Chapter 1
Introduction

The important mathematical problem of evaluating Feynman integrals arises quite naturally in elementary-particle physics when one treats various quantities in the framework of perturbation theory. Usually, it turns out that a given quantum-field amplitude that describes a process where particles participate cannot be completely treated in the perturbative way. However it also often turns out that the amplitude can be factorized in such a way that different factors are responsible for contributions of different scales. According to a factorization procedure a given amplitude can be represented as a product of factors some of which can be treated only non-perturbatively while others can be indeed evaluated within perturbation theory, i.e. expressed in terms of Feynman integrals over loop momenta.

A useful way to perform the factorization procedure is provided by solving the problem of asymptotic expansion of Feynman integrals in the corresponding limit of momenta and masses that is determined by the given kinematical situation. A universal way to solve this problem is based on the so-called strategy of expansion by regions [1, 21]. This strategy can be itself regarded as a (semi-analytical) method of evaluation of Feynman integrals according to which a given Feynman integral depending on several scales can be approximated, with increasing accuracy, by a finite sum of first terms of the corresponding expansion, where each term is written as a product of factors depending on different scales. The expansion by regions applicable to general limits as well as the expansion by subgraphs applicable to limits typical of Euclidean space are described in details in my other book [21] and, in a very brief way, in this book in Chap. 9. The main goal of this chapter is to present a general algorithm [9, 17] which has appeared after the publication of the book [21] and provides the possibility to find regions relevant to a given limit in a systematic way.

One needs to take into account various graphs that contribute to a given process. The number of graphs greatly increases when the number of loops gets large. For a given graph, the corresponding Feynman amplitude is represented as a Feynman integral over loop momenta, due to some Feynman rules. The Feynman integral, generally, has several Lorentz indices. The standard way to handle tensor quantities

V. A. Smirnov, *Analytic Tools for Feynman Integrals*, Springer Tracts in Modern Physics 250, DOI: 10.1007/978-3-642-34886-0_1,

is to perform a tensor reduction that enables us to write the given quantity as a linear combination of tensor monomials with scalar coefficients. Therefore we will imply that we deal with scalar Feynman integrals and consider only them in examples.

A given Feynman graph therefore generates various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators (indices). Let us observe that some powers can be negative, due to some initial polynomial in the numerator of the Feynman integral. A straightforward strategy is to evaluate, by some methods, every scalar Feynman integral resulting from the given graph. If the number of these integrals is small this strategy is quite reasonable. In non-trivial situations, where the number of different scalar integrals can be at the level of hundreds and thousands, this strategy looks too complicated. A well-known optimal strategy here is to derive, without calculation, and then apply some relations between the given family of Feynman integrals as *recurrence relations*. A well-known standard way to obtain such relations is provided by the method of integration by parts[1] (IBP) [7] which is based on putting to zero any integral of the form

$$\int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots \frac{\partial f}{\partial k_i^{\mu}}$$

over loop momenta $k_1, k_2, \ldots, k_i, \ldots$ within dimensional regularization with the space-time dimension $d = 4 - 2\varepsilon$ as a regularization parameter [5, 6, 8]. Here f is an integrand of a Feynman integral; it depends on the loop and external momenta. More precisely, one tries to use IBP relations in order to express a general dimensionally regularized integral from the given family as a linear combination of some irreducible integrals which are also called *master* integrals. Therefore the whole problem decomposes into two parts: solving the IBP relations and evaluating the master integrals. Observe that in such complicated situations, with the great variety of relevant scalar integrals, one really needs to know a *complete* solution of the recursion problem, i.e. to learn how an *arbitrary* integral with general integer powers of the propagators and powers of irreducible monomials in the numerator can be evaluated.

To illustrate the methods of evaluation that we are going to study in this book let us first orient ourselves at the evaluation of individual Feynman integrals, which might be master integrals, and take the simple scalar one-loop graph Γ shown in Fig. 1.1 as an example. The corresponding Feynman integral constructed with scalar propagators is written as

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[1] As is explained in textbooks on integral calculus, the method of IBP is applied with the help of the relation $\int_a^b \mathrm{d}x u v' = uv|_a^b - \int_a^b \mathrm{d}x u' v$ as follows. One tries to represent the integrand as uv' with some u and v in such a way that the integral on the right-hand side, i.e. of $u'v$ will be simpler. We do not follow this idea in the case of Feynman integrals. Instead we only use the fact that an integral of the derivative of some function is zero, i.e. we always neglect the corresponding surface terms. So the name of the method looks misleading. It is however unambiguously accepted in the physics community.

