Optimization of SIRMs Fuzzy Model Using Łukasiewicz Logic

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Abstract. The purpose of this study is to prove the existence of single input rule modules which minimize the performance functional of the feedback control using SIRMs fuzzy reasoning method. A bounded product (Łukasiewicz t-norm) and a bounded sum (Łukasiewicz t-conorm) are applied to the operations in SIRMs fuzzy reasoning for interpreting "ands" and "ors" respectively.

Keywords: SIRMs approximate reasoning method, Bounded product, Bounded sum, Calculus of variations.

1 Introduction

In inference process, Mamdani method applies min-max operation ($\wedge$, $\vee$). On the other the product-sum-gravity method proposed by Mizumoto applies product-sum operation for simplification of calculation. Łukasiewicz logical operation is one of the operations on fuzzy sets. The bounded product is t-norm, and the bounded sum is its dual t-conorm [1]. They are applied to fuzzy approximate reasoning [2].

Single input rule modules (SIRMs) connected fuzzy inference method proposed by Yubazaki et al., can decrease the number of fuzzy rules in comparison with fuzzy IF-THEN rules [3]. This model needs few rules and parameters, and the rules can be designed much easier. The functional type SIRMs method, in which a general function is used instead of membership function for consequent part, is a special case of the T-S reasoning method. It clarified that IF-THEN rules can be transformed to SIRMs by Seki [4].

Generally product-sum or min-max operations are used in SIRMs connected fuzzy inference. However the bounded product and the bounded sum are applied in this study. Because applying bounded product makes the universe (domain) of consequent fuzzy set smaller as different from the scaled method using "product" and the clipping method using "min". Then it is possible that the crisp value of inference result is accurate. Since bounded sum is dual operator of bounded

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product, it is applied to the operation which integrates the inference results of all SIRMs.

The optimization of fuzzy control discussed in this paper is different from conventional method such as classical control and modern control. We consider fuzzy optimal control problems as problems of finding the minimum (maximum) value of the performance function with feedback law constructed by fuzzy rules through fuzzy approximate reasoning [5,6,7]. To guarantee the convergence of optimal solution, the compactness of the set of membership functions is proved. And assuming fuzzy inference to be a functional on the set, its continuity is obtained. Then, it is shown that the system has an optimal feedback control by essential use of compactness of sets of fuzzy membership functions. The pair of membership functions, in other words SIRMs, which minimize an integral performance function of fuzzy logic control exists.

2 Fuzzy Control

In this study, it is assumed that the feedback part in the system is calculated by SIRMs connected fuzzy inference method.

2.1 Nonlinear Feedback Control

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the usual Euclidean norm. Consider a system given by the following state equation:

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where $x(t)$ is the state and the control input $u(t)$ of the system is given by the state feedback $u(t) = \rho(x(t))$. For a sufficiently large $r > 0$, $B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ denotes a bounded set containing all possible initial states x_0 of the system. Let T be a sufficiently large final time. Then, we have

Proposition 1. *Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function and $x_0 \in B_r$. Then, the state equation $\dot{x}(t) = f(x(t), \rho(x(t)))$ has a unique solution $x(t, x_0, \rho)$ on $[0, T]$ with the initial condition $x(0) = x_0$ such that the mapping $(t, x_0) \in [0, T] \times B_r \mapsto x(t, x_0, \rho)$ is continuous.*

For any $r_2 > 0$, denote by Φ the set of Lipschitz continuous functions $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfying $\sup_{u \in \mathbb{R}^n} |\rho(u)| \leq r_2$. Then, the following a) *and* b) *hold.*

a) *For any $t \in [0, T], x_0 \in B_r$ and $\rho \in \Phi$,*

$$\|x(t, x_0, \rho)\| \leq r_1,$$

where

$$r_1 = e^{M_f T} r + (e^{M_f T} - 1)(r_2 + 1). \tag{2}$$

b) *Let $\rho_1, \rho_2 \in \Phi$. Then, for any $t \in [0, T]$ and $x_0 \in B_r$,*

$$\|x(t, x_0, \rho_1) - x(t, x_0, \rho_2)\| \leq \frac{e^{\Delta_f (1+\Delta_{\rho_1})t} - 1}{1 + \Delta_{\rho_1}} \sup_{u \in [-r_1, r_1]^n} |\rho_1(u) - \rho_2(u)|, \tag{3}$$

where Δ_f and Δ_{ρ_1} are the Lipschitz constants of f and ρ_1 [5].

