Maximum Induced Multicliques and Complete Multipartite Subgraphs in Polygon-Circle Graphs and Circle Graphs

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Abstract. A graph is a *multiclique* if its connected components are cliques. A graph is a *complete multipartite graph* if it is the complement of a multiclique. A graph is a *multiclique-multipartite graph* if its vertex set has a partition U, W such that $G(U)$ is complete multipartite, $G(W)$ is a multiclique and every two vertices $u \in U$, $v \in W$ are adjacent. We describe a polynomial time algorithm to find in polygon-circle graphs a maximum induced complete multipartite subgraph containing an induced $K_{2,2}$. In addition, we describe polynomial time algorithms to find maximum induced multicliques and multiclique-multipartite subgraphs in circle graphs. These problems have applications for clustering of proteins by PPI criteria.

Keywords: polygon-circle graph, circle graph, induced multiclique, induced complete multipartite subgraph, Protein-Protein-Interaction.

1 Introduction

We consider only finite graphs $G(V,E)$ with no parallel edges and no self-loops, where V is the set of vertices and E the set of edges. For $U \subseteq V$, $G(U)$ is the subgraph induced by U. We denote $N(v)=\{ u \mid u \text{ adjacent to } v\}$ and $N[v]=N(v) \cup \{v\}$. The *complement* of a graph G is denoted *coG*. A graph is a *multiclique* if its connected components are cliques. A graph is a *complete multipartite graph* if its vertex set has a partition into independent sets such that every two vertices in different independent sets are adjacent, that is, its complement is a multiclique. A graph $G(V,E)$ is a *multiclique-multipartite graph* if its vertex set has a partition U, W such that $G(U)$ is complete multipartite, $G(W)$ is a multiclique and every two vertices $u \in U$, $v \in W$ are adjacent.

A graph G is an *intersection graph* of a family S of subsets of a set if there is a one-to-one correspondence between the vertices of G and the subsets in S such that two vertices are adjacent if and only if their corresponding subsets in S intersect [19]. Intersection graphs of intervals on a line are called *interval graphs* [19]. *Polygon-circle graphs* [15] are intersection graphs of families of convex polygons inscribed in a circle. *Circle graphs* are intersection graph of families of chords in a circle [3,6]. A transitively orientable graph is called a *comparability graph* [12,19]; a vertex is a *source* if all its edges are outgoing and is a *sink* if they are incoming.

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A *dissociation set of a graph* is a vertex set which induces a subgraph whose connected components are edges or single vertices. In a bipartite graph, finding a maximum induced multiclique is NP-complete since it is the problem of finding a maximum dissociation set, while finding a maximum induced complete multipartite subgraph is polynomial [22].

A graph is *weakly-chordal* if it has no holes or antiholes with five or more vertices. Cameron and Hell [1] described for these graphs a polynomial time algorithm for maximum weight dissociation sets, using the algorithm in [20] for maximum weight independent sets in weakly-chordal graphs.

In the present paper we describe a polynomial time algorithm to find in polygon-circle graphs a maximum induced complete multipartite subgraph containing an induced $K_{2,2}$. This algorithm can also be applied to the *circle n-gon graphs* and the *circle trapezoid graphs*, analyzed in [11]. In addition, we describe polynomial time algorithms to find maximum induced multicliques and multiclique-multipartite subgraphs in circle graphs. These problems are NP-complete for general graphs [5]. The partition of all the vertices of a graph into a given number of independent sets and cliques, with various restrictions on mutual interconnections, was discussed in [2,13]. Note that the recognition problem of polygon-circle graphs is NP-complete [18].

Gavril [10] described a polynomial time algorithm for maximum induced bicliques in polygon-circle graphs, using separation by chords. This algorithm can be extended to find maximum induced multicliques with a constant number k of cliques, by considering all combinations of k chords in the circle.

The above problems have applications when a given set of entities related by some property, must be clustered into cliques and independent sets by some strongly connected vs. non-connected or similarity vs. dissimilarity criteria. For example, in Protein-Protein-Interaction (PPI) problems, the proteins must be clustered into strongly interacting groups, with weak or no interaction between the groups. The criteria for clustering proteins are lock-and-key criteria [17], complementary domains criteria [21], domain-domain interaction criteria [14] or interacting motifs criteria [16].

