Hierarchical Bipolar Sugeno Integral Can Be Represented as Hierarchical Bipolar Choquet Integral

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Abstract. Several types of hierarchical fuzzy integral models in decision theory have been proposed and been investigated by many researchers. Some of them aim to simplify the models. Others aim to build a high modeling capability. Recently, the notion of *hierarchical bipolar Sugeno integral* has been proposed by Sugeno and Nakama in order to represent/model almost all *admissible* preference orderings. It is known that the ordinary Sugeno integral can be represented as a hierarchical Choquet integral. However, it has never been known whether the hierarchical bipolar Sugeno integral can be represented as some types of Choquet integrals, or not. This paper will show that the hierarchical bipolar Sugeno integral can be represented as a hierarchical bipolar Choquet integral.

Keywords: hierarchical Sugeno integral, bi-cooperative game, Choquet integral.

1 Introduction

The Choquet and Sugeno integrals are one of the most important integrals with respect to fuzzy measures. In decision theory, the Choquet integral model is well-known and used as a *cardinal* preference model and the Sugeno integral an *ordinal* one. Since 1995, hierarchical Choquet integrals have been studied by many researchers (e.g., Sugeno, Fujimoto, and Murofushi [12,15], Mesiar, Vivona, and Benvenuti [1,9]), Narukawa and Torra [14,17] ...). Recently, the notion of hierarchical bipolar Sugeno integral, i.e., hierarchical Sugeno integral with respect to bi-capacities, has been proposed by Sugeno and Nakama [16,13] in order to model/represent almost all *admissible*, in a natural/common sense, preference orderings. While, Narukawa and Torra [14] investigated relations with the Choquet and Sugeno integrals. They have proved that the Sugeno integral can be represented as a (2-step) hierarchical Choquet integral (with a constant). However, it has never been known whether the bipolar Sugeno integral can be represented as some types of the Choquet integrals, or not. In this paper, we shall show that the hierarchical bipolar Sugeno integral can be represented as a hierarchical bipolar Choquet integral.

2 Preliminaries

Throughout this paper we use the following notations and conventions. Let N be a non empty finite set, $N := \{1, \cdots, n\}$, U_m and B_m a finite set of integers, $U_m := \{0, 1, \cdots, m\}$

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and $B_m = \{-m, \cdots, -1, 0, 1, \cdots, m\}$. For a function $f : N \to U_m$ (resp., $f : N \to B_m$), we denote σ_f a permutation on N satisfying that

$$f(\sigma_f(1)) \le \cdots \le f(\sigma_f(n)) \quad \text{(resp., } |f(\sigma_f(1))| \le \cdots \le |f(\sigma_f(n))|)$$

with convention $f(\sigma_f(0)) = 0$, and $A_{\sigma_f(k)}$, $A^+_{\sigma_f(k)}$, and $A^-_{\sigma_f(k)}$ the subset of N such as

$$A_{\sigma_f(k)} := \{\sigma_f(k), \cdots, \sigma_f(n)\},$$
$$A^+_{\sigma_f(k)} := A_{\sigma_f(k)} \cap \{i \in N \mid f(i) \ge 0\}, \text{ and } A^-_{\sigma_f(k)} := A_{\sigma_f(k)} \setminus A^+_{\sigma_f(k)}$$

for any $k \in N$, respectively. we denote $\mathcal{P}(N) := 2^N = \{S \mid S \subseteq N\}$ and $\mathcal{Q}(N) := 3^N = \{(A_1, A_2) \mid A_1, A_2 \in \mathcal{P}(N), A_1 \cap A_2 = \emptyset\}$. Binary operators $\vee$, $\wedge$ on U_m and $\curlyvee$, $\curlywedge$ on B_m is defined as follows: For any $a, b \in U_m$ and $c, d \in B_m$,

$$a \vee b := \max\{a, b\} \quad \text{and} \quad a \wedge b := \min\{a, b\},$$
$$c \curlyvee d := sign(c + d) \cdot (|c| \vee |d|) \quad \text{and} \quad c \curlywedge d := sign(c \cdot d) \cdot (|c| \wedge |d|),$$

where $sign(c) := -1$ if $c < 0$, $:= 0$ if $c = 0$, and $:= 1$ if $c > 0$. These operators $\curlyvee$ and $\curlywedge$ have been introduced by Grabisch [4,5] in order to define the Sugeno integral with respect to a bipolar scale. Moreover, $\bigsqcup : B_m \times \cdots \times B_m \to B_m$ is defined as follows:

$$\bigsqcup_{i=1}^{n} b_i := \min_{i \in \{1, \cdots, n\}} b_i \curlyvee \max_{i \in \{1, \cdots, n\}} b_i.$$

2.1 Fuzzy Measures and Bi-Capacities

Definition 1 (fuzzy measure [13]). A function $\mu : \mathcal{P}(N) \to U_m$ is a *fuzzy measure* (or capacity) on $\mathcal{P}(N)$ to U_m if it satisfies the following two conditions:
(i) $\mu(\emptyset) = 0$,
(ii) $E, F \in \mathcal{P}(N), E \subseteq F \Rightarrow \mu(E) \le \mu(F)$.
μ is said to be *normalized* if μ satisfies that $\mu(N) = \max U_m = m$.

