Measure of Inconsistency for the Potential Method

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Abstract. The inconsistency of the decision maker's preferences may be measured as a number of violations of the transitivity rule. If the intensity of the preference is available, then the incosistency may be measured by measuring the inconsistency of each cycle of the preference graph. In the Potential Method, this may be accomplished by mesuring an angle (degree) between the preference flow and the column space of the incidence matrix.

In this article a random study is performed to determine the upper bound for admissible inconsistency. The degree distribution is recognized as the Gumbel distribution and the upper bound for admissible inconsistency measure is defined as a p-quantile ($p = 0.05$) of that distribution.

Keywords: decision making, preference graph, inconsistency measure, condition of order preservation, randomization.

Subject Classifications: 62C25, 90B50, 91B06.

1 Introduction

This paper is about the inconsistency in the decision maker's input data when it is in the form of the preferences obtained from pairwise comparisons. Inconsistency measure is a useful information which shows a degree of non-transitivity in the decision maker's preferences. The high inconsistency measure may suggests reconsidering the input again and again if necessary. For the Eigenvalue Method (EVM), proposed by Saaty [10], the *consistency index* CI is defined as

$$\text{CI}(A) = \frac{\lambda_{\max}(A) - n}{n - 1},$$

where A is the positive reciprocal matrix of order n and $\lambda_{\max}(A)$ is the Perron root of A. It is well-known that $\text{CI} \geq 0$ and $\text{CI}(A) = 0 \iff A$ is consistent. A positive reciprocal matrix A is of *admissible inconsistency* if

$$\text{CI}(A) \leq 0.1 \times \text{MRI}(n)$$

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where MRI(n) is the mean of the random CI. The random index study in AHP context was performed by several authors, from Crawford and Williams [4] to Alonso and Lamata [1]. A nice overview of the results is given in Alonso and Lamata [1, Table 1, p.449].

The Potential Method (PM) (Čaklović [5]) uses a *preference graph* to capture the results of pairwise comparisons. A *preference flow* $\mathcal{F}$ is the non-negative function defined on the set of arcs which captures the intensity of the preferences. An example of the preference graph in a voting procedure was considered by Condorcet [3]. He defined a *social preference* flow as

$$\mathcal{F}_C(u,v) := N(u,v) - N(v,u) \tag{1}$$

where $N(u,v)$ denotes the number of voters choosing u over v. We say that u is *socially preferred* to v if $\mathcal{F}_C(u,v) \geq 0$.

In the graph representation of the preferences, inconsistency is closely related to non-transitivity and may be defined even for incomplete graphs, which is not so straightforward for AHP. In simple words, the flow $\mathcal{F}$ is *consistent* if it is consistent along each cycle c, i.e. if the sum $\mathcal{F}_c$ of the algebraic components of the flow $\mathcal{F}$ along each cycle c is equal to zero, see Definition 1 and Theorem 1. In

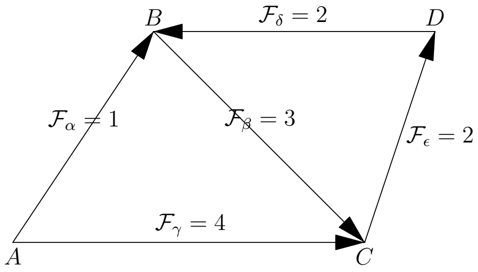

Fig. 1. An example of the inconsistent flow. The sum of the flow components along the cycle $CDBC$ is equal to $2+2+3=7$. The flow is consistent along the cycle $ABCA$.

Figure 1, the flow $\mathcal{F}$ is consistent along the cycle $ABCA$ and inconsistent along the cycle $CDBC$ because the sum $\mathcal{F}_c$ along this cycle is equal to $2+2+3=7$. Intuitively, the inconsistency measure of the flow may be defined as the sum $\sum_c \mathcal{F}_c$ over all independent cycles c in the preference graph divided by the 2-norm $\|\mathcal{F}\|_2$. The exact definition is slightly different: this is the angle between the flow $\mathcal{F}$ and the vector space of all consistent flows, see Definition 3. Please, note that the notion of the flow consistency, as defined here, is stronger than pure transitivity. This motivates the search for the upper bound of the admissible inconsistency of the given flow which is done in Section 4.