2.2 Performance Functional

We write x for a sequence of $x_1(t), x_2(t), \cdots, x_n(t)$ for simplicity, and put

$$x = (x_1, x_2, \cdots, x_n) = (x_1(t), x_2(t), \cdots, x_n(t)) = x(t).$$

In the following, assume the feedback law $u(t) = \rho(x(t))$ of the nonlinear system (1) is constructed based on fuzzy approximate reasoning. Then the operation ρ is a composite mapping of the membership functions and the inference calculations. On the other, put $\mathcal{M}$ be a set of membership functions and $\mathcal{F}$ be its element, we can say that the operation ρ is a composite functional on the set of membership functions $\mathcal{M}$. Then $\mathcal{F}$ is indexed to ρ.

The performance index of this fuzzy feedback control system is evaluated with following integral performance function:

$$J = \int_{B_r} \int_0^T w(x(t, \zeta, \rho_{\mathcal{F}}), \rho_{\mathcal{F}}(x(t, \zeta, \rho_{\mathcal{F}})))dtd\zeta \tag{4}$$

where $w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a positive continuous function. J depends on $\rho_{\mathcal{F}}$. Since admissible range of initial state B_r and final time T are known, J is a functional on the family of membership functions $\mathcal{M}$. The optimal control problem in this study is considered to be calculus of variations. The following lemma guarantees the existence of $\mathcal{F}$ on $\mathcal{M}$ which minimizes (maximizes) the previous performance function (4).

Lemma 1. *Suppose $\mathcal{M}$ to be a compact metric space and let $\{\mathcal{F}^k\}_{k\in\mathbb{N}} \subset \mathcal{M}$. If $\mathcal{F}^k \to \mathcal{F} \in \mathcal{M}$ $(k \to \infty)$ implies*

$$\sup_{x\in[-r_1,r_1]^n} |\rho_{\mathcal{F}^k}(x) - \rho_{\mathcal{F}}(x)| \to 0 \ (k \to \infty), \tag{5}$$

then the mapping

$$\mathcal{F} \in \mathcal{M} \mapsto J = \int_{B_r} \int_0^T w(x(t, \zeta, \rho_{\mathcal{F}}), \rho_{\mathcal{F}}(x(t, \zeta, \rho_{\mathcal{F}})))dtd\zeta$$

has a minimum (maximum) value on the compact metric space $\mathcal{M}$ on the basis of constants given by section 2.1.

Proof. It is sufficient to prove that the performance function J is continuous on $\mathcal{M}$. Fix $(t, \zeta) \in [0, T] \times B_r$, by b) of proposition 1, we have

$$\lim_{k\to\infty} \|x(t, \zeta, \rho_{\mathcal{F}^k}) - x(t, \zeta, \rho_{\mathcal{F}})\| = 0. \tag{6}$$

Further, it follows from (5), (6) and a) of proposition 1 that

$$\lim_{k\to\infty} \rho_{\mathcal{F}^k}(x(t, \zeta, \rho_{\mathcal{F}^k})) = \rho_{\mathcal{F}}(x(t, \zeta, \rho_{\mathcal{F}})). \tag{7}$$

Noting that $w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is positive and continuous, it follows from (6), (7) and the Lebesgue's dominated convergence theorem [8,9] that the mapping is continuous on the compact metric space $\mathcal{M}$. Thus it has a minimum (maximum) value on $\mathcal{M}$, and the proof is complete.

2.3 Single Input Rule Modules Connected Fuzzy Inference Model

Assume the feedback law ρ consists of the following single input rule modules (SIRMs) are given [3].

$$\text{SIRM-}i : \{R_j^i : \text{if } x_i = A_j^i \text{ then } y = C_j^i\}_{j=1}^{m_i} \quad (i = 1, 2, \ldots, n) \tag{8}$$

Here, n is the number of rule modules and premise variables, and $m_i (i = 1, 2, \ldots, n)$ are the numbers of single input rules in each rule module SIRM-i. $x_1, x_2, \ldots$ and x_n are premise variables, and y is consequent variable. x is the state of (1) and also input of this SIRMs connected fuzzy inference model.

