Chapter 4
Choi Matrices and Dual Functionals

In the theory of positive maps from the $n \times n$ matrices M_n ($=B(K)$ with $K = \mathbb{C}^n$) into $B(H)$, the Choi matrix corresponding to a map is very important. The present chapter is devoted to the close relationship between maps and their Choi matrices. In Sect. 4.1 we present the basic definitions and results. Then in Sect. 4.2 we introduce the dual functional to a map and show how its properties reflect the positivity properties of the map.

4.1 The Choi Matrix

In this section K is a finite dimensional Hilbert space. The vector space of linear maps of $B(K)$ into $B(H)$ can be identified with $B(K) \otimes B(H)$. In our treatment this identification will be done via the Choi matrix for a map.

Definition 4.1.1 Let $K = \mathbb{C}^n$ and let $\phi : B(K) \to B(H)$ be a linear map. Let (e_{ij}), $i, j = 1, \dots, n$ be a complete set of matrix units for $B(K)$. Then the *Choi matrix* for ϕ is the operator

$$C_\phi = \sum_{i,j=1}^{n} e_{ij} \otimes \phi(e_{ij}) \in B(K) \otimes B(H).$$

The map $\phi \to C_\phi$ is clearly linear and injective, and given an operator $\sum e_{ij} \otimes a_{ij} \in B(K) \otimes B(H)$, then we can define a linear map ϕ by $\phi(e_{ij}) = a_{ij}$. Thus the map $\phi \to C_\phi$ is surjective. This map is often called the *Jamiolkowski isomorphism*.

As defined the Choi matrix depends on the choice of matrix units (e_{ij}). The next lemma describes it with respect to another set of matrix units. Recall the notation $B(B(K), H)$ is the linear space of all linear maps from $B(K)$ into $B(H)$.

Lemma 4.1.2 *Let $\phi \in B(B(K), H)$ have Choi matrix C_ϕ with respect to a complete set of matrix units (e_{ij}). Let (f_{ij}) be another complete set of matrix units and w*

E. Størmer, *Positive Linear Maps of Operator Algebras*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-34369-8_4,

a unitary operator such that $w^* e_{ij} w = f_{ij}$. *Then the Choi matrix* C_ϕ^f *with respect to* (f_{ij}) *is given by*

$$C_\phi^f = Ad(w \otimes 1)(C_{\phi \circ Adw}).$$

Proof

$$\begin{aligned} C_{\phi \circ Adw} &= \sum e_{ij} \otimes \phi\big(w^* e_{ij} w\big) \\ &= \sum e_{ij} \otimes \phi(f_{ij}) \\ &= (w \otimes 1)\Big(\sum_{i,j} f_{ij} \otimes \phi(f_{ij})\big(w^* \otimes 1\big)\Big). \end{aligned}$$

Hence $C_\phi^f = (w^* \otimes 1) C_{\phi \circ Adw} (w \otimes 1)$. □

Two special cases are important.

Proposition 4.1.3 *Let* ω *be a linear functional on* $B(K)$ *with density operator* h, *viz.* $\omega(a) = Tr(ha)$, $a \in B(K)$. *Let* $a \in B(H)^+$, *and identify* $b\omega$ *with the map* $a \to \omega(a)b$ *of* $B(K)$ *into* $B(H)$. *Then*

$$C_{b\omega} = h^t \otimes b.$$

Proof

$$\begin{aligned} C_{b\omega} &= \sum e_{ij} \otimes \omega(e_{ij}) b \\ &= \sum \omega(e_{ij}) e_{ij} \otimes b \\ &= \sum Tr(h e_{ij}) e_{ij} \otimes b \\ &= \sum h_{ji} e_{ij} \otimes b \\ &= h^t \otimes b. \end{aligned}$$

□

Proposition 4.1.4 *Suppose* $\dim H = m < \infty$. *Let* $\xi_1, \ldots, \xi_n$ *(resp.* $\eta_1, \ldots, \eta_m$*) be an orthonormal basis for* K *(resp.* H*), and* (e_{ij}) *(resp.* (f_{kl})*) be the corresponding complete set of matrix units, so* $e_{ij}\xi_k = \delta_{jk}\xi_i$, *and similarly for* (f_{kl}). *Let* $V : H \to K$ *be defined by* $V\eta_k = \sum_i v_{ik}\xi_i$. *Let*

$$g_{(i,k),(j,l)} = e_{ij} \otimes f_{kl}.$$

Then the set $(g_{(i,k),(j,l)})$ *is a complete set of matrix units for* $B(K \otimes H)$, *and*

$$C_{AdV} = \sum v_{jl} \overline{v}_{ik} g_{(i,k),(j,l)}$$

is a positive scalar multiple of the projection onto $\omega = \sum \overline{v}_{ik} \xi_i \otimes \eta_k$.