Saaty's inconsistency [10, Saaty] and flow inconsistency are closely related. There is a theorem which states that a positive reciprocal matrix $A = (a_{ij})$ is consistent if and only if

$$a_{ij}a_{jk} = a_{ik}, \quad i,j,k = 1,\ldots,n \tag{2}$$

Taking the logarithm of this relation, one recognizes the inconsistency condition *4* from Theorem 1 for the flow

$$\mathcal{F}_{(i,j)} := \log(a_{ij}). \tag{3}$$

In the stochastic preference approach [6, French, p. 101] the author introduces a notion of the stochastic preference p_{ab} as a probability of choosing a when offered a choice between a and b. Then, it is easy to show that if the stochastic preference satisfies the consistency condition

$$\frac{p_{ab}}{p_{ba}} \cdot \frac{p_{bc}}{p_{cb}} = \frac{p_{ac}}{p_{ca}} \tag{4}$$

for all $a, b, c \in V$ then, it generates a weak preference order on the set of alternatives V. If we define a *stochastic flow* $\mathcal{F}$ by

$$\mathcal{F}_{(b,c)} := \log \frac{p_{bc}}{p_{cb}}, \tag{5}$$

then, the stochastic preference is consistent if and only if the flow $\mathcal{F}$ is consistent[1].

Another kind of inconsistency may be considered after the ranking procedure is over. This is the *number of violations of Condition of Order Preservation* (#VCOP) introduced in Costa-Vansnick [2]. This number indicates how far from the measurable value function is the calculated potential. The precise definition of COP is given in (15). The aim of this article is to investigate the correspondence of #VCOP and the flow inconsistency. This is done by performing a random study inspired by the random index study for EVM.

The paper is organized as follows. In Section 2 we introduce the basic notation, develop the idea of consistency and give some other equivalent conditions of consistency.

Section 3 describes the connection of PM with the Geometric Mean, the Ordinal value function and the Stochastic preference model. Some elementary facts about the PM as the social preference are mentioned.

In Section 4 we perform a randomization procedure to determine the admissible level of the flow inconsistency. The Analytic Hierarchy Process (AHP), more precisely the Eigenvalue Method (EVM), serves as a model. It is shown that the empirical distribution of the inconsistency measure DEG may be modeled as the Gumbel distribution. The upper bound for admissible inconsistency is defined as the 0.05-quantile of the theoretical distribution. The randomization is performed for complete graphs only because of the possibility to make a comparison with AHP.

In Section 5 the Condition of Order Preservation (COP) is considered and the number of violations #VCOP is calculated for the random graph and the random reciprocal matrix. It is shown that the *consistency index* (CI) is not correlated with #VCOP while the correlation between the random degree DEG and #VCOP is very good.

[1] Moreover, it may be shown that the corresponding potential X from formula (7) is a measurable value function which is true for every consistent flow.

2 Preference Graph

2.1 Consistent Preference Flow

A *preference graph* is a digraph $\mathcal{G} = (V, \mathcal{A})$ where V is the set of nodes and $\mathcal{A}$ is the set of arcs of $\mathcal{G}$. We say that the node a is *more preferred* than node b, in notation $a \succcurlyeq b$, if there is an arc (a, b) outgoing from b and in-going to a. A *preference flow* is a non-negative real function $\mathcal{F}$ defined on the set of arcs. The value $\mathcal{F}_\alpha$ on the arc α is the intensity of the preference on some scale[2]. For the arc $\alpha = (a, b)$, $\mathcal{F}_\alpha = 0$ means that the decision maker is indifferent to the pair $\{a, b\}$. In that case the orientation of the arc may be arbitrary.

The *incidence matrix* $A = (a_{\alpha,v})$ of the graph is defined as the $m \times n$ matrix, $m =$ Card $\mathcal{A}, n =$ Card V, where

$$a_{\alpha,v} = \begin{cases} -1, \text{if the arc } \alpha \text{ leaves the node } v \\ \ \ 1, \text{if the arc } \alpha \text{ enters the node } v \\ \ \ 0, \text{otherwise.} \end{cases}$$

It is more convenient to write a_{ij} where i is the index of i-th arc and j is the index of j-th node. The vector space $\mathbb{R}^m$ is called the *arc space* and the vector space $\mathbb{R}^n$ is called the *vertex space*. The incidence matrix[3] generates an orthogonal decomposition

$$N(A^\tau) \oplus R(A) = \mathbb{R}^m \tag{6}$$

where $R(A)$ is the column space of the matrix A and $N(A^\tau)$ is the null-space of the matrix A^τ. $N(A^\tau)$ is called the *cycle space* because it is generated by all cycles of the graph.

