Bend-Bounded Path Intersection Graphs: Sausages, Noodles, and Waffles on a Grill*

Steven Chaplick[1,**], Vít Jelínek[2], Jan Kratochvíl[3], and Tomáš Vyskočil[3]

[1] Department of Mathematics, Wilfrid Laurier University, Waterloo, CA
chaplick@cs.toronto.edu
[2] Computer Science Institute, Faculty of Mathematics and Physics, Charles University, Prague
jelinek@iuuk.mff.cuni.cz
[3] Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague
{honza,whisky}@kam.mff.cuni.cz

Abstract. In this paper we study properties of intersection graphs of k-bend paths in the rectangular grid. A k-bend path is a path with at most k 90 degree turns. The class of graphs representable by intersections of k-bend paths is denoted by B_k-VPG. We show here that for every fixed k, $B_k\text{-VPG} \subsetneq B_{k+1}\text{-VPG}$ and that recognition of graphs from B_k-VPG is NP-complete even when the input graph is given by a B_{k+1}-VPG representation. We also show that the class B_k-VPG (for $k \geq 1$) is in no inclusion relation with the class of intersection graphs of straight line segments in the plane.

1 Introduction

In this paper we continue the study of Vertex-intersection graphs of Paths in Grids[1] (VPG graphs) started by Asinowski et. al [1,2]. A *VPG representation* of a graph G is a collection of paths of the rectangular grid where the paths represent the vertices of G in such a way that two vertices of G are adjacent if and only if the corresponding paths share at least one vertex.

VPG representations arise naturally when studying circuit layout problems and layout optimization [15] where layouts are modelled as paths (wires) on grids. One approach to minimize the cost or difficulty of production involves minimizing the number of times the wires bend [3,13]. Thus the research has been focused on VPG representations parameterized by the number of times each path is allowed to *bend* (these representations are also the focus of [1,2]). In particular, a *k-bend path* is a path in the grid which contains at most k

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[1] The grids to which we refer are always rectangular.

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bends where a *bend* is when two consecutive edges on the path have different horizontal/vertical orientation. In this sense a B_k-VPG representation of a graph G is a VPG representation of G where each path is a k-bend path. A graph is B_k-VPG if it has a B_k-VPG representation.

Several relationships between VPG graphs and traditional graph classes (i.e., circle graphs, circular arc graphs, interval graphs, planar graphs, segment (SEG) graphs, and string (STRING) graphs) were observed in [1,2]. For example, the equivalence between string graphs (the intersection graphs of curves in the plane) and VPG graphs is formally proven in [2], but it was known as folklore result [6]. Additionally, the base case of this family of graph classes (namely, B_0-VPG) is a special case of segment graphs (the intersection graphs of line segments in the plane). Specifically, B_0-VPG is more well known as the 2-DIR[2]. The recognition problem for the VPG = string graph class is known to be NP-Hard by [9] and in NP by [14]. Similarly, it is NP-Complete to recognize 2-DIR = B_0-VPG graphs [11]. However, the recognition status of B_k-VPG for every $k > 0$ was given as an open problem from [2] (all cases were conjectured to be NP-Complete). We confirm this conjecture by proving a stronger result. Namely, we demonstrate that deciding whether a B_{k+1}-VPG graph is a B_k-VPG graph is NP-Complete (for any fixed $k > 0$) – see Section 4.

Furthermore, in [1,2] it is shown that B_0-VPG $\subsetneq$ B_1-VPG $\subsetneq$ VPG and it was conjectured that B_k-VPG $\subsetneq$ B_{k+1}-VPG for every $k > 0$. We confirm this conjecture constructively – see Section 3.

Finally, we consider the relationship between the B_k-VPG graph classes and segment graphs. In particular, we show that SEG and B_k-VPG are incomparable through the following pair of results (the latter of which is somewhat surprising): (1) There is a B_1-VPG graph which is not a SEG graph; (2) For every k, there is a 3-DIR graph which has no B_k-VPG representation.

