading of process of adoption and using this to optimize for I (see for instance [5,11,12,15]). However, this literature does not model the impact of price on the probability of adopting a service and does not attempt to quantify the revenue from adoption. Therefore it cannot be directly applied to answer our revenue-maximization question.

Our main technical contribution (Lemma 1) establishes a correspondence between the dynamics of our (price-sensitive) process and the dynamics of the general threshold model [11] from the influence maximization literature. We use it along with a recent result on optimizing non-negative submodular functions [3,7] to identify an algorithm that is a $\frac{1}{2}$-approximation for our problem (Theorem 1). It is worth noting that, although we prove our result through establishing a connection to the general threshold model [11], we cannot use the greedy $(1-\frac{1}{e})$-approximation algorithm of Nemhauser, Wolsey, and Fischer [16], and instead we need to use the recent $\frac{1}{2}$-approximation [3,7] for non-negative submodular maximization.

More Related Work. Besides the literature on influence maximization mentioned above, there is also an expanding literature on algorithms for revenue maximization with positive network externalities. Hartline, Mirrokni, and Sundararajan [9] study the marketing strategies where the seller can give the item for free to a set of buyers, and then visit the remaining buyers in a sequence offering each a buyer-specific price. Such strategies are hard to implement because the seller must control the time at which the transaction takes place. Further, there is also evidence that buyers may react negatively to price-discrimination as it generates a perception of unfairness. Oliver and Shor [17] discuss why such a negative reaction may arise. Partly in response to some of these issues, Akhlaghpour et al. [1] explore strategies that allow the seller to vary the price across time. Though these strategies do not perform price discrimination, there is some evidence that such strategies may also cause buyers to react negatively, especially if the prices vary significantly across time. For instance, there was some unhappiness when Apple dropped the price of an iPhone by 33% two months after an initial launch (http://www.apple.com/hotnews/openiphoneletter/). In contrast, our approach is to offer the good at a fixed price, albeit after giving the item for free to some set of users, a step which seems socially acceptable (see the examples in the Introduction.) This strategy can also increase the revenue to the seller above using a fixed price without an influence step. More recently, Haghpanah et al. [8] take an auction-theoretic (as opposed to a pricing) approach. This approach is applied only to some forms of positive externality where the temporal sequence of sales is not necessary for the externality to manifest (so it applies to the XBox example from the introduction, but not the settings where word-of-mouth is involved).

There is also a literature in economics that has studied equilibrium behavior in the adoption of goods with network externalities [2,4,6,10,14,18]. For instance, Carbal, Salant, and Woroch [4] show that in a social network the seller might decide to start with low *introductory* prices to attract a *critical mass* of players when the players are *large* (i.e, the network effect is significant). The focus here is to characterize the equilibrium that arises from buyer rationality, as opposed to optimizing the seller's strategy.

2 Model

Consider a seller who wants to sell a good to a set of potential buyers, V. Consider a digital good with zero marginal cost of manufacturing and assume that the seller has an unlimited supply of the good. We assume that the seller is a monopolist and is interested in maximizing its revenue.

Externality Model. We assume that a buyer i's value for the digital good depends on its own inherent valuation ω_i for the good and also on the influence from the set $S \subseteq V \setminus \{i\}$ of buyers who already own the good. More specifically, we consider the *graph model with concave influence* in which each buyer $i \in V$ is associated with a non-negative, non-decreasing, concave function $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $f_i(0) = 0$. The value of the digital item for a buyer $i \in V$ given that a set S of buyers have already bought the item is denoted by $v_i(S)$ and is equal to $\omega_i + f_i(\sum_{j\in S} w_{ij})$. Each inherent valuation ω_i is drawn independently from a uniform distribution (or more generally from a distribution G_i) and each w_{ij} is drawn from a distribution $\widetilde{G}_{ij}$ capturing the influence of buyer i over buyer j. We assume that a buyer i buys the item at a price p if and only if $v_i(S) \geq p$. We assume that the valuations and prices are in an interval $[0, M]$.

Fixed-Price Marketing. A *fixed-price marketing strategy* consists of two stages: in the first stage, the seller gives the item for *free* to a subset A of buyers (inital influence); in the second stage, the seller sets a fixed price p for the digital good (price setting). After setting the price p, buyers i with value $v_i(A) \geq p$ buy the item. Let set S_1 be the set of buyers whose value $v_i(A)$ after the influence step is greater than p, i.e., $S_1 = \{i \notin A | v_i(A) \geq p\}$. After buyers in set S_1 buy the item at price p, they may influence other buyers, and their value may increase and go above p. As a result, after set S_1 buys the item, some other buyers may have incentive to buy the item. Let set S_2 be this set of buyers, i.e., $S_2 = \{i \notin A \cup S_1 | v_i(A \cup S_1) \geq p\}$. As more buyers buy the digital good, more buyers have incentive to buy the item. This process continues and the dynamics propagates, i.e, for any i $(2 \leq i \leq k)$, S_i is the set of buyers not in $(\cup_{j<i} S_j) \cup A$ whose value is more than or equal to p given that set $(\cup_{j<i} S_j) \cup A$ of buyers already adopted the item. The seller's goal is to find a set A of buyers to influence and a fixed price p to maximize the total revenue he can extract from buyers, i.e., in the optimal *fixed-price marketing problem with positive network externalities*, the sellers's goal is to choose A and p to maximize $p(|\cup_{i \geq 1} S_i|)$.

3 Approximation Algorithm

In this section, we design a constant-factor approximation algorithm for the problem. We first observe that a simple $\frac{1}{8}$-approximation algorithm exists for the special case of the problem where weights are deterministic. Then we elaborate on an improved $\frac{1}{2}$-approximation algorithm for the graph model with concave influence function that explicitly exploits dynamics.

Sketch of a Simple $\frac{1}{8}$-Approximation Algorithm. For fixed ω_i's and w_{ij}'s, a randomized $\frac{1}{8}$-approximation algorithm is easily derived: Give the item for free to each buyer with probability 1/2 independently, then search for the highest revenue achievable given the freebies by considering all prices over a $1/\text{poly}(n)$-grid. Let A_*, p_* be an optimal solution to the problem and define $B_* = \{i \in V : \omega_i + f_i(\sum_j w_{ij}) \geq p_*\}$. In expectation, there are $|B_*|/2$ remaining potential buyers after the first stage. We claim that, for a fixed second-stage price of $p_*/2$, each of the remaining nodes in B_* has a probability $\frac{1}{2}$ of reaching value $p_*/2$ in the second stage—giving an expected revenue of $|B_*|p_*/8$ and proving the claim. Indeed, let $\mathcal{P}_i$ be the revenue earned from i when $p = p_*/2$ and note that, ignoring dynamics (i.e., considering only the first round following the influence stage), $\mathbb{E}[\mathcal{P}_i] \geq \frac{p_*}{4}\mathbb{P}\left[\omega_i + f_i\left(\sum_j \mathbb{1}_j w_{ij}\right) \geq \frac{p_*}{2}\right]$, where $\mathbb{1}_i$ is 1 if i gets the item for free, and 0 otherwise (and $w_{ii} = 0$). Noting that $\sum_j \mathbb{1}_j w_{ij} \geq \frac{1}{2}\sum_j w_{ij} \implies f_i\left(\sum_j \mathbb{1}_j w_{ij}\right) \geq \frac{1}{2} f_i\left(\sum_j w_{ij}\right) \geq \frac{1}{2}[p_* - \omega_i]$, where we used the concavity of f_i and the definition of B_*, we get $\mathbb{P}\left[\omega_i + f_i(\sum_j \mathbb{1}_j w_{ij}) \geq \frac{p_*}{2}\right] \geq \mathbb{P}\left[\sum_j \mathbb{1}_j w_{ij} \geq \frac{1}{2}\sum_j w_{ij}\right] \geq 1/2$, by symmetry.

A $\frac{1}{2}$-Approximation Algorithm. Now we present an improved $\frac{1}{2}$-approximation algorithm when the weights are *random* that explicitly exploits the *dynamics of the influence process*, unlike the simple algorithm above. We assume further that the prices are in an interval $[0, M]$ for some constant M, that the w_{ij}'s are drawn from arbitrary distributions and that the ω_i's are drawn from a uniform distribution over $[0, M]$. For convenience, we take $M = 1$. For any price $p \in [0, 1]$, consider the following set function $Y_p : 2^V \to \mathbb{R}_+$: for any subset $A \subset V$, $Y_p(A)$ is the expected revenue from giving the item for free to set A in the influence stage, and setting the price to p in the fixed-price stage. Our algorithm is as follows. Fix $\epsilon = o(n^{-1})$.

1. For every integer ρ where $0 \leq \rho \leq \epsilon^{-1}$ do:
   – Given that the price in the second stage is $p = \rho\epsilon$, using the approximation algorithm for non-negative submodular maximization in [3,7], find a set A_ρ of users to influence in the first stage. The algorithm in [3,7] uses oracle calls to the objective function. We simulate oracle calls to Y_p by running the influence process poly(n) times independently and averaging.
   – Let L_ρ be the revenue from giving the item to set A_ρ and setting price $p = \rho\epsilon$.
2. Output the set A_ρ and price $\rho\epsilon$ for which L_ρ is maximized.

