Chapter 6
Decision Making on the Basis of Fuzzy Geometry

6.1 Motivation

Decision making is conditioned by relevant information. This information very seldom has reliable numerical representation. Usually, decision relevant information is perception-based. A question arises of how to proceed from perception-based information to a corresponding mathematical formalism. When perception-based information is expressed in NL, the fuzzy set theory can be used as a corresponding mathematical formalism and then the theories presented in Chapters 3,4,5 can be applied for decision analysis. However, sometimes perception-based information is not sufficiently clear to be modeled by means of membership functions. In contrast, it remains at a level of some cloud images which are difficult to be caught by words. This imperfect information caught in perceptions cannot be precisiated by numbers or fuzzy sets and is referred to as *unprecisiated* information. In order to better understand a spectrum of decision relevant information ranging from numbers to unprecisiated information, let us consider a benchmark problem of decision making under imperfect information suggested Prof. Lotfi Zadeh. The problem is as follows.

Assume that we have two open boxes, A and B, each containing twenty black and white balls. A ball is picked at random. If I pick a white ball from A, I win a1 dollars; if I pick a black ball, I lose a2 dollars. Similarly, if I pick a white ball from B, I win b1 dollars; and if I pick a black ball, I lose b2 dollars. Then, we can formulate the five problems dependent on the reliability of the available information:

Case 1. I can count the number of white balls and black balls in each box. Which box should I choose?

Case 2. I am shown the boxes for a few seconds, not enough to count the balls. I form a perception of the number of white and black balls in each box. These perceptions lead to perception-based imprecise probabilities which allow to be described as fuzzy probabilities. The question is the same: which box should I choose.

Case 3. I am given enough time to be able to count the number of white and black balls, but it is the gains and losses that are perception-based and can be described as fuzzy numbers. The question remains the same.

Case 4. Probabilities, gains and losses are perception-based and can be described as fuzzy probabilities and fuzzy numbers. The question remains the same.

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Case 5. The numbers of balls of each color in each box cannot be counted. All a I have are visual perceptions which cannot be precisiated by fuzzy probabilities.

Let us discuss these cases. Case 1 can be successfully solved by the existing theories because it is stated in numerical information. Cases 2-4 are characterized by linguistic decision-relevant information, and therefore, can be solved by the decision theory suggested in Chapter 4. No theory can be used to solve Case 5 as it is stated, including the theory suggested in Chapter 4, because this case is initially stated in informational framework of visual perceptions for which no formal decision theory is developed. However, humans are able to make decisions based on visual perceptions. Modeling of this outstanding capability of humans, even to some limited extent, becomes a difficult yet a highly promising research area. This arises as a motivation of the research suggested in this chapter. In this chapter we use Fuzzy Geometry and the extended fuzzy logic [15] to cope with uncertain situations coming with unprecisiated information. In this approach, the objects of computing and reasoning are geometric primitives, which model human perceptions when the latter cannot be defined in terms of membership functions. The fuzzified axioms of Euclidean geometry are used and the main operations over fuzzy geometric primitives are introduced. A decision making method with outcomes and probabilities described by geometrical primitives is developed. In this method, geometrical primitives like fuzzy points and fuzzy lines represent the basic elements of the decision problem as information granules consisting of an imprecise value of a variable and the confidence degree for this value. The decision model considers a knowledge base with fuzzy geometric "if-then" rules.

All works on decision analysis assume availability of numeric or measurement-based information. In other words, the available imperfect information is always considered to admit required precision. The fundamental question remains: what if the information is not only imperfect and perception-based, but also unprecisiated?

As stated in [15] while fuzzy logic delivers an important capability to reason precisely in presence of imperfect information, the extended (or unprecisiated) fuzzy logic delivers a unique ability to reason imprecisely with imperfect information. The capability to reason imprecisely is used by human being when precise reasoning is infeasible, excessively costly or not required at all. A typical real-life case example is a case when the only available information is perception-based and no trustworthy precision or fuzzy numeric models (e.g. articulated through membership functions) are possible to obtain. As a model of unprecisiated fuzzy logic we consider fuzzy geometry [15].