In Section 2 we describe a representation of polygon-circle graphs on a line. In Section 3 we describe an algorithm for maximum induced complete multipartite subgraphs containing an induced $K_{2,2}$ in polygon-circle graphs. In Sections 4,5 we describe algorithms for maximum induced multiclique and multiclique-multipartite subgraphs in circle graphs.

2 Representation of Polygon-Circle Graphs on a Line

Consider an intersection representation of G by polygons on a circle CR and let Z be a point on CR distinct from any corner point (Figure 1(a)). For every polygon with more than one chord we delete its chord facing Z, that is, its chord delimiting the arc containing Z. The intersection relationship does not change since two intersecting polygons have two pairs of crossing chords. Now, we open CR at Z, straighten CR into a line L (Figure 1(b)), and transform every chord into a semicircle arc above L through the chord's endpoints on L. The intersection relationship does not change since two chords in *CR* are not crossing if and only if their corresponding semicircle arcs are not intersecting. The remaining boundary of every polygon becomes a sequence of semicircle arcs with their endpoints on *L*, called *polygon-filament*. The reverse process is also true, thus a graph is a polygon-circle graph if and only if it is the intersection graph of a family of polygon-filaments on a line.

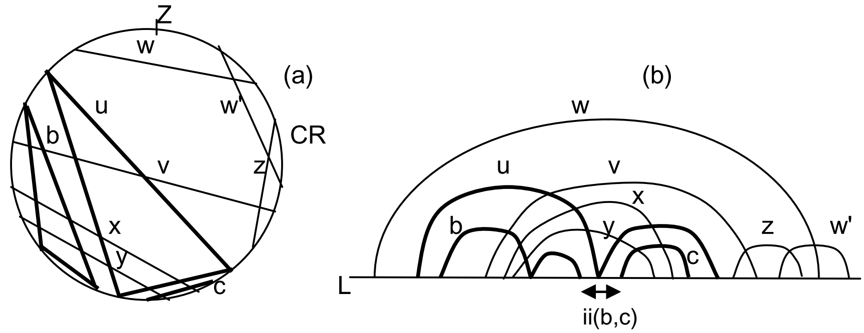

Fig. 1. A family of polygons in a circle and its representation as polygon-filaments on a line

These graphs are a subfamily of the interval-filament graphs defined by Gavril [7,8,9]. For a vertex *v*, we denote by $a(v)$ its corresponding polygon-filament and by $i(v)$ the interval on *L* delimiting $a(v)$. For two vertices u,v having $i(u)\cap i(v)=\phi$, we denote by $ii(u,v)$ the interval between $i(u)$ and $i(v)$. For a pair $p=(u,v)$ of adjacent vertices we denote $i(p)=i(u)\cup i(v)$, $N[p]=N[u]\cup N[v]$ and $U(p)=\{ x \mid i(x)\subseteq i(p)\}$. For a point *X* on *L* we denote $V_X=\{ v \mid X\in i(v)\}$. The *endpoints* of a polygon-filament $a(v)$, are the endpoints of its arcs. The *interval i of an arc* is the interval between its two endpoints. We denote by $R(G)$ the intersection representation of a polygon-circle graph *G* by polygon-filaments. The edges of $coG(V_X)$ represent containment of intervals of non-intersecting polygon-filaments, since the interval of every vertex of $G(V_X)$ contains *X*. We orient the edges u,v of $coG(V_X)$ from *u* to *v*, whenever $i(u)\subset i(v)$; all edges of $coG(V_X)$ become oriented. This orientation is acyclic and transitive since $i(u)\subset i(v)\subset i(w)$ implies $i(u)\subset i(w)$. The properties of families of polygon-filaments are the following:

Property 1. Two semicircle arcs of distinct polygon-filaments on *L* do not intersect (even when their polygon-filaments intersect) if and only if they have disjoint intervals or the interval of one arc appears between the endpoints of the other. This, because chords in *CR* corresponding to two non-intersecting arcs are non-crossing.

Property 2. Two polygon-filaments *b*, *c* do not intersect if and only if they have disjoint intervals, or the interval of one $i(b)$, is contained between the two endpoints of an arc of the other *c* (*b, u* in Figure 1(b)).

Lemma 1. In *G, R(G)*, for every pair $p=(u,v)$ of adjacent or identical vertices, there are no edges between a vertex $x\in U(p)$ and a vertex $w\in V-(N[p]\cup U(p))$ (Figure 1(b)).

