Chapter 1
Fuzzy Sets and Fuzzy Logic

1.1 Fuzzy Sets and Operations on Fuzzy Sets

Definition 1.1 Fuzzy Sets. Let X be a classical set of objects, called the universe, whose generic elements are denoted x. Membership in a classical subset A of X is **often** viewed as a characteristic function μ_A from A to $\{0,1\}$ such that

$$\mu_A(x) = \begin{cases} 1 \ \textit{iff} \ \ x \in A \\ 0 \ \textit{iff} \ \ x \notin A \end{cases}$$

where $\{0,1\}$ is called a valuation set; 1 indicates membership while 0 - non-membership.

If the valuation set is allowed to be in the real interval $[0,1]$, then A is called a fuzzy set denoted $\tilde{A}$ [2,3,6,8,57,58,114,133], $\mu_{\tilde{A}}(x)$ is the grade of membership of x in $\tilde{A}$

$$\mu_{\tilde{A}} : X \to [0,1]$$

As closer the value of $\mu_{\tilde{A}}(x)$ is to 1, so much x belongs to $\tilde{A}$.

$\tilde{A}$ is completely characterized by the set of pairs.

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), \ \ x \in X\}$$

Fuzzy sets with crisply defined membership functions are called ordinary fuzzy sets.

Properties of Fuzzy Sets

Definition 1.2. Equality of Fuzzy Sets. Two fuzzy sets $\tilde{A}$ and $\tilde{B}$ are said to be equal if and only if

$$\forall x \in X \quad \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x) \quad \tilde{A} = \tilde{B}.$$

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Definition 1.3. The Support and the Crossover Point of a Fuzzy Set. The Singleton. The support of a fuzzy set $\tilde{A}$ is the ordinary subset of X that has nonzero membership in $\tilde{A}$:

$$\text{supp}\tilde{A}=\tilde{A}^{+0}=\{x\in X,\mu_A(x)>0\}$$

The elements of x such as $\mu_{\tilde{A}}(x)=1/2$ are the crossover points of $\tilde{A}$.

A fuzzy set that has only one point in X with $\mu_{\tilde{A}}=1$ as its support is called a singleton.

Definition 1.4. The Height of a Fuzzy Set. Normal and Subnormal Sets. The height of $\tilde{A}$ is

$$hgt(\tilde{A})=\sup_{x\in X}\mu_{\tilde{A}}(X)$$

i.e., the least upper bound of $\mu_{\tilde{A}}(x)$.

$\tilde{A}$ is said to be normalized iff $\exists x\in X$, $\mu_{\tilde{A}}(x)=1$. This definition implies $hgt(\tilde{A})=1$. Otherwise $\tilde{A}$ is called subnormal fuzzy set.

The empty set $\varnothing$ is defined as

$$x\in X,\mu_{\varnothing}(x)=0,\text{of course}\,\forall x\in X\;\mu_X(x)=1$$

Definition 1.5. α-Level Fuzzy Sets. One of important way of representation of fuzzy sets is α-cut method. Such type of representation allows us to use properties of crisp sets and operations on crisp sets in fuzzy set theory.

The (crisp) set of elements that belongs to the fuzzy set $\tilde{A}$ at least to the degree α is called the α-level set:

$$A^{\alpha}=\{x\in X,\mu_{\tilde{A}}(x)\ge\alpha\}$$

$A^{\alpha}=\{x\in X,\mu_{\tilde{A}}(x)>\alpha\}$ is called "strong α-level set" or "strong α-cut".

Now we introduce fuzzy set A_{α}, defined as

$$A_{\alpha}(x)=\alpha A^{\alpha}(x) \tag{1.1}$$

Then the original fuzzy set $\tilde{A}$ may be defined as $\tilde{A}=\bigcup_{\alpha\in[0,1]}A_{\alpha}$. $\bigcup$ denotes the standard fuzzy union.

Definition 1.6. Convexity of Fuzzy Sets. A fuzzy set $\tilde{A}$ is convex iff

$$\mu_{\tilde{A}}(\lambda x_1+(1-\lambda)x_2)>\min(\mu_{\tilde{A}}(x_1),\mu_{\tilde{A}}(x_2)) \tag{1.2}$$

for all $x_1, x_2 \in R$, $\lambda \in [0,1]$, min denotes the minimum operator.

Alternatively, a fuzzy set $\tilde{A}$ on R is convex iff all its α-level sets are convex in the classical sense.

Definition 1.7. The Cardinality of a Fuzzy Set. When X is a finite set, the scalar cardinality $|\tilde{A}|$ of a fuzzy set $\tilde{A}$ on X is defined as

$$|\tilde{A}| = \sum_{x \in \tilde{A}} \mu_{\tilde{A}}(x)$$

Sometimes $|\tilde{A}|$ is called the power of $\tilde{A}$. $\|\tilde{A}\| = |\tilde{A}| / |X|$ is the relative cardinality.

When X is infinite, $|\tilde{A}|$ is defined as

$$|\tilde{A}| = \int_X \mu_{\tilde{A}}(x)\,dx$$

Definition 1.8. Fuzzy Set Inclusion. Given fuzzy sets $\tilde{A}, \tilde{B} \in \tilde{F}(X)$ $\tilde{A}$ is said to be included in $\tilde{B}\left(\tilde{A} \subseteq \tilde{B}\right)$ or $\tilde{A}$ is a subset of $\tilde{B}$ if $\forall x \in X, \mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x)$.

When the inequality is strict, the inclusion is said to be strict and is denoted as $\tilde{A} < \tilde{B}$.

Let consider representations and constructing of fuzzy sets. It was mentioned above that each fuzzy set is uniquely defined by a membership function. In the literature one can find different ways in which membership functions are represented.

List Representation. If universal set $X = \{x_1, x_2, \ldots, x_n\}$ is a finite set, membership function of a fuzzy set $\tilde{A}$ on X $\mu_{\tilde{A}}(x)$ can be represented as table. Such table lists all elements in the universe X and the corresponding membership grades as shown below

$$\tilde{A} = \mu_{\tilde{A}}(x_1)/x_1 + \ldots + \mu_{\tilde{A}}(x_n)/x_n = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)/x_i$$

Here symbol / (slash) does not denote division, it is used for correspondence between an element in the universe X (after slash) and its membership grade in the fuzzy set $\tilde{A}$ (before slash). The symbol + connects the elements (does not denote summation).

If X is a finite set then

$$\tilde{A} = \int_X \mu_{\tilde{A}}(x)/x\,.$$

Here symbol $\int_X$ is used for denoting a union of elements of set X.

Graphical Representation. Graphical description of a fuzzy set $\tilde{A}$ on the universe X is suitable in case when X is one or two-dimensional Euclidean space. Simple typical shapes of membership functions are usually used in fuzzy set theory and practice (Table 1.1).

Fuzzy n Cube Representation. All fuzzy sets on universe X with n elements can be represented by points in the n-dimensional unit cube – n-cube. Assume that universe X contains n elements $X = \{x_1, x_2, ..., x_n\}$. Each element x_i, $i = \overline{1,n}$ can be viewed as a coordinate in the n dimensional Euclidean space. A subset of this space for which values of each coordinate are restricted in $[0,1]$ is called n-cube. Vertices of the cube, i.e. bit list $(0,1,...,0)$ represent crisp sets. The points inside the cube define fuzzy subsets.

Analytical Representation. In case if universe X is infinite, it is not effective to use the above considered methods for representation of membership functions of a fuzzy sets. In this case it is suitable to represent fuzzy sets in an analytical form, which describes the shape of membership functions.

There are some typical formulas describing frequently used membership functions in fuzzy set theory and practice.

For example, bell-shaped membership functions often are used for representation of fuzzy sets. These functions are described by the formula:

$$\mu_{\tilde{A}}(x) = c\exp\left(-\frac{(x-a)^2}{b}\right)$$

which is defined by three parameters, a, b and c.

In general it is effective to represent the important typical membership functions by a parametrized family of functions.The following are formulas for describing the 6 families of membership functions

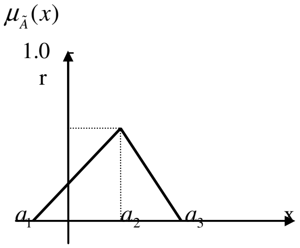
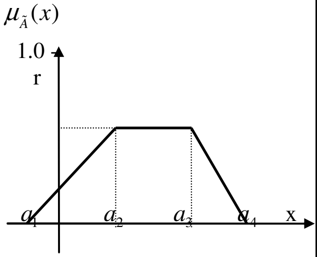
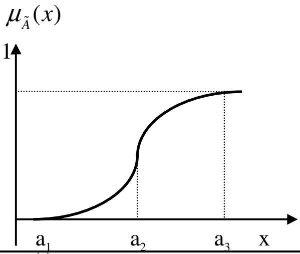
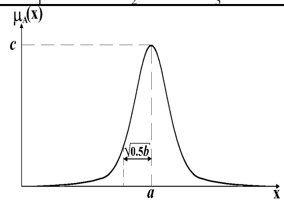
$$\mu_{\tilde{A}}(x, c_1) = [1 + c_1(x-a)^2]^{-1} \tag{1.3}$$

$$\mu_{\tilde{A}}(x, c_2) = \left[1 + c_2\left|x-a\right|\right]^{-1} \tag{1.4}$$

$$\mu_{\tilde{A}}(x, c_3, d) = \left[1 + c_3\left|x-a\right|^d\right]^{-1} \tag{1.5}$$

$$\mu_{\tilde{A}}(x, c_4, d) = \exp\left[-c_4\left|x-a\right|^d\right] \tag{1.6}$$

Table 1.1 Typical membership functions

Type of Membership function	Graphical Representation	Analytical Representation
Triangular MF		$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} r, & \text{if } a_1 \le x \le a_2 , \\ \frac{a_3-x}{a_3-a_2} r, & \text{if } a_2 \le x \le a_3 , \\ 0 , & \textit{otherwise} \end{cases}$
Trapezoidal MF		$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} r, & \text{if } a_1 \le x \le a_2 , \\ r , & \text{if } a_2 \le x \le a_3 , \\ \frac{a_4-x}{a_4-a_3} r, & \text{if } a_3 \le x \le a_4 , \\ 0 , & \textit{otherwise} \end{cases}$
S - shaped MF		$\mu_{\tilde{A}}(x) = \begin{cases} 0 , & \text{if } x \le a_1 , \\ 2 \left(\frac{x-a_1}{a_3-a_1} \right)^2 , & \text{if } a_1 < x < a_2 , \\ 1-2 \left(\frac{x-a_1}{a_3-a_1} \right)^2 , & \text{if } a_2 \le x < a_3 , \\ 1 , & \text{if } a_3 \le x \end{cases}$
Bell - shaped MF		$\mu_{\tilde{A}}(x) = c \cdot \exp \left(-\frac{(x-a)^2}{b} \right)$

$$\mu_{\tilde{A}}(x, c_5) = \max\left\{0, \left[1 - c_5 \left|x - a\right|\right]\right\} \tag{1.7}$$

$$\mu_{\tilde{A}}(x, c_6) = c_6 \exp\left[-\frac{(x-a)^2}{b}\right] \tag{1.8}$$

Here $c_i > 0,\ i = \overline{1,6},\ d > 1$ are parameters, a denotes the elements of corresponding fuzzy sets with the membership grade equal to unity.

Table 1.1 summarizes the graphical and analytical representations of frequently used membership functions (MF).

The problem of constructing membership functions is problem of knowledge engineering.

There are many methods for estimation of membership functions. They can be classified as follows:

1. Membership functions based on heuristics.
2. Membership functions based on reliability concepts with respect to the particular problem.
3. Membership functions based on more theoretical demand.
4. Membership functions as a model for human concepts.
5. Neural networks based construction of membership functions.

The estimation methods of membership functions based on more theoretical demand use axioms, probability density functions and so on.

Let consider operations on fuzzy sets. There exist three standard fuzzy operations: fuzzy intersection, union and complement which are generalization of the corresponding classical set operations.

Let's $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in X with the membership functions $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$ respectively. Then the operations of intersection, union and complement are defined as given below.

Definition 1.9. Fuzzy Standard Intersection and Union. The intersection ($\bigcap$) and union ($\bigcup$) of fuzzy sets $\tilde{A}$ and $\tilde{B}$ can be calculated by the following formulas:

$$\forall x \in X \quad \mu_{\tilde{A}\cap\tilde{B}}(x) = \min\ (\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

$$\forall x \in X \quad \mu_{\tilde{A}\cup\tilde{B}}(x) = \max\ (\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

where $\mu_{\tilde{A}\cap\tilde{B}}(x)$ and $\mu_{\tilde{A}\cup\tilde{B}}(x)$ are the membership functions of $\tilde{A}\bigcap\tilde{B}$ and $\tilde{A}\bigcup\tilde{B}$, respectively.

Definition 1.10. Standard Fuzzy Complement. The complement $\tilde{A}^c$ of $\tilde{A}$ is defined by the membership function:

$$\forall x \in X \quad \mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x).$$

As already mentioned $\mu_{\tilde{A}}(x)$ is interpreted as the degree to which x belongs to $\tilde{A}$. Then by the definition $\mu_{\tilde{A}^c}(x)$ can be interpreted as the degree to which x does not belong to $\tilde{A}$.

The standard fuzzy operations do not satisfy the law of excluded middle $\tilde{A} \bigcup \tilde{A}^c = X$ and the law of contradiction $\tilde{A} \bigcap \tilde{A}^c = \varnothing$ of classical set theory. But commutativity, associativity, idempotency, distributivity, and De Morgan laws are held for standard fuzzy operations.

For fuzzy union, intersection and complement operations there exist a broad class of functions. Function that qualify as fuzzy intersections and fuzzy unions are defined as t-norms and t-conorms.

Definition 1.11. t-Norms. t-norm is a binary operation in [0,1], i.e. a binary function t from [0,1] into [0,1] that satisfies the following axioms

$$t\left(\mu_{\tilde{A}}(x),1\right) = \mu_{\tilde{A}}(x) \tag{1.9}$$

if $\mu_{\tilde{A}}(x) \le \mu_{\tilde{C}}(x)$ and $\mu_{\tilde{B}}(x) \le \mu_{\tilde{D}}(x)$ then

$$t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \le t(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x)) \tag{1.10}$$

$$t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = t(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x)) \tag{1.11}$$

$$t(\mu_{\tilde{A}}(x), t(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = t(t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) \tag{1.12}$$

Here (1.9) is boundary condition, (1.10)-(1.12) are conditions of monotonicity, commutativity and associativity, respectively.

The function t takes as its arguments the pair consisting of the element membership grades in set $\tilde{A}$ and in set $\tilde{\mathrm{B}}$, and yields membership grades of the element in the $\tilde{A} \bigcap \tilde{B}$

$$(\tilde{A} \bigcap \tilde{B})(x) = t[\tilde{A}(x), \tilde{B}(x)] \qquad \forall \mathrm{x} \in \mathrm{X}.$$

The following are frequently used t-norm-based fuzzy intersection operations:

Standard Intersection

$$t_0(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \tag{1.13}$$

Algebraic Product

$$t_1(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x) \tag{1.14}$$

Bounded Difference

$$t_2(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \mu_{\tilde{A} \cap \tilde{B}}(x) = \max(0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1) \tag{1.15}$$

Drastic Intersection

$$t_3(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \begin{cases} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} & \text{if } \mu_{\tilde{A}}(x) = 1 \\ & \text{or } \mu_{\tilde{B}}(x) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.16}$$

For four fuzzy intersections the following is true

$$t_3(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) \le t_2(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) \le t_1(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) \le t_0(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) \quad (1.17)$$

Definition 1.12. t-Conorms. t-conorm is a binary operation in $[0,1]$, i.e. a binary function $S:[0,1]\times[0,1]\rightarrow[0,1]$ that satisfies the following axioms

$$S\left(\mu_{\tilde{A}}(x),0\right)=\mu_{\tilde{A}}(x) \text{ ;(boundary condition)} \quad (1.18)$$

if $\mu_{\tilde{A}}(x)\le\mu_{\tilde{C}}(x)$ and $\mu_{\tilde{B}}(x)\le\mu_{\tilde{D}}(x)$ then

$$S(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) \le S(\mu_{\tilde{C}}(x),\mu_{\tilde{D}}(x)) \text{ ; (monotonicity)} \quad (1.19)$$

$$S(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) = S(\mu_{\tilde{B}}(x),\mu_{\tilde{A}}(x)) \text{ ; (commutativity)} \quad (1.20)$$

$$S(\mu_{\tilde{A}}(x),S(\mu_{\tilde{B}}(x),\mu_{\tilde{C}}(x))) = S(S(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x),\mu_{\tilde{C}}(x))) \text{ ; (associativity).} \quad (1.21)$$

The function S yields membership grade of the element in the set $\tilde{A}\bigcup\tilde{B}$ on the argument which is pair consisting of the same elements membership grades in set $\tilde{A}$ and $\tilde{B}$

$$(A\bigcup B)(X)=S[A(x),B(x)] \quad (1.22)$$

The following are frequently used t-conorm based fuzzy union operations.

Standard Union

$$S_0(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) = \max\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\} \quad (1.23)$$

Algebraic Sum

$$S_1(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) = \mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x)-\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x) \quad (1.24)$$

Drastic Union

$$S_3(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) = \begin{cases} \max\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\} & \text{if } \mu_{\tilde{A}}(x)=0 \\ & \text{or } \mu_{\tilde{B}}(x)=0 \\ 1 & \text{otherwise} \end{cases} \quad (1.25)$$

For four fuzzy union operations the following is true

$$S_0(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) \le S_1(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) \le S_2(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) \le S_3(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)) \quad (1.26)$$

Definition 1.13. Cartesian Product of Fuzzy Sets. The Cartesian product of fuzzy sets $\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n$ on universes $X_1,X_2,\ldots,X_n$ respectively is a fuzzy set in the product space $X_1 \times X_2 \times \ldots \times X_n$ with the membership function $\mu_{\tilde{A}_1 \times \tilde{A}_2 \times \ldots \times \tilde{A}_n}(x) = \min\left\{\mu_{\tilde{A}_i}(x_i) \mid x = (x_1, x_2 \ldots, x_n), x_i \in X_i\right\}$.

Definition 1.14. Power of Fuzzy Sets. m-th power of a fuzzy set $\tilde{A}^m$ is defined as

$$\mu_{\tilde{A}^m}(x) = [\mu_{\tilde{A}}(x)]^m \ , \ \forall \mathrm{x} \in \mathrm{X}, \ \forall \mathrm{m} \in \mathrm{R}^+ \quad (1.27)$$

where R^+ is positively defined set of real numbers.

Definition 1.15. Concentration and Dilation of Fuzzy Sets

Let $\tilde{A}$ be fuzzy set on the universe:

$$\tilde{A} = \left\{x, \mu_{\tilde{A}}(x) / x \in X\right\}$$

Then the operator $Con_m \tilde{A} = \{(x, [\mu_{\tilde{A}}(x)]^m) / \mathrm{x} \in \mathrm{X}\}$ is called concentration of $\tilde{A}$ and the operator $Dil_n \tilde{A} = \{(x, \sqrt{\mu_{\tilde{A}}(x)}\ ) / \mathrm{x} \in \mathrm{X}\}$ is called dilation of $\tilde{A}$.

Definition 1.16. Difference of Fuzzy Sets. Difference of fuzzy sets is defined by the formula:

$$\forall x \in X, \ \ \mu_{\tilde{\mathrm{A}}|-|\tilde{B}}(x) = \max(0, \mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)) \quad (1.28)$$

$\tilde{A}|-|\tilde{B}$ is the fuzzy set of elements that belong to $\tilde{A}$ more than to $\tilde{\mathrm{B}}$.

Symmetrical difference of fuzzy sets $\tilde{A}$ and $\tilde{\mathrm{B}}$ is the fuzzy set $\tilde{\mathrm{A}} \nabla \tilde{\mathrm{B}}$ of elements that belong more to $\tilde{A}$ than to $\tilde{\mathrm{B}}$:

$$\forall x \in X \quad \mu_{\tilde{\mathrm{A}} \nabla \tilde{\mathrm{B}}}(x) = \left|\mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)\right| \quad (1.29)$$

Definition 1.17. Fuzzy Number. A fuzzy number is a fuzzy set $\tilde{A}$ on R which possesses the following properties: a) $\tilde{A}$ is a normal fuzzy set; b) $\tilde{A}$ is a convex fuzzy set; c) α-cut of $\tilde{A}$, A^α is a closed interval for every $\alpha \in (0,1]$; d) the support of $\tilde{A}$, A^{+0} is bounded.

In Fig. 1.1 some basic types of fuzzy numbers are shown. For comparison of a fuzzy number with a crisp number in Fig. 1.2 crisp number 2 is given.

Let consider arithmetic operation on fuzzy numbers. There are different methods for developing fuzzy arithmetic. In this section we present three methods.

Method based on the extension principle. By this method basic arithmetic operations on real numbers are extende to operations on fuzzy numbers. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers and $*$ denote any of four arithmetic operations $\{+, -, \cdot, :\}$.