Let $A_j^i(x_i)$ and $C_j^i(y)$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m_i)$ be fuzzy grade of each fuzzy set A_j^i and C_j^i for input x_i and consequent output y in the j-th rule R_j^i of the SIRM-i respectively. The membership functions of fuzzy set A_j^i and C_j^i are written as same character A_j^i and C_j^i in this paper. For simplicity, we write "if" and "then" parts in the rules by the following notation: $\mathcal{A}^i = (A_1^i, A_2^i, \ldots, A_{m_i}^i)$, $\mathcal{C}^i = (C_1^i, C_2^i, \ldots, C_{m_i}^i)$, $\mathcal{A} = (\mathcal{A}^1, \mathcal{A}^2, \ldots, \mathcal{A}^n)$, $\mathcal{C} = (\mathcal{C}^1, \mathcal{C}^2, \ldots, \mathcal{C}^n)$. Then, SIRMs (8) is named a fuzzy controller in this paper, and is denoted by $(\mathcal{A}, \mathcal{C})$ which is the pair of the membership functions.

When an input information $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ at certain time $t \in [0, T]$ is given to the fuzzy controller $(\mathcal{A}, \mathcal{C})$ (8), then one can obtain the amount of operation from the controller through the following procedures. Although product-sum or min-max operations are used in SIRMs connected fuzzy inference proposed by Yubazaki, bounded product and bounded sum are applied for interpreting "ands" and "ors" respectively.

Procedure 1: The inference result of j-th rule R_j^i in SIRM-i is calculated by

$$\alpha_j^i(x_i, y) = A_j^i(x_i) \odot C_j^i(y) \quad (j = 1, 2 \ldots, m_i;\ i = 1, 2, \ldots, n).$$

Here, $\odot$ means bounded product between premise and consequent part.

$$A_j^i(x_i) \odot C_j^i(y) = (A_j^i(x_i) + C_j^i(y) - 1) \vee 0.$$

That is to say, this operation presses graph of consequent membership function $C_j^i(y)$ down by $(1 - A_j^i(x_i))$.

Procedure 2: The inference result of rule group SIRM-i is calculated by integrating the inference results of each rules R_j^i as follows:

$$\beta_i(x_i, y) = \bigoplus_{j=1}^{m_i} \alpha_j^i(x_i, y) \quad (i = 1, 2, \ldots, n).$$

Here, $\oplus$ is bounded sum. For example, in case $j \le 2$,

$$\alpha_1^i(x_i, y) \oplus \alpha_2^i(x_i, y) = (\alpha_1^i(x_i, y) + \alpha_2^i(x_i, y)) \wedge 1.$$

Procedure 3: Defuzzification stage. The crisp outputs of each SIRM-i are obtained as centers of gravity of inference results $\beta_i(x_i, y)$.

$$\gamma_i(x_i) = \frac{\int y \beta_i(x_i, y) dy}{\int \beta_i(x_i, y) dy} \quad (i = 1, 2, \ldots, n).$$

In SIRMs inference method, importance degrees $d_i (i = 1, 2, \cdots, n)$ are introduced to give each SIRMs weight of contribution. In the same way as $\mathcal{A}$ and $\mathcal{C}$, put $d = (d_1, d_2, \ldots, d_n)$.

Procedure 4: The inference result of all rule modules is calculated as weighted sum of all $\gamma_i(x_i)$ using the importance degree $d = (d_1, d_2, \ldots, d_n)$.

$$\rho_{\mathcal{A}\mathcal{C}d}(x) = \sum_{i=1}^{n} d_i \gamma_i(x_i).$$

Here, since the inference result depends on the membership functions in the premise and consequent fuzzy sets, and the importance degrees, the subscripts $\mathcal{A}$, $\mathcal{C}$ and d are added to the function.

3 Compactness of a Family of Sets of Membership Functions

Fix a sufficiently large $r > 0$, $r_2 > 0$ and a final time T of the control (1) according to section 2.1. Put r_1 be the positive constant determined by (2). We also fix $\Delta_{ij} > 0$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m_i)$. Let $C[-r_1, r_1]$ and $C[-r_2, r_2]$ be the Banach space of all continuous real functions on $[-r_1, r_1]$ and $[-r_2, r_2]$ respectively. We consider the following two sets of fuzzy membership functions.

$$F_{\Delta_{ij}} = \{\mu \in C[-r_1, r_1];\ 0 \le \mu(x) \le 1 \text{ for } \forall x \in [-r_1, r_1],$$
$$|\mu(x) - \mu(x')| \le \Delta_{ij}|x - x'| \text{ for } \forall x, x' \in [-r_1, r_1]\},$$

$$G = \{\mu \in C[-r_2, r_2];\ 0 \le \mu(y) \le 1 \text{ for } \forall y \in [-r_2, r_2]\}.$$

The set $F_{\Delta_{ij}}$ above, which is more restrictive than G, contains triangular, trapezoidal and bell-shaped fuzzy membership functions with gradients less than positive value Δ_{ij}. Consequently, if $\Delta_{ij} > 0$ is taken large enough, $F_{\Delta_{ij}}$ contains almost all fuzzy membership functions which are used in practical applications. In this study, we shall assume that the fuzzy membership functions A^i_j in premise parts of the SIRMs (8) belong to the set $F_{\Delta_{ij}}$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m_i$. On the other hand, we assume that the membership function C^i_j in consequent part belongs to G.