Proof. Consider a pair $p=(u,v)$ of adjacent vertices and a vertex $w\in V-(N[p]\cup U(p))$. If $i(p)\cap i(w)=\phi$ then there are no edges between w and the vertices of $U(p)$. Otherwise, the interval $i(p)=i(u)\cup i(v)$ is contained between the endpoints of an arc of $a(w)$ and so is the interval of every $x\in U(p)$. Hence, by Property 2, $a(x)$ and $a(w)$ cannot intersect.

Lemma 2. In G, $R(G)$, consider four polygon-filaments b,c,x,y such that b,c have disjoint intervals, x,y are non-intersecting and both x,y intersect both b,c (Figure 1(b)). If there exists a polygon-filament u which intersects both x,y and does not intersect b,c, then u has an endpoint in $ii(b,c)$ and $i(b)\subset i(u)$ or $i(c)\subset i(u)$ or both.

Proof. Consider four such polygon-filaments b,c,x,y. Let X be the middle point of $ii(b,c)$. Both $i(x),i(y)$ must contain X, hence one of them must contain the other, say $i(y)\subset i(x)$. Consider a polygon-filament u which intersects both x,y and does not intersect b,c. If u has no endpoints in $ii(b,c)$, then it has no endpoints in $i(b)\cup ii(b,c)\cup i(c)$. Since $X\in i(y)$, the interval $i(y)$ must appear between the endpoints of $i(x)$ in $i(b)$ and $i(c)$. Therefore $i(y)\subset[i(b)\cup ii(b,c)\cup i(c)]$ implying that y and u cannot intersect. Thus, u has an endpoint in $ii(b,c)$. If $i(u)\subset ii(b,c)$, then by the above argument c has an endpoint in $ii(b,u)\subset ii(b,c)$ which is a contradiction. Therefore u has an endpoint in $ii(b,c)$ and $i(b)\subset i(u)$ or $i(c)\subset i(u)$.

Lemma 2 proves that in CR there are no three non-intersecting polygons b,c,u, each facing the other two, and two non-intersecting polygons x,y, intersecting b,c,u.

Theorem 3. In a representation $R(G)$ by polygon-filaments of a polygon-circle graph G, for every point $X\in L$, $G(V_X)$ is a weakly-chordal cocomparability graph.

Proof. As described earlier, we orient the edges of the comparability graph $coG(V_X)$ by containment of intervals to obtain an acyclic transitive orientation. Cocomparability graphs are perfect having no odd holes or antiholes with five or more vertices. Also [4], the cocomparability graphs cannot have holes with six or more vertices, since such holes contain asteroidal triples. Hence, $G(V_X)$ has no holes with five or more vertices.

Assume that $G(V_X)$ has an even antihole $h=\{v_1, v_2,\ldots, v_{2k}\}$ with six or more vertices. The hole coh is transitively oriented by the orientation of $coG(V_X)$. W.l.o.g. assume that the two edges of every v_{2i-1} are incoming and the two edges of every v_{2i} are outgoing. Let $arc(v)$ denote the arc of $a(v)$ containing X.

Let us prove that for two adjacent vertices v_{2i+1}, v_{2j+1} of h, the intervals $i(arc(v_{2i+1}))$, $i(arc(v_{2j+1}))$ are intersecting but are not contained one into another. Assume that $i(arc(v_{2i+1}))\subset i(arc(v_{2j+1}))$ and $2i<2i+1<2j+1$ (when $2i+1=1$, we take $2k$ for $2i$), otherwise we renumber the vertices. Hence, $i(v_{2i})\subset i(arc(v_{2i+1}))\subset i(arc(v_{2j+1}))$, implying, by Property 2, that $a(v_{2i})$ cannot intersect $a(v_{2j+1})$, contradicting the fact that v_{2i}, v_{2j+1} are adjacent.