Proof It is obvious that $(g_{(i,k),(j,l)})$ is a complete set of matrix units for $B(K \otimes H)$. Let $\xi = \sum a_k \eta_k \in H$. Then

$$\begin{aligned}(\xi, V^*\xi_i) = (V\xi, \xi_i) &= \sum_k a_k (V\eta_k, \xi_i) \\ &= \sum_k a_k v_{ik} = \sum_k a_k (\eta_k, \overline{v}_{ik}\eta_k) = \sum_k (\xi, \overline{v}_{ik}\eta_k).\end{aligned}$$

Thus

$$V^*\xi_i = \sum_k \overline{v}_{ik}\eta_k, \quad \text{for all } i. \tag{4.1}$$

It follows that

$$V^* e_{ij} V \eta_k = V^* e_{ij} \sum_s v_{sk}\xi_s = V^* v_{jk}\xi_i = \sum_l v_{jk}\overline{v}_{il}\eta_l.$$

Therefore we get

$$\begin{aligned}C_{AdV}(\xi_s \otimes \eta_t) &= \left(\sum_{ij} e_{ij} \otimes V^* e_{ij} V\right)(\xi_s \otimes \eta_t) \\ &= \sum \xi_i \otimes v_{st}\overline{v}_{ik}\eta_k \\ &= \left(\sum_{ik} e_{is} \otimes v_{st}\overline{v}_{ik} f_{kt}\right)(\xi_s \otimes \eta_t) \\ &= \left(\sum_{ik} v_{st}\overline{v}_{ik} g_{(i,k)(s,t)}\right)(\xi_s \otimes \eta_t).\end{aligned}$$

Thus

$$C_{AdV} = \sum_{i,j,k,l} v_{jl}\overline{v}_{ik} g_{(i,k)(j,l)}.$$

□

In the above proposition the rank of V is reflected in how ω is written as a tensor product of vectors.

Definition 4.1.5 Let $\xi \in K \otimes H$. Then ξ has *Schmidt rank r* denoted by $SR\xi$, if r is the smallest number m such that ξ can be written as $\xi = \sum_{i=1}^m \xi_i \otimes \eta_i$ with $\xi_i \in K$, $\eta_i \in H$.

Then we can find an orthonormal family $\omega_1, \dots, \omega_r \in H$ and vectors $\rho_i \in K$ such that $\xi = \sum_{i=1}^r \rho_i \otimes \omega_i$. To show this, note that the span of the η_i's must be r-dimensional by minimality of r, so we can write the η_i's as linear combinations of r orthonormal vectors $\omega_1, \dots, \omega_r$ in H. Using this we can give more specific information on V and ω in the last proposition.

Proposition 4.1.6 *Let $V : H \to K$ and ω be as in Proposition 4.1.4. Then $C_{AdV} = \lambda[\omega]$ for some $\lambda \geq 0$. ω has Schmidt rank r if and only if* rank $V = r$.

Proof Suppose rank $V = r$. Choose an orthonormal basis $\eta_1, \dots, \eta_m$ for H such that $V^*V\eta_k = \lambda_k \eta_k$ with $\lambda_1, \dots, \lambda_r > 0$ and $\lambda_k = 0$ for $k > r$. Let $\xi_1, \dots, \xi_n$ be an orthonormal basis for K. By Proposition 4.1.4

$$C_{AdV} = \lambda[\omega], \quad \omega = \sum_k \Big(\sum_i \overline{v}_{ik}\xi_i\Big) \otimes \eta_k$$

and

$$V\eta_k = \sum_i v_{ik}\xi_i.$$

Thus by (4.1)

$$\lambda_k \eta_k = V^*V\eta_k = \sum_i V^* v_{ik}\xi_i = \sum_{i,l} v_{ik}\overline{v}_{il}\eta_l, \tag{4.2}$$

hence $\overline{v}_{il} = 0$ for $l \neq k$, and $\sum_i |v_{ik}|^2 = \lambda_k$. Thus $v_{ik} \neq 0$ for some i when $k \leq r$, so that ω has Schmidt rank r.

Conversely, if $SR\omega = r$, choose and orthonormal basis $\eta_1, \dots, \eta_m$ in H such that

$$\omega = \sum_{k=1}^{r} \Big(\sum_i \overline{v_{ik}}\xi_i\Big) \otimes \eta_k = \sum_{ik} \overline{v}_{ik}\xi_i \otimes \eta_k.$$

If we define $V : H \to K$ by $V\eta_k = \sum_i v_{ik}\xi_i \neq 0$ if $k \leq r$, and $V\eta_k = 0$ for $k > r$, Proposition 4.1.4 shows us that C_{AdV} is a scalar multiplum of $[\omega]$. By construction V has rank r. Since $\phi \to C_\phi$ is an isomorphism, and $AdV = AdW$ if and only if $W = zV$, $|z| = 1$, the rank of V is uniquely defined whenever $C_{AdV} = \lambda[\omega]$ with $\lambda > 0$. Thus rank $V = r$. □

Remark 4.1.7 If dim $K = n$, and ι denotes the identity map of $B(K)$ into itself, then for $V = 1$ we get

$$C_\iota = C_{Ad1} = \sum e_{ij} \otimes e_{ij}$$

is n times the projection onto $\frac{1}{\sqrt{n}}\xi_i \otimes \xi_i$, called the *maximally entangled state*. For more on entanglement see the discussion after Proposition 4.1.11 and Sect. 7.4.

Note that by Proposition 4.1.6 rank $V = 1$ if and only if $C_{AdV} = \lambda[\xi] \otimes [\eta]$ if and only if $\omega = \xi \otimes \eta$ is a product vector.

