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Erling Størmer

Positive Linear Maps of Operator Algebras

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Introduction

The study of positive maps of C^*-algebras started around 1950 with Kadison's generalized Schwarz inequality and characterizations of isometries of C^*-algebras, [35, 36]. A few years later Stinespring introduced completely positive maps and showed his famous dilation theorem [70]. A little later Tomiyama proved some of the basic results on positive projections of von Neumann algebras onto von Neumann subalgebras, called conditional expectations [92]. After that the theory gradually developed, but with rather few people involved. A change came in the 1990's when it became clear that positive maps are important in the study of entanglement in quantum information theory. Since then the interest in the subject has increased considerably, as has the development of the theory.

The aim of the present book is to present the main part of the theory of positive maps as it stands today. We start with the basic results in Chap. 1 and prove in particular the Stinespring Theorem and the inequalities for positive maps that follow from it. It turns out that the order theory of C^*-algebras is closely related to Jordan algebras. In Chap. 2 we study positive maps from this point of view, and in particular study projection maps and their images. The unit ball of maps from one C^*-algebra into another is a convex set. As might be expected, many of the extreme points of this convex set have special properties. This topic will be treated in Chap. 3.

From Chap. 4 on much of the theory will be developed in finite dimensions. There are three main reasons for this; firstly, the main ideas come from finite dimensions, the extension to infinite dimensions are often unnecessarily technical, and the applications to quantum information theory are mostly in finite dimensions. The reader who is mainly interested in this part may skip Chaps. 2 and 3 on first reading.

Since we shall mainly be interested in properties of positive maps, and each C^*-algebra can be considered as a subalgebra of $B(H)$—the bounded linear operators on a Hilbert space H, we shall mostly restrict attention to maps into $B(H)$. If a map is from $B(K)$ into $B(H)$ with K finite dimensional, a very useful technique was introduced by Choi [7] and Jamiolkowski [30], namely the Choi matrix for a map. This matrix yields an isomorphism of the linear maps of $B(K)$ into $B(H)$ onto $B(K \otimes H)$, identified with $B(K) \otimes B(H)$. Thus problems on positive maps are reformulated in terms of matrices. Their basic properties will be studied in Chap. 4.

In Chap. 5 we introduce cones of maps in $P(H)$—the positive maps of $B(H)$ into itself—called mapping cones, and positivity of maps into $B(H)$ with respect to mapping cones. An important result in this connection is a Hahn-Banach type theorem for maps positive with respect to a mapping cone, which implies that we may restrict attention to maps of $B(K)$ into $B(H)$. This will be done in the three last chapters, which are all to a great extent inspired by quantum information theory. In Chap. 6 we study the dual cones of mapping cones, in Chap. 7 applications to states, and in Chap. 8 we consider different norms on positive maps.

In order to reach a more general audience the mathematical level of the book is kept as elementary as possible. We have therefore avoided proofs which require much of the theory of von Neumann algebras and the second dual of C^*-algebras. Therefore some results appear in less generality than is possible. In the Appendix we include some results and references which will be used in the text.

Since our main goal is the study of positive maps as such, we have omitted theory of more general type and closely related results concerning maps which are not positive.We have therefore not included results on completely bounded maps. For this see the book [59]. Furthermore, we have not included results on the facial structure of $P(H)$, nor the dual action on the state spaces of C^*-algebras given by unital positive maps. For a survey on the facial structure see [45]. For exposed maps see e.g. [13, 19]. Another area where positive maps appear, is in operator spaces. In that context the maps are usually completely positive, see [14]. However, there is a close connection with positive maps of C^*-algebras, which is shown in [34].

Most of the results in this book have not appeared in book form before. The exceptions are the basic theory of completely positive maps, which is well treated in Paulsen's book [59] and partly in Effros and Ruan's book [14]. Furthermore parts of the content of Chaps. 4, 6 and 7 are considered in the book [2] by Bengtsson and Zyczkowski, but then in a more descriptive form. For a survey of the theory as it was prior to 1974 see the survey article [75] by the author.

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Chapter 1
Generalities for Positive Maps

In this chapter we introduce the basic notions of positive maps. We show the main results on completely positive maps, inequalities and norm properties, plus the adjoint map.

1.1 Basic Definitions

It is expected that the reader knows the basic elements of operator algebra theory. But for the reader's convenience we shall state the main definitions and results we need, in the Appendix. Since there are different notations and conventions in use for the main concepts, we first introduce the basic ones used in this book.

The inner product on a complex Hilbert space H is denoted by

$$(\xi, \eta),\ \xi, \eta \in H.$$

The inner product is linear in the left variable and conjugate linear in the right.

We denote by $B(H)$ the bounded linear operators on H. $M_n = M_n(\mathbb{C})$ denotes the complex $n \times n$-matrices. It is identified with $B(\mathbb{C}^n)$. If A is a C^*-algebra we also use the notation $M_n(A)$ for the $n \times n$-matrices with entries in A. The transpose map is denoted by t, so that $t((a_{ij})) = (a_{ij})^t = (a_{ji})$. Tr will always denote the trace on M_n which takes the value 1 at minimal projections. The notation is independent of n, but will be clear from the context.

Definition 1.1.1 Let A and B be C^*-algebras. A linear map $\phi : A \to B$ is said to be *positive*, written $\phi \geq 0$, if $\phi(a) \geq 0$ whenever $a \geq 0$.

When we say a map is positive we always implicitly assume it is linear. Note that the definition makes sense in much more general circumstances, e.g. when A is an *operator system*, i.e. a linear subspace $A \subset B(H)$ such that $a \in A$ implies $a^* \in A$ and $1 \in A$. We shall often use the notation $B(A, H)$ for the linear space of bounded linear maps of A into $B(H)$ and $B(A, H)^+$ for the positive maps in $B(A, H)$.

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Since each self-adjoint operator is the difference of two positive operators with orthogonal supports, a positive map ϕ carries self-adjoint operators to self-adjoint operators. If $a = b + ic$ with b and c self-adjoint in A, we get

$$\phi(a^*) = \phi(b) - i\phi(c) = \phi(a)^*,$$

so ϕ preserves adjoints, and is often referred to as a *self-adjoint* linear map. If A has an identity 1 and $a \in A$ is self-adjoint, then $-\|a\|1 \le a \le \|a\|1$, so $-\|a\|\phi(1) \le \phi(a) \le \|a\|\phi(1)$. Thus the norm of the restriction $\phi|_{A_{sa}}$ of ϕ to the self-adjoint part of A, is $\|\phi(1)\|$. If A does not contain 1 then we can extend ϕ to $\tilde{A}$—the C^*-algebra A with 1 adjoined, i.e. $\tilde{A} = A + \mathbb{C}1$, and define $\phi(1)$ to be $\|\phi|_{A_{sa}}\|1$. If $a = b + ic$ as above we have

$$\|\phi(a)\| \le \|\phi(b)\| + \|\phi(c)\| \le 2\|\phi(1)\|\,\|a\|,$$

because $\|b\|, \|c\| \le \|a\|$. Thus every positive map is bounded and therefore continuous. We shall see later that $\|\phi\| = \|\phi(1)\|$.

A linear functional ρ on a C^*-algebra A is called a *state* if it is positive on positive operators and has norm 1. In particular if $1 \in A$, $\rho(1) = 1$. If $A = M_n$ the *density matrix* for ρ is the positive matrix h such that $\rho(a) = Tr(ha)$ for $a \in A$. If ρ is a state on $M_n \otimes M_m$, ρ is said to be a *product state* if there are states ρ_1 on M_n and ρ_2 on M_m such that $\rho = \rho_1 \otimes \rho_2$. ρ is said to be *separable* if it is a convex sum of product states.

1.2 Completely Positive Maps

Positive maps are divided into several classes of which the completely positive maps have been the most important. This has also been the case in applications to physics, see [49]. See the Appendix for a discussion of tensor products.

Definition 1.2.1 Let $\phi : A \to B$ be a linear map, and let $k \in \mathbb{N}$—the natural numbers. Then ϕ is *k-positive* if $\phi \otimes i_k : A \otimes M_k \to B \otimes M_k$ is positive, i_k denotes the identity map on M_k. ϕ is said to be *completely positive* if ϕ is k-positive for all $k \in \mathbb{N}$.

A restatement of the definition of k-positivity is that if $(a_{ij}) \in M_k(A)^+$—the positive elements in $M_k(A)$—then $(\phi(a_{ij})) \in M_k(B)^+$. Note also that since the flip map $A \otimes M_k \to M_k \otimes A$, defined by $a \otimes b \mapsto b \otimes a$, is an isomorphism, ϕ is also k-positive if and only if $i_k \otimes \phi : M_k \otimes A \to M_k \otimes B$ is positive.

We next list some properties of k-positive, and hence completely positive maps. But first recall that a $*$-*anti-homomorphism* is a self-adjoint linear map ϕ such that $\phi(ab) = \phi(b)\phi(a)$. We shall often drop the prefix $*$ when we say a map is a $*$-homomorphism or an $*$-anti-homomorphism. If K and H are Hilbert spaces, and

$V : H \to K$ is a bounded linear operator, then AdV denotes the map of $B(K)$ into $B(H)$ defined by

$$AdV(a) = V^* a V.$$

Since every C^*-algebra can be imbedded in $B(H)$ for some Hilbert space H, we can often replace the C^*-algebra B in Definition 1.2.1 by $B(H)$.

Lemma 1.2.2 *Let $A \subset B(K)$ and $B \subset B(H)$ be C^*-algebras and $\phi : A \to B$ a self-adjoint linear map.*

(i) *If $\phi = AdV$ for a bounded operator $V : H \to K$, then ϕ is completely positive.*
(ii) *If ϕ is a $*$-homomorphism then ϕ is completely positive.*
(iii) *If A_0 and B_0 are C^*-algebras, ϕ k-positive and $\alpha : A_0 \to A$, $\beta : B \to B_0$ are k-positive then $\beta \circ \phi \circ \alpha$ is k-positive.*
(iv) *If in* (iii) *α and β are $*$-anti-homomorphisms and ϕ k-positive, then $\beta \circ \phi \circ \alpha$ is k-positive.*

Proof (i) This follows since $AdV \otimes i_k = Ad(V \otimes 1_k)$ is positive, where i_k is the identity map on M_k.

(ii) Similarily if ϕ is a homomorphism, then so is $\phi \otimes i_k$, hence is positive.

(iii) Since

$$(\beta \circ \phi \circ \alpha) \otimes i_k = (\beta \otimes i_k) \otimes (\phi \otimes i_k) \otimes (\alpha \otimes i_k)$$

is a composition of positive maps, $\beta \circ \phi \circ \alpha$ is k-positive.

(iv) Similarily if α and β are anti-homomorphisms then

$$(\beta \circ \phi \circ \alpha) \otimes i_k = (\beta \otimes t) \circ (\phi \otimes i_k) \otimes (\alpha \otimes t),$$

as $t^2 = i_k$. Since $\alpha \otimes t$ and $\beta \otimes t$ are $*$-anti-homomorphisms they are positive maps, so again $\beta \circ \phi \circ \alpha$ is k-positive. □

If $(a_{ij}), (b_{ij}) \in M_n$ then their *Schur product* is the matrix $(a_{ij} b_{ij}) \in M_n$.

Lemma 1.2.3 *If $(a_{ij}) \in M_n^+$ then the Schur product $(b_{ij}) \mapsto (a_{ij} b_{ij})$ is a completely positive map $M_n \to M_n$.*

Proof By spectral theory we may assume (a_{ij}) is of rank 1, hence of the form $(\overline{a_i} a_j)$. Let V denote the diagonal matrix with diagonal entries $a_1, \ldots, a_n$. Then $(a_{ij} b_{ij}) = (\overline{a_i} b_{ij} a_j) = V^*(b_{ij})V$, so the lemma follows from Lemma 1.2.2 part (i). □

If either A or B is abelian then a positive map $\phi : A \to B$ is completely positive. In the next two theorems we prove this.

Theorem 1.2.4 *Let A and B be C^*-algebras with B abelian. Then every positive map $\phi : A \to B$ is completely positive. In particular, each state on A considered as a positive map of A into $\mathbb{C}$ is completely positive.*

Proof We first show that if ρ is a pure state on $B \otimes M_k$ then ρ is a product state. Indeed, since $B \otimes \mathbb{C}1$ is the center of $B \otimes M_k$, if $0 \leq b \leq 1$ in $B \otimes \mathbb{C}1$ then for all $a \geq 0$ in $B \otimes M_k$, $\rho(ba) \leq \rho(a)$. Since ρ is pure it follows that $\rho(ba) = \rho(b)\rho(a)$. Thus, if $\omega = \rho|_{B\otimes\mathbb{C}1}$ and $\eta = \rho|_{\mathbb{C}1\otimes M_k}$ then for $b \in B, a \in M_k$,

$$\begin{aligned}\rho(b \otimes a) &= \rho\big((b \otimes 1)(1 \otimes a)\big) = \rho(b \otimes 1)\rho(1 \otimes a) \\ &= \omega(b)\eta(a) = \omega \otimes \eta(b \otimes a),\end{aligned}$$

proving the assertion.

Let now $\phi : A \to B$ be a positive map. Let ρ be a pure state of $B \otimes M_k$. With ω and η as above

$$\rho \circ (\phi \otimes i_k) = (\omega \circ \phi) \otimes (\eta \circ i_k)$$

is the tensor product of two positive linear functionals, hence is positive. Since this holds for all pure states ρ, $\phi \otimes i_k$ is positive, so ϕ is completely positive. □

Theorem 1.2.5 *Let A and B be C^*-algebras with A abelian. Then every positive map $\phi : A \to B$ is completely positive.*

We first give a simple proof when A is finite dimensional. In that case let $e_1, \ldots, e_m$ be the minimal projections in A, so $Ae_i = \mathbb{C}e_i$. Define $\phi_i(a) = \phi(ae_i)$. Then ϕ_i is the composition of the homomorphism $a \mapsto ae_i$ and a positive map $\mathbb{C} \to B(H)$, so is clearly completely positive, hence so is $\phi = \sum_{i=1}^m \phi_i$.

Proof of Theorem 1.2.5 We may assume $A = C_0(X)$—the continuous complex functions vanishing at infinity on a locally compact Hausdorff space, or if A is unital that $A = C(X)$—the continuous functions on a compact Hausdorff space. Let $a = (f_{ij}) \in M_n(A)^+$. Assume $B \subset B(H)$, and let $\xi_1, \ldots, \xi_n$ be vectors in H. We wish to show

$$\sum_{i,j=1}^n \big(\phi(f_{ij})\xi_j, \xi_i\big) = \left(\big(\phi(f_{ij})\big) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \right) \geq 0. \tag{1.1}$$

By the Riesz-Markoff Theorem there exists a regular measure m on X such that $\sum_{i=1}^n (\phi(f)\xi_i, \xi_i) = \int f\, dm$ for all $f \in A$. Then by the Riesz-Markov and Radon-Nikodym theorems there exist measurable functions h_{ij} such that

$$\big(\phi(f)\xi_j, \xi_i\big) = \int f h_{ij} dm \quad \text{for all } f \in A. \tag{1.2}$$

Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then we have for $f \geq 0$,

$$\int f(\gamma)\left(\big(h_{ij}(\gamma)\big)\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix}\right) dm$$

$$= \int f(\gamma) \sum_{i,j} h_{ij}(\gamma)\lambda_j \overline{\lambda_i}\, dm$$

$$= \sum_{i,j} \big(\phi(f)\xi_j, \xi_i\big)\lambda_j \overline{\lambda_i}$$

$$= \left(\begin{pmatrix}\phi(f) & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \phi(f)\end{pmatrix}\begin{pmatrix}\lambda_1\xi_1\\ \vdots \\ \lambda_n\xi_n\end{pmatrix}, \begin{pmatrix}\lambda_1\xi_1\\ \vdots \\ \lambda_n\xi_n\end{pmatrix}\right)$$

$$\geq 0,$$

since $\phi(f) \geq 0$ when $f \geq 0$. It follows that for each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, $((h_{ij}(\gamma))\lambda, \lambda) \geq 0$ almost everywhere. Letting λ run through a countable dense set in $\mathbb{C}^n$ we conclude that $(h_{ij}(\gamma)) \geq 0$ almost everywhere.

