Chapter 3
Extremal Positive Maps

The unit ball of the set of positive maps from a C^*-algebra into another C^*-algebra is a convex set, and it is natural to expect that the maps which are extreme points, have special properties. We shall in the present chapter study different classes of extremal maps.

Section 3.1 is on general results and the most obvious extremal maps. Section 3.2 is devoted to Jordan homomorphisms, Sect. 3.3 to maps such that the composition with pure states are pure states, and Sect. 3.4 to maps called nonextendible maps, which have strong extremality properties.

Finally, in Sect. 3.5 we prove a Radon-Nikodym theorem for completely positive maps together with its applications to extremal maps.

3.1 General Properties of Extremal Maps

Definition 3.1.1 Let A and B be C^*-algebras and $\phi : A \to B$ a positive map. We say that ϕ is *extremal* if the only positive maps $\psi : A \to B$, such that $\phi - \psi$ is positive, are of the form $\lambda\phi$ with $0 \le \lambda \le 1$.

Thus if ϕ is positive with $\|\phi\| \le 1$, ϕ cannot be the convex combination $\lambda\psi_1 + (1-\lambda)\psi_2$ of two positive maps ψ_1 and ψ_2 of norms less than or equal to 1 unless both ψ_1 and ψ_2 are positive multiples of ϕ. We list some simple properties of extremal maps.

Lemma 3.1.2 *Let $\phi : A \to B$ be a positive map, A and B being C^*-algebras. Then we have*:

(i) *If e is a projection in A such that $\phi(e) = \phi(1)$, then the restriction of ϕ to eAe is an extremal map $eAe \to B$ if and only if ϕ is extremal.*
(ii) *If $\alpha : B \to C$ with C another C^*-algebra, is an order-isomorphism of B onto C, then $\alpha \circ \phi$ is extremal if and only if ϕ is extremal.*

E. Størmer, *Positive Linear Maps of Operator Algebras*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-34369-8_3,

Proof (i) Assume ϕ is extremal and $\psi : eAe \to B$ a positive map such that $0 \leq \psi \leq \phi|_{eAe}$. Extend ψ to a map ψ_0 on A defined by $\psi_0(a) = \psi(eae)$.

If $0 \leq a \in A$ then, since $\phi(a) = \phi(eae)$ from the assumption on ϕ,

$$0 \leq \psi_0(a) = \psi(eae) \leq \phi(eae) = \phi(a).$$

Since ϕ is extremal, $\psi_0 = \lambda\phi$, hence $\psi = \lambda\phi|_{eAe}$ for some $\lambda \geq 0$.

Conversely, if $0 \leq \psi \leq \phi$ then $0 \leq \psi|_{eAe} \leq \phi|_{eAe}$, so extremality of $\phi|_{eAe}$ implies $\psi|_{eAe} = \lambda\phi|_{eAe}$. Since $0 \leq \psi(1-e) \leq \phi(1-e) = 0$, it follows that

$$\psi(a) = \psi(eae) = \lambda\phi(eae) = \lambda\phi,$$

so ϕ is extremal.

(ii) This is obvious, since $0 \leq \psi \leq \alpha \circ \phi$ if and only if $0 \leq \alpha^{-1} \circ \psi \leq \phi$. □

As remarked in Sect. 1.1 we use the notation $B(A, H)$ (resp. $B(A, H)^+$) for the bounded linear (resp. positive) maps of A into $B(H)$.

Proposition 3.1.3 *Let H and K be Hilbert spaces and $V : H \to K$ a bounded linear operator. Then the map $AdV(a) = V^*aV$ is extremal in $B(B(K), H)^+$.*

Proof We first consider the case when $K = H$ and $V = 1$, so $AdV = \iota$—the identity map. Suppose ψ is a positive map of $B(H)$ into itself such that $\psi \leq \iota$. Let f be projection in $B(H)$. Then $\psi(f) \leq f$, hence by Lemma 2.3.5 applied to $M = B(H)$, $B = B(H)_{sa}$, $\psi(a) = \psi(1)a$ for all $a \in B(H)$. In particular $\psi(1)$ commutes with a for all $a \in B(H)$, so $\psi(1) = \lambda 1$, and $\psi = \lambda\iota$, proving that ι is extremal.

We next consider the case when V is invertible. Then AdV is an order-isomorphism, so by the above paragraph and Lemma 3.1.2, $AdV = \iota \circ AdV$ is extremal.

Let $e = \text{range}\, V^* = \text{support}\, V$, and $f = \text{range}\, V = \text{support}\, V^*$. Thus $AdV : fB(K)f \to eB(H)e$. If $V : eH \to fK$ is invertible, then $AdV : fB(K)f \to eB(H)e$ is extremal in $B(fB(K)f, eH)^+$ by the previous paragraph. Since any positive map $\psi \leq AdV$ maps $1 - f$ to 0 and $e\psi(a)e = \psi(a)$ for all a, it follows that AdV is extremal in $B(B(K), H)^+$.

Finally, if V is not invertible on eH choose an increasing net (e_γ) of projections converging strongly to e such that Ve_γ is invertible on $e_\gamma H$. Let $f_\gamma = \text{range}\, Ve_\gamma$. Then by Appendix A.1 $f_\gamma \to f$ strongly. If $\psi \leq AdV$ is a map in $B(B(K), H)^+$ then $\psi \circ Adf_\gamma \leq AdV \circ Adf_\gamma = Adf_\gamma V$, so by the previous paragraph, $\psi \circ Adf_\gamma = \lambda_\gamma Adf_\gamma V$ for a number $\lambda_\gamma \geq 0$. Let λ be a limit point for (λ_γ), then

$$\psi = \lim_\gamma \psi \circ Adf_\gamma = \lim_\gamma \lambda_\gamma Adf_\gamma V = \lambda AdV,$$

proving that AdV is extremal. □

Proposition 3.1.4 *Let A and B be C^*-algebras and $\phi : A \to B$ be an extreme point of the convex set of positive unital maps of A into B. Let $a \in A$ belong to the center of*

A and assume $\phi(a)$ belongs to the center of B. Then a belongs to the multiplicative domain for ϕ.

Proof We have

$$a = \frac{1}{2}\big(a + a^*\big) + \frac{1}{2i} i\big(a - a^*\big).$$

Since a^* satisfies the same assumptions as a, we may assume a is self-adjoint and $\|a\| < 1$. Then $\|\phi(a)\| < 1$, so $1 - a$ and $1 - \phi(a)$ are positive and invertible. Define $\psi : A \to B$ by

$$\psi(b) = \phi\big((1-a)b\big)\big(1 - \phi(a)\big)^{-1}.$$

Since $1 - a$ and $(1 - \phi(a))^{-1}$ belong to the centers of A and B respectively, there is $\lambda > 0$ such that $0 \le \psi \le \lambda\phi$. Furthermore

$$\psi(1) = \phi(1-a)\big(1 - \phi(a)\big)^{-1} = 1,$$

so by assumption on ϕ as an extreme point, $\psi = \phi$. Thus $(1 - \phi(a))\phi(b) = \phi(1 - a)b)$, hence $\phi(a)\phi(b) = \phi(ab)$ for all $b \in A$. □

Our next result is contained in Theorems 3.4.3 and 3.4.4 in Sect. 3.4, but will be needed in Sect. 3.3.

Proposition 3.1.5 *Let A and B be unital C^*-algebras and ϕ a Jordan homomorphism of A into B. Then ϕ is an extreme point of the unit ball of positive maps from $A \to B$.*

Proof We may assume $\phi(1) = 1$. Suppose $\phi = \frac{1}{2}(\psi + \eta)$ with ψ, η belonging to the unit ball of positive maps of A into B, and suppose there exists a self-adjoint operator $a \in A$ such that $\psi(a) \ne \eta(a)$. Then by the Kadison-Schwarz inequality, Theorem 1.3.1,

$$\begin{aligned}
\phi\big(a^2\big) = \phi(a)^2 = \frac{1}{4}\big(\psi(a) + \eta(a)\big)^2 &= \frac{1}{2}\big(\psi(a)^2 + \eta(a)^2\big) - \frac{1}{4}\big(\psi(a) - \eta(a)\big)^2 \\
&< \frac{1}{2}\big(\psi(a)^2 + \eta(a)^2\big) \le \frac{1}{2}\big(\psi\big(a^2\big) + \eta\big(a^2\big)\big) \\
&= \phi\big(a^2\big).
\end{aligned}$$

This is a contradiction so $\psi(a) = \eta(a)$, and hence $\psi = \eta = \phi$. □

Corollary 3.1.6 *Let A and B be unital abelian C^*-algebras. Let $\phi : A \to B$ be a unital positive map. Then ϕ is a homomorphism if and only if ϕ is an extreme point of the convex set of unital positive maps of A into B.*

Proof This is immediate from Propositions 3.1.4 and 3.1.5. □

We conclude this section with a characterization of automorphisms of $B(H)$. Recall the notation $[A\xi]$ for the projection onto the closed subspace generated by vectors $a\xi$, $a \in A$, $\xi \in H$. If $A = \mathbb{C}$ we use the notation $[\xi]$ instead of $[\mathbb{C}\xi]$ for the 1-dimensional projection on the subspace generated by the vector ξ.