Our approximation result follows from a mapping of the fixed-price strategy to a model of viral marketing introduced in [11,12]. In the viral marketing problem, one gives an item for free to a group of individuals as we do here but, in the subsequent influence stage, revenue is ignored (i.e., there is no price) and instead one aims to maximize the number of individuals who purchase the product. In [11,12], the general threshold model was introduced to model the influence process. Formally, the special case of the general threshold model relevant here is obtained from our influence process by setting $p = 0$ and letting ω_i be uniform in $[-1, 0]$. See [11,12] for more details on the general threshold model.

Theorem 1 (Approximation). *The above algorithm is a $\frac{1}{2}$-approximation algorithm for the optimal fixed-price marketing problem with positive network externalities in the graph model with concave influence.*

It is worth noting that, although we prove our result through establishing a connection to the general threshold model, the final set function that we need to maximize is *not necessarily monotone*. Therefore, unlike the viral marketing problem in [11,12], we cannot use the greedy $(1 - \frac{1}{e})$-approximation algorithm of Nemhauser, Wolsey, and Fischer [16] for monotone submodular maximization subject to cardinality constraints. Instead we use the local search $\frac{1}{2}$-approximation [3,7] for non-negative submodular maximization. Before stating the proof of this theorem, we note that the approximation algorithm applies to a more general setting for the distribution of inherent valuations ω_i's.

Remark 1. Our $\frac{1}{2}$-approximation algorithm holds more generally under the assumption that the inherent valuations ω_i are random with distribution G_i with positive, differentiable, non-decreasing density g_i on $(0, 1)$ and, further, that there is a constant $\overline{g} > 0$ such that the g_i's are bounded above by $\overline{g}$. Our proof is given under these assumptions. The obvious open question is to see if the assumption that g_i is non-decreasing can be relaxed to a more realistic assumption like the monotone hazard rate condition.

Proof. Note that it follows from Chebyshev's inequality and the fact that the revenue is bounded by n that our simulated oracle calls are accurate within $1/\text{poly}(n)$ with probability $1 - 1/\text{poly}(n)$. Let OPT be the optimal revenue. We first condition on the edge weights $\{w_{ij}\}_{ij}$.

Proposition 1 (Submodularity of Y_p). *Conditioned on the edge weights $\{w_{ij}\}_{ij}$, the function Y_p is a (not necessarily monotone) non-negative, submodular function.*

Proposition 2 (Continuity of Y_p). *Let δ_n be a vanishing function of n (possibly negative) with $|\delta_n| = o(n^{-k})$ with $k \geq 1$. Conditioned on the edge weights $\{w_{ij}\}_{ij}$, we have $|Y_p(S) - Y_{p+\delta_n}(S)| = o(n^{-k})\text{OPT}$, for any set S of buyers.*

By linearity, both propositions still hold after taking expectation over edge weights. Theorem 1 then follows from the main result in [3,7] where a $\frac{1}{2}$-approximation algorithm is derived for non-negative submodular maximization. The proof of Proposition 2 is omitted for space.

Proof. (of Proposition 1) For any price p and any buyer i, consider the following set function $h_p^i : 2^{V\setminus\{i\}} \to \mathbb{R}_+$: for any subset $A \subset V\setminus\{i\}$, $h_p^i(A)$ is the expected revenue *from user* i if we give the item for free to set A in the influence stage, and then set the price p in the second stage. For any set A, $Y_p(A) = \sum_{i \in V\setminus A} h_p^i(A)$. We need the following lemma

Lemma 1. *The set functions h_p^i for any buyer i are monotone and submodular.*

Proof. Fix $0 \leq p \leq 1$. Let S be a set of buyers. Note that $\omega_i + f_i\left(\sum_{j\in S} w_{ij}\right) \geq p$, if and only if $f_i\left(\sum_{j\in S} w_{ij}\right) \geq \max\{0, p - \omega_i\} \equiv \omega_{i,p}$. Denote by $Q_{i,p}$ the distribution function of $\omega_{i,p}$. Note that $Q_{i,p}(x) = 1 - G_i(p - x)$, for $0 \leq x < p$ and $Q_{i,p}(x) = 1$ for $x \geq p$. By assumption, on $(0, p)$, $Q'_{i,p}(x) = g_i(p - x) > 0$ and $Q''_{i,p}(x) = -g'_i(p - x) \leq 0$ so that $Q_{i,p}$ is increasing and concave. Further, since $Q_{i,p}$ is continuous at p and constant for $x \geq p$, $Q_{i,p}$ is non-decreasing and concave on $[0, +\infty)$.

Let U_i, $i \in V$, be independent uniform random variables. We now describe a mapping of our influence process to a special case of the general threshold model where a user i adopts a product as soon as $Z_i(\sum_{j\in S} w_{ij}) \geq U_i$ for a concave function Z_i. To transfer the randomness of our inherent valuation to the threshold side of the general threshold model, we use the inverse transform method where one simulates a random variable X with distribution function H by using $H^{-1}(U)$ where U is uniform in $[0, 1]$ and H^{-1} is a generalized inverse function. By definition of $Q_{i,p}$, $\mathbb{P}\left[Q_{i,p}\left(f_i\left(\sum_{j\in S} w_{ij}\right)\right) \geq U_i\right] = \mathbb{P}\left[f_i\left(\sum_{j\in S} w_{ij}\right) \geq \omega_{i,p}\right] = \mathbb{P}\left[\omega_i + f_i\left(\sum_{j\in S} w_{ij}\right) \geq p\right]$. Since $Q_{i,p}$ and f_i are non-decreasing and concave, the composition $Q_{i,p}(f_i(\cdot))$ is concave as well and $Q_{i,p}(f_i(\sum_{j\in S} w_{ij}))$ is submodular in S. Hence, we have shown that for any fixed p, the dynamics of the influence stage are equivalent to a submodular general threshold model. In particular, by the results in [15], we have that h_p^i is submodular.

Proposition 1 then follows from the following lemma whose proof is omitted (see [9] for a similar lemma).

Lemma 2. *If all set functions h_p^i for $i \in V$ are monotone and sub modular, then the set function Y_p is also sub modular (but not monotone).*

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The Competitive Facility Location Problem in a Duopoly: Connections to the 1-Median Problem

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Abstract. We consider a competitive facility location problem on a network, in which consumers are located on the vertices and wish to connect to the nearest facility. Knowing this, competitive players locate their facilities on vertices that capture the largest possible market share. In 1991, Eiselt and Laporte established the first relation between Nash equilibria of a facility location game in a duopoly and the solutions to the 1-median problem. They showed that an equilibrium always exists in a tree because a location profile is at equilibrium if and only if both players select a 1-median of that tree [4]. In this work, we further explore the relations between the solutions to the 1-median problem and the equilibrium profiles. We show that if an equilibrium in a cycle exists, both players must choose a solution to the 1-median problem. We also obtain the same property for some other classes of graphs such as quasi-median graphs, median graphs, Helly graphs, and strongly-chordal graphs. Finally, we prove the converse for the latter class, establishing that, as for trees, any median of a strongly-chordal graph is a winning strategy that leads to an equilibrium.

1 Introduction

Facility location problems deal with the optimal placement of facilities with respect to a set of customers. In the discrete version of this problem, a decision-maker needs to select a vertex of a graph whose vertices represent the potential locations where the facility may be placed. Vertices also represent customers and have weights that encode the demand at each location. Finally, distances are captured by the topology of the graph. In the centralized problem, a decision-maker has to select a vertex that minimizes the distance that customers need to travel to visit the facility, solution normally referred to as a 1-median [7].

In the competitive version of the facility location problem, a set of players is competing to attract customers and wish to maximize market share by locating their facilities strategically in the graph. This problem was first studied by Hotelling [8] in 1929, where two players select a location on a continuous and linear market with demand uniformly distributed along it. His prediction was that at equilibrium both players locate in the 1-median of that line because otherwise they can undercut the competitor and increase the market-share.

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We consider a discrete version of the competitive facility location problem in a duopoly. Given a graph with weights representing demands, both players must select a vertex to locate a facility. The utility of a player is given by the total demand among vertices closest to the selected facility. To break ties, demand is split evenly for vertices that are equidistant to the two facilities. The work of Eiselt and Laporte, the first among just a few references that study the facility location game as stated here, shows that trees always admit pure-strategy Nash equilibria [4]. Indeed, a selection of facilities is a Nash equilibrium if and only if both players select a (possibly different) 1-median of the tree, which always exists. Motivated by this result, our work establishes further links between Nash equilibria of the facility location problem in a duopoly and the 1-median problem, for various classes of topologies. To the best of our knowledge, with the exception of [4], we are not aware of other results in this direction.

