Hybrid Domain Decomposition Solvers for the Helmholtz and the Time Harmonic Maxwell's Equation

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Summary. We present hybrid finite element methods for the Helmholtz equation and the time harmonic Maxwell equations, which allow us to reduce the unknowns to degrees of freedom supported only on the element facets and to use efficient iterative solvers for the resulting system of equations. For solving this system, additive and multiplicative Schwarz preconditioners with local smoothers and a domain decomposition preconditioner with an exact subdomain solver are presented. Good convergence properties of these preconditioners are shown by numerical experiments.

1 Introduction

When solving the Helmholtz equation with a standard finite element method (FEM), due to the oscillatory behaviour of the solution and the pollution error [8] a large number of degrees of freedom (DoFs) is needed to resolve the wave, especially for high wave numbers. To overcome this difficulty, many methods have been developed during the last years. Apart from *hp* FEM [8], Galerkin Least Square Methods [7] or Discontinuous Galerkin Methods [6], some methods make use of problem adapted functions like plane waves. The most popular among them are the Partition of Unity Method [9], the Discontinuous Enrichment Approach [5] or the UWVF [2, 10]. All these techniques end up with large, complex valued, indefinite, possible symmetric linear systems. Although some advances have been made [3, 4], efficient preconditioners for wave type problems are still a big challenge.

In the present work the hybrid FEM from [11] is used for the Helmholtz equation and extended to the Maxwell case. This method allows us to use efficient iterative methods for solving the resulting linear system of equations. Following hybridization techniques from [1], the tangential continuity of the flux field is broken across element interfaces. In order to impose continuity again, Lagrange multipliers supported only on the facets, which can be interpreted as the tangential component of the unknown field, are introduced. Adding a second set of Lagrange multipliers,

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representing the tangential component of the flux field, allows us, due to local Robin boundary conditions, to eliminate the volume DoFs. Because, after hybridization, there is no coupling between volume basis functions of different elements, elimination of the volume DoFs can be done cheaply element by element, and the system of equation is reduced onto the smaller set of Lagrange multipliers. For the reduced system we present additive (AS) and multiplicative Schwarz (MS) block preconditioners with blocks related to DoFs of one facet and element, respectively. Additionally a domain decomposition (DD) preconditioner, which directly solves for the DoFs belonging to one subdomain, is investigated. This preconditioner is especially advantageous for domains contains cavity like structures. Numerical tests show, that a preconditioned CG iteration has good convergence properties combined with these preconditioners.

2 Hybridization of the Wave Equations

In the sequence, we will stick to the following settings. As computational domain we consider a Lipschitz polyhedron $\Omega \subset \mathbb{R}^d$ with $d = 2,3$ and the boundary $\Gamma = \partial\Omega$. In the scalar case, we search for a function $u : \Omega \to \mathbb{C}$ and a vector valued field $\mathbf{v} : \Omega \to \mathbb{C}^d$, which fulfills the Helmholtz equation in mixed form

$$\operatorname{grad} u = i\omega\mathbf{v} \qquad \text{and} \qquad \operatorname{div}\mathbf{v} = i\omega u \qquad \text{in } \Omega$$

with absorbing boundary conditions $\mathbf{v}\cdot\mathbf{n} + u = g$ on Γ, where ω is the angular frequency and $\mathbf{n}$ the outer normal vector. From [9] we know, that the solution u exists and is unique.

In the vectorial case, i.e. the harmonic Maxwell's equations, we search for a vector valued function $\mathbf{E} : \Omega \to \mathbb{C}^3$ and a flux field $\mathbf{H} : \Omega \to \mathbb{C}^3$, which solves

$$\operatorname{curl}\mathbf{H} + i\omega\mathbf{E} = 0 \qquad \text{and} \qquad \operatorname{curl}\mathbf{E} - i\omega\mathbf{H} = 0 \qquad \text{in } \Omega$$

under the boundary condition $-\mathbf{n}\times\mathbf{H} + \mathbf{E}_{\|} = \mathbf{g}$ on Γ, where $\mathbf{E}_{\|}$ represents the tangential component of $\mathbf{E}$, i.e. $\mathbf{n}\times\mathbf{E}\times\mathbf{n}$.