The concept of fuzzy geometry is not new. Many authors suggest various versions of fuzzy geometry. Some of well-known ones are the Poston's fuzzy geometry [8], coarse geometry [9], fuzzy geometry of Rosenfeld [10] , fuzzy geometry of Buckley and Eslami [2], fuzzy geometry of Mayburov [8], fuzzy geometry of Tzafestas [13], and fuzzy incidence geometry of Wilke [14]. Along this line of thought, many works are devoted to model spatial objects with fuzzy boundaries [3,4,12].

The study reported in [12] proposes a general framework to represent ill-defined information regarding boundaries of geographical regions by using the concept of relatedness measures for fuzzy sets. Regions are represented as fuzzy sets in a two–dimensional Euclidean space, and the notions of nearness and relative orientation are expressed as fuzzy relations. To support fuzzy spatial reasoning, the authors derive transitivity rules and provide efficient techniques to deal with the complex interactions between nearness and cardinal directions.

The work presented in [3] introduces a geometric model for uncertain lines that is capable of describing all the sources of uncertainty in spatial objects of linear type. Uncertain lines are defined as lines that incorporate uncertainty description both in the boundary and interior and can model all the uncertainty by which spatial data are commonly affected and allow computations in presence of uncertainty without oversimplification of the reality.

Qualitative techniques for spatial reasoning are adopted in [4]. The author formulates a computational model for defining spatial constraints on geographic regions, given a set of imperfect quantitative and qualitative constraints.

What is common in all currently known fuzzy geometries is that the underlying logic is the fuzzy logic. Fuzzy logic implies existence of valid numerical information (qualitative or quantitative) regarding the geometric objects under consideration. In situations, when source information is very unreliable to benefit from application of computationally-intensive mathematical computations of traditional fuzzy logic, some new method is needed. The new fuzzy geometry, the concept of which is proposed by Zadeh and referred to as F-Geometry, could be regarded as a highly suitable vehicle to model unprecisiated or extended fuzzy logic [15].

Of the geometries mentioned above, the fuzzy incidence geometry of Wilke [14] can form a starting point for developing the new F-Geometry. Thus fuzzy incidence geometry extends the Euclidean geometry by providing concepts of extended points and lines as subsets of coordinate space, providing fuzzy version of incidence axioms, and reasoning mechanism by taking into account the positional tolerance and truth degree of relations among primitives. To allow for partially true conclusions from partially true conditions, the graduated reasoning with Rational Pavelka Logic (RPL) is used [7].

The purpose of this chapter is to develop a concept and a technique that can be used to more adequately reflect the human ability to formally describe perceptions for which he/she could hardly suggest acceptable linguistic approximations due to their highly uncertain nature or for which such precision, if provided, would lead to a loss or degradation of available information. Such unprecisiatable perceptions many times form an underlying basis for everyday human reasoning as well as decision making in economics and business.

It is suggested that Fuzzy Geometry or F-geometry (or geometry for extended primitives) can be used to more adequately reflect the human ability to describe decision-relevant information by means of geometric primitives. Classical geometry is not useful in this case. As it was mentioned in [5], classical geometry fails to acknowledge that visual space is not an abstract one but its properties are defined by perceptions.

The main idea is to describe uncertain data (which are perceptions of human observer, researcher, or a decision maker) in geometric language using extended primitives: points, lines, bars, stripes, curves etc. to prevent possible loss of information due to the precision of such data to classical fuzzy sets based models (e.g. when using membership functions etc.).

6.2 Fuzzy Geometry Primitives and Operations

F-geometry is a simple and natural approach that can be used to express human perceptions in a visual form so that they can be used in further processing with minimal distortion and loss of information. In F-geometry, we use different primitive geometric concepts such as f-points, f-intervals, f-lines etc. as well as more complex f-transform concepts such as f-parallel, f-similar, f-convex, f-stable, etc. to express the underlying information. The primitive concepts can be entered by hand using simple graphic interface tool such as spray-pen or Z-mouse [15].

Pieces of information, describing the properties required for decision-making, are represented in forms of 2D geometric objects. For one-dimensional properties the second dimension can be used for expressing additional information.

For entering the information regarding a certain property, for example, to define a range of probabilities, the decision-maker (DM), instead of entering numbers, manually draws strips using a spray can. By doing so, the DM could also implicitly express his/her confidence degree about the entered information by drawing physically greater objects or thick lines for less confident information granules and more compact sized marks, e.g., points or thinner lines with strictly defined boundaries, to express more reliable information granules.