Proposition 3.1.7 *Let ϕ be an automorphism of $B(H)$. Then there exists a unitary operator U such that $\phi = AdU$.*

Proof Since ϕ maps minimal projections onto minimal projections, for each $\xi \in H$ there is $\eta \in H$ such that $\phi([\xi]) = [\eta]$. Composing ϕ by an inner automorphism AdU, we may assume $\phi([\xi]) = [\xi]$ for a unit vector ξ. Each unit vector in $B(H)$ is cyclic, so $[B(H)\xi] = 1$. Define an operator $V \in B(H)$ by

$$Va\xi = \phi(a)\xi, \quad a \in B(H). \tag{3.1}$$

Then

$$Vab\xi = \phi(ab)\xi = \phi(a)\phi(b)\xi = \phi(a)Vb\xi.$$

Thus

$$Va = \phi(a)V, \quad \text{for all } a \in B(H). \tag{3.2}$$

Since $\phi([\xi]) = [\xi]$,

$$\begin{aligned}\|Va\xi\|^2 &= (Va\xi, Va\xi) = \big(\phi(a)\xi, \phi(a)\xi\big) = \big(\phi\big(a^*a\big)\xi, \xi\big) = \big(\phi\big([\xi]a^*a[\xi]\big)\xi, \xi\big) \\ &= \big(a^*a\xi, \xi\big)\big(\phi\big([\xi]\big)\xi, \xi\big) = \big(a^*a\xi, \xi\big) = \|a\xi\|^2.\end{aligned}$$

Thus V is an isometry, which by (3.1) is surjective. Thus V is unitary, so by (3.2) $\phi(a) = VaV^*$. Let $U = V^*$. Then $\phi = AdU$. □

3.2 Jordan Homomorphisms

An important class of maps is that of Jordan homomorphisms. It follows from a result of Jacobson and Rickart [29] together with some structure theory for von Neumann algebras and second dual techniques for C^*-algebras, that each Jordan homomorphism of a C^* algebra into another is the sum of a homomorphism and an anti-homomorphism much like that of the proof of Theorem 1.2.11, see [72] hence they are not extremal, even though they are extreme points of the unit ball. To simplify our approach we shall restrict our attention to the simpler case of Jordan automorphisms of $B(H)$, where we can use more elementary techniques together with the extremality properties we have shown for Jordan homomorphisms. We start with the $n \times n$ matrices M_n and in particular M_2. Let $(e_{ij})_{i,j=1}^n$ denote a complete set of matrix units for M_n.

Lemma 3.2.1 *Let ρ be a linear functional on M_n. Then*

(i) *The density matrix for ρ is $(\rho(e_{ij}))^t$.*
(ii) *If ρ is a state then ρ is pure if and only if*

$$\left|\rho(e_{ij})\right|^2 = \rho(e_{ii})\rho(e_{jj}) \quad \textit{for all } 1 \leq i, j \leq n.$$

Proof (i) follows since $Tr((\rho(e_{ij}))^t e_{kl}) = \rho(e_{kl})$ for all k, l.

(ii) ρ is a pure state if and only if its density matrix is a 1-dimensional projection, hence by (i) if and only if $(\rho(e_{ij}))$ is a 1-dimensional projection, so (ii) follows. □

Lemma 3.2.2 *Denote by C_2 the convex set of unital positive maps of M_2 into itself. Let ϕ be an extreme point of C_2. Then there exists a pure state ρ of M_2 such that $\rho \circ \phi$ is a pure state.*

Proof Let ρ be a linear functional on M_2. Then its density operator is positive if and only if ρ is positive, hence by Lemma 3.2.1 if and only if $\rho(e_{11}) \geq 0$, $\rho(e_{22}) \geq 0$ and $|\rho(e_{12})|^2 \leq \rho(e_{11})\rho(e_{22})$. Suppose there is no pure state ρ such that $\rho \circ \phi$ is a pure state. Then for all pure states ρ, by Lemma 3.2.1(ii),

$$\rho\big(\phi(e_{11})\big)\rho\big(\phi(e_{22})\big) > \big|\rho\big(\phi(e_{12})\big)\big|^2.$$

Since the set of pure states on M_2 is compact there exists $\alpha > 0$ such that

$$\alpha \leq \rho\big(\phi(e_{11})\big)\rho\big(\phi(e_{22})\big) - \big|\rho\big(\phi(e_{12})\big)\big|^2$$

for all pure states ρ. Since $|\rho(\phi(e_{12}))|^2 \leq 1$

$$(1 \pm \alpha)\big|\rho\big(\phi(e_{12})\big)\big|^2 \leq \rho\big(\phi(e_{11})\big)\rho\big(\phi(e_{22})\big).$$

Define two maps ψ^+ and ψ^- of M_2 into itself as follows; $\psi^\pm$ is linear, $\psi^\pm(e_{ii}) = \phi(e_{ii})$, $i = 1, 2$, and

$$\psi^\pm(e_{12}) = (1 \pm i\delta)\phi(e_{12}), \qquad \psi^\pm(e_{21}) = (1 \mp i\delta)\phi(e_{21}),$$

where $0 < \delta < \alpha^{1/2}$, so that $|1 \pm i\delta|^2 = 1 + \delta^2 < 1 + \alpha$. By the characterization of positive linear functionals in the beginning of the proof $\rho \circ \psi^\pm$ is a positive linear functional for all states ρ, hence $\psi^\pm$ is a positive map. Furthermore $\psi^\pm(1) = \phi(1) = 1$, so $\psi^\pm \in C_2$. Since $\phi = \frac{1}{2}(\psi^+ + \psi^-)$, and ϕ is extreme, $\psi^+ = \psi^-$, so that $\phi(e_{12}) = 0$. Then $\phi(e_{22}) = 1 - \phi(e_{11})$, so the range of ϕ is an abelian subalgebra of M_2. Composing ϕ by AdV for a suitable unitary operator V, we can by an application of Lemma 3.1.2 assume the range of ϕ is contained in the diagonal algebra D_2. If $\phi(M_2) \subset \mathbb{C}1$, then ϕ is a state, so pure since ϕ is extreme, a case which is ruled out. Thus $\phi(M_2) = D_2$. Therefore $\phi(e_{11}) = xe_{11} + ye_{22}$, $\phi(e_{22}) = (1-x)e_{11} + (1-y)e_{22}$.

There are two cases. Assume first one of the four entries is 0; say $y = 0$. Then $1 - y = 1$. Thus $Tr(e_{22}\phi(e_{11})) = 0$, $Tr(e_{22}\phi(e_{22})) = 1$, so the state $\omega(a) = Tr(e_{22}\phi(a))$

is pure, a case which is ruled out. Assume next $0 < x < 1$, and $0 < y < 1$. Then there exists $\alpha > 0$ such that $\phi(e_{ii}) \geq \alpha 1$, $i = 1, 2$. Thus $\phi(a) \geq \alpha Tr(a) 1$ for all $a \geq 0$. By extremality $\phi(a) = \frac{1}{2} Tr(a)$ for all a, which is impossible since ϕ is extremal. We have thus obtained a contradiction to the assumption that $\rho \circ \phi$ is never pure for ρ a pure state. The proof is complete. □

Lemma 3.2.3 *Let ϕ be extreme in C_2. Then there is a unitary operator U such that*

$$AdU \circ \phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \alpha b + \beta c \\ \overline{\alpha} c + \overline{\beta} b & \gamma a + \varepsilon b + \overline{\varepsilon} c + \delta d \end{pmatrix},$$

where $0 \leq \gamma \leq 1$, $\delta = 1 - \gamma$.