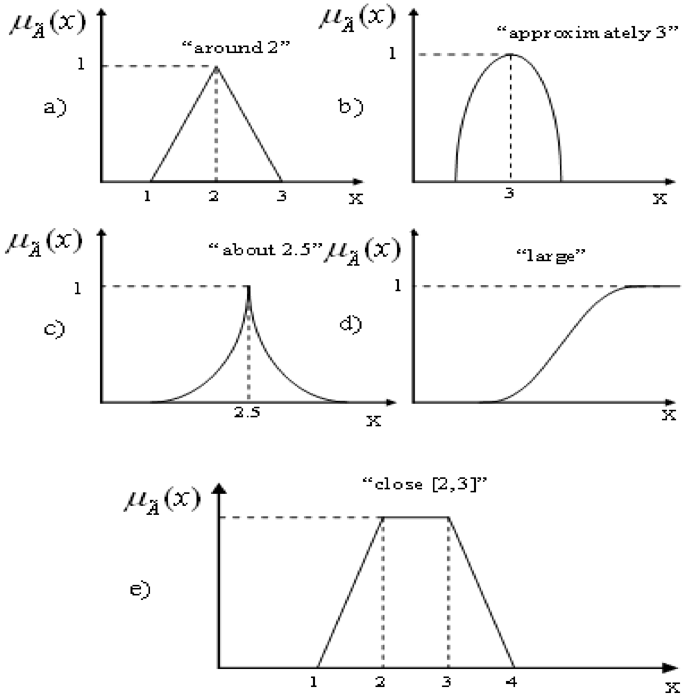


Fig. 1.1 Types of fuzzy numbers

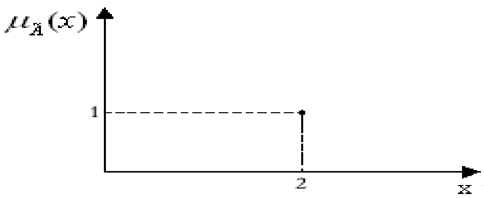


Fig. 1.2 Crisp number 2

A fuzzy set $\tilde{A} * \tilde{B}$ on R can be defined by the equation

$$\forall z \in \mathrm{R} \qquad \mu_{(\tilde{A}*\tilde{B})}(z) = \sup_{z=x*y} \min[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)] \tag{1.30}$$

It is shown in [57] that $\tilde{A} * \tilde{B}$ is fuzzy number and the following theorem has been formulated and proved.

Theorem 1.1. *Let* $* \in \{+, -, \cdot, :\}$*, and let* $\tilde{A}$*,* $\tilde{B}$ *denote continuous fuzzy numbers. Then, the fuzzy set* $\tilde{A} * \tilde{B}$ *defined by (1.30) is a continuous fuzzy number.*

Then for four basic arithmetic operations on fuzzy numbers we can write

$$\mu_{(\tilde{A}+\tilde{B})}(z) = \sup_{z=x+y} \min[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)] \tag{1.31}$$

$$\mu_{(\tilde{A}-\tilde{B})}(z) = \sup_{z=x-y} \min[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)] \tag{1.32}$$

$$\mu_{(\tilde{A}\cdot\tilde{B})}(z) = \sup_{z=x\cdot y} \min[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)] \tag{1.33}$$

$$\mu_{(\tilde{A}:\tilde{B})}(z) = \sup_{z=x:y} \min[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)] \tag{1.34}$$

Method Based on Interval Arithmetic and α-Cuts. This method is based on representation of arbitrary fuzzy numbers by their α-cuts and use interval arithmetic to the α-cuts. Let $\tilde{A}, \tilde{B} \subset R$ be fuzzy numbers and $*$ denote any of four operations. For each $\alpha \in (0,1]$, the α-cut of $\tilde{A} * \tilde{B}$ is expressed as

$$(\tilde{A} * \tilde{B})^{\alpha} = A^{\alpha} * B^{\alpha} \tag{1.35}$$

For $*$ we assume $0 \notin \mathrm{supp}(\tilde{B})$.

The resulting fuzzy number $\tilde{A} * \tilde{B}$ can be defined as

$$\tilde{A} * \tilde{B} = \bigcup_{\alpha \in [0,1]} \alpha (A * B)^{\alpha} \tag{1.36}$$

Next we using (1.35), (1.36) illustrate four arithmetic operations on fuzzy numbers.

Addition. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers and A^{α} and B^{α} their α-cuts

$$A^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}]; B^{\alpha} = [b_1^{\alpha}, b_2^{\alpha}] \tag{1.37}$$

Then we can write

$$A^{\alpha} + B^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}] + [b_1^{\alpha}, b_2^{\alpha}] = [a_1^{\alpha} + b_1^{\alpha}, a_2^{\alpha} + b_2^{\alpha}],\ \forall \alpha \in [0,1] \tag{1.38}$$

here

$$A^{\alpha} = \{x / \mu_{\tilde{A}}(x) \ge \alpha\}; B^{\alpha} = \{x / \mu_{\tilde{B}}(x) \ge \alpha\} \tag{1.39}$$

Subtraction. Subtraction of given fuzzy numbers $\tilde{A}$ and $\tilde{B}$ can be defined as

$$(A - B)^{\alpha} = A^{\alpha} - B^{\alpha} = [a_1^{\alpha} - b_2^{\alpha}, a_2^{\alpha} - b_1^{\alpha}], \forall \alpha \in [0,1] \tag{1.40}$$

We can determine (1.40) by addition of the image $\tilde{B}^{-}$ to $\tilde{A}$

$$\forall \alpha \in [0,1], B^{\alpha^-} = [-b_2^\alpha, -b_1^\alpha] \quad (1.41)$$

Multiplication. Let two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ be given. Multiplication $\tilde{A} \cdot \tilde{B}$ is defined as

$$(A \cdot B)^\alpha = A^\alpha \cdot B^\alpha = [a_1^\alpha, a_2^\alpha] \cdot [b_1^\alpha, b_2^\alpha] \forall \alpha \in [0,1] \quad (1.42)$$

Multiplication of fuzzy number $\tilde{A}$ in R by ordinary numbers $k \in R^+$ is performed as follows

$$\forall \tilde{A} \subset R \;\; kA^\alpha = [ka_1^\alpha, ka_2^\alpha] \quad (1.43)$$

Division. Division of two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is defined by

$$A^\alpha : B^\alpha = [a_1^\alpha, a_2^\alpha] : [b_1^\alpha, b_2^\alpha] \;\; \forall \alpha \in [0,1] \quad (1.44)$$

Definition 1.18. Absolute Value of a Fuzzy Number. Absolute value of fuzzy number is defined as:

$$abs(\tilde{A}) = \begin{cases} \max(\tilde{A}, -\tilde{A}), & \text{for} \quad R^+ \\ 0, & \text{for} \quad R^- \end{cases} \quad (1.45)$$

Let consider Z-number and operations on Z –numbers [128]. Decisions are based on decision-relevant information which must be reliable. Basically, the concept of a Z-number relates to the issue of reliability of information. A Z-number, Z, has two components, $Z=(\tilde{A}, \tilde{B})$. The first component, $\tilde{A}$, is a restriction (constraint) on the values which a real-valued uncertain variable, X, is allowed to take. The second component, $\tilde{B}$, is a measure of reliability (confidence)of the first component. Typically, $\tilde{A}$ and $\tilde{B}$ are described in a natural language.

The concept of a Z-number has a potential for many applications, especially in the realms of economics and decision analysis.

Much of the information on which decisions are based is uncertain. Humans have a remarkable capability to make rational decisions based on information which is uncertain, imprecise and or incomplete. Formalization of this capability, at least to some degree motivates the concepts Z-number [128].

The ordered triple $(X, \tilde{A}, \tilde{B})$ is referred to as a Z-valuation. A Z-valuation is equivalent to an assignment statement, X is $(\tilde{A}, \tilde{B})$. X is an uncertain random variable. For convenience, $\tilde{A}$ is referred to as a value of X, with the understanding that, $\tilde{A}$ is not a value of X but a restriction on the values which X can take. The second component, $\tilde{B}$, is referred to as confidence(certainty). When X is a random variable, certainty may be equated to probability. Typically, $\tilde{A}$ and $\tilde{B}$

are perception-based and are described in NL. A collection of Z-valuations is referred to as Z-information. It should be noted that much of everyday reasoning and decision-making is based on Z-information. For purposes of computation, when $\tilde{A}$ and $\tilde{B}$ are described in NL, the meaning of $\tilde{A}$ and $\tilde{B}$ is precisiated through association with membership functions, $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$, respectively. Simple examples of Z-valuations are:

(anticipated budget deficit, about 3 million dollars, likely);

(price of oil in the near future, significantly over 50 dollars/barrel, veri likely).

The Z-valuation $(X, \tilde{A}, \tilde{B})$ may be viewed as a restriction on X defined by:

Prob (X is $\tilde{A}$) is $\tilde{B}$.

In a Z-number, $(\tilde{A}, \tilde{B})$, the underlying probability distribution p_X, is not known. What is known is a restriction on p_X which may be expressed as [128]:

$$\int_R \mu_{\tilde{A}}(u) p_X(u) du \text{ is } \tilde{B}$$

An important qualitative attribute of a Z-number is informativeness. Generally, but not always, a Z-number is informative if its value has high specificity, that is, is tightly constrained [110], and its certainty is high. Informativeness is a desideratum when a Z-number is a basis for a decision. A basic question is: When is the informativeness of a Z-number sufficient to serve as a basis for an intelligent decision?

The concept of a Z-number is based on the concept of a fuzzy granule [120,121,124]. A concept which is closely related to the concept of a Z-number is the concept of a Z^+-number. Basically, a Z^+-number, Z^+, is a combination of a fuzzy number, $\tilde{A}$, and a random number, R, written as an ordered pair $Z^+ = (\tilde{A}, \tilde{B})$. In this pair, $\tilde{A}$ plays the same role as it does in a Z-number, and R is the probability distribution of a random number. Equivalently, R may be viewed as the underlying probability distribution of X in the Z-valuation $(X, \tilde{A}, \tilde{B})$. Alternatively, a Z^+-number may be expressed as $(\tilde{A}, p_X)$ or $(\mu_{\tilde{A}}, p_X)$, where $\mu_{\tilde{A}}$ is the membership function of $\tilde{A}$. A Z^+-valuation is expressed as $(X, \tilde{A}, p_X)$ or, equivalently, as $(X, \mu_{\tilde{A}}, p_X)$, where p_X is the probability distribution (density) of X.

The scalar product of $\mu_{\tilde{A}}$ and p_X, $\mu_{\tilde{A}} p_X$ is the probability measure, $P_{\tilde{A}}$, of $\tilde{A}$. More concretely,

$$\mu_{\tilde{A}} p_X = P_{\tilde{A}} = \int_R \mu_{\tilde{A}}(u) p_X(u) du \tag{1.46}$$

It is this relation that links the concept of a Z-number to that of a Z^+-number.

More concretely,

$$Z(\tilde{A},\tilde{B}) = Z^{+}(\tilde{A},\mu_{\tilde{A}}\; p_X \; is\; \tilde{B})$$

What should be underscored is that in the case of a Z-number what is known is not p_X but a restriction on p_X expressed as $\mu_{\tilde{A}}\; p_X$ is $\tilde{B}$.

Let X be a real-valued variable taking values in U. For our purposes, it will be convenient to assume that U is a finite set $U=\{u_1,...,u_n\}$. We can associate with X a possibility distribution μ, and a probability distribution p, expressed as:

$$\mu=\mu_1 / u_1 + ..., + \mu_n / u_n$$

$$p=\mu_1 \setminus p_1 + ..., + \mu_n \setminus p_n$$

Here μ_i / u_i means that μ_i, $i = 1,...n$, is the possibility that $X = u_i$. Similarly, $p_i \setminus u_i$ means that p_i is the probability that $X = u_i$.

The possibility distribution, μ, may be combined with the probability distribution, p, through what is referred to as confluence. More concretely,

$$\mu : p = (\mu_1, p_1)/u_1 + ... + (\mu_n, p_n)/u_n$$

As was noted earlier, the scalar product, expressed as $\mu \cdot p$, is the probability measure of $\tilde{A}$. In terms of the bimodal distribution, the Z^{+}-valuation and the Z-valuation associated with X may be expressed as:

$$(X, \tilde{A}, p_X)$$

$$(X, \tilde{A}, \tilde{B}),\; \mu_{\tilde{A}}\; p_X \; is\; \tilde{B},$$

respectively, with the understanding that $\tilde{B}$ is a possibilistic restriction on $\mu_{\tilde{A}}\; p_X$.

A key idea which underlies the concept of a Z-mouse [128] is that visual interpretation of uncertainty is much more natural than its description in natural language or as a membership function of a fuzzy set. This idea is closely related to the remarkable human capability to precisiate (graduate) perceptions, that is, to associate perceptions with degrees.

Using a Z-mouse, a Z-number is represented as two f-marks on two different scales. The trapezoidal fuzzy sets which are associated with the f-marks serve as objects of computation.

Let us consider computation with Z-numbers. Computation with Z^{+}-numbers is much simpler than computation with Z-numbers. Assume that $*$ is a binary operation whose operands are Z^+-numbers, $\mathrm{Z}^+{}_X = (\tilde{A}_X, R_X)$ and $\mathrm{Z}^+{}_Y = (\tilde{A}_Y, R_Y)$ By definition,

$$\mathrm{Z}^+{}_X * Z^+{}_Y = (\tilde{A}_X * \tilde{A}_Y \, R_X * R_Y) \tag{1.47}$$

with the understanding that the meaning of $*$ in $R_X * R_Y$ is not the same as the meaning of $*$ in $\tilde{A}_X * \tilde{A}_Y$. In this expression, the operands of $*$ in $\tilde{A}_X * \tilde{A}_Y$ are fuzzy numbers; the operands of $*$ in $R_X * R_Y$ are probability distributions.

Assume that $*$ is sum. In this case, $\tilde{A}_X + \tilde{A}_Y$ is defined by:

$$\mu_{(\tilde{A}_X + \tilde{A}_Y)}(v) = \sup_u (\mu_{\tilde{A}_X}(u) \wedge \mu_{\tilde{A}_Y}(v-u)), \wedge = \min \tag{1.48}$$

Similarly, assuming that R_X and R_Y are independent, the probability density function of $R_X * R_Y$ is the convolution, $\circ$, of the probability density functions of R_X and R_Y. Denoting these probability density functions as p_{R_X} and p_{R_Y}, respectively, we have:

$$p_{R_X + R_Y}(v) = \int_R p_{R_X}(u) p_{R_Y}(v-u) du \tag{1.49}$$

Thus,

$$Z^+{}_X + Z^+{}_Y = (\tilde{A}_X + \tilde{A}_Y, p_{R_X} \circ p_{R_Y}) \tag{1.50}$$

More generally, to compute $\mathrm{Z}_X * Z_Y$ what is needed is the extension principle of fuzzy logic [114,115].

Turning to computation with Z-numbers, assume for simplicity that $* = \mathrm{sum}$. Assume that $Z_X = (\tilde{A}_X, \tilde{B}_X)$ and $Z_Y = (\tilde{A}_Y, \tilde{B}_Y)$. Our problem is to compute the sum $\mathrm{Z{=}X{+}Y}$. Assume that the associated Z-valuations are $(X, \tilde{A}_X, \tilde{B}_X)$, $(Y, \tilde{A}_Y, \tilde{B}_Y)$ and $(Z, \tilde{A}_Z, \tilde{B}_Z)$.

The first step involves computation of p_Z. To begin with, let us assume that p_X and p_Y are known, and let us proceed as we did in computing the sum of Z^+-numbers. Then

$$p_Z = p_X \circ p_Y$$

or more concretely

$$p_Z(v) = \int_R p_X(u) p_Y(v-u) du$$

In the case of Z-numbers what we know are not $\mathrm{p_X}$ and $\mathrm{p_Y}$ but restrictions on $\mathrm{p_X}$ and $\mathrm{p_Y}$

$$\int_R \mu_{\tilde{A}_X}(u)\,p_X(u)du \text{ is } \tilde{B}_X$$

$$\int_R \mu_{\tilde{A}_Y}(u)\,p_Y(u)du \text{ is } \tilde{B}_Y$$

In terms of the membership functions of $\tilde{B}_X$ and $\tilde{B}_Y$, these restrictions may be expressed as:

$$\mu_{\tilde{B}_X}(\int_R \mu_{\tilde{A}_X}(u)\,p_X(u)du)$$

$$\mu_{\tilde{B}_Y}(\int_R \mu_{\tilde{A}_Y}(u)\,p_Y(u)du)$$

Additional restrictions on p_X and p_Y are:

$$\int_R p_X(u)du = 1$$

$$\int_R p_Y(u)du = 1$$

$$\int_R u p_X(u)du = \frac{\int_R u\mu_{\tilde{A}_X}(u)du}{\int_R \mu_{\tilde{A}_X}(u)du} \quad \text{(compatibility)}$$

$$\int_R u p_Y(u)du = \frac{\int_R u\mu_{\tilde{A}_Y}(u)du}{\int_R \mu_{\tilde{A}_Y}(u)du} \quad \text{(compatibility)}$$

Applying the extension principle, the membership function of p_Z may be expressed as:

$$\mu_{p_Z}(p_Z) = \sup_{p_X, p_Y}(\mu_{\tilde{B}_X}(\int_R \mu_{\tilde{A}_X}(u)\,p_X(u)du) \wedge \mu_{\tilde{B}_Y}(\int_R \mu_{\tilde{A}_Y}(u)\,p_Y(u)du))$$

subject to

$$p_Z = p_X \circ p_Y$$

$$\int_R p_X(u)du = 1$$

$$\int_R p_Y(u)du = 1$$

$$\int_R u p_X(u)du = \frac{\int_R u\mu_{\tilde{A}_X}(u)du}{\int_R \mu_{\tilde{A}_X}(u)du}$$

$$\int_R u p_Y(u)du = \frac{\int_R u\mu_{\tilde{A}_Y}(u)du}{\int_R \mu_{\tilde{A}_Y}(u)du}$$

The second step involves computation of the probability of the fuzzy event, Z is $\tilde{A}_Z$, given p_Z. As was noted earlier, in fuzzy logic the probability measure of the fuzzy event X is $\tilde{A}$, where $\tilde{A}$ is a fuzzy set and X is a random variable with probability density p_X, is defined as:

$$\int_R \mu_{\tilde{A}}(u) p_X(u)du$$

Using this expression, the probability measure of $\tilde{A}_Z$ may be expressed as:

$$B_Z = \int_R \mu_{\tilde{A}_Z}(u) p_Z(u)du,$$

where

$$\mu_{\tilde{A}_Z}(u) = \sup_v(\mu_{\tilde{A}_X}(v) \wedge \mu_{\tilde{A}_Y}(u-v))$$

It should be noted that B_Z is a number when p_Z is a known probability density function. Since what we know about p_Z is its possibility distribution, $\mu_{p_Z}(p_Z)$, $\tilde{B}_Z$ is a fuzzy set with membership function $\mu_{\tilde{B}_Z}$. Applying the extension principle, we arrive at an expression for $\mu_{\tilde{B}_Z}$. More specifically,

$$\mu_{\tilde{B}_Z}(w) = \sup_{p_Z} \mu_{p_Z}(p_Z)$$

subject to

$$w = \int_R \mu_{\tilde{A}_Z}(u) p_Z(u)du$$

Where $\mu_{p_Z}(p_Z)$ is the result of the first step. In principle, this completes computation of the sum of Z-numbers, Z_X and Z_Y.

In a similar way, we can compute various functions of Z-numbers. The basic idea which underlies these computations may be summarized as follows. Suppose that our problem is that of computing $f(Z_X, Z_Y)$, where Z_X and Z_Y are Z-numbers, $Z_X = (\tilde{A}_X, \tilde{B}_X)$ and $Z_Y = (\tilde{A}_Y, \tilde{B}_Y)$ respectively, and $f(Z_X, Z_Y) = (\tilde{A}_Z, \tilde{B}_Z)$. We begin by assuming that the underlying probability distributions p_X and p_Y are known. This assumption reduces the computation of $f(Z_X, Z_Y)$ to computation of $f(Z_X^+, Z_Y^+)$, which can be carried out through the use of the version of the extension principle which applies to restrictions which are Z^{+}-numbers. At this point, we recognize that what we know are not p_X and p_Y but restrictions on p_X and p_Y. Applying the version of the extension principle which relates to probabilistic restrictions, we are led to $f(Z_X, Z_Y)$. We can compute the restriction, $\tilde{B}_Z$, of the scalar product of $f(\tilde{A}_X, \tilde{A}_Y)$ and $f(p_X, p_Y)$. Since $\tilde{A}_Z = f(\tilde{A}_X, \tilde{A}_Y)$, computation of $\tilde{B}_Z$ completes the computation of $f(Z_X, Z_Y)$.

There are many important directions which remain to be explored, especially in the realm of calculi of Z-rules and their application to decision analysis and modeling of complex systems.

Computation with Z-numbers may be viewed as a generalization of computation with numbers, intervals, fuzzy numbers and random numbers. More concretely, the levels of generality are: computation with numbers (ground level1);computation with intervals (level1); computation with fuzzy numbers (level 2); and computation with Z-numbers (level3). The higher the level of generality, the greater is the capability to construct realistic models of real-world systems, especially in the realms of economics and decision analysis.