Since the evaluation $f \mapsto f(\gamma)$ is a state for $\gamma \in X$, it is completely positive by Theorem 1.2.4, so the matrix $(f_{ij}(\gamma)) \in M_n^+$ for all $\gamma \in X$. Therefore by Lemma 1.2.3 the Schur product

$$\big(f_{ij}(\gamma)h_{ij}(\gamma)\big) \geq 0$$

almost everywhere, and so

$$\sum_{i,j} f_{ij}(\gamma)h_{ij}(\gamma) \geq 0 \quad \text{almost everywhere.}$$

Thus from (1.1) and (1.2)

$$\sum_{i,j} \big(\phi(f_{ij})\xi_j, \xi_i\big) = \int \sum_{i,j} f_{ij}(\gamma)h_{ij}(\gamma)\, dm \geq 0,$$

completing the proof. □

Remark 1.2.6 An alternative proof of the above theorem would be to show that the cone $(A \otimes M_k)^+$ of positive operators in $A \otimes M_k$ equals the cone $A^+ \otimes M_k^+$ generated by operators $a \otimes b$ with $a \in A^+, b \in M_k^+$. In that case, if $a \in (A \otimes M_k)^+$ then a is of the form $a = \sum a_i \otimes b_i$, $a_i \in A^+$, $b_i \in M_k^+$, so $(\phi \otimes \iota_k)(a) = \sum \phi(a_i) \otimes b_i \geq 0$, and therefore ϕ is completely positive.

The main result on completely positive maps is the Stinespring Theorem, which is an extension of the GNS construction for states to completely positive maps.

Theorem 1.2.7 *Let A be a unital C^*-algebra and $\phi : A \to B(H)$. Then ϕ is completely positive if and only if there exist a Hilbert space K, a bounded linear operator $V : H \to K$ and a $*$-homomorphism $\pi : A \to B(K)$ such that*

$$\phi(a) = V^*\pi(a)V \quad \textit{for all } a \in A.$$

Furthermore $\|V\|^2 \le \|\phi(1)\|$.

Proof If ϕ is of the above form then ϕ is completely positive by Lemma 1.2.2.

The proof of the converse is a generalization of the proof of the GNS-representation for a state. We define a sesquilinear form on $A \otimes H$ by

$$\left\langle \sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i \right\rangle = \sum \big(\phi\big(a_i^* b_j\big)\eta_j, \xi_i\big),$$

for $a_i, b_j \in A$, $\eta_j, \xi_i \in H$, $i = 1, \ldots, k$, $j = 1, \ldots, l$. In particular, if $\eta = (\eta_1, \ldots, \eta_l)$, then

$$\begin{aligned}\left\langle \sum_i^l b_j \otimes \eta_j, \sum_j b_i \otimes \eta_i \right\rangle &= \sum_i^l \big(\phi\big(b_i^* b_j\big)\eta_j, \eta_i\big) \\ &= \big(\big(\phi\big(b_i^* b_j\big)\big)\eta, \eta\big) \ge 0,\end{aligned}$$

since ϕ is in particular l-positive, and

$$\big(b_i^* b_j\big) = \begin{pmatrix} b_1^* & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ b_l^* & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} b_1 & \cdots & b_l \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in M_l(A)^+.$$

We therefore have a positive semidefinite sesquilinear form, and if we let

$$N = \big\{u \in A \otimes H \, : \, \langle u, u \rangle = 0\big\}$$

then $\langle\, , \,\rangle$ induces a Hilbert space inner product on $(A \otimes H)/N$. We let K denote the completion of the pre-Hilbert space $(A \otimes H)/N$.

For each $a \in A$ we let $\pi(a)$ be the linear map on $A \otimes H$ defined by

$$\pi(a)\Big(\sum a_j \otimes \xi_j\Big) = \sum a a_j \otimes \xi_j.$$

If $\sum a_j \otimes \xi_j \in A \otimes H$ let $\xi = (\xi_1, \ldots, \xi_k)$. We then have

$$\begin{aligned}&\left\langle \pi(a)\Big(\sum_j a_j \otimes \xi_j\Big), \pi(a)\Big(\sum_i a_i \otimes \xi_i\Big)\right\rangle \\ &= \sum \big(\phi\big(a_i^* a^* a a_j\big)\xi_j, \xi_i\big)\end{aligned}$$

$$\begin{aligned}&= \big(\big(\phi\big(a_i^* a^* a a_j\big)\big)\xi, \xi\big)\\ &\le \|a\|^2 \big(\big(\phi\big(a_i^* a_j\big)\big)\xi, \xi\big)\\ &= \|a\|^2 \Big\langle \sum a_j \otimes \xi_j, \sum a_i \otimes \xi_i \Big\rangle.\end{aligned}$$

In particular $\pi(a)$ maps N into itself. $\pi(a)$ therefore determines a bounded linear operator, also denoted by $\pi(a)$, of $A \otimes H/N$ into itself. It is clear that $\|\pi(a)\| \le \|a\|$. Thus $\pi(a)$ extends to a linear operator on K, which we again denote by $\pi(a)$. It is easy to check that $\pi : A \to B(K)$ is a unital $*$-homomorphism.

Define $V : H \to K$ by

$$V\xi = 1 \otimes \xi + N.$$

Then

$$\begin{aligned}\|V\xi\|^2 = \langle V\xi, V\xi\rangle = \langle 1 \otimes \xi, 1 \otimes \xi\rangle = \big(\phi(1)\xi, \xi\big)\\ \le \big\|\phi(1)\big\| \|\xi\|^2,\end{aligned}$$

so in particular, V is bounded and $\|V\|^2 \le \|\phi(1)\|$.

Finally, if $a \in A$ and $\xi, \eta \in H$ then

$$\begin{aligned}\big(V^*\pi(a)V\xi, \eta\big) &= \big\langle \pi(a)(1 \otimes \xi), 1 \otimes \eta\big\rangle\\ &= \langle a \otimes \xi, 1 \otimes \eta\rangle\\ &= \big(\phi(a)\xi, \eta\big).\end{aligned}$$

Thus $\phi(a) = V^*\pi(a)V$, completing the proof. □

For more on the Stinespring Theorem see Sect. 3.5.

The Stinespring Theorem has immediate formulations to other classes of maps, as we shall now see.

Definition 1.2.8 Let A be a C^*-algebra and $\phi : A \to B(H)$. We say that ϕ is *copositive* if $t \circ \phi$ is completely positive, where t is the transpose map on $B(H)$. ϕ is *decomposable* if ϕ is the sum of a completely positive and a copositive map. Otherwise ϕ is *indecomposable*.

Remark 1.2.9 Note that if t' is the transpose map on $B(H)$ with respect to another orthonormal basis, then there is a unitary operator $u \in B(H)$ such that $t' = Adu \circ t$. Thus by Lemma 1.2.2 the definition of copositive maps is independent of the choice of basis, and thus of t.

The same is the situation with maps of the form $t \circ \phi$ with ϕ k-positive. We shall come back to these and k-positive maps in Chaps. 6 and 8, where it will be shown that they fit well into the classification scheme for positive maps. The remaining class which is very poorly understood, is that of atomic maps, where a positive map

$\phi : A \to B(H)$ is said to be *atomic* if it cannot be written as a sum $\phi = \phi_1 + t \circ \phi_2$ with ϕ_1 and ϕ_2 two 2-positive maps.

Definition 1.2.10 Let A and B be C^*-algebras and $\phi : A \to B$ a self-adjoint linear map. We say that ϕ is a *Jordan homomorphism* if $\phi(a^2) = \phi(a)^2$ for all self-adjoint operators $a \in A$.

Note that since $ab + ba = (a + b)^2 - a^2 - b^2$, a Jordan homomorphism preserves the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.

Theorem 1.2.11 *Let A be a unital C^*-algebra and $\phi : A \to B(H)$ a positive map. Then*

(i) *If ϕ is copositive then there exist a Hilbert space K, a bounded linear operator $V : H \to K$ and an anti-homomorphism $\pi : A \to B(K)$ such that $\phi(a) = V^*\pi(a)V$ for $a \in A$.*
(ii) *If ϕ is decomposable then there exist K and V as in (i) and a Jordan homomorphism $\pi : A \to B(K)$ such that $\phi(a) = V^*\pi(a)V$ for $a \in A$.*

Proof (i) Let K_0 be a Hilbert space such that $A \subset B(K_0)$, and let t_0 denote the transpose on $B(K_0)$ and t the transpose on $B(H)$. Let $B = t_0(A)$. Then B is a C^*-algebra anti-isomorphic to A. Define $\phi' : B \to B(H)$ by $\phi'(b) = \phi(t_0(b))$. This is well-defined because $t_0 = t_0^{-1}$. Since ϕ is copositive, $t \circ \phi$ is completely positive, hence by Lemma 1.2.2 (iv) $\phi' = t \circ (t \circ \phi) \circ t_0$ is completely positive. Therefore by the Stinespring Theorem, 1.2.7, there exist V and π' as in the statement of the Stinespring Theorem such that $\phi' = V^*\pi'V$. Then if $a \in A$,

$$\phi(a) = \phi \circ t_0 \circ t_0(a) = \phi'(t_0 a) = V^*\pi'\big(t_0(a)\big)V = V^*\pi(a)V,$$

where π is a $*$-anti-homomorphism, proving (i).

(ii) Suppose $\phi = \phi_1 + \phi_2$ with $\phi_1 : A \to B(H)$ completely positive and $\phi_2 : A \to B(H)$ copositive. By the Stinespring Theorem and (i) there exist Hilbert spaces K_i, a homomorphism $\pi_1 : A \to B(K_1)$, and an anti-homomorphism $\pi_2 : A \to B(K_2)$ such that $\phi_i = V_i^*\pi_i V_i$ where $V_i : H \to K_i$. Let $V : H \to K_1 \oplus K_2$ by

$$V\xi = V_1\xi \oplus V_2\xi.$$

Define $\pi : A \to B(K_1 \oplus K_2)$ by

$$\pi(a) = \pi_1(a) + \pi_2(a).$$

Thus π is a Jordan homomorphism, and for $a \in A$,

$$V^*\pi(a)V = V_1^*\pi_1(a)V_2 + V_2^*\pi_2(a)V_2 = \phi_1(a) + \phi_2(a) = \phi(a),$$

completing the proof of the theorem. □

1.3 Inequalities

The Stinespring Theorem, 1.2.7, yields several inequalities for positive maps. From the theorem it follows that if $\phi = V^*\pi V$, then $\|V\|^2 \le \|\phi\|$, hence if $\|\phi\| \le 1$ then $\|V\| \le 1$.

Theorem 1.3.1 *Let A be a C^*-algebra and $\phi : A \to B(H)$ be a positive map with $\|\phi\| \le 1$. Then for $a \in A$ we have:*

(i) *If A is unital and ϕ is completely positive then $\phi(a^*a) \ge \phi(a)^*\phi(a)$.*
(ii) *If a is a normal operator then $\phi(a^*a) \ge \phi(a)^*\phi(a)$.*
(iii) *If a is a self-adjoint operator then $\phi(a^2) \ge \phi(a)^2$.*
(iv) *$\phi(a^*a + aa^*) \ge \phi(a)^*\phi(a) + \phi(a)\phi(a)^*$.*

Proof (i) If ϕ is completely positive, $\phi = V^*\pi V$ as in the Stinespring Theorem with $\|V\| \le 1$. Thus

$$\begin{aligned}\phi\big(a^*a\big) &= V^*\pi\big(a^*a\big)V = V^*\pi(a)^*\pi(a)V \ge V^*\pi(a)^*VV^*\pi(a)V \\ &= \phi(a)^*\phi(a).\end{aligned}$$

(ii) If a is a normal operator in A then the C^*-algebra $C(a)$ generated by a and 1 is abelian and ϕ has a positive extension to $C(A)$ with norm $\|\phi\| \le 1$. Then the restriction of ϕ to $C(a)$ is completely positive by Theorem 1.2.5. Hence (ii) follows from (i). (iii) is immediate from (ii). (iv) The operators $a + a^*$ and $i(a - a^*)$ are self-adjoint. Thus by (iii)

$$\phi\big(\big(a + a^*\big)^2\big) + \phi\big(\big(i\big(a - a^*\big)\big)^2\big) \ge \phi\big(a + a^*\big)^2 + \phi\big(i\big(a - a^*\big)\big)^2.$$

A straightforward computation now yields the desired result. □

Corollary 1.3.2 *Let A be a unital C^*-algebra and ϕ a 2-positive map with $\|\phi\| \le 1$ of A into $B(H)$. Then $\phi(a^*a) \ge \phi(a)^*\phi(a)$ for all $a \in A$.*

Proof Let ι_2 denote the identity map on M_2. Since $\phi \otimes \iota_2$ is positive, Theorem 1.3.1 (iii) implies

$$\begin{aligned}\begin{pmatrix}\phi(a^*a) & 0 \\ 0 & \phi(aa^*)\end{pmatrix} &= \phi \otimes \iota_2\begin{pmatrix}a^*a & 0 \\ 0 & aa^*\end{pmatrix} = \phi \otimes \iota_2\left(\begin{pmatrix}0 & a^* \\ a & 0\end{pmatrix}^2\right) \\ &\ge \phi \otimes \iota_2\begin{pmatrix}0 & a^* \\ a & 0\end{pmatrix}^2 = \begin{pmatrix}\phi(a)^*\phi(a) & 0 \\ 0 & \phi(a)\phi(a)^*\end{pmatrix},\end{aligned}$$

proving that $\phi(a^*a) \ge \phi(a)^*\phi(a)$. □

In Sect. 1.1 we showed that if ϕ is positive $\|\phi\| \le 2\|\phi(1)\|$. Using Theorem 1.3.1 we can now improve this.

Theorem 1.3.3 *Let A be a unital C^*-algebra and $\phi : A \to B(H)$ a self-adjoint linear map. If ϕ is positive then $\|\phi\| = \|\phi(1)\|$. Conversely, if $\phi(1) = 1$ and $\|\phi\| = 1$, then ϕ is positive.*

Proof Multiplying ϕ by a scalar we may assume $\|\phi\| = 1$. By the Russo-Dye Theorem, see Appendix A1.5, the unit ball of A is the closed convex hull of the unitary operators in A. Thus $1 = \sup \|\phi(u)\|$, where the sup is taken over all unitary operators in A. But by Theorem 1.3.1(ii) if ϕ is positive, then

$$\left\|\phi(u)\right\|^2 = \left\|\phi(u)^*\phi(u)\right\| \le \left\|\phi\bigl(u^*u\bigr)\right\| = \left\|\phi(1)\right\| \le 1.$$

Thus $1 = \sup \|\phi(u)\|^2 \le \|\phi(1)\| \le 1$, so $\|\phi(1)\| = 1 = \|\phi\|$. Conversely if ϕ is a self-adjoint linear map such that $\phi(1) = 1$, and $\|\phi\| = 1$, then for each state ρ of $B(H)$, $\rho \circ \phi$ is a state, hence is positive. Since this holds for all states, ϕ is positive. □

1.4 The Adjoint Map

If K and H are finite dimensional Hilbert spaces, then $B(K)$ and $B(H)$ with the Hilbert-Schmidt inner product $\langle a, b\rangle = Tr(ab^*)$ are Hilbert spaces. Thus a linear map $\phi : B(K) \to B(H)$ can be considered as a bounded operator between Hilbert spaces and therefore has an adjoint map defined by

$$Tr\bigl(\phi(a)b\bigr) = Tr\bigl(a\phi^*(b)\bigr), \quad a \in B(K), b \in B(H). \tag{1.3}$$

In the infinite dimensional case we must assume ϕ is *normal*, i.e. if $(a_\alpha)_{\alpha\in I}$ is an increasing net in $B(K)^+$ with least upper bound a, so $a_\alpha \nearrow a \in B(K)$, implies $\phi(a_\alpha) \nearrow \phi(a)$, then ϕ is weakly continuous on bounded sets, see Appendix A.1. Since every normal state on $B(H)$ is defined by a density operator, which is a positive trace class operator, a normal positive map ϕ has an *adjoint map* ϕ^* mapping the trace class operators $\mathscr{T}(H)$ on H into $\mathscr{T}(K)$, defined by (1.3).