For example, the incidence matrix of the preference graph in Fig. 1 is given in Table 1 (left). The arcs $\alpha, \beta, \gamma, \delta, \epsilon$ form the basis of the ars space. In the last column are the components of the preference flow. Please, note that the cycles c_1 and c_2 (the basis of the cycle space) are orthogonal to the columns of the incidence matrix according to (6). The columns of the incidence matrix span the space $R(A)$ of the consistent flows.

Table 1. Incidence matrix of the preference graph from Table 1 and the cycle space

	nodes$_n$				flow
arcs$_m$	A	B	C	D	$\mathcal{F}$
α	−1	1	0	0	1
β	0	−1	1	0	3
γ	−1	0	1	0	4
δ	0	1	0	−1	2
ϵ	0	0	−1	1	2

	cycle space	
arcs$_m$	c_1	c_2
α	1	0
β	1	1
γ	−1	0
δ	0	1
ϵ	0	1

[2] For subjective pairwise comparisons the scale is $\{0, 1, 2, 3, 4\}$.
[3] And matrix in general.

Definition 1. *A preference flow $\mathcal{F}$ is consistent if there is no component of the flow in the cycle-space.*

The following theorem is evident (the proof is left to the reader).

Theorem 1. *The following statements are equivalent:*
1. $\mathcal{F}$ *is* consistent.
2. *$\mathcal{F}$ is a linear combination of the columns of the incidence matrix A.*
3. *There exists $X \in \mathbb{R}^n$ such that $AX = \mathcal{F}$.*
4. *The scalar product $y^\tau \mathcal{F} = 0$ for each cycle y, i.e. $\mathcal{F}$ is orthogonal to the cycle space.*

We may test the consistency of the given flow $\mathcal{F}$ by solving the equation

$$AX = \mathcal{F}. \tag{7}$$

Definition 2. *A solution of the equation $AX = \mathcal{F}$, if it exists, is called the potential of $\mathcal{F}$.*

Evidently, X is not unique because the constant column $\mathbb{1}^\tau = [\ 1\ \cdots\ 1\ ]^\tau$ is an element of the kernel $N(A)$. For the consistent flow the equation (7) may be rewritten as

$$\mathcal{F}_\alpha = X(a) - X(b),\ \forall \alpha = (a, b) \in \mathcal{A} \tag{8}$$

which means that X is a measurable value function i.e. it measures the preference on the interval scale. For the consistent flow it is easy to find a potential X using a spanning tree of the preference graph (if it is connected). The details are left to the reader.

PM calculates the weights of the nodes in the following way. If X denotes the potential of the flow, then the weights w are obtained using the formula

$$w = \frac{a^X}{\|a^X\|_1} \tag{9}$$

where $\|\cdot\|_1$ represents l_1-norm. The exponential function $X \mapsto a^X$ is defined by the components and $a > 1$ is a positive constant. Currently, we use the value $a = 2$ but the user may precise some other value. The arguments for such definition is that the flow $\mathcal{F}$, and the potential X, are the logarithms of the data on the ratio scale and we should go back on that scale by exponential function.

2.2 Potential of the Inconsistent Preference Flow

In practice, a decision maker, while performing pairwise comparisons, does not give the flow which is necessarily consistent. The best approximation of that flow by the column space of the incidence matrix may be calculated in this situation. The *approximative potential* or *potential* X is a solution of the Laplace equation

$$A^\tau AX = A^\tau \mathcal{F}, \qquad \sum_{v \in V} X(v) = 0 \tag{10}$$

where the second requirement is for uniqueness.

Because of the linearity of the equations it is evident that X is invariant on the multiplication of $\mathcal{F}$ by a positive number, i.e. $X(\alpha\mathcal{F}) = \alpha X(\mathcal{F}),\ \alpha > 0$. In other words $\mathcal{F}$ is measured on the ratio scale.