When deriving the hybrid formulation, we use a regular finite element mesh $\mathscr{T}$ with elements T, and the set of facets is called $\mathscr{F}$. The vector $\mathbf{n}_T$ is the outer normal vector of the element T, and $\mathbf{n}_F$ represents the normal vector onto a facet F. Furthermore, we denote a volume integral as $\left(u,v\right)_T := \int_T uv\,\mathrm{d}\mathbf{x}$, and a surface integral as $\left\langle u,v\right\rangle_{\partial T} := \int_{\partial T} uv\,\mathrm{d}s$.

2.1 The Mixed Hybrid Formulation for the Helmholtz Equation

The mixed hybrid formulation for the Helmholtz equation was already introduced in [11]. For completeness, we repeat the problem formulation:

Find $(u,\mathbf{v},u^F,v^F) \in L^2(\Omega)\times H(\operatorname{div},T)\times L^2(\mathscr{F})\times L^2(\mathscr{F}) =: X\times\tilde{Y}\times X^F\times Y^F$, such that for all $(\sigma,\mathbf{w},\sigma^F,w^F) \in X\times\tilde{Y}\times X^F\times Y^F$

$$\sum_{T\in\mathscr{T}}\Big(\big(i\omega u,\sigma\big)_T-\big(i\omega\mathbf{v},\mathbf{w}\big)_T-\big(\operatorname{div}\mathbf{v},\sigma\big)_T-\big(u,\operatorname{div}\mathbf{w}\big)_T+\big\langle u^F,\mathbf{n}_T\cdot\mathbf{w}\big\rangle_{\partial T}$$
$$+\big\langle \mathbf{n}_T\cdot\mathbf{v},\sigma^F\big\rangle_{\partial T}+\big\langle \mathbf{n}_F\cdot\mathbf{v}-v^F,\mathbf{n}_F\cdot\mathbf{w}-w^F\big\rangle_{\partial T}\Big)+\big\langle u^F,\sigma^F\big\rangle_\Gamma=\big\langle g,\sigma^F\big\rangle_\Gamma.$$

2.2 The Mixed Hybrid Formulation for the Maxwell Problem

We will now concentrate on the derivation of the mixed hybrid formulation for the vectorial wave equation. We start from the mixed system of equations from above, multiply the first equation with a test function $\mathbf{e}\in U:=(L^2(\Omega))^3$ and the second one with a function $\mathbf{h}\in V:=H(\text{curl},\Omega)$ and integrate over the domain Ω. Performing integration by parts elementwise leads to

$$\sum_{T\in\mathscr{T}}\Big(\big(\operatorname{curl}\mathbf{H},\mathbf{e}\big)_T+\big(i\omega\mathbf{E},\mathbf{e}\big)_T\Big)=0 \qquad \forall\mathbf{e}\in U$$
$$\sum_{T\in\mathscr{T}}\Big(\big(\mathbf{E},\operatorname{curl}\mathbf{h}\big)_T-\big(i\omega\mathbf{H},\mathbf{h}\big)_T-\big\langle\mathbf{E},\mathbf{n}_T\times\mathbf{h}\big\rangle_{\partial T}\Big)=0 \qquad \forall\mathbf{h}\in V.$$

Note that for a tangential continuous field $\mathbf{E}$, i.e. $\mathbf{n}\times\mathbf{E}\times\mathbf{n}$ is continuous on element interfaces, the boundary integrals for inner facets cancel due to the tangential continuity of $\mathbf{h}$, and inserting the absorbing boundary condition into the boundary facet integrals leads to the standard mixed finite element formulation for our problem.