Definition 1.4.1 Let M be a von Neumann algebra and $\phi : M \to B(H)$ be a normal positive map. Then the *null space of* ϕ is the sup of all projections $e \in M$ such that $\phi(e) = 0$. If f is the null space of ϕ then $1 - f$ is the *support of* ϕ, denoted by $\operatorname{supp}\phi$. We say ϕ is *faithful* if the null space of ϕ is 0, i.e. if $a \ge 0$ and $\phi(a) = 0$ then $a = 0$.

Proposition 1.4.2 *Let K and H be Hilbert spaces and $\phi : B(K) \to B(H)$ a normal positive map. Then we have*:

(i) *$\phi^* : \mathscr{T}(H) \to B(K)$ is positive.*
(ii) *$\phi(1) = 1$ if and only if $Tr_K \circ \phi^* = Tr_H$, where Tr_K and Tr_H are the traces on $B(K)$ and $B(H)$ respectively.*

(iii) *Let* $e = \operatorname{supp}\phi$. *Then* $e\phi^*(b)e = \phi^*(b)$ *for all* $b \in \mathscr{T}(H)$.
(iv) *If* $V : H \to K$ *is linear then* $(AdV)^* = AdV^*$.

Proof (i) This follows since $a \in B(K)$ is positive if and only if $Tr(ab) \geq 0$ for all positive $b \in \mathscr{T}(K)$.

(ii) $Tr_H(\phi(1)b) = Tr_K(\phi^*(b))$ for all $b \in \mathscr{T}(H)$. Thus (ii) follows.

(iii) Since $\phi(e) = \phi(1)$ we have for $b \in \mathscr{T}(H)$

$$0 = Tr\big((1-e)\phi^*(b)\big) = Tr\big((1-e)\phi^*(b)(1-e)\big).$$

Hence for all $b \geq 0$ in $\mathscr{T}(H)$, $(1-e)\phi^*(b)(1-e) = 0$, so that $(1-e)\phi^*(b) = 0$ for all positive b, and therefore $\phi^*(b) = e\phi^*(b) = (e\phi^*(b))^* = \phi^*(b)e$. Thus (iii) follows easily, since the positive operators span $\mathscr{T}(H)$.

(iv) This follows since

$$Tr\big(AdV(a)b\big) = Tr\big(V^*aVb\big) = Tr\big(aVbV^*\big) = Tr\big(aAdV^*(b)\big). \qquad \square$$

If H is finite dimensional then $\mathscr{T}(H) = B(H)$, so $1 \in \mathscr{T}(H)$. Then we can add the following to Proposition 1.4.2.

Proposition 1.4.3 *Let* H *be finite dimensional and* $\phi : B(K) \to B(H)$ *be weakly continuous on bounded sets. Then we have*:

(i) ϕ *is positive if and only if* ϕ^* *is positive.*
(ii) *If* $f = \operatorname{supp}\phi^*$ *then* $\phi^* : fB(H)f \to eB(K)e$ *is faithful, where* $e = \operatorname{supp}\phi$.
(iii) ϕ *is* k*-positive if and only if* ϕ^* *is* k*-positive. Hence* ϕ *is completely positive if and only if* ϕ^* *is completely positive.*
(iv) *If* ϕ *is faithful then the range projection of* $\phi^*(1)$ *equals* 1.

Proof (i) This follows by the argument of Proposition 1.4.2(i).

(ii) If $f = \operatorname{supp}\phi^*$ then $\phi^*(b) = \phi^*(fbf)$, so by Proposition 1.4.2(iii) $\phi^*(fbf) = e\phi^*(b)e$, and (ii) follows by definition of $\operatorname{supp}\phi^*$.

(iii) We have $(\phi \otimes \iota_k)^* = \phi^* \otimes \iota_k^* = \phi^* \otimes \iota_k$. Thus by (i) ϕ is k-positive if and only if ϕ^* is k-positive.

(iv) If the range projection of $\phi^*(1)$ is not the identity then there exists a 1-dimensional projection p orthogonal to $\phi^*(1)$. Then $Tr(\phi(p)) = Tr(p\phi^*(1)) = 0$. Since ϕ is faithful $p = 0$, completing the proof. $\square$

1.5 Notes

The main results in the present chapter are closely related to completely positive maps. The definition is due to Stinespring [70], who proved Theorem 1.2.7. As the reader can see, the proof is a generalization of the proof of the GNS construction for states. Our proof follows closely the proof in the book by Effros and Ruan [14]. The theorem has been extended to $*$-algebras of unbounded operators by Timmermann [90].

The ideas of the Stinespring Theorem go back to Naimark [52], who proved the theorem in the case when the map is from an abelian C^*-algebra into $B(H)$. Thus Theorem 1.2.5 is due to him. Our proof follows closely the one due to Stinespring. The other theorem on positive maps being automatically completely positive, Theorem 1.2.4, is due to the author [71], see also [85]. Among the inequalities in Theorem 1.3.1 the most famous is the third, $\phi(a^2) \geq \phi(a)^2$. This inequality was proved by Kadison [36] and is usually referred to as the Kadison-Schwarz inequality. Corollary 1.3.2 is due to Choi [6], and Theorem 1.3.3 to Russo and Dye [65].

Chapter 2
Jordan Algebras and Projection Maps

The order structure in C^*-algebras is closely related to Jordan algebras. In this chapter we shall study this connection. In the first part we shall study general positive maps, and in the second and third projection maps, i.e. positive idempotent maps of C^*-algebras into themselves.

2.1 Jordan Properties of Positive Maps

The class of Jordan algebras which we shall encounter, are contained in C^*-algebras.

Definition 2.1.1 A *JC-algebra* J is a norm closed real linear subspace of the self-adjoint operators in $B(H)$ for a Hilbert space H, such that $a, b \in J$ implies $a \circ b = \frac{1}{2}(ab + ba) \in J$.

$a \circ b$ is called the *Jordan product* of a and b. Since $2a \circ b = (a + b)^2 - a^2 - b^2$, one could equivalently just require that $a \in J$ implies that $a^2 \in J$. Thus the self-adjoint part A_{sa} of a C^*-algebra A is a JC-algebra.

Definition 2.1.2 Let A and B be C^*-algebras and $\phi : A \to B$ a self-adjoint linear map. Then ϕ is an *order-isomorphism* if ϕ is bijective and $\phi(a) \geq 0$ if and only if $a \geq 0$.

The close relation between the order-structure and the Jordan structure is clear from the following theorem.

Theorem 2.1.3 *Let A and B be unital C^*-algebras and $\phi : A \to B$ a unital self-adjoint linear map. Then ϕ is an order-isomorphism if and only if ϕ is a Jordan isomorphism.*

Proof Since a self-adjoint operator is positive if and only if it is of the form a^2 with a self-adjoint, it is clear that a Jordan isomorphism is an order-isomorphism.

E. Størmer, *Positive Linear Maps of Operator Algebras*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-34369-8_2,

Conversely assume ϕ is an order-isomorphism. By the Kadison-Schwarz inequality, Theorem 1.3.1(iii), $\phi(a^2) \geq \phi(a)^2$ for all self-adjoint $a \in A$. Since the inverse map ϕ^{-1} is also positive and unital, it also satisfies the Kadison-Schwarz inequality, hence for a self-adjoint in A,

$$a^2 = \phi^{-1}\big(\phi\big(a^2\big)\big) \geq \phi^{-1}\big(\phi(a)^2\big) \geq \phi^{-1}\big(\phi(a)\big)^2 = a^2,$$

so that $\phi(a^2) = \phi(a)^2$, hence ϕ is a Jordan isomorphism. □

Definition 2.1.4 Let A and B be C^*-algebras and $\phi : A \to B$ a positive map. The *definite set* for ϕ is the set $D = \{a \in A_{sa} : \phi(a^2) = \phi(a)^2\}$. The *multiplicative domain* for ϕ is the set $M_\phi = \{a \in A : \phi(ba) = \phi(b)\phi(a)\}$ for all $b \in A$.

We say ϕ is a *Schwarz map* if it satisfies the Schwarz inequality $\phi(a^*a) \geq \phi(a)^*\phi(a)$ for all $a \in A$. Then ϕ is in particular a contraction, since $\phi(1) \geq \phi(1)^2$. By Corollary 1.3.2 each 2-positive contraction is a Schwarz map.

Proposition 2.1.5 *Let A and B be C^*-algebras and $\phi : A \to B$ a Schwarz map. Suppose $a \in A$ satisfies*

$$\phi\big(a^*\big)\phi(a) = \phi\big(a^*a\big).$$

Then

$$\phi\big(b^*a\big) = \phi(b)^*\phi(a) \quad \text{and} \quad \phi\big(a^*b\big) = \phi(a)^*\phi(b)$$

for all $b \in A$. Hence a belongs to the multiplicative domain M_ϕ for ϕ.

Proof With a and b as above and $t \in \mathbb{R}$ we have, using the assumption on a,

$$\begin{aligned} & t\big(\phi(a)^*\phi(b) + \phi(b)^*\phi(a)\big) \\ &\quad = \phi(ta+b)^*\phi(ta+b) - t^2\phi(a)^*\phi(a) - \phi(b)^*\phi(b) \\ &\quad \leq \phi\big((ta+b)^*(ta+b)\big) - t^2\phi(a)^*\phi(a) - \phi(b)^*\phi(b) \\ &\quad \leq t\phi\big(a^*b + b^*a\big) + \big(\phi\big(b^*b\big) - \phi(b)^*\phi(b)\big). \end{aligned}$$

Since this holds for all $t \in \mathbb{R}$,

$$\phi(a)^*\phi(b) + \phi(b)^*\phi(a) = \phi\big(a^*b + b^*a\big). \tag{2.1}$$

Replacing b by $-ib$ and then multiplying by i gives

$$\phi(a)^*\phi(b) - \phi(b)^*\phi(a) = \phi\big(a^*b - b^*a\big). \tag{2.2}$$

Adding (2.1) and (2.2) and then subtracting one from the other yields the two equations in the proposition.

The last statement is obvious from the first of the two equations. □

Corollary 2.1.6 *Let A and B be C^*-algebras and $\phi : A \to B$ a Schwarz map. Then the multiplicative domain M_ϕ for ϕ is a subalgebra of A.*

Proof If $a, b \in M_\phi$ and $c \in A$, then

$$\begin{aligned}\phi\big((ab)c\big) &= \phi\big(a(bc)\big) = \phi(a)\phi(bc) = \phi(a)\phi(b)\phi(c)\\ &= \phi(ab)\phi(c),\end{aligned}$$

hence $ab \in M_\phi$. Since M_ϕ is clearly a linear set it is an algebra. □

Proposition 2.1.7 *Let A and B be C^*-algebras and $\phi : A \to B$ positive with $\|\phi\| \le 1$. Suppose a belongs to the definite set D for ϕ. Then for all $b \in A_{sa}$ we have*

(i) $\phi(a \circ b) = \phi(a) \circ \phi(b)$.
(ii) $\phi(aba) = \phi(a)\phi(b)\phi(a)$.

Furthermore, D is a JC-subalgebra of A_{sa}.

Proof In the proof of Proposition 2.1.5 we use the Schwarz inequality only for operators a, b, and $ta+b$, so when they are self-adjoint we only needed the Kadison-Schwarz inequality. Therefore (i) follows from (2.1).

(ii) follows from (i) via the identity

$$aba = 2(a \circ b) \circ a - a^2 \circ b.$$

To show D is a JC-algebra let $a, b \in D$. Then by (i) and (ii)

$$\begin{aligned}4\phi\big((a \circ b)^2\big) &= \phi\big(abab + ab^2a + ba^2b + baba\big)\\ &= 2\phi\big(a \circ (bab)\big) + \phi\big(ab^2a\big) + \phi\big(ba^2b\big)\\ &= 2\phi(a) \circ \phi(bab) + \phi\big(ab^2a\big) + \phi\big(ba^2b\big)\\ &= 2\phi(a) \circ \phi(b)\phi(a)\phi(b) + \phi(a)\phi(b)^2\phi(a) + \phi(b)\phi(a)^2\phi(b)\\ &= 4\big(\phi(a) \circ \phi(b)\big)^2\\ &= 4\phi(a \circ b)^2,\end{aligned}$$

so that $a \circ b \in D$. We have

$$\begin{aligned}\phi\big((a+b)^2\big) = \phi\big(a^2 + 2a \circ b + b^2\big) &= \phi(a)^2 + 2\phi(a) \circ \phi(b) + \phi(b)^2\\ &= \big(\phi(a) + \phi(b)\big)^2 = \phi(a+b)^2,\end{aligned}$$

hence $a + b \in D$. □

Proposition 2.1.7 raises a natural problem, namely, what kind of JC-algebra is the definite set D for different kinds of positive maps. In the finite dimensional case the irreducible JC-algebras are: $(M_n)_{sa}$, the real symmetric matrices in M_n; if the quaternions $\mathbb{Q}$ are represented by 2×2-matrices, the self-adjoint block matrices $M_n(\mathbb{Q})_{sa}$ in $M_{2n}(\mathbb{C})$ with entries in $\mathbb{Q}$, and the spin factors to be defined in Sect. 2.3.

A JC-algebra J is said to be *reversible* if it is closed under symmetric products, i.e. if $a_1, \dots, a_k \in J$ then

$$a_1 a_2 \dots a_k + a_k a_{k-1} \dots a_1 \in J.$$

In this case if R is the real algebra generated by J then $R_{sa} = J$. In the above examples only the spin factors are not reversible.

Proposition 2.1.8 *Let A be a C^*-algebra and $\phi : A \to B(H)$ a unital positive map. Let D be the definite set of ϕ. Then we have*:

(i) *If ϕ is decomposable then D is a reversible JC-algebra of A_{sa}.*
(ii) *If ϕ is completely positive then D is the self-adjoint part of a C^*-subalgebra of A.*

Proof (i) By Proposition 2.1.7 D is a JC-subalgebra of A_{sa}. By Theorem 1.2.11 there exist a Hilbert space K, a bounded linear operator $V : H \to K$, and a Jordan homomorphism $\pi : A \to B(K)$ such that $\phi(a) = V^*\pi(a)V$ for $a \in A$. Then for $a \in D$,

$$V^*\pi(a)^2 V = V^*\pi\big(a^2\big)V = \phi\big(a^2\big) = \phi(a)^2 = \big(V^*\pi(a)V\big)^2.$$

Since $V^*V = \phi(1) = 1$, $e = VV^*$ is a projection, and if we set $\pi(a) = x$, we have $ex^2e = exexe$, so

$$\big((1-e)xe\big)^*(1-e)xe = ex^2e - exexe = 0,$$

hence $(1-e)xe = 0$, so that $xe = exe = (exe)^* = ex$, hence $\pi(a) = x \in \{e\}'$, the commutant of e.

Conversely, if $a \in A_{sa}$ with $\pi(a)e = e\pi(a)$, then

$$\phi\big(a^2\big) = V^*\pi(a)^2 V = V^*\pi(a)e\pi(a)V = V^*\pi(a)VV^*\pi(a)V = \phi(a)^2.$$

Then $D = \pi^{-1}(\{e\}' \cap \pi(A_{sa}))$.

Since π is the sum of a homomorphism and an anti-homomorphism, as was shown in the proof of Theorem 1.2.11, we have for $a_1, \dots, a_n \in A_{sa}$

$$\pi\left(\prod_1^n a_i + \prod_n^1 a_i\right) = \prod_1^n \pi(a_i) + \prod_n^1 \pi(a_i).$$

In particular if $a_i \in D$, by the above characterization of D, $\pi(\prod_1^n a_i + \prod_n^1 a_i)$ commutes with e and hence belongs to $\pi(D)$, so $\prod_1^n a_i + \prod_n^1 a_i \in D$, hence D is reversible, proving (i).