It is easy to prove that for the complete flow (the proof is left to the reader):

$$X(v) = \frac{1}{n}\Bigg(\sum_{\alpha \in \mathrm{In}(v)} \mathcal{F}_\alpha - \sum_{\alpha \in \mathrm{Out}(v)} \mathcal{F}_\alpha \Bigg), \tag{11}$$

where $\mathrm{Out}(v)$ and $\mathrm{In}(v)$ denote the set of all outgoing and in-going arcs for v. Formula (11) may be simplified by introducing the *flow matrix* F

$$F_{ij} = \begin{cases} \mathcal{F}_{(i,j)} & \text{if } (i,j) \in \mathcal{A}, \\ -\mathcal{F}_{(j,i)} & \text{if } (j,i) \in \mathcal{A}, \end{cases}$$

with the convention $F_{ii} = 0$. The matrix F is anti symmetric and the potential X, defined by (11), is the arithmetic mean of the columns of F, i.e.

$$x_i = \frac{1}{n}\sum_{j=1}^{n} F_{ij},\ i = 1, \ldots, n. \tag{12}$$

Definition 3. *Measure of inconsistency of the flow $\mathcal{F}$, in notation* DEG($\mathcal{F}$), *is defined as the angle between $\mathcal{F}$ and the column space of the incidence matrix.*

Evidently, DEG($\mathcal{F}$) is the angle between $\mathcal{F}$ and AX, where AX is the consistent approximation of $\mathcal{F}$ and DEG($\mathcal{F}$) $= 0$ if and only if $\mathcal{F}$ is consistent. In case when DEG($\mathcal{F}$) $= \pi/2$ then, there is no transitivity at all in the preference graph.

3 Potential and Other Methods

3.1 Potential and Geometric Mean

In AHP the results of pairwise comparisons are measured on the ratio scale and stored in a positive reciprocal matrix A. The logarithm of A, taken by components,

$$F_{ij} = \log_a a_{ij}, \quad a > 0$$

is an anti-symmetric matrix F which is the flow matrix of some flow $\mathcal{F}$. The potential X of $\mathcal{F}$ may be expressed in terms of the matrix A, using the formula (12), as

$$x_i = \frac{1}{n}\sum_j F_{ij} = \frac{1}{n}\sum_j \log_a a_{ij} = \log_a \Big(\prod_j a_{ij}\Big)^{\frac{1}{n}},$$

and the weight w_i, using (9), may be written as the row geometric mean

$$w_i = \Big(\prod_j a_{ij}\Big)^{\frac{1}{n}},\ i = 1, \ldots, n.$$

3.2 Potential as Ordinal Value Function

Suppose, for the moment, that $\mathcal{F}$ is an uni-modular flow, i.e. $\mathcal{F}_\alpha \in \{0,1\}, \forall \alpha \in \mathcal{A}$. In that case, we may define the relation

$$u \succcurlyeq_{\mathcal{F}} v \Leftrightarrow \mathcal{F}_{(u,v)} \geq 0.$$

If $\succcurlyeq_{\mathcal{F}}$ is a weak preference relation, then the potential X is an ordinal value function, i.e.

$$\mathcal{F}_{(a,b)} \geq 0 \Leftrightarrow X(a) - X(b) \geq 0.$$

The proof may be found in Čaklović [5].

3.3 Potential of the Stochastic Flow

For the complete stochastic flow defined by the formula (4) we may calculate the potential X using the formula (12) and formula (5).

$$X(a) = \frac{1}{n} \sum_{b \neq a} F_{ab} = \log \left(\prod_{b \neq a} \frac{p_{ab}}{p_{ba}} \right)^{\frac{1}{n}}$$

and the weight of the node a is, by formula (9),

$$w_a = \left(\prod_{b \neq a} \frac{p_{ab}}{p_{ba}} \right)^{\frac{1}{n}}.$$

3.4 Potential as the Social Preference

Let us give a few comments about the social preference and the PM. The starting point is the Condorcet flow $\mathcal{F}_C$ defined by (1). There are two possibilities how PM may be used for ranking the candidates. One of them is *direct PM ranking*, and another one is *indirect PM ranking*. The first one calculates the potential X of the Condorcet flow $\mathcal{F}_C$, and another one calculates the potential X^u of the unimodular flow $\mathcal{F}_C^u$, where the uni-modular flow of a given flow is obtained by taking the sign of the given intensity. A candidate with the maximal X^u value we call the PM-winner.