Since it is natural for players to locate in a central location in the market, we seek to understand under what circumstances when an equilibrium exists, players have the incentive to select solutions to the 1-median problem. We provide a proof of this result for cycles, which combined with the results of [4] can be extended to cacti and other more general, but specific, topologies. This extension relies on a decomposition technique that allows one to focus in the subgraph that contains the equilibria [6]. The idea is to represent the graph as a tree of maximal bi-connected components. This representation conserves some of the relevant information about the original graph and allows one to find the components where equilibria might be located. In addition, we show that for an arbitrary graph topology, when an equilibrium exists, both players select vertices that are local optima to the 1-median problem. This result automatically translates to proving that equilibria can only be located at a 1-median for different classes of graphs where no local optima exist, such as median graphs, quasi-median graphs and Helly graphs. Those families of graphs include grids and latices, which capture the topology of many real urban networks. Finally, we generalize the result that trees always have equilibria, which are located in medians, to the class of strongly chordal graphs. That family of graphs includes trees but also other topologies such as interval graphs and block graphs.

To decide if an instance of this game admits an equilibrium by exhaustive search, it is necessary to evaluate all possible deviations from each possible outcome of the game. There are $O(|V|^2)$ outcomes, $O(|V|)$ deviations, and for each we must evaluate a shortest path tree to compute the market share for each player. Our results imply that it is not necessary to check every possible outcome of the game but just the combinations of winning strategies, or the 1-medians if the former are not available and a fast algorithm to compute them is available for the specific instance.

To conclude, various versions of facility locations games have been studied over the last decades, differing in the number of players, the splitting techniques and the space considered to locate the facilities. For details and references, the reader is referred to [5]. Intimately related to discrete facility location games are the Voronoi games, which have been recently visited by [3,11]. In a Voronoi game on a graph with several players, each player chooses a vertex and achieves a utility equal to the number of vertices that are closer to the chosen vertex than to those of the other players.

2 The Facility Location Game

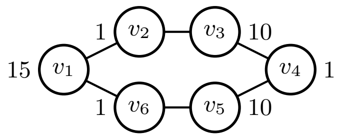
Let $G = (V, E)$ be an undirected connected weighted graph, in which each vertex represents a location and has an associated weight $w(v) > 0$ that quantifies demand. We denote the demand in a set $S \subseteq V$ of vertices by $W(S) = \sum_{v \in S} w(v)$, and let $W = W(V)$ be the total demand. Two players compete for market share by selecting a vertex each to locate their facilities. We refer to the vertices selected by the players as $x_1, x_2 \in V$, respectively. Given a profile $\bar{x} = (x_1, x_2)$, each vertex v will split its demand evenly among the set of facilities that are closest to it; i.e., $F(v, \bar{x}) := \arg\min_{i \in \{1,2\}} d(v, x_i)$, where $d(\cdot, \cdot)$ is the distance function induced by the topology of the graph where edges have unit length (this restriction is without much loss of generality since other distances can be achieved by subdividing edges). Similarly, letting $V_i(\bar{x}) := \{v \in V : d(x_i, v) \leq d(x_j, v)\ \forall j \neq i\}$, a player i will receive utility u_i composed by the full demand from vertices in V_i where the inequality is strict plus half of the demand from vertices in V_i where there is equality. Since $u_1 + u_2 = W$, this is a zero-sum game.

We say that a profile $\bar{x}$ is a *pure-strategy Nash equilibrium* (PSNE) of this facility location game if $u_i(x_i, x_{-i}) \geq u_i(y, x_{-i})$ for any $y \in V$, for $i = 1, 2$. The main property of an equilibrium is that both players must obtain equal utility; otherwise, the player with the lowest utility would prefer to emulate the other player's strategy and get a utility of $W/2$.

Although there are always equilibria in mixed strategies, [6] provides examples that show that not every facility location game with two players has a PSNE. They characterized equilibria for different topologies using ad-hoc techniques. We unify some of those results, considering vertices that ensure a big-enough market share. Indeed, we say that a vertex $w \in V$ is a *winning strategy* if the utility obtained by a player when choosing vertex w guarantees *winning the game*, regardless of the selection of the other player; i.e., $u_1(w, v) \geq W/2$ for all $v \in V$. There is a one-to-one relationship between the location of winning strategies and that of equilibria. In fact, any equilibrium must consist of each player choosing a winning strategy.

Lemma 1. *For arbitrary topologies, an equilibrium of a facility location game with two players exists if and only if there exists at least a winning strategy.*

Proof. The result follows from the definition of a winning strategy. If $\bar{x}$ is at equilibrium, $W/2 = u_2(x_1, x_2) \geq u_2(x_1, v)$ for all $v \in V$, which implies that x_1 is a winning strategy because, since the game is zero-sum, $W/2 \leq u_1(x_1, v)\ \forall v \in V$. To prove the converse, take a winning strategy w and consider $\bar{x} = (w, w)$. By definition $W/2 = u_1(\bar{x}) \leq u_1(w, v)$ for all $v \in V$. Using again that the game is zero-sum proves the equilibrium condition.



(a) Vertices are annotated with demands. For each pair of vertices, one player can always deviate to obtain more than half of the demand.

v	v_1	v_2	v_3	v_4	v_5	v_6
$D(v)$	45	59	55	69	55	59

(b) Total distances to vertices. Vertex v_1 is the unique 1-median, while v_3 and v_5 are local medians.

Fig. 1. Instance with no equilibria and its medians

Note that winning strategies are related to *dominant strategies*, which refer to selections that are always optimal regardless of the opponent's choice. Although our game does not necessarily have dominant-strategy equilibria and winning strategies are not necessarily dominant, playing a winning strategy guarantees that the player is not worse than the opponent *even if the opponent deviates from the equilibrium* (hence the name). To illustrate, consider the path $(v_1, v_2, v_3, v_4, v_5)$ of 5 vertices with unit weight. The unique winning strategy is to choose v_3 (therefore, the only equilibrium is (v_3, v_3)). However, a best response to an opponent that chooses v_5 is to choose v_4 and hence v_3 is not a dominant strategy.

Winning strategies, though, are not guaranteed to exist. In the instance shown in Fig. 1, no vertex can guarantee a player a utility of $W/2$. Indeed, a best response to selecting a vertex with demand larger than one is to select the opposite vertex, whereas a best response to selecting a vertex with unit demand is to select the adjacent vertex. Therefore, an equilibrium for this instance does not exist.

As discussed in the introduction, it is natural to select the vertex that is nearest to the demand. Hence, for a single facility located at vertex $y \in V$, we compute the total distance to it as $D(y) = \sum_{v \in V} d(y, v)w(v)$. A vertex is called a 1-*median* of G if it minimizes $D(\cdot)$. In the rest of the paper, we sometimes just write *median* to refer to the 1-median of a graph, and we use the term *median-set* to refer to the set of vertices that are 1-medians.

3 Cycles and the 1-Median Problem

Although we already saw that cycles do not always admit equilibria, when demands are sufficiently large an equilibrium must exist. For instance, [6] showed that an equilibrium of a cycle can use a vertex v if and only if the demand of any subpath of cardinality $|V|/2$ that excludes v does not exceed $W/2$. That condition can be interpreted as saying that the corresponding vertices are winning strategies. To see this, note that for an arbitrary profile each player gets the demand from exactly half of the vertices (one or more vertices may be split equally depending on the parity of the cycle and on the location of both facilities). Therefore, the condition of [6] is equivalent to v being a winning strategy.

For trees, it is known that medians and equilibria (and hence winning strategies) of the facility location game in duopolies coincide [4]. The relation of winning strategy and medians for trees follows from a result by Kariv and Hakimi that establishes that a vertex is a median if and only if removing it induces components of weight not larger than $W/2$ [9]. We now show a similar result for cycles: every winning strategy must be a median. Note that the converse is not true: medians always exist but winning strategies may not.

Theorem 1. *If a winning strategy w of a cycle exists, it must solve the* 1*-median problem.*

To prove this, we compute the difference in total distance from w to any other vertex v, representing it as a weighted sum of paths of cardinality $|V|/2$. Using that w is a winning strategy, we can prove that the difference is non-negative. Due to lack of space, the full proof is omitted.

Combining the results for trees and for cycles, we extend the previous property to more general topologies. A *cactus* is a graph where every edge belongs to at most one cycle. Reducing an arbitrary cactus to a tree representing its components, as explained in [6], and then using the result for cycles, winning strategies must also solve the 1-median problem. In addition, because one can compute winning strategies for cacti in $O(|V|)$-time, this also provides an efficient algorithm to compute the medians of cacti that admit equilibria.

4 Local Medians

Even though it is natural to think that if an equilibrium of the facility location game exists, players will choose a 1-median solution, we do not know if this is true for arbitrary topologies. Nevertheless, we can prove that equilibria of this game translate into a local median property. Indeed, whenever a winning strategy exists, it must be a local minimum of the 1-median problem with respect to neighboring vertices. To illustrate the definition of local median, in the example of Fig. 1 there is a global median and 2 local ones (the result does not apply to the example because there are no winning strategies in it).