(ii) If in the above proof ϕ is completely positive, by the Stinespring Theorem 1.2.7, π is a homomorphism. Thus

$$D = \{a \in A_{sa} : \pi(a)e = e\pi(a)\},$$

hence if $a, b \in D$ then $\pi(ab)e = \pi(a)\pi(b)e = e\pi(ab)$. It follows that each a in the C^*-algebra $C^*(D)$ generated by D satisfies $\pi(a)e = e\pi(a)$. Thus $D = C^*(D)_{sa}$ proving (ii). □

2.2 Projection Maps

Definition 2.2.1 Let A be a C^*-algebra and $P : A \to A$ a positive map with $\|P\| \le 1$. Then P is a *projection map* if $P^2 = P \circ P = P$. If $P(A)$ is a C^*-subalgebra of A then P is called a *conditional expectation*.

These maps, and especially conditional expectations, have been very important in the theory of von Neumann algebras. We shall mainly be interested in their structure and as examples of positive maps. For simplicity of the arguments we shall mostly consider faithful projection maps.

Theorem 2.2.2 *Let A be a C^*-algebra and $P : A \to A$ a faithful projection map. Then*

(i) *$P(A_{sa})$ is a JC-subalgebra of A_{sa} contained in the definite set for P.*
(ii) *If P is a Schwarz map then $P(A)$ is a C^*-subalgebra of A contained in the multiplicative domain for P.*

Proof We first show (ii), because (i) follows by the same arguments. So assume P is a Schwarz map, and let $a \in P(A)$. Then

$$P\big(P(a^*a) - a^*a\big) = P(a^*a) - P(a^*a) = 0.$$

From the Schwarz inequality

$$P(a^*a) - a^*a \ge P(a)^*P(a) - a^*a = a^*a - a^*a = 0,$$

so by faithfulness of P, $P(a^*a) = P(a)^*P(a) = a^*a$, so by Proposition 2.1.5 a belongs to the multiplicative domain for P. Thus $P(ba) = P(b)a$ for all $b \in A$. In particular, if $b \in P(A)$, then $ab = P(ab) \in P(A)$, so $P(A)$ being closed under the $*$-operation, is a C^*-subalgebra of the multiplicative domain.

To show (i) apply the above arguments to $a \in A_{sa}$. Then it follows that $P(a^2) = a^2 = P(a)^2$, so a belongs to the definite set for P, and as in the proof of (ii) it follows from Proposition 2.1.7 that $P(A_{sa})$ is a JC-subalgebra of A_{sa}, and $P(a \circ b) = a \circ P(b)$ for all $b \in A$. □

We make the following observation.

Lemma 2.2.3 *If A is a unital C^*-algebra and $P : A \to A$ is a faithful projection map, then $P(1) = 1$.*

Proof Since $\|P\| \leq 1$, $P(1) \leq 1$. But $P(1 - P(1)) = 0$, so by faithfulness of P, $P(1) = 1$. □

Theorem 2.2.4 *Let A be a unital C^*-algebra, $A \subset B(H)$, and $P : A \to A$ a faithful decomposable projection map. Then $P(A_{sa})$ is a reversible JC-algebra.*

Proof By Proposition 2.1.8 and Lemma 2.2.3 the definite set D of P is a reversible JC-subalgebra, and by Theorem 2.2.2, $P(A_{sa}) \subset D$. By the definition of D, the restriction $P|_D$ is a Jordan homomorphism.

Let $P = V^*\pi V$ as in the proof of Proposition 2.1.8. Since π is the sum of a homomorphism and an anti-homomorphism, so is the restriction of P to D. Hence P preserves symmetric products, so that if $a_1, \ldots, a_k \in D$ then

$$\prod_1^n P(a_i) + \prod_n^1 P(a_i) = P\left(\prod_1^n a_i + \prod_n^1 a_i\right) \in P(D) = P(A_{sa}).$$

Thus $P(A_{sa})$ is a reversible JC-algebra. □

We have now seen how the image of a projection map depends on positivity properties of the map. A natural problem is whether there are results in the converse direction. This is true for Theorem 2.2.4, i.e. if $P(A_{sa})$ is reversible then P is decomposable, see [76], but we shall not prove this because the proof is too much of a detour into Jordan algebra theory to belong here. It was shown by Robertson [64] that the assumption in Theorem 2.2.4 can be weakened, because if P is the sum of a 2-positive and a 2-copositive map, then P is automatically decomposable.

However, if the image is a C^*-algebra, a converse is easier to prove. Remember, since each completely positive map is a Schwarz map by Theorem 1.3.1 it follows by Theorem 2.2.2 that the image of a faithful completely positive projection map is a C^*-algebra. We first prove a simple lemma.

Lemma 2.2.5 *Let A be a C^*-algebra. Then every positive operator in $M_n(A)$ is a sum of n positive operators of the form $(a_i^* a_j)$ for $a_1, \ldots, a_n \in A$.*

Proof Let $b \in M_n(A)$ be the matrix whose kth row is $a_1, \ldots, a_n$ and the other entries are 0. Then $b^*b = (a_i^* a_j)$, so each operator $(a_i^* a_j)$ is positive. Now let $a \in M_n(A)^+$. Then $a = b^*b$ for $b \in M_n(A)$. Write $b = b_1 + \cdots + b_n$ where b_k is the kth row of b and 0 elsewhere. Then $b_i^* b_j = 0$ when $i \neq j$, so $a = b^*b = \sum_{i=1}^n b_i^* b_i$, is of the form desired. □

If $B \subset B(H)$ is a C^*-algebra and $\xi \in H$ we denote by $[B\xi]$ the orthogonal projection of H onto the closure of the subspace of H consisting of vectors $b\xi, b \in B$. Since $ab\xi \in B\xi$ for a and $b \in B$, $[B\xi]$ is invariant under B, hence belongs to the commutant B' of B. If $[B\xi] = 1$ then ξ is said to be a *cyclic vector* for B.

Theorem 2.2.6 *Let $B \subset A$ be unital C^*-algebras. Suppose $P : A \to B$ is a surjective projection map. Then P is completely positive.*

Proof We may assume $A \subset B(H)$. Assume first there exists a unit vector η_0 in H cyclic for B. We have to show that if $a \in M_n(A)^+$ then $P \otimes \iota_n(a) \in M_n(A)^+$, where we identify $M_n(A)$ with $A \otimes M_n$. By Lemma 2.2.5 we may assume $a = (a_i^* a_j)$ with $a_1, \dots, a_n \in A$. Let $\xi_1, \dots, \xi_n \in H$. We have to show

$$\sum_{i,j=0}^{n} \big(P\big(a_i^* a_j\big)\xi_j, \xi_i\big) \ge 0, \tag{2.3}$$

see (1.1).

Let $\varepsilon > 0$. Since η_0 is cyclic for B there exist $b_i \in B$ such that

$$\|b_i \eta_0 - \xi_i\| < \varepsilon / n^2 \max \|\xi_i\|, \quad i = 1, \dots, n.$$

By Proposition 2.1.5 applied to the b_i's we get

$$\begin{aligned} \sum_{i,j} \big(P\big(a_i^* a_j\big)\xi_j, \xi_i\big) &\ge \sum_{i,j} \big(P\big(a_i^* a_j\big) b_j \eta_0, b_i \eta_0\big) - \varepsilon \\ &= \sum_{i,j} \big(P\big(b_i^* a_i^* a_j b_j\big)\eta_0, \eta_0\big) - \varepsilon \\ &= (P\Big(\sum_{i,j} (a_i b_i)^* (a_j b_j)\eta_0, \eta_0\Big) - \varepsilon \\ &\ge -\varepsilon. \end{aligned}$$

Since ε is arbitrary, (2.3) follows.

In the general case there exists a sequence (η_k) in H such that $\sum_k [B\eta_k] = 1$, where $[B\eta_k]$ denotes the projection onto the closure of the set $\{b\eta_k : b \in B\}$. Then we have by the above, since $[B\eta_k] \in B'$,

$$\begin{aligned} \sum_{i,j} \big(P\big(a_i^* a_j\big)\xi_j, \xi_i\big) &= \sum_{i,j,k} \big([B\eta_k] P\big(a_i^* a_j\big)[B\eta_k]\xi_j, \xi_i\big) \\ &= \sum_{i,j,k} \big(P\big(a_i^* a_j\big)[B\eta_k]\xi_j, [B\eta_k]\xi_i\big), \end{aligned}$$

which is nonnegative by the first part of the proof, since η_k is cyclic for $[B\eta_k]B[B\eta_k]$ as acting on $[B\eta_k]H$. □

Corollary 2.2.7 *Let A be a unital C^*-algebra and $P : A \to A$ a faithful projection which is a Schwarz map. Then P is completely positive.*

Proof By Theorem 2.2.2 $P(A)$ is a C^*-subalgebra of A. By Lemma 2.2.3 P is unital, and by Theorem 2.2.6 P is completely positive. □

When we described the ranges of projection maps we assumed the maps were faithful. We shall now see what happens when they are not faithful. It is then simplest to replace the C^*-algebras by von Neumann algebras and assume the projection maps to be normal. By a *JW-algebra* we mean a weakly closed JC-algebra.

Proposition 2.2.8 *Let M be a von Neumann algebra and $P : M \to M$ be a normal unital projection map. Let e denote the support of P (Definition 1.4.1). Then P_e defined by $P_e(a) = eP(eae)e$ is a faithful projection map of eMe onto $eP(M)e$, hence $eP(M_{sa})e$ is a JW-algebra.*

Proof We first show P_e is a projection map. Let $a \in M$. Then since $P(eae) = P(a)$,

$$P_e^2(a) = eP\big(eP(eae)e\big)e = eP\big(P(a)\big)e = eP(a)e = eP(eae)e = P_e(a),$$

so P_e is a projection map. To show P_e is faithful on eMe assume $a \geq 0$ and $P_e(a) = 0$. Then, using that P is faithful on eMe, we have

$$0 = eP(eae)e = P\big(eP(eae)e\big) = P\big(P(eae)\big) = P(eae),$$

so that $eae = 0$, and $P_e(eM_{sa}e)$ is a JC-algebra by Theorem 2.2.2. Since P is normal, P is weakly continuous on bounded sets, see Appendix A.1, hence $P(M)$ is weakly closed. Thus $eP(M_{sa})e$ is a JW-algebra. □

Proposition 2.2.9 *Let M be a von Neumann algebra and $P : M \to M$ a normal unital projection map. Let e be the support of P and $N = P(M_{sa})$. Then e belongs to the commutant N' of N, and $N + fM_{sa}f$ is a JW-subalgebra of M_{sa}, where $f = 1 - e$.*

Proof Let $a \in N$. By Proposition 2.1.7,

$$P(aea) = aP(e)a = a^2 = P\big(a^2\big),$$

so $P(a(1-e)a) = 0$. Hence by definition of the support $ea(1-e)ae = 0$. Therefore $ea(1-e) = 0$, and so $ea = eae = ae$.

By Proposition 2.2.8 eNe is a JW-subalgebra of $eM_{sa}e$. Thus by the above

$$N = Ne + Nf \subset eNe + fM_{sa}f,$$

so that $N + fM_{sa}f$ is a JW-subalgebra of M_{sa}. □

It should be remarked that in both of the last two propositions we could have assumed M to be a JW-algebra rather than a von Neumann algebra. The proofs would be the same.

There are many theorems in the literature showing the existence of projection maps of C^*- or von Neumann algebras into themselves. We shall need one, which we for simplicity state for finite dimensional algebras, even though the result is true under much more general circumstances.

Proposition 2.2.10 *Let A be a C^*-algebra acting on a finite dimensional Hilbert space. Let Tr be a faithful trace on A, and let B be a JC-subalgebra of A_{sa}. Then there exists a faithful projection map $P : A \to B + iB$ given by the formula $Tr(ab) = Tr(P(a)b)$ for all $b \in B$.*

Proof With the inner product $< a, b >= Tr(ab^*)$, A becomes a pre-Hilbert space, and $B + iB$ is a complex subspace. If $a \in A$ the map $b \mapsto Tr(ab)$ is a continuous linear functional on $B + iB$, so by the Riesz representation theorem there exists an operator $P(a) \in B + iB$ such that

$$Tr(ab) = Tr\big(P(a)b\big), \quad b \in B.$$

Clearly P so defined is linear, unital, and idempotent. If $a \geq 0$ then $Tr(P(a)b) \geq 0$ for all $b \in B^+$. If $P(a)$ were not positive, by spectral theory there would exist non zero projections commuting with $P(a)$, $e, f \in B$ with $e + f = 1$, such that $P(a)e \geq 0$, $0 \neq P(a)f \leq 0$. But then

$$0 \leq Tr\big(P(a)f\big) < 0,$$

a contradiction. Thus $P(a) \geq 0$, and P is a projection map. Finally, if $a \geq 0$ and $P(a) = 0$, then $Tr(a) = Tr(P(a)) = 0$, so $a = 0$, since Tr is faithful, and therefore P is faithful. □

If ϕ is a unital positive map of a C^*-algebra into itself, then its fixed point set has Jordan structure. Our next result describes this in more detail.

Theorem 2.2.11 *Let M be a von Neumann algebra and $\phi : M \to M$ a normal unital positive map. Let $M^\phi = \{a \in M : \phi(a) = a\}$ be the fixed point set for ϕ. Then we have*:

(i) *There exists a projection map $P : M \to M^\phi$.*

Assume that there exists a faithful normal state on M such that $\omega \circ \phi = \omega$. Then we have:

(ii) *P is normal, faithful, and M^ϕ_{sa} is a JW-subalgebra of M_{sa}.*
(iii) *If ϕ is 2-positive then M^ϕ is a von Neumann subalgebra of M.*

Proof For each $n \in \mathbb{N}$ let $\phi_n = \frac{1}{n}\sum_{k=1}^{n} \phi^k$. Since the unit ball in the set of positive maps of M into itself is BW-compact, see Appendix A.1.1, there is a subnet (ϕ_{n_α}) of (ϕ_n) which converges pointwise weakly to a positive unital map $P : M \to M$. Then we have for all $n \in \mathbb{N}$,

$$\begin{aligned}\phi^n\big(P(a)\big) &= \phi^n\left(\lim_\alpha \frac{1}{n_\alpha}\sum_{k=1}^{n_\alpha}\phi^k(a)\right)\\ &= \lim_\alpha \frac{1}{n_\alpha}\sum_1^{n_\alpha}\phi^{n+k}(a)\\ &= \lim_\alpha \frac{1}{n_\alpha}\left(\sum_1^{n_\alpha}\phi^k(a) - \sum_1^{n}\phi^k(a) + \sum_1^{n}\phi^{k+n_\alpha}(a)\right)\\ &= \lim_\alpha \frac{1}{n_\alpha}\sum_1^{n_\alpha}\phi^k(a)\\ &= P(a).\end{aligned}$$

In particular, $\phi_n(P(a)) = P(a)$, and we have

$$P^2(a) = P\big(P(a)\big) = \lim_\alpha \phi_{n_\alpha}\big(P(a)\big) = P(a),$$

so P is a projection. Clearly $\phi(a) = a$ implies $P(a) = a$. Conversely, if $P(a) = a$, then by the above, $a = P(a) = \phi(P(a)) = \phi(a)$, so $a \in M^\phi$. Thus $P(M) = M_\phi$, and we have proved (i).