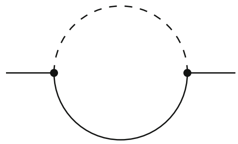

Fig. 1.1 One-loop self-energy graph. The *dashed line* denotes a massless propagator

$$F_\Gamma(q^2, m^2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)(q-k)^2}, \tag{1.1}$$

where the usual $+\mathrm{i}0$ is implied in the propagators.

The same picture Fig. 1.1 can also denote the Feynman integral with general powers of the two propagators,

$$F_\Gamma(q^2, m^2; a_1, a_2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^{a_1}[(q-k)^2]^{a_2}}. \tag{1.2}$$

Suppose, one needs to evaluate the Feynman integral $F_\Gamma(q^2, m^2; 2, 1; d) \equiv F(2, 1; d)$ which is finite in four dimensions, $d = 4$. (It can also be depicted by Fig. 1.1 with a dot on the massive line.) There is a lot of ways to evaluate it. For example, a straightforward way is to take into account the fact that the given function of q is Lorentz-invariant so that it depends on the external momentum through its square, q^2. One can choose a frame $q = (q_0, \mathbf{0})$, introduce spherical coordinates for $\boldsymbol{k}$, integrate over angles, then over the radial component and, finally, over k_0. This strategy can be, however, hardly generalized to multi-loop[2] Feynman integrals.

Another way is to use a dispersion relation that expresses Feynman integrals in terms of a one-dimensional integral of the imaginary part of the given Feynman integral, from the value of the lowest threshold to infinity. This dispersion integral can be expressed by means of the well-known Cutkosky rules. We will not apply this method, which was, however, very popular in the early days of perturbative quantum field theory, and only briefly comment on it in Appendix E.

Let us now turn to the methods that will be indeed actively used in this book. To illustrate them all let me use this very example of Feynman integrals (1.2) and present main ideas of these methods, with the obligation to present the methods in great details in the rest of the book.

First, we will exploit the well-known technique of alpha or Feynman parameters. In the case of $F(2, 1; d)$, one writes down the following Feynman-parametric formula:

$$\frac{1}{(k^2 - m^2)^2(q-k)^2} = 2\int_0^1 \frac{\xi \mathrm{d}\xi}{[(k^2 - m^2)\xi + (1-\xi)(q-k)^2 + \mathrm{i}0]^3}. \tag{1.3}$$

[2] Since the Feynman integrals are rather complicated objects the word ‘multi-loop’ means the number of loops greater than one ;-)

Then one can change the order of integration over ξ and k, perform integration over k with the help of the formula (10.1) (which we will derive in Chap. 3) and obtain the following representation:

$$F(2,1;d) = -\mathrm{i}\pi^{d/2}\Gamma(1+\varepsilon)\int_0^1 \frac{\mathrm{d}\xi\, \xi^{-\varepsilon}}{[m^2 - q^2(1-\xi) - \mathrm{i}0]^{1+\varepsilon}}. \tag{1.4}$$

This integral can easily be evaluated at $d = 4$ with the following result:

$$F(2,1;4) = \mathrm{i}\pi^2 \frac{\ln\left(1 - q^2/m^2\right)}{q^2}. \tag{1.5}$$

In principle, any given Feynman integral $F(a_1, a_2; d)$ with concrete numbers a_1 and a_2 can similarly be evaluated by Feynman parameters. In particular, $F(1, 1; d)$ reduces to

$$F(1,1;d) = \mathrm{i}\pi^{d/2}\Gamma(\varepsilon)\int_0^1 \frac{\mathrm{d}\xi\, \xi^{-\varepsilon}}{[m^2 - q^2(1-\xi) - \mathrm{i}0]^{\varepsilon}}. \tag{1.6}$$

There is an ultraviolet (UV) divergence which manifests itself in the first pole of the function $\Gamma(\varepsilon)$, i.e. at $d = 4$. The integral can be evaluated in expansion in a Laurent series in ε, for example, up to ε^0. We obtain

$$\begin{aligned} F(1,1;d) = \mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}\Bigg[\frac{1}{\varepsilon} - \ln m^2 + 2 \\ - \left(1 - \frac{m^2}{q^2}\right)\ln\left(1 - \frac{q^2}{m^2}\right) + O(\varepsilon)\Bigg], \end{aligned} \tag{1.7}$$

where γ_{E} is the Euler's constant.