Generally F-geometry primitives can be defined as two-dimensional sets, which are subsets of R^2. F-marks are primitives of F-geometry that can be used in arithmetic, comparison, and set-theoretic operations. Therefore, we require that F-marks (but not necessarily their F-transforms) are convex sets. A concept named Z-number has been suggested in [16], which also could be used to approximately represent F-marks (see Section 1.1). A Z-number consists of a pair of fuzzy sets, basically trapezoidal fuzzy numbers, entered by using a specialized graphical interface tool. The fundamental difference between a Z-number and an F-mark suggested here is that a Z-number is still explicitly based on membership functions whereas an F-mark is not.

Definition 6.1 [6,14]**. F-mark.** An ***F-mark*** is a bounded subset of R^2, representing a graphical hand-mark drawn by human being to indicate visually a value of a perception-based information granule.

So, formally an F-mark *A* can be represented as a bounded subset of $R^2 : A \subset R^2$. But an F-mark is more than just a physical area as it is meant to hold a perception of a measurable value. Therefore, we will use two notations: *A* (when meaning a perception, an unprecisiated value) and A (when meaning an area or a variable to hold a measurable value). Usually, an area A, representing an F-mark is assumed to be a convex set [14].

If required, we should be able to approximately represent, i.e. precisiate, F-marks by using two-dimensional membership functions (e.g. of truncated pyramidal or con form) based on density, intensity, or width of the spray pen (Z-mouse) used for the drawing [15].

Let us define some basic primitives that we will use in context of decision making.

Any F-mark, which represents a convex subset A of R^2, can be approximately defined by its center $c = (x_c, y_c)$ (which is a Euclidean point) and two diameters ($\phi_{\min}, \phi_{\max}$) [14]:

$$A = P(c, \phi_{\min}, \phi_{\max}) \tag{6.1}$$

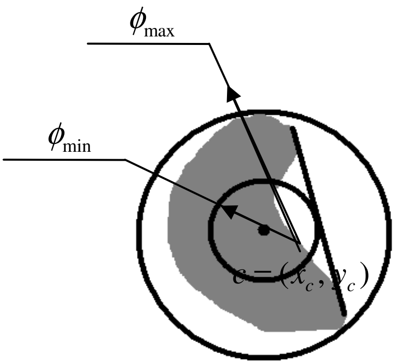

Fig. 6.1 An F-mark with two diameters and its convex hull

The center c can be computed as a center of gravity of convex hull: c=C(ch(A)) while the two diameters are [14]:

$$\phi_{\min} = \min_t \left| ch(A) \cap \left\{ c + t \cdot R_\alpha (0,1)^T \right\} \right|$$

$$\phi_{\max} = \max_t \left| ch(A) \cap \left\{ c + t \cdot R_\alpha (0,1)^T \right\} \right|$$

where $t \in R$ and R_α is the rotation matrix describing rotation by angle α.

The illustration of the concept is presented in Fig. 6.1.

Definition 6.2. F-point. The degree to which an F-mark $A = P(c, \phi_{\min}, \phi_{\max})$ is an ***F-point*** is determined as follows [14]:

$$p(A) = \phi_{\min} / \phi_{\max} \tag{6.2}$$

Definition 6.3. The degree to which an F-mark A is an ***F-line*** is determined as:

$$l(A) = 1 - p(A).$$

Definition 6.4. Truth Degree of an Incidence of Two F-marks. The truth degree of predicate for the incidence of F-marks A and B is determined as [14]:

$$inc(A,B) = \max\left(\frac{|ch(A)\cap ch(B)|}{|ch(A)|}, \frac{|ch(A)\cap ch(B)|}{|ch(B)|}\right), \tag{6.3}$$

here $ch(A)$ is a convex hull of an f-mark A, $|ch(A)|$ is the area covered by $ch(A)$.

Definition 6.5. Truth Degree of an Equality of Two F-marks. The truth degree of a predicate defining the equality of F-marks A and B, is determined as follows [14]:

$$eq(A,B) = \min\left(\frac{|ch(A)\cap ch(B)|}{|ch(A)|}, \frac{|ch(A)\cap ch(B)|}{|ch(B)|}\right) \tag{6.4}$$

Definition 6.6. Measure of Distinctness of Two F-marks. The measure of distinctness of f-marks A and B is determined as [14]:

$$dp(A,B) = \max\left(0, 1 - \frac{\max\left(\phi_{\max}(A), \phi_{\max}(B)\right)}{\phi_{\max}(ch(A\cup B))}\right) \tag{6.5}$$

Two f-points A and B can generate an f-line L as follows (Fig. 6.2):

$$L = ch(A\cup B)$$

Axioms of the Fuzzy Incidence Geometry

The following axioms formalize the behavior of points and lines in incident geometry [14]:

(A1) For every two distinct points *p* and *q*, at least one line *l* exists that is incident with *p* and *q*.