Now assume there is a faithful normal state ω such that $\omega \circ \phi = \omega$. Then clearly $\omega \circ \phi_n = \omega$, and since ω is weakly continuous on the unit ball of M by the Appendix A.1, $\omega \circ P = \omega$. Let (a_α) be an increasing net in M^+ such that $a_\alpha \nearrow a \in M$. Then

$$0 = \lim \omega(a - a_\alpha) = \omega\big(P(a) - P(a_\alpha)\big).$$

Since P is positive, $P(a_\gamma) \le P(a)$, so $x = \sup_\alpha P(a_\alpha) \le P(a)$, hence $P(a) = \sup_\alpha P(a_\alpha)$, proving that P is normal. If $a \ge 0$ and $P(a) = 0$ then $0 = \omega(P(a)) = \omega(a)$, so $a = 0$, thus P is faithful. Since the support of P is 1, $M_{sa}^\phi = P(M_{sa})$ is a JW-algebra by Proposition 2.2.8, proving (ii).

(iii) If ϕ is 2-positive, then, since the composition of two 2-positive maps is 2-positive, it follows that P is 2-positive, hence by Corollary 1.3.2, P is a Schwarz map. But then by Theorem 2.2.2 $M^\phi = P(M)$ is a von Neumann subalgebra of M. □

2.3 Spin Factors

The canonical anticommutation relations give rise to an interesting class of JC-algebras, called spin factors. Algebraically they are quite different from the reversible ones we have encountered so far. We shall in the present section study projections onto spin factors and show they have properties which are very different from the others we have considered.

Definition 2.3.1 Let H be a Hilbert space. A *spin system* in $B(H)$ is a collection $\mathscr{P}$ of at least two symmetries, i.e. self-adjoint unitary operators different from ± 1 such that $s \circ t = \frac{1}{2}(st + ts) = 0$ whenever $s \neq t$ in $\mathscr{P}$. A JC-algebra A is called a *spin factor* if it is the real linear span of 1 and a spin system.

Given a spin system $\mathscr{P}$ let H_0 be its real linear span. Then any two elements $a, b \in H_0$ can be written as $a = \sum_i \alpha_i s_i$, $b = \sum_i \beta_i s_i$, $\alpha_i, \beta_i \in \mathbb{R}$, $s_i \in \mathscr{P}$ distinct. From this we get

$$a \circ b = \left(\sum_i \alpha_i \beta_i \right) 1,$$

from which it follows that H_0 is a real pre-Hilbert space with inner product defined by

$$\langle a, b \rangle 1 = a \circ b.$$

It is clear that $H_0 + \mathbb{R}1$ is a Jordan subalgebra of $B(H)_{sa}$, whose norm closure is the spin factor obtained from $\mathscr{P}$. It is also clear that if $\mathscr{P}_1$ and $\mathscr{P}_2$ are two spin systems with the same number of symmetries, then the spin factors are Jordan isomorphic; just take a bijection between $\mathscr{P}_1$ and $\mathscr{P}_2$ and extend it linearly.

In order to give an example of a spin factor let

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

be the Pauli spin matrices in M_2. Let $\sigma_3^{\otimes k}$ denote the k-fold tensor product $\sigma_3 \otimes \cdots \otimes \sigma_3$ of σ_3 with itself k times in M_{2^k}, and let similarily $1^{\otimes k}$ denote the k-fold tensor product of 1 with itself in M_{2^k}. Let

$$\begin{aligned} s_1 &= \sigma_1 \otimes 1^{\otimes n-1}, \\ s_2 &= \sigma_2 \otimes 1^{\otimes n-1}, \\ s_3 &= \sigma_3 \otimes \sigma_1 \otimes 1^{\otimes n-2}, \\ s_4 &= \sigma_3 \otimes \sigma_2 \otimes 1^{\otimes n-2}, \\ &\vdots \\ s_{2n-1} &= \sigma_3^{\otimes n-1} \otimes \sigma_1, \\ s_{2n} &= \sigma_3^{\otimes n-1} \otimes \sigma_2. \end{aligned} \tag{2.4}$$

Then $\mathscr{P}_k = \{s_1, \ldots, s_k\}$, $k \in \{2n-1, 2n\}$ is a spin system in M_{2^n}, and the real linear span V_k of $\mathscr{P}_k$ and 1 is a $k+1$ dimensional spin factor. We say a JC-algebra is *irreversible* if it is not reversible.

Lemma 2.3.2 *The spin factors V_4 and $V_k, k \geq 6$, are irreversible.*

Proof V_4 is the span of $s_1, \dots, s_4$ and 1, so it is of dimension 5. Suppose $\frac{1}{2}(s_1s_2s_3s_4 + s_4s_3s_2s_1) \in V_4$, and let $s = s_1s_2s_3s_4$. Since $s_is_j = -s_js_i$ for $i \neq j$, $s^* = s_4s_3s_2s_1 = s$, and $s^2 = 1$, so s is a symmetry in V_4. Furthermore $s \circ s_i = 0$, so $\mathscr{P} = \{s_1, s_2, s_3, s_4, s\}$ is a spin system such that the span of $\mathscr{P}$ and 1 is of dim 6, contradicting the fact that $\dim V_4 = 5$, hence $s = \frac{1}{2}(s_1s_2s_3s_4 + s_4s_3s_2s_1) \notin V_4$, so V_4 is not reversible.

Let $\mathscr{P}_k$ and V_k be as above with $k \geq 6$, and suppose $s \in V_k$. Let $l > k$. Then $s_l \circ s_i = 0$ for all $s_i \in \mathscr{P}_k$, and thus

$$s_ls = s_ls_1s_2s_3s_4 = s_1s_2s_3s_4s_l = ss_l.$$

Since $s \in V_k$, $s = \alpha 1 + \sum_{i=1}^k \alpha_i s_i$ with $\alpha, \alpha_i \in \mathbb{R}$. Thus

$$\alpha s_l - \sum \alpha_i s_i s_l = \alpha s_l + \sum \alpha_i s_l s_i = s_l s = s s_l = \alpha s_l + \sum \alpha_i s_i s_l.$$

Thus $\sum \alpha_i s_i s_l = 0$. Since s_l is a symmetry, $\sum \alpha_i s_i = 0$. But $s_1, \dots, s_k$ are linearly independent, so $\alpha_i = 0$ for all i, hence $s = \alpha 1$, contradicting the fact that $\{s, s_1, \dots, s_4\}$ is a spin system. It follows that $s \notin V_k$, so V_k is irreversible. □

By Proposition 2.2.10 if $V_k \subset M_n$, $k \geq 2$, then there exists a faithful projection P of M_n onto V_k. By Theorem 2.2.4 and Lemma 2.3.2 this projection cannot be decomposable unless $k \in \{2, 3, 5\}$. We thus have

Proposition 2.3.3 *Let $k = 4$ or $k \geq 6$, and $V_k \subset M_n$. Then the projection map $P : M_n \to V_k + iV_k$ given by $Tr(P(a)b) = Tr(ab), a \in M_n, b \in V_k$, is indecomposable.*

We thus have an infinite family of indecomposable maps. However, a stronger result is true. Recall that a map is atomic if it is not of the form $\phi_1 + \phi_2 \circ t$ for ϕ_1 and ϕ_2 both 2-positive.

Theorem 2.3.4 *Let $P : M_n \to M_n$ be a faithful projection map such that $P(M_n)_{sa}$ is a spin factor of dimension 5 or greater than or equal to 7. Then P is atomic.*

In order to prove the theorem we need the following lemma.

Lemma 2.3.5 *Let M be a von Neumann algebra and B a JW-subalgebra of M_{sa}. Suppose $\phi : M \to M$ is a positive map such that $\phi(x) \leq x$ for all $x \in B^+$. Then*

$$\phi(b) = \phi(1)b = b\phi(1) \quad \textit{for all } b \in B.$$

Proof Given a projection $e \in B$ we have $0 \leq \phi(e) \leq e$, so that $(1 - e)\phi(e) = 0$. Replacing e by $1 - e$ gives $e\phi(1 - e) = 0$, and subtraction of these two equations results in $\phi(e) = e\phi(1)$, and taking adjoints $\phi(e) = \phi(1)e$. Since B is the weakly closed linear span of its projections, the lemma follows. □

Proof of Theorem 2.3.4 We can assume $P(M_n)_{sa} = V_k$ with $k = 4$ or $k \geq 6$, and the spin system is the one defined in (2.4). Let A be the C^*-algebra generated by V_4. Then A is isomorphic to $M_4 = M_2 \otimes M_2$. Let t denote the transpose on M_2 such that $\sigma_i^t = \sigma_i, i = 1, 2$, and let $\beta = Ad\sigma_3$. Since $\sigma_3^t = -\sigma_3$ and $\beta(\sigma_i) = -\sigma_i, i = 1, 2$, it follows that the map $\alpha(a) = (t \otimes t \circ \beta)(a)$ is a $*$-anti-automorphism of A such that $\alpha(a) = a$ for $a \in V_4$.

In order to prove the theorem we assume P is not atomic and will produce a contradiction. So assume $P = \phi + \psi$ with ϕ 2-positive and $\psi = \psi' \circ t'$, with ψ' 2-positive, t' being the transpose on M_n extending t. By Theorem 2.2.6 and Proposition 2.2.10 there exists a completely positive projection map $P_1 : M_n \to A$. Then $Q = \alpha \circ P_1$ is a projection map of M_n onto A such that $Q \circ t'$ is 2-positive and $Q(a) = a$ for all $a \in V_4$.

Let $0 < \varepsilon < 1/2$, and let

$$P_\varepsilon = (1 - 2\varepsilon)P + \varepsilon\iota + \varepsilon Q,$$

where ι is the identity map on M_n. Then

$$P_\varepsilon = \phi_0 + \psi_0,$$

where $\phi_0 = (1 - 2\varepsilon)\phi + \varepsilon\iota$ is 2-positive, and $\psi_0 = (1 - 2\varepsilon)\psi + \varepsilon Q$ is such that $\psi_0 \circ t'$ is 2-positive. Moreover, $h = \phi_0(1)^{1/2}$, $k = \psi_0(1)^{1/2}$ are invertible. We then have unital positive maps $\phi_1, \psi_1 : M_n \to M_n$ such that

$$\phi_1(a) = h^{-1}\phi_0(a)h^{-1}, \qquad \psi_1 = k^{-1}\psi_0(1)k^{-1}.$$

Then ϕ_1 and $\psi_1 \circ t'$ are 2-positive, and

$$P_\varepsilon(a) = h\phi_1(a)h + k\psi_1(a)k = \phi_0(a) + \psi_0(a).$$

Now $P_\varepsilon(a) = a$ for all $a \in V_4$. Thus by Lemma 2.3.5

$$\phi_0(a) = h^2 a = ah^2, \qquad \psi_0(a) = k^2 a = ak^2, \quad \text{for all } a \in V_4.$$

It follows that $ha = ah$ and $ka = ak$ for all $a \in V_4$. Therefore

$$\phi_1(a) = a = \psi_1(a) \quad \text{for all } a \in V_4.$$

Since ϕ_1 is positive and unital and $\phi_1(s_i) = s_i$, $i = 1, \ldots, 4$, $\phi_i(s_i^2) = \phi_i(1) = 1 = s_i^2$, s_i belongs to the multiplicative domain for ϕ_1. Hence by iterated use of Proposition 2.1.5, since ϕ_1 is a Schwarz map by Corollary 1.3.2, we have for $s = s_1 s_2 s_3 s_4$,

$$\phi_1(s) = \phi_1(s_1 s_2 s_3 s_4) = s_1 s_2 s_3 s_4 = s.$$

Similarly, by the same result for maps satisfying the inequality $\phi(a^* a) \geq \phi(a)\phi(a)^*$, we get $\psi_1(s) = s$. Now h and k commute with all operators in V_4, hence with s. Thus we get

$$P_\varepsilon(s) = h\phi_1(s)h + k\psi_1(s)k = hsh + ksh = \big(h^2 + k^2\big)s = s,$$

since $h^2 + k^2 = P_\varepsilon(1) = 1$. Letting $\varepsilon \to 0$ we get

$$P(s) = \lim_{\varepsilon \to 0} P_\varepsilon(s) = s \in V_k.$$

But from the proof of Lemma 2.3.2, $s \notin V_k$, so we have obtained the desired contradiction. □

2.4 Notes

Some of the results in this chapter have been part of the theory of C^*-algebras for several years. Theorem 2.1.3 was proved by Kadison [35] already in 1952. Definite sets and multiplicative domains appeared later. Proposition 2.1.7 on definite sets was shown by Broise [3] in 1967. Multiplicative domains were introduced by Choi [6] and Proposition 2.1.5 is due to him. Proposition 2.1.8 is due to Robertson [62]. For further work on multiplicative domains see [33].

Projection maps, and especially conditional expectations, have been important in von Neumann algebra theory since the paper of Tomiyama [92] in 1957. In our treatment of projection maps we have avoided the applications of von Neumann algebras, because that would divert our attention more than desired from the emphasis on positivity properties of the maps. See Takesaki's book [87] for some of this theory. In the case of automorphism groups of C^*-algebras there often exist invariant projection maps onto the fixed point algebra, see e.g. [40, 61, 74].

Among the results in Sect. 2.2, Theorem 2.2.2 can be traced back to [92], while Theorem 2.2.4 is due to the author [76]. Theorem 2.2.6 is due to Nakamura, Takesaki and Umegaki [54]. Proposition 2.2.9 can be found in [15] and the same with Proposition 2.2.10, but that result and its generalizations were known before, see for example [86].

For the theory of JC-algebras, and in particular spin factors see the book of Hanche-Olsen and the author [22]. Proposition 2.3.3 appeared in [76], and is the first example of an infinite family of indecomposable map in different dimensions found in the literature. Other such families were later exhibited by Terhal [89], see Theorem 7.4.8 below and Tanashashi and Tomiyama [88], see Remark 7.3.7. Theorem 2.3.4 is due to Robertson [63], see also [18].

Chapter 3
Extremal Positive Maps

The unit ball of the set of positive maps from a C^*-algebra into another C^*-algebra is a convex set, and it is natural to expect that the maps which are extreme points, have special properties. We shall in the present chapter study different classes of extremal maps.

Section 3.1 is on general results and the most obvious extremal maps. Section 3.2 is devoted to Jordan homomorphisms, Sect. 3.3 to maps such that the composition with pure states are pure states, and Sect. 3.4 to maps called nonextendible maps, which have strong extremality properties.

Finally, in Sect. 3.5 we prove a Radon-Nikodym theorem for completely positive maps together with its applications to extremal maps.

3.1 General Properties of Extremal Maps

Definition 3.1.1 Let A and B be C^*-algebras and $\phi : A \to B$ a positive map. We say that ϕ is *extremal* if the only positive maps $\psi : A \to B$, such that $\phi - \psi$ is positive, are of the form $\lambda\phi$ with $0 \leq \lambda \leq 1$.

Thus if ϕ is positive with $\|\phi\| \leq 1$, ϕ cannot be the convex combination $\lambda\psi_1 + (1 - \lambda)\psi_2$ of two positive maps ψ_1 and ψ_2 of norms less than or equal to 1 unless both ψ_1 and ψ_2 are positive multiples of ϕ. We list some simple properties of extremal maps.

Lemma 3.1.2 *Let $\phi : A \to B$ be a positive map, A and B being C^*-algebras. Then we have*:

(i) *If e is a projection in A such that $\phi(e) = \phi(1)$, then the restriction of ϕ to eAe is an extremal map $eAe \to B$ if and only if ϕ is extremal.*
(ii) *If $\alpha : B \to C$ with C another C^*-algebra, is an order-isomorphism of B onto C, then $\alpha \circ \phi$ is extremal if and only if ϕ is extremal.*

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Proof (i) Assume ϕ is extremal and $\psi : eAe \to B$ a positive map such that $0 \le \psi \le \phi|_{eAe}$. Extend ψ to a map ψ_0 on A defined by $\psi_0(a) = \psi(eae)$.

If $0 \le a \in A$ then, since $\phi(a) = \phi(eae)$ from the assumption on ϕ,

$$0 \le \psi_0(a) = \psi(eae) \le \phi(eae) = \phi(a).$$

Since ϕ is extremal, $\psi_0 = \lambda\phi$, hence $\psi = \lambda\phi|_{eAe}$ for some $\lambda \ge 0$.