As an immediate consequence of Proposition 4.1.4 we have

Theorem 4.1.8 *Let K and H be finite dimensional and $\phi \in B(B(K), H)$. Then the following conditions are equivalent*:

(i) *ϕ is completely positive.*
(ii) $C_\phi \geq 0$.
(iii) $\phi = \sum_{i=1}^{m} AdV_i$ *with* $V_i : H \to K$ *linear, and* $m \leq \dim K \cdot \dim H$.
(iv) $\phi = \sum_{i=1}^{k} AdW_i$, *with* $W_i : H \to K$ *linear and* $k \in \mathbb{N}$.

Proof (i) $\Rightarrow$ (ii). Let $n = \dim K$, $m = \dim H$. If ϕ is completely positive then $\iota_n \otimes \phi : M_n \otimes B(K) \to M_n \otimes B(H)$ is positive, where $\iota_n$is the identity map on M_n. Hence

$$C_\phi = \sum_{ij} e_{ij} \otimes \phi(e_{ij}) = \iota_n \otimes \phi\left(\sum_{ij} e_{ij} \otimes e_{ij}\right) \geq 0.$$

(ii) $\Rightarrow$ (iii). If $C_\phi \geq 0$ then $C_\phi = \sum_{i=1}^{mn} \lambda_i [\omega_i]$ with ω_i an orthonormal basis for $K \otimes H$, $1 \leq i \leq mn$, $\lambda_i \geq 0$. By Proposition 4.1.4 $[\omega_i] = C_{AdV_i}$ for an operator $V_i : H \to K$. Thus $\phi = \sum_1^{mn} \lambda_i AdV_i$. If we replace V_i by $\lambda_i^{-1/2} V_i$ whenever $\lambda_i \neq 0$, we have (iii).

(iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i). This follows since $\iota_n \otimes AdV = Ad(\iota_n \otimes V)$ is positive, so AdV is completely positive (see also Lemma 1.2.2). □

The decomposition (iii) in the above theorem is usually called the *Kraus decomposition* for ϕ.

Corollary 4.1.9 *Let* $\phi : B(K) \to B(H)$, *with* $K = \mathbb{C}^n$, $H = \mathbb{C}^m$, *and let* $k = \min(m, n)$. *Then ϕ is completely positive if and only if ϕ is k-positive.*

Proof Suppose ϕ is k-positive. Assume first $k = n$. Then $\iota_n \otimes \phi$ is positive, so $C_\phi = \iota_n \otimes \phi(\sum_{ij} e_{ij} \otimes e_{ij}) \geq 0$. Thus by Theorem 4.1.8 ϕ is completely positive. If $k = m$ then $\phi^* : B(H) \to B(K)$ is k-positive from Proposition 1.4.3, hence by the first part ϕ^* is completely positive. Then by the same proposition ϕ is completely positive. The converse is obvious. □

The above corollary can be extended to maps of C^*-algebras. Then it states that every k-positive map of a C^*-algebra A into another B is completely positive if and only if either A or B has all its irreducible representations on Hilbert spaces of dimension less than or equal to k, see [93].

We shall need to know the Choi matrix for ϕ^* when $\phi \in P(H)$, the cone of positive maps of $B(H)$ into itself.

Lemma 4.1.10 *Let* $\dim H = n$ *and* $\xi_1, \ldots, \xi_n$ *be an orthonormal basis for* H. *Let* J *be the conjugation of* $H \otimes H$ *defined by*

$$Jz\xi_i \otimes \xi_j = \bar{z}\xi_j \otimes \xi_i$$

with $z \in \mathbb{C}$. *Let* $\phi \in P(H)$. *Then* $C_{\phi^*} = JC_\phi J$.

Proof Let $V = (v_{ij})_{i,j \leq n} \in B(H)$, and let e_{ij} denote the matrix units such that $e_{ij}\xi_k = \delta_{jk}\xi_i$. Then a straightforward computation yields

$$AdV(e_{kl}) = V^* e_{kl} V = (\overline{v}_{ki} v_{lj})_{ij}.$$

Since $V^* = (\overline{v}_{ji})$ it follows that

$$AdV^*(e_{kl}) = V e_{kl} V^* = (v_{ik}\overline{v}_{jl})_{ij}.$$

From the definition of J it thus follows that

$$\begin{aligned} JC_{AdV}J(z\xi_p \otimes \xi_q) &= J\Big(\sum_{ijkl} e_{kl} \otimes \overline{v}_{ki} v_{lj} e_{ij}\Big)\overline{z}\xi_q \otimes \xi_p \\ &= \sum v_{ki}\overline{v}_{lj} e_{ij}\xi_p \otimes z e_{kl}\xi_q \\ &= \Big(\sum_{ik} v_{ik}\overline{v}_{jl} e_{kl} \otimes e_{ij}\Big)(z\xi_p \otimes \xi_q) \\ &= \Big(\sum e_{kl} \otimes V e_{kl} V^*\Big)(z\xi_p \otimes \xi_q) \\ &= C_{AdV^*}(z\xi_p \otimes \xi_q), \end{aligned}$$

where we at the third equality sign exchanged (i, j) with (k, l). Since the vectors $\xi_p \otimes \xi_q$ form a basis for $H \otimes H$, $JC_{AdV}J = C_{AdV^*}$. Now, if ϕ is a positive map then C_ϕ is self-adjoint, hence the difference between two positive operators, which both are Choi matrices for completely positive maps by Theorem 4.1.8. Hence by Theorem 4.1.8 again ϕ is a real linear sum of maps AdV. By Proposition 1.4.2 the adjoint map of AdV is AdV^*. Applying this to each summand AdV, we thus get $JC_\phi J = C_{\phi^*}$. □