The paper is organized as follows. In Section 2 we introduce the Noodle-Forcing Lemma, which is the key to restricting the topological structure of VPG representations[3]. In Section 3 we introduce the "sausage" structure which is the crucial gadget that we use for the hardness reduction and which by itself shows that B_k-VPG is strict subset of B_{k+1}-VPG[4]. We also demonstrate the incomparability of B_k-VPG and SEG in Section 3. The NP-hardness reduction is presented in Section 4. We end the paper with some remarks and open problems.

2 Noodle-Forcing Lemma

In this section, we present the key lemma of this paper (see Lemma 1). Essentially, we prove that, for "proper" representations R of a graph G, there is a graph G' where G is an induced subgraph of G' and R is "sub-representation"

[2] Note: a k-DIR graph is an intersection graph of straight line segments in the plane with at most k distinct directions (slopes).

[3] This was inspired by the order forcing lemma of [12].

[4] This gadget is named due to its VPG representation resembling sausage links.

of every representation of G' (i.e., all representations of G' require the part corresponding to G to have the "topological structure" of R). We begin this section with several definitions.

Let $G = (V, E)$ be a graph. A *representation* of G is a collection $R = \{R(v), v \in V\}$ of piecewise linear curves in the plane, such that $R(u) \cap R(v)$ is nonempty iff uv is an edge of G.

An *intersection point* of a representation R is a point in the plane that belongs to (at least) two distinct curves of R. Let $\text{In}(R)$ denote the set of intersection points of R.

A representation is *proper* if

1. each $R(v)$ is a simple curve, i.e., it does not intersect itself,
2. R has only finitely many intersection points (in particular no two curves may overlap) and finitely many bends, and
3. each intersection point p belongs to exactly two curves of R, and the two curves cross in p (in particular, the curves may not touch, and an endpoint of a curve may not belong to another curve).

Let R be a proper representation of $G = (V, E)$, let R' be another (not necessarily proper) representation of G, and let ϕ be a mapping from $\text{In}(R)$ to $\text{In}(R')$. We say that ϕ is *order-preserving* if it is injective and has the property that for every $v \in V$, if $p_1, p_2, \ldots, p_k$ are all the distinct intersection points on $R(v)$, then $\phi(p_1), \ldots, \phi(p_k)$ all belong to $R'(v)$ and they appear on $R'(v)$ in the same relative order as the points $p_1, \ldots, p_k$ on $R(v)$. (If $R'(v)$ visits the point $\phi(p_i)$ more than once, we may select one visit of each $\phi(p_i)$, such that the selected visits occur in the correct order $\phi(p_1), \ldots, \phi(p_k)$.)

For a set P of points in the plane, the *ε-neighborhood of P*, denoted by $\mathcal{N}_\varepsilon(P)$, is the set of points that have distance less than ε from P.

Lemma 1 (Noodle-Forcing Lemma). *Let $G = (V, E)$ be a graph with a proper representation $R = \{R(v), v \in V\}$. Then there exists a graph $G' = (V', E')$ containing G as an induced subgraph, which has a proper representation $R' = \{R'(v), v \in V'\}$ such that $R(v) = R'(v)$ for every $v \in V$, and $R'(w)$ is a horizontal or vertical segment for $w \in V' \setminus V$. Moreover, for any $\varepsilon > 0$, any (not necessarily proper) representation of G' can be transformed by a homeomorphism of the plane and by circular inversion into a representation $R^\varepsilon = \{R^\varepsilon(v), v \in V'\}$ with these properties:*

1. *for every vertex $v \in V$, the curve $R^\varepsilon(v)$ is contained in the ε-neighborhood of $R(v)$, and $R(v)$ is contained in the ε-neighborhood of $R^\varepsilon(v)$.*
2. *there is an order-preserving mapping $\phi\colon In(R) \to In(R^\varepsilon)$, with the additional property that for every $p \in In(R)$, the point $\phi(p)$ coincides with the point p.*

Due to space limitations, we only sketch the proof of the lemma. Suppose R is a proper representation of a graph G. The main idea is to overlay the representation R with a sufficiently fine grid-like configuration C of short horizontal and vertical segments, so that the position of a curve $R(v) \in R$ is well approximated by the set of segments of C that are intersected by $R(v)$. We refer to this step as 'grilling' of the representation R, since the segments of C form a structure resembling a grill.