Conversely, if $0 \le \psi \le \phi$ then $0 \le \psi|_{eAe} \le \phi|_{eAe}$, so extremality of $\phi|_{eAe}$ implies $\psi|_{eAe} = \lambda\phi|_{eAe}$. Since $0 \le \psi(1-e) \le \phi(1-e) = 0$, it follows that

$$\psi(a) = \psi(eae) = \lambda\phi(eae) = \lambda\phi,$$

so ϕ is extremal.

(ii) This is obvious, since $0 \le \psi \le \alpha \circ \phi$ if and only if $0 \le \alpha^{-1} \circ \psi \le \phi$. □

As remarked in Sect. 1.1 we use the notation $B(A, H)$ (resp. $B(A, H)^+$) for the bounded linear (resp. positive) maps of A into $B(H)$.

Proposition 3.1.3 *Let H and K be Hilbert spaces and $V : H \to K$ a bounded linear operator. Then the map $AdV(a) = V^*aV$ is extremal in $B(B(K), H)^+$.*

Proof We first consider the case when $K = H$ and $V = 1$, so $AdV = \iota$—the identity map. Suppose ψ is a positive map of $B(H)$ into itself such that $\psi \le \iota$. Let f be projection in $B(H)$. Then $\psi(f) \le f$, hence by Lemma 2.3.5 applied to $M = B(H)$, $B = B(H)_{sa}$, $\psi(a) = \psi(1)a$ for all $a \in B(H)$. In particular $\psi(1)$ commutes with a for all $a \in B(H)$, so $\psi(1) = \lambda 1$, and $\psi = \lambda\iota$, proving that ι is extremal.

We next consider the case when V is invertible. Then AdV is an order-isomorphism, so by the above paragraph and Lemma 3.1.2, $AdV = \iota \circ AdV$ is extremal.

Let $e = \text{range}\, V^* = \text{support}\, V$, and $f = \text{range}\, V = \text{support}\, V^*$. Thus $AdV : fB(K)f \to eB(H)e$. If $V : eH \to fK$ is invertible, then $AdV : fB(K)f \to eB(H)e$ is extremal in $B(fB(K)f, eH)^+$ by the previous paragraph. Since any positive map $\psi \le AdV$ maps $1 - f$ to 0 and $e\psi(a)e = \psi(a)$ for all a, it follows that AdV is extremal in $B(B(K), H)^+$.

Finally, if V is not invertible on eH choose an increasing net (e_γ) of projections converging strongly to e such that Ve_γ is invertible on $e_\gamma H$. Let $f_\gamma = \text{range}\, Ve_\gamma$. Then by Appendix A.1 $f_\gamma \to f$ strongly. If $\psi \le AdV$ is a map in $B(B(K), H)^+$ then $\psi \circ Adf_\gamma \le AdV \circ Adf_\gamma = Adf_\gamma V$, so by the previous paragraph, $\psi \circ Adf_\gamma = \lambda_\gamma Adf_\gamma V$ for a number $\lambda_\gamma \ge 0$. Let λ be a limit point for (λ_γ), then

$$\psi = \lim_\gamma \psi \circ Adf_\gamma = \lim_\gamma \lambda_\gamma Adf_\gamma V = \lambda AdV,$$

proving that AdV is extremal. □

Proposition 3.1.4 *Let A and B be C^*-algebras and $\phi : A \to B$ be an extreme point of the convex set of positive unital maps of A into B. Let $a \in A$ belong to the center of*

A and assume $\phi(a)$ belongs to the center of B. Then a belongs to the multiplicative domain for ϕ.

Proof We have

$$a = \frac{1}{2}\left(a + a^*\right) + \frac{1}{2i} i\left(a - a^*\right).$$

Since a^* satisfies the same assumptions as a, we may assume a is self-adjoint and $\|a\| < 1$. Then $\|\phi(a)\| < 1$, so $1 - a$ and $1 - \phi(a)$ are positive and invertible. Define $\psi : A \to B$ by

$$\psi(b) = \phi\big((1-a)b\big)\big(1 - \phi(a)\big)^{-1}.$$

Since $1 - a$ and $(1 - \phi(a))^{-1}$ belong to the centers of A and B respectively, there is $\lambda > 0$ such that $0 \le \psi \le \lambda\phi$. Furthermore

$$\psi(1) = \phi(1-a)\big(1 - \phi(a)\big)^{-1} = 1,$$

so by assumption on ϕ as an extreme point, $\psi = \phi$. Thus $(1 - \phi(a))\phi(b) = \phi((1 - a)b)$, hence $\phi(a)\phi(b) = \phi(ab)$ for all $b \in A$. □

Our next result is contained in Theorems 3.4.3 and 3.4.4 in Sect. 3.4, but will be needed in Sect. 3.3.

Proposition 3.1.5 *Let A and B be unital C^*-algebras and ϕ a Jordan homomorphism of A into B. Then ϕ is an extreme point of the unit ball of positive maps from $A \to B$.*

Proof We may assume $\phi(1) = 1$. Suppose $\phi = \frac{1}{2}(\psi + \eta)$ with ψ, η belonging to the unit ball of positive maps of A into B, and suppose there exists a self-adjoint operator $a \in A$ such that $\psi(a) \ne \eta(a)$. Then by the Kadison-Schwarz inequality, Theorem 1.3.1,

$$\begin{aligned}
\phi\big(a^2\big) = \phi(a)^2 = \frac{1}{4}\big(\psi(a) + \eta(a)\big)^2 &= \frac{1}{2}\big(\psi(a)^2 + \eta(a)^2\big) - \frac{1}{4}\big(\psi(a) - \eta(a)\big)^2 \\
&< \frac{1}{2}\big(\psi(a)^2 + \eta(a)^2\big) \le \frac{1}{2}\big(\psi\big(a^2\big) + \eta\big(a^2\big)\big) \\
&= \phi\big(a^2\big).
\end{aligned}$$

This is a contradiction so $\psi(a) = \eta(a)$, and hence $\psi = \eta = \phi$. □

Corollary 3.1.6 *Let A and B be unital abelian C^*-algebras. Let $\phi : A \to B$ be a unital positive map. Then ϕ is a homomorphism if and only if ϕ is an extreme point of the convex set of unital positive maps of A into B.*

Proof This is immediate from Propositions 3.1.4 and 3.1.5. □

We conclude this section with a characterization of automorphisms of $B(H)$. Recall the notation $[A\xi]$ for the projection onto the closed subspace generated by vectors $a\xi$, $a \in A$, $\xi \in H$. If $A = \mathbb{C}$ we use the notation $[\xi]$ instead of $[\mathbb{C}\xi]$ for the 1-dimensional projection on the subspace generated by the vector ξ.

Proposition 3.1.7 *Let ϕ be an automorphism of $B(H)$. Then there exists a unitary operator U such that $\phi = AdU$.*

Proof Since ϕ maps minimal projections onto minimal projections, for each $\xi \in H$ there is $\eta \in H$ such that $\phi([\xi]) = [\eta]$. Composing ϕ by an inner automorphism AdU, we may assume $\phi([\xi]) = [\xi]$ for a unit vector ξ. Each unit vector in $B(H)$ is cyclic, so $[B(H)\xi] = 1$. Define an operator $V \in B(H)$ by

$$Va\xi = \phi(a)\xi, \quad a \in B(H). \tag{3.1}$$

Then

$$Vab\xi = \phi(ab)\xi = \phi(a)\phi(b)\xi = \phi(a)Vb\xi.$$

Thus

$$Va = \phi(a)V, \quad \text{for all } a \in B(H). \tag{3.2}$$

Since $\phi([\xi]) = [\xi]$,

$$\begin{aligned} \|Va\xi\|^2 &= (Va\xi, Va\xi) = \big(\phi(a)\xi, \phi(a)\xi\big) = \big(\phi\big(a^*a\big)\xi, \xi\big) = \big(\phi\big([\xi]a^*a[\xi]\big)\xi, \xi\big) \\ &= \big(a^*a\xi, \xi\big)\big(\phi\big([\xi]\big)\xi, \xi\big) = \big(a^*a\xi, \xi\big) = \|a\xi\|^2. \end{aligned}$$

Thus V is an isometry, which by (3.1) is surjective. Thus V is unitary, so by (3.2) $\phi(a) = VaV^*$. Let $U = V^*$. Then $\phi = AdU$. □

3.2 Jordan Homomorphisms

An important class of maps is that of Jordan homomorphisms. It follows from a result of Jacobson and Rickart [29] together with some structure theory for von Neumann algebras and second dual techniques for C^*-algebras, that each Jordan homomorphism of a C^* algebra into another is the sum of a homomorphism and an anti-homomorphism much like that of the proof of Theorem 1.2.11, see [72] hence they are not extremal, even though they are extreme points of the unit ball. To simplify our approach we shall restrict our attention to the simpler case of Jordan automorphisms of $B(H)$, where we can use more elementary techniques together with the extremality properties we have shown for Jordan homomorphisms. We start with the $n \times n$ matrices M_n and in particular M_2. Let $(e_{ij})_{i,j=1}^n$ denote a complete set of matrix units for M_n.

Lemma 3.2.1 *Let ρ be a linear functional on M_n. Then*

(i) *The density matrix for ρ is $(\rho(e_{ij}))^t$.*
(ii) *If ρ is a state then ρ is pure if and only if*

$$\left|\rho(e_{ij})\right|^2 = \rho(e_{ii})\rho(e_{jj}) \quad \textit{for all } 1 \le i, j \le n.$$

Proof (i) follows since $Tr((\rho(e_{ij}))^t e_{kl}) = \rho(e_{kl})$ for all k, l.

(ii) ρ is a pure state if and only if its density matrix is a 1-dimensional projection, hence by (i) if and only if $(\rho(e_{ij}))$ is a 1-dimensional projection, so (ii) follows. □

Lemma 3.2.2 *Denote by C_2 the convex set of unital positive maps of M_2 into itself. Let ϕ be an extreme point of C_2. Then there exists a pure state ρ of M_2 such that $\rho \circ \phi$ is a pure state.*

Proof Let ρ be a linear functional on M_2. Then its density operator is positive if and only if ρ is positive, hence by Lemma 3.2.1 if and only if $\rho(e_{11}) \ge 0$, $\rho(e_{22}) \ge 0$ and $|\rho(e_{12})|^2 \le \rho(e_{11})\rho(e_{22})$. Suppose there is no pure state ρ such that $\rho \circ \phi$ is a pure state. Then for all pure states ρ, by Lemma 3.2.1(ii),

$$\rho\big(\phi(e_{11})\big)\rho\big(\phi(e_{22})\big) > \left|\rho\big(\phi(e_{12})\big)\right|^2.$$

Since the set of pure states on M_2 is compact there exists $\alpha > 0$ such that

$$\alpha \le \rho\big(\phi(e_{11})\big)\rho\big(\phi(e_{22})\big) - \left|\rho\big(\phi(e_{12})\big)\right|^2$$

for all pure states ρ. Since $|\rho(\phi(e_{12}))|^2 \le 1$

$$(1 \pm \alpha)\left|\rho\big(\phi(e_{12})\big)\right|^2 \le \rho\big(\phi(e_{11})\big)\rho\big(\phi(e_{22})\big).$$

Define two maps ψ^+ and ψ^- of M_2 into itself as follows; $\psi^\pm$ is linear, $\psi^\pm(e_{ii}) = \phi(e_{ii})$, $i = 1, 2$, and

$$\psi^\pm(e_{12}) = (1 \pm i\delta)\phi(e_{12}), \qquad \psi^\pm(e_{21}) = (1 \mp i\delta)\phi(e_{21}),$$

where $0 < \delta < \alpha^{1/2}$, so that $|1 \pm i\delta|^2 = 1 + \delta^2 < 1 + \alpha$. By the characterization of positive linear functionals in the beginning of the proof $\rho \circ \psi^\pm$ is a positive linear functional for all states ρ, hence $\psi^\pm$ is a positive map. Furthermore $\psi^\pm(1) = \phi(1) = 1$, so $\psi^\pm \in C_2$. Since $\phi = \frac{1}{2}(\psi^+ + \psi^-)$, and ϕ is extreme, $\psi^+ = \psi^-$, so that $\phi(e_{12}) = 0$. Then $\phi(e_{22}) = 1 - \phi(e_{11})$, so the range of ϕ is an abelian subalgebra of M_2. Composing ϕ by AdV for a suitable unitary operator V, we can by an application of Lemma 3.1.2 assume the range of ϕ is contained in the diagonal algebra D_2. If $\phi(M_2) \subset \mathbb{C}1$, then ϕ is a state, so pure since ϕ is extreme, a case which is ruled out. Thus $\phi(M_2) = D_2$. Therefore $\phi(e_{11}) = xe_{11} + ye_{22}$, $\phi(e_{22}) = (1 - x)e_{11} + (1 - y)e_{22}$.

There are two cases. Assume first one of the four entries is 0; say $y = 0$. Then $1 - y = 1$. Thus $Tr(e_{22}\phi(e_{11})) = 0$, $Tr(e_{22}\phi(e_{22})) = 1$, so the state $\omega(a) = Tr(e_{22}\phi(a))$

is pure, a case which is ruled out. Assume next $0 < x < 1$, and $0 < y < 1$. Then there exists $\alpha > 0$ such that $\phi(e_{ii}) \geq \alpha 1$, $i = 1, 2$. Thus $\phi(a) \geq \alpha Tr(a) 1$ for all $a \geq 0$. By extremality $\phi(a) = \frac{1}{2} Tr(a)$ for all a, which is impossible since ϕ is extremal. We have thus obtained a contradiction to the assumption that $\rho \circ \phi$ is never pure for ρ a pure state. The proof is complete. □

Lemma 3.2.3 *Let ϕ be extreme in C_2. Then there is a unitary operator U such that*

$$AdU \circ \phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \alpha b + \beta c \\ \overline{\alpha} c + \overline{\beta} b & \gamma a + \varepsilon b + \overline{\varepsilon} c + \delta d \end{pmatrix},$$

where $0 \leq \gamma \leq 1$, $\delta = 1 - \gamma$.

Proof Write ϕ in the form $\phi(a) = \sum \phi_{ij}(a) e_{ij}$, where ϕ_{ij} is a linear functional on M_2. By Lemma 3.2.2 we can compose ϕ by AdU for a suitable unitary U so we can assume ϕ_{11} is the pure state $\phi_{11}((a_{ij})) = a_{11}$. Thus $\phi_{11}(e_{22}) = 0$, so $\phi_{12}(e_{22}) = 0 = \phi_{12}(e_{11})$. Thus ϕ is of the form described in the lemma. □

Theorem 3.2.4 *Let ϕ be a normal Jordan automorphism of $B(H)$. Then ϕ is either an automorphism or an anti-automorphism, hence is of the form AdU or $AdU \circ t$ for a unitary operator U.*

Proof We first assume $\dim H = 2$, so $B(H) = M_2$. By Proposition 3.1.5 ϕ is extreme in C_2, hence we can assume ϕ is of the form described in Lemma 3.2.3, i.e.

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \alpha b + \beta c \\ \overline{\alpha} c + \overline{\beta} b & \gamma a + \varepsilon b + \overline{\varepsilon} c + \delta d \end{pmatrix},$$

with $\gamma + \delta = 1$. In particular

$$\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \overline{\beta} & \varepsilon \end{pmatrix},$$

hence

$$0 = \phi\left(\begin{pmatrix} & 1 \\ 0 & \end{pmatrix}^2 \right) = \phi \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}^2 = \begin{pmatrix} \alpha\overline{\beta} & \alpha\varepsilon \\ \varepsilon\overline{\beta} & \alpha\overline{\beta} + \varepsilon^2 \end{pmatrix}.$$

Thus, $\alpha\overline{\beta} = \alpha\varepsilon = \varepsilon\overline{\beta} = \alpha\overline{\beta} + \varepsilon^2 = 0$. There are three cases.