Definition 2 (bi-capacity [6]). When equipped with the following order $\sqsubseteq$: for arbitrary $(A_1, A_2), (B_1, B_2) \in \mathcal{Q}(N)$,

$$(A_1, A_2) \sqsubseteq (B_1, B_2) \text{ iff } A_1 \subseteq B_1, A_2 \supseteq B_2,$$

a function $v : \mathcal{Q}(N) \to B_m$ is a *bi-capacity* on $\mathcal{Q}(N)$ to B_m if it satisfies the following two conditions:
(i) $v(\emptyset, \emptyset) = 0$,
(ii) $A, B \in \mathcal{Q}(N), A \sqsubseteq B \Rightarrow v(A) \le v(B)$.

If v satisfies that $v(N, \emptyset) = \max B_m = m$ and $v(\emptyset, N) = \min B_m = -m$, v is said to be *normalized*. If v satisfies the condition (i), v is said to be a *bi-cooperative game* [2].

Definition 3 (the Möbius transform [6]). To any bi-cooperative game $v : \mathcal{Q}(N) \to B_m$, another function (bi-cooperative game) $m^v : \mathcal{Q}(N) \to B_l$ for some integer l can be associated by

$$v(A_1, A_2) = \sum_{(B_1, B_2) \sqsubseteq (A_1, A_2)} m^v(B_1, B_2) \quad \forall (A_1, A_2) \in \mathcal{Q}(N).$$

This correspondence proves to be one-to-one, since conversely, for $(A_1, A_2) \in \mathcal{Q}(N)$,

$$m^v(A_1, A_2) = \sum_{\substack{B_1 \subseteq A_1 \\ A_2 \subseteq B_2 \subseteq A_1^c}} (-1)^{|A_1 \setminus B_1| + |B_2 \setminus A_2|} v(B_1, B_2).$$

Definition 4 (support, null set). A subset $S \subseteq N$ is said to be a *support* of N with respect to a bi-cooperative game v on $\mathcal{Q}(N)$ if

$$v(A, B) = v(A \cap S, B \cap S) \quad \forall (A, B) \in \mathcal{Q}(N).$$

So, $N \setminus S$ is said to be a *null set*.

Definition 5 (relative bi-cooperative game). For a bi-cooperative game v on $\mathcal{Q}(N)$ and a subset S on N, the bi-cooperative game w on $\mathcal{Q}(S)$ is said to be the *relative bi-cooperative game* of v on $\mathcal{Q}(S)$ if

$$w(A, B) = v(A, B) \quad \forall (A, B) \in \mathcal{Q}(S).$$

Proposition 1. [3] *Let v be a bi-cooperative game on $\mathcal{Q}(N)$. The following two conditions are equivalent to each other:*

(i) v is a bi-capacity,

(ii) $$\sum_{(C_1, C_2) \sqsubseteq (B_1, B_2) \sqsubseteq (A_1, A_2)} m^v(B_1, B_2) \geq 0$$

for all $(C_1, C_2) \in \mathcal{Q}(N)$ such as $|C_2| = n - 1$ and all $(A_1, A_2) \in \mathcal{Q}(N)$.

Proposition 2. *For any bi-cooperative game v on $\mathcal{Q}(N)$, there exist two bi-capacities v_1 and v_2 on $\mathcal{Q}(N)$ such that*

$$v(A_1, A_2) = v_1(A_1, A_2) - v_2(A_1, A_2) \quad \forall (A_1, A_2) \in \mathcal{Q}(N).$$

Proof. Let m^v be the Möbius transform of v. Then, we define two functions m_1 and m_2 on $\mathcal{Q}(N)$ via v as follows:

$$m_1(A_1, A_2) := \begin{cases} m^v(A_1, A_2) & \text{if } m^v(A_1, A_2) \geq 0 \text{ and } A_2 \neq N. \\ 0 & \text{if } m^v(A_1, A_2) < 0 \text{ and } A_2 \neq N, \end{cases}$$

$$m_2(A_1, A_2) := \begin{cases} -m^v(A_1, A_2) & \text{if } m^v(A_1, A_2) < 0 \text{ and } A_2 \neq N. \\ 0 & \text{if } m^v(A_1, A_2) \geq 0 \text{ and } A_2 \neq N, \end{cases}$$

$$m_1(\emptyset, N) := -\sum_{\substack{(B_1, B_2) \sqsubseteq (\emptyset, \emptyset) \\ B_2 \neq N}} m_1(B_1, B_2), \quad m_2(\emptyset, N) := -\sum_{\substack{(B_1, B_2) \sqsubseteq (\emptyset, \emptyset) \\ B_2 \neq N}} m_2(B_1, B_2).$$

Through these m_1 and m_2, we define v_1 and v_2 as

$$v_i(A_1, A_2) := \sum_{(B_1, B_2) \sqsubseteq (A_1, A_2)} m_i(B_1, B_2) \quad \forall i \in \{1, 2\}.$$