Next, the tangential continuity of the flux field $\mathbf{H}$ is broken across element interfaces, thus we search for $\mathbf{H}\in\tilde{V}:=\big\{\mathbf{v}\in(L^2(\Omega))^3\;:\;\mathbf{v}|_T\in H(\text{curl},T)\;\forall T\in\mathscr{T}\big\}$. In order to reinforce continuity, Lagrange multipliers $\mathbf{E}^F$, which are only supported on the element facets, i.e. they are from the space $U^F:=(L^2(\mathscr{F}))^3$, are introduced. The continuity of the tangential fluxes is reached via an additional equation, which forces the jump of $[\mathbf{n}\times\mathbf{H}]:=\mathbf{n}_{T_1}\times\mathbf{H}|_{T_1}+\mathbf{n}_{T_2}\times\mathbf{H}|_{T_2}$ for inner facets $F\in\mathscr{F}_I$ with adjacent elements T_1 and T_2 to zero, thus

$$\sum_{F\in\mathscr{F}_I}\big\langle[\mathbf{n}\times\mathbf{H}],\mathbf{e}\big\rangle_F=\sum_{T\in\mathscr{T}}\Big(\big\langle\mathbf{n}_T\times\mathbf{H},\mathbf{e}\big\rangle_{\partial T}-\big\langle\mathbf{n}_T\times\mathbf{H},\mathbf{e}\big\rangle_{\partial T\cap\Gamma}\Big)=0,\quad\forall\mathbf{e}\in U^F.$$

The resulting system of equations for $(\mathbf{E},\mathbf{H},\mathbf{E}^F)\in U\times\tilde{V}\times U^F$ reads as

$$\sum_{T\in\mathscr{T}}\Big(\big(\operatorname{curl}\mathbf{H},\mathbf{e}\big)_T+\big(i\omega\mathbf{E},\mathbf{e}\big)_T\Big)=0 \qquad \forall\mathbf{e}\in U$$
$$\sum_{T\in\mathscr{T}}\Big(\big(\mathbf{E},\operatorname{curl}\mathbf{h}\big)_T-\big(i\omega\mathbf{H},\mathbf{h}\big)_T-\big\langle\mathbf{E}^F,\mathbf{n}_T\times\mathbf{h}\big\rangle_{\partial T}\Big)=0 \qquad \forall\mathbf{h}\in\tilde{V}$$
$$-\sum_{T\in\mathscr{T}}\big\langle\mathbf{n}_T\times\mathbf{H},\mathbf{e}^F\big\rangle_{\partial T}+\big\langle\mathbf{E}^F,\mathbf{e}^F\big\rangle_\Gamma=\big\langle\mathbf{g},\mathbf{e}^F\big\rangle_\Gamma \qquad \forall\mathbf{e}^F\in U^F.$$

In this system of equations, the Lagrange parameter $\mathbf{E}^F$ plays the role of the tangential component of $\mathbf{E}$, evaluated on the facets. Because there is no coupling between volume DoFs belonging to different elements, it is possible to eliminate the volume unknowns $\mathbf{E}$ and $\mathbf{H}$, cheaply by static condensation (compare [1]). The resulting system of equations needs now to be solved only for the Lagrange multipliers.

In order to eliminate the inner DoFs, one has to solve the first two equations of the system from above for some function $\mathbf{E}^F$ element by element. But this is equivalent to solving a Dirichlet problem, and uniqueness of the solution can not be guaranteed. This drawback can be compensated by adding a new facet unknown $\mathbf{H}^F \in V^F := (L^2(\mathscr{F}))^3$ representing $\mathbf{n}_F \times \mathbf{H}$ on the facets via a consistent stabilization term $\sum_T \langle \mathbf{n}_F \times \mathbf{H} - \mathbf{H}^F, \mathbf{n}_F \times \mathbf{h} - \mathbf{h}^F \rangle_{\partial T}$. We obtain

$$\sum_{T \in \mathscr{T}} \Big((\operatorname{curl} \mathbf{H}, \mathbf{e})_T + (i\omega \mathbf{E}, \mathbf{e})_T \Big) = 0 \qquad \forall \mathbf{e} \in U \tag{1}$$