Let us prove that for every three $i(arc(v_{2i-1}))$, $i(arc(v_{2i+1}))$, $i(arc(v_{2i+3}))$, we have $i(arc(v_{2i+1}))\subset[i(arc(v_{2i-1}))\cup i(arc(v_{2i+3}))]$ (when $2i+1=1$, we take $2k-1$ for $2i-1$). Since every two of the three intervals are intersecting but are not contained one into another, one of the three is contained in the union of the two others. Assume that $i(arc(v_{2i+3}))\subset[i(arc(v_{2i-1}))\cup i(arc(v_{2i+1}))]$. By above, neither $i(arc(v_{2i+1})$ nor $i(arc(v_{2i-1})$ can contain $i(arc(v_{2i+3}))$. Thus $[i(arc(v_{2i-1}))\cap i(arc(v_{2i+1}))]\subset i(arc(v_{2i+3}))$ since the three intervals contain X. Then, $X\in i(v_{2i})\subset[i(arc(v_{2i-1}))\cap i(arc(v_{2i+1}))]\subset i(arc(v_{2i+3}))$ and by Property 2, $a(v_{2i})$ cannot intersect $a(v_{2i+3})$, contradicting the fact that v_{2i}, v_{2i+3} are adjacent.

Hence, the left (right) endpoint of $i(arc(v_{2i+1}))$ appears on L between the left (right, respectively) endpoints of $i(arc(v_{2i-1}))$ and $i(arc(v_{2i+3}))$. Assume that the left endpoints of $i(arc(v_1))$, $i(arc(v_3))$, $i(arc(v_5))$ appear from left to right on L in the order: left endpoint of $i(arc(v_1))$, left endpoint of $i(arc(v_3))$, left endpoint of $i(arc(v_5))$. Then, by induction, we obtain that the left endpoints and the right endpoints of $i(arc(v_1))$, $i(arc(v_3)),\ldots,i(arc(v_{2k-1}))$ appear on L in this order from left to right. Hence, $i(v_{2k})\subset i(arc(v_1))\cap i(arc(v_{2k-1}))\subset i(arc(v_3))$ and by Property 2, $a(v_{2k})$ cannot intersect $a(v_3)$, contradicting the fact that v_3, v_{2k} are adjacent. Therefore, $G(V_X)$ has no holes and antiholes with five or more vertices, and is weakly-chordal.

3 Algorithm for Complete Multipartite Subgraphs Containing an Induced $K_{2,2}$, in Polygon-Circle Graphs

Consider a polygon-filament representation $R(G)$ of a polygon-circle graph $G(V,E)$.

Lemma 4. For two non-adjacent vertices u,v having $i(u)\subset i(v)$ let

$$V(u,v)=\{\, w \mid i(u)\subset i(w)\subset i(v),\ w\notin N(u)\cup N(v)\}.$$

Then, every vertex $z\in N(u)\cap N(v)$ is adjacent to every vertex $w\in V(u,v)$ (Figure 2).

Proof. Assume that there are two non-adjacent vertices $z\in N(u)\cap N(v)$ and $w\in V(u,v)$. If $i(w)\subset i(z)$ then $i(u)\subset i(w)\subset i(z)$ and $a(z)$ cannot intersect $a(u)$. If $i(z)\subset i(w)$ then $i(z)\subset i(w)\subset i(v)$ and $a(z)$ cannot intersect $a(v)$. If $i(w)\cap i(z)=\phi$, then $i(u)\cap i(z)=\phi$, since $i(u)\subset i(w)$. All three cases contradict the fact that z is adjacent to both u and v but not to w.

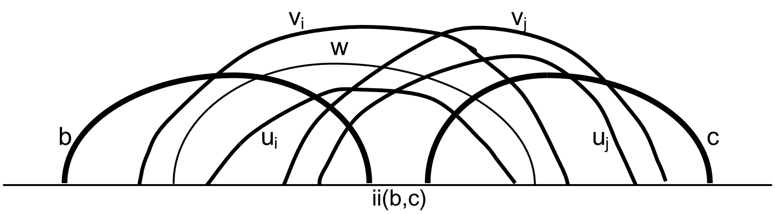

Fig. 2. For $w\notin N(u_i)\cup N(v_i)$ and $i(u_i)\subset i(w)\subset i(v_i)$, every vertex $v_j\in N(u_i)\cap N(v_i)$ is adjacent to w

The algorithm to find a maximum induced complete multipartite subgraph of a polygon-circle graph G solves separately the following two cases:

Case 1: The solution $B(IND_1,\ldots,IND_k,E)$, where every IND_j is an independent set, contains two vertices with disjoint intervals. W.l.o.g. assume that IND_1 has two vertices b,c having $i(b)\cap i(c)=\phi$, and their intervals are minimal in IND_1 (Figure 2). Consider some IND_j, $2\leq j\leq k$: for every $x\in IND_j$, $i(x)$ contains the middle point X of $ii(b,c)$; hence $IND_j\subseteq V_X$. In addition, every two vertices $u,v\in IND_j$, being non-adjacent and their intervals containing the point X, fulfill $i(u)\subset i(v)$. Thus, IND_j cannot contain two vertices both with minimal or both with maximal intervals, since the interval of one must be contained into the other. Let u_j, v_j be the unique vertices of IND_j with minimal and maximal intervals; we may have $u_j=v_j$. Then, by Lemma 4 (Figure 2), $IND_j\subseteq V(u_j,v_j)\cup\{u_j,v_j\}$ and for every $s\neq j$ $IND_s\subseteq N(u_j)\cap N(v_j)$; we assign the weight $|IND_j|$ to the pair u_j,v_j. The vertices of every pair u_j,v_j, $2\leq j\leq k$, are not adjacent while every two vertices in different pairs are adjacent. Thus, the set of pairs u_j,v_j, $2\leq j\leq k$, forms a weighted dissociation set of the complement $coG(V_X)$ of $G(V_X)$.

Since B contains an induced $K_{2,2}$, some IND_j, $2\leq j\leq k$, contains at least two vertices. Hence by Lemma 2, the polygon-filament of every vertex $d\in IND_1$, $d\neq b,c$, has an endpoint in $ii(b,c)$ and $i(b)\subset i(d)$ or $i(c)\subset i(d)$ or both. The above implies that $IND_1-\{b,c\}$ is a subset of

$$S(b,c)=\{\, d \mid d\notin N(b)\cup N(c),\ a(d) \text{ has an endpoint in } ii(b,c) \text{ and } i(b)\subset i(d) \text{ or } i(c)\subset i(d)\}$$

and for every $2\leq j\leq k$, IND_j is a subset of $V_X(b,c)=V_X\cap N(b)\cap N(c)$. The two sets $S(b,c)\cup\{b,c\}$ and $V_X(b,c)$ are disjoint. Therefore, the family IND_j, $1\leq j\leq k$, is defined by the family of pairs $\{(b,c)\}\cup\{(u_j,v_j)\mid 2\leq j\leq k\}$ fulfilling that the vertices of every pair are not adjacent while, every two vertices in different pairs are adjacent. Thus, for Case 1, the algorithm considers every two non-adjacent vertices b,c fulfilling $i(b)\cap i(c)=\phi$, and the middle point X of $ii(b,c)$. The algorithm finds by the algorithm in [7] a maximum independent set in $S(b,c)$ to obtain $IND_1-\{b,c\}$, for every pair u,v of non-adjacent vertices in $V_X(b,c)$ finds a maximum independent set IND in $V(u,v)$ and assigns the weight $|IND\cup\{u,v\}|$ to the pair u,v. Now, the algorithm finds a maximum weight dissociation set in $coG(V_X(b,c))$, by the algorithm in [1,20], since by Theorem 3, $coG(V_X(b,c))$ is weakly-chordal. By the above explanation, the independent sets corresponding in G to this maximum weight dissociation set together with IND_1 form a maximum induced complete multipartite subgraph of G.

Case 2: No independent set in the solution $B(IND_1,\ldots,IND_k,E)$ has two vertices b,c having $i(b)\cap i(c)=\phi$. Then every two intervals corresponding to vertices in B have a non-empty intersection and by the Helly property there is a point X on L contained in all these intervals. Therefore, as in Case 1, the problem is reduced to finding a maximum weight dissociation set in the weakly-chordal comparability graph $coG(V_X)$.

The algorithm works in time $O(|V|^5+|V|^2F(|V|))$, where $F(|V|)$ is the time required to find a maximum weight dissociation set in a weakly-chordal comparability graph.

In the special case when B contains no induced $K_{2,2}$ implying that every IND_j, $2\leq j\leq k$, contains one vertex, the problem is to find a maximum induced subgraph with a vertex partition into an independent set IND and a clique C, completely interconnected. If IND

contains at least three vertices b,c,d, with mutually disjoint intervals, the above algorithm cannot be applied, and the problem remains open. Note that in such a case, by Lemma 2, $N(b)\cap N(c)\cap N(d)$ is a clique.