Then, it follows from Proposition 1 that both v_1 and v_2 are bi-capacities. Moreover, for any $(A_1, A_2) \in \mathcal{Q}(N)$,

$$\begin{aligned} v_1(A_1, A_2) - v_2(A_1, A_2) &= \sum_{(B_1,B_2)\sqsubseteq(A_1,A_2)} m_1(B_1, B_2) - \sum_{(B_1,B_2)\sqsubseteq(A_1,A_2)} m_2(B_1, B_2) \\ &= \sum_{(B_1,B_2)\sqsubseteq(A_1,A_2)} m^v(B_1, B_2) \;=\; v(A_1, A_2), \end{aligned}$$

from the definition of the Möbius transform (See, Definition 3). □

2.2 The Sugeno Integral and the Choquet Integral

Let $\mathfrak{F}(N, U_m)$ be the set of all fuzzy measures on $\mathcal{P}(N)$ to U_m, $\mathfrak{B}(N, B_m)$ the set of all bi-capacities on $\mathcal{Q}(N)$ to B_m, and $\mathfrak{B}'(N, B_m)$ the set of all bi-cooperative games on $\mathcal{Q}(N)$ to B_m.

Definition 6 (the Sugeno and Choquet integral). The *Sugeno* (resp., *Choquet*) *integral*, $S_\mu(f)$ (resp., $C_\mu(f)$), of a function $f : N \to U_m$ with respect to $\mu \in \mathfrak{F}(N, U_m)$ is given by

$$S_\mu(f) := \bigvee_{k=1}^{n} \left[f(\sigma_f(k)) \wedge \mu(A_{\sigma_f(k)}) \right], \tag{1}$$

$$C_\mu(f) := \sum_{k=1}^{n} \Big(f(\sigma_f(k)) - f(\sigma_f(k-1)) \Big) \cdot \mu(A_{\sigma_f(k)}). \tag{2}$$

Definition 7 (the bipolar Sugeno and Choquet integral [4,6]). The *Sugeno* (resp., *Choquet*) *integral*, $BS_v(f)$ (resp., $BC_v(f)$), of a function $f : N \to B_m$ with respect to $v \in \mathfrak{B}'(N, B_m)$ is given by

$$BS_v(f) := \bigsqcup_{k=1}^{n} \left[|f(\sigma_f(k))| \wedge v(A^+_{\sigma_f(k)}, A^-_{\sigma_f(k)}) \right], \tag{3}$$

$$BC_v(f) := \sum_{k=1}^{n} \Big(|f(\sigma_f(k))| - |f(\sigma_f(k-1))| \Big) \cdot v(A^+_{\sigma_f(k)}, A^-_{\sigma_f(k)}). \tag{4}$$

Proposition 3. *Suppose that $S \subseteq N$ is a support of N with respect to $v \in \mathfrak{B}'(N, B_m)$ and that w is the relative bi-cooperative game of v on $\mathcal{Q}(S)$. Then for any $f : N \to B_m$,*

$$BC_v(f) = BC_w(f|_S)$$

where $f|_S$ is the function on S such that $f|_S(i) = f(i)\ \forall i \in S$.

Proof. It suffice to prove the case where supports S are represented as

$$S = N \setminus \{k\}$$

for some null $k \in N$ with respect to v. When we denote $g(i)$ instead of $f|_S(i)$, we have a permutation σ_g on $S = N \setminus \{k\}$ such as

$$\sigma_g(i) = \begin{cases} \sigma_f(i) & \text{if } i < \sigma_f^{-1}(k), \\ \sigma_f(i+1) & \text{if } i > \sigma_f^{-1}(k). \end{cases}$$

Then, it is easy to verify, from the fact that $\{k\}$ is a null set, that

$$v(A^+_{\sigma_f(\sigma_f^{-1}(k))}, A^-_{\sigma_f(\sigma_f^{-1}(k))}) = v(A^+_{\sigma_f(\sigma_f^{-1}(k)+1)}, A^-_{\sigma_f(\sigma_f^{-1}(k)+1)}).$$

Under above notations with convention $(A^+_{\sigma_f(n+1)}, A^-_{\sigma_f(n+1)}) = (A^+_{\sigma_g(n)}, A^-_{\sigma_g(n)}) = (\emptyset, \emptyset)$,