Theorem 2. *If w is a winning strategy for the the facility location game, then $D(w) \leq D(v)$ for all $v \in N(w)$, where $N(w) := \{v \in V | vw \in E\}$.*

Proof. Let w be a winning strategy and let $v \in N(w)$. Let $d_z := d(v,z) - d(w,z)$ for all $z \in V$. Because w and v are neighbors, $d_z \in \{-1, 0, 1\}$ $\forall z$. We consider $\bar{x} = (w, v)$. Since (w, w) is at equilibrium, $W/2 = u_2(w, w) \geq u_2(\bar{x})$, from where $u_1(\bar{x}) - u_2(\bar{x}) \geq 0$ because the game is zero-sum. Let M_w (resp. M_v) be the set of vertices that are strictly closer to w than to v (resp. v to w). The result follows using that $d_z = 1$ for $z \in M_w$ and $d_z = -1$ for $z \in M_v$, because

$$D(v) - D(w) = \sum_{z \in V} w(z)(d(v,z) - d(w,z)) = W(M_v) - W(M_w) = u_1(\bar{x}) - u_2(\bar{x}).$$

In light of this result, one would like to characterize the local minima of the 1-median problem to understand the possible locations of the winning strategies. For certain families of graphs, these minima coincide with the (global) 1-medians. Indeed, [1] proved that if G is a connected graph, then the following conditions are equivalent: (a) The median-set is connected for arbitrary weights w, and, (b) The set of local medians coincide with the median-set for arbitrary rational weights w. Based on this equivalence, we obtain the following corollary.

Corollary 1. *Let G be a graph that belongs to a family for which, for any rational weights w, the solutions to the* 1*-median problem induce a connected subgraph of G. Then, every winning strategy of G solves the* 1*-median problem.*

Families of graphs satisfying this property include *median graphs*, *quasi-median graphs*, *pseudo-median graphs*, *Helly graphs* and *strongly chordal graphs*. A complete characterization of graphs with connected median-sets can be found in [1]. Among graphs in this family, median graphs are particularly important in location applications because they represent cities well. Median graphs satisfy that any three vertices a, b, and c have a unique median (which is a vertex that belongs to shortest paths between any two of a, b, and c). This class includes lattices, meshes, and grids, which encode the topology of many realistic networks.

Notice that not all graphs have connected medians (e.g., Fig. 1a). Furthermore, median-sets may have arbitrary topologies; that is, given a graph G, there exists a graph H for which the subgraph of H induced by the median vertices is isomorphic to G [12]. This result implies that the median-set can induce a disconnected subgraph.

5 Strongly Chordal Graphs

In this section we focus on strongly chordal graphs, which are relevant because they generalize many well-known classes of graphs such as trees, block graphs and interval graphs. A graph is *chordal* if every cycle with more than three vertices has a *chord*, i.e., an edge joining two non-consecutive vertices of the cycle. A *p-sun* is a chordal graph with a vertex set $x_1, \ldots, x_p, y_1, \ldots, y_p$ such that $y_1, \ldots, y_p$ is an independent set, $(x_1, \ldots, x_p, x_1)$ is a cycle, and each vertex y_i has exactly two neighbors $x_i - 1$ and x_i, where $x_0 = x_p$. A graph is *strongly chordal* if it is chordal and contains no p-sun for $p \geq 3$.

We knew from the previous section that winning strategies in strongly chordal graphs solve the 1-median problem. We prove the converse result, establishing that graphs in that family always admit equilibria and that the equilibrium locations and 1-medians coincide. This completely extends the results for trees of [4] to this family, which is a strict superclass of trees.

Theorem 3. *Every connected strongly chordal graph has an equilibrium. Furthermore, there is a one-to-one correspondence between winning strategies and the solutions to the* 1*-median problem.*

The proof of this theorem, which is omitted due to lack of space, uses induction by identifying a vertex that cannot be a winning strategy and removing it to reduce the problem. This approach follows the methodology of Theorem 1 in [10], where it is shown that the median-set of a connected strongly chordal graph is a clique. While we use the same inductive idea, we need to rely on more complex structures. As a corollary, the set of winning strategies of a connected strongly chordal graph is, not only connected as previously discussed, but also a clique.

6 Concluding Remarks

We have explored the relations between winning strategies and solutions to the 1-median problem in duopolistic facility location games. For several families of graphs, we have shown that the locations of both sets coincide or that one is inside another. We believe that both sets should coincide for some of the classes of graphs considered, and others as well. Identifying when it holds or providing counterexamples remains as open problems. In particular, it would be interesting to further extend our results to some super-class of the strongly-chordal graphs. We would have liked to prove related results for graphs of bounded treewidth, but we could not adapt the decomposition technique used to prove our result for cacti, so other ideas may be needed for that result.

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The Ring Design Game with Fair Cost Allocation
[Extended Abstract]*

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Abstract. In this paper we study the network design game when the underlying network is a ring. In a network design game we have a set of players, each of them aims at connecting nodes in a network by installing links and sharing the cost of the installation equally with other users. The ring design game is the special case in which the potential links of the network form a ring. It is well known that in a ring design game the price of anarchy may be as large as the number of players. Our aim is to show that, despite the worst case, the ring design game always possesses good equilibria. In particular, we prove that the price of stability of the ring design game is at most 3/2, and such bound is tight. We believe that our results might be useful for the analysis of more involved topologies of graphs, e.g., planar graphs.

1 Introduction

In a network design game, we are given an undirected graph $G = (V, E)$ and edge costs given by a function $c : E \to \mathbb{R}^+$. The edge cost function naturally extends to any subset of edges, that is $c(B) = \sum_{e \in B} c(e)$ for any $B \subseteq E$. We define $c(\emptyset) = 0$. There is a set of n players $[n] = \{1, \ldots, n\}$; each player $i \in [n]$ wishes to establish a connection between two nodes $s_i, t_i \in V$ called

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the source and destination node of player i, respectively. The set of strategies available to player i consists of all paths connecting nodes s_i and t_i in G. We call a *state* of the game a set of strategies $\sigma \in \Sigma$ (where Σ is the set of all the states of the game), with one strategy per player, i.e., $\sigma = (\sigma_1, \ldots, \sigma_n)$ where σ_i denotes the strategy of player i in σ . Given a state σ, let $n_\sigma(e)$ be the number of players using edge e in σ. Then, the cost of player i in σ is defined as $c_\sigma(i) = \sum_{e \in \sigma_i} \frac{c(e)}{n_\sigma(e)}$. Let $E(\sigma)$ be the set of edges that are used by at least one player in state σ. The social cost $C(\sigma)$ is simply the total cost of the edges used in state σ which coincides with the sum of the costs of the players, i.e., $C(\sigma) = \sum_{e \in E(\sigma)} c(e) = \sum_{i \in [n]} c_\sigma(i) = c(E(\sigma))$.

Let (σ_{-i}, σ'_i) denote the state obtained from σ by changing the strategy of player i from σ_i to σ'_i. Given a state $\sigma = (\sigma_1, \ldots, \sigma_n)$, an *improving move* of player i in σ is a strategy σ'_i such that $c_{(\sigma_{-i}, \sigma'_i)}(i) < c_\sigma(i)$. A state of the game is a *Nash equilibrium* if and only if no player can perform any improving move. An *improvement dynamics* (shortly *dynamics*) is a sequence of improving moves. A game is said to be *convergent* if, given any initial state σ, any dynamics leads to a Nash equilibrium. It is well known, as it has been proved by Rosenthal [6] for the more general class of congestion games, that any network design game is convergent. We denote by NE the set of states that are Nash equilibria. A Nash equilibrium can be different from the socially optimal solution. Let Opt be a state of the game minimizing the social cost. The *price of anarchy* (PoA) of a network design game is defined as the ratio of the maximum social cost among all Nash equilibria over the optimal cost, i.e., $\text{PoA} = \frac{\max_{\sigma \in \text{NE}} C(\sigma)}{C(\text{Opt})}$. It is trivial to observe that the PoA in a network design game may be as large as the number of players n, and such bound is tight. The *price of stability* (PoS) is defined as the ratio of the minimum social cost among all Nash equilibria over the optimal cost, i.e., $\text{PoS} = \frac{\min_{\sigma \in \text{NE}} C(\sigma)}{C(\text{Opt})}$. Anshelevich *et al.* [1] proved that the price of stability is at most $H_n = 1 + 1/2 + \ldots + 1/n$. Although the upper bound proof has been shown to be tight for directed networks, the problem is still open for undirected networks. There have been several attempts to give a significant lower bound for the undirected case, e.g., [5,4,2,3]. The best known lower bound so far of $348/155 \approx 2.245$, has been recently shown in [2].

The aim of the current paper is to analyze the network design game when the underlying graph is a ring. We refer to this special case as *ring design game*. For the sake of clarity, by a ring we mean an undirected graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_k\}$, $E = \{e_1, e_2, \ldots, e_k\}$, and $e_i = v_i v_{i+1}$, $i = 1, \ldots, k$ (where $v_{k+1} = v_1$). Note that this simple case captures the whole spectra of interesting behavior, i.e., PoA remains equal the number of players. Moreover, the ring is crucial in the sense that it is the first non-trivial topology to analyze in the context of network design and it is the first step in order to cope with more involved topologies, like planar graphs. Hence, we believe that giving a tight bounds here could give some insight for studying more general settings.

Let us first point out that, in a ring design game, any improvement dynamics starting from the optimal state leads to an equilibrium at most 2 times the cost of the optimal state. In fact, either the optimal state is a Nash equilibrium, or there is a player j wishing to switch from his optimal strategy to the alternative path. At the optimum, the cost of player j is at most $C(\text{OPT})$, and thus the cost of the alternative path cannot be more than this quantity. Since the alternative path of j contains edges of the ring not belonging to $E(\text{OPT})$, it implies that $C(\text{OPT})$ is also an upper bound to $c(E \setminus E(\text{OPT}))$. Consequently, the cost of the entire ring, and thus the cost of any state, is at most $2C(\text{OPT})$. As we show here, by doing a more careful analysis, we are actually able to prove a tight bound of $3/2 \cdot C(\text{OPT})$ on the cost of any equilibrium reachable from the optimum.