4 Algorithm for Multicliques in Circle Graphs

Consider a polygon-filament representation $R(G)$ of a circle graph $G(V,E)$: a vertex is represented by one semicircle. For a clique C let $i(C)=\cup_{w\in C}\, i(w)$ and $a(C)=\cup_{w\in C}\, a(w)$. The pair of (not necessarily distinct) vertices $u,v\in C$ to which the endpoints of $i(C)$ belong, fulfils $i(C)=i(u)\cup i(v)$; we say that the pair $p=(u,v)$ *delimits* $i(C)$ (Figure 3). By Lemma 1, there are no edges between vertices in $U(p)$ and vertices in $V-(N[p]\cup U(p))$. Let H be the graph whose vertices are pairs of adjacent or identical vertices of G, two pairs p,q being connected by an edge if and only if they have a vertex in common, or two vertices one in p one in q are adjacent. The graph H is an intersection graph, in which every vertex $p=(u,v)$ is represented by the union $a(p)$ of the two intersecting polygon-filaments $a(u)$ and $a(v)$. Let $E2$ be the oriented edge subset $\{q\rightarrow p\}$ of the edge set of coH given by the relation $i(q)\subset i(p)$ and $a(q)\cap a(p)=\phi$; this orientation of $E2$ is transitive. By Lemma 1, for an edge $q\rightarrow p$ in $E2$, there are no edges in E between $U(q)$ and $U(p)-(N[q]\cup U(q))$.

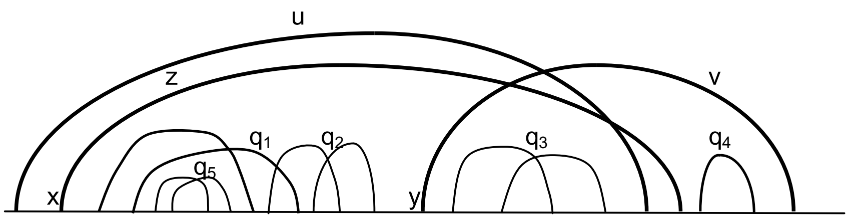

Fig. 3. For the pair $p=(u,v)$ we have $C_p=\{u,z,v\}$, $P_M(p)=\{q_1,q_3,q_4,q_5\}$, sinks are q_1, q_3, q_4, and $M(p)=C_p\cup M(q_1)\cup M(q_3)\cup M(q_4)$; in the interval $i_{j,p}=[x,y]$, $s_1=q_1$ is the unique sink of $coH(P_M\cap W(i_{j,p}),E2)$

For a multiclique M, let P_M be the vertex set of H corresponding to the pairs delimiting the cliques in M; P_M is an independent set of H. For a pair $p(u,v)\in P_M$ (Figure 3), let $P_M(p)=\{\, q \mid q\in P_M,\ i(q)\subseteq i(p)\}$ and let $M(p)$ be the partial multiclique of M defined by $P_M(p)$ in the subgraph $G(U(p))$. $M(p)$ is composed of a clique $C_p\subseteq N[u]\cap N[v]\cap U(p)$ and of the cliques defined by $P_M(p)-\{p\}$. Every pair q in $P_M(p)-\{p\}$ fulfils $i(q)\subset i(p)$ and $a(q)\cap a(p)=\phi$, implying that $q\rightarrow p\in E2$ and p is a sink of $P_M(p)$ in $coH(P_M,E2)$. Similarly, for every two pairs s,q in $P_M(p)-\{p\}$ either $s\rightarrow q\in E2$ or $i(s)\cap i(q)=\phi$. Let $q_1,\dots,q_k$ be the sinks of $P_M(p)-\{p\}$ in the transitive orientation of $coH(P_M,E2)$. Then, $M(p)=C_p\cup M(q_1)\cup\dots\cup M(q_k)$ and for every $q\in M(q_1)\cup\dots\cup M(q_k)$, $i(q)$ appears between consecutive endpoints of $a(C_p)$. When M is a maximum induced multiclique, $M(p)$ is a maximum induced multiclique of $G(U(p))$, otherwise we can replace $M(p)$ by a maximum one. We assign to p the weight $weight(p)=|M(p)|=|C_p|+|M(q_1)|+\dots+|M(q_k)|$. Consider an interval $i_{j,p}$ between two consecutive endpoints of $a(C_p)$. Let $W(i_{j,p})=\{ q \mid i(q)\sqsubset i_{j,p}\}$; the polygon-filaments corresponding to the vertices of $W(i_{j,p})$ do not intersect the polygon-filaments corresponding to the vertices of C_p or of the cliques of M delimited by intervals of $a(C_p)$ disjoint from $i_{j,p}$. Consider the weighted interval graph $I(W(i_{j,p}))$ in which every vertex q in $W(i_{j,p})$ is represented by $i(q)$ with $weight(q)$. Let $s_1,\dots,s_r$ be the sinks of $coH(P_M \cap W(i_{j,p}),E2)$. When M is a maximum induced multiclique, $s_1,\dots,s_r$ is a maximum weight independent set of the interval graph $I(W(i_{j,p}))$, otherwise we could obtain a larger induced multiclique by replacing $s_1,\dots,s_r$ by a maximum weight independent set.