$$\sum_{T \in \mathscr{T}} \Big((\mathbf{E}, \operatorname{curl} \mathbf{h})_T - (i\omega \mathbf{H}, \mathbf{h})_T - \langle \mathbf{E}^F, \mathbf{n}_T \times \mathbf{h} \rangle_{\partial T} - \langle \mathbf{n}_T \times \mathbf{H}, \mathbf{n}_T \times \mathbf{h} \rangle_{\partial T} + \langle \mathbf{H}^F, \mathbf{n}_F \times \mathbf{h} \rangle_{\partial T} \Big) = 0 \qquad \forall \mathbf{h} \in \tilde{V} \tag{2}$$

$$\sum_{T \in \mathscr{T}} \Big(\langle \mathbf{n}_F \times \mathbf{H}, \mathbf{h}^F \rangle_{\partial T} - \langle \mathbf{H}^F, \mathbf{h}^F \rangle_{\partial T} \Big) = 0 \qquad \forall \mathbf{h}^F \in V^F \tag{3}$$

$$-\sum_{T \in \mathscr{T}} \langle \mathbf{n}_T \times \mathbf{H}, \mathbf{e}^F \rangle_{\partial T} + \langle \mathbf{E}^F, \mathbf{e}^F \rangle_\Gamma = \langle \mathbf{g}, \mathbf{e}^F \rangle_\Gamma \qquad \forall \mathbf{e}^F \in U^F. \tag{4}$$

Now, by static condensation the time harmonic Maxwell's equation with absorbing boundary conditions has to be solved on the element level, where uniqueness is guaranteed, and the resulting system contains only the facet unknowns $\mathbf{E}^F$ and $\mathbf{H}^F$. Thus we search for a function $\mathbf{w} \in W := U^F \times V^F$ such that

$$s(\mathbf{w}, \mathbf{v}) = f(\mathbf{v}) \qquad \forall \mathbf{v} \in W,$$

where the Schur complement bilinearform s and the linearform f are obtained from (1) to (4) by eliminating the unknowns $\mathbf{E}$ and $\mathbf{H}$. Elimination of the inner DoFs can be also seen as calculating for a given incoming impedance trace $\mathbf{E}^F - \mathbf{H}^F$ the resulting outgoing impedance trace $\mathbf{E}^F + \mathbf{H}^F$ on the element level. By exchanging the Dirichlet and Neumann traces $\mathbf{E}^F, \mathbf{H}^F$ by incoming and outgoing impedance traces, one obtains an equivalent formulation which fits well into the context of the UWVF of [2].

3 Iterative Solvers

In this section, we focus on solving the system of equations. As already mentioned, the volume DoFs can be eliminated cheaply element by element, and the resulting system of equation just has to be solved for the much smaller number of facet DoFs. Because volume DoFs of one element couple apart from themselves only to facet DoFs of the surrounding facets, the Schur complement matrix S obtained by static condensation is sparse, and it just has nonzero entries between facet DoFs belonging to facets of the same element. Due to the hybrid formulation, efficient iterative solvers can be used for the reduced system of equations.

Because the Schur complement matrix is complex symmetric, a preconditioned CG-iteration together with an AS or MS block preconditioner, M_{AS} and M_{MS} is used, although convergence for complex symmetric matrices is not guaranteed. The iteration matrices of these two preconditioners are given as

$$I - M_{AS}^{-1} S = I - \sum_{i=1}^{n} P_i,$$

$$I - M_{MS}^{-1} S = \left(\prod_{i=n}^{1} (I - P_i) \right) \left(\prod_{i=1}^{n} (I - P_i) \right),$$

where P_i is the matrix representation of the variational projector $\mathscr{P}_i : W \to W_i \subset W$ with respect to the bilinearform s. In the scalar case $W = X^F \times Y^F$. We will use two different choices of subspaces W_i, functions supported on the facet F_i or on facets, which are boundary facets of the element T_i. Note that the first strategy leads to nonoverlapping blocks, while the blocks of the second choice overlap.