$$\begin{aligned}
BC_v(f) &= \sum_{i=1}^{n} |f(\sigma_f(i))| \left[v(A^+_{\sigma_f(i)}, A^-_{\sigma_f(i)}) - v(A^+_{\sigma_f(i+1)}, A^-_{\sigma_f(i+1)})\right] \\
&= \sum_{i<\sigma_f^{-1}(k)-1} |f(\sigma_f(i))| \left[v(A^+_{\sigma_f(i)}, A^-_{\sigma_f(i)}) - v(A^+_{\sigma_f(i+1)}, A^-_{\sigma_f(i+1)})\right] \\
&\quad + |f(\sigma_f(\sigma_f^{-1}(k)-1))| \left[v(A^+_{\sigma_f(\sigma_f^{-1}(k)-1)}, A^-_{\sigma_f(\sigma_f^{-1}(k)-1)}) - v(A^+_{\sigma_f(\sigma_f^{-1}(k))}, A^-_{\sigma_f(\sigma_f^{-1}(k))})\right] \\
&\quad + |f(\sigma_f(\sigma_f^{-1}(k)))| \left[v(A^+_{\sigma_f(k)}, A^-_{\sigma_f(k)}) - v(A^+_{\sigma_f(\sigma_f^{-1}(k)+1)}, A^-_{\sigma_f(\sigma_f^{-1}(k)+1)})\right] \\
&\quad + \sum_{i>\sigma_f^{-1}(k)} |f(\sigma_f(i))| \left[v(A^+_{\sigma_f(i)}, A^-_{\sigma_f(i)}) - v(A^+_{\sigma_f(i+1)}, A^-_{\sigma_f(i+1)})\right] \\
&= \sum_{i<\sigma_f^{-1}(k)-1} |f(\sigma_f(i))| \left[v(A^+_{\sigma_f(i)}, A^-_{\sigma_f(i)}) - v(A^+_{\sigma_f(i+1)}, A^-_{\sigma_f(i+1)})\right] \\
&\quad + |f(\sigma_f(\sigma_f^{-1}(k)-1))| \left[v(A^+_{\sigma_f(\sigma_f^{-1}(k)-1)}, A^-_{\sigma_f(\sigma_f^{-1}(k)-1)}) - v(A^+_{\sigma_f(\sigma_f^{-1}(k)+1)}, A^-_{\sigma_f(\sigma_f^{-1}(k)+1)})\right] \\
&\quad + \sum_{i>\sigma_f^{-1}(k)} |f(\sigma_f(i))| \left[v(A^+_{\sigma_f(i)}, A^-_{\sigma_f(i)}) - v(A^+_{\sigma_f(i+1)}, A^-_{\sigma_f(i+1)})\right] \\
&= \sum_{i<\sigma_f^{-1}(k)-1} |g(\sigma_g(i))| \left[v(A^+_{\sigma_g(i)}, A^-_{\sigma_g(i)}) - v(A^+_{\sigma_g(i+1)}, A^-_{\sigma_g(i+1)})\right] \\
&\quad + |g(\sigma(\sigma_f^{-1}(k)-1))| \left[v(A^+_{\sigma_g(\sigma_f^{-1}(k)-1)}, A^-_{\sigma_g(\sigma_f^{-1}(k)-1)}) - v(A^+_{\sigma_g(\sigma_f^{-1}(k)+1)}, A^-_{\sigma_g(\sigma_f^{-1}(k)+1)})\right] \\
&\quad + \sum_{n-1\geq i\geq\sigma_f^{-1}(k)} |g(\sigma_g(i))| \left[v(A^+_{\sigma_g(i)}, A^-_{\sigma_g(i)}) - v(A^+_{\sigma_g(i+1)}, A^-_{\sigma_g(i+1)})\right] \\
&= \sum_{i=1}^{n-1} |g(\sigma_g(i))| \left[w(A^+_{\sigma_g(i)}, A^-_{\sigma_g(i)}) - w(A^+_{\sigma_g(i+1)}, A^-_{\sigma_g(i+1)})\right] = BC_w(g) = BC_w(f|_S). \ \square
\end{aligned}$$

Definition 8 (hierarchical bipolar Sugeno and Choquet integral [13]). Let L be a positive integer and $\{N_j\}_{j\in\{1,\cdots,L+1\}}$ a family of non empty sets with $N_1 = N$, $N_{L+1} = \{1\}$, and $N_j := \{1, \cdots, n_j\}$ for an integer n_j for all $j \in \{1, \cdots, L+1\} \setminus \{1, L+1\}$. For each $j \in \{1, \cdots, L\}$, let $\{v_i^j\}_{i\in N_{j+1}}$ be a set of bi-cooperative games on $\mathcal{Q}(N_j)$ to B_{m_j}, i.e., $v_i^j \in \mathfrak{B}'(N_j, B_{m_j})$ for any $i \in N_{j+1}$. Then,

$$\mathfrak{H}^L := (\{N_j\}_{j\in\{1,\cdots,L+1\}},\ \{v_i^j\}_{i\in N_{j+1},\ j\in\{1,\cdots,L\}})$$

is called a *system of L-step hierarchical bi-cooperative games*. Then, the L-step hierarchical bipolar Sugeno (resp., Choquet) integral, $HBS_{\mathfrak{H}^L}(f)$ (resp., $HBC_{\mathfrak{H}^L}(f)$), of a function $f : N \to B_m$ with respect to $\mathfrak{H}^L$ is given by the following procedures (e.g., see Fig. 1.):

(i) $F_1(i) := f(i) \ \forall i \in N(= N_1)$,
(ii) $F_j(i) := BS_{v_i^{j-1}}(F_{j-1})$ (resp., $F_j(i) := BC_{v_i^{j-1}}(F_{j-1})$)
$\forall i \in N_j, \quad j \in \{1, \cdots, L+1\} \setminus \{1, L+1\}$.
(iii) $HBS_{\mathfrak{H}^L}(f) := BS_{v_1^L}(F_L)$ (resp., $HBC_{\mathfrak{H}^L}(f) := BC_{v_1^L}(F_L)$).