In the following, we endow the space $F_{\Delta_{ij}}$ and G with norm topology on the space of continuous functions. Then, for all $i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m_i$, $F_{\Delta_{ij}}$ and G are compact [5]. Put

$$\mathcal{L}' = \prod_{i=1}^{n} \left\{ \prod_{j=1}^{m_i} \left(F_{\Delta_{ij}} \times G\right) \right\}.$$

Then, every element $(\mathcal{A}, \mathcal{C})$ of $\mathcal{L}'$ is fuzzy controller given by the SIRMs (8). By the Tychonoff theorem [9], we can have following proposition.

Proposition 2. *$\mathcal{L}'$ is compact and metrizable with respect to the product topology on* $\prod_{i=1}^{n} (C[-r_1, r_1] \times C[-r_2, r_2])^{m_i}$.

Let n-tuple of importance degrees d join with fuzzy controller $(\mathcal{A}, \mathcal{C})$. Then it is denoted by $(\mathcal{A}, \mathcal{C}, d)$. We can consider $(\mathcal{A}, \mathcal{C}, d)$ as the pair of SIRMs and importance degree and newly call it SIRMs fuzzy controller.

$$D = \left\{ d = (d_1, d_2, \ldots, d_n) \in \mathbb{R}^n ;\ \forall i = 1, 2, \ldots, n,\ d_i \in (0, 1), \sum_{i=1}^{n} d_i \leq 1 \right\}$$

In this paper, it is assumed that each $d_i (i = 1, 2, \cdots, n)$ is belonging to closed interval $(0, 1)$, and satisfies $\sum_{i=1}^{n} d_i \leq 1$. Yubazaki does not need this condition [3], but for the existence of solution of the state equation (1) it is needed in this study.

Put $\mathcal{L} = \mathcal{L}' \times D$. Then $\mathcal{L}$ is Cartesian product and consists of SIRMs and importance degrees of inference calculations. And it is obvious that $(\mathcal{A}, \mathcal{C}, d) \in \mathcal{L}$.

To avoid making the denominator of the fractional expressions in the defuzzification stage in the previous section equal to 0, for any $\delta > 0$, consider the set:

$$\mathcal{L}_\delta = \left\{ (\mathcal{A}, \mathcal{C}, d) \in \mathcal{L};\ \forall i = 1, \ldots, n, \forall x \in [-r_1, r_1]^n, \int_{-r_2}^{r_2} \beta_i(x_i, y) dy \geq \delta \right\}, \quad (9)$$

which is a slight modification of $\mathcal{L}$. If δ is taken small enough, it is possible to consider $\mathcal{L} = \mathcal{L}_\delta$ for practical applications. An element $(\mathcal{A}, \mathcal{C}, d)$ of $\mathcal{L}_\delta$ is an admissible fuzzy controller. Since $\mathcal{L}_\delta$ is a closed subset of $\mathcal{L}$, we can have the following proposition.

Proposition 3. *The set $\mathcal{L}_\delta$ of all admissible fuzzy controllers is compact and metrizable with respect to the product topology.*

4 Continuity of Approximate Reasoning and Its Application

For any $(\mathcal{A}, \mathcal{C}, d)$ in $\mathcal{L}_\delta$ from (9), we define the feedback control low $u(x) = \rho_{\mathcal{AC}d}(x)$ of the state equation (1) at certain time $t \in [0, T]$ on the basis of the SIRMs (8).

$$\rho_{\mathcal{AC}d}(x) = \sum_{i=1}^{n} d_i \frac{\int_{-r_2}^{r_2} y \beta_i(x_i, y) dy}{\int_{-r_2}^{r_2} \beta_i(x_i, y) dy}, \quad \text{where} \quad \beta_i(x_i, y) = \bigoplus_{j=1}^{m_i} A_j^i(x_i) \odot C_j^i(y).$$

Then, the following proposition about $\rho_{\mathcal{AC}d}$ is obtained.