Our results. In this paper we show that in a ring design game, differently from what the classical bound of n on the price of anarchy suggests, there always exist good performing Nash equilibria. In particular, we show that there always exists a Nash equilibrium of cost at most 3/2 times the cost of an optimal state, thus giving a bound on the PoS. We show that such equilibrium can be reached by a dynamics having as initial state an optimal configuration. Such result also gives some insight on the problem of computing an equilibrium in a ring design game. In fact, it reveals that if the cost of the entire ring is larger than 3/2 times the cost of an optimal state, then the dynamics starting from an optimal state converges quickly, within at most 3 steps, to an equilibrium. We also show that such bound on the PoS is tight, by showing an instance for which $\text{PoS} = 3/2 - \epsilon$.

2 Upper and Lower Bounds on the Price of Stability

We start by upper bounding the price of stability. Our technique to prove the bound on the PoS is different from the previously used ones. Previous techniques used potential function arguments and proved that any equilibrium reached by any dynamics starting by an optimal state has potential value at most $H_n \cdot C(\text{OPT})$. Here we also bound the cost of a Nash equilibrium reachable by a dynamics from the optimal state but without using potential function arguments. In particular, the analysis is made by cases on the number of moves, and for each such case we write a linear program that captures the most important inequalities. The most important observation we use is that one needs to consider the cases when at most 4 players move. We prove that for higher number of moves the PoS can only be smaller.

Our notation includes the number m representing the amount of steps in which some fixed dynamics reaches a Nash equilibrium starting from an optimal state OPT, the Nash equilibrium N obtained after m steps, as well as players making a move in the dynamics, meaning that π_j denotes the player that made the move at step $j = 1, \ldots, m$ during the dynamics. Note that a player could make a move at many different steps of the dynamics. Let $\sigma^0, \ldots, \sigma^j, \ldots, \sigma^m$ be the states corresponding to the considered dynamics, where $\sigma^0 = \text{OPT}$ and $\sigma^m = \text{N}$. Also, let f be a set of players of interest. The set f will be composed by a subset of the players moving in the dynamics. The usage of f will be clear in the proof of Theorem 1. For any $A \subseteq f$ the set D_A will denote the edges in OPT which are used by exactly the players in A, and R_A will denote the edges used in OPT which are used by exactly the players in A and at least one player from outside of f, formally:

$$D_A^f = \{e \in E \mid (\forall i \in f.\ e \in \text{OPT}_i \iff i \in A) \wedge \neg\exists i \notin f.\ e \in \text{OPT}_i\},$$
$$R_A^f = \{e \in E \mid (\forall i \in f. e \in \text{OPT}_i \iff i \in A) \wedge \exists i \notin f.\ e \in \text{OPT}_i\}.$$

For the sake of simplicity in the sequel we will omit the superscript f when it is clear from the context. Notice that D_A andR_A naturally define a partition of the edges of the ring, and that for any f we have $E(\text{OPT}) = \left(\bigcup_{A \subseteq f} D_A \cup R_A\right) \setminus D_\emptyset$. Moreover, let $\lambda > 0$ be such that $c(D_\varnothing) \leq \lambda C(\text{OPT})$. Since OPT and $D_\emptyset$ is a partition of E and the cost of any equilibrium N can be at most $c(E)$, then:

$$PoS \leq \frac{C(\text{N})}{C(\text{OPT})} \leq \frac{C(\text{OPT}) + c(D_\varnothing)}{C(\text{OPT})} \leq \frac{C(\text{OPT}) + \lambda \cdot C(\text{OPT})}{C(\text{OPT})} \leq 1 + \lambda. \quad (1)$$

Now let us write the necessary conditions for the fact that player π_j can move in step j of the dynamics, for any $j = 1, \ldots, m$. Such conditions will be expressed by using the above defined variables D_A and R_A. Unfortunately, we do not know the exact usage of edges in sets R_A. Let us define functions $\text{left}_k, \text{right}_k : \Sigma \to \mathbb{R}$ for any players $k \in f$. Set $d_\sigma(k)$ and $r_\sigma(k)$ to be the collection of subsets A of f such that player k is using (all) edges of D_A and R_A respectively in state σ, i.e., $d_\sigma(k) = \{A \in 2^f \mid k \text{ is using edges of } D_A \text{ in } \sigma\}$, $r_\sigma(k) = \{A \in 2^f \mid k \text{ is using edges of } R_A \text{ in } \sigma\}$. Also, define the edges' usage by the players' of interest (i.e., players belonging to f) $\hat{n}_\sigma : 2^E \to \mathbb{N}$ as $\hat{n}_\sigma(H) = \#\{i \in f \mid H \subseteq \sigma_i\}$. Let us define:

$$\text{left}_\sigma(k) = \sum_{A \in d_\sigma(k)} \sum_{e \in D_A} \frac{c(e)}{n_\sigma(e)} + \sum_{A \in r_\sigma(k)} \sum_{e \in R_A} \frac{c(e)}{n} = \sum_{A \in d_\sigma(k)} \frac{c(D_A)}{\hat{n}_\sigma(D_A)} + \sum_{A \in r_\sigma(k)} \frac{c(R_A)}{n}.$$

In the following the function left will be used as a lower bound on the cost of a player. Then wlog we can consider $\frac{c(R_A)}{n}$ to be 0 for any R_A. Therefore in the following we will omit such terms. Moreover, let us define:

$$\text{right}_\sigma(k) = \sum_{A \in d_\sigma(k)} \sum_{e \in D_A} \frac{c(e)}{n_\sigma(e)} + \sum_{A \in r_\sigma(k)} \sum_{e \in R_A} \frac{c(e)}{\hat{n}_\sigma(R_A) + 1}$$
$$= \sum_{A \in d_\sigma(k)} \frac{c(D_A)}{\hat{n}_\sigma(D_A)} + \sum_{A \in r_\sigma(k)} \frac{c(R_A)}{\hat{n}_\sigma(R_A) + 1}.$$

Then the following inequalities hold for any state $\sigma \in \Sigma$:

$$\text{left}_\sigma(k) \leq c_\sigma(k) \leq \text{right}_\sigma(k).$$

The role of functions left_k and right_k is to weaken the inequalities between player's utilities in some neighbour states, so that they become manageable. As we do not know the exact usage of edges in sets R_A, it would be hard to derive the precise bounds. This means that on the lower-hand side we introduce the maximum possible number (i.e., n) of players using edges of sets R_A in σ and on the upper-hand side we introduce the minimum number of players using edges of R_A in σ, i.e., $\hat{n}_\sigma(R_A) + 1$.

The proof of the following lemma will be given in the full version of this paper. This lemma will become useful in the proof of the main theorem.

Lemma 1. *In the ring design game, if in state* OPT *there are at least two players able to perform an improving move (both starting from state* OPT*) then the cost of the whole ring is at most $\frac{3}{2}$ times the cost of an optimal solution, that is $c(E) \le \frac{3}{2}C(\text{OPT})$.*

Theorem 1. *The price of stability for the ring design game is at most $\frac{3}{2}$.*

Proof. The proof is split into five different cases, depending on the amount m of steps in which some fixed dynamics reaches a Nash equilibrium starting from an optimal state OPT. Moreover notice that since in a ring design game the strategy set of each player i is composed by exactly 2 different strategies, i.e., the clockwise and counterclockwise paths connecting s_i and t_i. This implies that $\pi_j \neq \pi_{j+1}$, for any $j = 1, \dots, m-1$. We remark that in some cases we get the bound by solving a linear program where constraints are naturally defined by using left and right functions, and where objective functions are proper defined in each of the case.

Case $m = 0$. The equality $m = 0$ trivializes the instance into an example where OPT is a Nash equilibrium, thus PoS $= 1$.