It should be noted that many numbers, especially in fields such as economics and decision analysis are in reality Z-numbers, but they are not treated as such because it is much simpler to compute with numbers than with Z-numbers. Basically, the concept of a Z-number is a step toward formalization of the remarkable human capability to make rational decisions in an environment of imprecision and uncertainty.

We now consider fuzzy relations, linguistic variables. In modeling systems the internal structure of a system must be described first. An internal structure is characterized by connections (associations) among the elements of system. As a rule these connections or associations are represented by means of relation. We will consider here fuzzy relations which gives us the representation about degree or strength of this connection.

There are several definitions of fuzzy relation [54,113,117]. Each of them depends on various factors and expresses different aspects of modeling systems.

Definition 1.19. Fuzzy Relation. Let $X_1, X_2, ..., X_n$ be nonempty crisp sets. Then, a $\tilde{R}(X_1, X_2, ..., X_n)$ is called a fuzzy relation of sets $X_1, X_2, ..., X_n$, if $\tilde{R}(X_1, X_2, ..., X_n)$ is the fuzzy subset given on Cartesian product $X_1 \times X_2 \times ... \times X_n$.

If $n = 2$, then fuzzy relation is called binary fuzzy relation, and is denoted as $\tilde{R}(X, Y)$. For three, four, or n sets the fuzzy relation is called ternary, quaternary, or n-ary, respectively.

In particular, if $X_1 = X_2 = ... = X_n = X$ we say that fuzzy relation R is given on set X among elements $x_1, x_2, ..., x_n \in X$.

Notice, that fuzzy relation can be defined in another way. Namely, by two ordered fuzzy sets.

Assume, two fuzzy sets $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(y)$ are given on crisp sets X and Y, respectively. Then, it is said, that fuzzy relation $R_{\tilde{A}\tilde{B}}(X,Y)$ is given on sets X and Y, if it is defined in the following way

$$\mu_{R_{\tilde{A}\tilde{B}}}(x,y) = \min_{x,y}[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)]$$

for all pairs (x,y), where $x \in X$ and $y \in Y$. As above, fuzzy relation $R_{\tilde{A}\tilde{B}}$ is defined on Cartesian product.

Let fuzzy binary relation on set X be given. Consider the following three properties of relation $\tilde{R}$:

1. Fuzzy relation $\tilde{R}$ is reflexive, if

$$\mu_{\tilde{R}}(x,x) = 1$$

for all $x \in X$. If there exist $x \in X$ such that this condition is violated, then relation $\tilde{R}$ is irreflexive, and if $\tilde{R}(x,x) = 0$ for all $x \in X$, the relation $\tilde{R}$ is antireflective;

2. A fuzzy relation $\tilde{R}$ is symmetric if it satisfies the following condition:

$$\mu_{\tilde{R}}(x,y) = \mu_{\tilde{R}}(y,x)$$

for all $x, y \in X$. If from $\tilde{R}(x,y) > 0$ and $\tilde{R}(y,x) > 0$ follows $x = y$ for all $x, y \in X$ relation $\tilde{R}$ is called antisymmetric;

3. A fuzzy relation $\tilde{R}$ is transitive (or, more specifically, max-min transitive) if

$$\mu_{\tilde{R}}(x,z) \geq \max_{y \in Y} \min(\mu_{\tilde{R}}(x,y), \mu_{\tilde{R}}(y,z))$$

is satisfied for all pairs $(x,z) \in X$.

Definition 1.20. Fuzzy Proximity. A fuzzy relation is called a fuzzy proximity or fuzzy tolerance relation if it is reflexive and symmetric. A fuzzy relation is called a fuzzy similarity relation if it is reflexive, symmetric, and transitive.

Definition 1.21. Fuzzy Composition. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets on $X \times Y$ and $Y \times Z$, respectively. A fuzzy relation $\tilde{R}$ on $X \times Z$ is defined as

$$\tilde{R} = \{((x,z), \mu_{\tilde{R}}(x,z) \mid (x,z) \in X \times Z\} \quad (1.51)$$

here

$$\mu_{\tilde{R}} : X \times Y \to [0,1]$$

$$(x,z) \mapsto \mu_{\tilde{R}}(x,z) = \mu_{\tilde{A}\circ\tilde{B}}(x,z) = \underset{y\in Y}{S}(\mathrm{T}(\mu_{\tilde{A}}(x,y), \mu_{\tilde{B}}(y,z))) \quad (1.52)$$

For $x \in X$ and $z \in Z$, T and S are triangular norms and triangular conorms, respectively.

Definition 1.22. Equivalence (Similarity) Relation. If fuzzy relation $\tilde{R}$ is reflexive, symmetric and transitive then relation $\tilde{R}$ is an equivalence relation or similarity relation.

A fuzzy relation $\tilde{R}$ is a fuzzy compatibility relation if it is reflexive and symmetric. This relation is cutworthy. Compatibility classes are defined by means of α-cut. In fact, using α-cut a class of compatibility relation is represented by means of crisp subset.

Therefore a compatibility relation can also be represented by reflexive undirected graph.

Now consider fuzzy partial ordering.

Let X be nonempty set. It is well known, that to order a set it is necessary to give an order relation on this set. But sometimes our knowledge and estimates of the elements of a set are not accurate and complete. Thus, to order such set the fuzzy order on set must be defined.

Definition 1.23. Fuzzy Partial Ordering Relation. Let $\tilde{R}$ be binary fuzzy relation on X . Then fuzzy relation $\tilde{R}$ is called fuzzy partial ordering, if it satisfies the following conditions:

1.Fuzzy relation $\tilde{R}$ is reflexive;
2.Fuzzy relation $\tilde{R}$ is antisymmetric;
3.Fuzzy relation $\tilde{R}$ is fuzzy transitive.

If fuzzy partial order is given on set X then we will say that set X is fuzzy partially ordered.

Next we consider projections and cylindric extension.

Let $\tilde{R}$ be n-dimensional fuzzy relation on Cartesian product $X = X_1 \times X_2 \times ... \times X_n$ of nonempty sets $X_1, X_2, ..., X_n$ and $(i_1, i_2, ..., i_k)$ be a subsequence of $(1, 2, ..., n)$.

The practice and experimental evidence have shown that decision theories developed for a perfect decision-relevant information and 'well-defined' preferences are not capable of adequate modeling of real-world decision making. The reason is that real decision problems are characterized by imperfect decision-relevant information and vaguely defined preferences. This leads to the fact that when solving real-world decision problems we need to move away from traditional decision approaches based on exact modeling which is good rather for decision analysis of thought experiments.

More concretely, the necessity to sacrifice the precision and determinacy is by the fact that real-world problems are characterized by perception-based information and choices, for which natural language is more covinient and close than precise formal approaches. Modeling decision making from this perspective is impossible without dealing with fuzzy categories near to human notions and imaginations. In this connection, it is valuable to use the notion of linguistic variable first introduced by L.Zadeh [119]. Linguistic variables allow an adequate reflection of approximate in-word descriptions of objects and phenomena in the case if there is no any precise deterministic description. It should be noted as well that many fuzzy categories described linguistically even appear to be more informative than precise descriptions.

Definition 1.24. Linguistic Variable. A linguistic variable is characterized by the set (u,T,X,G,M), where u is the name of variable; T denotes the term-set of u that refer to as base variable whose values range over a universe X; G is a syntactic rule (usually in form of a grammar) generating linguistic terms; M is a semantic rule that assigns to each linguistic term its meaning, which is a fuzzy set on X.

A certain $t \in T$ generated by the syntactic rule G is called a term. A term consisting of one or more words, the words being always used together, is named an atomary term. A term consisting of several atomary terms is named a composite term. The concatenation of some components of a composite term (i.e. the result of linking the chains of components of the composite term) is called a subterm. Here $t_1, t_2, \ldots$ are terms in the following expression

$$T = t_1 + t_2 + \ldots$$

The meaning of $M(t)$ of the term t is defined as a restriction $R(t;x)$ on the basis variable x conditioned by the fuzzy variable $\tilde{X}$:

$$M(t) \equiv R(t;x)$$

it is assumed here that $R(t;x)$ and, consequently, $M(t)$ can be considered as a fuzzy subset of the set X named as t.

The assignment equation in case of linguistic variable takes the form in which t-terms in T are names generated by the grammar G, where the meaning assigned to the term t is expressed by the equality

$$M(t) = R(term\, in\, T)$$

In other words the meaning of the term t is found by the application of the semantic rule M to the value of term t assigned according to the right part of equation.

Moreover, it follows that $M(t)$ is identical to the restriction associated with the term t.

It should be noted that the number of elements in T can be unlimited and then for both generating elements of the set T and for calculating their meaning, the application of the algorithm, not simply the procedure for watching term-set, is necessary.

We will say that a linguistic variable u is structured if its term-set T and the function M, which maps each element from the term-set into its meaning, can be given by means of algorithm. Then both syntactic and semantic rules connected with the structured linguistic variable can be considered algorithmic procedures for generating elements of the set T and calculating the meaning of each term in T, respectively.

However in practice we often encounter term-sets consisting of a small number of terms. This makes it easier to list the elements of term-set T and establishes a direct mapping from each element to its meaning. For axample, an intuitive description of possible economic conditions may be represented by linguistic terms like “strong econonmic growth”, “weak economic growth” etc. Then the term set of linguistic variable “state of economy” can be written as follows:

T(state of economy) = “strong growth” + “moderate growth” + “static situation” + “recession”.

The variety of economic conditions may also be described by ranges of the important economic indicators. However, numerical values of indicators may not be sufficiently clear even for experts and may arise questions and doubts. In contrast, linguistic description is well perceived by human intuition as qualitative and fuzzy.

1.2 Classical and Extented Fuzzy Logic

First we consider classical fuzzy logic. We will consider the logics with multi-valued and continuous values (fuzzy logic). Let’s define the semantic truth function of this logic. Let P be statement and $T(P)$ its truth value, where

$$T(P) \in [0,1]$$

Negation values of the statement P are defined as:

$$T(\neg P) = 1 - T(P).$$

Hence

$$T(\neg\neg P) = T(P).$$

The implication connective is always defined as follows:

$$T(P \to Q) = T(\neg P \vee Q),$$

and the equivalence as

$$T(P \leftrightarrow Q) = T\left[(P \rightarrow Q) \wedge (Q \rightarrow P)\right].$$

It should be noted that exclusive disjunction ex, disjunction of negations (Shiffer's connective) $|$, conjunction of negations $\downarrow$ and $\sim\rightarrow$ (has no common name) are defined as negation of equivalence $\leftrightarrow$, conjunction $\wedge$, disjunction $\vee$, and implication $\rightarrow$, respectively.

The tautology denoted $\bullet$ and contradiction denoted $\circ$ will be, respectively:

$$T\left(\overset{\bullet}{P}\right) = T(P \vee \neg P); T\left(\overset{\circ}{P}\right) = T(P \wedge \neg P).$$

More generally

$$T\left(\overset{\bullet}{P}Q\right) = T\left((P \vee \neg P) \vee (Q \vee Q)\right)$$

$$T\left(\overset{\circ}{P}Q\right) = T\left((P \wedge \neg P) \wedge (Q \wedge Q)\right)$$

Semantic Analysis of Different Fuzzy Logics. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy sets of the subsets of non-fuzzy universe U ; in fuzzy set theory it is known that $\tilde{A}$ is a subset of $\tilde{B}$ iff

$$\mu_{\tilde{A}} \leq \mu_{\tilde{B}}, \text{i.e. } \forall x \in U, \qquad \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x).$$

Definition 1.25. Power Fuzzy Set. For given fuzzy implication $\rightarrow$ and fuzzy set $\tilde{B}$ from the universe U , the power fuzzy set $\tilde{P}\tilde{B}$ from $\tilde{B}$ is given by membership function $\mu_{\tilde{P}\tilde{B}}$ [3,19]:

$$\mu_{\tilde{P}\tilde{B}}\tilde{A} = \underset{X \in U}{\wedge}(\mu_{\tilde{A}}(x) \rightarrow \mu_{\tilde{B}}(x))$$

Then the degree to which $\tilde{A}$ is subset of $\tilde{B}$, is

$$\pi(\tilde{A} \subseteq \tilde{B}) = \mu_{\tilde{P}\tilde{B}}\tilde{A}$$

Definition 1.26. If fuzzy implication operator [3,19] is given on the closed unit interval [0,1] then

$$a \leftarrow b = b \rightarrow a$$

$$a \leftrightarrow b = (a \rightarrow b) \wedge (a \leftarrow b) = (a \rightarrow b) \wedge (a \leftarrow b)$$

Definition 1.27. Degree of ″Equivalency″. Under the conditions of the definition $P\tilde{B}$ the degree to which fuzzy sets $\tilde{A}$ and $\tilde{B}$ are equivalent is:

$$\pi\left(\tilde{A} \equiv \tilde{B}\right) = \pi\left(\tilde{A} \subseteq \tilde{B}\right) \wedge \pi\left(\tilde{B} \subseteq \tilde{A}\right);$$

or

$$\pi\left(\tilde{A} \equiv \tilde{B}\right) = \underset{x \in U}{\wedge} (\mu_{\tilde{A}} x \rightarrow \mu_{\tilde{B}} x)$$

For practical purposes [3,19] in most cases it is advisable to work with multi-valued logics in which logical variable takes values from the real interval $I = [0,1]$ divided into 10 subintervals, i.e. by using set $V_{11} = [0, 0.1, 0.2, ..., 1]$.

We denote the truth values of premises $\tilde{A}$ and $\tilde{B}$ through $T(\tilde{A}) = a$ and $T(\tilde{B}) = b$. The implication operation in analyzed logics [2,3,88] has the following form:

1) min-logic

$$a \underset{\min}{\rightarrow} b = \begin{cases} a, & if \quad a \le b \\ b, & otherwise. \end{cases}$$

2) $S^{\#}$ - logic

$$a \underset{s^{\#}}{\rightarrow} b = \begin{cases} 1, & if \quad a \ne 1 \quad or \quad b = 1, \\ 0 & , otherwise. \end{cases}$$

3) S - logic (″Standard sequence″)

$$a \underset{S}{\rightarrow} b = \begin{cases} 1, & if \quad a \le b, \\ 0, & otherwise. \end{cases}$$

4) G - logic (″Gödelian sequence″)

$$a \underset{G}{\rightarrow} b = \begin{cases} 1, & if \quad a \le b, \\ b, & otherwise. \end{cases}$$

5) $G43$ - logic

$$a \underset{G43}{\rightarrow} b = \begin{cases} 1, & if \quad a = 0, \\ \min(1, b/a), & otherwise. \end{cases}$$

6) L - logic (Lukasiewicz's logic)

$$a \underset{L}{\rightarrow} b = \min\left(1, 1 - a + b\right).$$

7) KD - logic

$$a \underset{KD}{\rightarrow} b = ((1-a) \vee b = \max(1-a, b).$$

In turn ALI1-ALI4 - logics, suggested by us, which will be used in further chapters are characterized by the following implication operations [4,5]:

8) ALI1 – logic

$$a \underset{ALI1}{\rightarrow} b = \begin{cases} 1-a, & if \quad a < b, \\ 1, & if \quad a = b, \\ b, & if \quad a > b \end{cases}$$

9) ALI2 - logic

$$a \underset{ALI2}{\rightarrow} b = \begin{cases} 1, & if \quad a \le b, \\ (1-a) \wedge b, & if \quad a > b \end{cases}$$

10) ALI3 - logic

$$a \underset{ALI3}{\rightarrow} b = \begin{cases} 1, & if \quad a \le b, \\ b/[a + (1-b)], & otherwise. \end{cases}$$

11) ALI4 – logic

$$a \xrightarrow[ALI4]{} b = \begin{cases} \dfrac{1 - \mathrm{a} + b}{2}, & \mathrm{a} > \mathrm{b}, \\ 1, & \mathrm{a} \le \mathrm{b}. \end{cases}$$

A necessary observation to be made in the context of this discussion is that with the only few exceptions for S *-logic* (3) and G *-logic* (4), and ALI1-ALI4 (8)-(11), all other known fuzzy logics (1)-(2), (5)-(7) do not satisfy either the classical "modus-ponens" principle, or other criteria which appeal to the human perception of mechanisms of a decision making process being formulated in [74]. The proposed fuzzy logics ALI1-ALI4 come with an implication operators, which satisfy the classical principle of "modus-ponens" and meets some additional criteria being in line with human intuition.

The comparative analysis of the first seven logics has been given in [19]. The analysis of these seven logics has shown that only S - and G - logics satisfy the classical principle of Modus Ponens and allow development of improved rule of fuzzy conditional inference. At the same time the value of truthness of the implication operation in G -logic is equal either to 0 or 1; and only the value of truthness of logical conclusion is used in the definition of the implication operation in S-logic. Thus the degree of "fuzziness" of implication is decreased, which is a considerable disadvantage and restricts the use of these logics in approximate reasoning.

Definition 1.28. Top of a Fuzzy Set. The top of fuzzy set $\tilde{B}$ is

$$H\tilde{B} = \vee_U \mu_{\tilde{B}}(x).$$

Definition 1.29. Bottom of a Fuzzy Set. The bottom of fuzzy set $\tilde{B}$ is

$$p\tilde{B} = \wedge_U \mu_{\tilde{B}}(x).$$

Definition 1.30. Nonfuzziness. Nonfuzziness $a \in U$ is $ka = a \vee (1-a)$. Then nonfuzziness of fuzzy set $\tilde{B}$ is defined as:

$$k\tilde{B} = \wedge_U k\mu_{\tilde{B}}(x)$$

Let us give a brief semantic analysis of the proposed fuzzy logics ALI1-ALI3 by using the terminology accepted in the theory of power fuzzy sets. For this purpose we formulate the following.

Proposal. Possibility degree of the inclusion of set $\pi(\tilde{A} \subseteq \tilde{B})$ in fuzzy logic ALI1-ALI3 is determined as:

$$\pi_1\left(\tilde{A} \subseteq \tilde{B}\right) = \begin{cases} 1-\mu_{\tilde{A}}(x), & if \quad \mu_{\tilde{A}}(x) < \mu_{\tilde{B}}(x), \\ 1, & if \quad \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x), \\ \mu_{\tilde{B}}(x), & if \quad \mu_{\tilde{A}}(x) > \mu_{\tilde{B}}(x); \end{cases}$$

$$\pi_2\left(\tilde{A} \subseteq \tilde{B}\right) = \begin{cases} 1, & if \quad \mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x), \\ \left(1-\mu_{\tilde{A}}(x)\right) \wedge \mu_{\tilde{B}}(x), & if \quad \mu_{\tilde{A}}(x) > \mu_{\tilde{B}}(x); \end{cases}$$

$$\pi_3\left(\tilde{A} \subseteq \tilde{B}\right) = \begin{cases} 1, & if \quad \mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x), \\ \dfrac{\mu_{\tilde{B}}(x)}{\mu_{\tilde{A}}(x) + \left(1-\mu_{\tilde{B}}(x)\right)}, & if \quad \mu_{\tilde{A}}(x) > \mu_{\tilde{B}}(x). \end{cases}$$

We note, that if $\mu_{\tilde{A}}(x) = 0$ or $\tilde{A} \ne \varnothing$, then the crisp inclusion is possible for fuzzy logic ALI1. Below we consider the equivalence of fuzzy sets.

Proposal. Possibility degree of the equivalence of the sets $\pi(\tilde{A} \equiv \tilde{B})$ is determined as:

$$\pi_1\left(\tilde{A} \equiv \tilde{B}\right) = \begin{cases} 1 - \left[\left(1 - \mu_{\tilde{A}}(x)\right) \vee \mu_{\tilde{B}}(x)\right], & if \quad \mu_{\tilde{A}}(x) < \mu_{\tilde{B}}(x), \\ 1, & if \quad \tilde{A} = \tilde{B}, \\ 1 - \left[\left(1 - \mu_{\tilde{B}}(x)\right) \vee \mu_{\tilde{A}}(x)\right], & if \quad \mu_{\tilde{A}}(x) > \mu_{\tilde{B}}(x), \end{cases}$$

$$\pi_2\left(\tilde{A} \equiv \tilde{B}\right) = \begin{cases} 1, & if \quad \tilde{A} = \tilde{B}, \\ \wedge_{T}\left\{\left[\left(1 - \mu_{\tilde{A}}(x)\right) \wedge \mu_{\tilde{B}}(x)\right], \left[\left(1 - \mu_{\tilde{B}}(x)\right) \wedge \mu_{\tilde{A}}(x)\right]\right\} & if \quad \tilde{A} \neq \tilde{B}, \\ 0, \quad if\ \exists x \,|||\, \mu_{\tilde{A}}(x) = 0, \mu_{\tilde{B}}(x) \neq 0 & (or\ vice\ versa), \\ and\ \ also\ \ \exists x \,|||\, \mu_{\tilde{A}}(x) = 1,\ \mu_{\tilde{B}}(x) \neq 1 & (or\ vice\ versa), \end{cases}$$

$$\pi_3\left(\tilde{A} \subseteq \tilde{B}\right) = \begin{cases} 1, & if \quad \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \\ \dfrac{\mu_{\tilde{B}}(x)}{\mu_{\tilde{A}}(x) + \left(1 - \mu_{\tilde{B}}(x)\right)} & ,if \quad \mu_{\tilde{A}}(x) > \mu_{\tilde{B}}(x). \end{cases}$$

Here the set $T = \{x \in U \,|\, \mu_{\tilde{A}} x \neq \mu_{\tilde{B}} x\}$ and $\tilde{A} = \tilde{B}$ means that $\forall x$

$\mu_{\tilde{A}}\left(x\right) = \mu_{\tilde{B}}\left(x\right)$ or in other words, $T = \varnothing$.