It is easy to prove that the Condorcet winner[4] is the PM-winner. This may not be true if the social ranking is taken to be direct PM-ranking. It can be also proved the PM winner is in the *minimal domination set*, and that the indirect PM ranking is *clone independent*. The exhaustive list of the social choice properties of PM is under the reconstruction.

[4] The Condorcet winner is defined as the winner in all pairwise confrontations with other candidates.

4 Admissible Inconsistency

In this section we shall determine the distribution of the inconsistency measure of the random flow (Definition 3). For $n \geq 4$ this distribution is recognized as the Gumbel distribution

$$e(x) = \frac{e^{-e^{\frac{-x+\alpha}{\beta}} + \frac{-x+\alpha}{\beta}}}{\beta}, \tag{13}$$

which parameters depend[5] upon the number of nodes in the graph, see Table 2. For instance, if the randomization is made as a log-normal perturbation of the random consistent flow defined in formula (14), the inconsistency measure is the Gumbel Distribution $E(\alpha = 17.61, \beta = 7.03)$ (Figure 4).

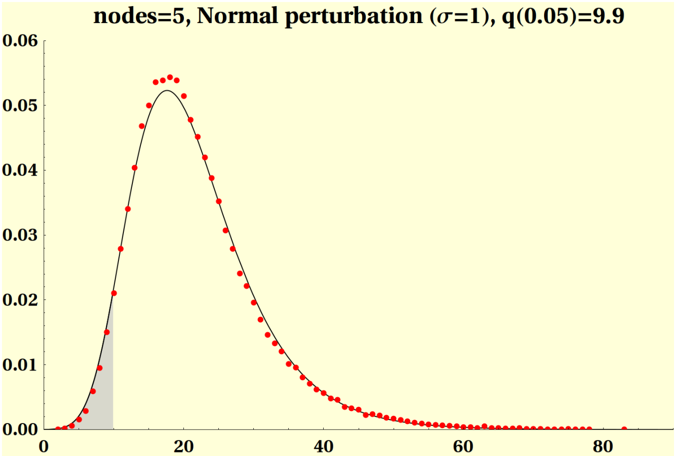

Fig. 2. The simulated distribution of the inconsistency measure (dots). 0.05-quantile (9.9) is taken as the upper bound for admissible inconsistency (10^5 simulations).

4.1 Randomization

A random index study in the AHP context was performed by several authors, from Crawford and Williams [4] to Alonso and Lamata [1]. An overview of the results is given in [1, Table 1, p. 449].

The randomization of the preferences may be designed, generally speaking, as: random *perturbation* (of the consistent situation) and random *distribution*. We performed the following randomizations:

1. normal perturbation of the consistent flow (reciprocal matrix),

[5] We also performed a uniform perturbation of the consistent flow and the results are slightly different.

Table 2. Quantiles of random degree as a function of the nodes number (10^5 simulations)

		The Gumbel Distribution $E(\alpha, \beta)$			
number	perturbation	0.05-quantile		Parameters	
of nodes	($\sigma = 1$)	from data	theoretical	α	β
4	normal	6	5.3	15.01	8.83
5	normal	10	9.9	17.61	7.03
6	normal	13	12.7	19.18	5.91
7	normal	15	14.7	20.24	5.07
8	normal	16	16.1	21.03	4.47
9	normal	17	17.2	21.64	4.02
10	normal	18	18.0	22.06	3.67
11	normal	18	18.8	22.49	3.38
12	normal	19	19.3	22.77	3.16
13	normal	19	19.8	23.04	2.96
14	normal	20	20.2	23.28	2.79
15	normal	20	20.6	23.47	2.66

2. uniform perturbation of the consistent flow (reciprocal matrix),
3. unrestricted randomization of the flow (reciprocal matrix).

We present here only the results of the first type of perturbation. The results of the uniform perturbation are just slightly different and the unrestricted randomization generates highly inconsistent flows with the average greater than 50^o. We believe that the decision maker's preferences in real life are well described by the first process.