Case $m = 1$. In this case the dynamics reaches a Nash Equilibrium N after one step starting from OPT. Since player π_1 can perform an improving move starting by state OPT, the following inequalities hold: $\text{left}_{\text{N}}(\pi_1) \le c_{\text{N}}(\pi_1) < c_{\text{OPT}}(\pi_1) \le \text{right}_{\text{OPT}}(\pi_1)$. Therefore, by setting $f = \{\pi_1\}$ we have that: $\frac{c\left(D_\varnothing\right)}{1} < \frac{c\left(D_{\{\pi_1\}}\right)}{1} + \frac{c\left(R_{\{\pi_1\}}\right)}{2}$. The last inequality directly implies that:

$$\frac{C(\text{N})}{C(\text{OPT})} = \frac{c(D_\varnothing) + c(R_\varnothing) + c(R_{\{\pi_1\}})}{c(D_{\{\pi_1\}}) + c(R_\varnothing) + c(R_{\{\pi_1\}})} \le \frac{c(D_{\{\pi_1\}}) + c(R_\varnothing) + \frac{3}{2}c(R_{\{\pi_1\}})}{c(D_{\{\pi_1\}}) + c\left(R_\varnothing\right) + c\left(R_{\{\pi_1\}}\right)} \le \frac{3}{2}.$$

Case $m = 2$. When $m = 2$, the player π_1 leads the dynamic from OPT to σ^1 and player π_2 leads the dynamics from σ^1 to N. Therefore the following must hold:

$$\text{left}_{\sigma^1}(\pi_1) \le c_{\sigma^1}(\pi_1) < c_{\text{OPT}}(\pi_1) \le \text{right}_{\text{OPT}}(\pi_1),$$
$$\text{left}_{\text{N}}(\pi_2) \le c_{\text{N}}(\pi_2) < c_{\sigma^1}(\pi_2) \le \text{right}_{\sigma^1}(\pi_2).$$

Therefore, by setting $f = \{\pi_1, \pi_2\}$ we have that:

$$\frac{c\left(D_\varnothing\right)}{1} + \frac{c\left(D_{\{\pi_2\}}\right)}{2} < \frac{c\left(D_{\{\pi_1\}}\right)}{1} + \frac{c\left(R_{\{\pi_1\}}\right)}{2} + \frac{c\left(D_{\{\pi_1,\pi_2\}}\right)}{2} + \frac{c\left(R_{\{\pi_1,\pi_2\}}\right)}{3}$$

$$\frac{c\left(D_\varnothing\right)}{2} + \frac{c\left(D_{\{\pi_1\}}\right)}{1} < \frac{c\left(D_{\{\pi_2\}}\right)}{2} + \frac{c\left(R_{\{\pi_2\}}\right)}{2} + \frac{c\left(D_{\{\pi_1,\pi_2\}}\right)}{1} + \frac{c\left(R_{\{\pi_1,\pi_2\}}\right)}{2}.$$

Without loss of generality we can add the following constraints:

$$\sum_{e \in OPT} c(e) \leq 1, \qquad \forall e \in E.\ c(e) \geq 0.$$

We need to bound the value of $c(D_\varnothing) - \frac{1}{2}c(D_{\{\pi_1,\pi_2\}})$ with respect to the above inequalities. Such a bound can be obtained by forming a linear program from all the above equations including the appropriate objective function. We have solved this linear program on a computer using a standard LP solver. This way we have obtained the following bound: $c(D_\varnothing) - \frac{1}{2}c(D_{\{\pi_1,\pi_2\}}) \leq \frac{5}{11} < \frac{1}{2}$.

In the remainder of the proof similar bounds have been obtained in the same way by using a LP solver. Further, the cost of states N and OPT are:

$$C(\mathrm{N}) = c(D_\varnothing) + c(R_\varnothing) + c(D_{\{\pi_1\}}) + c(R_{\{\pi_1\}}) + c(D_{\{\pi_2\}}) + c(R_{\{\pi_2\}}) + c(R_{\{\pi_1,\pi_2\}}),$$

and

$$C(\mathrm{OPT}) = c(D_{\{\pi_1,\pi_2\}}) + c(R_\varnothing) + c(D_{\{\pi_1\}}) + c(R_{\{\pi_1\}}) + c(D_{\{\pi_2\}}) + c(R_{\{\pi_2\}}) + c(R_{\{\pi_1,\pi_2\}}),$$

respectively. Therefore, by using the upper bound on $c(D_\varnothing) - \frac{1}{2}c(D_{\{\pi_1,\pi_2\}})$ we obtain that: $\frac{C(\mathrm{N})}{C(\mathrm{OPT})} \leq \frac{16}{11} < \frac{3}{2}$.

Case $m = 3$. Similarly to the previous case, we will construct a suitable linear program. We know that $\pi_1 \neq \pi_2$ and $\pi_2 \neq \pi_3$. If $\pi_1 = \pi_3$ then by Lemma 1 we have that $PoS \leq \frac{3}{2}$. Therefore we can assume that $\pi_1 \neq \pi_3$. The following must hold:

$$\begin{aligned}
\mathrm{left}_{\sigma^1}(\pi_1) \leq c_{\sigma^1}(\pi_1) < c_{\mathrm{OPT}}(\pi_1) \leq \mathrm{right}_{\mathrm{OPT}}(\pi_1),\\
\mathrm{left}_{\sigma^2}(\pi_2) \leq c_{\sigma^2}(\pi_2) < c_{\sigma^1}(\pi_2) \leq \mathrm{right}_{\sigma^1}(\pi_2),\\
\mathrm{left}_{\mathrm{N}}(\pi_3) \leq c_{\mathrm{N}}(\pi_3) < c_{\sigma^2}(\pi_3) \leq \mathrm{right}_{\sigma^2}(\pi_3).
\end{aligned}$$

By setting $f = \{\pi_1, \pi_2, \pi_3\}$ we obtain a set of constraints that along with $C(\mathrm{OPT}) \leq 1$ and maximization target $c(D_\varnothing) - \frac{1}{2}c(D_{\{\pi_1,\pi_2,\pi_3\}})$ constitute a linear program with a solution $c(D_\varnothing) - \frac{1}{2}c(D_{\{\pi_1,\pi_2,\pi_3\}}) \leq \frac{198}{487} < \frac{1}{2}$. Substituting it into the ratio of the costs of N and OPT we get that: $\frac{C(\mathrm{N})}{C(\mathrm{OPT})} \leq \frac{685}{487} < \frac{3}{2}$.

Case $m \geq 4$. Here it is enough to consider the case $m = 4$. This is due to the fact that the inequalities obtained by the dynamics of the first 4 players are strong enough to bound the cost of the whole ring. This gives the bound on the cost of any Nash equilibrium the dynamics will converge to, because even if more players move the cost of the final state will be smaller than the cost of the whole ring. We show that if $m = 4$ then $c(D_\varnothing) < \frac{1}{2} \cdot C(\mathrm{OPT})$. Clearly, adding new constraints for $m > 4$ cannot increase this bound. Then let us consider $m = 4$. As in the previous case we have that $\pi_1 \neq \pi_2$ and $\pi_2 \neq \pi_3$ and $\pi_1 \neq \pi_3$, anyway we are not able to derive any conclusion about π_4. It follows that we have to consider 3 subcases, i.e., $\pi_4 = \pi_1$, $\pi_4 = \pi_2$ and $\pi_4 \neq \pi_z$ for $z = 1, 2, 3$. As usually in this proof, we are going to derive sets of constraints that must hold at every step of the dynamics by using functions left and right.

By summarizing we get three different sets of constraints corresponding to three different linear programs. In each of them it suffices to consider maximization target $c\big(D_\varnothing\big)$ assuming that $C(\text{OPT}) \leq 1$ wlog. In all cases the maximum value of $c\big(D_\varnothing\big)$ turns out to be smaller than $\frac{1}{2}$. Hence, by (1) for all these cases we know that PoS is bounded by $\frac{3}{2}$. □

Corollary 1. *In a ring design game, if the cost of the entire ring is larger than* 3/2 *times the cost of an optimal state, then the improvement dynamics starting from an optimal state converges quickly, within at most 3 steps, to a Nash equilibrium.*

The following theorem will be given in the full version of this paper and constructs an example (Figure 1) when the above upper bound is reached.

Theorem 2. *Given any* $\epsilon > 0$*, there exists an instance of the ring design game such that the price of stability is at least* $\frac{3}{2} - \epsilon$*.*

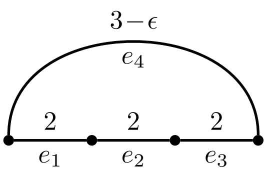

Fig. 1. The lower bound example for PoS on the ring. There is a player associated with each edge. The optimum uses three edges of cost 2 whereas the only Nash equilibrium uses the whole ring.

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Tight Lower Bounds on Envy-Free Makespan Approximation

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Abstract. In this work we give a tight lower bound on makespan approximation for envy-free allocation mechanisms dedicated to scheduling tasks on unrelated machines. Specifically, we show that no mechanism exists that can guarantee an envy-free allocation of jobs to m machines with a makespan less than a factor of $O(\log m)$ of the minimal makespan. Combined with previous results, this paper definitively proves that the optimal algorithm for obtaining a minimal makespan for any envy-free division can at best approximate the makespan to a factor of $O(\log m)$.

Keywords: makespan approximation, locally-efficient, envy-freeness, mechanism design, scheduling.

1 Introduction

Consider the scenario in which there is a set of tasks and a workforce that is commissioned to complete it. The tasks we are interested in are *indivisible*, that is, we can assign more than one job to each worker but two workers cannot both work on the same task. One goal is to complete all the tasks in the shortest period of time. However, each worker is specialized in his own way and ranks the difficulty of performing each task differently from his colleagues. We would also like to allocate the tasks in such a way that no worker would prefer to complete the workload of a colleague over his own. This problem is the focus of this paper.