The symbol $|||$ means "such as ". From the expression $\pi_i\left(\tilde{A} \equiv \tilde{B}\right)$, $i = \overline{1,3}$, it follows that for ALI1 fuzzy logic the equivalency $\pi_1\left(\tilde{A} \equiv \tilde{B}\right) = 1$ takes place only when $\tilde{A} = \tilde{B}$. It is obvious that the equivalence possibility is equal to 0 only in those cases when one of the statements is crisp, i.e. either true or false, while the other is fuzzy.

Proposal. Degree to which fuzzy set $\tilde{B}$ is empty $\pi(\tilde{B} \equiv \varnothing)$ is determined as

$$\pi_1\left(\tilde{B} \equiv \varnothing\right) = \begin{cases} 1, & if \quad \tilde{B} = \varnothing, \\ 0, & otherwise; \end{cases}$$

$$\pi_2\left(\tilde{B} \equiv \varnothing\right) = \begin{cases} 1, & if \quad H\tilde{B} < 1 \ \ or \ \ \tilde{B} = \varnothing, \\ 0, & otherwise; \end{cases}$$

$$\pi_3\left(\tilde{B} \equiv \varnothing\right) = \begin{cases} 1, & if \quad \tilde{B} = \varnothing, \\ 0, & otherwise. \end{cases}$$

Here $\tilde{B}=\varnothing$ means that for $\forall x\, \mu_{\tilde{B}}(x)=0$, or equivalently $H\tilde{B}=0$.

We introduce the concept of disjointness of fuzzy sets. There are two kinds of the disjointness. For a set $\tilde{A}$ the first kind is defined by degree to which set $\tilde{A}$ is a subset of the complement of $\tilde{B}^c$. The second kind is the degree to which the intersection of sets is empty. Therefore, we formulate the following.

Proposal. Degree of disjointness of sets $\tilde{A}$ and $\tilde{B}$ is degree to which $\tilde{A}$ and $\tilde{B}$ are disjoint

$$\pi\left(\tilde{A}\ \ disj_1\ \ \tilde{B}\right)=\pi\left(\tilde{A}\subseteq\tilde{B}^C\right)\wedge\pi\left(\tilde{B}\subseteq\tilde{A}^C\right),$$

$$\pi\left(\tilde{A}\ \ disj_2\ \ \tilde{B}\right)=\pi\left(\left(\tilde{A}\cap\tilde{B}\right)=\varnothing\right)\ .$$

Proposal. Disjointness grade of sets $\tilde{A}$ and $\tilde{B}$ is determined as

$$\pi_1\left(\tilde{A}\ \ disj_1\ \ \tilde{B}\right)=\begin{cases}1, & if\quad \exists x\,|||\,\mu_{\tilde{A}}(x)=1-\mu_{\tilde{B}}(x),\\ \left(1-\mu_{\tilde{A}}(x)\right)\wedge\left(1-\mu_{\tilde{B}}(x)\right), & otherwise,\\ 0, & never;\end{cases}$$

$$\pi_2\left(\tilde{A}\ disj_1\ \tilde{B}\right)=\begin{cases}1, & if\ \ \mu_{\tilde{A}}(x)\le 1-\mu_{\tilde{B}}(x),\\ 0, & if\ \ \exists x\,|||\,\mu_{\tilde{A}}(x)=1,\quad but\quad \mu_{\tilde{B}}(x)\ne 0,\\ & or\ \ \mu_{\tilde{B}}(x)=1,\ but\ \ \mu_{\tilde{A}}(x)\ne 0,\\ \underset{T}{\wedge}\left[\left(1-\mu_{\tilde{A}}(x)\right),\left(1-\mu_{\tilde{B}}(x)\right)\right], & otherwise;\end{cases}$$

$$\pi_3\left(\tilde{A}\ disj_1\ \tilde{B}\right)=\begin{cases}1, & if\quad \mu_{\tilde{A}}(x)=\mu_{\tilde{B}}(x)\quad or\ \ \mu_{\tilde{B}}(x)=0,\\ \underset{T}{\wedge}\left[\dfrac{1-\mu_{\tilde{B}}(x)}{\mu_{\tilde{A}}(x)+\left(1-\mu_{\tilde{B}}(x)\right)},\dfrac{1-\mu_{\tilde{A}}(x)}{\mu_{\tilde{B}}(x)+\left(1-\mu_{\tilde{A}}(x)\right)}\right], & otherwise,\\ 0, & never.\end{cases}$$

here $T=\{x\,|||\,\mu_{\tilde{A}}(x)>1-\mu_{\tilde{B}}(x)\}$.

We note that, the disjointness degree of the set is equal to 0 only for fuzzy logic ALI2, when under the condition that one of the considered fuzzy sets is normal, the other is subnormal.

Proposal. Degree to which set is a subset of its complement for the considered fuzzy logics $\pi_i(\tilde{A}\subseteq\tilde{B}^C)$ takes the following form

$$\pi_1(\tilde{A} \subseteq \tilde{A}^C) = \begin{cases} 1, & if\ H\tilde{A} = 0, \\ 0, & if\ H\tilde{A} = 1, \\ 1 - H\tilde{A}, & otherwise; \end{cases}$$

$$\pi_2(\tilde{A} \subseteq \tilde{A}^C) = \begin{cases} 1, & if\ H\tilde{A} \le 0, \\ 0, & if\ H\tilde{A} = 1, \\ 1 - H\tilde{A}, & otherwise; \end{cases}$$

$$\pi_3(\tilde{A} \subseteq \tilde{A}^C) = \begin{cases} 1, & if\ H\tilde{A} \le 0.5, \\ 0, & if\ H\tilde{A} = 1, \\ (1 - H\tilde{A})/(2H\tilde{A}), & otherwise; \end{cases}$$

It is obvious that for the fuzzy logic ALI1 the degree to which a set is the subset of its complement is equal to the degree to which this set is empty. It should also be mentioned that the semantic analysis given in [6,8,9] as well as the analysis given above show a significant analogy between features of fuzzy logics ALI1 and KD. However, the fuzzy logic ALI1, unlike the KD logic, has a number of advantages. For example, ALI1 logic satisfies the condition $\mu_{\tilde{A}}x \wedge (\mu_{\tilde{A}}x \to \mu_{\tilde{B}}x) \le \mu_{\tilde{B}}x$ necessary for development of fuzzy conditional inference rules. ALI2 and ALI3 logics satisfy this inequality as well. This allows them to be used for the formalization of improved rules of fuzzy conditional inference and for the modeling of relations between main elements of a decision problem under uncertainty and interaction among behavioral factors.

Extended Fuzzy Logic [127]

Fuzzy logic adds to bivalent logic an important capability—a capability to reason precisely with imperfect information. In classical fuzzy logic, results of reasoning are expected to be provably valid, or p-valid for short. Extended fuzzy logic adds an equally important capability—a capability to reason imprecisely with imperfect information. This capability comes into play when precise reasoning is infeasible, excessively costly or unneeded. In extended fuzzy logic, p-validity of results is desirable but not required. What is admissible is a mode of reasoning which is fuzzily valid, or f-valid for short. Actually, much of everyday human reasoning is f-valid reasoning. What is important to note is that f-valid reasoning based on a realistic model may be more useful than p-valid reasoning based on an unrealistic model. As John Maynard Keynes states, "*It is better to be roughly right than precisely wrong*" In constructing better models of reality, a problem that has to be faced is that as the complexity of a system, increases, it becomes increasingly difficult to construct a model, which is both cointensive, that is, close-fitting, and precise. This applies, in particular, to systems in which human judgment, perceptions and emotions play a prominent role. Economic systems, legal systems and political systems are cases in point. As the complexity of a system increases further, a point is reached at which construction of a model which is both cointensive and precise is not merely difficult—it is impossible. It is at this point that extended fuzzy logic comes into play. Actually, extended fuzzy logic is not the only formalism that comes into play at this point. The issue of what to do when an exact solution cannot be found or is excessively costly is associated with a vast literature. Prominent in this literature are various approximation theories [16], theories centered on bounded rationality [100], qualitative reasoning [106], commonsense reasoning [65,78] and theories of argumentation [101]. Extended fuzzy logic differs from these and related theories both in spirit and in substance. The difference will become apparent in Section 1.3, in which the so-called f-geometry is used as an illustration. To develop an understanding of extended fuzzy logic, it is expedient to start with the following definition of classical fuzzy logic. Classical fuzzy logic is a precise conceptual system of reasoning, deduction and computation in which the objects of discourse and analysis are, or are allowed to be, associated with imperfect information. In fuzzy logic, the results of reasoning, deduction and computation are expected to be provably valid (p-valid) within the conceptual structure of fuzzy logic. In fuzzy logic precision is achieved through association of fuzzy sets with membership functions and, more generally, association of granules with generalized constraints [126]. What this implies is that classical fuzzy logic is what may be called precisiated logic.

At this point, a key idea comes into play. The idea is that of constructing a fuzzy logic, which, in contrast to classical, is unprecisiated. What this means is that in unprecisiated fuzzy logic UFL membership functions and generalized constraints are not specified, and are a matter of perception rather than measurement. A question which arises is: What is the point of constructing UFL - a logic in which provable validity is off the table? But what is not off the table is what may be called fuzzy validity, or f-validity for short. As will be shown in section 1.3 a model of UFL is f-geometry. Actually, everyday human reasoning is preponderantly f-valid reasoning. Humans have a remarkable capability to perform a wide variety of physical and mental tasks without any measurements and any computations. In this context, f-valid reasoning is perception-based. The concept of unprecisiated fuzzy logic provides a basis for the concept of extended fuzzy logic, EFL. More specifically, EFL is the result of adding UFL to classical fuzzy logic. Basically, extended fuzzy logic. Effect, extended fuzzy logic adds to fuzzy logic a capability to deal imprecisely with imperfect information when precision is infeasible, carries a high cost or is unneeded. This capability is a necessity when repeated attempts at constructing a theory which is both realistic and precise fail to achieve success. Cases in point are the theories of rationality, causality and decision-making under second order uncertainty, that is, uncertainty about uncertainty. There is an important point to be made. f-Validity is a fuzzy concept and hence is a matter of degree. When a chain of reasoning leads to a conclusion, a natural question is: What is the possibly fuzzy degree of validity, call it the validity index, of the conclusion? In most applications involving f -valid reasoning a high validity index is a desideratum. How can it be achieved? Achievement of a high validity index is one of the principal objectives of extended fuzzy logic. The importance of extended fuzzy logic derives from the fact that it adds to fuzzy logic an essential capability—the capability to deal with unprecisiated imperfect information.

1.3 Fuzzy Analyses and Fuzzy Geometry

In this section we concern with the necessary concepts related to the calculus of fuzzy set-valued mappings, for short fuzzy functions. Let X be an arbitrary set.A family τ of fuzzy sets in X is called a fuzzy topology for X and the pair (X,τ) a fuzzy topological space if: (i) $\mu_X \in \tau$ and $\mu_\phi \in \tau$; (ii) $\bigcup_{i\in I} \tilde{A}_i \in \tau$ whenever each $\tilde{A}_i \in \tau (i \in I)$; and (iii) $\tilde{A} \cap \tilde{B} \in \tau$ whenever $\tilde{A}, \tilde{B} \in \tau$ [25].

Definition 1.31. Fuzzy Function [25]**.** A fuzzy function $\tilde{f}$ from a set X into a set Y assigns to each x in X a fuzzy subset $\tilde{f}(x)$ of Y. We denote it by $\tilde{f}: X \to Y$. We can identify $\tilde{f}$ with a fuzzy subset $G_{\tilde{f}}$ of $X \times Y$ and $\tilde{f}(x)(y) = G_{\tilde{f}}(x, y)$.

If $\tilde{A}$ is a fuzzy subset of X, then the fuzzy set $\tilde{f}(\tilde{A})$ in Y is defined by

$$\tilde{f}(\tilde{A})(y) = \sup_{x\in X}[G_{\tilde{f}}(x, y) \wedge \tilde{A}(x)]$$

The graph $\tilde{G}_{\tilde{f}}$ of $\tilde{f}$ is the fuzzy subset of X×Y associated with $\tilde{f}$,

$$\tilde{G}_{\tilde{f}} = \left\{(x, y) \in X \times Y : [\tilde{f}(x)](y) \neq 0 \right\}$$

Let X be a fuzzy topological space. Neighborhood of a fuzzy set $\tilde{A} \subset X$ is any fuzzy set $\tilde{B}$ for which there is an open fuzzy set $\tilde{V}$ satisfying $\tilde{A} \subset \tilde{V} \subset \tilde{B}$. Any open fuzzy set $\tilde{V}$ that satisfies $\tilde{A} \subset \tilde{V}$ is called an open neighborhood of $\tilde{A}$.

A fuzzy function $\tilde{f}: X \to Y$ between two fuzzy topological spaces X and Y is: upper semicontinuous at the point x, if for every open neighborhood U of $\tilde{f}(x), \tilde{f}^u(U)$ is a neighborhood of x in X; lower semicontinuous at x, if for every open fuzzy set $\tilde{V}$ which intersects $\tilde{f}(x)$, $f^l(U)$ is a neighborhood of x; and continuous if it is both upper and lower semicontinuous.

Let $\mathcal{E}^n$ [34,62] be a space of all fuzzy subsets of $\mathcal{R}^n$. These subsets satisfy the conditions of normality, convexity, and are upper semicontinuous with compact support.

Definition 1.32. Fuzzy Closeness [25]**.** A function $\tilde{f}: X \to Y$ between two fuzzy topological spaces is fuzzy closed or has fuzzy closed graph if its graph is a closed fuzzy subset of $X \times Y$

Definition 1.33. Composition [25]**.** Let $\tilde{f}: X \to Y$ and $\tilde{g}: Y \to Z$ be two fuzzy functions. The composition $\tilde{g} \circ \tilde{f}: X \to Z$ $\tilde{f}: X \to Z$ is defined by $(g \circ f)(x) = \ = \cup\{g(y) : [f(x)](y) \neq 0\}$.

Theorem 1.2. Convex Hull of a Fuzzy Set [25]**.** *Let X, Y and Z be three fuzzy topological spaces. Let $\tilde{f}: X \to Y$ and $\tilde{g}: Y \to Z$ be two fuzzy functions. Then*

$$(i)\ (\tilde{g}_0\tilde{f})^u(\tilde{A}) = \tilde{f}^u(\tilde{g}^u(\tilde{A}))$$

and

$$(ii)\ (\tilde{g}_0\tilde{f})^l(\tilde{A}) = \tilde{f}^l(\tilde{g}^l(\tilde{A}))$$

where $\tilde{A}$ is an open fuzzy subset of Z.
A fuzzy set $\tilde{A}$ in $\tilde{E}$ is called convex if for each $t \in [0,1], [tA + (1-t)A](x) \leq A(x)$. The convex hull of a fuzzy set $\tilde{B}$ is smallest convex fuzzy set containing $\tilde{B}$ and is denoted by $\tilde{c}_0(\tilde{B})$.

Definition 1.34. Fuzzy Topological Vector Space [25]**.** A fuzzy linear topology on a vector space E over K is a fuzzy topology τ on E such that the two mappings:

$$f: E \times E \to E,\ f(x, y) = x + y,$$

$$h: K \times E \to E,\ h(t, x) = tx,$$

are continuous when K has the usual fuzzy topology and $K \times E$, $E \times E$ the corresponding product fuzzy topologies. A linear space with a fuzzy linear topology is called a fuzzy topological vector space. A fuzzy topological vector space E is called locally convex if it has a base at origin of convex fuzzy sets.

Definition 1.35. Fuzzy Multivalued Functions [25]**.** If $\tilde{f}, \tilde{g}: X \to Y$ are two fuzzy multivalued functions, where Y is a vector space, then we define:

(1) The sum fuzzy multivalued function $\tilde{f} + \tilde{g}$ by

$$(\tilde{f} + \tilde{g})(x) = \tilde{f}(x) + \tilde{g}(x) = \{y + z : y \in \tilde{f}(x)\, and\ z \in \tilde{g}(x)\}$$

(2) The convex hull of a fuzzy multivalued function $\tilde{c}_0(\tilde{f})$ of $\tilde{f}$ by

$$(\tilde{c}_0(\tilde{f}))(x) = \tilde{c}_0(\tilde{f}(x)).$$

(3) If Y is a fuzzy topological vector space, the closed convex hull of a fuzzy multivalued function $cl(\tilde{c}_0(\tilde{f}))$ of $\tilde{f}$ by

$$(cl(\tilde{c}_0(\tilde{f})))(x) = cl(\tilde{c}_0(\tilde{f}(x)))$$

Below a definition of measurability of fuzzy mapping $\tilde{F}: T \to \mathcal{E}^n$ is given.

Definition 1.36. Measurability of Fuzzy Mapping [34,62]**.** We say that a mapping $\tilde{F}: T \to \mathcal{E}^n$ is strongly measurable if for all $\alpha \in [0,1]$ the set-valued mapping $F_\alpha : T \to P_K(\mathcal{R}^n)$ defined by

$$F_\alpha(t) = [F(t)]^\alpha$$

is (Lebesgue) measurable , when $P_K(\mathcal{R}^n)$ is endowed with the topology generated by the Hausdorff metric d_H .

If $\tilde{F}: T \to \mathcal{E}^n$ is continuous with respect to the metric d_H then it is strongly measurable [34,62].

A mapping $\tilde{F}: T \to \mathcal{E}^n$ is called integrably bounded if there exists an integrable function h such that $\| x \| \le h(t)$ for all $x \in \tilde{F}_0(t)$.

Definition 1.37. Integrability of Fuzzy Mapping [34,62]**.** Let $\tilde{F}: T \to \mathcal{E}^n$. The integral of $\tilde{F}$ over T , denoted $\int_T \tilde{F}(t)dt$ or $\int_a^b \tilde{F}(t)dt$, is defined levelwise by the equation

$$[\int_T \tilde{F}(t)dt] = \int_T F_\alpha(t)dt = \{\int_T f(t)dt \mid f: T \to \mathcal{R}^n \text{ is a measurable selection for } F_\alpha\}$$

for all $0 < \alpha \le 1$. A strongly measurable and interably bounded mapping $\tilde{F}: T \to \mathcal{E}^n$ is said to be integrable over T if $\int_T \tilde{F}(t)dt \in \mathcal{E}^n$.

Hausdorff Distance [34,62]**.** Let $P_K(R^n)$ denote the family of all nonempty compact convex subsets of R^n and define the addition and scalar multiplication in $P_K(R^n)$ as usual. Let C and D be two nonempty bounded subsets of R^n . The distance between C and D is defined by using the Hausdorff metric

$$d_H(C,D) = \left\{ \max(\sup_{c \in C} \inf_{d \in D} \|c-d\|, \sup_{d \in D} \inf_{c \in C} \|c-d\|) \right\} \tag{1.53}$$

where $\|\cdot\|$ denotes the usual Euclidean norm in R^n. Then it becomes clear that $(P_K(R^n), d_H)$ becomes a metric space.

The next necessary concept that will be used in the sequel is the concept of difference of two elements of $\mathcal{E}^n$ referred to as Hukuhara difference:

Definition 1.38. Hukuhara Difference [34,62]**.** Let $\tilde{X}, \tilde{Y} \in \mathcal{E}^n$. If there exists $\tilde{Z} \in \mathcal{E}^n$ such that $\tilde{X} = \tilde{Y} + \tilde{Z}$, then $\tilde{Z}$ is called a Hukuhara difference of $\tilde{X}$ and $\tilde{Y}$ and is denoted as $\tilde{X} -_h \tilde{Y}$.

Note that with the standard fuzzy difference for $\tilde{Z}$ produced of $\tilde{X}$ and $\tilde{Y}$, $\tilde{X} \neq \tilde{Y} + \tilde{Z}$. We use Hukuhara difference when we need $\tilde{X} = \tilde{Y} + \tilde{Z}$.

Let us consider and example. Let $\tilde{X}$ and $\tilde{Y}$ be triangular fuzzy sets $\tilde{X} = (3,7,11)$ and $\tilde{Y} = (1,2,3)$. Then Hukuhara difference of $\tilde{X}$ and $\tilde{Y}$ is $\tilde{X} -_h \tilde{Y} \quad = (3,7,11) -_h (1,2,3) = (3-1, 7-2, 11-3) = (2,5,8)$ Indeed, $\tilde{Y} + \left(\tilde{X} -_h \tilde{Y}\right) = (1,2,3) + (2,5,8) = (3,7,11) = \tilde{X}$.