Proposition 4. *Let $(\mathcal{A}, \mathcal{C}, d) \in \mathcal{L}_\delta$. Then, the following* a) *and* b) *hold.*
a) *$\rho_{\mathcal{AC}d}$ is Lipschitz continuous on $[-r_1, r_1]^n$.*
b) *$|\rho_{\mathcal{AC}d}(x)| \leq r_2$ for all $x \in [-r_1, r_1]^n$.*

Proof. ajSince $\rho_{\mathcal{A}\mathcal{C}d}$ is the composite mapping of α_j^i, β_i and γ_i in section 2.3, Lipschitz continuity of each function is proved separately. For all $x=(x_1,\ldots,x_n), x'=(x_1',\ldots,x_n') \in [-r_1,r_1]^n$, $i=1,2,\ldots,n; j=1,2,\ldots,m_i$,

$$\begin{aligned}
&|\alpha_j^i(x_i,y)-\alpha_j^i(x_i',y)| = |A_j^i(x_i)\odot C_j^i(y) - A_j^i(x_i')\odot C_j^i(y)| \\
&\leq \frac{1}{2}\left\{|A_j^i(x_i)-A_j^i(x_i')| + \left||A_j^i(x_i)+C_j^i(y)-1| - |A_j^i(x_i')+C_j^i(y)-1|\right|\right\} \\
&\leq |A_j^i(x_i)-A_j^i(x_i')| \leq \Delta_{ij}|x_i-x_i'|,
\end{aligned}$$

where Δ_{ij} is Lipschitz constant defined by previous section. This inequality means Lipschitz continuity of the mapping α_j^i.

Mathematical induction for mapping β_i is employed. For each $i=1,2,\ldots,n$, assume that

$$\left|\bigoplus_{j=1}^{m_i-1}\alpha_j^i(x_i,y) - \bigoplus_{j=1}^{m_i-1}\alpha_j^i(x_i',y)\right| \leq \Delta_{(m_i-1)}|x_i-x_i'|.$$

Here $\Delta_{(m_i-1)}$ is Lipschitz constant. Then we have

$$\begin{aligned}
&\left|\bigoplus_{j=1}^{m_i-1}\alpha_j^i(x_i,y)\oplus\alpha_{m_i}^i(x_i,y) - \bigoplus_{j=1}^{m_i-1}\alpha_j^i(x_i',y)\oplus\alpha_{m_i}^i(x_i',y)\right| \\
&\leq \left|\bigoplus_{j=1}^{m_i-1}\alpha_j^i(x_i,y) - \bigoplus_{j=1}^{m_i-1}\alpha_j^i(x_i',y)\right| + |\alpha_{m_i}^i(x_i,y)-\alpha_{m_i}^i(x_i',y)| \\
&\leq \Delta_{(m_i-1)}|x_i-x_i'| + \Delta_{im_i}|x_i-x_i'| = (\Delta_{(m_i-1)}+\Delta_{im_i})|x_i-x_i'|.
\end{aligned}$$

Thus $\beta_i(x_i,y)=\bigoplus_{j=1}^{m_i}\alpha_j^i(x_i,y)$ is Lipschitz continuous on $[-r_1,r_1]^n$. Note that

$$|\gamma_i(x_i)-\gamma_i(x_i')| \leq \frac{4{r_2}^3}{\delta^2}m_i|\beta_i(x_i,y)-\beta_i(x_i',y)|,$$

by [7] and $d_i<1$, we find that

$$\begin{aligned}
&|\rho_{\mathcal{A}\mathcal{C}d}(x)-\rho_{\mathcal{A}\mathcal{C}d}(x')| \leq \sum_{i=1}^n d_i|\gamma_i(x_i)-\gamma_i(x_i')| \\
&\leq \frac{4{r_2}^3}{\delta^2}\sum_{i=1}^n m_i|\beta_i(x_i,y)-\beta_i(x_i',y)| \leq \frac{4{r_2}^3}{\delta^2}\sum_{i=1}^n\left\{m_i(\Delta_{(m_i-1)}+\Delta_{im_i})\right\}\|x-x'\|.
\end{aligned}$$

Then the proof of Lipschitz continuity of $\rho_{\mathcal{A}\mathcal{C}d}$ on $[-r_1,r_1]^n$ is complete.
b) The proof is omitted.

It needs that the domain of bounded Lipschitz function $\rho_{\mathcal{A}\mathcal{C}d}$ is expanded from $[-r_1,r_1]^n$ to $\mathbb{R}^n$ to adopt the previous proposition 4 to the proposition 1.