The algorithm works as follows: Using the topological ordering defined by the transitive orientation of $E2$ on coH, we go from sources to sinks on $E2$ and construct for every pair $p=(u,v)$ a maximum induced multiclique $M(p)$ of $G(U(p))$, using the maximum induced multicliques $M(q)$ of the pairs q having $q \rightarrow p \in E2$. For a given p, we must find a clique $C_p \subseteq N[u] \cap N[v] \cap U(p)$, and a maximum weight independent set IND which is the union of maximum weight independent sets (with sinks $s_1,\dots,s_r$) in the intervals between consecutive endpoints of $a(C_p)$ such that $|C_p|+weight(s_1)+\dots+weight(s_r)$ is maximum. Then, $C_p \cup M(s_1) \cup \dots \cup M(s_r)$ is a maximum induced multiclique of $G(U(p))$.

By the Helly property of intervals on a line, every clique in $N[u] \cap N[v] \cap U(p)$ is contained in a vertex set V_X for a point X in $i(u) \cap i(v)$: we must consider every subinterval between consecutive endpoints of polygon-filaments, in $i(u) \cap i(v)$, in each a point X, and for each X we must construct C_p and IND. For the semicircle $a(u)$ of a vertex u, let l_u, r_u denote its left and right endpoints.

Lemma 5. In a circle graph G, the vertices of a clique C_p, $p=(u,v)$, $C_p \subseteq V_X$, are represented by semicircles whose endpoints at the left and the right of X are in the same order.

Proof. Assume that the endpoints of $a(x)$, $a(y)$ representing $x,y \in C_p$ are in order $l_x < l_y$ at the left of X and in $r_y < r_x$ at the right of X. Then, the endpoints of $a(y)$ are contained in $i(x)$ and $a(x),a(y)$ cannot intersect, contradicting the fact that $x,y \in C_p$.

For a pair $p=(u,v)$ and a point X in $i(u) \cap i(v)$ we denote by $l_u = l_1 < l_2 < \dots < l_s = l_v < X$ the left endpoints of the semicircles representing the vertices in $U(p) \cap V_X$. For a vertex w_i whose left endpoint of $i(w_i)$ is l_i, we denote the right endpoint by r_i; note that $X < r_u < r_i < r_v$. We now go on the left endpoints from left to right and for every i we find a maximum multiclique $M(p,1,i)$ within the intervals $[l_1,l_i] \cup [r_1,r_i]$; let $C_{p,i}$ denote its clique containing u and w_i. Assume that we found such a solution for $1,\dots,i-1$ and we want to find one for i and w_i. We consider every $l_j < l_i$, such that w_j, w_i are adjacent, hence by Lemma 5 $r_j < r_i$, and we evaluate a maximum weight independent set $IM(p,j,i)$ in the weighted interval graph $I(W([l_j,l_i]) \cup W([r_j,r_i]))$ in which every vertex q is represented by $i(q)$ with $weight(q)$. By Lemma 5, every vertex w_k of the clique $C_{p,j}$ in the partial solution $M(p,1,j)$ has $l_k \le l_j < l_i$ and $r_k \le r_j < r_i$, hence w_k is adjacent to w_i implying that $C_{p,j} \cup \{w_i\}$ is a clique, Among all j, we take the solution with maximum $|M(p,1,j)|+weight(IM(p,j,i))+1$ and assign it to i as $M(p,1,i)$; by induction

$\{w_i\} \cup M(p,1,j) \cup \{ M(q) \mid q \in IM(p,j,i)\}$ is a maximum induced multiclique $M(p,1,i)$. For the final solution, we find a maximum weight independent set in the interval graph defined by all the pairs of adjacent vertices in G.