Apart from an AS or MS Preconditioner, a DD preconditioner compareable to [12] was used, which is based on a partitioning of the domain Ω into N subdomains Ω_i. The iteration matrix of this preconditioner can be described by

$$I - M_{DD}^{-1} S = \left(\prod_{i=n}^{1} (I - P_{I,i}) \right) \left(I - \sum_{i=1}^{N} P_{\Omega_i} \right) \left(\prod_{i=1}^{n} (I - P_{I,i}) \right),$$

where P_{Ω_i} and $P_{I,i}$ are matrices corresponding to variational projection operators which project to the spaces W_{Ω_i} and $W_{I,i}$. The space W_{Ω_i} contains functions which are supported only on facets in the interior of the subdomain Ω_i, while the space $W_{I,i}$ is choosen such that it contains functions which are only supported on facets of an element T_i such that $\partial T_i \cap \partial \Omega_j \neq \emptyset$. Again a nonoverlapping option is to collect the functions supported on a facet F_i which is located on Γ or the subdomain interfaces in $W_{I,i}$. Thus, in each preconditioner step a forward block Gauss Seidel iteration is carried out, followed by a direct inversion of each subdomain block and a backward block Gauss Seidel step. Note that solving directly for the unknowns in a subdomain is equivalent to solve a problem with robin boundary conditions on the subdomain, and uniqueness and existence are guaranteed.

One big advantage of the DD preconditioner is, that it can cope with problems containing cavity like structures. For such problems other preconditioners suffer from internal reflections, which leads to high iteration numbers. If the whole cavity is contained in one single subdomain Ω_i , the DD preconditioner inverts the whole matrix block related to the cavity, and internal reflections are treated exactly. Thus they do not influence the iteration number.

4 Numerical Results

In order to demonstrate the dependence of the number of iterations on polynomial order, wavelength and meshsize h for the presented preconditioners, we choose a simple two dimensional model problem with a wave of Gaussian amplitude and wavelength λ propagating through a unit square domain (compare Fig. 1). For a meshsize $h = \lambda = 0.1$ the lefthand plot shows the number of iterations for different polynomial orders. For the three preconditioners, the DoFs of an element were collected in one block. In addition, for the DD preconditioner, the computational domain was divided into nine subdomains. If the polynomial order is large enough to resolve the wave, i.e. larger than four, the number of iterations stays constant or is only slightly growing with growing polynomial order, while the number of facet unknowns grows linearly in 2D.

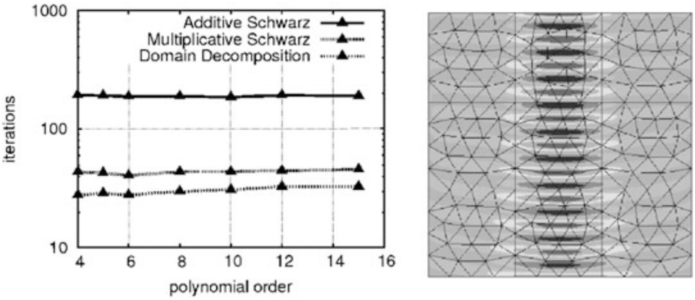

Fig. 1. Iterations depending on the polynomial order (*left*) for the 2D model problem (*right*)

Table 1. Iterations depending on wavelength and mesh size for the MS/DD Preconditioner ($p = 6$).

λ	0.64	0.32	0.16	0.08	0.04	0.02	0.01
$h = 0.16$	35/40	35/38	32/33	31/31			
$h = 0.08$	52/42	48/38	50/36	47/33	50/38		
$h = 0.04$	88/55	76/47	74/43	76/39	65/35	97/59	
$h = 0.02$	147/75	129/55	113/48	117/44	118/42	115/38	199/82
$h = 0.01$	246/107	236/80	226/60	203/53	228/49	271/50	291/45

Next we investigate the dependence on h and λ for a fixed polynomial order of 6. The results are presented in Table 1. For λ smaller than $\frac{h}{2}$, which corresponds to less than three unknowns per wavelength, the solution can not be resolved, and the solvers show large iteration numbers. Fixing h, the iteration number is minimal at about $h \approx \lambda$, i.e. at about six unknowns per wavelength, and it increases for growing wavelength. For $h = 0.16$ every subdomain consists of only a small number of elements, and an inversion of the DoFs subdomain by subdomain is compareable to an inversion element by element. Therefore the two preconditioners show about the same performance. If h decreases, it is more and more advantageous to collect the unknowns in subdomain blocks. While the iteration number almost doubles for the MS preconditioner if the mesh size is divided by 2, the increase is much less for the DD preconditioner. Table 2 shows, that the DD preconditioner also performs better than the MS preconditioner with respect to time, although one iteration is more expensive.