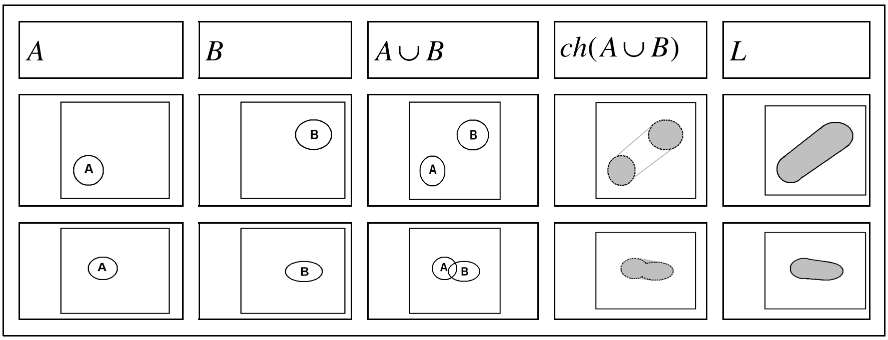

Fig. 6.2 Generation of an f-line from two f-points

(A2) Such a line is unique.

(A3) Every line is incident with at least two points.

(A4) At least three points exist that are not incident with the same line.

For fuzzy version of incident geometry each of the above axioms may not evaluate to absolute truth for all possible inputs.

A fuzzy version of the incident geometry, which is suitable to work with f-marks can be axiomatized as follows [14]:

$$\text{(A1')}\ \left(dp(x,y) \to \sup_{z}\left[l(z) \otimes inc(x,z) \otimes inc(y,z) \right], r_1 \right)$$

$$\text{(A2')}\ \Big(dp(x,y) \to \\ \to \left[l(z) \to \left[inc(x,z) \to \left[\begin{array}{l} inc(y,z) \to l(z') \to \\ \to \left[inc(x,z') \to \left[inc(y,z') \to eq(z,z') \right] \right] \end{array} \right] \right] \right], r_3 \Big)$$

$$\text{(A3')}\ \left(l(z) \to \sup_{x,y} \left\{ \begin{array}{l} p(x) \otimes p(y) \otimes \neg \\ \neg eq(x,y) \otimes inc(x,z) \otimes inc(y,z) \end{array} \right\}, r_3 \right) \tag{6.6}$$

$$\text{(A4')}\left(\sup_{u,v,w,z} \left[\begin{array}{l} p(u) \otimes p(v) \otimes p(w) \otimes l(z) \to \\ \to \neg\left(inc(u,z) \otimes inc(v,z) \otimes inc(w,z) \right) \end{array} \right], r_4 \right),$$

where x, y, z, z', u, v, and w are measurable variables to hold F-marks, $\otimes$ denotes Lukasiewicz t-norm, r_1, r_2, r_3, and r_4 are truth values of the associated axioms.

In this study, we consider two basic types of F-marks: F-points and F-lines.

When it is needed for a concise representation or fast computation, any convex F-mark can be approximately represented as (6.1) [14]. Instead of it, we suggest an approximation illustrated in Fig. 6.3.

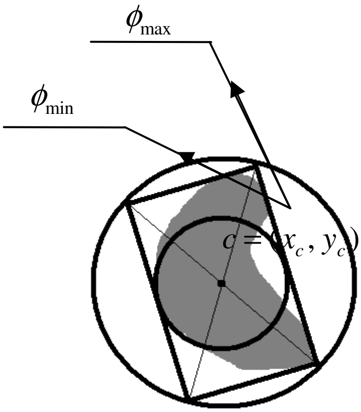

Fig 6.3 The suggested approximation of an F-mark

The idea behind this method is to find a parallelogram with a minimum square, the intersection with f-mark of which is the same f-mark. Then the sides of the parallelogram are the two diameters, the shorter one is $\phi_{\min}$ and the longer one is $\phi_{\max}$.