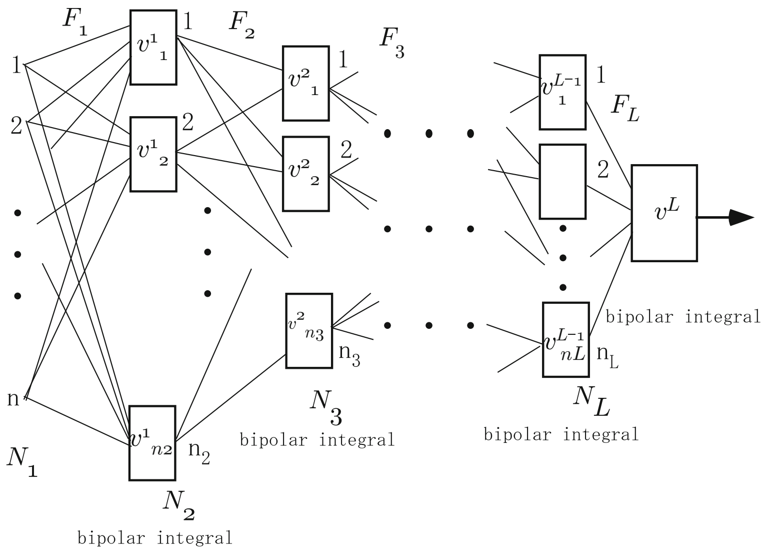

Fig. 1. Hierarchical bipolar Sugeno and/or Choquet integral

3 Lemmas

3.1 Binary Operators ∧, ∨, ⋏, ⋎ and the Bipolar Choquet Integral

Lemma 1. *Let* $N := \{1, 2\}$. *For any* $a, b \in B_m$, $a > 0$, *we define a function* $f : N \to B_m$ *as* $f(1) = a$ *and* $f(2) = b$. *Then*

$$a \curlywedge b = BC_v(f),$$

where

$$v(A, B) = \begin{cases} 1 & \text{if } A = N, \\ 0 & \text{if } A = \emptyset \text{ or } A \neq N,\ B = \emptyset, \\ -1 & \text{otherwise}. \end{cases}$$

Proof. In the case that $b > a$, i.e., $a \curlywedge b = a$, $BC_v(f) = a \cdot v(N, \emptyset) + (b - a)v(2, \emptyset) = a$.
In the case that $a \geq b \geq 0$, i.e., $a \curlywedge b = b$, $BC_v(f) = b \cdot v(N, \emptyset) + (a - b)v(1, \emptyset) = b$.
In the case that $-a \leq b \leq 0$, i.e., $a \curlywedge b = b$, $BC_v(f) = -b \cdot v(1, 2) + (a + b)v(1, \emptyset) = b$.
In the case that $b \leq -a$, i.e., $a \curlywedge b = -a$, $BC_v(f) = a \cdot v(1, 2) + (-b - a)v(\emptyset, 2) = -a$. □

Lemma 2. *Let $N := \{1, 2\}$. For any $a, b \in B_m$, we define a function $f : N \to B_m$ as $f(1) = a$ and $f(2) = b$. Then $a \vee b$ can be represented as a hierarchical bipolar Choquet integral of f as follows:*

(i) Define a set of non empty set N_1 and N_2 as $N_1 = N$ and $N_2 = \{1, 2, 3\}$.
(ii) Define a set of bi-cooperative games $\{v_j^1\}_{j \in N_2}$ in $\mathfrak{B}'(N, B_{2m})$ and v^2 in $\mathfrak{B}'(N, B_1)$ as

$$v_1^1(A,B) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & otherwise, \end{cases}, \quad v_2^1(A,B) = \begin{cases} 0 & \text{if } B = \emptyset, \\ -1 & otherwise, \end{cases}$$

$$v_3^1(A,B) = (|A| - |B|) \cdot m, \quad v^2(A,B) = \begin{cases} 1 & \text{if } A = \{1,3\}, \\ -1 & \text{if } B = \{2,3\}, \\ 0 & otherwise, \end{cases}$$

respectively.
(iii) Define the function $F_2 : N_2 \to B_m$ as $F_2(i) := BC_{v_i^1}(f) \quad \forall i \in N_2$.

Then, we have that $a \vee b = BC_{v^2}(F_2)$.

Proof. Let $f^+(i) := \max\{f(i), 0\}$ and $f^-(i) := \min\{f(i), 0\}$, for $i \in N_1$.