In fact, the integration in (1.6) can straightforwardly be performed at general ε with the result

$$-\mathrm{i}\pi^{d/2} m^{-2\varepsilon}\Gamma(\varepsilon - 1)\,{}_2F_1\left(1, \varepsilon; 2-\varepsilon; q^2/m^2\right) \tag{1.8}$$

which can then be expanded in ε. However, for sufficiently complicated Feynman integrals, this strategy of evaluating at general ε and expanding results is hardly feasible.

Alpha parameters are closely related to Feynman parameters. For usual propagators, one starts from the representation

$$\frac{1}{k^2 - m^2} = -\mathrm{i}\int_0^\infty \mathrm{d}\alpha\, \mathrm{e}^{\mathrm{i}(k^2 - m^2)\alpha}, \tag{1.9}$$

changes the order of integration over alpha parameters and loop momenta and takes d-dimensional integrals over the loop momenta. For example, one obtains

$$F(1,1;d) = \mathrm{e}^{-\mathrm{i}\pi(1+d/2)/2}\pi^{d/2}\int_0^\infty\int_0^\infty \mathrm{e}^{\mathrm{i}q^2\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}-\mathrm{i}m^2\alpha_1}\frac{\mathrm{d}\alpha_1\mathrm{d}\alpha_2}{(\alpha_1+\alpha_2)^{d/2}}. \quad (1.10)$$

Then one can turn to Feynman parameters, i.e. to (1.6), by changing variables by $\alpha_1 = \eta\xi$, $\alpha_2 = \eta(1-\xi)$ and integrating over η.

We will study the method of Feynman and alpha parameters in Chap. 3, by deriving a lot of useful formulae and considering various examples. The next chapter also deals with parametric representations which are used there to resolve the singularity structure in ε. In contrast to examples of Chap. 3, where some subtractions are used for this purpose when analytically evaluating Feynman integrals, here the goal is to do this in an algorithmic way by introducing so-called sector decompositions which can be used either for analysis of convergence of regularized or renormalized Feynman integrals, or for numerical evaluation.

To illustrate the basic idea of sector decompositions let us turn again to the integral (1.1) which can be represented by (1.10) and reveal its UV divergence. (And let us forget that we did this in a simple way using Feynman parameters (1.6) where the divergence manifested itself as a pole of the overall gamma function.) We cannot expand the integral in ε under the integral sign because the initial term, i.e. its value at $d = 4$ is divergent. In alpha parameters, UV divergences manifest themselves as divergences at small values, so that let us consider just the integral

$$I(\varepsilon) = \int_0^1\int_0^1 \mathrm{d}\alpha_1\mathrm{d}\alpha_2(\alpha_1+\alpha_2)^{\varepsilon-2}f(\alpha_1,\alpha_2), \quad (1.11)$$

where f is regular at the origin.

To reveal the analytic structure in ε near $\varepsilon = 0$ let us decompose the integration domain into two sectors, $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$ and represent I as $I_1 + I_2$. The two integrals are similar so that let us consider only I_1. Let us introduce sector variables by $\alpha_1 = t_1 t_2,\ \alpha_2 = t_2$. We have again an integration over the unit square:

$$I_1(\varepsilon) = \int_0^1\int_0^1 \mathrm{d}t_1\mathrm{d}t_2\ t_2^{\varepsilon-1}\ g(t_1,t_2), \quad (1.12)$$

where $g(t_1,t_2) = (1+t_1)^{\varepsilon-2}f(t_1t_2,t_2)$. Such a form of the integral easily provides the possibility of expanding under the integral sign. To reveal the pole in ε we then write down $g(t_1,t_2)$ as $g(t_1,0)$ plus $g(t_1,t_2) - g(t_1,0)$. Taking explicitly the integration over t_2 in the first term we arrive at

$$I_1(\varepsilon) = \frac{1}{\varepsilon}\int_0^1 \mathrm{d}t_1\ g(t_1,0) + \int_0^1\int_0^1 \mathrm{d}t_1\mathrm{d}t_2\ t_2^{\varepsilon-1}\ [g(t_1,t_2)-g(t_1,0)]. \quad (1.13)$$

We see that we have achieved our goal because the first term is just a simple pole in ε while the second term can be expanded in ε. In Chap. 4, such procedure will be extended to general Feynman integrals and various methods of sector decompositions [2–4, 10, 19] will be described.