If a parallelogram can be specified as $S\left((c_x, c_y), \alpha, h, w\right)$, where (c_x, c_y) is its center (centroid), α is rotation angle (e.g. counter-clockwise vs. Y axis, which is 0), and *h* and *w* are its sides.

Then $\phi_{\min} = \min(h, w)$ and $\phi_{\max} = \max(h, w)$, where *h* and *w* are found by solving the optimization task:

$$\begin{gathered} h \cdot w \rightarrow \min \\ \text{s.t. } S\left((c_x, c_y), \alpha, h, w\right) \cap A = A\,. \end{gathered} \tag{6.7}$$

As it can be seen $A_{\min} \subseteq A \subseteq A_{\max}$, where $A_{\min}$ and $A_{\max}$ are 2D disks with diameters $\phi_{\min} = \min(h, w)$ and $\phi_{\max} = \max(h, w)$, respectively.

As we pointed above, all F-marks are 2D sets, which are bounded subsets of R^2.

$$A \equiv \int_{(x,y)\in A} \{(x, y)\},$$

where $A \subset R^2$ and the integral sign $\int$ does not mean integration but denotes a collection of all points $(x, y) \in A$. With any desired accuracy an F-mark can be represented as a discrete set:

$$A \equiv \sum_{(x_i, y_i)\in A} \{(x_i, y_i)\},$$

where $A \subset R^2$, and a summation sign $\sum$ is used to represent a collection of elements of a discrete set.

We define arithmetic operations of summation and subtraction as follows:

$$A + B = \int_{\substack{(x_1, y_1)\in A \\ (x_2, y_2)\in B}} \{(x_1 + x_2, y_1 + y_2)\} \tag{6.8}$$

$$A - B = \int_{\substack{(x_1, y_1)\in A \\ (x_2, y_2)\in B}} \{(x_1 - x_2, y_1 - y_2)\} \tag{6.9}$$

Also we define the above arithmetic operations with respect to one of the axes *X* or *Y*. As we see it below, the operation is done with respect to axis *X*. For example, when the entered data is a scalar value (i.e. 1D) and the axis *Y* is used to represent the user's confidence degree (or vice versa):

$$A \underset{X}{+} B = \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{(x_1 + x_2, y_1)\} \quad \cup \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{(x_1 + x_2, y_2)\} \tag{6.10}$$

$$A \underset{Y}{+} B = \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{(x_1, y_1 + y_2)\} \quad \cup \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{(x_2, y_1 + y_2)\} \tag{6.11}$$

$$A \underset{X}{-} B = \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{(x_1 - x_2, y_1)\} \quad \cup \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{(x_1 - x_2, y_2)\} \tag{6.12}$$

$$A \underset{Y}{-} B = \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{(x_1, y_1 - y_2)\} \quad \cup \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{(x_2, y_1 - y_2)\} \tag{6.13}$$

Let us define an operation of multiplication of an F-mark A by a numeric value k :

$$A \underset{X}{\cdot} k = \int_{(x,y)\in A} \{(x \cdot k, y)\} \tag{6.14}$$

We define the Max and Min operations:

$$\underset{X}{Max}(A,B) = \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{\max(x_1,x_2), y_1\} \cup \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{\max(x_1,x_2), y_2\} \tag{6.15}$$

$$\underset{X}{Min}(A,B) = \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{\min(x_1,x_2), y_1\} \quad \cup \int_{\substack{(x_1,y_1)\in A \\ (x_2,y_2)\in B}} \{\min(x_1,x_2), y_2\} \tag{6.16}$$

To compare f-marks A and B, we use the method adopted from the Jaccard compatibility measure to compare degree to which A exceeds B: $g_{\geq}(A,B)$. We assume that $A > B$ if $g_{\geq}(A,B) > g_{\geq}(B,A)$, $A < B$ if $g_{\geq}(A,B) < g_{\geq}(B,A)$ and $A = B$ otherwise.