Claim 1 : $F_2(1) = \max_{i \in N} f^+(i)$.
Now we will prove the claim 1. In the case that $b \geq a \geq 0$, i.e., $\max f^+ = b$,
$F_2(1) = BC_{v_1^1}(f) = (b - a) \cdot v_1^1(2, \emptyset) + a \cdot v_1^1(12, \emptyset) = b$.
In the case that $a > b \geq 0$, i.e., $\max f^+ = a$,
$F_2(1) = BC_{v_1^1}(f) = (a - b) \cdot v_1^1(1, \emptyset) + b \cdot v_1^1(12, \emptyset) = a$.
In the case that $a > -b \geq 0$, i.e., $\max f^+ = a$,
$F_2(1) = BC_{v_1^1}(f) = (a + b) \cdot v_1^1(1, \emptyset) + (-b) \cdot v_1^1(1, 2) = a$.
In the case that $-b \geq a \geq 0$, i.e., $\max f^+ = a$,
$F_2(1) = BC_{v_1^1}(f) = (-b - a) \cdot v_1^1(\emptyset, 2) + a \cdot v_1^1(1, 2) = a$.
In the case that $b \geq -a \geq 0$, i.e., $\max f^+ = b$,
$F_2(1) = BC_{v_1^1}(f) = (b + a) \cdot v_1^1(2, \emptyset) + (-a) \cdot v_1^1(2, 1) = b$.
In the case that $-a > b \geq 0$, i.e., $\max f^+ = b$,
$F_2(1) = BC_{v_1^1}(f) = (-a - b) \cdot v_1^1(\emptyset, 1) + b \cdot v_1^1(2, 1) = b$.
In the case that $-a > -b \geq 0$, i.e., $\max f^+ = 0$,
$F_2(1) = BC_{v_1^1}(f) = (-a + b) \cdot v_1^1(\emptyset, 1) + (-b) \cdot v_1^1(\emptyset, 12) = 0$.
In the case that $-b > -a \geq 0$, i.e., $\max f^+ = 0$,
$F_2(1) = BC_{v_1^1}(f) = (-b + a) \cdot v_1^1(\emptyset, 2) + a \cdot v_1^1(\emptyset, 12) = 0$.

Claim 2 : $F_2(2) = \min_{i \in N} f^-(i)$.
This claim can be verified similarly to the proof of Claim 1.

Claim 3 : $F_2(3) = (a + b) \cdot m$.
This claim can also be verified similarly to the proof of Claim 1.

Here, we consider the following three cases:

$$|F_2(1)| > |F_2(2)|, \quad |F_2(1)| < |F_2(2)|, \quad \textit{and} \quad |F_2(1)| = |F_2(2)|.$$

In the case that $|F_2(1)| > |F_2(2)|$,
i.e., $|\max f^+| > |\min f^-|$ and $F_2(3) \geq F_2(1) > -F_2(2) \geq 0$. Then,

$$\begin{aligned} BC_{v^2}(F_2) &= (F_2(3) - F_2(1)) \cdot v^2(3, \emptyset) + (F_2(1) + F_2(2)) \cdot v^2(13, \emptyset) - F_2(2) \cdot v^2(13, 2) \\ &= F_2(1) = \max f^+ = a \vee b, \quad \text{since } \max f^+ \geq 0 \text{ and } |\max f^+| > |\min f^-|. \end{aligned}$$

In the case that $|F_2(1)| < |F_2(2)|$,
i.e., $|\max f^+| < |\min f^-|$ and $-F_2(3) \geq -F_2(2) > F_2(1) \geq 0$. Then,

$$\begin{aligned} BC_{v^2}(F_2) &= (-F_2(3) + F_2(2)) \cdot v^2(\emptyset, 3) + (-F_2(2) - F_2(1)) \cdot v^2(\emptyset, 23) + F_2(1) \cdot v^2(1, 23) \\ &= F_2(2) = \min f^- = a \vee b, \quad \text{since } \min f^- \leq 0 \text{ and } |\max f^+| < |\min f^-|. \end{aligned}$$

In the case that $|F_2(1)| = |F_2(2)|$,
i.e., $|\max f^+| = |\min f^-|$ and $F_2(1) = -F_2(2) \geq F_2(3) = 0$. Then,

$$BC_{v^2}(F_2) = F_2(1) \cdot v^2(1, 2) = 0 = a \vee b, \quad \text{since } a = \max f^+ = -\min f^- = -b. \quad \square$$

Lemma 3. *There exist bi-cooperative games v and $w \in \mathfrak{B}'(N, B_m)$ such that, for any function $f : N \to B_m$,*

$$\max_{i \in N} f^+(i) = BC_v(f) \quad \textit{and} \quad \min_{i \in N} f^-(i) = BC_w(f),$$

where $f^+(i) = \max\{f(i), 0\}$ and $f^-(i) = \min\{f(i), 0\}$ for $i \in N$.

Proof. We put v and w as

$$v(A, B) = \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad w(A, B) = \begin{cases} -1 & \text{if } B \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then, it is easy to verify that

$$\max_{i \in N} f^+(i) = BC_v(f), \quad \min_{i \in N} f^-(i) = BC_w(f). \quad \square$$