Determining fair division is at the heart of a large body of research in computer science. One of its earliest occurrences in literature was in 1947 when Neyman, Steinhaus, Banach and Bronislaw modeled it as the problem of how to find a fair partitioning of cake ([13], [14]). Since then, several books including [1], [2], [3], [11], [9], [12] have been dedicated to the subject. The problem is generally described as a way of assigning n jobs to be processed by m machines or agents in a fair manner. One of the reasons that this area of research is so rich is that there are multiple ways to characterize a fair allocation. One way to do so is to consider divisions that preserve *envy-freeness*, the notion that no agent would be better off if he were assigned the set of jobs given to another ([5], [6]). In the scenario where jobs can be divided among more than one machine, one solution would be to divide all jobs equally among all agents (although depending on the

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set-up of the problem this division might be ill-defined, i.e. if one agent takes an infinitely long time to process a specific job).

Determining a fair allocation when jobs are not divisible is less straightforward. In order to furnish a solution we must define a *mechanism* that determines an allocation as well as payments either to or from the agents, or between agents and the mechanism or agents among themselves. We consider the utility of each agent to be quasi-linear, i.e. the difference between the payment he receives and the cost to process his assignment of jobs.

When determining an optimal envy-free solution other goals can be considered such as revenue maximization or economic efficiency. The additional goal we described in our earlier example is *makespan* minimization, or the intention to minimize the longest processing time of jobs on any one machine. In their paper [8], Hartline et. al. considers a task schedule for m machines in which the minimum makespan for any indivisible allocation is 1. They then go on to show that no mechanism exists that can provide an envy-free indivisible allocation with a makespan less than $2-1/m$. In addition they provide an algorithm to find an envy-free indivisible allocation that upper bounds the makespan by $(m+1)/2$. Cohen et. al. generalized and tightened the bounds on makespan approximation [4]. In their paper, they show that there does not exist a mechanism that provides an envy-free division with a makespan less than $O(\log m/\log\log m)$ times the optimal, and demonstrate a polynomial time algorithm that finds an envy-free scheduling that approximates the minimal makespan by a factor of $O(\log m)$.

Our contribution is to tighten the lower bound on makespan approximation to the upper bound. Specifically, we show that no mechanism exists that can guarantee an envy-free allocation of jobs to m machines with a makespan less than a factor of $O(\log m)$ of the minimal makespan. The technique we use is a refined variation of the one employed by Cohen et. al., but the more intricate construction gives us a better bound. This result definitively proves that the optimal algorithm for obtaining a minimal makespan for any envy-free division can at best approximate the makespan to a factor of $O(\log m)$.

2 Preliminaries

The scheduling problem that we are interested in is the following: We have n indivisible jobs and m machines. We are given a *cost matrix* $\mathbf{c} = (c_{i,j})_{1\leq i\leq m, 1\leq j\leq n}$ where $c_{i,j}$ is the cost of running job j on machine i. The *allocation matrix* $\mathbf{a} = (a_{i,j})_{1\leq i\leq m, 1\leq j\leq n}$ specifies which jobs are assigned to run on which machines, so that $a_{i,j} = 1$ if we run job j on the ith machine and $a_{i,j} = 0$ otherwise. Since our focus is on indivisible jobs, if $a_{i,j} = 1$, then $a_{i',j} = 0$ for all $i' \neq i$. In the case where jobs are divisible $a_{i,j} \in [0,1]$. In both the divisible and indivisible job cases, $\sum_{i=1}^{m} a_{i,j} = 1$, i.e. we always find an allocation of jobs to machines where every job is processed in its entirety.

Let $\bar{c}_i = (c_{i1},\ldots,c_{in})$ denote the ith row of the cost matrix $\mathbf{c}$ and let $\bar{a}_i = (a_{i1},\ldots,a_{in})$ denote the ith row of the allocation matrix $\mathbf{a}$. Then the *load* on machine i, or the cost of running the jobs assigned to machine i is $\bar{c}_i \cdot \bar{a}_i =$

$\sum_{j=1}^{n} c_{i,j} a_{i,j}$. The *makespan* of an assignment is the maximum load on any machine, or $\max_{1 \leq i \leq m} \bar{c}_i \cdot \bar{a}_i$.

We can formulate the problem of finding the minimum makespan for indivisible jobs as an integer programming problem and for divisible jobs as a linear programming problem. In 1990, Lenstra, et. al. introduced a 2-approximation algorithm for finding the minimum makespan for indivisible jobs, and showed that there does not exist a ρ-approximation algorithm for finding the minimum makespan for $\rho < 3/2$ unless $P = NP$ [10].

In this formulation we consider each of the m machines as a selfish agent. An allocation function a is mapped to the $m \times n$ cost matrix $\mathbf{c}$ so that $a(\mathbf{c}) = \mathbf{a}$. As before let $\bar{c}_i = (c_{i1}, \ldots, c_{in})$ and $\bar{a}(\mathbf{c})_i = (a(\mathbf{c})_{i1}, \ldots, a(\mathbf{c})_{in})$ denote the ith row of $\mathbf{c}$ and $a(\mathbf{c})$, respectively. Let p denote a payment function that is a mapping from $\mathbf{c}$ to $\mathbb{R}^m$, and let $p(\mathbf{c})_i$ denote the ith coordinate of $p(\mathbf{c})$.

We define a *mechanism* as a pair of functions, $M =< a, p >$ where a is the allocation function and p is the payment function. Given a mechanism $< a, p >$ with a cost matrix $\mathbf{c}$, the *utility* of agent i is $p(\mathbf{c})_i - \bar{c}_i \cdot \bar{a}_i$. A mechanism is considered *envy-free* if no agent can increase his utility by trading his job allocation and payment with another player. More formally, a mechanism is envy-free if, for all $j \in 1..n$,

$$p(\mathbf{c})_i - \bar{c}_i \cdot \bar{a}_i \geq p(\mathbf{c})_j - \bar{c}_i \cdot \bar{a}_j. \tag{1}$$

We call an allocation function *envy-free implementable* (*EF*-implementable) if there exists a payment function p such that mechanism $< a, p >$ is envy-free.

An allocation function is *locally-efficient* if for all cost matrices c and permutations π of $1, \ldots, m$, we have

$$\sum_{i=1}^{m} \bar{c}_i \cdot \bar{a}_i \leq \sum_{i=1}^{m} \bar{c}_i \cdot \bar{a}_{\pi(i)}. \tag{2}$$

Hartline, et. al. introduced the following useful theorem [8].

Theorem 1. *An allocation is EF-implementable if and only if it is locally-efficient.*

3 Main Result: Lower Bound on Envy-Free Makespan Approximation

We give a lower bound of $\Omega(\log m)$ on the makespan achievable by any envy-free allocation of jobs.

Let $n = \frac{\tilde{n}}{\log \tilde{n}} + 1$ be the number of jobs for some $\tilde{n} \in \mathbb{Z}^+$. The number of machines is $m = n + l$ where $2^l = \log \tilde{n}$. Let $\mathbf{c}$ denote a cost matrix where $c_{i,j}$ gives the cost of running job j on machine i. For this cost matrix, we have

$$\mathbf{c} = \left(\begin{array}{llllll} 1 & \infty & \infty & \dots & \infty & \infty \\ 1-\frac{\log\tilde{n}}{2\tilde{n}} & 1 & \infty & \dots & \infty & \infty \\ 1-\frac{2\log\tilde{n}}{2\tilde{n}} & 1-\frac{\log\tilde{n}}{2(\tilde{n}-1)} & 1 & \dots & \infty & \infty \\ 1-\frac{3\log\tilde{n}}{2\tilde{n}} & 1-\frac{2\log\tilde{n}}{2(\tilde{n}-1)} & 1-\frac{\log\tilde{n}}{2(\tilde{n}-2)} & \dots & \infty & \infty \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-\left(\frac{\tilde{n}}{\log\tilde{n}}-1\right)\frac{\log\tilde{n}}{2\tilde{n}} & 1-\left(\frac{\tilde{n}}{\log\tilde{n}}-2\right)\frac{\log\tilde{n}}{2(\tilde{n}-1)} & 1-\left(\frac{\tilde{n}}{\log\tilde{n}}-3\right)\frac{\log\tilde{n}}{2(\tilde{n}-2)} & \dots & 1 & \infty \\ 1/2 & 1-\left(\frac{\tilde{n}}{\log\tilde{n}}-1\right)\frac{\log\tilde{n}}{2(\tilde{n}-1)} & 1-\left(\frac{\tilde{n}}{\log\tilde{n}}-2\right)\frac{\log\tilde{n}}{2(\tilde{n}-2)} & \dots & \geq 1/2 & 1 \\ \hline 2 & 2 & 2 & \dots & 2 & 2 \\ 4 & 4 & 4 & \dots & 4 & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2^l & 2^l & 2^l & \dots & 2^l & 2^l \end{array}\right) \tag{3}$$

Each row i for $1 \leq i \leq n+l$ gives the costs for the ith machine and each entry $c_{i,j}$ in the matrix denotes the cost of running job j on machine i. The horizontal line lies between machines n and $n+1$. For $1 \leq i \leq n$, the cost of running job j on machine i is given by

$$c_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 1 - \frac{(i-j)\log\tilde{n}}{2(\tilde{n}-j+1)} & \text{if } i > j \\ \infty & \text{if } i < j \end{cases} . \tag{4}$$

Note that for $1 \leq i < n$ and $i > j$, we have $c_{i,j} - c_{i+1,j} = \frac{\log\tilde{n}}{2(\tilde{n}-j+1)}$. For $n < i \leq n+l$, the cost to process any job on machine i is 2^{i-n}.