A powerful method of evaluating Feynman integrals is based on the Mellin–Barnes (MB) representation [20, 22]. The underlying idea is to replace a sum of terms raised to some power by the product of these terms raised to certain powers, at the cost of introducing an auxiliary integration that goes from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$ in the complex plane. The most obvious way to apply this representation is to write down a massive propagator in terms of massless ones. For $F(2, 1; 4)$, we obtain

$$\frac{1}{(m^2-k^2)^2} = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{(m^2)^z}{(-k^2)^{2+z}} \Gamma(2+z)\Gamma(-z). \tag{1.14}$$

Applying (1.14) to the first propagator in (1.2), changing the order of integration over k and z and evaluating the internal integral over k by means of the one-loop formula (10.7) (which we will derive in Chap. 3) we arrive at the following onefold MB integral representation:

$$\begin{aligned} F(2,1;d) = & -\frac{\mathrm{i}\pi^{d/2}\Gamma(1-\varepsilon)}{(-q^2)^{1+\varepsilon}} \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \\ & \times \frac{\Gamma(1+\varepsilon+z)\Gamma(-\varepsilon-z)\Gamma(-z)}{\Gamma(1-2\varepsilon-z)}. \end{aligned} \tag{1.15}$$

The contour of integration is chosen in the standard way: the poles with a $\Gamma(\cdots+z)$ dependence are to the left of the contour and the poles with a $\Gamma(\cdots-z)$ dependence are to the right of it. If $|\varepsilon|$ is small enough we can choose this contour as a straight line parallel to the imaginary axis with $-1 < \mathrm{Re}z < 0$. For $d=4$, we obtain

$$F(2,1;4) = -\frac{\mathrm{i}\pi^2}{q^2}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(z)\Gamma(-z). \tag{1.16}$$

By closing the integration contour to the right and taking a series of residues at the points $z = 0, 1, \ldots$, we reproduce (1.5). Using the same technique, any integral from the given family can similarly be evaluated.

We will study the method of MB representation in Chap. 5. This method provides the possibility to resolve singularities in ε in an easy way. We will see, through various examples, how one can analytically evaluate rather complicated Feynman integrals. Moreover, this method can be applied almost in an automatic way because various public computer codes for the application of this method are available.

Let us, however, think about a more economical strategy based on IBP relations which would enable us to evaluate any integral (1.2) as a linear combination of some master integrals. Putting to zero dimensionally regularized integrals of $\frac{\partial}{\partial k}\cdot k f(a_1, a_2)$ and $q\cdot\frac{\partial}{\partial k} f(a_1, a_2)$, where $f(a_1, a_2)$ is the integrand in (1.2), and writing down obtained relations in terms of integrals of the given family we obtain the following two IBP relations:

$$d - 2a_1 - a_2 - 2m^2 a_1 \mathbf{1}^+ - a_2 \mathbf{2}^+ (\mathbf{1}^- - q^2 + m^2) = 0, \tag{1.17}$$

$$a_2 - a_1 - a_1 \mathbf{1}^+ (q^2 + m^2 - \mathbf{2}^-) - a_2 \mathbf{2}^+ (\mathbf{1}^- - q^2 + m^2) = 0, \tag{1.18}$$

in the sense that they are applied to the general integral $F(a_1, a_2)$. Here the standard notation for increasing and lowering operators has been used, e.g. $\mathbf{1}^+ \mathbf{2}^- F(a_1, a_2) = F(a_1 + 1, a_2 - 1)$.