Table 2. Iteration times for $\lambda = 0.08$ and a polynomial order of 6.

h	DoFs	MS	DD
0.16	69980	0.35	0.37
0.08	217900	1.73	1.33
0.04	701228	9.30	5.15
0.02	2518524	53.5	22.4
0.01	9857920	367	111

Table 3. Iteration numbers and computational times for the cavity and the square.

	cavity		square	
	its.	time(s)	its.	time(s)
DD (element)	35	40.4	34	31.2
DD (facet)	64	69.7	61	59.7
MS (element)	1612	1720	102	88.9
AS (element)	$> 10^5$	$>$ 1h	575	186

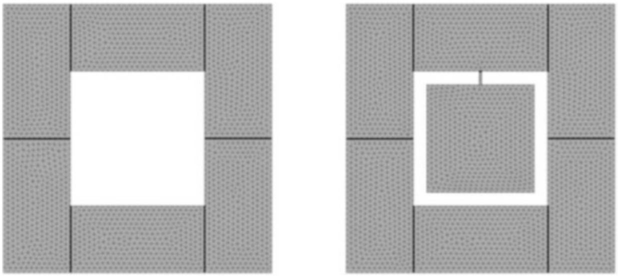

Fig. 2. A resonator (*right*) is compared with the domain without cavity (*left*)

Now we compare the preconditioners for a resonator and the domain without cavity (compare Fig. 2). From the top of the square an incident wave with $\lambda = 0.01$ is prescribed. The DD-preconditioner uses, depending on the presence of the cavity six and seven subdomains, respectively, where all cavity DoFs, including the cavity boundary are collected in one single block. Table 3 shows the iteration numbers and computational times for different preconditioners and for the two examples. For the domain without cavity the performance of the preconditioners is compareable. When the cavity is added, reflections inside the cavity lead to an enormous increase in iteration numbers and computational times for the AS and the MS preconditioner. Because of direct inversion of the cavity DoFs, the DD preconditioner does not suffer from internal reflections and the iteration number stays almost constant, which leads together with a larger number of unknowns to a moderate increase in computational time.

We finish the numerical results section with an example from optics. A small sphere with radius 0.3 and refractive index 2 is placed (not exactly in the center) in a spherical computational domain with radius 1 and background refractive index 1.

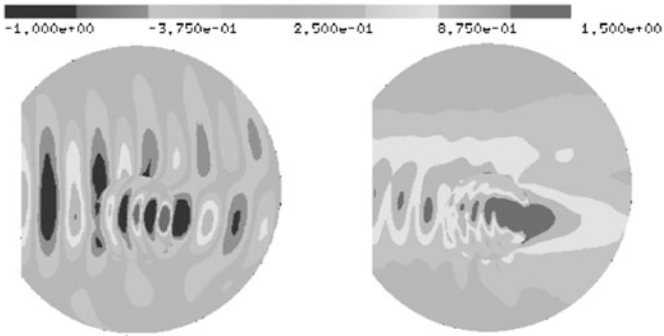

Fig. 3. Real part of E_y (*left*) and $|E|$ (*right*) evaluated at a cross section parallel to the xy plane

We prescribe an incident wave from the left with a Gaussian amplitude and wavelength 0.35, such that the diameter of the computational domain is approximately six wavelength in free space. In order to resolve the wave we used 3,256 elements with a polynomial order of 6, which results in 1.66 millions of unknowns. The solution (compare Fig. 3) was obtained by 258 cg-iterations with a Block AS preconditioner.

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