For the two extended points A and B one has [1,6]

$$g_{\geq}(A,B) = \frac{1}{2}\left(\frac{\|\mathrm{Max}(A,B) \cap A\|}{\|\mathrm{Max}(A,B) \cup A\|} + \frac{\|\mathrm{Min}(A,B) \cap B\|}{\|\mathrm{Min}(A,B) \cup B\|} \right) \tag{6.17}$$

An F-point A can approximately be represented parametrically as $A = M((c_x, c_y), h, w)$. We use the notation $M()$ to denote parametrically a general

F-mark, while the set of parameters in parentheses depends on chosen approximation model. Without any loss of generality, we assume that $0 < h \le w$, and, hence, h and w are convenient replacements for $\phi_{\min}$ and $\phi_{\max}$, respectively. An F-line L can be (approximately) produced from a convex hull of two F-points $A_1 = M((c_{1x}, c_{1y}), h_1, w_1)$ and $A_2 = M((c_{2x}, c_{2y}), h_2, w_2)$:

$$L = ch\left(M((c_{1x}, c_{1y}), h_1, w_1), M((c_{2x}, c_{2y}), h_2, w_2)\right).$$

Let $h = \max\left(h_1, w_1, h_2, w_2\right)$, then an F-line can be represented approximately as an F-point:

$$\tilde{M}\left(\left((c_{1x} + c_{2x})/2, (c_{1y} + c_{2y})/2\right), h, w\right),$$

where $w \sim h + \sqrt{(c_{2x} - c_{1x})^2 + (c_{2y} - c_{1y})^2}$.

Therefore, we can parametrically represent both F-line and F-point either as $M((c_x, c_y), h, w)$ or as $M((c_{1x}, c_{1y}), (c_{2x}, c_{2y}), h)$.

The parameter h could visually be interpreted as the height or thickness of an F-mark. Likewise the parameter w can be regarded as the width or length of an F-mark. When F-marks representing information regarding a value of a scalar (1D) uncertain variable (e.g. a probability of an event or expected profit) are accepted from the user (e.g. a decision maker, a DM), the second dimension is assumed to express the degree of confidence (or belief or trust) of the user in the entered data. For example, thicker (or long) lines would mean less trustworthy data than the one associated with the thinner (or short) lines.

Without loss of generality, we relate the parameter h with the degree of confidence of DM in the value of specified F-mark (either F-point or F-line).

To do so we define a decreasing function $\sigma(h)$ expressing a relationship between the height and the associated confidence degree for which the following conditions hold true:

$$\lim_{h \to +0} \sigma(h) = 1$$

$$M(c_1, c_2, \sigma^{-1}(0)) = \varnothing,$$

where $\sigma^{-1}(d)$ is the reciprocal function producing a confidence degree d and associated value of h. A suitable function could be $\sigma(h) - 1 - h / h_{\max}$ for which $\sigma^{-1}(d) = h_{\max}(1 - d)$, where $h \in (0, h_{\max}]$, $h_{\max} > 0$.

Let us also introduce a function that for any F-mark $A = M((c_{1x}, c_{1y}), (c_{2x}, c_{2y}), h)$ returns its parameter h ($\phi_{\min}$):

$$H(A) = H\left(M((c_{1x}, c_{1y}), (c_{2x}, c_{2y}), h)\right) = h$$

F-geometry can be effectively used in decision-making. The decision-making "if-then" rules for an uncertain environment can be composed on the basis of F-geometry concepts used to more adequately reflect the perceived information granules and relationships. F-geometry based decision-making allows for better modeling of the knowledge of human observer, researcher, or a DM, thereby making the inference system's output more realistic (through minimizing losses of meaning and distortion of source information).

Let us start with a formal problem statement.

In an unprecisiated perception-based information setting, we consider a decision making problem as a 4-tuple $\left(\mathcal{S}, \mathcal{Y}, \mathcal{A}, \underset{\sim}{\succ}\right)$ where the set of states of nature $\mathcal{S} = \{S_1, S_2, ..., S_n\}$, corresponding probability distribution P and set of outcomes $\mathcal{Y}$ are generally considered as spaces of F-marks. The set of actions $\mathcal{A}$ is considered as a set of mappings from $\mathcal{S}$ to $\mathcal{Y}$. In turn, preferences $\underset{\sim}{\succ}$ are to be implicit in some knowledge base described as some "if-then" rules, which include $\mathcal{S}, \mathcal{Y}, \mathcal{A}$ - based description of various decision making situations faced before and a DM's or experts' opinion-based evaluations of actions' assessment U (combined outcome) which are also to be described by f-marks.