We let R' be the representation $R \cup C$ and G' be the graph represented by R'. Moreover, let G_C be the graph whose intersection representation is C. The configuration of C has the property that any representation C' of the graph G_C can be transformed into the representation C by a homeomorphism and a circular inversion, followed possibly by a truncation of some of the curves of C'. In particular, any representation of G' can be transformed by a homeomorphism and a circular inversion into a representation R'' that essentially contains a copy of C. The segments of C then constrain the relative positions of the curves representing the vertices of G in R''.

This allows us to argue that the curve $R''(v) \in R''$ representing a vertex v of G can be deformed to be arbitrarily close to the corresponding curve $R(v)$ of R, and conversely, every point of $R(v)$ is close to a point of $R''(v)$. In fact, for every ε, we may deform $R''(v)$ into a curve $R^\varepsilon(v)$ which is confined to the the ε-neighborhood of the original curve $R(v)$, without affecting the intersections between this curve and the curves of C'. We call the ε-neighborhood of $R(v)$ *the noodle* of $R(v)$, denoted by $N(v)$.

It now remains to provide the order-preserving mapping ϕ. Suppose that $R(u)$ and $R(v)$ are two curves of R that cross at a point p. Assuming ε is small enough, $N(u)$ and $N(v)$ intersect in a parallelogram-shaped region surrounding the point p. We call this region *the zone* of p. We may assume that distinct intersection points of R have disjoint zones.

Assume from now on that all the curves of R and R^ε have a prescribed orientation, i.e., a fixed beginning and end. Suppose that a curve $R(v)$ contains k intersection points $p_1, \dots, p_k$ appearing in this order, with zones $P_1, \dots, P_k$. We may assume that the two endpoints of $R^\varepsilon(v)$ are ε-close to the corresponding endpoints of $R(v)$, otherwise we only consider a truncated part of $R^\varepsilon(v)$ that has this property. Following the curve $R^\varepsilon(v)$ from beginning to end, we eventually encounter all the zones $P_1, \dots, P_k$. Of course, a given zone P_i may be intersected several times by $R^\varepsilon(v)$, since the curve $R^\varepsilon(v)$ may be folded inside $N(v)$ in a complicated way. For every zone P_i, we fix the first occurrence when $R^\varepsilon(v)$ enters inside P_i and then exits through the opposite side of P_i. The subcurve of $R^\varepsilon(v)$ inside P_i that corresponds to this occurrence will be called *the representative of* $R^\varepsilon(v)$ *inside* P_i, and denoted by $r_i(v)$. Note that the representatives appear on $R^\varepsilon(v)$ in the 'correct' order, i.e., $r_1(v), r_2(v), \dots, r_k(v)$.

We now define the order-preserving mapping ϕ. Let $p \in \mathrm{In}(R)$ be an intersection point of two curves $R(u)$ and $R(v)$, and let P be its zone. Let $r(u)$ and $r(v)$ be the representatives of $R^\varepsilon(u)$ and $R^\varepsilon(v)$ inside this zone, and let p'' be an arbitrary intersection of $r(u)$ and $r(v)$. We then put $\phi(p) = p''$. This mapping is order-preserving by the construction of the representatives. Deforming the curves of R^ε inside each zone, we may even assume that p'' coincides with p. This completes the sketch of proof of the Noodle-Forcing Lemma.

3 Relations between Classes

With the Noodle-Forcing Lemma, we can prove our separation results.

Theorem 1. *For any $k \geq 1$, there is a graph G' that has a proper representation using k-bend axis-parallel curves, but has no representation using $(k-1)$-bend axis-parallel curves.*

Proof. Consider the graph K_2 consisting of a single edge uv, with a representation R in which both u and v are represented by weakly increasing k-bend staircase curves that have $k+1$ common intersections $p_1, \ldots, p_{k+1}$, in left-to-right order, see Fig.1. We refer to this representation as a *sausage* due to it resembling sausage links.