3.2 Ordinary Bipolar Sugeno Integral and Hierarchical Bipolar Choquet Integral

Lemma 4. *There exists a system of 4-step hierarchical bi-cooperative games $\mathfrak{H}^4$ such that, for any function $f : N \to B_m$, $\bigsqcup_{i \in N} f(i)$ can be represented as a hierarchical bipolar Choquet integral of f with respect to $\mathfrak{H}^4$, i.e.,*

$$\bigsqcup_{i \in N} f(i) = HBC_{\mathfrak{H}^4}(f).$$

Proof. Let $N_1 = N$, $N_2 = \{1, \cdots, 2n\}$, and $N_3 = \{1, 2\}$. Put $\{v_j^1\}_{j \in N_2}$ as

$$v_j^1(A, B) = \begin{cases} 1 & \text{if } A \ni j, \\ 0 & \text{otherwise} \end{cases} \quad \text{if } j \leq n, \quad v_j^1(A, B) = \begin{cases} -1 & \text{if } B \ni j - n, \\ 0 & \text{otherwise} \end{cases} \quad \text{if } j > n$$

and put $\{v_j^2\}_{j \in N_3}$ as

$$v_1^2(A, B) = \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad v_2^2(A, B) = \begin{cases} -1 & \text{if } B \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Here, we define a function F_2 on N_2 as $F_2(j) := BC_{v_j^1}(f)$. Then

$$F_2(j) = \begin{cases} f^+(j) & \text{if } j \le n, \\ f^-(j-n) & \text{if } j > n. \end{cases}$$

Next, define F_3 on N_3 as $F_3(j) = BC_{v_j^2}(F_2)$. Then, it follows from Lemma 3 that

$$F_3(1) = \max_{i \in N} f^+ \quad \text{and} \quad F_3(2) = \min_{i \in N} f^-.$$

Thus,

$$\bigsqcup_{i \in N} f(i) = F_3(1) \vee F_3(2) \quad \text{since} \quad \bigsqcup_{i \in N} f(i) = \max_{i \in N} f^+ \vee \min_{i \in N} f^-.$$

Then, it follows from Lemma 2 that $\bigsqcup_{i \in N} f(i)$ can be represented as a 4-step bipolar Choquet integral. □

Lemma 5. *Let $N := \{1, \cdots, n\}$, and $v \in \mathfrak{B}(N, B_m)$. Then, for any $k \in N$, there exists a system of 4-step bi-cooperative games, $\mathfrak{H}_k^4$, such that*

$$v(A^+_{\sigma_f(k)}, A^-_{\sigma_f(k)}) = HBC_{\mathfrak{H}_k^4}(g^f)$$

for any function $f : N \to B_m$, where g^f is the function on $N \cup \{0\}$ defined by

$$g^f(i) = \begin{cases} f(i) & \textit{if } i \in N, \\ 1 & \textit{if } i = 0. \end{cases}$$

Proof. We consider a 4-step bipolar Choquet integral of g^f. Let $N_1 := N \cup \{0\} = \{0, \cdots, n\}$, $N_2 := \{0, \cdots, 2n\}$, $N_3 := \{0, \cdots, n\}$, and $N_4 := \{1, \cdots, n\}$. Here, we use the notation v_i^{sign}, on the domain considered, to denote, for any $p \in N$,

$$v_p^{sign}(A, B) := \begin{cases} 1 & \text{if } A \ni p, \\ -1 & \text{if } B \ni p, \\ 0 & \textit{otherwise}. \end{cases}$$

We put a set of bi-cooperative games $\{v_i^1\}_{i \in N_2}$ as follows:

$$v_i^1(A, B) := v_i^{sign}(A, B) \qquad \text{if } i \le n,$$

$$v_i^1(A, B) := \begin{cases} 1 & \text{if } |A \cup B| > n + 1 - k \text{ and } A \ni i - n, \\ -1 & \text{if } |A \cup B| > n + 1 - k \text{ and } B \ni i - n, \\ 0 & \textit{otherwise}, \end{cases} \qquad \text{if } i > n.$$

Here, we define a function $F_2 : N_2 \to B_m$ on N_2 as

$$F_2(i) := BC_{v_i^1}(g^f) \quad \forall i \in N_2.$$

Then,

$$F_2(i) = \begin{cases} g^f(i) & \text{if } i \le n, \\ g^f(i-n) \wedge |g^f(\sigma_{g^f}(k-1))| & \text{if } i > n. \end{cases}$$

Next, we would define $F_3 : N_3 \to B_m$ as

$$F_3(i) := \begin{cases} F_2(0), \ \ i.e., \ 1 & \text{if } i = 0, \\ F_2(i) - F_2(n+i), \ \ i.e., \ g^f(i) - |g^f(\sigma_{g^f}(k-1))| & \text{if } f(i) \geq 0 \text{ and } \sigma_f(i) \geq \sigma_f(k), \\ F_2(i) - F_2(n+i), \ \ i.e., \ g^f(i) + |g^f(\sigma_{g^f}(k-1))| & \text{if } f(i) < 0 \text{ and } \sigma_f(i) \geq \sigma_f(k), \\ F_2(i) - F_2(n+i), \ \ i.e., \ 0 & \text{otherwise,} \end{cases}$$

via some bipolar Choquet integral. To do this, $\{v_i^2\}_{i \in N_3}$ should be $v_0^2 := v_0^{sign}$ and

$$v_i^2(A, B) = \begin{cases} 1 & \text{if } A \ni i \text{ and } A \not\ni n+i, \\ -1 & \text{if } B \ni i \text{ and } B \not\ni n+i, \\ 0 & \text{otherwise,} \end{cases}$$