The algorithm works in time $O(|V|^6)$. Polygon-circle graphs do not fulfill Lemma 5 and their problem remains open.

5 Algorithm for Multipartite-Multiclique Subgraphs in Circle Graphs

Consider a circle graph $G(V,E)$ represented as an intersection graph of chords in a circle CR.

Let $M(U,W)$ be an induced multiclique-multipartite subgraph of $G(V,E)$ where $M(U)$ is complete multipartite, $M(W)$ is a multiclique and every two vertices $u \in U$, $v \in W$ are adjacent. Consider a vertex u in U with chord X_uY_u (Figure 4). Let x_v,x_z be the endpoints of chords of vertices v,z in W closest to X_u. Let y_w,y_s be the endpoints of chords of vertices w,s in W closest to Y_u. We denote all arcs counterclockwise. Every vertex in W is adjacent to $u \in U$, hence its chord intersects the chord X_uY_u and has its endpoints in the disjoint arcs x_vy_w and y_sx_z. The vertices v,z can be identical or must have intersecting chords; similarly for w,s. Let $Z(v,z,w,s)$ be the set of vertices of G whose chords have the endpoints one in each arc x_vy_w and y_sx_z. Thus $W \subseteq Z(v,z,w,s)$. Let $Q(v,z,w,s)$ be the set of vertices of G whose chords have the endpoints one in each arc x_z,x_v and y_w,y_s.

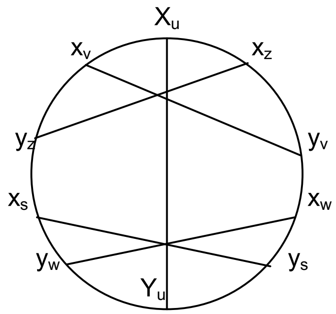

Fig. 4. The chords on CR of the vertices $u \in U$ and $v,z,y,s \in W$; the pairs v,z and y,s are in different cliques of W.

Case 1: Assume that $M(W)$ has at least two cliques. This implies that the chords of v,z do not intersect the chords of w,s. Since the chord of every vertex in U intersects the chords of all the vertices in W, it has its endpoints one in each arc x_z,x_v and y_w,y_s. Hence, $U \subseteq Q(v,z,w,s)$. The algorithm works as follows: We consider every two pairs v,z and w,s of adjacent (or identical) vertices with no interconnecting edges (Figure 4). Let $Z(v,z,w,s)$ and $Q(v,z,w,s)$ be defined as above. By the algorithms in Sections 3,4, we find a maximum induced multiclique $G(W)$ in $G(Z(v,z,w,s))$, and a maximum induced multipartite subgraph $G(U)$ in $G(Q(v,z,w,s))$. Among these pairs v,z and w,s we chose the induced multipartite-multiclique subgraph with a maximum number of vertices.

Case 2: Assume that $M(W)$ has only one clique C. Hence $v=s$, $z=w$ and v,z are adjacent. Therefore, among the four arcs defined by the chords of v,z on CR, there is a pair of opposite arcs such that the chords corresponding to the vertices of U have the endpoints one in each arc of this pair. The algorithm works as follows: For every pair v,z of adjacent vertices and for every pair of their opposite arcs we find a maximum clique $G(W)$ for the pair v,z, and a maximum induced multipartite subgraph $G(U)$ for their opposite pairs of arcs. To find $G(U)$: we use the algorithm in Section 3, to cover the case that it contains an induced $K_{2,2}$; we use an algorithm to find a maximum independent set in a permutation graph to cover the case that $G(U)$ has no induced $K_{2,2}$ and thus has only one independent set. Among all pairs v,z of adjacent vertices, we chose the induced multipartite-multiclique subgraph with a maximum number of vertices. Note that this covers the case that $M(U,W)$ has one independent set and one clique, unsolved in Section 4.

The algorithm works in time $O(|V|^4F_1(V)+|V|^4F_2(V))$ where $F_1(V)$ is the time required to find a maximum induced multiclique in $G(Z(v,z,w,s))$, and $F_2(V)$ the time required to find a maximum induced multipartite subgraph in $G(Q(v,z,w,s))$.

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