A random positive reciprocal matrix is obtained as the normal perturbation of the random consistent reciprocal matrix with elements

$$a_{ij} = w_{ij} * \exp(N(0, \sigma)) \tag{14}$$

where $\sigma = 1$ and $w_{ij} := int(rand(1\text{-}9))^{\alpha}$ $(i < j)$, is the random choice from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, powered by α which is the random choice from $\{-1, 1\}$, and $w_{ij} := w_{ji}^{-1}$, for $j < i$, and $w_{ii} = 1$, $\forall i = 1, \ldots, n$.

The random consistent flow is made by random choice of the orientation of the arc and by random choice of the flow value in the set $\{0, 1, 2, 3, 4\}$. Normal perturbation has a standard deviation $\sigma = 1$. The randomization procedure was performed by `Perl` and data analysis was done by `R`.

In the AHP context our results are exactly the same as in Noble [9].

4.2 Admissible Inconsistency

It seems reasonable to determine the upper bound for admissible inconsistency as a p-quantile of the theoretical random degree distribution. Those values for

$p = 0.05$ are given in Table 2 in the column *theoretical*. The quantiles of the generated data are given in the column *from data*.

If the number of vertices in the preference graph is $n = 3$ the distribution is not the Gumbel distribution. The reason may be in the severe restriction on the stochastic flow values, i.e $\mathcal{F}_\alpha \in \{0, 1, 2, 3, 4\}$. The randomization in this case should be recalculated in a slightly different way, perhaps with less restrictions.

5 Condition of Order Preservation

We say that the potential X satisfies the Condition of Order Preservation (COP) if

$$\mathcal{F}_{(i,j)} > \mathcal{F}_{(k,l)} \implies X_i - X_j > X_k - X_l. \tag{15}$$

In contrast to the measure of inconsistency $\text{DEG}(\mathcal{F})$, which is an 'a priori' inconsistency measure, the number of violations of COP may be regarded as an 'a posteriori' measure of inconsistency which shows 'how far' the calculated potential X is from the measurable value function.

For a reciprocal positive matrix A, we say that COP is satisfied if

$$(a_{ij} > 1 \;\&\; a_{kl} > 1 \;\&\; a_{ij} > a_{kl}) \implies \frac{w_i}{w_j} > \frac{w_k}{w_l},$$

where w is the Perron eigenvector of A.

In this section we present the results of a statistical comparison of the number of violations of the COP (#VCOP) between EVM and PM. For this purpose we performed 10^4 simulations of 5×5 positive reciprocal matrix. For each randomly generated reciprocal matrix we calculate its consistency index CI and #VCOP(EVM) for EVM. Then, we calculate the measure of inconsistency $\text{DEG}(\mathcal{F})$ of the flow $\mathcal{F}$ defined by formula (3), and #VCOP(PM) generated by PM. The correlation matrix of the random vector (DEG, #VCOP(PM), CI, #VCOP(EVM)) is given in Table 3. The correlation between CI and #VCOP equals 0.460128 while the correlation between DEG and #VCOP equals 0.811266 which suggests that DEG may better predict the #VCOP than CI (in average). We do not impose the zero value of #VCOP as a standard, we just want to say that #VCOP gives some new information about the inconsistency from the metric topology's point of view.

Table 3. Correlation matrix of the random vector (DEG, #VCOP(PM), CI, #VCOP(EVM))

	DEG	#VCOP(PM)	CI	#VCOP(EVM)
DEG	1.	0.811266	0.55142	0.817288
#VCOP(PM)		1.	0.449875	0.950875
CI			1.	0.460128
#VCOP(EVM)				1.

5.1 Post Festum

The consistency ratio has been criticized because it allows contradictory judgments in matrices (Bana e Costa and Vansnick [2]) or rejects reasonable matrices (Karapetrovic and Rosenbloom [7]). Several authors (Wang-Chin-Luo [11], Korhonen [8]) argued that the implicit information about priority judgments in the AHP matrix should be taken into account and that the above criticism is not justified. That implicit information, according to them, is of the form $a_{ij}a_{jk}$ which is the element of A^2. But A^2 is 'more consistent'[6] than A regarding the iterative procedure $w_n = A^n e / \|A^n e\|, n \in \mathbb{N}$ of obtaining the Perron vector. The consistency of A should not be measured by the 'consistency' of A^2. It seems that the criticism of the criticism is not well-founded either.