for $i \in N_2 \setminus \{0\}$. Next, we would define $F_4 : N \to B_m$ as $F_4(i) := F_3(0) \wedge F_3(i)$, i.e.,

$$F_4(i) = \begin{cases} 1 & \text{if } g^f(i) \geq 0 \text{ and } \sigma_{g^f}(i) \geq \sigma_{g^f}(k), \quad \text{i.e, if } f(i) \geq 0 \text{ and } \sigma_f(i) \geq \sigma_f(k), \\ -1 & \text{if } g^f(i) < 0 \text{ and } \sigma_{g^f}(i) \geq \sigma_{g^f}(k), \quad \text{i.e, if } f(i) < 0 \text{ and } \sigma_f(i) \geq \sigma_f(k), \\ 0 & \text{otherwise,} \end{cases}$$

via some bipolar Choquet integral. It is easy, from Lemma 1, to demonstrate the fact. Finally, we have that

$$BC_v(F_4) = v(A^+_{\sigma_f(k)}, A^-_{\sigma_f(k)}).$$

□

Lemma 6. *Let $N := \{1, \cdots, n\}$, For any $k \in N$, there exists a bi-cooperative game $v_k \in \mathfrak{B}'(N, B_m)$ such that $|f(\sigma_f(k))| = BC_{v_k}(f)$ for any function $f : N \to B_m$.*

Proof. We have that $|f(\sigma(k))| = BC_{v^k}(f)$ for any $f : N \to B_m$ via

$$v^k(A, B) := \begin{cases} 1 & \text{if } |A \cup B| > n - k, \\ 0 & \text{otherwise.} \end{cases}$$

□

The next lemma is obtained immediately from Lemmas 1, 5, and 6.

Lemma 7. *Let $N := \{1, \cdots, n\}$, and $v \in \mathfrak{B}(N, B_m)$. Then, for any $k \in N$, there exists a system of 5-step bi-cooperative games, $\mathfrak{H}_k^5$, such that*

$$|f(\sigma_f(k))| \wedge v(A^+_{\sigma_f(k)}, A^-_{\sigma_f(k)}) = HBC_{\mathfrak{H}_k^5}(g^f)$$

for any function $f : N \to B_m$, where g^f is the function on $N \cup \{0\}$ defined by

$$g^f(i) = \begin{cases} f(i) & \text{if } i \in N, \\ 1 & \text{if } i = 0. \end{cases}$$

The next lemma, obtained immediately from Lemmas 4 and 7, shows that the bipolar Sugeno integral can be represented as a hierarchical bipolar Choquet integral.

Lemma 8. *Let $N := \{1, \cdots, n\}$, and $v \in \mathfrak{B}(N, B_m)$. Then, there exists a system of 9-step bi-cooperative games, $\mathfrak{H}_k^9$, such that*

$$\bigsqcup_{k \in N} \left[|f(\sigma_f(k))| \wedge v(A^+_{\sigma_f(k)}, A^-_{\sigma_f(k)}) \right] = HBC_{\mathfrak{H}_k^9}(g^f),$$

i.e.,

$$BS_v(f) = HBC_{\mathfrak{H}_k^9}(g^f)$$

for any function $f : N \to B_m$, where g^f is the function on $N \cup \{0\}$ defined by $f^g(i) := f(i)$ if $i \in N$ and $:= 1$ if $i = 0$.

4 Theorem

Theorem 1. *Let $N := \{1, \cdots, n\}$ and $\mathfrak{H}$ a system of hierarchical bi-cooperative games. Then, there exists another system of hierarchical bi-cooperative games, $\mathfrak{H}'$, such that*

$$HBS_{\mathfrak{H}}(f) = HBC_{\mathfrak{H}'}(g^f)$$

for any function $f : N \to B_m$, where g^f is the function on $N \cup \{0\}$ defined by $f^g(i) := f(i)$ if $i \in N$ and $:= 1$ if $i = 0$.

That is, the hierarchical bipolar Sugeno integral, of any functions with respect to any systems of hierarchical bi-cooperative games, can be represented as a corresponding hierarchical bipolar Choquet integral.

Proof. It is easy to verify from the fact that the hierarchical bipolar Sugeno integral is represented as a hierarchical combination of bipolar Sugeno integrals and that any bipolar Sugeno integral can be represented as a hierarchical bipolar Choquet integral. □

5 Concluding Remarks

In this paper, we show that the hierarchical bipolar Sugeno integral of any function f can be represented as a hierarchical bipolar Choquet integral of g^f which is obtained by extending the domain N of f to $N \cup \{0\}$ (i.e., $g^f(i) = f(i)$ if $i \in N$ and $=1$ if $i = 0$). The Choquet integral of this g^f is essentially the same as *the Choquet integral with constant*, introduced by Narukawa and Torra [14], of f. Moreover, Torra and Narukawa [17] have demonstrated that the bipolar Choquet integral can be represented as a hierarchical CPT-type Choquet integral. That is, the hierarchical bipolar Sugeno integral can be represented as a hierarchical Choquet integral without using bi-capacities (bi-cooperative games).

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