Proposition 4.1.11 *Let H be a Hilbert space of arbitrary dimension. Let $\phi \in B(B(K), H)$. Then ϕ is positive if and only if $Tr(C_\phi\, a \otimes b) \geq 0$ for all $a \in B(K)^+$ and b a positive trace class operator on H.*

Proof Computing we get

$$\begin{aligned} Tr(C_\phi a \otimes b) &= \sum_{ij} Tr\big((e_{ij} \otimes \phi(e_{ij}))(a \otimes b)\big) \\ &= \sum_{ij} Tr(e_{ij}a)Tr\big(\phi(e_{ij})b\big) \\ &= \sum a_{ji} Tr\big(\phi(e_{ij})b\big) \\ &= Tr\big(\phi(a^t)b\big). \end{aligned}$$

Since this holds for all positive trace class operators b, and $a \geq 0$ if and only if $a^t \geq 0$, $\phi(a) \geq 0$ if and only if $Tr(C_\phi a \otimes b) \geq 0$ for all positive a and b. □

In quantum information theory C_ϕ is often called an *entanglement witness* when ϕ is not completely positive, because the proposition shows that if $h = \sum a_i \otimes b_i \geq 0$ is the density operator for a state ω on $B(K \otimes H)$, then ω is *entangled*, i.e. h cannot be written in the above form with all $a_i, b_i \geq 0$, if there exists a positive map $\phi : B(K) \to B(H)$ such that $Tr(C_\phi h) < 0$.

Let $\phi \in B(B(K), H)$ be a self-adjoint linear map, so $\phi(a)$ is self-adjoint when a is self-adjoint. Then it is easily seen that C_ϕ is a self-adjoint operator, hence is the difference of two positive operators C_ϕ^+ and C_ϕ^- such that $C_\phi^+ C_\phi^- = 0$.

We shall see later, Theorem 7.4.3, that C_ϕ^- contains much information. Presently we concentrate on C_ϕ^+. Let $c \geq 0$ be the smallest positive number such that $c1 \geq C_\phi$. Then $c = \|C_\phi^+\|$. Hence, if $c \neq 0$ there exists a map $\phi_{cp} : B(K) \to B(H)$ such that its Choi matrix $C_{\phi_{cp}} = 1 - \frac{1}{c} C_\phi$ is a positive operator. Thus if Tr is identified with the positive map $a \to Tr(a)1$, it is straightforward to show that $C_{Tr} = 1$, so $\frac{1}{c}\phi = Tr - \phi_{cp}$. By Theorem 4.1.8, ϕ_{cp} is completely positive. We have

Theorem 4.1.12 *Let $\phi \in B(B(K), H)$ be a self-adjoint linear map such that $-\phi$ is not completely positive. Then there exists a completely positive map $\phi_{cp} : B(K) \to B(H)$ such that*

$$\left\|C_\phi^+\right\|^{-1} \phi = Tr - \phi_{cp}.$$

Furthermore, ϕ is positive if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all product states $\rho = \omega_1 \otimes \omega_2$ on $B(K) \otimes B(H)$.

Proof The existence of ϕ_{cp} was shown above. To show the second part let $\rho(x) = Tr \otimes Tr((a \otimes b)x)$ be a product state on $B(K) \otimes B(H)$ with density operator $a \otimes b$. Then

$$\rho(C_{\phi_{cp}}) = Tr \otimes Tr(C_{\phi_{cp}} a \otimes b),$$

so that $Tr(C_\phi a \otimes b) \geq 0$ if and only if $\rho(C_{\phi_{cp}}) \leq 1$. Hence the theorem follows from Proposition 4.1.11. □

Recall from Definition 1.2.1 that a map ϕ is k-positive if $\iota_k \otimes \phi$ is positive, where ι_k denotes the identity map on M_k. We now give several characterizations of k-positive maps, one of them in terms of the Choi matrix.

Definition 4.1.13 An operator C on $K \otimes H$ is called *k-block positive* if $(C \sum_{i=1}^k \xi_i \otimes \eta_i, \sum_{i=1}^k \xi_i \otimes \eta_i) \geq 0$ for all choices of vectors $\xi_1, \ldots, \xi_k \in K$, and $\eta_1, \ldots, \eta_k \in H$.

Remark 4.1.14 Note that a vector $\xi \in K \otimes H$ is of the form $\sum_{i=1}^k \xi_i \otimes \eta_i$ if and only if $\xi = (1 \otimes q)\psi$ for a vector $\psi \in K \otimes H$ and projection $q \in B(H)$ of dimension k.