A typical knowledge base may look as follows:

If $S_1, S_2, ..., S_n$ and $\left(\mathrm{P}_2 \text{ is } P_{i2}\right)$ and…and $\left(\mathrm{P}_n \text{ is } P_{in}\right)$

Then $\mathrm{U}_1 = U_{i1}$ and $\mathrm{U}_2 = U_{i2}$ and ... and $\mathrm{U}_{\mathrm{m}} = U_{im}$, α_i) ... , $(i = \overline{1, q})$.

Here P_{i} is the variable describing user entered F-mark for probability of the state of nature S_j and P_{ij} is an F-mark describing the probability of the state of nature S_j used in rule i $(i = \overline{1, q})$, $\mathrm{S}_j \in \mathcal{S}$ $(j = \overline{1, n})$, n is the number of states, U_{ik} is an F-mark describing the assessment of k-th action $(k = \overline{1, m})$ in rule i $(i = \overline{1, q})$, m is the number of considered alternative actions, α_i is the degree of confidence of the expert (designer of the knowledge base) in the rule i $(i = \overline{1, q})$.

The purpose of reasoning is to produce the vector of aggregated assessments $U_1, U_2, ..., U_m$ for different actions f_k $(k = \overline{1, m})$.

The best action then can be selected by ranking of F-marks describing the respective integrated assessments. For integrated assessments' f-marks $U_{f_{k_1}}$ and $U_{f_{k_2}}$ (corresponding to actions f_{k_1} and f_{k_1} respectively), f_{k_1} is the better action if $U_{f_{k_1}} > U_{f_{k_2}}$.

6.3 Fuzzy Geometry Gased If-Then Rules and the Reasoning Method

For simplicity, let us consider that the states of nature ($S_i, i = \overline{1,n}$) remain unchanged and thus could be removed from consideration in the rules. Then the above mentioned knowledge base takes on the following form:

If $(\mathrm{P}_1 \text{ is } P_{11})$ and $(\mathrm{P}_2 \text{ is } P_{12})$ and … and $(\mathrm{P}_n \text{ is } P_{1n})$
Then $\mathrm{U}_1 = U_{11}, \mathrm{U}_2 = U_{12}, ..., \mathrm{U}_\mathrm{m} = U_{1m}, \alpha_1$

If $(\mathrm{P}_1 \text{ is } P_{i1})$ and $(\mathrm{P}_2 \text{ is } P_{i2})$ and … and $(\mathrm{P}_\mathrm{n} \text{ is } P_{in})$
Then $\mathrm{U}_1 = U_{i1}, \mathrm{U}_2 = U_{i2}, ..., \mathrm{U}_\mathrm{m} = U_{im}, \alpha_i$

If and $(\mathrm{P}_1 \text{ is } P_{q1})$ and $(\mathrm{P}_2 \text{ is } P_{q2})$ and … and $(\mathrm{P}_\mathrm{n} \text{ is } P_{qn})$
Then $\mathrm{U}_1 = U_{q1}, \mathrm{U}_2 = U_{q2}, ..., \mathrm{U}_\mathrm{m} = U_{qm}, \alpha_q$

Assume that based on available cases or expert data, the knowledge base in form of F-geometry based "If-Then" rules shown above has been formulated.

The following steps describe the essence of the underlying methodology and reasoning procedure of decision-making using the suggested F-geometry based approach.

1. Obtain the F-lines P_j from the user and apply them for P_j, $j = \overline{1,n}$.
2. Obtain the minimum value of satisfaction of the fuzzy incidence axioms (A1')-(A4') for all the F-lines P_j generated by the user: $r(j)$, $j = \overline{1,n}$. If $r = \min_j(r(j))$ is lower than a predefined minimum threshold value (e.g., 0.3), ask the user to resubmit the primitives (go to step 1).
3. For each rule compute $\theta_i = \left(\wedge_j(\theta_{ij})\right)$, where $\theta_{ij} = \min\left(r, inc(\mathrm{P}_j, P_{ij})\right)$, $i = \overline{1,q}$
4. For each rule compute $R_i = \theta_i \cdot \alpha_i$
5. Find the indexes i of the rules for which $R_i \geq R_{\min}$, where $R_{\min}$ is the minimum creditability value that a rule must exhibit to be activated. For all such rules $R_{i'} \geq R_{\min}, i' = \overline{1,q'}, q' \leq q$, where i' are new indexes for the rules after removing those for which the above condition fails. If there are no such rules repeat the process starting from step 1.