Proof Write ϕ in the form $\phi(a) = \sum \phi_{ij}(a) e_{ij}$, where ϕ_{ij} is a linear functional on M_2. By Lemma 3.2.2 we can compose ϕ by AdU for a suitable unitary U so we can assume ϕ_{11} is the pure state $\phi_{11}((a_{ij})) = a_{11}$. Thus $\phi_{11}(e_{22}) = 0$, so $\phi_{12}(e_{22}) = 0 = \phi_{12}(e_{11})$. Thus ϕ is of the form described in the lemma. □

Theorem 3.2.4 *Let ϕ be a normal Jordan automorphism of $B(H)$. Then ϕ is either an automorphism or an anti-automorphism, hence is of the form AdU or $AdU \circ t$ for a unitary operator U.*

Proof We first assume $\dim H = 2$, so $B(H) = M_2$. By Proposition 3.1.5 ϕ is extreme in C_2, hence we can assume ϕ is of the form described in Lemma 3.2.3, i.e.

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \alpha b + \beta c \\ \overline{\alpha} c + \overline{\beta} b & \gamma a + \varepsilon b + \overline{\varepsilon} c + \delta d \end{pmatrix},$$

with $\gamma + \delta = 1$. In particular

$$\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \overline{\beta} & \varepsilon \end{pmatrix},$$

hence

$$0 = \phi\left(\begin{pmatrix} & 1 \\ 0 & \end{pmatrix}^2\right) = \phi\begin{pmatrix} & 1 \\ 0 & \end{pmatrix}^2 = \begin{pmatrix} \alpha\overline{\beta} & \alpha\varepsilon \\ \varepsilon\overline{\beta} & \alpha\overline{\beta} + \varepsilon^2 \end{pmatrix}.$$

Thus, $\alpha\overline{\beta} = \alpha\varepsilon = \varepsilon\overline{\beta} = \alpha\overline{\beta} + \varepsilon^2 = 0$. There are three cases.

(i) $\alpha = 0$. Then $\varepsilon\overline{\beta} = \varepsilon^2 = 0$, so

$$\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \overline{\beta} & 0 \end{pmatrix}, \qquad \phi \begin{pmatrix} & 0 \\ 1 & \end{pmatrix} = \begin{pmatrix} & \beta \\ 0 & \end{pmatrix}.$$

(ii) $\beta = 0$. Then similarily

$$\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \qquad \phi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & 0 \\ \overline{\alpha} & \end{pmatrix}.$$

(iii) $\varepsilon = 0$. Then $\alpha\overline{\beta} = 0$, so one of the two cases (i) or (ii) occurs. In case (i) $\phi\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & \beta \\ \overline{\beta} & \end{pmatrix}$, so the square is 1, hence $|\beta| = 1$. In case (ii) $|\alpha| = 1$. It follows that in case (i) ϕ is an anti-automorphism, and in case (ii) an automorphism.

Now consider the general case. Let p be a 1-dimensional projection. Then p is a minimal projection, so $\phi(p)$ is a minimal projection, hence is a 1-dimensional projection. Let e be a 2-dimensional projection. Then it is the sum of two 1-dimensional projections, so $\phi(e)$ is a 2-dimensional projection, and $\phi : eB(H)e \to \phi(e)B(H)\phi(e)$ is a Jordan isomorphism, hence by the first part of the proof applied to the composition of ϕ by an isomorphism of $\phi(e)B(H)\phi(e)$ onto $eB(H)e$, ϕ is either an isomorphism or an anti-isomorphism. Let now p and q be distinct 1-dimensional projections in $B(H)$ and $e = \mathrm{span}(p, q)$. Then e is a 2-dimensional projection, and so is $\phi(e)$. By the above applied to e, if ϕ is an isomorphism, $\phi(pq) = \phi(p)\phi(q)$, and in the anti-isomorphism case $\phi(pq) = \phi(q)\phi(p)$.

Let X_p (resp. Y_p) be the set of 1-dimensional projections q in $B(H)$ such that $0 \neq pq \neq p$ and $\phi(pq) = \phi(p)\phi(q)$ (resp. $\phi(pq) = \phi(q)\phi(p)$). Then either X_p or Y_p is non-empty, say $X_p \neq \emptyset$. Let $q \in X_p$. Then q is an interior point of X_p. Indeed, let $\gamma = \|pq\|$,

$$c = \big\|\phi(pq) - \phi(q)\phi(p)\big\|.$$

Then $\gamma > 0$, $c > 0$. Let f be a 1-dimensional projection such that $f \neq p$ and

$$\|f - q\| \leq \delta = \min(c/4, \gamma/2).$$

Then $\|fp\| \geq \|qp\| - \|(f - q)p\| \geq \gamma/2$. Furthermore,

$$\begin{aligned} c &= \big\|\phi(pq) - \phi(q)\phi(p)\big\| \\ &\leq \big\|\phi(pq) - \phi(pf)\big\| + \big\|\phi(pf) - \phi(f)\phi(p)\big\| + \big\|\big(\phi(f) - \phi(q)\big)\phi(p)\big\| \\ &\leq \delta + \big\|\phi(pf) - \phi(f)\phi(p)\big\| + \delta. \end{aligned}$$

Hence

$$\big\|\phi(pf) - \phi(f)\phi(p)\big\| \geq c - c/2 = c/2.$$

Then $f \in X_p$, proving that q is an interior point of X_p.

Let $g \neq p$ be a 1-dimensional projection such that $gp \neq 0$. Let ψ, ξ, η be unit vectors such that $p = [\psi]$, $g = [\xi]$, $q = [\eta]$. Multiplying ξ and η by scalars we may assume $(\xi, \psi) > 0$, $(\eta, \psi) > 0$. Let

$$\xi(t) = (1 - t)\eta + t\xi, \quad t \in [0, 1],$$

be the line segment in H from η to ξ. Then $\|\xi(t)\| \leq 1$, and $(\xi(t), \psi) = (1 - t)(\eta, \psi) + t(\xi, \psi) > 0$, so $p[\xi(t)] \neq 0$. It follows from the previous paragraph applied to $q = [\xi(0)]$ and thus to each $[\xi(t)]$ that the set of t such that $[\xi(t)] \in X_p$ is open. Since the set is trivially closed, it follows that $g = [\xi(1)] \in X_p$.

We have thus shown that every 1-dimensional projection with $gp \neq 0$ belongs to X_p. Since each projection $g \perp p$ obviously satisfies the identity $\phi(pg) = \phi(p)\phi(g)$, this identity is therefore shown for all 1-dimensional projections g. Since p was arbitrary, it follows by linearity and normality of ϕ that ϕ is an isomorphism. Similarly, if $Y_p \neq \emptyset$, ϕ is an anti-isomorphism.

The last statement follows from Proposition 3.1.7, and the fact that the transpose t is an anti-automorphism of $B(H)$, and the composition of two anti-isomorphisms is an isomorphism. □

3.3 Maps which Preserve Vector States

In Lemma 3.2.2 we saw that for each extreme point ϕ of the convex set of unital maps of M_2 into itself, there is a pure state ϕ of M_2 such that $\rho \circ \phi$ is a pure state. A natural problem is to study maps in the extreme converse direction, i.e. maps $\phi : A \to B$, with A, B C^*-algebras, such that $\rho \circ \phi$ is a pure state for all pure states ρ of B. It was shown in [71] that for all such maps $\pi \circ \phi$ is either a pure state, or an anti-homomorphism or homomorphism of A for all irreducible representations of B. We shall in the present section restrict ourselves to maps of $B(K)$ into $B(H)$ for which $\omega_\xi \circ \phi$ is a vector state of $B(K)$ for all vector states ω_ξ of $B(H)$ defined by $\omega_x(a) = (a\xi, \xi)$. We then apply this to maps which carry positive rank 1 operators to positive rank 1 operators.

Lemma 3.3.1 *Let K and H be Hilbert spaces and $\phi \in B(B(K), H)$ a unital positive map such that for each vector state ω_η of $B(H)$ there is a vector state ω_ξ of $B(K)$ such that $\omega_\xi \circ \phi = \omega_\eta$. For such a pair ξ, η, either $\phi([\eta]) = [\xi]$ or $\phi([\eta]) = 1$. In the latter case $\phi(a) = \omega_\eta(a)1$ for all $a \in B(H)$. Furthermore ϕ is weakly continuous.*

Proof We first show ϕ is weakly continuous. Let $(a_\alpha)_{\alpha \in J}$ be a net in $B(K)$ such that $a_\alpha \to a$ is weakly. Let ξ be a unit vector in H and η a unit vector in K such that $\omega_\xi \circ \phi = \omega_\eta$. Then $\omega_\xi(\phi(a_\alpha)) = \omega_\eta(a_\alpha) \to \omega_\eta(a) = \omega_\xi(\phi(a))$.

Since each weakly continuous linear functional on $B(H)$ is a linear combination of vector states, $(\phi(a_\alpha))_{\alpha \in J}$ converges weakly to $\phi(a)$, so ϕ is weakly continuous.