Lemma 1. *For $1 \leq i \leq n+l$ and $1 \leq j \leq n$, we have $c_{i,j} \geq 1/2$.*

Proof. The statement is trivially true for $i \leq j$ and $i > n$. For $j < i \leq n$ we prove the equivalent statement that $(i-j)\log\tilde{n} < \tilde{n} - j + 1$. The LHS is maximized when i is maximized so we have:

$$(i-j)\log\tilde{n} \leq \left(\frac{\tilde{n}}{\log\tilde{n}} + 1 - j\right)\log\tilde{n} = \tilde{n} - (j-1)\log\tilde{n} \leq \tilde{n} - (j-1) \tag{5}$$

for $\tilde{n} > 2$.

For this cost matrix, the optimal makespan is 1. We reach this makespan when $i = j$. Since the cost of running any job on any machine is strictly greater than 1/2, if more than one job is run per machine the makespan will be more than 1. Any other permutation of jobs would require at least one job j to be run on some machine i for $i < j$ or for $n < i \leq n+l$. Either of these scenarios would give us a makespan of at least 2.

We show that any envy-free makespan for this cost matrix has a lower bound of $\log n$. More specifically, we show that no matter how we partition the n jobs into $n+l$ bundles, any locally-efficient assignment of the bundles has a makespan of at least $\log\tilde{n} \geq \log n$. This establishes that there does not exist an algorithm that can always find a makespan less than $\log n$.

Theorem 2. *For any partition of n jobs into bundles, the makespan for every locally efficient assignment of bundles is at least $2^l = \log \tilde{n}$.*

Before we prove this theorem, we introduce the following useful lemma.

Lemma 2. *Any allocation for the cost matrix* $\mathbf{c}$ *that has makespan fewer than $2^l = \log \tilde{n}$ has the following properties:*

1. *Fewer than 2^{l+1} jobs run on each machine.*
2. *Fewer than $2^l/2^{i-n}$ jobs run on each machine $n+i$ for $n < i \leq n+l$.*
3. *The total number of jobs running on machines $n+1, \ldots, n+l$ is fewer than 2^l.*

Proof. Property (1) follows directly from Lemma 1; Property (2) holds since $c_{i,j} = 2^{i-n}$ for $n < i \leq n+l$; and (3) follows from (2) because $\sum_{i=n+1}^{n+1} c_{i,j} < \sum_{i=n+1}^{n+1} 2^l/2^{i-n} = 2^l$.

Proof (Proof of Theorem 2). Consider an arbitrary partition of the n jobs into $n+l$ bundles with a makespan less than 2^l. Suppose that this assignment is locally-efficient. In order to prove this theorem by contradiction, we must provide a permutation of the assignment that decreases the total cost over all jobs. Since the cost of running a job on machine $n+l$ is 2^l, there are no jobs assigned to run on machine $n+l$. Therefore, the permutation we will provide is the one in which each bundle of jobs assigned to machine i is moved to machine $i+1$.

By Lemma 2 (1), less than 2^{l+1} jobs run on machine n, so the increase in cost from moving the bundle of jobs from machine n to machine $n+1$ is less than $2^{l+1}(2-1/2) = 3 \cdot 2^l$. For $n < i < n+l$, we have $c_{i+1,j} = 2c_{i,j}$, and so moving each bundle from machine i to $i+1$ in this range increases the cost by a factor of 2. By Lemma 2 (3), fewer than 2^l jobs run on this set of l machines, and so moving each bundle to the next machine would increase the total cost by less than $l \cdot 2^l$. Therefore, moving each bundle on machine i to machine $i+1$ for $n \leq i < n+l$ increases the cost of the assignment by less than $(l+3)2^l = (\log\log\tilde{n} + 3)\log\tilde{n}$.

By Lemma 2 (3), there are fewer than 2^l jobs running on machines $n+1, \ldots, n+l$, which implies that the total number of jobs running on machines $1, \ldots, n$ is greater than $n - 2^l$. Pairing this with Lemma 2 (1), we know that the total number of jobs running on machines $1, \ldots, n-1$ is greater than $n - 2^l - 2^{l+1} = n - 3 \cdot 2^l$. As noted earlier, moving any job j from machine i to machine $i+1$ in this range decrease the cost of the job by $\frac{\log \tilde{n}}{2(\tilde{n}-j+1)}$. Therefore, the total cost of moving all the bundles on machines $1, \ldots, n-1$ decreases by at least $(\frac{\log \tilde{n}}{2})(H_{\tilde{n}/\log \tilde{n}} - H_{3 \cdot 2^l}) \approx (\frac{\log \tilde{n}}{2})(\ln \tilde{n} - \ln \log \tilde{n} - \ln(3 \log \tilde{n}))$, where H_k is the kth harmonic number.

The decrease in cost from the first $n-1$ machines is larger than the increase in cost from the last $l+1$ machines and so the makespan for any locally efficient assignment must be greater than 2^l.

Corollary 3 *For any partition of n jobs into bundles, the makespan for every envy-free assignment of bundles is $\Omega(\log m)$.*

Proof. By Theorem 2, every locally efficient partition has a makespan of at least $\log \tilde{n} = \log((n-1)\log \tilde{n}) \geq \log n$. Since $m = n + l = O(n)$ for the cost matrix defined, it holds that it is an $\Omega(\log m)$ approximation.

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Biased Assimilation, Homophily, and the Dynamics of Polarization
(Working Paper)

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Abstract. Are we as a society getting more polarized, and if so, why? We try to answer this question through a model of opinion formation. Empirical studies have shown that homophily results in polarization. However, we show that DeGroot's well-known model of opinion formation based on repeated averaging can never be polarizing, even if individuals are arbitrarily homophilous. We generalize DeGroot's model to account for a phenomenon well-known in social psychology as *biased assimilation*: when presented with mixed or inconclusive evidence on a complex issue, individuals draw undue support for their initial position thereby arriving at a more extreme opinion. We show that in a simple model of homophilous networks, our biased opinion formation process results in either polarization, persistent disagreement or consensus depending on how biased individuals are. In other words, homophily alone, without biased assimilation, is not sufficient to polarize society. Quite interestingly, biased assimilation also provides insight into the following related question: do internet based recommender algorithms that show us personalized content contribute to polarization? We make a connection between biased assimilation and the polarizing effects of some random-walk based recommender algorithms that are similar in spirit to some commonly used recommender algorithms.

A full version of this paper is available at `http://arxiv.org/abs/1209.5998`.

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Generalized Weighted Model Counting: An Efficient Monte-Carlo meta-algorithm (Working Paper)

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Abstract. In this paper, we focus on computing the prices of securities represented by logical formulas in combinatorial prediction markets when the price function is represented by a Bayesian network. This problem turns out to be a natural extension of the weighted model counting (WMC) problem [1], which we call *generalized weighted model counting (GWMC)* problem. In GWMC, we are given a logical formula F and a polynomial-time computable weight function. We are asked to compute the total weight of the valuations that satisfy F.

Based on importance sampling, we propose a Monte-Carlo meta-algorithm that has a good theoretical guarantee for formulas in disjunctive normal form (DNF). The meta-algorithm queries an oracle algorithm that computes marginal probabilities in Bayesian networks, and has the following theoretical guarantee. When the weight function can be approximately represented by a Bayesian network for which the oracle algorithm runs in polynomial time, our meta-algorithm becomes a *fully polynomial-time randomized approximation scheme (FPRAS)*.

NOTE: A full version of this paper is available at
http://people.seas.harvard.edu/~lxia/Files/GWMC-WINE.pdf
This paper is part of a longer working paper, which we plan to submit to a journal that may not accept papers published previously in conferences.

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The AND-OR Game: Equilibrium Characterization*
(Working Paper)

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1 Introduction and Results Overview

Walrasian equilibrium is one of the basic notions in economic theory. Items are priced in such a way that the market clears i.e. the supply for each item equals the demand for it (or there may be items with excess supply priced at zero.) When there is a Walrasian equilibrium, it captures nicely the "right" pricing of items. Unfortunately, Walrasian equilibria are guarantee to exists only for limited classes of agents' valuations, namely gross-substitute valuations.

An alternative approach that we looked at in a previous work is to auction the items simultaneously, and analyze the resulting equilibria. For the simultaneous first price auction, the resulting pure Nash equilibria turn out to be in one-to-one correspondence with the Walrasian equilibria. Moreover, even when there is no Walrasian equilibrium, there is always a mixed Nash equilibrium for the simultaneous first price auction (with some tie breaking rule).

Walrasian equilibria may fail to exist even with two agents and two items when one of the agents views the items as complements while the other views them as substitutes. Here we consider the prototypical game of this form with a "pure" complement player, `AND`, and a pure substitute player `OR`. The `AND` valuation is 1 if it gets both items and zero otherwise, while the `OR` has a value of v for any single item (or both) and zero otherwise. For $v > 1/2$ there is no Walrsian equilibrium (or equivalently, pure Nash equilibrium) and in our previous work we presented a specific mixed Nash equilibrium for the `AND-OR` game.

In this work we completely characterize the mixed Nash equilibria of the `AND-OR` game, showing that they are all slight variants of a single one.

The full version of this paper is available on the arXiv.

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