During the randomization process we found an 4×4 AHP matrix with $\lambda_{max} = 4.107$ and the consistency ratio CR $= 0.04$, while its inconsistency degree is DEG $= 73.278$. According to Table 2, the upper bound for admissible inconsistency is 5.3 degrees. Here is the matrix:

$$A = \begin{pmatrix} 1. & 1.024 & 0.852 & 1.521 \\ 0.976 & 1. & 1.41 & 0.719 \\ 1.174 & 0.709 & 1. & 1.197 \\ 0.658 & 1.391 & 0.835 & 1. \end{pmatrix}$$

The preference graph associated with this matrix is given in Figure 3.

The reader who is more familiar with graphs may immediately conclude from Figure 3 that the inconsistency degree of the flow is high. First, a spanning tree should be chosen. In our example, the maximal spanning tree of the graph is drawn (solid line), together with the corresponding chords (dashed lines) which generate the base in the cycle space. Here we have 3 basic cycles, one for each chord. For example, the chord $2 \to 4$ generates the cycle $2 \to 4 \to 3 \to 1 \to 2$, and the sum of the flow components along this cycle equals $0.476 + 0.259 - 0.231 - 0.034 = 0.47$. This number is written in the parentheses beside the flow component (Σ-value). This value is also the scalar product of the flow with this cycle.

According to the decomposition (6), the arc space may be decomposed as the orthogonal sum of the cycle space and the range space of the incidence matrix M which is given bellow. To be a bit more precise let us fix the canonical base in the arc space: $(2 \to 1, 1 \to 3, 4 \to 3, 3 \to 2, 2 \to 4, 4 \to 1)$. The first 3 elements of the base are the arcs of the spanning tree, the rest are the chords. The cycle space is generated by the first three columns of the matrix B

[6] It is meaningless to speak about the consistency index of A^2 because it is not reciprocal and its Perron root is generaly smaller that the dimension of A. But its columns are closer to the Perron eigenvector than those of A and from this point of view we may say that it is more consistent than A.

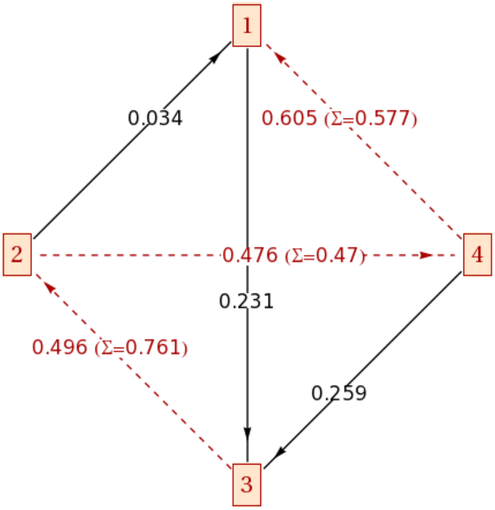

Fig. 3. The flow obtained from the matrix A

$$M = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad B = \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

and the other three columns form the base of the column space of the incidence matrix. For example, the first column of B is the cycle $3 \to 2 \to 1 \to 3$ determined by the chord $3 \to 2$. Σ-values are the scalar products of the flow

$$\mathcal{F} = (0.034, 0.231, 0.259, 0.496, 0.476, 0.605)$$

with the first 3 columns of the matrix B. Because of the high Σ-values it is obvious that the flow inconsistency is also high. The precise calculation gives the inconsistency of 73.278 degrees.

6 Conclusion

This paper explains the properties of the Potential Method and the randomization procedure for obtaining the upper bound for admissible inconsistency of the input data. The upper bound is determined as a 0.5-quantile of the theoretical distribution if DEG which is recognized as the Gumbel distribution (Table 2).

A comparison with the Eigenvalue Method is given, regarding the correlation of the inconsistency measure and the number of violations of the Condition of Order Preservation (COP). The Potential Method and the Eigenvalue Method are 'equally good' from the point of view of the number of violations (#VCOP) of the COP. On the other side, the inconsistency measure of PM correlates better with #VCOP than the consistency index of AHP (Table 3).

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