Let ξ and η be as above. Then $0 \leq \phi([\eta]) \leq 1$ and $\omega_\xi(\phi([\eta])) = 1$. Thus $\phi([\eta])[\xi] = [\xi] \leq \phi([\eta])$. To prove the lemma we first assume $n = \dim H < \infty$, and use induction on n. If $n = 1$ the lemma is trivial.

Suppose $n = 2$ and $\phi([\eta]) \neq [\xi]$. We may then assume $B(H) = M_2$ and

$$\phi([\eta]) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \tag{3.3}$$

with $0 < p \leq 1$. Let μ be a unit vector in K orthogonal to $[\eta]$. Let $f = [\eta] + [\mu]$. Then $fB(K)f \cong M_2$. Let $(e_{ij}), i, j = 1, 2$, denote the matrix units in M_2 such that $[\eta] = e_{11}, [\mu] = e_{22}$. If ω_ρ is a vector state of M_2 then $\omega_\rho \circ \phi = \omega_\tau$ for a unit vector

$\tau \in K$, so its restriction to $fB(K)f$ is $\omega_{f\tau}$, which is a scalar multiple of a vector state, so by Lemma 3.2.1 satisfies the equality

$$\omega_\rho \circ \phi(e_{11})\omega_\rho \circ \phi(e_{22}) = \left|\omega_\rho \circ \phi(e_{12})\right|^2. \tag{3.4}$$

In particular this holds for $\rho = \eta$. Since also $0 \le \phi(e_{11} + e_{22}) \le 1$, we have

$$\phi(e_{22}) = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}, \qquad \phi(e_{12}) = \begin{pmatrix} 0 & r \\ s & t \end{pmatrix}.$$

Since $\rho = (\rho_1, \rho_2)$ is a vector in $\mathbb{C}^2$ the following equations hold, cf. (3.3):

$$\omega_\rho \circ \phi(e_{11}) = |\rho_1|^2 + p|\rho_2|^2,$$

$$\omega_\rho \circ \phi(e_{22}) = q|\rho_2|^2,$$

$$\omega_\rho \circ \phi(e_{12}) = t|\rho_2|^2 + r\overline{\rho_1}\rho_2 + s\rho_1\overline{\rho_2}.$$

Thus, using (3.4)

$$\begin{aligned}
&|t|\rho_2|^2 + r\overline{\rho_1}\rho_2 + s\rho_1\overline{\rho_2}|^2 \\
&\quad = |t|^2|\rho_2|^4 + \left(|r|^2 + |s|^2\right)|\rho_1|^2|\rho_2|^2 + 2\Re\left((r\overline{t} + \overline{s}t)|\rho_2|^2\overline{\rho_1}\rho_2\right) + 2\Re\left(r\overline{s}(\overline{\rho_1}\rho_2)^2\right) \\
&\quad = q|\rho_2|^2\left(|\rho_1|^2 + p|\rho_2|^2\right).
\end{aligned} \tag{3.5}$$

Now, if f_1, f_2, f_3 are complex valued functions of the two complex variables ρ_1 and ρ_2 such that

$$f_1\left(|\rho_1|, |\rho_2|\right) = \Re\left(f_2\left(|\rho_1|, |\rho_2|\right)\overline{\rho_1}\rho_2 + f_3\left(|\rho_1|, |\rho_2|\right)(\overline{\rho_1}\rho_2)^2\right),$$

then it is easily verified that $f_1 = f_2 = f_3 = 0$. With

$$f_1\left(|p_1|, |\rho_2|\right) = \left(|\rho_1|^2 + p|\rho_2|^2\right)q|\rho_2|^2 - |t|^2|\rho_2|^4 - \left(|r|^2 + |s|^2\right)|\rho_1|^2|\rho_2|^2$$

and f_2 and f_3 the two real parts in (3.5), we get

$$r\overline{t} + \overline{s}t = 0 = r\overline{s}, \qquad |t|^2 = pq, \qquad |r|^2 + |s|^2 = q.$$

Thus $q = 0$, and $\phi([\mu]) = \phi(e_{22}) = 0$. Since this holds for every unit vector $[\mu]$ orthogonal to η, and since ϕ is weakly continuous, $\phi([\eta]) = 1$, as asserted.

Suppose $n \ge 3$, and assume the lemma is proved whenever $\dim H \le n - 1$. Let e be a projection in $B(H)$ containing ξ, and $\dim e = k < n$. Then $Ade \circ \phi$ has the same properties as ϕ with respect to composition with vector states,

$$Ade \circ \phi : B(K) \to eB(H)e,$$

and $\omega_\xi \circ \phi = \omega_\eta$. By induction assumption $e\phi([\eta])e$ equals $[\xi]$ or e. If $e\phi([\eta])e = [\xi]$ then

$$0 = e\left(\phi\left([\eta]\right) - [\xi]\right)e = \left(\left(\phi\left([\eta]\right) - [\xi]\right)^{1/2}e\right)^*\left(\left(\phi\left([\eta]\right) - [\xi]\right)^{1/2}e\right),$$

so $(\phi([\eta]) - [\xi])e = 0$, hence $\phi([\eta])e = [\xi] = e\phi([\eta])$, taking adjoints. Similarily, if $e\phi([\eta])e = e$, then $e(1 - \phi([\eta]))e = 0$, and $e\phi([\eta]) = \phi([\eta])e = e$. Thus $\phi([\eta])$ commutes with every projection containing ξ. Since $n \geq 3$ this is possible only if $\phi([\eta])$ equals $[\xi]$ or 1.

If H is not finite dimensional it follows from the above that $\phi([\eta])$ commutes with every finite dimensional projection containing $[\xi]$. Hence $\phi([\eta])$ equals $[\xi]$ or 1. □

Theorem 3.3.2 *Let K and H be Hilbert spaces and $\phi \in B(B(K), H)$ be a positive unital map such that for each unit vector $\xi \in H$ there is a unit vector $\eta \in K$ such that $\omega_\xi \circ \phi = \omega_\eta$. Then either $\phi(a) = \omega_\rho(a)1$ for a vector $\rho \in K$, or there is a linear isometry $V : H \to K$ such that $\phi = AdV$ or $\phi = AdV \circ t$, t being the transpose on $B(K)$.*

Proof By Lemma 3.3.1 ϕ is weakly continuous. Let Tr denote the trace on either $B(K)$ or $B(H)$. Thus if $\omega_\xi \circ \phi = \omega_\eta$ we have for $a \in B(K)$

$$\begin{aligned} Tr\big(\phi^*\big([\xi]\big)a\big) &= Tr\big([\xi]\phi(a)\big) = \omega_\xi \circ \phi(a) \\ &= \omega_\eta(a) = Tr\big([\eta]a\big). \end{aligned}$$

Thus $\phi^*([\xi]) = [\eta]$, and $\phi^* : B(H) \to B(K)$ is faithful and maps 1-dimensional projections to 1-dimensional projections. Let ξ and μ be mutually orthogonal unit vectors in H. Let η and ρ be unit vectors in K such that $\omega_\xi \circ \phi = \omega_\eta$, and $\omega_\mu \circ \phi = \omega_\rho$. By Lemma 3.3.1 either $\phi([\eta]) = 1$, in which case support $\phi = [\eta]$, so that $\phi(a) = \phi([\eta]a[\eta]) = \omega_\eta(a)1$, so ϕ is a vector state, or $\phi([\eta]) = [\xi]$, $\phi([\rho]) = [\mu]$. In the latter case

$$0 \leq \omega_\eta\big([\rho]\big) = \omega_\xi\big(\phi\big([\rho]\big)\big) = \omega_\xi\big([\mu]\big) = 0,$$

so η and ρ are orthogonal. Since $\phi^*([\xi]) = [\eta]$ and $\phi^*([\mu]) = [\rho]$, it follows that ϕ^* maps mutually orthogonal 1-dimensional projections onto mutually orthogonal projections. Thus ϕ^* is a Jordan isomorphism on finite rank operators in $B(H)$ into those of $B(K)$. Thus for each finite dimensional projection $e \in B(H)$, ϕ^* is a Jordan isomorphism of $eB(H)e$ into $\phi^*(e)B(K)\phi^*(e)$, and onto, since they have the same dimensions. It follows from Theorem 3.2.4 that ϕ^* is either an isomorphism or anti-isomorphism of $eB(H)e$ onto $\phi^*(e)B(K)\phi^*(e)$, and implemented by a unitary operator $U : eK \to \phi^*(e)H$. By Proposition 1.4.2 the adjoint map of AdU is AdU^*, and the adjoint of the transpose map t is t. Thus $\phi : \phi^*(e)B(K)\phi^*(e) \to eB(H)e$ is either an isomorphism or an anti-isomorphism. Let $f = \vee_e \phi^*(e)$, where the span is over all finite dimensional projections in $B(H)$. Since ϕ is weakly continuous it is either an isomorphism or anti-isomorphism of $fB(K)f$ onto $B(H)$. □

Remark 3.3.3 Theorem 3.3.2 has a generalization to C^*-algebras. Recall that if ρ is a state of a C^*-algebra B then there are a Hilbert space H_ρ, a $*$-representation π_ρ of B on H_ρ and a vector $\xi_\rho \in H_\rho$ such that $\rho(a) = \omega_{\xi_\rho} \circ \pi_\rho(a)$ for $a \in B$.