6. Compute aggregated output components from all rules:

$$\mathrm{U}_{\mathrm{f}_\mathrm{k}} = U_{f_k} = ch\left(\frac{\sum_{i'=\overline{1,q'}} \left(U_{i'k} \cdot R_{i'} \right)}{\sum_{i'=\overline{1,q'}} R_{i'}} \right), (k = \overline{1,m}).$$

7. Do ranking of the output F-marks U_k, $(k = \overline{1,m})$, and choose the best action depending on the index i_{best} such that $\mathrm{U}_{i_{best}} \geq \mathrm{U}_k$, $(k = \overline{1,m})$.

References

1. Aliev, R.A., Alizadeh, A.V., Guirimov, B.G.: Unprecisiated information-based approach to decision making with imperfect information. In: Proc. of the 9th International Conference on Application of Fuzzy Systems and Soft Computing (ICAFS 2010), Prague, Czech Republic, pp. 387–397 (2010)
2. Buckley, J.J., Eslami, E.: Fuzzy plane geometry I: Points and lines. Fuzzy Sets and Systems 86(2), 179–187 (1997)
3. Clementini, E.: A model for uncertain lines. Journal of Visual Languages and Computing 16, 271–288 (2005)
4. Dutta, S.: Qualitative Spatial Reasoning: A Semi-Quantitative Approach Using Fuzzy Logic. In: Buchmann, A., Smith, T.R., Wang, Y.-F., Günther, O. (eds.) SSD 1989. LNCS, vol. 409, pp. 345–364. Springer, Heidelberg (1990)
5. Ferrato, M., Foster, D.H.: Elements of a fuzzy geometry for visual space. In: O, Y.L., Toet, A., Foster, D.H., Heijmans, H.J.A.M., Meer, P. (eds.) Shape in Picture. Mathematical Description of Shape in Grey-level Images, pp. 333–342. Springer, Berlin (1994)
6. Guirimov, B.G., Gurbanov Ramiz, S., Aliev Rafik, A.: Application of fuzzy geometry in decision making. In: Proc. of the Sixth International Conference on Soft Computing and Computing with Words in System Analysis, Decision and Control (ICSCCW 2011), Antalya, pp. 308–316 (2011)
7. Hajek, P., Paris, J., Shepherdson: Rational Pavelka Predicate Logic is a Conservative Extension of Lukasiewicz Predicate Logic. The Journal of Syumbolic Logi. Association for Syumbolic Logic 65(2), 669–682 (2000), http://www.jstor.org/stable/258660
8. Mayburov, S.: Fuzzy geometry of phase space and quantization of massive fields. Journal of Physics A: Mathematical and Theoretical 41, 1–10 (2008)
9. Poston, T.: Fuzzy geometry. Ph.D Thesis, University of Warwick (1971)
10. Roe, J.: Index theory, coarse geometry, and topology of manifolds. In: CBMS: Regional Conf. Ser. in Mathematics. The American Mathematical Society, Rhode Island (1996)
11. Rosenfeld, A.: Fuzzy geometry: an updated overview. Information Science 110(3-4), 127–133 (1998)
12. Schockaert, S., De Cock, M., Kerre, E.: Modelling nearness and cardinal directions between fuzzy regions. In: Proceedings of the IEEE World Congress on Computational Intelligence (FUZZ-IEEE), pp. 1548–1555 (2008)

13. Tzafestas, S.G., Chen, C.S., Fokuda, T., Harashima, F., Schmidt, G., Sinha, N.K., Tabak, D., Valavanis, K. (eds.): Fuzzy logic applications in engineering science. Microprocessor Based and Intelligent Systems Engineering, vol. 29, pp. 11–30. Springer, Netherlands (2006)
14. Wilke, G.: Approximate Geometric Reasoning with Extended Geographic Objects. In: Proceedings of the Workshop on Quality, Scale and Analysis Aspects of City Models, Lund, Sweden, December 3-4 (2009), `http://www.isprs.org/proceedings/XXXVIII/2-W11/Wilke.pdf`
15. Zadeh, L.A.: Toward extended fuzzy logic. A first step. Fuzzy Sets and Systems 160, 3175–3181 (2009)
16. Zadeh, L.A.: A Note on Z-numbers. Information Sciences 181, 2923–2932 (2010)