Definition 1.39. Fuzzy Hausdorff Distance [10,11]**.** Let $\tilde{A}, \tilde{B} \in \mathcal{E}^n$. The fuzzy Hausdorff distance $\tilde{d}_{fH}$ between $\tilde{A}$ and $\tilde{B}$ is defined as

$$\tilde{d}_{fH}(\tilde{A}, \tilde{B}) = \bigcup_{\alpha \in [0,1]} \alpha \left[d_H(A^1, B^1), \sup_{\alpha \le \bar{\alpha} \le 1} d_H(A^{\bar{\alpha}}, B^{\bar{\alpha}}) \right], \tag{1.54}$$

where d_H is the Hausdorff distance [34,62] and A^1, B^1 denote the cores ($\alpha = 1$ level sets) of fuzzy sets $\tilde{A}, \tilde{B}$ respectively. Let us denote by $\left\|\tilde{A} -_h \tilde{B}\right\| = d_{fH}(\tilde{A} -_h \tilde{B}, \hat{0})$ a fuzzy norm of the Hukuhara difference. We note that $d_{fH}(\tilde{A} -_h \tilde{B}, \hat{0}) = d_{fH}(\tilde{A}, \tilde{B})$. We will be using this difference in further considerations.

Let us consider a small example. Let $\tilde{A}$ and $\tilde{B}$ be triangular fuzzy sets $\tilde{A} = (2,3,4)$ and $\tilde{B} = (6,8,12)$. Then the fuzzy Hausdorff distance d_{fH} between $\tilde{A}$ and $\tilde{B}$ is defined as a triangular fuzzy set $\tilde{d}_{fH}(\tilde{A}, \tilde{B}) = (5,5,8)$.

Fuzzy Norms. Let $\tilde{x}, \tilde{y} \in E^n$. We denote by $\left\|\tilde{x} -_h \tilde{y}\right\|_{fH}$ a fuzzy norm defined as

$$\left\|\tilde{x} -_h \tilde{y}\right\|_{fH} = d_{fH}(\tilde{x}, \tilde{y}). \tag{1.55}$$

It is the fuzzy Haussdorf distance mentioned above.

Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_n) \in E^n$. We denote by $\|\tilde{u}\|_f$ a fuzzy norm defined by the formula

$$\|\tilde{u}\|_f = |\tilde{u}_1| + |\tilde{u}_2| + ... + |\tilde{u}_n|. \quad (1.56)$$

where $|.|$ is the absolute value of a fuzzy number [3,7].

Derivatives of Fuzzy Functions and Fuzzy Derivatives [46,52]. It is necessary to distinguish between the following cases:

–we are given a fuzzy function and our interest is to determine its derivative at a particular point a (see Fig 1.3 (a));

–we have a function but the information about the point $\tilde{a}$ at which we are to consider the derivative is vague (uncertain) (see Fig 1.3 (b));

–we have a fuzzy function and we are interested in its derivative at a vague point $\tilde{a}$ (see Fig 1.3 (c)).

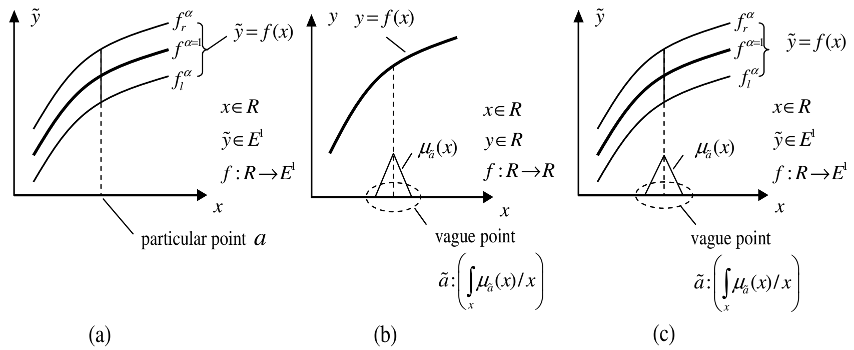


Fig. 1.3 Derivatives of fuzzy functions and fuzzy derivatives

In this paper, we analyze the situations in which the points are not exactly known, and therefore they need to be substituted by subjective and vague estimates, viz. could be treated as fuzzy sets (numbers) defined over some interval.

Strongly Generalized Differentiability [24]. Let $f : (a, b) \to E^n$ and $t_0 \in (a,b)$. We say that f is strongly generalized differentiable at t_0 if there exists an element $f'(t_0) \in E^n$, such that

a) for all $h > 0$ sufficiently small, $\exists\ f(t_0 + h) -_h f(t_0)$, $f(t_0) -_h f(t_0 - h)$ (i.e. the length of $diam\left((f(t))^\alpha\right)$ increases) and the limits (in the supremum metric [34])

$$\lim_{h \to 0^+} \frac{f(t_0 + h) -_h f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) -_h f(t_0 - h)}{h} = f'(t_0),$$

or

b) for all $h > 0$ sufficiently small, $\exists\ f(t_0) -_h f(t_0 + h)$, $f(t_0 - h) -_h f(t_0)$ (i.e. the length of $diam\left((f(t))^\alpha\right)$ decreases) and the limits (in the supremum metric [34]) interval

$$\lim_{h \to 0^+} \frac{f(t_0) -_h f(t_0 + h)}{(-h)} = \lim_{h \to 0^+} \frac{f(t_0 - h) -_h f(t_0)}{(-h)} = f'(t_0),$$

(h and $(-h)$ shown in the denominators mean $1/h$ and $1/(-h)$ respectively).

Let $f : (a,b) \to E^1$ be a differentiable function. We introduce the notation $f^\alpha(t) = \left[f_l^\alpha(t), f_r^\alpha(t)\right]$. Then $f_l^\alpha(t)$ and $f_r^\alpha(t)$ are differentiable and $\left(f'(t)\right)^\alpha = \left[\min\left(\left(f_l^\alpha(t)\right)', \left(f_r^\alpha(t)\right)'\right), \max\left(\left(f_l^\alpha(t)\right)', \left(f_r^\alpha(t)\right)'\right)\right]$.

If f is continuous then $g(t) = \int_a^t f(\tau)d\tau$ is differentiable on (a,b) and $g'(t) = f(t)$, $\forall t \in (a,b)$. Moreover, if f is differentiable on (a,b) and $f'(\cdot)$ is integrable on (a,b) then for all $t \in (a,b)$ we have $f(t) = f(t_0) + \int_{t_0}^t f'(\tau)d\tau$, $a < t_0 \le t < b$.

Possibility Measure [3,110,121]. Given two fuzzy sets defined in the same universe of discourse $\mathbf{X}$, a fundamental question arises as to their similarity or proximity. There are several well-documented approaches covered in the literature. One of them concerns a *possibility measure*. The possibility measure, denoted by $Poss(\tilde{A}, \tilde{X})$ describes a level of overlap between two fuzzy sets and is expressed in the form

$$Poss(\tilde{A}, \tilde{X}) = \sup_{x \in X} \left[\tilde{A}(x) t \tilde{X}(x) \right],$$

where t is a t-norm. Computationally, we note that the possibility measure is concerned with the determination of the intersection between $\tilde{A}$ and $\tilde{X}$, $\tilde{A}(x)t\tilde{X}(x)$, that is followed by the optimistic assessment of this intersection. It is done by picking up the highest values among the intersection grades of $\tilde{A}$ and $\tilde{X}$ that are taken over all elements of the universe of discourse $\mathbf{X}$. For example, let $\tilde{a}$ and $\tilde{b}$ be fuzzy sets with trapezoidal membership functions:

$$\mu_{\tilde{a}}(x) = \begin{cases} 1 - \dfrac{a_1 - x}{\alpha_l}, & if\ a_1 - \alpha_l \le x \le a_1 \\ 1, & if\ a_1 \le x \le a_2 \\ 1 - \dfrac{x - a_2}{\alpha_r}, & if\ a_2 - \alpha_r \le x \le a_2 \\ 0, & otherwise \end{cases} \qquad \mu_{\tilde{b}}(x) = \begin{cases} 1 - \dfrac{b_1 - x}{\beta_l}, & if\ b_1 - \beta_l \le x \le b_1 \\ 1, & if\ b_1 \le x \le b_2 \\ 1 - \dfrac{x - b_2}{\beta_r}, & if\ b_2 - \beta_r \le x \le b_r \\ 0, & otherwise \end{cases}$$

The graphs of the corresponding membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ are shown in Fig.1.4.

Then the possibility measure of the proposition "$\tilde{a}$ is equal to $\tilde{b}$" is defined as follows:

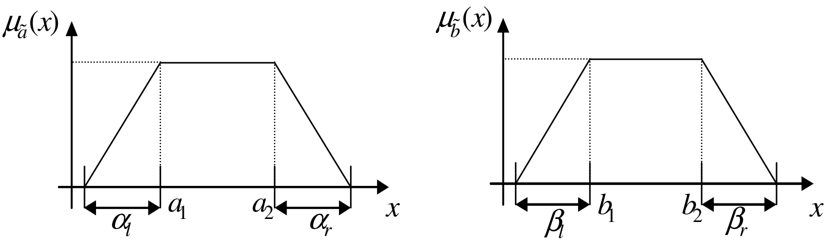


Fig. 1.4 Trapezoidal fuzzy numbers $\tilde{a}$ and $\tilde{b}$

$$(\tilde{a} = \tilde{b}) = Poss(\tilde{a}/\tilde{b}) = \max\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(x)) = \begin{cases} 1 - \dfrac{a_1 - b_2}{\alpha_l + \beta_r}, & if\ 0 < a_1 - b_2 < \alpha_l + \beta_r \\ 1, & if\ \max(a_1, b_1) \le \min(a_2, b_2) \\ 1 - \dfrac{b_1 - a_2}{\alpha_r + \beta_l}, & if\ 0 < b_1 - a_2 < \alpha_r + \beta_l \\ 0, & otherwise \end{cases} \quad (1.57)$$

Fuzzy Geometry

In general, fuzzy geometry may be considered as extension of conventional geometry to the fuzzy case [29,77,90-94,96,107]. Fuzzy geometry includes the topological concepts of area, perimeter, compactness, length, adjacency etc. These measures can be used to reflect the ambiguity in decision relevant information.

Definition 1.40. Fuzzy Point. Fuzzy point $\tilde{x}_0$ is a convex fuzzy subset of R^i. Fuzzy point in R is characterized by kernel x_0 whose precise location is only approximately known.

A crisp point $x_0 \in R^i$ is the kernel, from which membership function decreases in all directions monotonically [17]. In Fig. 1.5 and Fig. 1.6 fuzzy points with hyperpyramidal (Fig.1.5) and hyperparaboloidal (Fig. 1.6) membership functions are shown. In first case imprecision of location of fuzzy point is expressed by intervals for the components, in second case by definite matrix in all directions of the space.

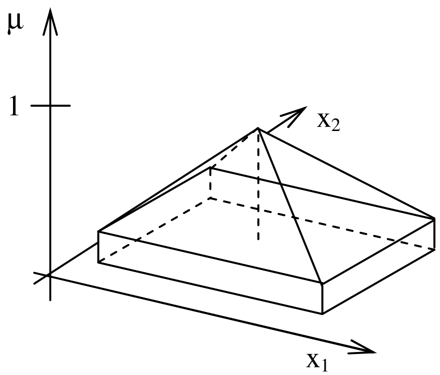


Fig. 1.5 Fuzzy points with hyperpyramidal membership

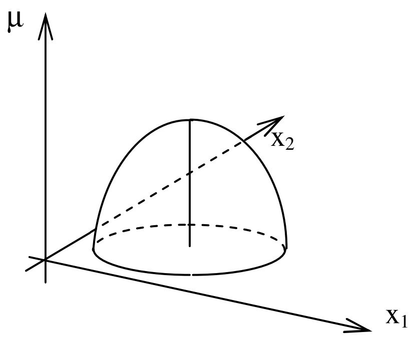


Fig. 1.6 Fuzzy points with hyperparaboloidal membership

Definition 1.41. Fuzzy Interval. If fuzzy domain I of the real line R is bounded by two normalized convex fuzzy sets then it is called fuzzy interval. In Fig. 1.7 fuzzy interval with fuzzy ends $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ is given. A crisp interval [a, b] is the kernel, from which the membership function decreases to zero [17]. Analogously, fuzzy region in R^i is represented as a crisp region, which is surrounded by a fuzzy transition zone, in which the membership function decreases monotonically to zero [18].

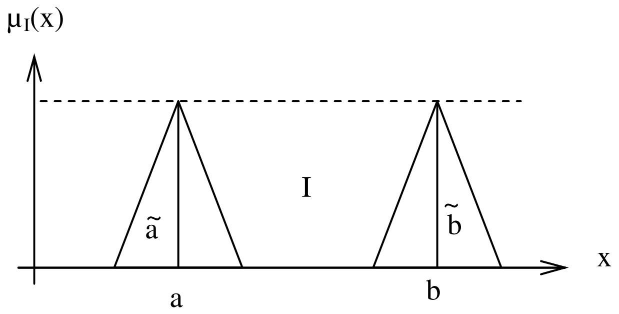


Fig. 1.7 Fuzzy interval

Definition 1.42. Length of a Fuzzy Interval. Length of fuzzy interval I is defined as

$$L(I) = \int_{I_0} \mu_I(x)dx$$

Here $I_0 = cl\{x \mid \mu_I(x) > 0\}$ is support of fuzzy interval.

Definition 1.43. Distance between Fuzzy Points. A distance between two points $d(x_1,x_2)$ by using the extension principle translates to a fuzzy distance between fuzzy sets.

The fuzzy distance between two fuzzy sets $\tilde{A}$ and $\tilde{B}$ on X (X is metric space) is defined as [18,81]

$$\mu_{d(\tilde{A},\tilde{B})}(y) = \sup_{\substack{(x_1,x_2)\in X\times X \\ d(x_1,x_2)=y}} \min(\mu_{\tilde{A}}(x_1),\ \mu_{\tilde{B}}(x_2))$$

Example. Fuzzy distance in case when $X = R^1$, $d(x_1,x_2) = |x_1 - x_2|$ is shown in Fig 1.8.

Some distances frequently used in practical problems are given below (for R^1):

$d(x_1,x_2)= (x_1-x_2)^2$ Euclidean distance

$d(x_1,x_2)= (|x_1-x_2|^p)^{1/p}$ Minkowski metric

$d(x_1,x_2)= c\ |x_1-x_2|$ Tschebyscheff metric

$d(x_1,x_2)= |x_1-x_2|$ Hamming distance

Definition 1.44. Fuzzy Area. The area of fuzzy subset is defined as the area of fuzzy subset $\tilde{A}$ given on R^2 is defined as

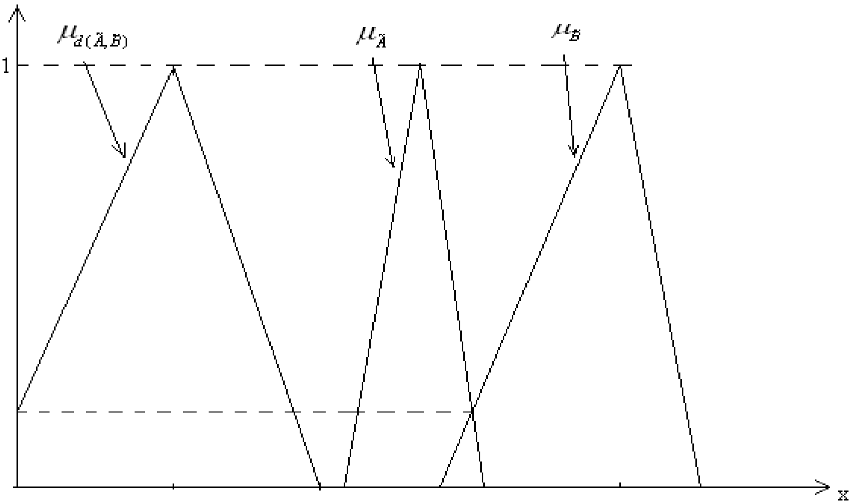


Fig 1.8 Fuzzy distance between fuzzy sets

$$S(\tilde{A}) = \iint_{\tilde{A}_0} \mu_{\tilde{A}}(x,y)dxdy \tag{1.58}$$

Here $\tilde{A}_0 = \{(x,y) \mid m_{\tilde{A}}(x,y) > 0\}$ is support of fuzzy region $\tilde{A}$.

For fuzzy region, represented by piecewise membership function, the area is defined as [96]

$$S(\tilde{A}) = \sum_i \mu(i) \tag{1.59}$$

Definition 1.45. Perimeter. In case of fuzzy set $\tilde{A}$ when $\mu_{\tilde{A}}$ is piecewise constant, the perimeter of fuzzy set $\tilde{A}$ is defined as

$$\mathrm{P}_{\tilde{A}} = \sum_{i,j,k} \left|\mu(i) - \mu(j)\right| * L(i,j,k) \tag{1.60}$$

Here $\mu(i)$ and $\mu(j)$ are the membership values of two adjacent regions, $L(i, j, k)$ is length of a k-th arc of these regions.

Definition 1.46. Compactness. The compactness of a fuzzy set $\tilde{A}$ with area $S_{\tilde{A}}$ and perimeter $P_{\tilde{A}}$ is defined as

$$C\left(\tilde{A}\right) = \frac{S_{\tilde{A}}}{\mathrm{P}_{\tilde{A}}^2} \tag{1.61}$$

Definition 1.47. Length and Breadth of a Fuzzy Set. The length of a fuzzy set $\tilde{A}$ is defined as

$$l\left(\tilde{A}\right) = \max_x \left\{ \int \mu_{\tilde{A}}(x, y)\, dy \right\}, \tag{1.62}$$

where the integral is taken over the region with $\mu_{\tilde{A}}(x, y) > 0$. For discrete case formula (1.62) takes form

$$l\left(\tilde{A}\right) = \max_x \left\{ \sum_y \mu_{\tilde{A}}(x, y) \right\} \tag{1.63}$$

The breadth of a fuzzy set $\tilde{A}$ is defined as

$$b\left(\tilde{A}\right) = \max_y \left\{ \int \mu_{\tilde{A}}(x, y)\, dx \right\} \tag{1.64}$$

or

$$b\left(\tilde{A}\right) = \max_y \left\{ \sum_x \mu_{\tilde{A}}(x, y) \right\} \tag{1.65}$$

Definition 1.48. Index of Area Coverage (IOAC). IOAC of a fuzzy set $\tilde{A}$ is defined as

$$IOAC\left(\tilde{A}\right) = \frac{S_{\tilde{A}}}{l\left(\tilde{A}\right) \cdot b\left(\tilde{A}\right)} \tag{1.66}$$

This index for fuzzy region represents the fraction of the maximum area (covered by the length and breadth of the region) actually covered by the region.

Now let us consider f -Geometry and f-transformation suggested by Zadeh [127].

In the described above geometry the underlying logic is precisiated fuzzy logic. In the world of f -geometry, suggested by Zadeh [127] the underlying logic is unprecisiated fuzzy logic, UFL. This f -Geometry differs both in spirit and in substance from Poston's fuzzy geometry [87], coarse geometry [89], fuzzy geometry of Rosenfeld [94], fuzzy geometry of Buckley and Eslami [29], fuzzy geometry of Mayburov [71], and fuzzy geometry of Tzafestas [102].

The counterpart of a crisp concept in Euclidean geometry is a fuzzy concept in this fuzzy geometry. Fuzzy concept may be obtained by fuzzy transformation (f -transform) of a crisp concept.

For example, the f -transform of a point is an f -point, the f -transform of a line is an f -line, the f-transform of a triangle is an f -triangle, the f -transform of a circle is an f -circle and the f -transform of parallel is f -parallel (Fig. 1.9). In summary, f -geometry may be viewed as the result of application of f -transformation to Euclidean geometry.

Fig. 1.9 Examples of f -transformation

A key idea in f -geometry is the following: if C is p-valid then its f -transform, f - C , is f-valid with a high validity index.

An important f -principle in f -geometry, referred to as the validation principle, is the following. Let p be a p-valid conclusion drawn from a chain of premises $*p_1,...,*p_n$. Then, using the star notation, $*p$ is an f -valid conclusion drawn from $*p_1,...,*p_n$, and $*p$ has a high validity index. It is this principle that is employed to derive f -valid conclusions from a collection of f-premises.

A basic problem which arises in computation of f -transforms is the following. Let g be a function, a functional or an operator. Using the star notation, let an f-transform , $*C$, be an argument of g . The problem is that of computing $g(*C)$. Generally, computing $g(*C)$ is not a trivial problem.

An f -valid approximation to $g(*C)$ may be derived through application of an f -principle which is referred to as precisiation / imprecisiation principle or P/I principle, for short [123]. More specifically, the principle may be expressed as

$$g(*C)* = *g(C)$$

where*=should be read as approximately equal. In words, $g(*C)$ is approximately equal to the f -transform of $g(C)$.