Let us observe that any integral with $a_1 \leq 0$ is zero because it is a massless tadpole which is naturally put to zero within dimensional regularization. Moreover, any integral with $a_2 \leq 0$ can be evaluated in terms of gamma functions for general d with the help of (10.3) (which we will derive in Chap. 3). The number a_2 can be reduced either to one or to a non-positive value using the following relation which is obtained as the difference of (1.17) multiplied by $q^2 + m^2$ and (1.18) multiplied by $2m^2$:

$$\begin{aligned}(q^2 - m^2)^2 a_2 \mathbf{2}^+ = (q^2 - m^2) a_2 \mathbf{1}^- \mathbf{2}^+ \\ - (d - 2a_1 - a_2) q^2 - (d - 3a_2) m^2 + 2m^2 a_1 \mathbf{1}^+ \mathbf{2}^- .\end{aligned} \tag{1.19}$$

Indeed, when the left-hand side of (1.19) is applied to $F(a_1, a_2)$, we obtain integrals with reduced a_2 or, due to the first term on the right-hand side, reduced a_1.

Suppose now that $a_2 = 1$. Then we can use the difference of relations (1.17) and (1.18),

$$d - a_1 - 2a_2 - a_1 \mathbf{1}^+ (\mathbf{2}^- - q^2 + m^2) = 0, \tag{1.20}$$

and rewrite it down, at $a_2 = 1$, as

$$(q^2 - m^2) a_1 \mathbf{1}^+ = a_1 + 2 - d + a_1 \mathbf{1}^+ \mathbf{2}^- . \tag{1.21}$$

This relation can be used to reduce the index a_1 to one or the index a_2 to zero. We see that we can now express any integral of the given family as a linear combination of the integral $F(1, 1)$ and simple integrals with $a_2 \leq 0$ which can be evaluated for general d in terms of gamma functions. In particular, we have

$$F(2, 1) = \frac{1}{m^2 - q^2} \left[(1 - 2\varepsilon) F(1, 1) - F(2, 0) \right] . \tag{1.22}$$

At this point, we might stop our activity because we have already essentially solved the problem. However, mathematically (and aesthetically), it is natural to be more curious and wonder about the minimal number of master integrals which form a linearly independent basis in the family of integrals $F(a_1, a_2)$. We will do this in

Chap. 6. We will also consider other simple examples where IBP relations can be solved 'by hand', as in this example.

The are two news. The bad news is that solving IBP relations by hand is hardly possible at the high modern complexity level of practical calculations. The good news is that one can solve IBP relations automatically using various algorithms. The most popular one is the Laporta's algorithm [14, 15] based on solving overconstrained systems of linear equations. There are public codes where this algorithm is implemented. We will turn to this algorithm in Chap. 6 where some other algorithms will be also briefly presented.

Two powerful methods described in this book are based on equations: differential equations for one of them and difference equations for the other one. Within both of them, it is assumed that one can solve IBP relations for the family of Feynman integrals to which a given integral belongs. Practically, these methods are used to evaluate master integrals.

Let us illustrate the method of differential equations (DE) [11–13, 18] again with the help of our favourite example. To evaluate the master integral $F(1,1)$ let us observe that its derivative in m^2 is nothing but $F(2,1)$ (because $\left(\partial/(\partial m^2)\right)\left(1/(k^2-m^2)\right) = 1/(k^2-m^2)^2$) which is expressed, according to our reduction procedure, by (1.22). Therefore we arrive at the following differential equation for $f(m^2) = F(1,1)$:

$$\frac{\partial}{\partial m^2} f(m^2) = \frac{1}{m^2-q^2}\left[(1-2\varepsilon) f(m^2) - F(2,0)\right], \tag{1.23}$$

where the quantity $F(2,0)$ is a simpler object because it can be evaluated in terms of gamma functions for general ε. The general solution to this equation can easily be obtained by the method of the variation of the constant, with fixing the general solution from the boundary condition at $m = 0$. Eventually, the above result (1.7) can successfully be reproduced.

As we will see in Chap. 7, the strategy of the method of DE in much more nontrivial situations is similar: one takes derivatives of a master integral in some arguments, expresses them in terms of original Feynman integrals, by means of some variant of solution of IBP relations, and solves resulting differential equations.