Indeed, if $\xi = \sum_{i=1}^{k} \xi_i \otimes \eta_i$ let q denote the projection onto the span of $\eta_1, \dots, \eta_k$, then $\xi = (1 \otimes q)\xi$. Conversely, if $\xi = (1 \otimes q)\psi$ with $\psi = \sum_{i=1}^{n} \xi_i \otimes \eta_i$, q as above, we can choose a basis $\gamma_1, \dots, \gamma_k$ for qH such that $q\eta_i = \sum \alpha_{ij}\gamma_j$. Then

$$1 \otimes q(\psi) = \sum \xi_i \otimes q\eta_i = \sum \alpha_{ij}\xi_i \otimes \gamma_j = \sum_{j=1}^{k} \Big(\sum_i \alpha_{ij}\xi_i \Big) \otimes \gamma_j .$$

The same argument also yields that $\xi = \sum_1^k \xi_i \otimes \eta_i$ if and only if $\xi = p \otimes q(\psi)$ for $\psi \in K \otimes H$, and p and q k-dimensional projections in $B(K)$ and $B(H)$ respectively.

Theorem 4.1.15 *Let $\phi \in B(B(K), H)$ and $k \leq \min(\dim K, \dim H)$. Then the following conditions are equivalent.*

(i) *ϕ is k-positive.*
(ii) *$\phi \circ AdV$ is completely positive for all $V \in B(K)$ with* $\operatorname{rank} V \leq k$.
(iii) *$AdW \circ \phi$ is completely positive for all $W \in B(H)$ with* $\operatorname{rank} W \leq k$.
(iv) *C_ϕ is k-block positive.*

Proof The proof goes as follows. (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Let ϕ be k-positive and $V \in B(K)$ with rank $V \leq k$. Let $e = $ support V. Then $\dim e \leq k$. Thus

$$\phi \circ AdV = \phi \circ AdV \circ Ade : eB(K)e \to B(H).$$

Since $eB(K)e \cong M_l$ with $l \leq k$, and ϕ is k-positive, $\phi \circ AdV$ is completely positive by Corollary 4.1.9.

(ii) $\Rightarrow$ (i). Let $(e_{ij})_{i,j \leq k}$ be a complete set of matrix units for M_k. Let $a = \sum_{i,j \leq k} e_{ij} \otimes a_{ij} \in (M_k \otimes B(K))^+$. Again by Corollary 4.1.9 $a = C_\psi$ for a completely positive map $\psi : M_k \to B(K)$. By Theorem 4.1.8 $\psi = \sum AdV_i$ with $V_i : K \to \mathbb{C}^k$. Since $k \leq \dim K$ we may assume $\mathbb{C}^k \subset K$, hence $V_i \in B(K)$ with rank $V_i \leq k$ for all i. Thus by (ii) $\phi \circ \psi$ is completely positive, hence by Theorem 4.1.8

$$\iota_k \otimes \phi(a) = \iota_k \otimes \phi(C_\psi) = C_{\phi \circ \psi} \geq 0,$$

so that ϕ is k-positive.

(ii) $\Rightarrow$ (iv). Let $\xi = \sum_1^k \xi_i \otimes \eta_i \in K \otimes H$ have Schmidt rank k. Let q be a k-dimensional projection in $B(H)$ such that $q\eta_i = \eta_i$ for all i. Let (e_{ij}) be a complete set of matrix units in $B(K)$ such that $C_\phi = \sum e_{ij} \otimes \phi(e_{ij})$. Then we have

$$C_{Adq \circ \phi} = \sum e_{ij} \otimes Adq\big(\phi(e_{ij})\big) = Ad(1 \otimes q)(C_\phi).$$

Thus by (ii) and Theorem 4.1.8 $Ad(1 \otimes q)(C_\phi) \geq 0$. It follows that

$$(C_\phi \xi, \xi) = \big(C_\phi (1 \otimes q)\xi, (1 \otimes q)\xi\big) = \big(Ad(1 \otimes q)(C_\phi)\xi, \xi\big) \geq 0.$$

Thus C_ϕ is k-block positive.

(iv) $\Rightarrow$ (iii). Let $W \in B(H)$ with rank $W \leq k$. Let $\xi = \sum \xi_i \otimes \eta_i \in K \otimes H$. Let e support W, so $\dim e \leq k$. Then there exist k vectors $\alpha_1, \dots, \alpha_k \in H$ such that $e\eta_i = \sum_1^n c_{ij}\alpha_j$, $c_{ij} \in \mathbb{C}$. We can therefore write $1 \otimes W\xi = \sum_1^k \xi_j' \otimes \beta_j$ with $\xi_j' \in K$, $\beta_j \in H$. Thus $1 \otimes W\xi$ has Schmidt rank$\leq k$, hence by the assumption that C_ϕ is k-block positive $(C_\phi(1 \otimes W)\xi, (1 \otimes W)\xi) \geq 0$. Thus

$$C_{Adw\circ\phi} = (1 \otimes AdW)(C_\phi) \geq 0,$$

so that $AdW \circ \phi$ is completely positive.