(i) $\alpha = 0$. Then $\varepsilon\overline{\beta} = \varepsilon^2 = 0$, so

$$\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \overline{\beta} & 0 \end{pmatrix}, \qquad \phi \begin{pmatrix} & 0 \\ 1 & \end{pmatrix} = \begin{pmatrix} & \beta \\ 0 & \end{pmatrix}.$$

(ii) $\beta = 0$. Then similarily

$$\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \qquad \phi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & 0 \\ \overline{\alpha} & \end{pmatrix}.$$

(iii) $\varepsilon = 0$. Then $\alpha\overline{\beta} = 0$, so one of the two cases (i) or (ii) occurs. In case (i) $\phi\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right) = \left(\begin{smallmatrix} & \beta \\ \overline{\beta} & \end{smallmatrix}\right)$, so the square is 1, hence $|\beta| = 1$. In case (ii) $|\alpha| = 1$. It follows that in case (i) ϕ is an anti-automorphism, and in case (ii) an automorphism.

Now consider the general case. Let p be a 1-dimensional projection. Then p is a minimal projection, so $\phi(p)$ is a minimal projection, hence is a 1-dimensional projection. Let e be a 2-dimensional projection. Then it is the sum of two 1-dimensional projections, so $\phi(e)$ is a 2-dimensional projection, and $\phi : eB(H)e \to \phi(e)B(H)\phi(e)$ is a Jordan isomorphism, hence by the first part of the proof applied to the composition of ϕ by an isomorphism of $\phi(e)B(H)\phi(e)$ onto $eB(H)e$, ϕ is either an isomorphism or an anti-isomorphism. Let now p and q be distinct 1-dimensional projections in $B(H)$ and $e = \text{span}(p, q)$. Then e is a 2-dimensional projection, and so is $\phi(e)$. By the above applied to e, if ϕ is an isomorphism, $\phi(pq) = \phi(p)\phi(q)$, and in the anti-isomorphism case $\phi(pq) = \phi(q)\phi(p)$.

Let X_p (resp. Y_p) be the set of 1-dimensional projections q in $B(H)$ such that $0 \neq pq \neq p$ and $\phi(pq) = \phi(p)\phi(q)$ (resp. $\phi(pq) = \phi(q)\phi(p)$). Then either X_p or Y_p is non-empty, say $X_p \neq \emptyset$. Let $q \in X_p$. Then q is an interior point of X_p. Indeed, let $\gamma = \|pq\|$,

$$c = \big\|\phi(pq) - \phi(q)\phi(p)\big\|.$$

Then $\gamma > 0$, $c > 0$. Let f be a 1-dimensional projection such that $f \neq p$ and

$$\|f - q\| \leq \delta = \min(c/4, \gamma/2).$$

Then $\|fp\| \geq \|qp\| - \|(f - q)p\| \geq \gamma/2$. Furthermore,

$$\begin{aligned} c &= \big\|\phi(pq) - \phi(q)\phi(p)\big\| \\ &\leq \big\|\phi(pq) - \phi(pf)\big\| + \big\|\phi(pf) - \phi(f)\phi(p)\big\| + \big\|\big(\phi(f) - \phi(q)\big)\phi(p)\big\| \\ &\leq \delta + \big\|\phi(pf) - \phi(f)\phi(p)\big\| + \delta. \end{aligned}$$

Hence

$$\big\|\phi(pf) - \phi(f)\phi(p)\big\| \geq c - c/2 = c/2.$$

Then $f \in X_p$, proving that q is an interior point of X_p.

Let $g \neq p$ be a 1-dimensional projection such that $gp \neq 0$. Let ψ, ξ, η be unit vectors such that $p = [\psi]$, $g = [\xi]$, $q = [\eta]$. Multiplying ξ and η by scalars we may assume $(\xi, \psi) > 0$, $(\eta, \psi) > 0$. Let

$$\xi(t) = (1 - t)\eta + t\xi, \quad t \in [0, 1],$$

be the line segment in H from η to ξ. Then $\|\xi(t)\| \leq 1$, and $(\xi(t), \psi) = (1 - t)(\eta, \psi) + t(\xi, \psi) > 0$, so $p[\xi(t)] \neq 0$. It follows from the previous paragraph applied to $q = [\xi(0)]$ and thus to each $[\xi(t)]$ that the set of t such that $[\xi(t)] \in X_p$ is open. Since the set is trivially closed, it follows that $g = [\xi(1)] \in X_p$.

We have thus shown that every 1-dimensional projection with $gp \neq 0$ belongs to X_p. Since each projection $g \perp p$ obviously satisfies the identity $\phi(pg) = \phi(p)\phi(g)$, this identity is therefore shown for all 1-dimensional projections g. Since p was arbitrary, it follows by linearity and normality of ϕ that ϕ is an isomorphism. Similarly, if $Y_p \neq \emptyset$, ϕ is an anti-isomorphism.

The last statement follows from Proposition 3.1.7, and the fact that the transpose t is an anti-automorphism of $B(H)$, and the composition of two anti-isomorphisms is an isomorphism. □

3.3 Maps which Preserve Vector States

In Lemma 3.2.2 we saw that for each extreme point ϕ of the convex set of unital maps of M_2 into itself, there is a pure state ϕ of M_2 such that $\rho \circ \phi$ is a pure state. A natural problem is to study maps in the extreme converse direction, i.e. maps $\phi : A \to B$, with A, B C^*-algebras, such that $\rho \circ \phi$ is a pure state for all pure states ρ of B. It was shown in [71] that for all such maps $\pi \circ \phi$ is either a pure state, or an anti-homomorphism or homomorphism of A for all irreducible representations of B. We shall in the present section restrict ourselves to maps of $B(K)$ into $B(H)$ for which $\omega_\xi \circ \phi$ is a vector state of $B(K)$ for all vector states ω_ξ of $B(H)$ defined by $\omega_x(a) = (a\xi, \xi)$. We then apply this to maps which carry positive rank 1 operators to positive rank 1 operators.

Lemma 3.3.1 *Let K and H be Hilbert spaces and $\phi \in B(B(K), H)$ a unital positive map such that for each vector state ω_η of $B(H)$ there is a vector state ω_ξ of $B(K)$ such that $\omega_\xi \circ \phi = \omega_\eta$. For such a pair ξ, η, either $\phi([\eta]) = [\xi]$ or $\phi([\eta]) = 1$. In the latter case $\phi(a) = \omega_\eta(a)1$ for all $a \in B(H)$. Furthermore ϕ is weakly continuous.*

Proof We first show ϕ is weakly continuous. Let $(a_\alpha)_{\alpha \in J}$ be a net in $B(K)$ such that $a_\alpha \to a$ is weakly. Let ξ be a unit vector in H and η a unit vector in K such that $\omega_\xi \circ \phi = \omega_\eta$. Then $\omega_\xi(\phi(a_\alpha)) = \omega_\eta(a_\alpha) \to \omega_\eta(a) = \omega_\xi(\phi(a))$.

Since each weakly continuous linear functional on $B(H)$ is a linear combination of vector states, $(\phi(a_\alpha))_{\alpha \in J}$ converges weakly to $\phi(a)$, so ϕ is weakly continuous.

Let ξ and η be as above. Then $0 \leq \phi([\eta]) \leq 1$ and $\omega_\xi(\phi([\eta])) = 1$. Thus $\phi([\eta])[\xi] = [\xi] \leq \phi([\eta])$. To prove the lemma we first assume $n = \dim H < \infty$, and use induction on n. If $n = 1$ the lemma is trivial.

Suppose $n = 2$ and $\phi([\eta]) \neq [\xi]$. We may then assume $B(H) = M_2$ and

$$\phi([\eta]) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \tag{3.3}$$

with $0 < p \leq 1$. Let μ be a unit vector in K orthogonal to $[\eta]$. Let $f = [\eta] + [\mu]$. Then $fB(K)f \cong M_2$. Let $(e_{ij}), i, j = 1, 2$, denote the matrix units in M_2 such that $[\eta] = e_{11}, [\mu] = e_{22}$. If ω_ρ is a vector state of M_2 then $\omega_\rho \circ \phi = \omega_\tau$ for a unit vector

$\tau \in K$, so its restriction to $fB(K)f$ is $\omega_{f\tau}$, which is a scalar multiple of a vector state, so by Lemma 3.2.1 satisfies the equality

$$\omega_\rho \circ \phi(e_{11})\omega_\rho \circ \phi(e_{22}) = \left|\omega_\rho \circ \phi(e_{12})\right|^2. \tag{3.4}$$

In particular this holds for $\rho = \eta$. Since also $0 \le \phi(e_{11} + e_{22}) \le 1$, we have

$$\phi(e_{22}) = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}, \qquad \phi(e_{12}) = \begin{pmatrix} 0 & r \\ s & t \end{pmatrix}.$$

Since $\rho = (\rho_1, \rho_2)$ is a vector in $\mathbb{C}^2$ the following equations hold, cf. (3.3):

$$\omega_\rho \circ \phi(e_{11}) = |\rho_1|^2 + p|\rho_2|^2,$$
$$\omega_\rho \circ \phi(e_{22}) = q|\rho_2|^2,$$
$$\omega_\rho \circ \phi(e_{12}) = t|\rho_2|^2 + r\overline{\rho_1}\rho_2 + s\rho_1\overline{\rho_2}.$$

Thus, using (3.4)

$$\begin{aligned} &|t|\rho_2|^2 + r\overline{\rho_1}\rho_2 + s\rho_1\overline{\rho_2}|^2 \\ &\quad = |t|^2|\rho_2|^4 + \left(|r|^2 + |s|^2\right)|\rho_1|^2|\rho_2|^2 + 2\Re\left((r\bar{t} + \bar{s}t)|\rho_2|^2\overline{\rho_1}\rho_2\right) + 2\Re\left(r\bar{s}(\overline{\rho_1}\rho_2)^2\right) \\ &\quad = q|\rho_2|^2\left(|\rho_1|^2 + p|\rho_2|^2\right). \end{aligned} \tag{3.5}$$

Now, if f_1, f_2, f_3 are complex valued functions of the two complex variables ρ_1 and ρ_2 such that

$$f_1\left(|\rho_1|, |\rho_2|\right) = \Re\left(f_2\left(|\rho_1|, |\rho_2|\right)\overline{\rho_1}\rho_2 + f_3\left(|\rho_1|, |\rho_2|\right)(\overline{\rho_1}\rho_2)^2\right),$$

then it is easily verified that $f_1 = f_2 = f_3 = 0$. With

$$f_1\left(|p_1|, |\rho_2|\right) = \left(|\rho_1|^2 + p|\rho_2|^2\right)q|\rho_2|^2 - |t|^2|\rho_2|^4 - \left(|r|^2 + |s|^2\right)|\rho_1|^2|\rho_2|^2$$

and f_2 and f_3 the two real parts in (3.5), we get

$$r\bar{t} + \bar{s}t = 0 = r\bar{s}, \qquad |t|^2 = pq, \qquad |r|^2 + |s|^2 = q.$$

Thus $q = 0$, and $\phi([\mu]) = \phi(e_{22}) = 0$. Since this holds for every unit vector $[\mu]$ orthogonal to η, and since ϕ is weakly continuous, $\phi([\eta]) = 1$, as asserted.

Suppose $n \ge 3$, and assume the lemma is proved whenever $\dim H \le n - 1$. Let e be a projection in $B(H)$ containing ξ, and $\dim e = k < n$. Then $Ade \circ \phi$ has the same properties as ϕ with respect to composition with vector states,

$$Ade \circ \phi : B(K) \to eB(H)e,$$

and $\omega_\xi \circ \phi = \omega_\eta$. By induction assumption $e\phi([\eta])e$ equals $[\xi]$ or e. If $e\phi([\eta])e = [\xi]$ then

$$0 = e\left(\phi\left([\eta]\right) - [\xi]\right)e = \left(\left(\phi\left([\eta]\right) - [\xi]\right)^{1/2}e\right)^*\left(\left(\phi\left([\eta]\right) - [\xi]\right)^{1/2}e\right),$$

so $(\phi([\eta]) - [\xi])e = 0$, hence $\phi([\eta])e = [\xi] = e\phi([\eta])$, taking adjoints. Similarily, if $e\phi([\eta])e = e$, then $e(1 - \phi([\eta]))e = 0$, and $e\phi([\eta]) = \phi([\eta])e = e$. Thus $\phi([\eta])$ commutes with every projection containing ξ. Since $n \geq 3$ this is possible only if $\phi([\eta])$ equals $[\xi]$ or 1.

If H is not finite dimensional it follows from the above that $\phi([\eta])$ commutes with every finite dimensional projection containing $[\xi]$. Hence $\phi([\eta])$ equals $[\xi]$ or 1. □

Theorem 3.3.2 *Let K and H be Hilbert spaces and $\phi \in B(B(K), H)$ be a positive unital map such that for each unit vector $\xi \in H$ there is a unit vector $\eta \in K$ such that $\omega_\xi \circ \phi = \omega_\eta$. Then either $\phi(a) = \omega_\rho(a)1$ for a vector $\rho \in K$, or there is a linear isometry $V : H \to K$ such that $\phi = AdV$ or $\phi = AdV \circ t$, t being the transpose on $B(K)$.*

Proof By Lemma 3.3.1 ϕ is weakly continuous. Let Tr denote the trace on either $B(K)$ or $B(H)$. Thus if $\omega_\xi \circ \phi = \omega_\eta$ we have for $a \in B(K)$

$$Tr\big(\phi^*\big([\xi]\big)a\big) = Tr\big([\xi]\phi(a)\big) = \omega_\xi \circ \phi(a)$$
$$= \omega_\eta(a) = Tr\big([\eta]a\big).$$

Thus $\phi^*([\xi]) = [\eta]$, and $\phi^* : B(H) \to B(K)$ is faithful and maps 1-dimensional projections to 1-dimensional projections. Let ξ and μ be mutually orthogonal unit vectors in H. Let η and ρ be unit vectors in K such that $\omega_\xi \circ \phi = \omega_\eta$, and $\omega_\mu \circ \phi = \omega_\rho$. By Lemma 3.3.1 either $\phi([\eta]) = 1$, in which case support $\phi = [\eta]$, so that $\phi(a) = \phi([\eta]a[\eta]) = \omega_\eta(a)1$, so ϕ is a vector state, or $\phi([\eta]) = [\xi]$, $\phi([\rho]) = [\mu]$. In the latter case

$$0 \leq \omega_\eta\big([\rho]\big) = \omega_\xi\big(\phi\big([\rho]\big)\big) = \omega_\xi\big([\mu]\big) = 0,$$

so η and ρ are orthogonal. Since $\phi^*([\xi]) = [\eta]$ and $\phi^*([\mu]) = [\rho]$, it follows that ϕ^* maps mutually orthogonal 1-dimensional projections onto mutually orthogonal projections. Thus ϕ^* is a Jordan isomorphism on finite rank operators in $B(H)$ into those of $B(K)$. Thus for each finite dimensional projection $e \in B(H)$, ϕ^* is a Jordan isomorphism of $eB(H)e$ into $\phi^*(e)B(K)\phi^*(e)$, and onto, since they have the same dimensions. It follows from Theorem 3.2.4 that ϕ^* is either an isomorphism or anti-isomorphism of $eB(H)e$ onto $\phi^*(e)B(K)\phi^*(e)$, and implemented by a unitary operator $U : eK \to \phi^*(e)H$. By Proposition 1.4.2 the adjoint map of AdU is AdU^*, and the adjoint of the transpose map t is t. Thus $\phi : \phi^*(e)B(K)\phi^*(e) \to eB(H)e$ is either an isomorphism or an anti-isomorphism. Let $f = \vee_e \phi^*(e)$, where the span is over all finite dimensional projections in $B(H)$. Since ϕ is weakly continuous it is either an isomorphism or anti-isomorphism of $fB(K)f$ onto $B(H)$. □

Remark 3.3.3 Theorem 3.3.2 has a generalization to C^*-algebras. Recall that if ρ is a state of a C^*-algebra B then there are a Hilbert space H_ρ, a $*$-representation π_ρ of B on H_ρ and a vector $\xi_\rho \in H_\rho$ such that $\rho(a) = \omega_{\xi_\rho} \circ \pi_\rho(a)$ for $a \in B$.