Furthermore, ρ is a pure state if and only if π_ρ is irreducible. Then the generalization of Theorem 3.3.2 states, see [71]: Let A and B be unital C^*-algebras and $\phi : A \to B$ a positive unital map. Then $\rho \circ \phi$ is a pure state of A and for all pure states ρ of B if and only if for each irreducible representation ψ of B on a Hilbert space H, $\psi \circ \phi$ is either a pure state of A or $\psi \circ \phi = V^*\pi V$, where V is a linear isometry of H into a Hilbert space K, and π is an irreducible $*$-homomorphism or $*$-anti-homomorphism of A into $B(K)$.

Many problems on maps of operator algebras are what are called preserver problems. Then one studies maps which preserve selected properties. For a treatment on this topic we refer the reader to the book [51] of Molnár. Our next result, which is close to Theorem 3.3.2, is of this type.

Theorem 3.3.4 *Let K and H be finite dimensional Hilbert spaces and $\phi \in B(B(K), H)$ a positive map such that rank $\phi(p) \leq 1$ for all 1-dimensional projections $p \in B(K)$. Then one of the following three conditions holds*:

(i) *There exist a state ω on $B(K)$ and a positive rank 1 operator $q \in B(H)$ such that $\phi(a) = q\omega(a)$ for $a \in B(K)$.*
(ii) *$\phi = AdU$ with $U : H \to K$ a bounded linear operator.*
(iii) *$\phi(a) = (AdU(a))^t$ for $a \in B(K)$, t is the transpose on $B(H)$.*

Proof Let $e =$ support of ϕ. Then $\phi : eB(K)e \to B(H)$ is faithful, so we may restrict attention to $eB(K)e$ and assume ϕ is faithful. By Proposition 1.4.3(iv) $\phi^*(1)$ is invertible. Let $h = \phi^*(1)^{-1/2}$. Then $h\phi^*(1)h = 1$, so the map $\psi(a) = h\phi^*(a)h$ is unital and positive. Then for $a \in B(K), b \in B(H)$ we have

$$Tr\big(a\psi(b)\big) = Tr\big(hah\phi^*(b)\big) = Tr\big(\phi(hah)b\big).$$

If p is a 1-dimensional projection in $B(K)$ then $hph = \lambda q$ for a 1-dimensional projection q, so by the assumption on ϕ, $\phi(hph) = \lambda\phi(q)$ is positive of rank 1. It follows that the functional

$$\omega'(a) = Tr\big(p\psi(a)\big) = Tr\big(ph\phi^*(a)h\big) = Tr\big(\phi(hph)a\big) = \lambda Tr\big(\phi(q)a\big),$$

for $a \in B(H)$, is a scalar multiple of a pure state on $B(H)$. Furthermore, $\omega'(1) = Tr(p\psi(1)) = Tr(p) = 1$, so ω' is a pure state. Thus $\psi : B(H) \to B(K)$ preserves vector states. By Theorem 3.3.2 and 3.2.4 ψ is either

(i) a vector state, i.e. $\psi(a) = \omega_\xi(a)1$.
(ii) $\psi(a) = V^*aV$, $V : K \to H$ is a linear isometry of K into H.
(iii) $\psi(a) = V^*a^tV$, with V as in (ii).

If ρ is a state on $B(K)$ with density operator d then for $a \in B(H)$

$$Tr\big(a\rho^*(b)\big) = Tr\big(\rho(a)b\big) = Tr\big(Tr(da)b\big) = Tr\big(daTr(b)\big),$$

so that $\rho^*(b) = dTr(b)$. By construction, $\phi^* = h^{-1}\psi h^{-1}$. Thus we have in case (i), $\psi(a) = Tr(qa)$ for a 1-dimensional projection q, so that

$$\begin{aligned} Tr\big(\phi(a)b\big) &= Tr\big(ah^{-1}\psi(b)h^{-1}\big) \\ &= Tr\big(ah^{-1}Tr(qb)h^{-1}\big) \\ &= Tr\big(ah^{-2}\big)Tr(qb) \\ &= Tr\big(qTr\big(ah^{-2}\big)b\big), \end{aligned}$$

so that $\phi(a) = qTr(ah^{-2})$ is as in (i) in the theorem.

In case (ii)

$$Tr\big(\phi(a)b\big) = Tr\big(ah^{-1}\psi(b)h^{-1}\big) = Tr\big(h^{-1}ah^{-1}V^*bV\big) = Tr\big(\big(Vh^{-1}\big)a\big(Vh^{-1}\big)^*b\big),$$

so that $\phi(a) = AdU$ with $U^* = Vh^{-1} : H \to K$.

In case (iii) we similarly have

$$Tr\big(\phi(a)b\big) = Tr\big(h^{-1}ah^{-1}V^*b^tV\big) = Tr\big(\big(AdU(a)\big)^t b\big),$$

so that $\phi(a) = t \circ AdU$. □

It turns out that 2-positive and 2-copositive extremal maps in $B(B(K), H)^+$ are of the form described in Theorem 3.3.4. We conclude the section with a proof of this. Assume for simplicity that K and H are finite dimensional. Recall that if ξ is a vector in an n-dimensional Hilbert space, $\xi = (\xi_1, \ldots, \xi_n)$ then ξ can be identified with the $1 \times n$ column matrix

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}.$$

Then $\xi^* = [\overline{\xi_1}, \ldots, \overline{\xi_n}]$. If η is another vector we get

$$\xi^*\eta = \langle \eta, \xi \rangle,$$

and if they are unit vectors, $\xi\eta^*$ is the partial isometry from η to ξ. In particular $\xi\xi^*$ is the projection $[\xi]$.

Lemma 3.3.5 *Let $\phi \in B(B(K), H)$ be of the form $\phi(x) = AxA^*$ with $A : K \to H$ non-zero. Choose unit vectors $\xi \in K, \omega \in H$ and $\lambda > 0$ such that*

$$\phi\big(\xi\xi^*\big)\omega = \lambda\omega.$$

Define $B : K \to H$ by

$$B\eta = \lambda^{-1/2}\phi\big(\eta\xi^*\big)\omega.$$

Then $B = e^{it}A$ for some $t \in [0, 2\pi)$.

Proof By assumption

$$\lambda\omega = A\xi\xi^* A^*\omega = A\xi(A\xi)^*\omega = A\xi\langle\omega, A\xi\rangle.$$

Thus $A\xi = z\omega$ for some $z \in \mathbb{C}$. Since

$$|z|^2\omega = z\omega\langle\omega, z\omega\rangle = A\xi\{\omega, A\xi\rangle = \lambda\omega,$$

$|z| = \lambda^{1/2}$. Let $\eta \in K$. Then

$$B\eta = \lambda^{-1/2} A\eta\xi^* A^*\omega = \lambda^{-1/2} A\eta\langle\omega, A\xi\rangle = \lambda^{-1/2}\bar{z}A\eta = e^{it}A\eta,$$

where t satisfies $\lambda^{-1/2}\bar{z} = e^{it}$. Thus $B = e^{it}A$. □

Proposition 3.3.6 *Let $\phi \in B(B(K), H)^+$. Let λ, ξ, ω, B be defined by ϕ as in Lemma 3.3.5. Let $\psi \in B(B(K), H)^+$ be the map $\psi(x) = BxB^*$. Then $\psi \le \phi$ if and only if for all $\eta \in K$, $\rho \in H$ we have the inequality*

$$\left|\langle\phi(\eta\xi^*)\omega, \rho\rangle\right|^2 \le \langle\phi(\xi\xi^*)\omega, \omega\rangle\langle\phi(\eta\eta^*)\rho, \rho\rangle.$$

Proof Clearly $\psi \le \phi$ if and only if for all $\eta \in K$, $\rho \in H$

$$\langle\psi(\eta\eta^*)\rho, \rho\rangle \le \langle\phi(\eta\eta^*\rho, \rho\rangle.$$

The left hand side of the above inequality is equal to

$$\begin{aligned}\langle B\eta\eta^* B^*\rho, \rho\rangle &= \langle B\eta(B\eta)^*\rho, \rho\rangle\\ &= \langle B\eta\langle\rho, B\eta\rangle, \rho\rangle\\ &= \left|\langle B\eta, \rho\rangle\right|^2\\ &= \lambda^{-1}\left|\langle\phi(\eta\xi^*)\omega, \rho\rangle\right|^2,\end{aligned}$$

by definition of B. If the inequality in the proposition is satisfied it follows that

$$\begin{aligned}\langle\psi(\eta\eta^*)\rho, \rho\rangle &\le \lambda^{-1}\langle\phi(\xi\xi^*)\omega, \omega\rangle\langle\phi(\eta\eta^*)\rho, \rho\rangle\\ &= \langle\phi(\eta\eta^*)\rho, \rho\rangle,\end{aligned}$$

by choice of λ. Thus $\psi \le \phi$.