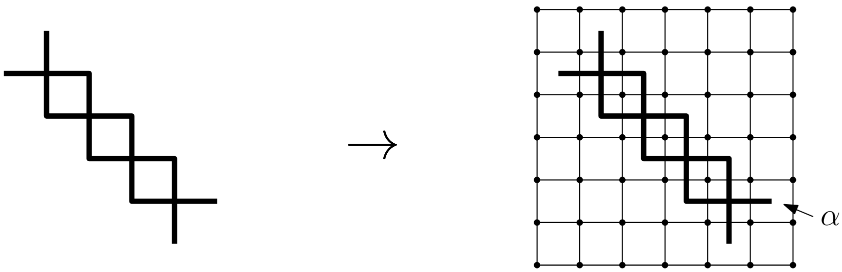

Fig. 1. The sausage representation for $k = 3$ and its grilled version

We now grill the sausage (i.e., we apply the Noodle-Forcing Lemma to K_2 and R) to obtain a graph G' with a k-bend representation R'. We claim that G' has no $(k-1)$-bend representation. Assume for contradiction that there is a $(k-1)$-bend representation R'' of G'. Lemma 1 then implies that there is an order-preserving mapping ϕ: $\text{In}(R) \to \text{In}(R'')$. Let $s_i(u)$ be the subcurve of $R''(u)$ between the points $\phi(p_i)$ and $\phi(p_{i+1})$, and similarly for $s_i(v)$ and $R''(v)$. Consider, for each $i = 1, \ldots, k$, the union $c_i = s_i(u) \cup s_i(v)$. We know from Lemma 1 that $s_i(u)$ and $s_i(v)$ cannot completely overlap, and therefore the closed curve c_i must surround at least one nonempty bounded region of the plane. Therefore c_i contains at least two bends different from $\phi(p_i)$ and $\phi(p_{i+1})$. We conclude that $R''(u)$ and $R''(v)$ together have at least $2k$ bends, a contradiction.

A straightforward consequence is the following.

Corollary 1. *For every k, B_k-VPG $\subsetneq$ B_{k+1}-VPG.*

Because two straight-line segments in the plane cross at most once, the Noodle-Forcing Lemma also implies the following.

Corollary 2. *For every $k \geq 1$, B_k-VPG $\not\subseteq$ SEG.*

This raises a natural question: Is there some k such that every SEG graph is contained in B_k-VPG? The following theorem answers it negatively.

Theorem 2. *For every k, there is a graph which belongs to 3-DIR but not to B_k-VPG.*

Proof. We fix an arbitrary k. Consider, for an integer n, a representation $R \equiv R(n)$ formed by $3n$ segments, where n of them are horizontal, n are vertical and n have a slope of 45 degrees. Suppose that every two segments of R with different slopes intersect, and their intersections form the regular pattern depicted in Figure 2 (with a little bit of creative fantasy this pattern resembles a waffle, especially when viewed under a linear transformation).

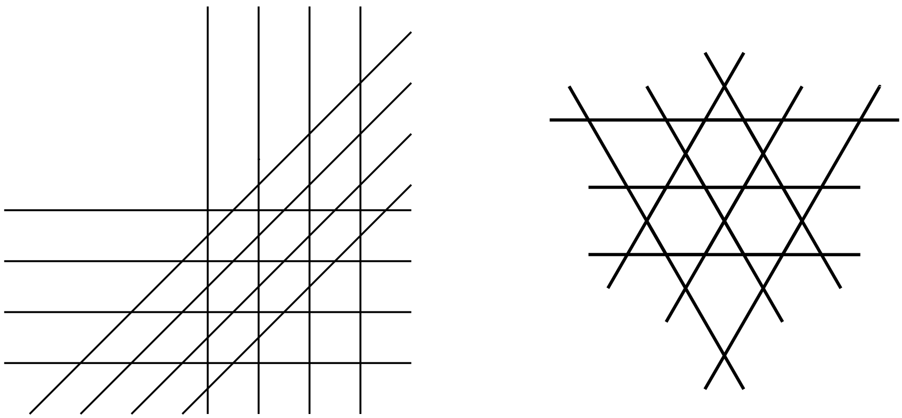

Fig. 2. The 'waffle' representation R from Theorem 2 and its transformed representation

Note that the representation R forms $\Omega(n^2)$ empty internal triangular faces bounded by segments of R, and the boundaries of these faces intersect in at most a single point. Suppose that n is large enough, so that there are more than $3kn$ such triangular faces. Let G be the graph represented by R.