1.4 Approximate Reasoning

In our daily life we often make inferences where *antecedents* and *consequents* a represented by fuzzy sets. Such inferences cannot be realized adequately by the methods, which are based either on two-valued logic or many-valued logic. In order to facilitate such an inference, Zadeh [114,118,119,122,123,125] suggested an inference rule called a "compositional rule of inference". Using this inference rule, Zadeh, Mamdani [68], Mizumoto et al [38,74,75], R.Aliev and A.Tserkovny [7,9,12,13] suggested several methods for fuzzy reasoning in which the antecedent contain a conditional proposition involving fuzzy concepts:

$$\begin{array}{l} \text{Ant 1: If } x \text{ is } \tilde{P} \text{ then } y \text{ is } \tilde{Q} \\ \text{Ant2: } x \text{ is } \tilde{P}' \\ \hline \text{Cons: } y \text{ is } \tilde{Q}' . \end{array} \tag{1.67}$$

Those methods are based on implication operators present in various fuzzy logics. This matter has been under a thorough discussion for the last couple decades. Some comparative analysis of such methods was presented in [20-23,38,40,47,50,51,53,69,70,74,75,98,111,112]. A number of authors proposed to use a certain suite of fuzzy implications to form fuzzy conditional inference rules [7,9,38,39,59,68,74,75]. The implication operators present in the theory of fuzzy sets were investigated in [7,9,14,26-28,30-33,35,36,41,42,45,48,55,60,61,63,66, 67,69,72,73,76,79,80,82,84,86,99,103,104,108,109,112,129,131,132].On the other hand, statistical features of fuzzy implication operators were studied in [83,105] In turn, the properties of stability and continuity of fuzzy conditional inference rules were investigated in [37,39,49,56]. We will begin with a *formation* of a fuzzy logic regarded as an *algebraic system closed under all its operations*. In the sequel an investigation of statistical characteristics of the proposed fuzzy logic will be presented. Special attention will be paid to building a set of fuzzy conditional inference rules on the basis of the fuzzy logic proposed in this study. Next, continuity and stability features of the formalized rules will be investigated. Lately in fuzzy sets research the great attention is paid to the development of Fuzzy Conditional Inference Rules (CIR) [1,5,36,56,64,72,80,95]. This is connected with the feature of the natural language to contain a certain number of fuzzy concepts (F-concepts), therefore we have to make logical inference in which the preconditions and conclusions contain such F-concepts. The practice shows that there is a huge variety of ways in which the formalization of rules for such kind of inferences can be made. However, such inferences cannot be satisfactorily formalized using the classical Boolean Logic, i.e. here we need to use multi-valued logical systems. The development of the conditional logic rules embraces mainly three types of fuzzy propositions:

$$P_1 = IF\ x\ is\ \tilde{A}\ THEN\ y\ is\ \tilde{B}$$

$$P_2 = IF\ x\ is\ \tilde{A}\ THEN\ y\ is\ \tilde{B}$$

$$OTHERWISE\ \tilde{C}$$

$$P_3 = IF\ x_1\ is\ \tilde{A}_1\ AND\ x_2\ is\ \tilde{A}_2 ... AND ... AND\ x_n\ is\ \tilde{A}_n$$

$$THEN\ y\ is\ \tilde{B}$$

The conceptual principle in the formalization of fuzzy rules is the Modus Ponens inference (separation) rule that states:

$IF\,(\alpha \rightarrow \beta)$ is true and α is true $THEN\ \beta$ is true.

The methodological base for this formalization is the compositional rule suggested by L.Zadeh [114,116]. Using this rule, he formulated some inference rules in which both the logical preconditions and consequences are conditional propositions including F-concepts. Later E.Mamdani [68] suggested inference rule, which like Zadeh's rule was developed for the logical proposition of type P_1. In other words the following type F-conditional inference is considered:

Proposition 1: $IF\ x\ is\ \tilde{A}\ THEN\ y\ is\ \tilde{B}$

Proposition 2: $x\ is\ \tilde{A}'$ (1.68)

Conclusion: y is $\tilde{B}$,

where $\tilde{A}$ and $\tilde{A}'$ are F-concepts represented as F-sets in the universe U; $\tilde{B}$ is F-conceptions or F-set in the universe V. It follows that B' is the consequence represented as a F-set in V. To obtain a logical conclusion based on the CIR, the Propositions 1 and 2 must be transformed accordingly to the form of binary F-relation $\tilde{R}(A_1(x)), A_2(y))$ and unary F-relation $\tilde{R}(A_1(x))$.

Here $A_1(x)$ and $A_2(y)$ are defined by the attributes x and y which take values from the universes U and V, respectively. Then

$$\tilde{R}\ (A_1(x)) = \tilde{A}' \tag{1.69}$$

According to Zadeh-Mamdani's inference rule $\tilde{R}(A_1(x)), A_2(y))$ is defined as follows.

The maximin conditional inference rule

$$\tilde{R}_m(A_1(x),\ A_2(y)) = (\tilde{A}\times\tilde{B})\cup(\neg\tilde{A}\times V) \tag{1.70}$$

The arithmetic conditional inference rule

$$\tilde{R}_a(A_1(x), A_2(y)) = (\neg\tilde{A}\times V)\oplus(U\times\tilde{B}) \tag{1.71}$$

The mini-functional conditional inference rule

$$\tilde{R}_c(A_1(x),\ A_2(y)) = \tilde{A}\times\tilde{B} \tag{1.72}$$

where $\times$, $\cup$ and $\neg$ are the Cartesian product, union, and complement operations, respectively; $\oplus$ is the limited summation.

Thus, in accordance with [68,114,116] the logical consequence $\tilde{R}(A_2(y))$, ($\tilde{B}'$ in (1.72)) can be derived as follows:

$$\tilde{R}(A_2(y)) = \tilde{A}'\circ[(\tilde{A}\times\tilde{B})]\cup[\neg\tilde{A}\times U)]$$

$$\tilde{R}(A_2(y)) = \tilde{A}'\circ[(\neg\tilde{A}\times V)]\oplus[\neg U\times\tilde{B})]$$

or

$$\tilde{R}(A_2(y)) = \tilde{A}'\circ(\tilde{A}\times\tilde{B})$$

where $\circ -$ is the F-set maximin composition operator.

On the base of these rules the conditional inference rules for type P_2 were suggested in [15]:

$$\begin{aligned}&\tilde{R}_4(\tilde{A}_1(x), \tilde{A}_2(y)) =\\ &= [(\tilde{A}\times V)\oplus(U\times\tilde{B})]\cap[(\tilde{A}\times V)\oplus(U\times\tilde{C})]\end{aligned} \tag{1.73}$$

$$\begin{aligned}&\tilde{R}_5(A_1(x), A_2(y)) =\\ &= [(\neg\tilde{A}\times V)\cup(U\times\tilde{B})]\cap[(\tilde{A}\times V)\cup(U\times\tilde{C})]\end{aligned} \tag{1.74}$$

$$\tilde{R}_6(A_1(x),\ A_2(y)) = [(\tilde{A}\times\tilde{B})\cup(\neg\tilde{A}\times\tilde{C})] \tag{1.75}$$

Note that in [15] also the fuzzy conditional inference rules for type P_3 were suggested:

$$\tilde{R}_7(A_1(x),\ A_2(y)) = \left[\bigcap_{i=1,n}(\neg\tilde{A}_i \times V)\right] \oplus [(U \times \tilde{B})] \tag{1.76}$$

$$\tilde{R}_8(A_1(x),\ A_2(y)) = \left[\bigcap_{i=1,n}(\neg\tilde{A}_i \times V)\right] \cup [(U \times \tilde{B})] \tag{1.77}$$

$$\begin{aligned}\tilde{R}_9(A_1(x),\ A_2(y)) &= (\neg\tilde{A} \times V) \oplus (U \times \tilde{B}) = \\ &= \int_{U \times V} 1 \wedge (1 - \mu_{\tilde{A}}(u) + \mu_{\tilde{B}}(v))/(u,v)\end{aligned} \tag{1.78}$$

In order to analyze the effectiveness of rules (1.68)-(1.78) we use some criteria for F-conditional logical inference suggested in [38]. The idea of these criteria is to compare the degree of compatibility of some fuzzy conditional inference rules with the human intuition when making approximate reasoning. These criteria are the following:

Criterion I
Precondition 1: IF x is $\tilde{A}$ THEN y is $\tilde{B}$
Precondition 2: x is $\tilde{A}$

Conclusion: y is $\tilde{B}$

Criterion II-1 Precondition 1: IF x is $\tilde{A}$ THEN y is $\tilde{B}$
Precondition 2: x is very $\tilde{A}$

Conclusion: y is very $\tilde{B}$

Criterion II-2 Precondition 1: IF x is $\tilde{A}$ THEN y is $\tilde{B}$
Precondition 2: x is very $\tilde{A}$

Conclusion: y is B

Criterion III
Precondition 1: IF x is $\tilde{A}$ THEN y is $\tilde{B}$
Precondition 2: x is more or less $\tilde{A}$

Conclusion: y is more or less $\tilde{B}$

Criterion IV-1
Precondition 1: IF x is $\tilde{A}$ THEN y is $\tilde{B}$
Precondition 2: x is not $\tilde{A}$

Conclusion: y is unknown

Criterion IV-2
Precondition 1: IF x is $\tilde{A}$ THEN y is $\tilde{B}$
Precondition 2: x is not $\tilde{A}$

Conclusion: y is not $\tilde{B}$

In [38] it was shown that in Zadeh-Mamdani's rules the relations $\tilde{R}_m$, $\tilde{R}_c$ and $\tilde{R}_c$ do not always satisfy the above criteria. For the case of mini-operational rule $\tilde{R}_c$ it has been found that criteria I and II-2 are satisfied while criteria II-1 and III are not.

In [38] an important generalization was made that allows some improvement to the mentioned F-conditional logical inference rules. It was shown there that for the conditional proposition arithmetical rule defined by Zadeh

$$P_1 = IF\ x\ is\ \tilde{A}\ THEN\ y\ is\ \tilde{B}$$

the following takes place

$$\begin{aligned}
&\tilde{R}_9(A_1(x),\ \ A_2(y)) = (\neg\tilde{A}\times V)\oplus(U\times\tilde{B}) = \\
&= \int_{U\times V} 1\wedge(1-\mu_{\tilde{A}}(u)+\mu_{\tilde{B}}(v))/(u,v)
\end{aligned}$$

The membership function for this F-relation is

$$1\wedge(1-\mu_{\tilde{A}}(u)+\mu_{\tilde{B}}(v))$$

that obviously meets the implication operation or the Ply-operator for the multi-valued logic L (by Lukasiewicz), i.e.

$$T(P \underset{L}{\rightarrow} Q), T(P) \tag{1.79}$$

where $T(P \underset{L}{\rightarrow} Q), T(P)$ and $T(Q)$ - are the truth values for the logical propositions $P \underset{L}{\rightarrow} Q, P$ and Q respectively.

In other words, these expressions can be considered as adaptations of implication in the L-logical system to a conditional proposition.

Having considered this fact, the following expression was derived:

$$\begin{aligned}
&\tilde{R}_a(A_1(x),\ \ A_2(y)) = (\neg\tilde{A}\times V)\oplus(U\times\tilde{B}) = \\
&= \int_{U\times V} 1\wedge(1-\mu_{\tilde{A}}(u)+\mu_{\tilde{B}}(v))/(u,v) = \\
&= \int_{U\in V} (\mu_{\tilde{A}}(u) \underset{L}{\rightarrow} \mu_{\tilde{B}}(v))/(u,v) = (\tilde{A}\times V)\rightarrow(U\times\tilde{B})
\end{aligned} \tag{1.80}$$

In [38] an opinion was expressed that the implication operation or the Ply-operator in the expression (1.80) may belong to any multi-valued logical system.

The following are guidelines for deciding which logical system to select for developing F-conditional logical inference rules [38]. Let F-sets $\tilde{A}$ from U and B from V are given in the form:

$$\tilde{A} = \int_V \mu_{\tilde{A}}(u)/u, \tilde{B} = \int_V \mu_{\tilde{B}}(v)/v$$

Then, as mentioned above, the conditional logical proposition P_1 can be transformed to the F-relation $\tilde{R}(A_1(x), A_2(y))$ by adaptation of the Ply-operator in multi-valued logical system, i.e.

$$\tilde{R}(A_1(x), A_2(y)) = \tilde{A} \times V \to U \times \tilde{B} = \int_{U \times V} (\mu_{\tilde{A}}(u) \to \mu_{\tilde{B}}(v))/(u,v) \tag{1.81}$$

where the values $\mu_{\tilde{A}}(u) \to \mu_{\tilde{B}}(v)$ are depending on the selected logical system.

Assuming $\tilde{R}(A_1(x)) = \tilde{A}$ we can conclude a logical consequence $\tilde{R}(A_2(y))$, then using the CIR for $\tilde{R}(A_1(x))$ and $\tilde{R}(A_1(x), A_2(y))$, then

$$\begin{aligned} &\tilde{R}(A_2(y)) = \tilde{A} \circ \tilde{R}(A_1(x), A_2(y)) = \\ &= \int_U \mu_{\tilde{A}}(u)/u \circ \int_{U \times V} \mu_{\tilde{A}}(u) \to \mu_{\tilde{B}}(v))/(u,v) = \\ &= \int_V \vee_{u \in V} [\mu_{\tilde{A}}(u) \wedge (\mu_{\tilde{A}}(u) \to \mu_{\tilde{B}}(v))] \end{aligned} \tag{1.82}$$

For the criterion I to be satisfied, one of the following equalities must hold true

$$\tilde{R}(A_2(y)) = \tilde{B},$$

$$\vee_{u \in V} [\mu_{\tilde{A}}(u) \wedge (\mu_{\tilde{A}}(u) \to \mu_{\tilde{B}}(v))] = \mu_{\tilde{B}}(v),$$

or

$$[\mu_{\tilde{A}}(u) \wedge (\mu_{\tilde{A}}(u) \to \mu_{\tilde{B}}(v))] \le \mu_{\tilde{B}}(v) \tag{1.83}$$

the latter takes place for any u∈ U and v∈ V or in terms of truth values:

$$T(P \wedge (P \to Q)) \le T(Q) \tag{1.84}$$

The following two conditions are necessary for formalization of F-conditional logical inference rules: the conditional logical inference rules (CIR) must meet the criteria I-IV; the conditional logical inference rules (CIR) satisfy the inequality (1.84). Now we consider formalization of the fuzzy conditional inference for a different type of conditional propositions. As was shown above, the logical inference for conditional propositions of type P_1 is of the following form:

$$\begin{array}{l} \text{Proposition 1: } IF\ x\ is\ \tilde{A}\ THEN\ y\ is\ \tilde{B} \\ \text{Proposition 2: } x\ is\ \tilde{A}' \\ \hline \text{Conclusion: } y\ is\ \tilde{B}' \end{array} \tag{1.85}$$

where $\tilde{A}$, $\tilde{B}$, and $\tilde{A}'$ are F-concepts represented as F-sets in U, V, and V, respectively, which should satisfy the criteria I, II-1, III, and IV-1.

For this inference if the Proposition 2 is transformed to an unary F-relation in the form $\tilde{R}(A_1(x)) = \tilde{A}'$ and the Proposition 1 is transformed to an F-relation $\tilde{R}(A_1(x), \tilde{R}(A_2(y))$ defined below, then the conclusion $\tilde{R}(A_2(y))$ is derived by using the corresponding F-conditional logical inference rule, i.e.

$$\tilde{R}(A_2(y) = \tilde{R}(A_1(x)) \circ \tilde{R}(A_1(x)) \tag{1.86}$$

where $\tilde{R}(A_2(y))$ is equivalent to $\tilde{B}'$ in (1.85).

Fuzzy Conditional Inference Rule 1

Theorem 1.3. *If the F-sets $\tilde{A}$ from U and $\tilde{B}$ from V are given in the traditional form:*

$$\tilde{A} = \int_U \mu_{\tilde{A}}(u)/u, \tilde{B} = \int_V \mu_{\tilde{B}}(v)/v \tag{1.87}$$

and the relation for the multi-valued logical system ALI1

$$\begin{aligned} \tilde{R}_1(A_1(x), A_2(y)) &= \tilde{A} \times V \xrightarrow[ALI1]{} U \times \tilde{B} = \\ &= \int_{U\times V} \mu_{\tilde{A}}(u)/(u,v) \xrightarrow[ALI1]{} \int_{U\times V} \mu_{\tilde{B}}(v)/(u,v) = \\ &= \int_{U\times V} (\mu_{\tilde{A}}(u) \xrightarrow[ALI1]{} \mu_{\tilde{B}}(v))/(u,v) \end{aligned} \tag{1.88}$$

where

$$\mu_{\tilde{A}}(u) \xrightarrow[ALI1]{} \mu_{\tilde{B}}(v) = \begin{cases} 1 - \mu_{\tilde{A}}(u), \mu_{\tilde{A}}(u) < \mu_{\tilde{B}}(v) \\ 1, \qquad \mu_{\tilde{A}}(u) = \mu_{\tilde{B}}(v) \\ \mu_{\tilde{B}}(v), \quad \mu_{\tilde{A}}(u) > \mu_{\tilde{B}}(v) \end{cases}$$

then the criteria I-IV are satisfied.

We will consider ALI4 in detailes.

Consider a continuous function $F(p,q) = p - q$ which defines a distance between p and q where p, q assume values in the unit interval. Notice that $F(p,q) \in [-1,1]$, where $F(p,q)^{\min} = -1$ and $F(p,q)^{\max} = 1$. The normalized version of $F(p,q)$ is defined as follow

$$F(p,q)^{norm} = \frac{F(p,q) - F(p,q)^{\min}}{F(p,q)^{\max} - F(p,q)^{\min}} = \frac{F(p,q)+1}{2} = \frac{p-q+1}{2} \tag{1.89}$$

It is clear that $F(p,q)^{norm} \in [0,1]$. This function quantifies a concept of "closeness" between two values (potentially the ones for the truth values of *antecedent* and *consequent*), defined within unit interval, which therefore could play significant role in the formulation of the implication operator in a fuzzy logic.

Definition 1.49. An implication is a continuous function I from $[0,1]\times[0,1]$ into $[0,1]$ such that for $\forall p,\ p',\ q,\ q'\ r \in [0,1]$ the following properties are satisfied

(I1) If $p \le p'$, then $I(p,q) \ge I(p',q)$ (Antitone in first argument),
(I2) If $q \le q'$, then $I(p,q) \le I(p,q')$ (Monotone in second argument),
(I3) $I(0,q) = 1$, (Falsity),
(I4) $I(1,q) \le q$ (Neutrality),
(I5) $I(p,I(q,r)) = I(q,I(p,r))$ (Exchange),
(I6) $I(p,q) = I(n(q),n(p))$ (Contra positive symmetry), where $n()$ - is a negation, which could be defined as $n(q) = T(\neg Q) = 1 - T(Q)$

Let us define the implication operation

$$I(p,q) = \begin{cases} 1 - F(p,q)^{norm}, & p > q \\ 1, & p \le q \end{cases} \tag{1.90}$$

where $F(p,q)^{norm}$ is expressed by (1.89). Before showing that operation $I(p,q)$ satisfies axioms *(I1)-(I6),* let us show some basic operations encountered in proposed fuzzy logic.

Let us designate the truth values of the *antecedent* P and *consequent* Q as $T(P) = p$ and $T(P) = q$, respectively. The relevant set of proposed fuzzy logic operators is shown in Table 1.2.

To obtain the truth values of these expressions, we use well known logical properties such as

$$p \to q = \neg p \vee q,\ p \wedge q = \neg(\neg p \vee \neg q) \text{ and alike.}$$

In other words, we propose a new many-valued system, characterized by the set of *union* $(\cup)$ and *intersection* $(\cap)$ operations with relevant *complement*, defined as $T(\neg Q) = 1 - T(Q)$. In addition, the operators $\downarrow$ and $\uparrow$ are expressed as negations of the $\cup$ and $\cup$, respectively. It is well known that the *implication* operation in fuzzy logic supports the foundations of decision-making exploited in numerous schemes of approximate reasoning. Therefore let us prove that the proposed *implication* operation in (1.90) satisfies axioms *(I1)-(I6)*. For this matter, let us emphasize that we are working with a many-valued system, whose values for our purposes are the elements of the real interval $R=[0,1]$. For our discussion the set of truth values $V_{11}=\{0,0.1,0.2,...,0.9,1\}$ is sufficient. In further investigations, we use this particular set V_{11}.

Table 1.2 Fuzzy logic operators

Name	Designation	Value
Tautology	$\overset{\bullet}{P}$	1
Controversy	$\overset{\circ}{P}$	0
Negation	$\neg P$	$1-P$
Disjunction	$P \vee Q$	$\begin{cases} \frac{p+q}{2}, p+q \neq 1, \\ 1, p+q=1 \end{cases}$
Conjunction	$P \wedge Q$	$\begin{cases} \frac{p+q}{2}, p+q \neq 1, \\ 0, p+q=1 \end{cases}$
Implication	$P \rightarrow Q$	$\begin{cases} \frac{1-p+q}{2}, p \neq q, \\ 1, p=q \end{cases}$
Equivalence	$P \leftrightarrow Q$	$\begin{cases} \min((p-q),(q-p)), p \neq q, \\ 1, p=q \end{cases}$
Pierce Arrow	$P \downarrow Q$	$\begin{cases} 1-\frac{p+q}{2}, p+q \neq 1, \\ 0, p+q=1 \end{cases}$
Shaffer Stroke	$P \uparrow Q$	$\begin{cases} 1-\frac{p+q}{2}, p+q \neq 1, \\ 1, p+q=1 \end{cases}$

Theorem 1.4. *Let a continuous function $I(p,q)$ be defined by (1.90) i.e.*

$$I(p,q)=\begin{Bmatrix} 1-F(p,q)^{norm}, p>q \\ 1, \qquad p \leq q \end{Bmatrix}, p>q=\begin{cases} \frac{1-p+q}{2}, p>q \\ 1, p \leq q \end{cases} \tag{1.91}$$

where $F(p,q)^{norm}$ is defined by (1.89). Then axioms (I1)-(I6) are satisfied and, therefore (1.91) is an implication operation.