(iii) $\Rightarrow$ (i). Let $V \in B(H)$ with rank $V \leq k$. Then $(AdV)^* = AdV^*$. Hence

$$\phi^* \circ AdV^* = (AdV \circ \phi)^* : B(H) \rightarrow B(K).$$

Since by assumption $AdV \circ \phi$ is completely positive, so is $\phi^* \circ AdV^*$. We have therefore shown that $\phi^* \circ AdW$ is completely positive for all $W \in B(H)$ with rank $W \leq k$. Therefore by the equivalence (i) $\Leftrightarrow$ (ii) applied to $\phi^* : B(H) \rightarrow B(K)$, ϕ^* is k-positive, hence so is ϕ. □

4.2 The Dual Functional of a Map

In the previous section we studied the duality between positive maps of $B(K)$ into $B(H)$ as matrices via the Jamiolkowski isomorphism $\phi \rightarrow C_\phi \in B(K \otimes H)$. In this section we consider the duality between maps and linear functionals on $B(K) \otimes \mathcal{T}(H)$, or more generally $A \otimes \mathcal{T}(H)$, where $\mathcal{T}(H)$ denotes the trace class operators on $B(H)$, and A is an *operator system*, i.e. a unital linear subspace of $B(K)$ such that $a \in A$ implies $a^* \in A$.

Definition 4.2.1 Let A be an operator system and $\phi : A \rightarrow B(H)$ a bounded linear map. Then its *dual functional* $\widetilde{\phi}$ on $A \otimes \mathcal{T}(H)$ is the functional defined by

$$\widetilde{\phi}(a \otimes b) = Tr\big(\phi(a)b^t\big),$$

where t is the transpose on $B(H)$ defined by a fixed orthonormal basis.

$\widetilde{\phi}$ is well defined because $\phi(a)$ is a bounded operator in $B(H)$, and b is a trace class operator. Let the *projective norm* on the algebraic tensor product of A and $\mathcal{T}(H)$ be defined by

$$\|x\|_\wedge = \inf\left\{\sum \|a_i\|\|b_i\|_1 \, : \, x = \sum_{i=1}^n a_i \otimes b_i, a_i \in A, b_i \in \mathcal{T}(H)\right\}$$

where $\|b\|_1$ is the trace norm $\|b\|_1 = Tr(|b|)$. We denote by $A\widehat{\otimes}\mathcal{T}(H)$ the completion of the algebraic tensor product with respect to the projective norm, and by $A^+\widehat{\otimes}\mathcal{T}(H)^+$ the closed cone generated by operators $\sum_i a_i \otimes b_i$ with $a_i \in A^+, b_i \in \mathcal{T}(H)^+$. $A\widehat{\otimes}\mathcal{T}(H)$ is called the *projective tensor product* of A and $\mathcal{T}$.

Lemma 4.2.2 *Let A be an operator system. Then the map $\phi \to \widetilde{\phi}$ is an isometric isomorphism of the space of bounded linear maps of A into $B(H)$ and $(A\widehat{\otimes}\mathscr{T}(H))^*$. Furthermore ϕ is positive if and only if $\widetilde{\phi}$ is positive on $A^+\widehat{\otimes}\mathscr{T}(H)^+$.*

Proof Let $x = \sum_1^n a_i \otimes b_i \in A\widehat{\otimes}\mathscr{T}(H)$ be a finite tensor. Then

$$\begin{aligned}
\left|\widetilde{\phi}(x)\right| &= \left|\sum_i Tr\big(\phi(a_i)b_i^t\big)\right| \\
&\leq \sum \left|Tr\big(\phi(a_i)b_i^t\big)\right| \\
&\leq \sum \left\|\phi(a_i)\right\| \|b_i\|_1 \\
&\leq \|\phi\| \sum \|a_i\| \|b_i\|_1 .
\end{aligned}$$

Thus $\|\widetilde{\phi}\| \leq \|\phi\|$.

Conversely, since $\|\ \|_\wedge$ is a cross norm,

$$\begin{aligned}
\|\phi\| = \sup_{\|a\|=1} \left\|\phi(a)\right\| &= \sup_{\|a\|=1, \|b\|_1=1} \left|Tr\big(\phi(a)b^t\big)\right| \\
&= \sup\left|\widetilde{\phi}(a \otimes b)\right| \\
&= \sup \|\widetilde{\phi}\| \|a \otimes b\| \\
&\leq \sup \|\widetilde{\phi}\| \|a\| \|b\|_1 \\
&\leq \|\widetilde{\phi}\|.
\end{aligned}$$

Thus the map $\phi \to \widetilde{\phi}$ is an isometry. The last part of the lemma follows from the proof of Proposition 4.1.11. □

The connection between the Choi matrix C_ϕ and $\widetilde{\phi}$ is given by the following result.

Lemma 4.2.3 *Let K be finite dimensional and $\phi \in B(B(K), H)$. Then C_ϕ^t is the density operator for $\widetilde{\phi}$.*

Proof Since the transpose is Tr-invariant, if $a \otimes b \in B(K) \otimes \mathscr{T}(H)$,

$$\begin{aligned}
Tr\big(C_\phi^t a \otimes b\big) &= Tr\big(C_\phi a^t \otimes b^t\big) \\
&= \sum_{ij} Tr\big(e_{ij}a^t \otimes \phi(e_{ij})b^t\big) \\
&= \sum_{ij} Tr\big(e_{ij}a^t\big) Tr\big(\phi(e_{ij})b^t\big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum a_{ij} Tr\big(e_{ij}\phi^*\big(b^t\big)\big) \\
&= Tr\big(a\phi^*\big(b^t\big)\big) \\
&= \widetilde{\phi}(a \otimes b),
\end{aligned}
$$

proving the lemma. □

We shall often encounter the situation when we compose a map by the transpose map both in the domain and the range of ϕ. Let as before t denote the transpose both of $B(K)$ and $B(H)$.