The representation R is proper, so we can apply the Noodle-Forcing Lemma to R and G, obtaining a graph G' together with its 3-DIR representation R'. We claim that G' has no B_k-VPG representation.

Suppose for contradiction that there is a B_k-VPG representation R'' of G'. We will show that the $3n$ curves of R'' that represent the vertices of G contain together more than $3kn$ bends.

From the Noodle-Forcing Lemma, we deduce that there exists an order-preserving mapping ϕ: $\mathrm{In}(R_n) \to \mathrm{In}(R''_n)$. Let T be a triangular face of the representation R. The boundary of T consists of three intersection points $p, q, r \in \mathrm{In}(R)$ and three subcurves a, b, c. The three intersection points $\phi(p)$, $\phi(q)$ and $\phi(r)$ determine the corresponding subcurves a'', b'' and c'' in R''.

The Noodle-Forcing Lemma implies that there is a homeomorphism h which sends a'', b'', and c'' into small neighborhoods of a, b and c, respectively. This shows that each of the three curves a'', b'' and c'' contains a point that does not belong to any of the other two curves. This in turn shows that at least one of the three curves is not a segment, i.e., it has a bend in its interior.

Since the triangular faces of R have non-overlapping boundaries, and since ϕ is order-preserving, we see that for each triangular face of R there is at least one bend in R'' belonging to a curve representing a vertex of G. Since G has $3n$ vertices and R determines more than $3kn$ triangular faces, we conclude that at least one curve of R'' has more than k bends, a contradiction.

4 Hardness Results

In this section we strengthen the separation result of Corollary 1 by showing that not only are the classes B_k-VPG and B_{k+1}-VPG different, but providing a B_{k+1}-VPG representation does not help in deciding B_k-VPG membership. This also settles the conjecture on NP-hardness of recognition of these classes stated in [2], in a considerably stronger form than it was asked.

Theorem 3. *For every $k \geq 0$, deciding membership in B_k-VPG is NP-complete even if the input graph is given with a B_{k+1}-VPG representation.*

Proof. It is not difficult to see that recognition of B_k-VPG is in NP and therefore we will be concerned in showing NP-hardness only. We use the NP-hardness reduction developed in [11] for showing that recognizing grid intersection graphs is NP-complete. Grid intersection graphs are intersection graphs of vertical and horizontal segments in the plane with additional restriction that no two segments of the same direction share a common point. Thus these graphs are formally close but not equal to B_0-VPG graphs (where paths of the same direction are allowed to overlap). However, bipartite B_0-VPG graphs are exactly grid intersection graphs. This follows from a result of Bellantoni et al. [4] who proved that bipartite intersection graphs of axes parallel rectangles are exactly grid intersection graphs.

The reduction in [11] constructs, given a Boolean formula Φ, a graph G_Φ which is a grid intersection graph if and only if Φ is satisfiable. In arguing about this, a representation by vertical and horizontal segments is described for a general layout of G_Φ for which it is also shown how to represent its parts corresponding to the clauses of the formula, referred to as *clause gadgets*, if at least one literal is true. The clause gadget is reprinted with a generous approval of the author in Fig. 3, while Fig. 4 shows the grid intersection representations of satisfied clauses, and Fig. 5 shows the problem when all literals are false. In Fig. 6, we show that in the case of all false literals, the clause gadget can be represented by grid paths with at most 1 bend each. It follows that $G_\Phi \in B_1$-VPG and a 1-bend representation can be constructed in polynomial time. Thus, recognition of B_0-VPG is NP-complete even if the input graph is given with a B_1-VPG representation.