It should be mentioned that the proposed fuzzy logic could be characterized by yet some other three features:

$p \wedge 0 \equiv 0, p \leq 1,$ whereas $p \wedge 1 \equiv p, p \geq 0$ and $\neg\neg p = p$.

As a conclusion, we should admit that all above features confirm that *resulting system* can be applied to V_{11} for every finite and infinite n up to that $(V_{11}, \neg, \wedge, \vee, \rightarrow)$ is then *closed* under all its operations.

Let us investigate Statistical Properties of the Fuzzy Logic. In this section, we discuss some properties of the proposed fuzzy implication operator (1.91), assuming that the two propositions (*antecedent/consequent*) in a given compound proposition are independent of each other and the truth values of the propositions are uniformly distributed [64] in the unit interval. In other words, we assume that the propositions P and Q are independent from each other and the truth values $v(P)$ and $v(Q)$ are uniformly distributed across the interval $[0,1]$. Let $p = v(P)$ and $q = v(Q)$. Then the value of the implication $I = v(p \rightarrow q)$ could be represented as the function $I = I(p,q)$.

Because p and q are assumed to be uniformly and independently distributed across $[0,1]$, the expected value of the implication is

$$E(I) = \iint_R I(p,q)dpdq, \tag{1.92}$$

Its variance is equal to

$$Var(I) = E[(I - E(I))^2] = \iint_R (I(p,q) - E(I))^2 dpdq = E[I^2] - E[I^2] \tag{1.93}$$

where $R = \{(p,q): 0 \leq p \leq 1, 0 \leq q \leq 1\}$ From (1.92) and given (1.93) as well as the fact that

$I(p,q) = \begin{cases} I_1(p,q), p > q, \\ I_2(p,q), p \leq q, \end{cases}$ we have the following

$$E(I_1) = \iint_{\Re} I_1(p,q)dpdq, = \int_0^1\int_0^1 \frac{1-p+q}{2} dpdq = \frac{1}{2}\int_0^1 (\int_0^1 (1-p+q)dp)dp = \\ = \frac{1}{2}\left[\int_0^1 \left((p - \frac{p^2}{2} + p) \Bigg|_{p=0}^{p=1} \right) dq \right] = \frac{1}{2}\left[\frac{1}{2} + \frac{q^2}{2} \Bigg|_{q=0}^{q=1} \right] = \frac{1}{2} \tag{1.94}$$

Whereas $E(I_2) = 1$ Therefore $E(I) = (E(I_1) + E(I_2))/2 = 0.75$

From (1.93) we have

$$I_1^2(p,q) = \frac{1}{4}(1 - p + q)^2 = \frac{1}{4}(1 - 2p + 2q + p^2 - 2pq + q^2)$$

$$E(I_1^2) = \iint_{\Re} I_1^2(p,q)dpdq, = \frac{1}{4}\int_0^1 (\int_0^1 (1-2p+2q+p^2-2pq+q^2)dp)dq =$$

$$\frac{1}{4}\int_0^1 [p - 2\frac{p^2}{2} + \frac{p^3}{3} - 2\frac{p^2}{2}q + 2q + q^2]\Big|_{p=0}^{p=1} dq = \frac{1}{4}\int_0^1 (\frac{1}{3} + q + q^2)dq =$$

$$= \frac{1}{4}\left[\frac{q}{3} + \frac{q^2}{2} + \frac{q^3}{3}\right]\Big|_{q=0}^{q=1} = \frac{7}{24}$$

Here $E(I_2^2) = 1$ Therefore $E(I^2) = (E(I_1^2) + E(I_2^2))/2 = \frac{31}{48}$ From (1.93) and (1.94) we have $Var(I) = \frac{1}{12} = 0.0833$

Both values of $E(I)$ and $Var(I)$ demonstrate that the proposed fuzzy implication operator could be considered as one of the fuzziest from the list of the exiting implications [45]. In addition, it satisfies the set of important Criteria I-IV, which is not the case for the most implication operators mentioned above.

As it was mentioned in [38] "in the semantics of natural language there exist a vast array of concepts and humans very often make inferences antecedents and consequences of which contain fuzzy concepts". A formalization of methods for such inferences is one of the most important issues in fuzzy sets theory. For this purpose, let U and V (from now on) be two *universes of discourses* and P and Q are corresponding fuzzy sets:

$$\tilde{P} = \int_U \mu_{\tilde{P}}(u)/u \, , \; \tilde{Q} = \int_V \mu_{\tilde{Q}}(v)/v \tag{1.95}$$

Given (1.95), a *binary relationship* for the fuzzy conditional proposition of the type: "*If* x *is* $\tilde{P}$ *then* y *is* $\tilde{Q}$ " for proposed fuzzy logic is defined as

$$\begin{aligned}\tilde{R}(A_1(x), A_2(y)) &= \tilde{P} \times V \to U \times \tilde{B} = \int_{U\times V} \mu_{\tilde{P}}(u)/(u,v) \to \int_{U\times V} \mu_{\tilde{Q}}(v)/(u,v) = \\ &= \int_{U\times V} (\mu_{\tilde{P}}(u) \to \mu_{\tilde{Q}}(v))/(u,v)\end{aligned} \tag{1.96}$$

Given (1.90), expression (1.96) reads as

$$\mu_{\tilde{P}}(u) \to \mu_{\tilde{Q}}(v) = \begin{cases} \dfrac{1 - \mu_{\tilde{P}}(u) + \mu_{\tilde{Q}}(v)}{2}, \mu_{\tilde{P}}(u) > \mu_{\tilde{Q}}(v) \\ 1, \mu_{\tilde{P}}(u) \le \mu_{\tilde{Q}}(v) \end{cases} \tag{1.97}$$

It is well known that given a *unary relationship* $\tilde{R}(A_1(x))$ one can obtain the consequence $\tilde{R}(A_2(y))$ by applying a compositional rule of inference (CRI) to $\tilde{R}(A_1(x))$ and $\tilde{R}(A_1(x), A_2(y))$ of type (1.91):

$$\tilde{R}(A_2(y)) = \tilde{P} \circ \tilde{R}(A_1(x), A_2(y)) = \int_U \mu_{\tilde{P}}(u)/u \circ \int_{U\times V} \mu_{\tilde{P}}(u) \to \mu_{\tilde{Q}}(v)/(u,v) =$$
$$\int_V \bigcup_{u\in V} [\mu_{\tilde{P}}(u) \wedge (\mu_{\tilde{P}}(u) \to \mu_{\tilde{Q}}(v))]/v \qquad (1.98)$$

In order to have Criterion I satisfied, that is $\tilde{R}(A_2(y)) = \tilde{Q}$ from (1.98), the equality

$$\int_V \bigcup_{u\in V} [\mu_{\tilde{P}}(u) \wedge (\mu_{\tilde{P}}(u) \to \mu_{\tilde{Q}}(v))] = \mu_{\tilde{Q}}(v) \qquad (1.99)$$

has to be satisfied for any arbitrary v in V. To satisfy (1.99), it becomes necessary that the inequality

$$\mu_{\tilde{P}}(u) \wedge (\mu_{\tilde{P}}(u) \to \mu_{\tilde{Q}}(v)) \le \mu_{\tilde{Q}}(v) \qquad (1.100)$$

holds for arbitrary $u \in U$ and $v \in V$. Let us define a new method of fuzzy conditional inference of the following type:

$$\begin{array}{l} \text{Ant 1: If } x \text{ is } \tilde{P} \text{ then } y \text{ is } \tilde{Q} \\ \text{Ant 2: } x \text{ is } \tilde{P}' \\ \hline \text{Cons: } y \text{ is } \tilde{Q}'. \end{array} \qquad (1.101)$$

where $\tilde{P}, \tilde{P}' \subseteq U$ and $\tilde{Q}, \tilde{Q}' \subseteq V$. Fuzzy conditional inference in the form given by (1.101) should satisfy Criteria I-IV. It is clear that the inference (1.100) is defined by the expression (1.98), when $\tilde{R}(A_2(y)) = \tilde{Q}'$.

Theorem 1.5. *If fuzzy sets* $\tilde{P} \subseteq U$ *and* $\tilde{Q} \subseteq V$ *are defined by (1.96) and (1.97), respectively and*

$\tilde{R}(A_1(x), A_2(y))$ *is expressed as*

$$\tilde{R}(A_1(x), A_2(y)) = \tilde{P} \times V \xrightarrow[ALI4]{} U \times \tilde{Q} =$$
$$= \int_{U\times V} \mu_{\tilde{P}}(u)/(u,v) \xrightarrow[ALI4]{} \int_{U\times V} \mu_{\tilde{Q}}(v)/(u,v) =$$
$$= \int_{U\times V} (\mu_{\tilde{P}}(u) \xrightarrow[ALI4]{} \mu_{\tilde{Q}}(v))/(u,v)$$

where

$$\mu_{\tilde{P}}(u) \xrightarrow[ALI4]{} \mu_{\tilde{Q}}(v) = \begin{cases} \dfrac{1 - \mu_{\tilde{P}}(u) + \mu_{\tilde{Q}}(v)}{2}, \mu_{\tilde{P}}(u) > \mu_{\tilde{Q}}(v) \\ 1, \mu_{\tilde{P}}(u) \le \mu_{\tilde{Q}}(v) \end{cases} \qquad (1.102)$$

then Criteria I, II, III and IV-1 [38] are satisfied [13].

Theorem 1.6. *If fuzzy sets $\tilde{P} \subseteq U$ and $\tilde{Q} \subseteq V$ are defined by (1.96) and (1.97), respectively, and*
$\tilde{R}(A_1(x), A_2(y))$ *is defined as*

$$\tilde{R}_1(A_1(x), A_2(y)) = (\tilde{P} \times V \xrightarrow[ALI4]{} U \times \tilde{Q}) \cap (\neg\tilde{P} \times V \xrightarrow[ALI4]{} U \times \neg\tilde{Q}) =$$
$$= \int_{U \times V} (\mu_{\tilde{P}}(u) \xrightarrow[ALI4]{} \mu_{\tilde{Q}}(v)) \wedge ((1 - \mu_{\tilde{P}}(u)) \xrightarrow[ALI4]{} (1 - \mu_{\tilde{Q}}(v))) / (u, v) \tag{1.103}$$

where

$$(\mu_{\tilde{P}}(u) \xrightarrow[ALI4]{} \mu_{\tilde{Q}}(v)) \wedge ((1 - \mu_{\tilde{P}}(u)) \xrightarrow[ALI4]{} (1 - \mu_{\tilde{Q}}(v))) =$$
$$= \begin{cases} \dfrac{1 - \mu_{\tilde{P}}(u) + \mu_{\tilde{Q}}(v)}{2}, \mu_{\tilde{P}}(u) > \mu_{\tilde{Q}}(v), \\ 1, \mu_{\tilde{P}}(u) = \mu_{\tilde{Q}}(v), \\ \dfrac{1 - \mu_{\tilde{P}}(u) + \mu_{\tilde{Q}}(v)}{2}, \mu_{\tilde{P}}(u) < \mu_{\tilde{Q}}(v), \end{cases}$$

Then Criteria I, II, III and IV-2 [38] are satisfied.

Theorems 1.4 and 1.5 show that fuzzy conditional inference rules, defined in (1.103) could adhere with human intuition to the higher extent as the one defined by (1.102). The major difference between mentioned methods of inference might be explained by the difference between *Criteria IV-1* and *IV-2*. In particular, a satisfaction of the *Criterion IV-1* means that in case of logical negation of an original antecedent we achieve an ambiguous result of an inference, whereas for the case of the *Criterion IV-2* there is a certainty in a logical inference. Let us to investigate stability and continuity of fuzzy conditional inference in this section. We revisit the fuzzy conditional inference rule (1.101). It will be shown that when the membership function of the observation $\tilde{P}$ is continuous, then the conclusion $\tilde{Q}$ depends continuously on the observation; and when the membership function of the relation $\tilde{R}$ is continuous then the observation $\tilde{Q}$ has a continuous membership function. We start with some definitions. A fuzzy set $\tilde{A}$ with membership function $\mu_{\tilde{A}} : \mathcal{R} \to [0,1] = I$ is called a fuzzy number if $\tilde{A}$ is normal, continuous, and convex. The fuzzy numbers represent the continuous possibility distributions of fuzzy terms of the following type

$$\tilde{A} = \int_{\mathcal{R}} \mu_A(x) / x$$

Let $\tilde{A}$ be a fuzzy number, then for any $\theta \geq 0$ we define $\omega_A(\theta)$ the modulus of continuity of $\tilde{A}$ by

$$\omega_{\tilde{A}}(\theta) = \max_{|x_1 - x_2| \leq \theta} \left|\mu_{\tilde{A}}(x_1) - \mu_{\tilde{A}}(x_2)\right| \tag{1.104}$$

An α -level set of a fuzzy interval $\tilde{A}$ is a non-fuzzy set denoted by $[A]^{\alpha}$ and is defined by $[A]^{\alpha} = \{t \in R \left|\mu_{\tilde{A}}(t) \geq \alpha\right.\}$ for $\alpha = (0,1]$ and $[A]^{\alpha=0} = cl\left(\bigcup_{\alpha \in (0,1]} [A]^{\alpha}\right)$ for $\alpha = 0$. Here we use a metric of the following type

$$D(\tilde{A}, \tilde{B}) = \sup_{\alpha \in [0,1]} d([A]^{\alpha}, [B]^{\alpha}) \tag{1.105}$$

where d denotes the classical Hausdorff metric expressed in the family of compact subsets of R^2, i.e.

$$d([A]^{\alpha}, [B]^{\alpha}) = \max\{\left|a_1(\alpha) - b_1(\alpha)\right|, \left|a_2(\alpha) - b_2(\alpha)\right|\},$$

whereas

$[A]^{\alpha} = [a_1(\alpha), a_2(\alpha)], [B]^{\alpha} = b_1(\alpha), b_2(\alpha).$ When the fuzzy sets $\tilde{A}$ and $\tilde{B}$ have finite support $\{x_1, ..., x_n\}$ then their Hamming distance is defined as

$$H(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} \left|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)\right|$$

In the sequel we will use the following lemma.

Lemma 1.1 [28]. Let $\delta \geq 0$ be a real number and let $\tilde{A}$, $\tilde{B}$ be fuzzy intervals. If $D(\tilde{A}, \tilde{B}) \leq \delta$, *Then*

$$\sup_{t \in \mathcal{R}} \left|\mu_{\tilde{A}}(t) - \mu_{B^{`}}(t)\right| \leq \max\{\omega_{\tilde{A}}(\delta), \omega_{\tilde{B}}(\delta)\}$$

Consider the fuzzy conditional inference rule with different observations $\tilde{P}$ and $\tilde{P}'$:

Ant 1: If x is $\tilde{P}$ then y is $\tilde{Q}$
Ant2: x is $\tilde{P}$

Cons: y is $\tilde{Q}$.

Ant 1: If x is $\tilde{P}$ then y is $\tilde{Q}$
Ant2: x is $\tilde{P}'$

Cons: y is $\tilde{Q}'$.

According to the fuzzy conditional inference rule, the membership functions of the conclusions are computed as

$$\mu_{\tilde{Q}}(v) = \bigcup_{u\in R}[\mu_{\tilde{P}}(u) \wedge (\mu_{\tilde{P}}(u) \to \mu_{\tilde{Q}}(v))],$$

$$\mu_{\tilde{Q}'}(v) = \bigcup_{u\in R}[\mu_{\tilde{P}'}(u) \wedge (\mu_{\tilde{P}}(u) \to \mu_{\tilde{Q}}(v))],$$

or

$$\mu_{\tilde{Q}}(v) = \sup[\mu_{\tilde{P}}(u) \wedge (\mu_{\tilde{P}}(u) \to \mu_{\tilde{Q}}(v))],$$

$$\mu_{\tilde{Q}'}(v) = \sup[\mu_{\tilde{P}'}(u) \wedge (\mu_{\tilde{P}}(u) \to \mu_{\tilde{Q}}(v))], \tag{1.106}$$

The following theorem shows the fact that when the observations are closed to each other in the metric $D(.)$ of (1.105) type, then there can be only a small deviation in the membership functions of the conclusions.

Theorem 1.7. *(Stability theorem) Let* $\delta \ge 0$ *and let* $\tilde{P}$ *,* $\tilde{P}'$ *be fuzzy intervals and an implication operation in the fuzzy conditional inference rule (1.106) is of type (1.97). If* $D(\tilde{P},\tilde{P}') \le \delta$*, then*

$$\sup_{v\in R}\left|\mu_{\tilde{P}}(v) - \mu_{\tilde{P}'}(v)\right| \le \max\{\omega_{\tilde{P}}(\delta), \omega_{\tilde{P}'}(\delta)\}$$

Theorem 1.8. *(Continuity theorem) Let binary relationship* $\tilde{R}(u,v) = = \mu_{\tilde{P}}(u) \xrightarrow[ALI4]{} \mu_{\tilde{Q}}(v)$ *be continuous. Then* $\tilde{Q}$ *is continuous and* $\omega_{\tilde{Q}}(\delta) \le \omega_{\tilde{R}}(\delta)$ *for each* $\delta \ge 0$.

While we use extended fuzzy logic to reason with partially true statements we need to extend logics (6) for partial truth. We consider here only extension at the Lukasewicz logic for partial truth. In order to deal with partial truth Pavelka [85] extended this logic by adding truth constants for all reals in $[0,1]$ Hajek [43] simplified it by adding these truth constants $\overline{r}$ only for each rational $r \in [0,1]$ (so $\overline{r}$ is an atomic formula with truth value *r*). They also added 'book - keeping axioms'

$$\overline{r \Rightarrow s} \equiv \overline{r} \to \overline{s} \text{ for r, s rational} \in [0,1] \ .$$

This logic is called Rational Pavelka logic (RPL). RPL was introduced in order to reason with partially true statements. In this section we note that this can already be done in Lukasiewicz logic, and that the conservative extension theorems allow us to lift the completeness theorem, that provability degree equals truth degree from RPL to Lukasiewicz logic. This may be regarded as an additional conservative extension theorem, confirming that, even for partial truth, Rational Pavelka logic deals with exactly the same logic as Lukasiewicz logic - but in a very much more convenient way. RPL extends the language of infinite valued Łukasiewicz logic by adding to the truth constants 0 and 1 all rational numbers r of the unit interval $[0,1]$ A *graded formula* is a pair (φ, r) consisting of a formula φ of Łukasiewicz logic and a rational element $r \in [0,1]$, indicating that the truth value of φ is at least r, $\varphi \geq r$ [107]. For example, $\left(p(x), \frac{1}{2}\right)$ expresses the fact that the truth value of $p(x),\ x \in Dom$, is at least $\frac{1}{2}$. The inference rules of RPL are the *generalization rule*

$$\frac{\varphi}{(\forall x)(\varphi)}, \tag{1.107}$$

and a modified version of the *modus ponens rule*,

$$\frac{(\varphi, r), (\varphi \to \psi, s)}{(\psi, r \otimes s)} \tag{1.108}$$

Here $\otimes$ denotes the Łukasiewicz t-norm. Rule (1.108) says that if formula φ holds at least with truth value *r*, and the implication $\varphi \to \psi$ holds at least with truth value *s*, then formula ψ holds at least with truth value $r \otimes s$. The modified modus ponens rule is derived from the so-called *book-keeping axioms* for the rational truth constants *r*. The book-keeping axioms add to the axioms of Łukasiewicz logic and provide rules for evaluating compound formulas involving rational truth constants [44]. The use of fuzzy reasoning trades accuracy against speed, simplicity and interpretability for lay users. In the context of ubiquitous computing, these characteristics are clearly advantageous.