Definition 4.2.4 Let $\phi \in B(B(K), H)$. Then we denote by

$$\phi^t = t \circ \phi \circ t.$$

The basic properties are given in

Lemma 4.2.5 *Let $\phi \in B(B(K), H)$. Then we have*

(i) *If ϕ is k-positive (resp. completely positive), so is ϕ^t.*
(ii) *If $\phi = AdV$ then $\phi^t = AdV^{t*}$.*
(iii) *If $\dim K < \infty$ then $C_{\phi^t} = C_\phi^t$, where C_ϕ^t is the transpose on $B(K \otimes H)$.*

Proof (i) Let $\iota = \iota_k$ be the identity map on M_k. Then

$$\iota \otimes \phi^t = (\iota \otimes t) \circ (\iota \otimes \phi) \circ (\iota \otimes t) = (t \otimes t) \circ (\iota \otimes \phi) \circ (t \otimes t),$$

is positive, since $t \otimes t$ is the transpose on $B(K \otimes H)$, so is a positive map, and $\iota \otimes \phi$ is a positive map when ϕ is k-positive.

(ii) $(AdV)^t(a) = (AdV(a^t))^t = (V^* a^t V)^t = V^t a V^{*t}$.

(iii) $C_{\phi^t} = \sum e_{ij} \otimes \phi^t(e_{ij}) = \sum e_{ij} \otimes \phi(e_{ji})^t = (\sum e_{ji} \otimes \phi(e_{ji}))^t = C_\phi^t$. □

The relationship between $\widetilde{\phi}$ and ϕ^t is given in the next result.

Lemma 4.2.6 *Let K and H be finite dimensional. Let $\pi : B(K) \otimes B(K) \to B(K)$ be defined by $\pi(a \otimes b) = b^t a$. Then $Tr \circ \pi$ is positive and linear. Let $\phi \in B(B(K), H)$. Then*

$$\widetilde{\phi} = Tr \circ \pi \circ \big(\iota \otimes \phi^{*t}\big).$$

Proof Linearity of $Tr \circ \pi$ is clear. To show positivity let $x = \sum a_i \otimes b_i \in B(K) \otimes B(K)$. Then

$$
\begin{aligned}
Tr \circ \pi\big(xx^*\big) &= \sum_{ij} Tr \circ \pi\big(a_i a_j^* \otimes b_i b_j^*\big) \\
&= \sum Tr\big(b_j^{*t} b_i^t a_i a_j^*\big)
\end{aligned}
$$

$$= \sum_{ij} Tr\big((b_j^t a_j)^*(b_i^t a_i)\big)$$

$$= Tr\Big(\Big(\sum b_j^t a_j\Big)^*\Big(\sum b_i^t a_i\Big)\Big) \geq 0,$$

so $Tr \circ \pi$ is positive. The formula in the lemma follows from the computation

$$\widetilde{\phi}(a \otimes b) = Tr\big(\phi(a)b^t\big) = Tr\big(a\phi^*(b^t)\big) = Tr\big(a\phi^{*t}(b)^t\big) = Tr \circ \pi\big(\iota \otimes \phi^{*t}(a \otimes b)\big).$$

□

In the finite dimensional case we showed in Theorem 4.1.8 that $\phi \in B(B(K), H)$ is completely positive if and only if $C_\phi \geq 0$, hence by Lemma 4.2.3 if and only if $\widetilde{\phi}$ is positive. We now show a generalization of this. When H is infinite dimensional we define the positive cone $(A\widehat{\otimes}\mathscr{T}(H))^+$ in $A\widehat{\otimes}\mathscr{T}(H)$ for A an operator system, to be the closure of the positive cone in the algebraic tensor product $A \otimes \mathscr{T}(H)$.

Theorem 4.2.7 *Let A be an operator system and $\phi : A \to B(H)$. Then ϕ is completely positive if and only if $\widetilde{\phi}$ is a positive linear functional on $A\widehat{\otimes}\mathscr{T}(H)$.*

Proof We first assume H is finite dimensional. Then we have

$$\widetilde{\phi}^t(a \otimes b) = \widetilde{\phi}^*(b \otimes a), \quad a \in A, b \in B(H). \tag{4.3}$$

This follows from the computation

$$\widetilde{\phi}^t(a \otimes b) = Tr\big(\phi\big(a^t\big)^t b^t\big) = Tr\big(a^t \phi^*(b)\big) = \widetilde{\phi}^*(b \otimes a).$$

Assume $\widetilde{\phi}$ is a positive linear functional on $A \otimes B(H)$. Since $1 \otimes 1$ is an interior point of the positive cone $(A \otimes B(H))^+$ in the algebraic tensor product $A \otimes B(H)$, and $\widetilde{\phi}$ is positive on $(A \otimes B(H))^+$, it follows from Appendix A.3.1 that $\widetilde{\phi}$ has an extension to a positive linear functional ρ on $B(K) \otimes B(H)$. Since $\rho(1 \otimes 1) = \widetilde{\phi}(1 \otimes 1)$, ρ is bounded, and by the definition of the dual functional and Lemma 4.2.2, ρ is of the form $\rho = \widetilde{\psi}$ for a positive map $\psi \in B(B(K), H)$.