Conversely, if $\psi \le \phi$, then by the above computations

$$\lambda^{-1}\left|\langle\phi(\eta\xi^*)\omega, \rho\rangle\right|^2 \le (\phi(\eta\eta^*)\rho, \rho\rangle,$$

so the inequality in the proposition follows from the definition of λ. □

Theorem 3.3.7 *Let $\phi \in B(B(K), H)^+$ be an extremal map. Assume ϕ is 2-positive (resp. 2-copositive). Then ϕ is a completely positive of the form $\phi = AdV$ with $V : H \to K$ (resp. ϕ is copositive of the form $AdV \circ t$).*

Proof Let ξ, ω, λ be as in Lemma 3.3.5. Let $\eta \in K$. Consider the positive matrix

$$X = \begin{pmatrix} \xi\xi^* & \xi\eta^* \\ \eta\xi^* & \eta\eta^* \end{pmatrix} = \begin{pmatrix} 0 & \xi \\ \eta & 0 \end{pmatrix} \begin{pmatrix} 0 & \xi \\ \eta & 0 \end{pmatrix}^* \in M_2\big(B(K)\big).$$

Since ϕ is 2-positive the matrix

$$\phi_2(X) = \begin{pmatrix} \phi(\xi\xi^*) & \phi(\xi\eta^*) \\ \phi(\eta\xi^*) & \phi(\eta\eta^*) \end{pmatrix} \in M_2\big(B(H)\big)^+.$$

Thus for each $\rho \in H$ we have

$$\begin{pmatrix} \langle\phi(\xi\xi^*)\omega, \omega\rangle & \langle\phi(\xi\eta^*)\rho, \omega\rangle \\ \langle\phi(\eta\xi^*)\omega, \rho\rangle & \langle\phi(\eta\eta^*)\rho, \rho\rangle \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & \rho \end{pmatrix}^* \phi_2(X) \begin{pmatrix} \omega & 0 \\ 0 & \rho \end{pmatrix} \geq 0.$$

Thus the inequality in Proposition 3.3.6 is satisfied, so by the theorem $\psi \leq \phi$. Since ϕ is extremal $\psi = AdB^* = \mu\phi$ for some $\mu > 0$. Hence $\phi = AdV$ with $V = \mu^{-1/2}B^*$.

If ϕ is 2-copositive then $\phi \circ t$ is 2-positive and still extremal by Lemma 3.1.2, so $\phi \circ t = AdV$, hence $\phi = AdV \circ t$. □

3.4 Nonextendible Maps

If A is a C^*-algebra and $\phi \in B(A, H)$ is a unital completely positive map the Stinespring Theorem, 1.2.7, states that there are a Hilbert space K, an isometry $V : H \to K$, and a representation $\pi : A \to B(K)$ such that $\phi = V^*\pi V$.

Since $V^*V = 1$, VV^* is a projection, which we can look at as the projection $P : K \to H$, where we consider H as a subspace of K. Then ϕ has the form $P\pi P$. We can thus consider π as an extension of ϕ to a map $\pi : A \to B(K)$. We therefore make the following definition.

Definition 3.4.1 Let A be a unital C^*-algebra, and $H \subset K$ two Hilbert spaces. Let P be the orthogonal projection of K onto H. Let $\phi \in B(A, H)$ and $\Phi \in B(A, K)$ be positive unital maps. We say

(i) Φ is an *extension of* ϕ and write $\Phi \supset \phi$ if $\phi(a) = P\Phi(a)P$ for all $a \in A$.
(ii) $\Phi \supset \phi$ is *trivial* if H is invariant under the action of $\Phi(a)$ for all $a \in A$, i.e. $\Phi(a)\xi = \phi(a)\xi$ for $a \in A$ and $\xi \in H$.
(iii) ϕ is called *nonextendible* if all extensions $\Phi \supset \phi$ are trivial.

Note that if $\Phi \supset \phi$ is an extension as above, and $\sum_1^n a_i \otimes \xi_i \in A \otimes H$, consider the element

$$\sum_k \phi(a_i)\xi_i = P\Big(\sum \Phi(a_i)\xi_i\Big) \in H.$$

Then

$$\left\| \sum \phi(a_i)\xi_i \right\| \leq \left\| \sum \Phi(a_i)\xi_i \right\|. \tag{3.6}$$

If the extension $\Phi \supset \phi$ is trivial then $\sum \Phi(a_i)\xi_i \in H$, so we have equality in (3.6). Conversely, if for all $\sum_i a_i \otimes \xi_i \in A \otimes H$ we have equality in (3.6), then $\sum_i \phi(a_i)\xi_i = \sum_i \Phi(a_i)\xi_i$, so the extension $\Phi \supset \phi$ is trivial. We have shown:

Lemma 3.4.2 *Let $\phi \in B(A, H)$ be a positive unital map. Then ϕ is nonextendible if and only if*

$$\left\| \sum \phi(a_i)\xi_i \right\| = \left\| \sum \Phi(a_i)\xi_i \right\|$$

for all extensions $\Phi \supset \phi$ and $a_i \in A$, $\xi_i \in H$.

We say a positive map $\phi : A \to B(H)$ is *irreducible* if the commutant of $\phi(A)$ is the scalar operators, i.e. the only operators which commute with $\phi(a)$ for all $a \in A$, are the scalar multiples of the identity operator 1.

Theorem 3.4.3 *Let A be a C^*-algebra and $\phi \in B(A, H)$ be a unital positive map. Then*

(i) *If ϕ is nonextendible then ϕ is an extreme point of the convex set of positive unital maps of A into $B(H)$.*
(ii) *If ϕ is both nonextendible and irreducible then ϕ is an extremal map.*

Proof Assume $\phi \in B(A, H)^+$ is nonextendible and $\phi = \lambda\phi_1 + \mu\phi_2$ with $\phi_i : A \to B(H)$ positive linear maps, $\lambda, \mu > 0$ and $\lambda + \mu = 1$. The operators $\phi_i(1)$ are invertible on the subspace $\phi_i(1)H$. Let H_i denote the closure of $\phi_i(1)H$.

Let

$$\psi_i(a) = \phi_i(1)^{-1/2}\phi_i(a)\phi_i(1)^{-1/2}, \quad a \in A.$$

Then $\psi_i(a)$ defines an operator on H_i, which we still denote by $\psi_i(a)$. Let

$$K = H_1 \oplus H_2, \qquad \Phi = \psi_1 \oplus \psi_2.$$

Then

$$\Phi : A \to B(K)$$

is unital and positive. Let $V : H \to K$ be the linear operator

$$V(\xi) = \big(\lambda\phi_1(1)\big)^{1/2}\xi \oplus \big(\mu\phi_2(1)\big)^{1/2}\xi.$$

Then a straightforward computation yields

$$\big(\phi(a)\xi, \eta\big) = \big(\Phi(a)V\xi, V\eta\big)$$

for $\xi, \eta \in H$ and $a \in A$. In particular, if we put $a = 1$, we see that V is an isometric imbedding of H into K. Thus $\Phi \supset \phi$ is an extension of ϕ. By assumption ϕ is nonextendible. Thus Φ is a trivial extension. In our definition we considered H as a subspace of K. In the general case one must consider the case when H is imbedded in K as it is here, with $V : H \to K$. Thus we have

$$\Phi(a)V\xi = V\phi(a)\xi \quad \text{for } a \in A, \xi \in H.$$

By the definitions of V and $\Phi = \psi_1 \oplus \psi_2$ we get

$$\Phi(a)V\xi = \lambda^{1/2}\phi_1(1)^{-1/2}\phi_1(a)\xi \oplus \mu^{1/2}\phi_2(1)^{-1/2}\phi_2(a)\xi.$$

This is equal to

$$V\phi(a)\xi = \lambda^{1/2}\phi_1(1)^{1/2}\phi(a)\xi \oplus \mu^{1/2}\phi_2(a)^{1/2}\phi(a)\xi,$$

so that

$$\phi_i(1)\phi(a)\xi = \phi_i(a)\xi, \quad \text{for all } \xi \in H,$$

hence $\phi_i = \phi_i(1)\phi$.