It is possible without increasing its Lipschitz constant and bound though details are omitted. Then the state equation (1) for the feedback law $\rho_{\mathcal{AC}d}$ has a unique solution $x(t, x_0, \rho_{\mathcal{AC}d})$ with the initial condition $x(0) = x_0$ [10]. Consequently, in the following the extension Lipschitz function is written as $\rho_{\mathcal{AC}d}$ without confusion.

The optimal control problem is considered to be the calculus of variations by treating $\rho_{\mathcal{AC}d}$ as functional on the family of sets of membership functions $\mathcal{L}_\delta$. The following proposition is obtained.

Proposition 5. *The mapping*

$$J = \int_{B_r} \int_0^T w(x(t, \zeta, \rho_{\mathcal{AC}d}), \rho_{\mathcal{AC}d}(x(t, \zeta, \rho_{\mathcal{AC}d})))dtd\zeta$$

has a minimum (maximum) value on the compact space $\mathcal{L}_\delta$ *defined by* (9).

Proof. It suffices to prove the continuity of $\rho_{\mathcal{AC}d}$ on $\mathcal{L}_\delta$ as functional. For all $i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m_i$, note that

$$\left| {A_j^i}^k(x_i) \odot {C_j^i}^k(y) - A_j^i(x_i) \odot C_j^i(y) \right| \leq \left| {A_j^i}^k(x_i) - A_j^i(x_i) \right| + \left| {C_j^i}^k(y) - C_j^i(y) \right|,$$

and

$$\left| \bigoplus_{j=1}^{m_i} {\alpha_j^i}^k(x_i, y) - \bigoplus_{j=1}^{m_i} \alpha_j^i(x_i, y) \right| = \left| \sum_{j=1}^{m_i} {\alpha_j^i}^k(x_i, y) \wedge 1 - \sum_{j=1}^{m_i} \alpha_j^i(x_i, y) \wedge 1 \right|.$$

Routine calculation gives the estimate

$$\begin{aligned} &|\rho_{\mathcal{A}^k\mathcal{C}^k d^k}(x) - \rho_{\mathcal{AC}d}(x)| \\ &\leq {r_2}^2 \sum_{i=1}^{n} \left| {d_i}^k - d_i \right| + \frac{1}{\delta^2} \sum_{i=1}^{n} \left(2{r_2}^3 \sum_{j=1}^{m_i} \left| {A_j^i}^k(x_i) - A_j^i(x_i) \right| \right. \\ &\left. + {r_2}^2 \int_{-r_2}^{r_2} \sum_{j=1}^{m_i} \left| {C_j^i}^k(y) - C_j^i(y) \right| dy + 2r_2 \int_{-r_2}^{r_2} |y| \sum_{j=1}^{m_i} \left| {C_j^i}^k(y) - C_j^i(y) \right| dy \right). \end{aligned}$$

Assume that $(\mathcal{A}^k, \mathcal{C}^k, d^k) \to (\mathcal{A}, \mathcal{C}, d)$ $(k \to \infty)$ in $\mathcal{F}_\delta$. Then it follows from the estimate above that

$$\lim_{k\to\infty} \sup_{x\in[-r_1, r_1]^n} |\rho_{\mathcal{A}^k\mathcal{C}^k d^k}(x) - \rho_{\mathcal{AC}d}(x)| = 0.$$

Thus $\rho_{\mathcal{AC}d}$ is continuous on $\mathcal{L}_\delta$. Then the existence of minimum (maximum) value of J is proved by applying lemma 1.

This proposition guarantees the existence of SIRMs fuzzy controller $(\mathcal{A}, \mathcal{C}, d) \in \mathcal{L}_\delta$ which minimizes (maximizes) the previous performance functional.

5 Conclusion

A mathematical analysis of SIRMs connected fuzzy inference method, in which bounded product and bounded sum are applied as operations, has been suggested. The set of membership functions which compose SIRMs is considered as variable of the performance functional, and the approximate reasoning calculation of SIRMs method is considered as functional on the family of sets of membership functions. Since the performance functional is composite mapping of the approximate reasoning calculation, the family of sets of membership functions is its domain. The compactness of the family of sets of membership functions and continuity of performance functional on it have been mainly discussed. Based on the above mathematical analysis, we conclude that there exists minimum value of performance function on the family of sets of membership function. Generally speaking, the SIRMs which give optimal control exist. Future research will focus on method of solution by the calculus of variations.

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