References

1. Aguilo, I., Suner, J., Torrens, J.: A characterization of residual implications derived from left-continuous uninorms. Information Sciences 180, 3992–4005 (2010)
2. Aliev, R.A.: Fuzzy knowledge based Intelligent Robots. Radio i svyaz, Moscow (1995) (in Russian)
3. Aliev, R.A., Aliev, R.R.: Soft Computing and its Application. World Scientific, New Jersey (2001)
4. Aliev, R.A., Aliev, F.T., Babaev, M.D.: Fuzzy process control and knowledge engineering. Verlag TUV Rheinland, Koln (1991)
5. Aliev, R.A., Aliev, R.R.: Soft Computing, vol. I, II, III. ASOA Press, Baku (1997-1998) (in Russian)
6. Aliev, R.A., Bonfig, K.W., Aliev, F.T.: Messen, Steuern und Regeln mit Fuzzy-Logik. Franzis-Verlag, München (1993)
7. Aliev, R.A., Fazlollahi, B., Aliev, R.R.: Soft Computing and its Application in Business and Economics. Springer, Heidelberg (2004)

8. Aliev, R.A., Mamedova, G.A., Aliev, R.R.: Fuzzy Sets Theory and its application. Tabriz University Press, Tabriz (1993)
9. Aliev, R.A., Mamedova, G.A., Tserkovny, A.E.: Fuzzy control systems. Energoatomizdat, Moscow (1991)
10. Aliev, R.A., Pedrycz, W.: Fundamentals of a fuzzy-logic-based generalized theory of stability. IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetic 39(4), 971–988 (2009)
11. Aliev, R.A., Pedrycz, W., Fazlollahi, B., Huseynov, O.H., Alizadeh, A.V., Guirimov, B.G.: Fuzzy logic-based generalized decision theory with imperfect information. Information Sciences 189, 18–42 (2012)
12. Aliev, R.A., Tserkovny, A.: The knowledge representation in intelligent robots based on fuzzy sets. Soviet Math. Doklady 37, 541–544 (1988)
13. Aliev, R.A., Tserkovny, A.E.: A systemic approach to fuzzy logic formalization for approximate reasoning. Information Sciences 181, 1045–1059 (2011)
14. Azadeh, I., Fam, I.M., Khoshnoud, M., Nikafrouz, M.: Design and implementation of a fuzzy expert system for performance assessment of an integrated health, safety, environment (HSE) and ergonomics system: The case of a gas refinery. Information Sciences 178(22), 4280–4300 (2008)
15. Baldwin, J.F., Pilsworth, B.W.: A model of fuzzy reasoning through multivalued logic and set theory. Int. J. Man-Machines Studies 11, 351–380 (1979)
16. Ban, A.I., Gal, S.G.: Defects of Properties in Mathematics. Quantitative Characterizations. Series on Concrete and Applicable Mathematics, vol. 5. World Scientific, Singapore (2002)
17. Bandemer, H., Gottwald, S.: Fuzzy sets, Fuzzy logic, Fuzzy methods with applications. John Wiley and Sons, England (1995)
18. Bandemer, H., Nather, W.: Fuzzy data analysis. Kluwer Academic Publishers, Boston (1992)
19. Bandler, W., Kohout, L.: Fuzzy power sets and fuzzy implications operators. Fuzzy Sets and Systems 1, 13–30 (1980)
20. Bandler, W., Kohout, L.J.: Fuzzy relational products as a tool for analysis of complex artificial and natural systems. In: Wang, P.P., Chang, S.K. (eds.) Fuzzy Sets; Theory and Applications to Policy Analysis and Information Systems, p. 311. Plenum Press, New York (1980)
21. Bandler, W., Kohout, L.J.: Semantics of fuzzy implication operators and relational products. International Journal of Man-Machine Studies 12(1), 89–116 (1980)
22. Bandler, W., Kohout, L.J.: The identification of hierarchies in symptoms and patients through computation of fuzzy relational products. In: Parslow, R.D. (ed.) BCS 1981: Information Technology for the Eighties, P. 191. Heyden & Sons (1980)
23. Bandler, W., Kohout, L.J.: The four modes of inference in fuzzy expert systems. In: Trappl, R. (ed.) Cybernetics and Systems Research 2, pp. 581–586. North Holland, Amsterdam (1984)
24. Bede, B., Gal, S.G.: Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. Fuzzy Sets and Systems 151, 581–599 (2005)
25. Beg, I.: Fuzzy multivalued functions. Centre for Advanced Studies in Mathematics, and Department of Mathematics, Lahore University of Management Sciences (LUMS), 54792-Lahore, Pakistan (2012), http://wenku.baidu.com/view/71a84c136c175f0e7cd1372d.html

26. Belohlavek, R., Sigmund, E., Zacpal, J.: Evaluation of IPAQ questionnaires supported by formal concept analysis. Information Science 181(10), 1774–1786 (2011)
27. Bloch, I.: Lattices of fuzzy sets and bipolar fuzzy sets, and mathematical morphology. Information Sciences 181(10), 2002–2015 (2011)
28. Bobillo, F., Straccia, U.: Reasoning with the finitely many-valued Łukasiewicz fuzzy Description Logic SROIQ. Information Sciences 181(4), 758–778 (2011)
29. Buckley, J.J., Eslami, E.: Fuzzy plane geometry I: Points and lines. Fuzzy Sets and Systems 86(2), 179–187 (1997)
30. Bustince, H., Barrenechea, E., Fernandez, J., Pagola, M., Montero, J., Guerra, C.: Contrast of a fuzzy relation. Information Sciences 180, 1326–1344 (2010)
31. Chajda, I., Halas, R., Rosenberg, I.G.: On the role of logical connectives for primality and functional completeness of algebras of logics. Information Sciences 180(8), 1345–1353 (2010)
32. Chen, T.: Optimistic and pessimistic decision making with dissonance reduction using interval valued fuzzy sets. Information Sciences 181(3), 479–502 (2010)
33. Davvaz, B., Zhan, J., Shum, K.P.: Generalized fuzzy Hv-submodules endowed with interval valued membership functions. Information Sciences 178, Nature Inspired Problem-Solving 1, 3147–3159 (2008)
34. Diamond, P., Kloeden, P.: Metric spaces of fuzzy sets. Theory and applications. World Scientific, Singapoure (1994)
35. Dian, J.: A meaning based information theory - inform logical space: Basic concepts and convergence of information sequences. Information Sciences 180: Special Issue on Modelling Uncertainty 15, 984–994 (2010)
36. Fan, Z.-P., Feng, B.: A multiple attributes decision making method using individual and collaborative attribute data in a fuzzy environment. Information Sciences 179, 3603–3618 (2009)
37. Fedrizzi, M., Fuller, R.: Stability in Possibilistic Linear Programming Problems with Continuous Fuzzy Number Parameters. Fuzzy Sets and Systems 47, 187–191 (1992)
38. Fukami, S., Mizumoto, M., Tanaka, K.: Some considerations of fuzzy conditional inference. Fuzzy Sets and Systems 4, 243–273 (1980)
39. Fuller, R., Zimmermann, H.J.: On Zadeh's compositional rule of inference. In: Lowen, R., Roubens, M. (eds.) Fuzzy Logic: State of the Art, Theory and Decision Library, Series D, pp. 193–200. Kluwer Academic Publishers, Dordrecht (1993)
40. Gerhke, M., Walker, C.L., Walker, E.A.: Normal forms and truth tables for fuzzy logics. Fuzzy Sets and Systems 138, 25–51 (2003)
41. Grabisch, M., Marichal, J., Mesiar, R., Pap, E.: Aggregation functions: Construction methods, conjunctive, disjunctive and mixed classes. Information Sciences 181, 23 (2011)
42. Grzegorzewski, P.: On possible and necessary inclusion of intuitionistic fuzzy sets. Information Sciences 181, 342 (2011)
43. Hajek, P.: Fuzzy Logic from the Logical Point of View. In: Bartosek, M., Staudek, J., Wiedermann, J. (eds.) SOFSEM 1995. LNCS, vol. 1012, pp. 31–49. Springer, Heidelberg (1995)
44. Hajek, P.: Metamathematics of Fuzzy Logic. Trends in Logic. Kluwer Academic Publishers (1998)
45. Hu, Q., Yu, D., Guo, M.: Fuzzy preference based rough sets. Information Sciences 180: Special Issue on Intelligent Distributed Information Systems 15, 2003–2022 (2010)

46. Buckley, J.J., Feuring, T.: Fuzzy differential equations. Fuzzy Sets and Systems 151, 581–599 (2005)
47. Jantzen, J.: Array approach to fuzzy logic. Fuzzy Sets and Systems 70, 359–370 (1995)
48. Jayaram, B., Mesiar, R.: I-Fuzzy equivalence relations and I-fuzzy partitions. Information Sciences 179, 1278–1297 (2009)
49. Jenei, S.: Continuity in Zadeh's compositional rule of inference. Fuzzy Sets and Systems 104, 333–339 (1999)
50. Kallala, M., Kohout, L.J.: The use of fuzzy implication operators in clinical evaluation of neurological movement disorders. In: International Symposium on Fuzzy Information Processing in Artificial Intelligence and Operational Research, Christchurch College, Cambridge University (1984)
51. Kallala, M., Kohout, L.J.: A 2-stage method for automatic handwriting classification by means of norms and fuzzy relational inference. In: Proc. of NAFIPS 1986 (NAFIPS Congress), New Orleans (1986)
52. Kalina, M.: Derivatives of fuzzy functions and fuzzy derivatives. Tatra Mountains Mathematical Publications 12, 27–34 (1997)
53. Kandel, A., Last, M.: Special issue on advances in Fuzzy logic. Information Sciences 177, 329–331 (2007)
54. Kaufman, A.: Introduction to theory of fuzzy sets, vol. 1. Academic Press, Orlando (1973)
55. Kehagias, A.: Some remarks on the lattice of fuzzy intervals. Information Sciences 181(10), 1863–1873 (2010)
56. Kiszka, J.B., Kochanska, M.E., Sliwinska, D.S.: The influence of some fuzzy implication operators on the accuracy of a fuzzy model. Fuzzy Sets and Systems 15, (Part1) 111–128, (Part2) 223–240 (1985)
57. Klir, G.J., Yuan, B.: Fuzzy sets and fuzzy logic. Theory and Applications. PRT Prentice Hall, NJ (1995)
58. Klir, G.J., Clair, U.S., Yuan, B.: Fuzzy Set Theory. Foundations and Applications. PTR Prentice Hall, NJ (1997)
59. Kohout, L.J. (ed.): Perspectives on Intelligent Systems: A Framework for Analysis and Design. Abacus Press, Cambridge (1986)
60. Kolesarova, A., Mesiar, R.: Lipschitzian De Morgan triplets of fuzzy connectives. Information Sciences 180, 3488–3496 (2010)
61. Lai, J., Xu, Y.: Linguistic truth-valued lattice-valued propositional logic system lP(X) based on linguistic truth-valued lattice implication algebra. Information Sciences 180: Special Issue on Intelligent Distributed Information Systems, 1990–2002 (2010)
62. Lakshmikantham, V., Mohapatra, R.: Theory of fuzzy differential equations and inclusions. Taylor & Francis, London (2003)
63. Levy, P.: From social computing to reflexive collective intelligence: The IEML research program. Information Sciences 180: Special Issue on Collective Intelligence 2, 71 (2010)
64. Li, C., Yi, J.: Sirms based interval type–2 fuzzy inference systems properties and application. International Journal of Innovative Computing. Information and Control 6(9), 4019–4028 (2010)
65. Lifschitz, V. (ed.): Formalizing Common Sense, Papers by John McCarthy. Greenwood Publishing Group Inc., NJ (1990)

66. Long, Z., Liang, X., Yang, L.: Some approximation properties of adaptive fuzzy systems with variable universe of discourse. Information Sciences 180, 2991–3005 (2010)
67. Ma, H.: An analysis of the equilibrium of migration models for biogeography-based optimization. Information Sciences 180, 3444–3464 (2010)
68. Mamdani, E.H.: Application of fuzzy logic to approximate reasoning using linguistic syntheses. IEEE Transactions on Computers C-26(12), 1182–1191 (1977)
69. Mas, M., Monserrat, M., Torrens, J.: The law of importation for discrete implications. Information Sciences 179, 4208–4218 (2009)
70. Mas, M., Monserrat, M., Torrens, J., Trillas, E.: A Survey on Fuzzy Implication Functions. IEEE Transactions on Fuzzy Systems 15(6), 1107–1121 (2007)
71. Mayburov, S.: Fuzzy geometry of phase space and quantization of massive Fields. Journal of Physics A: Mathematical and Theoretical 41, 1–10 (2008)
72. Medina, J., Ojeda Aciego, M.: Multi-adjoint t-concept lattices. Information Sciences 180, 712–725 (2010)
73. Mendel, J.M.: On answering the question “Where do I start in order to solve a new problem involving interval type-2 fuzzy sets?”. Information Sciences 179, 3418–3431 (2009)
74. Mizumoto, M., Fukami, S., Tanaka, K.: Some methods of fuzzy reasoning. In: Gupta, R., Yager, R. (eds.) Advances in Fuzzy Set Theory Applications. North-Holland, New York (1979)
75. Mizumoto, M., Zimmermann, H.-J.: Comparison of fuzzy reasoning methods. Fuzzy Sets and Systems 8, 253–283 (1982)
76. Molai, A.A., Khorram, E.: An algorithm for solving fuzzy relation equations with max-T composition operator. Information Sciences 178, 1293–1308 (2008)
77. Mordeson, J.N., Nair, P.S.: Fuzzy mathematics: an introduction for engineers and Scientists. Physica-Verlag, Heidelberg (2001)
78. Mueller, E.: Commonsense Reasoning. Morgan Kaufmann, San Francisco (2006)
79. Munoz-Hernandez, S., Pablos-Ceruelo, V., Strass, H.: R Fuzzy: Syntax, semantics and implementation details of a simple and expressive fuzzy tool over Prolog. Information Sciences 181(10), 1951–1970 (2011)
80. Nachtegael, M., Sussner, P., Melange, T., Kerre, E.E.: On the role of complete lattices in mathematical morphology: From tool to uncertainty model. Information Sciences (2010); corrected proof, available online 15 (in Press)
81. Nguyen, H.T., Walker, E.A.: A first Course in Fuzzy logic. CRC Press, Boca Raton (1996)
82. Noguera, C., Esteva, F., Godo, L.: Generalized continuous and left-continuous t-norms arising from algebraic semantics for fuzzy logics. Information Sciences 180, 1354–1372 (2010)
83. Oh, K.W., Bandler, W.: Properties of fuzzy implication operators, Florida State University, Tallahassee, FL, U. S. A. Department of Computer Science, pp. 24–33 (1988)
84. Ouyang, Y., Wang, Z., Zhang, H.: On fuzzy rough sets based on tolerance relations. Information Sciences 180, 532 (2010)
85. Pavelka, J.: On fuzzy logic I, II, III. Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik 25(45-52), 119–134, 447–464 (1979)
86. Pei, D.: Unified full implication algorithms of fuzzy reasoning. Information Sciences 178, 520 (2008)
87. Poston, T.: Fuzzy geometry. Ph.D. Thesis, University of Warwick (1971)

88. Rescher, N.: Many-Valued Logic. McGraw–Hill, NY (1969)
89. Roe, J.: Index theory, coarse geometry, and topology of manifolds. In: CBMS: Regional Conf. Ser. in Mathematics. The American Mathematical Society, Rhode Island (1996)
90. Rosenfeld, A.: The diameter of a fuzzy set. Fuzzy Sets and Systems 13, 241–246 (1984)
91. Rosenfeld, A.: Distances between fuzzy sets. Pattern Recognition Letters 3(4), 229–233 (1985)
92. Rosenfeld, A.: Fuzzy rectangles. Pattern Recognition Letters 11(10), 677–679 (1990)
93. Rosenfeld, A.: Fuzzy plane geometry: triangles. Pattern Recognition Letters 15, 1261–1264 (1994)
94. Rosenfeld, A.: Fuzzy geometry: an updated overview. Information Science 110(3-4), 127–133 (1998)
95. Rutkowski, L., Cpalka, K.: Flexible Neuro-Fuzzy Systems. IEEE Transactions on Neural Networks 14(3), 554–573 (2003)
96. Sankar, K.P., Ghosh, A.: Fuzzy geometry in image analysis. Fuzzy Sets and Systems 48, 23–40 (1992)
97. Sankar, K.P.: Fuzzy geometry, entropy, and image information. In: Proceedings of the Second Join Technology Workshop on Neural Networks and Fuzzy Logic, vol. 2, pp. 211–232 (1991)
98. Serruier, M., Dubois, D., Prade, H., Sudkamp, T.: Learning fuzzy rules with their implication operator. Data & Knowledge Engineering 60, 71–89 (2007)
99. Shieh, B.S.: Infinite fuzzy relation equations with continuous t-norms. Information Sciences 178, 1961–1967 (2008)
100. Simon, H.: Models of Bounded Rationality: Empirically Grounded Economic Reason, vol. 3. MIT Press, Cambridge (1997)
101. Toulmin, S.: The Uses of Argument. Cambridge University Press, UK (2003)
102. Tzafestas, S.G., Chen, C.S., Fokuda, T., Harashima, F., Schmidt, G., Sinha, N.K., Tabak, D., Valavanis, K. (eds.): Fuzzy logic applications in engineering science. Microprocessor based and Intelligent Systems Engineering, vol. 29, pp. 11–30. Springer, Netherlands (2006)
103. Valle, M.E.: Permutation-based finite implicative fuzzy associative memories. Information Sciences 180, 4136–4152 (2010)
104. Valverde Albacete, F.J., Pelaez Moreno, C.: Extending conceptualization modes for generalized Formal Concept Analysis. Information Sciences 181(10), 1888–1909 (2011)
105. Wenstop, F.: Quantitative analysis with linguistic values. Fuzzy Sets and Systems 4(2), 99–115 (1980)
106. Werthner, H.: Qualitative Reasoning, Modeling and the Generation of Behavior. Springer (1994)
107. Wilke, G.: Approximate Geometric Reasoning with Extended Geographic Objects. In: Proceedings of the Workshop on Quality, Scale and Analysis Aspects of City Models, Lund, Sweden, December 3-4 (2009), `http://www.isprs.org/proceedings/XXXVIII/2W11/Wilke.pdf`
108. Xie, A., Qin, F.: Solutions to the functional equation I(x, y) = I(x, I(x, y)) for three types of fuzzy implications derived from uninorms. Information Sciences 186(1), 209–221 (2012)
109. Xu, Y., Liu, J., Ruan, D., Li, X.: Determination of [alpha]-resolution in lattice-valued first-order logic LF(X). Information Sciences 181(10), 1836–1862 (2010)

110. Yager, R.R.: On measures of specificity. In: Kaynak, O., Zadeh, L.A., Turksen, B., Rudas, I.J. (eds.) Computational Intelligence: Soft Computing and Fuzzy-Neuro Integration with Applications, pp. 94–113. Springer, Berlin (1998)
111. Yager, R.R.: On global requirements for implication operators in fuzzy modus ponens. Fuzzy Sets and Systems 106, 3–10 (1999)
112. Yager, R.R.: A framework for reasoning with soft information. Information Sciences 180(8), 1390–1406 (2010)
113. Yeh, R.T., Bang, S.Y.: Fuzzy relations, fuzzy graphs, and their applications to clustering analysis. In: Zadeh, L.A., Fu, K.S., Shimura, M.A. (eds.) Fuzzy Sets and Their Applications, pp. 125–149. Academic Press, NY (1975)
114. Zadeh, L.A.: Fuzzy Sets. Information and Control 8, 338–353 (1965)
115. Zadeh, L.A.: Probability measures of fuzzy events. Journal of Mathematical Analysis and Applications 23(2), 421–427 (1968)
116. Zadeh, L.A.: Fuzzy orderings. Information Sciences 3, 117–200 (1971)
117. Zadeh, L.A.: Similarity relations and Fuzzy orderings. Information Sciences 3, 177–200 (1971)
118. Zadeh, L.A.: Outline of a new approach to the analysis of complex system and decision processes. IEEE Trans. Systems, Man, and Cybernetics 3, 28–44 (1973)
119. Zadeh, L.A.: The concept of a linguistic variable and its applications in approximate reasoning. Information Sciences 8, 43–80, 301–357; 9, 199–251 (1975)
120. Zadeh, L.A.: Fuzzy sets and information granularity. In: Gupta, M., Ragade, R., Yager, R. (eds.) Advances in Fuzzy Set Theory and Applications, pp. 3–18. North-Holland Publishing Co., Amsterdam (1979)
121. Zadeh, L.A.: Possibility theory, soft data analysis. In: Cobb, L., Thrall, R.M. (eds.) Mathematical Frontiers of the Social and Policy Sciences, pp. 69–129. Westview Press, Boulder (1981)
122. Zadeh, L.A.: Fuzzy logic. IEEE Computer 21(4), 83–93 (1988)
123. Zadeh, L.A.: Toward a generalized theory of uncertainty — an outline. Information Sciences 172, 1–40 (2005)
124. Zadeh, L.A.: Generalized theory of uncertainty (GTU) – principal concepts and ideas. Computational statistics & Data Analysis 51, 15–46 (2006)
125. Zadeh, L.A.: Is there a need for fuzzy logic? Information Sciences 178, 2751–2779 (2008)
126. Zadeh, L.A.: Fuzzy logic. In: Encyclopedia of Complexity and Systems Science, pp. 3985–4009. Springer, Berlin (2009)
127. Zadeh, L.A.: Toward extended fuzzy logic. A first step. Fuzzy Sets and Systems 160, 3175–3181 (2009)
128. Zadeh, L.A.: A Note on Z-numbers. Information Sciences 181, 2923–2932 (2010)
129. Zhang, J., Yang, X.: Some properties of fuzzy reasoning in propositional fuzzy logic systems. Information Sciences 180, 4661–4671 (2010)
130. Zhang, L., Cai, K.Y.: Optimal fuzzy reasoning and its robustness analysis. Int. J. Intell. Syst. 19, 1033–1049 (2004)
131. Zhang, X., Yao, Y., Yu, H.: Rough implication operator based on strong topological rough algebras. Information Sciences 180, 3764–3780 (2010)
132. Zhao, S., Tsang, E.C.C.: On fuzzy approximation operators in attribute reduction with fuzzy rough sets. Information Sciences 178, Including Special Issue: Recent Advances in Granular Computing (2008); Fifth International Conference on Machine Learning and Cybernetics, vol. 15, pp. 3163–3176
133. Zimmermann, H.J.: Fuzzy Set Theory and its applications. Kluwer Academic Publishers, Norwell (1996)