In case (i) in the theorem $\phi_i(1) = 1$, so $\phi_i = \phi$, and the conclusion in (i) follows.

In case (ii) $\phi_i(a) = \phi_i(1)\phi(a)$ for all a. Taking adjoints for a self-adjoint we see that $\phi_i(1)$ commutes with the self-adjoint operator $\phi(a)$, and therefore $\phi_i(1) \in \phi(A)'$, which we assumed is the scalar operators. Thus ϕ_i is a scalar multiple of ϕ, and thus ϕ is extremal. □

It is a quite special property to be a nonextendible map. Our next result is an example of a nonextendible map. It is an extension of Proposition 3.1.5, where it was shown that Jordan homomorphisms were extremal in the set of positive unital maps.

Theorem 3.4.4 *Let A be a C^*-algebra and $\phi \in B(A, H)$ a unital Jordan homomorphism. Then ϕ is nonextendible.*

Proof Since $\phi(1)$ is always a projection the assumption that ϕ is unital is just made for convenience. Let $\Phi \supset \phi$ be an extension, so $\phi(a) = P\Phi(a)P$, where P is the projection of K onto H, $\Phi : A \to B(K)$ positive and unital. If $a \in A$ is self-adjoint then the Kadison-Schwarz inequality, Theorem 1.3.1, applied to Φ, implies with 1 the identity in $B(K)$,

$$\begin{aligned} 0 &\le P\Phi(a)(1-P)\Phi(a)P \\ &= P\Phi(a)^2P - \phi(a)^2 \\ &= P\Phi(a)^2P - \phi\big(a^2\big) \\ &= P\big(\Phi(a)^2 - \Phi\big(a^2\big)\big)P \le 0. \end{aligned}$$

It follows that $(1 - P)\Phi(a)P = 0$, hence $\Phi(a)\xi \in H$ for all $\xi \in H$. Thus Φ is a trivial extension of ϕ. □

In the converse direction we see that if ϕ is a nonextendible unital completely positive map, then the Stinespring Theorem, 1.2.7, shows that ϕ has an extension which is a representation, hence by nonextendibility ϕ, is itself a homomorphism. It is interesting that this conclusion holds in much more generality. Recall from Definition 1.2.1 that a map $\phi \in B(A, H)$ is 2-positive if $\phi \otimes \iota$ is positive, where ι is the identity map of M_2 onto itself. This means that

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in M_2(A)^+ \Rightarrow \begin{pmatrix} \phi(a) & \phi(b) \\ \phi(b)^* & \phi(c) \end{pmatrix} \in M_2\big(B(H)\big)^+.$$

Theorem 3.4.5 *Let A be a C^*-algebra and $\phi \in B(A, H)$ a unital 2-positive nonextendible map. Then ϕ is a homomorphism.*

Proof Let $a, b \in A$ with $a \geq 0$. Then

$$\begin{pmatrix} a & ab^* \\ ba & bab^* \end{pmatrix} = \begin{pmatrix} a^{1/2} & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & a^{1/2} \\ a^{1/2} & a \end{pmatrix} \begin{pmatrix} a^{1/2} & 0 \\ 0 & b^* \end{pmatrix} \geq 0.$$

Let b be fixed, and, then since ϕ is 2-positive,

$$\psi(a) = \begin{pmatrix} \phi(a) & \phi(ab^*) \\ \phi(ba) & \phi(bab^*) \end{pmatrix}$$

defines a positive map of A into $B(H \oplus H)$. Then $\psi(1)$ is invertible on $\psi(1)H \oplus H$. Let K denote the closure of $\psi(1)H \oplus H$. Define a map $\Phi : A \to B(H \oplus H)$ by

$$\Phi(a) = \psi(1)^{-1/2}\psi(a)\psi(1)^{-1/2}.$$

Then Φ is a positive unital map of A into $B(K)$. Let $V : H \to K$ be the linear operator defined by

$$V\xi = \psi(1)^{1/2}(\xi \oplus 0).$$

Thus for $\xi, \eta \in H$ we immediately get

$$\big(\phi(a)\xi, \eta\big) = \big(\Phi(a)V\xi, V\eta\big).$$

In particular, if $a = 1$, so $\phi(a) = 1$, we see that $V : H \to K$ is an isometric imbedding, and so

$$\phi(a) = V^*\Phi(a)V.$$

Thus Φ is an extension of ϕ, and since ϕ is nonextendible, $\Phi \supset \phi$ is a trivial extension. Therefore

$$\Phi(a)V\xi = V\phi(a)\xi.$$

Using the defining formulas for Φ and V we then get

$$\psi(1)^{-1/2}\begin{pmatrix}\phi(a) & \phi(ab^*)\\ \phi(ba) & \phi(bab^*)\end{pmatrix}\begin{pmatrix}\xi\\ 0\end{pmatrix} = \Phi(a)V\xi = V\phi(a)\xi = \psi(1)^{1/2}\begin{pmatrix}\phi(a)\xi\\ 0\end{pmatrix}.$$

If we multiply on the left by $\psi(1)^{1/2}$, we get

$$\begin{pmatrix}\phi(a) & \phi(ab^*)\\ \phi(ba) & \phi(bab^*)\end{pmatrix}\begin{pmatrix}\xi\\ 0\end{pmatrix} = \begin{pmatrix}1 & \phi(b)^*\\ \phi(b) & \phi(bb^*)\end{pmatrix}\begin{pmatrix}\phi(a)\xi\\ 0\end{pmatrix},$$

hence $\phi(ba)\xi = \phi(b)\phi(a)\xi$ for all $\xi \in H$, proving that ϕ is a homomorphism. □

3.5 A Radon-Nikodym Theorem

One version of the classical Radon-Nikodym theorem for measures states that if μ and η are finite measures on a measure space, and $\eta \le \mu$, then there exists a measurable function $0 \le f \le 1$ such that

$$\int g\, d\eta = \int f g\, d\mu$$

for all integrable functions g. We shall in the present section prove an analogous result for completely positive maps and then apply this to characterize maps which are extremal among the completely positive ones. We first show a sharpening of the Stinespring Theorem 1.2.7.

Lemma 3.5.1 *Let A be a C^*-algebra and $\phi : A \to B(H)$ a completely positive map. Then there exist a Hilbert space K, a representation π of A on K, a bounded operator $V : H \to K$ with the property that the closed subspace*

$$\big[\pi(A)VH\big] = \big\{\pi(a)V\xi \,:\, a \in A, \xi \in H\big\}^-$$

equals K, and such that $\phi = V^*\pi V$.

Proof Let $W^*\pi_0 W$ be a Stinespring decomposition of ϕ as in Theorem 1.2.7 with π_0 a representation of A on a Hilbert space K_0, and $W : H \to K_0$ a bounded operator. Let e be the projection onto $[\pi_0(A)WH]$. Then e belongs to the commutant $\pi_0(A)'$ of $\pi_0(A)$, because if $a, b \in A$ then

$$\pi_0(b)\big(\pi_0(a)W\xi\big) = \pi_0(ba)W\xi \in \big[\pi_0(A)WH\big].$$

Let $K = eK_0$, $\pi = e\pi_0$ and $V = eW$, then

$$V^*\pi(a)V = V^*e\pi_0(a)eV = W\pi_0(a)W = \phi(a),$$

and

$$\big[\pi(A)VH\big] = \big[e\pi_0(A)WH\big] = eK_0 = K.$$ □

Lemma 3.5.2 *Let ϕ_1 and ϕ_2 be completely positive maps of A into $B(H)$ such that $\phi_2 - \phi_1$ is completely positive. Let $\phi_i(a) = V_i^* \pi_i(a) V_i$ be the Stinespring decompositions such that $[\pi_i(A) V_i H] = K_i, i = 1, 2$. Then there exists an operator $T : K_2 \to K_1$ with $\|T\| \le 1$ such that*

(i) $T V_2 = V_1$.
(ii) $T \pi_2(a) = \pi_1(a) T, a \in A$.