The recently developed method based on difference equations [16] uses relations between Feynman integrals in shifted dimension, d. To illustrate it let us turn again to our favourite example. To evaluate the master integral $F_1(d) \equiv F(1,1;d)$ let us use its alpha representation (1.10) and consider $F_1(d-2)$. Up to simple changes of exponents in the prefactors, the most essential change is the appearance of the extra factor $(\alpha_1+\alpha_2)$ in the integrand. Then each of these two terms can be described as a Feynman integral with a shifted index, i.e. either $F(2,1;d)$ or $F(1,2;d)$. As we will see in Chap. 6 any integral of this family can be reduced to the two master integrals, $F_1(d)$, and $F_2(d) = F(1,0;d)$. (A partial reduction, where $F(2,0;d)$ can be reduced further, to $F_2(d)$, is given by (1.22).) This is how one obtains the following dimensional recurrence relation for the master integral $F_1(d)$:

$$F_1(d-2) = 2\frac{(d-3)x}{(1-x)^2}F_1(d) - \frac{(d-2)(1+x)}{2(1-x)^2}F_2(d), \tag{1.24}$$

where we set $x = q^2/m^2$ and $m = 1$.

We will see in Chap. 8 how this and other similar equations can be systematically solved. In this example, one can arrive at a result in terms of a hypergeometric function which, after using some identity, can be reduced to (1.8). Within this method, one obtains solutions in terms of multiple series with excellent convergence. For one-scale integrals, this provides the possibility to evaluate each term in an ε expansion with a big accuracy and then obtain analytic results in very high orders of this expansion.

As promised in the beginning of this introduction, the semi-analytic method of expansions in limits of momenta and masses [1, 21] is briefly presented in Chap. 9. Let us take again the integral $F(2, 1; d)$ given by (1.2) as an example and study it in the limit $m^2/q^2 \to 0$. As explained in Chap. 9, one can proceed either by expansion by regions, or using an explicit formula for the expansion written in graph-theoretical language. In both cases, one has the sum of two contributions to the expansion. One of them is obtained by expanding the propagator $1/(k^2 - m^2)^2$ in a Taylor series in the mass m and the other one is obtained by expanding the propagator $1/(q-k)^2$ in a Taylor series in the loop momentum k. This and other typical examples are studied in Chap. 9. It will be also explained how to find regions relevant to a given limit by a geometrical algorithm [9, 17].

Before studying these methods, basic definitions are presented in Chap. 2 where tools for dealing with Feynman integrals are also introduced. Methods for evaluating individual Feynman integrals are studied in Chaps. 3–5, 7–9 and the reduction problem is studied in Chap. 6. In Appendix A, one can find a table of basic one-loop and two-loop Feynman integrals as well as some useful auxiliary formulae. Appendix B contains definitions and properties of special functions that are used in this book. A table of summation formulae for onefold series is given in Appendix C. In Appendix D, a table of onefold MB integrals is presented.

Some other methods are briefly characterized in Appendix E. These are mainly old methods whose details can be found in the literature. If I do not present some methods, this means that either I do not know about them, or I do not know physically important situations where they work not worse than the methods I present.

I will use almost the same examples in Chaps. 3–9 and Appendix E to illustrate all the methods. On the one hand, this is done in order to have the possibility to compare them. On the other hand, the methods often work together: for example, MB representation can be used in alpha or Feynman parametric integrals, the methods based on differential and difference equations require a solution of the reduction problem, boundary conditions within the method of DE can be obtained by means of the method of MB representation, etc.

Basic notational conventions are presented below. The notation is described in more detail in the List of Symbols. In the Index, one can find numbers of pages where definitions of basic notions are introduced.

1.1 Notation

We use Greek and Roman letters for four-indices and spatial indices, respectively:

$$x^{\mu} = (x^0, \boldsymbol{x}),$$
$$q \cdot x = q^0 x^0 - \boldsymbol{q} \cdot \boldsymbol{x} \equiv g_{\mu\nu} q^{\mu} x^{\nu}.$$

The parameter of dimensional regularization is

$$d = 4 - 2\varepsilon.$$

The d-dimensional Fourier transform and its inverse are defined as

$$\tilde{f}(q) = \int d^d x \, \mathrm{e}^{\mathrm{i}q \cdot x} f(x),$$
$$f(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d q \, \mathrm{e}^{-\mathrm{i}x \cdot q} \tilde{f}(q).$$

In order to avoid Euler's constant γ_{E} in Laurent expansions in ε, we usually pull out the factor $\mathrm{e}^{-\gamma_{\mathrm{E}}\varepsilon}$ per loop.

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