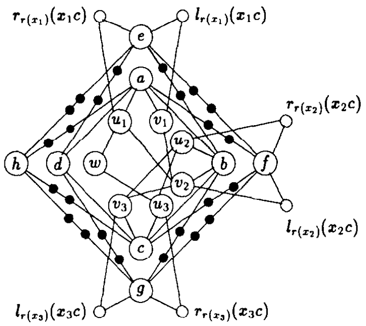

Fig. 3. The clause gadget reprinted from [11]

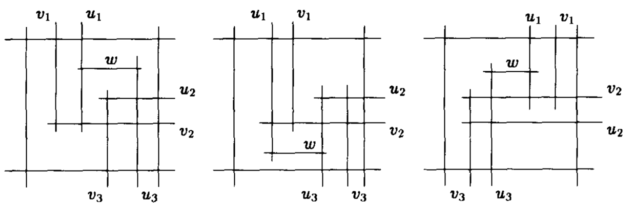

Fig. 4. The representations of satisfied clauses reprinted from [11]

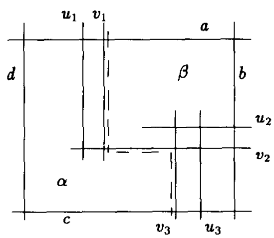

Fig. 5. The problem preventing the representation of an unsatisfied clause reprinted from [11]

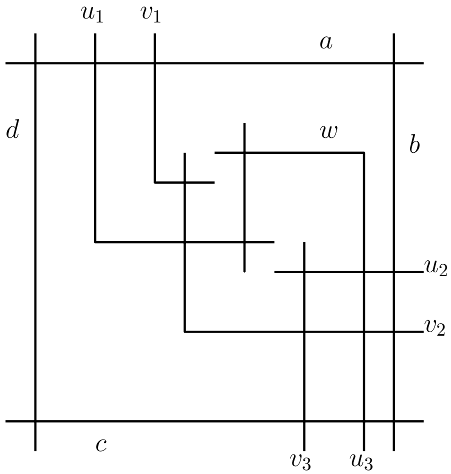

Fig. 6. The representation of an unsatisfied clause gadget via curves with one bend

We use a similar approach for arbitrary $k > 0$ with a help of the Noodle-Forcing Lemma. We grill the same representation R of K_2 as in the proof of Theorem 1. We call the resulting graph $P(u)$ where u is one of the vertices of the K_2, the one whose curve in R is ending in a boundary cell denoted by α in the schematic Fig. 1. We call this graph the *pin* since it follows from Lemma 1 that it has a B_k-VPG representation such that the bounding paths of the cell α wrap around the grill and the last segment of $R(u)$ extends arbitrarily far (see the schematic Fig. 7). We will refer to this extending segment as the *tip* of the pin. It is crucial to observe that in any B_k-VPG representation R' of $P(u)$ all bends of $R'(u)$ are consumed between the crossing points with the curve representing the other vertex of K_2 and hence the part of $R'(u)$ that lies in the α cell of R' is necessarily straight.

Fig. 7. Construction of a pin

Next we combine two pins together to form a *clothespin*. The construction is illustrated in the schematic Fig. 8. We start with a K_4 whose edges are subdivided by one vertex each. Every STRING representation of this graph contains

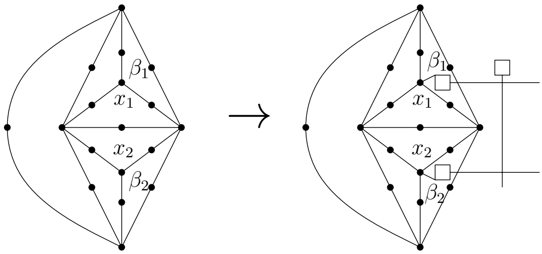

Fig. 8. Construction of a clothespin

4 basic regions which correspond to the faces of a drawing of the K_4 (this is true for every 3-connected planar graph and it is seen by contracting the curves corresponding to the degree 2 vertices, the argument going back to Sinden [15]). We add two vertices x_1, x_2 that are connected by paths of length 2 to the boundary vertices of two triangles, say β_1 and β_2. The curves representing x_1 and x_2 must lie entirely inside the corresponding regions. Then we add two pins, say $P(u_1)$ and $P(u_2)$, connect the vertices of the boundary of α_i to x_i by paths of length 2 and make u_i adjacent to a vertex on the boundary of β_i (for $i = 1, 2$). Finally, we add a third pin $P(u_3)$ and make u_3 adjacent to u_1 and u_2. We denote the resulting graph by $CP(u)$.

It is easy to check that the clothespin has a B_k-VPG representation $\check{R}$ such that the tips of $\check{R}(u_1)$ and $\check{R}(u_2)$ are parallel and extend arbitrarily far from the rest of the representation, as indicated in Fig. 8.