Proof Let $\xi_1, \ldots, \xi_n \in H$, $a_1, \ldots, a_n \in A$. Then

$$\begin{aligned}\Big\| \sum_j \pi_1(a_j) V_1 \xi_j \Big\|^2 &= \sum_{ij} \big(V_1^* \pi_1\big(a_i^* a_j\big) V_1 \xi_j, \xi_i \big) \\ &= \sum_{ij} \big(\phi_1\big(a_i^* a_j\big) \xi_j, \xi_i \big) \\ &\le \sum_{ij} \big(\phi_2\big(a_i^* a_j\big) \xi_j, \xi_i \big) \\ &= \Big\| \sum \pi_2(a_j) V_2 \xi_j \Big\|^2,\end{aligned}$$

since $\phi_2 - \phi_1$ is completely positive and $(a_i^* a_j) \in (A \otimes M_n)^+$. Therefore there exists a unique contraction T defined on $[\pi_2(A) V H] = K_2$ which satisfies $T \pi_2(a) V_2 \xi = \pi_1(a) V_1 \xi$ for all $a \in A, \xi \in H$. Taking $a = 1$, we have $T V_2 = V_1$. If $a, b \in A$ then

$$T \pi_2(a) \pi_2(b) V_2 \xi = T \pi_2(ab) V_2 \xi = \pi_1(ab) V_1 \xi = \pi_1(a) T \pi_2(b) V_2 \xi,$$

so that $T \pi_2(a) = \pi_1(a) T$, using that $[\pi_2(A) V_2 H] = K_2$. □

Let ϕ be a completely positive map of A into $B(H)$ with Stinespring decomposition $\phi = V^* \pi V$. If $0 \le T \le 1$ is an operator in $\pi(A)'$ then the map $\phi_T(a) = V^* T \pi(a) V$ is a completely positive map of A into $B(H)$, because if $W = T^{1/2} V$, then $\phi_T(a) = W^* \pi(a) W$, so is completely positive by the Stinespring theorem, 1.2.7. If we apply this to $1 - T$, we see that $\phi - \phi_T = \phi_{1-T}$ is also completely positive.

Theorem 3.5.3 *Let A be a C^*-algebra and ϕ and ψ completely positive maps of A into $B(H)$ such that $\phi - \psi$ is completely positive. Let $\phi = V^* \pi V$ be the Stinespring decomposition of ϕ with $[\pi(A) V H] = K$. Then there is a unique operator $T \in \pi(A)'$ with $0 \le T \le 1$ such that $\psi(a) = \phi_T(a) = V^* T \pi(a) V$.*

Proof The map $T \to \phi_T$ is clearly linear, and if $\phi_T = 0$ then for all $a, b \in A$ and $\xi, \eta \in H$ we have

$$\big(T \pi(a) V \xi, \pi(b) V \eta \big) = \big(V^* T \pi\big(b^* a\big) V \xi, \eta \big) = \big(\phi_T\big(b^* a\big) \xi, \eta \big) = 0.$$

Since $[\pi(A) V H] = K$, $T = 0$, so we have uniqueness in the theorem.

It remains to show that $\psi = \phi_T$ for $0 \le T \le 1, T \in \pi(A)'$. By Lemma 3.5.1 ψ has a Stinespring decomposition, $\psi = W^*\sigma W$, where $W : H \to K_1$ and $K_1 = [\sigma(A)WH]$. By Lemma 3.5.2 there is a contraction $X : K \to K_1$ such that $XV = W$ and $X\pi(a) = \sigma(a)X$ for all $a \in A$, and taking adjoints, $\pi(a)X^* = X^*\sigma(a)$ for $a \in A$. Let $T = X^*X$. Then clearly $0 \le T \le 1$, and $T\pi(a) = X^*\sigma(a)X = \pi(a)T$, so that $T \in \pi(A)'$. Finally, we have for $\xi, \eta \in H$,

$$\begin{aligned}\big(\phi_T(a)\xi, \eta\big) &= \big(X^*X\pi(a)V\xi, V\eta\big)\\ &= \big(X\pi(a)V\xi, XV\eta\big)\\ &= \big(\sigma(a)XV\xi, XV\eta\big)\\ &= \big(\sigma(a)W\xi, W\eta\big)\\ &= \big(\psi(a)\xi, \xi\big),\end{aligned}$$

completing the proof of the theorem. □

We can now show the promised characterization of maps extremal in the cone of completely positive maps. For this we make the following,

Definition 3.5.4 Let $\phi : A \to B(H)$ be completely positive. We say ϕ is *pure* if every completely positive map $\psi : A \to B(H)$ with $\phi - \psi$ completely positive is a scalar multiple of ϕ.

It is well known that a state is pure if and only if its GNS-representation is irreducible. This extends to completely positive maps as follows.

Corollary 3.5.5 *Let $\phi : A \to B(H)$ be completely positive with Stinespring decomposition $\phi = V^*\pi V$, such that $V : H \to K$ and $[\pi(A)VH] = K$. Then ϕ is pure if and only if π is irreducible.*

Proof Let ϕ be pure. By the comments before Theorem 3.5.3 the set $\{T \in \pi(A)' : 0 \le T \le 1\}$ consists of scalar multiple of the identity, which implies that $\pi(A)$ is irreducible.

Conversely, if π is irreducible and $\psi : A \to B(H)$ is a map such that ψ and $\phi - \psi$ are completely positive, then by Theorem 3.5.3 $\psi = \phi_T$ for some $T \in \pi(A)', 0 \le T \le 1$. Since $\pi(A)'$ consists of scalar operators, $T = \lambda 1$ for some $0 \le \lambda \le 1$, so ψ is a scalar multiple of ϕ, hence ϕ is pure. □

In the finite dimensional case we get a stronger extremality result for pure maps. The result can easily be extended to maps $\phi : A \to B(H)$, where A is a C^*-algebra all of whose irreducible representations are finite dimensional.

Corollary 3.5.6 *Let K_0 be a finite dimensional Hilbert space and $\phi : B(K_0) \to B(H)$ completely positive. Then ϕ is pure if and only if it is an extremal positive map in $B(B(K_0), H)^+$.*

Proof It is clear that if ϕ is extremal then it is in particular pure. Conversely, assume ϕ is pure with Stinespring decomposition $\phi = V^*\pi V$, where by Corollary 3.5.5 π is irreducible. Since K_0 is finite dimensional, $\pi(B(K_0)) = B(K)$, K as in Corollary 3.5.5, and by finiteness π is an isomorphism. By Proposition 3.1.3 $AdV : B(K) \to B(H)$ is extremal. Let $\psi \in B(B(K_0), H)^+$, with $\psi \leq \phi$. Then $\psi \circ \pi^{-1} \leq AdV$, so by extremality of AdV, $\psi \circ \pi^{-1} = \lambda AdV$ for $0 \leq \lambda \leq 1$. Thus $\psi = \lambda AdV \circ \pi = \lambda\phi$, so ϕ is extremal. □

3.6 Notes

Extreme points of the convex set of unital positive maps were studied in [71]. The results in Sect. 3.1, except Proposition 3.1.7, are mostly variations of results in [71]. Proposition 3.1.7 is a special case of well known results on automorphisms of von Neumann algebras.

As mentioned in the introduction to Sect. 3.2 Jacobson and Rickart [29] showed that Jordan homomorphisms of matrix algebras over certain rings are sums of homomorphisms and anti-homomorphisms. Their result was used by Kadison [35] to show that surjective Jordan homomorphisms between C^*-algebras were sums of homomorphisms and anti-homomorphims, and finally the author [72] showed the same result for Jordan homomorphisms of a C^*-algebra into another C^*-algebra. Theorem 3.2.4 is a special case of Kadison's result, but the proof is quite different from the proofs in the papers referred to above. In [9] surjective Jordan homomorphisms were characterized as those positive maps which map invertible operators onto invertible operators.

Theorem 3.3.2 and its proof is taken from [71], but its followup, Theorem 3.3.4 is, with a different proof, due to Marciniak [50]. For a closely related result for maps which are not necessarily positive, see [31, 46–48]. Theorem 3.3.7 is also due to Marciniak [50]. For further work on nonextendible maps see [95, 96].

The contents of Sect. 3.4 on nonextendible maps are all due to Woronowicz [99], see also [42].

The Radon-Nikodym type theorem, Theorem 3.5.3 is due to Arveson [1].

If K and H are finite dimensional the facial structure of the cone $B(B(K), H)^+$ has been studied by several authors; see [45] for a survey. In this context maps which generate exposed rays in $B(B(K), H)^+$, called exposed maps have attracted much attention as they form a dense subset of the extremal maps, see e.g. [13, 19].