Let $\sum_i a_i \otimes b_i \geq 0$ in $B(K) \otimes B(H)$. Then $\sum a_i^t \otimes b_i^t = (\sum a_i \otimes b_i)^t \geq 0$, hence $\sum b_i^t \otimes a_i^t \geq 0$. Thus by (4.3)

$$\widetilde{\psi}^*\Big(\sum b_i \otimes a_i\Big) = \widetilde{\psi}^t\Big(\sum a_i \otimes b_i\Big) = \widetilde{\psi}\Big(\sum a_i^t \otimes b_i^t\Big) \geq 0,$$

so $\widetilde{\psi}^*$ is positive.

To continue the proof assume first K finite dimensional. Then, $\widetilde{\psi}^* \geq 0$ implies $C_{\psi^{*t}} = C_{\psi^*}^t \geq 0$ by Lemma 4.2.3, hence ψ^* is completely positive by Theorem 4.1.8. In the general case let e be a finite dimensional projection in $B(K)$ such that $e^t = e$. Then

$$\big(Ade \circ \psi^*\big)^{\widetilde{}} = \widetilde{\psi^{*t}} \circ Ad(1 \otimes e), \tag{4.4}$$

which is positive, so $\psi^* : B(H) \to eB(K)e$ is completely positive by the finite dimensional case. Since this holds for all e as above, ψ^* is completely positive. But then ψ is completely positive by Proposition 1.4.3. Since $\widetilde{\psi}$ is an extension of $\widetilde{\phi}$, ψ is an extension of ϕ. Thus ϕ is completely positive.

If $\dim H = \infty$, we use the same argument, and let (e_γ) be a net of finite dimensional projections in $B(H)$, such that $e_\gamma = e_\gamma^t$, and $e_\gamma \to 1$. Then as in (4.4)

$$(Ade_\gamma \circ \phi)\widetilde{} = \widetilde{\phi}^t \circ Ad(1 \otimes e_\gamma) \tag{4.5}$$

is positive, so by the first part of the proof $Ade_\gamma \circ \phi$ is completely positive, and finally by taking limits ϕ is completely positive.

Conversely suppose ϕ is completely positive. Assume first that $\dim H = n < \infty$, so $B(H) = M_n$. Let $\phi_n = \phi \otimes \iota_n$. Then ϕ_n is a positive map $A \otimes M_n \to M_n \otimes M_n$. Let $\pi : M_n \otimes M_n \to M_n$ be defined by $\phi(a \otimes b) = b^t a$. By Lemma 4.2.6 $Tr \circ \pi$ is positive. Let $\sum_i a_i \otimes b_i \in (A \otimes M_n)^+$. Then we have

$$\begin{aligned}
\widetilde{\phi}\Big(\sum_i a_i \otimes b_i\Big) &= \sum_i Tr\big(\phi(a_i) b_i^t\big) \\
&= \sum_i Tr \circ \pi\big(\phi(a_i) \otimes b_i\big) \\
&= Tr \circ \pi\Big(\phi_n\Big(\sum_i a_i \otimes b_i\Big)\Big) \geq 0,
\end{aligned}$$

so $\widetilde{\phi}$ is positive. In the general case let (e_γ) be an increasing net in $B(H)$ as in the previous paragraph. Then $Ade_\gamma \circ \phi : A \to e_\gamma B(H) e_\gamma$ is completely positive, so by the above $(Ade_\gamma \circ \phi)\widetilde{}$ is positive.

For each $a \in B(H)$, $e_\gamma a e_\gamma \to a$ strongly. Thus for each trace class operator b,

$$Tr(ae_\gamma b e_\gamma) = Tr(e_\gamma a e_\gamma b) \to Tr(ab).$$

Hence $e_\gamma b e_\gamma \to b$ as trace class operators. Thus if $\sum a_i \otimes b_i \in (A \otimes \mathscr{T}(H))^+$ we get

$$\begin{aligned}
\widetilde{\phi}\Big(\sum_i a_i \otimes b_i\Big) &= \sum_i Tr\big(\phi(a_i) b_i^t\big) \\
&= \lim \sum_i Tr\big(\phi(a_i) e_\gamma b_i^t e_\gamma\big)
\end{aligned}$$

is positive, since $\sum_i a_i \otimes e_\gamma b_i e_\gamma = Ad(1 \otimes e_\gamma)(\sum a_i \otimes b_i) \geq 0$. Thus $\widetilde{\phi} \geq 0$. □

4.3 Notes

The results in Sect. 4.1 are due to several authors. The Kraus decomposition was noted by Kraus [41] and the Jamiolkowski isomorphism by Jamiolkowski [30] a year later. Then Choi introduced the Choi matrix [7] and showed Theorem 4.1.8. Propositions 4.1.4, 4.1.6, and Theorem 4.1.15 can be found in [67–69], but some of these results were previously known in the literature in one form or the other, see [2, Sect. 10.3].

The results in Sect. 4.2 can be found in [78], except for Lemma 4.2.6, which is taken from [80].