On the other hand, in any B_k-VPG representation R' of $CP(u)$, if a curve crosses $R'(u_1)$ and $R'(u_2)$ and no other path of $R'(CP(u))$, then it must cross the tips of $R'(u_1)$ and $R'(u_2)$. This follows from the fact that for $i = 1, 2$, $R'(x_i)$ must lie in α_i (to be able to reach all its bounding curves), and hence, by circle inversion, all bends of $R'(u_i)$ are trapped inside β_i. If a curve crosses both $R'(u_1)$ and $R'(u_2)$, it must cross them outside $\beta_1 \cup \beta_2$, and hence it only may cross their tips.

Now we are ready to describe the construction of G'_Φ. We take G_Φ as constructed in [11] replace every vertex u by a clothespin $CP(u)$, and whenever $uv \in E(G_\Phi)$, we add edges $u_i v_j, i, j = 1, 2$. Now we claim that $G'_\Phi \in B_k$-VPG if and only if Φ is satisfiable, while $G'_\Phi \in B_{k+1}$-VPG is always true.

On one hand, if $G'_\Phi \in B_k$-VPG and R' is a B_k-VPG representation of G'_Φ, then the tips of $R'(u_1), u \in V(G_\Phi)$ form a 2-DIR representation of G_Φ ($R'(u_1)$ and $R'(v_1)$ may only intersect in their tips) and Φ is satisfiable.

On the other hand, if Φ is satisfiable, we represent G_Φ as a grid intersection graph and replace every segment of the representation by a clothespin with slim parallel tips and the body of the pin tiny enough so that does not intersect anything else in the representation. Similarly, if Φ is not satisfiable, we modify a 1-bend representation of G_Φ by replacing the paths of the representation by clothespins with 1-bend on the tips, thus obtaining a B_{k+1}-VPG representation of G_Φ. The representation consists of a large part inherited from the representation of G_Φ and tiny parts representing the heads of the pins, but these can be made all of the same constant size and thus providing only a constant ratio refinement of the representation of G_Φ. The representation is thus still of linear size and can be constructed in polynomial time.

5 Concluding Remarks

In this paper we have affirmatively settled two main conjectures of Asinowski et al [2] regarding VPG graphs. We have also demonstrated the relationship between B_k-VPG graphs and segment graphs.

The first conjecture that we settled claimed that B_k-VPG is a strict subset of B_{k+1}-VPG for all k. We have proven this constructively. Previously only the following separation was known: B_0-VPG $\subsetneq$ B_1-VPG $\subsetneq$ VPG.

The second conjecture claimed that the B_k-VPG recognition problem is NP-Complete for all k. We have actually proven a stronger result; namely, that the B_k-VPG recognition problem is NP-Complete for all k even when the input graph is a B_{k+1}-VPG graph. Previously only the NP-Completeness of B_0-VPG (from 2-DIR [11]) and VPG (from STRING [9,14]) were known.

Finally due to the close relationship between VPG graphs and segment graphs (i.e., since B_0-VPG = 2-DIR, and SEG $\subsetneq$ STRING = VPG) we have considered the relationship between these classes. In particular, we have shown that:

- There is no k such that 3-DIR is contained in B_k-VPG (i.e., SEG is not contained in B_k-VPG for any k).
- B_1-VPG is not contained in SEG.

Thus, to obtain polynomial time recognition algorithms, one would need to restrict attention to specific cases with (potentially) useful structure. In this respect, in [8], certain subclasses of B_0-VPG graphs have been characterized and shown to admit polynomial time recognition; namely split, chordal claw-free, and chordal bull-free B_0-VPG graphs are discussed in [8]. Additionally, in [5], B_0-VPG chordal and 2-row B_0-VPG[5] have been shown to have polynomial time recognition algorithms. In particular, we conjecture that applying similar restrictions to the B_k-VPG graph class will also yield polynomial time recognition algorithms. It is interesting to note that since our separating examples are not chordal it is also open whether B_k-VPG chordal $\subsetneq$ B_{k+1}-VPG chordal.

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[5] Where the VPG representation has at most two rows.

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