Furthermore, ρ is a pure state if and only if π_ρ is irreducible. Then the generalization of Theorem 3.3.2 states, see [71]: Let A and B be unital C^*-algebras and $\phi : A \to B$ a positive unital map. Then $\rho \circ \phi$ is a pure state of A and for all pure states ρ of B if and only if for each irreducible representation ψ of B on a Hilbert space H, $\psi \circ \phi$ is either a pure state of A or $\psi \circ \phi = V^*\pi V$, where V is a linear isometry of H into a Hilbert space K, and π is an irreducible $*$-homomorphism or $*$-anti-homomorphism of A into $B(K)$.

Many problems on maps of operator algebras are what are called preserver problems. Then one studies maps which preserve selected properties. For a treatment on this topic we refer the reader to the book [51] of Molnár. Our next result, which is close to Theorem 3.3.2, is of this type.

Theorem 3.3.4 *Let K and H be finite dimensional Hilbert spaces and $\phi \in B(B(K), H)$ a positive map such that rank $\phi(p) \leq 1$ for all 1-dimensional projections $p \in B(K)$. Then one of the following three conditions holds*:

(i) *There exist a state ω on $B(K)$ and a positive rank 1 operator $q \in B(H)$ such that $\phi(a) = q\omega(a)$ for $a \in B(K)$.*
(ii) *$\phi = AdU$ with $U : H \to K$ a bounded linear operator.*
(iii) *$\phi(a) = (AdU(a))^t$ for $a \in B(K)$, t is the transpose on $B(H)$.*

Proof Let $e =$ support of ϕ. Then $\phi : eB(K)e \to B(H)$ is faithful, so we may restrict attention to $eB(K)e$ and assume ϕ is faithful. By Proposition 1.4.3(iv) $\phi^*(1)$ is invertible. Let $h = \phi^*(1)^{-1/2}$. Then $h\phi^*(1)h = 1$, so the map $\psi(a) = h\phi^*(a)h$ is unital and positive. Then for $a \in B(K)$, $b \in B(H)$ we have

$$Tr\big(a\psi(b)\big) = Tr\big(hah\phi^*(b)\big) = Tr\big(\phi(hah)b\big).$$

If p is a 1-dimensional projection in $B(K)$ then $hph = \lambda q$ for a 1-dimensional projection q, so by the assumption on ϕ, $\phi(hph) = \lambda\phi(q)$ is positive of rank 1. It follows that the functional

$$\omega'(a) = Tr\big(p\psi(a)\big) = Tr\big(ph\phi^*(a)h\big) = Tr\big(\phi(hph)a\big) = \lambda Tr\big(\phi(q)a\big),$$

for $a \in B(H)$, is a scalar multiple of a pure state on $B(H)$. Furthermore, $\omega'(1) = Tr(p\psi(1)) = Tr(p) = 1$, so ω' is a pure state. Thus $\psi : B(H) \to B(K)$ preserves vector states. By Theorem 3.3.2 and 3.2.4 ψ is either

(i) a vector state, i.e. $\psi(a) = \omega_\xi(a)1$.
(ii) $\psi(a) = V^*aV$, $V : K \to H$ is a linear isometry of K into H.
(iii) $\psi(a) = V^*a^tV$, with V as in (ii).

If ρ is a state on $B(K)$ with density operator d then for $a \in B(H)$

$$Tr\big(a\rho^*(b)\big) = Tr\big(\rho(a)b\big) = Tr\big(Tr(da)b\big) = Tr\big(daTr(b)\big),$$

so that $\rho^*(b) = dTr(b)$. By construction, $\phi^* = h^{-1}\psi h^{-1}$. Thus we have in case (i), $\psi(a) = Tr(qa)$ for a 1-dimensional projection q, so that

$$\begin{aligned} Tr\big(\phi(a)b\big) &= Tr\big(ah^{-1}\psi(b)h^{-1}\big) \\ &= Tr\big(ah^{-1}Tr(qb)h^{-1}\big) \\ &= Tr\big(ah^{-2}\big)Tr(qb) \\ &= Tr\big(qTr\big(ah^{-2}\big)b\big), \end{aligned}$$

so that $\phi(a) = qTr(ah^{-2})$ is as in (i) in the theorem.

In case (ii)

$$Tr\big(\phi(a)b\big) = Tr\big(ah^{-1}\psi(b)h^{-1}\big) = Tr\big(h^{-1}ah^{-1}V^*bV\big) = Tr\big(\big(Vh^{-1}\big)a\big(Vh^{-1}\big)^*b\big),$$

so that $\phi(a) = AdU$ with $U^* = Vh^{-1} : H \to K$.

In case (iii) we similarily have

$$Tr\big(\phi(a)b\big) = Tr\big(h^{-1}ah^{-1}V^*b^tV\big) = Tr\big(\big(AdU(a)\big)^tb\big),$$

so that $\phi(a) = t \circ AdU$. □

It turns out that 2-positive and 2-copositive extremal maps in $B(B(K), H)^+$ are of the form described in Theorem 3.3.4. We conclude the section with a proof of this. Assume for simplicity that K and H are finite dimensional. Recall that if ξ is a vector in an n-dimensional Hilbert space, $\xi = (\xi_1, \ldots, \xi_n)$ then ξ can be identified with the $1 \times n$ column matrix

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}.$$

Then $\xi^* = [\overline{\xi_1}, \ldots, \overline{\xi_n}]$. If η is another vector we get

$$\xi^*\eta = \langle \eta, \xi \rangle,$$

and if they are unit vectors, $\xi\eta^*$ is the partial isometry from η to ξ. In particular $\xi\xi^*$ is the projection $[\xi]$.

Lemma 3.3.5 *Let $\phi \in B(B(K), H)$ be of the form $\phi(x) = AxA^*$ with $A : K \to H$ non-zero. Choose unit vectors $\xi \in K, \omega \in H$ and $\lambda > 0$ such that*

$$\phi\big(\xi\xi^*\big)\omega = \lambda\omega.$$

Define $B : K \to H$ by

$$B\eta = \lambda^{-1/2}\phi\big(\eta\xi^*\big)\omega.$$

Then $B = e^{it}A$ for some $t \in [0, 2\pi)$.

Proof By assumption

$$\lambda\omega = A\xi\xi^* A^*\omega = A\xi(A\xi)^*\omega = A\xi\langle\omega, A\xi\rangle.$$

Thus $A\xi = z\omega$ for some $z \in \mathbb{C}$. Since

$$|z|^2\omega = z\omega\langle\omega, z\omega\rangle = A\xi\{\omega, A\xi\rangle = \lambda\omega,$$

$|z| = \lambda^{1/2}$. Let $\eta \in K$. Then

$$B\eta = \lambda^{-1/2}A\eta\xi^* A^*\omega = \lambda^{-1/2}A\eta\langle\omega, A\xi\rangle = \lambda^{-1/2}\overline{z}A\eta = e^{it}A\eta,$$

where t satisfies $\lambda^{-1/2}\overline{z} = e^{it}$. Thus $B = e^{it}A$. □

Proposition 3.3.6 *Let $\phi \in B(B(K), H)^+$. Let λ, ξ, ω, B be defined by ϕ as in Lemma 3.3.5. Let $\psi \in B(B(K), H)^+$ be the map $\psi(x) = BxB^*$. Then $\psi \le \phi$ if and only if for all $\eta \in K$, $\rho \in H$ we have the inequality*

$$\left|\langle\phi(\eta\xi^*)\omega, \rho\rangle\right|^2 \le \langle\phi(\xi\xi^*)\omega, \omega\rangle\langle\phi(\eta\eta^*)\rho, \rho\rangle.$$

Proof Clearly $\psi \le \phi$ if and only if for all $\eta \in K$, $\rho \in H$

$$\langle\psi(\eta\eta^*)\rho, \rho\rangle \le \langle\phi(\eta\eta^*\rho, \rho\rangle.$$

The left hand side of the above inequality is equal to

$$\begin{aligned}\langle B\eta\eta^* B^*\rho, \rho\rangle &= \langle B\eta(B\eta)^*\rho, \rho\rangle\\ &= \langle B\eta\langle\rho, B\eta\rangle, \rho\rangle\\ &= \left|\langle B\eta, \rho\rangle\right|^2\\ &= \lambda^{-1}\left|\langle\phi(\eta\xi^*)\omega, \rho\rangle\right|^2,\end{aligned}$$

by definition of B. If the inequality in the proposition is satisfied it follows that

$$\begin{aligned}\langle\psi(\eta\eta^*)\rho, \rho\rangle &\le \lambda^{-1}\langle\phi(\xi\xi^*)\omega, \omega\rangle\langle\phi(\eta\eta^*)\rho, \rho\rangle\\ &= \langle\phi(\eta\eta^*)\rho, \rho\rangle,\end{aligned}$$

by choice of λ. Thus $\psi \le \phi$.

Conversely, if $\psi \le \phi$, then by the above computations

$$\lambda^{-1}\left|\langle\phi(\eta\xi^*)\omega, \rho\rangle\right|^2 \le (\phi(\eta\eta^*)\rho, \rho\rangle,$$

so the inequality in the proposition follows from the definition of λ. □

Theorem 3.3.7 *Let $\phi \in B(B(K), H)^+$ be an extremal map. Assume ϕ is 2-positive (resp. 2-copositive). Then ϕ is a completely positive of the form $\phi = AdV$ with $V : H \to K$ (resp. ϕ is copositive of the form $AdV \circ t$).*

Proof Let ξ, ω, λ be as in Lemma 3.3.5. Let $\eta \in K$. Consider the positive matrix

$$X = \begin{pmatrix} \xi\xi^* & \xi\eta^* \\ \eta\xi^* & \eta\eta^* \end{pmatrix} = \begin{pmatrix} 0 & \xi \\ \eta & 0 \end{pmatrix} \begin{pmatrix} 0 & \xi \\ \eta & 0 \end{pmatrix}^* \in M_2\big(B(K)\big).$$

Since ϕ is 2-positive the matrix

$$\phi_2(X) = \begin{pmatrix} \phi(\xi\xi^*) & \phi(\xi\eta^*) \\ \phi(\eta\xi^*) & \phi(\eta\eta^*) \end{pmatrix} \in M_2\big(B(H)\big)^+.$$

Thus for each $\rho \in H$ we have

$$\begin{pmatrix} \langle\phi(\xi\xi^*)\omega, \omega\rangle & \langle\phi(\xi\eta^*)\rho, \omega\rangle \\ \langle\phi(\eta\xi^*)\omega, \rho\rangle & \langle\phi(\eta\eta^*)\rho, \rho\rangle \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & \rho \end{pmatrix}^* \phi_2(X) \begin{pmatrix} \omega & 0 \\ 0 & \rho \end{pmatrix} \geq 0.$$

Thus the inequality in Proposition 3.3.6 is satisfied, so by the theorem $\psi \leq \phi$. Since ϕ is extremal $\psi = AdB^* = \mu\phi$ for some $\mu > 0$. Hence $\phi = AdV$ with $V = \mu^{-1/2}B^*$.

If ϕ is 2-copositive then $\phi \circ t$ is 2-positive and still extremal by Lemma 3.1.2, so $\phi \circ t = AdV$, hence $\phi = AdV \circ t$. □

3.4 Nonextendible Maps

If A is a C^*-algebra and $\phi \in B(A, H)$ is a unital completely positive map the Stinespring Theorem, 1.2.7, states that there are a Hilbert space K, an isometry $V : H \to K$, and a representation $\pi : A \to B(K)$ such that $\phi = V^*\pi V$.

Since $V^*V = 1$, VV^* is a projection, which we can look at as the projection $P : K \to H$, where we consider H as a subspace of K. Then ϕ has the form $P\pi P$. We can thus consider π as an extension of ϕ to a map $\pi : A \to B(K)$. We therefore make the following definition.

Definition 3.4.1 Let A be a unital C^*-algebra, and $H \subset K$ two Hilbert spaces. Let P be the orthogonal projection of K onto H. Let $\phi \in B(A, H)$ and $\Phi \in B(A, K)$ be positive unital maps. We say

(i) Φ is an *extension of* ϕ and write $\Phi \supset \phi$ if $\phi(a) = P\Phi(a)P$ for all $a \in A$.
(ii) $\Phi \supset \phi$ is *trivial* if H is invariant under the action of $\Phi(a)$ for all $a \in A$, i.e. $\Phi(a)\xi = \phi(a)\xi$ for $a \in A$ and $\xi \in H$.
(iii) ϕ is called *nonextendible* if all extensions $\Phi \supset \phi$ are trivial.

Note that if $\Phi \supset \phi$ is an extension as above, and $\sum_1^n a_i \otimes \xi_i \in A \otimes H$, consider the element

$$\sum_k \phi(a_i)\xi_i = P\Big(\sum \Phi(a_i)\xi_i\Big) \in H.$$

Then

$$\left\|\sum \phi(a_i)\xi_i\right\| \le \left\|\sum \Phi(a_i)\xi_i\right\|. \tag{3.6}$$

If the extension $\Phi \supset \phi$ is trivial then $\sum \Phi(a_i)\xi_i \in H$, so we have equality in (3.6). Conversely, if for all $\sum_i a_i \otimes \xi_i \in A \otimes H$ we have equality in (3.6), then $\sum_i \phi(a_i)\xi_i = \sum_i \Phi(a_i)\xi_i$, so the extension $\Phi \supset \phi$ is trivial. We have shown:

Lemma 3.4.2 *Let $\phi \in B(A, H)$ be a positive unital map. Then ϕ is nonextendible if and only if*

$$\left\|\sum \phi(a_i)\xi_i\right\| = \left\|\sum \Phi(a_i)\xi_i\right\|$$

for all extensions $\Phi \supset \phi$ and $a_i \in A$, $\xi_i \in H$.

We say a positive map $\phi : A \to B(H)$ is *irreducible* if the commutant of $\phi(A)$ is the scalar operators, i.e. the only operators which commute with $\phi(a)$ for all $a \in A$, are the scalar multiples of the identity operator 1.

Theorem 3.4.3 *Let A be a C^*-algebra and $\phi \in B(A, H)$ be a unital positive map. Then*

(i) *If ϕ is nonextendible then ϕ is an extreme point of the convex set of positive unital maps of A into $B(H)$.*
(ii) *If ϕ is both nonextendible and irreducible then ϕ is an extremal map.*

Proof Assume $\phi \in B(A, H)^+$ is nonextendible and $\phi = \lambda\phi_1 + \mu\phi_2$ with $\phi_i : A \to B(H)$ positive linear maps, $\lambda, \mu > 0$ and $\lambda + \mu = 1$. The operators $\phi_i(1)$ are invertible on the subspace $\phi_i(1)H$. Let H_i denote the closure of $\phi_i(1)H$.

Let

$$\psi_i(a) = \phi_i(1)^{-1/2}\phi_i(a)\phi_i(1)^{-1/2}, \quad a \in A.$$

Then $\psi_i(a)$ defines an operator on H_i, which we still denote by $\psi_i(a)$. Let

$$K = H_1 \oplus H_2, \qquad \Phi = \psi_1 \oplus \psi_2.$$

Then

$$\Phi : A \to B(K)$$

is unital and positive. Let $V : H \to K$ be the linear operator

$$V(\xi) = \big(\lambda\phi_1(1)\big)^{1/2}\xi \oplus \big(\mu\phi_2(1)\big)^{1/2}\xi.$$

Then a straightforward computation yields

$$\big(\phi(a)\xi, \eta\big) = \big(\Phi(a)V\xi, V\eta\big)$$

for $\xi, \eta \in H$ and $a \in A$. In particular, if we put $a = 1$, we see that V is an isometric imbedding of H into K. Thus $\Phi \supset \phi$ is an extension of ϕ. By assumption ϕ is nonextendible. Thus Φ is a trivial extension. In our definition we considered H as a subspace of K. In the general case one must consider the case when H is imbedded in K as it is here, with $V : H \to K$. Thus we have

$$\Phi(a)V\xi = V\phi(a)\xi \quad \text{for } a \in A, \xi \in H.$$

By the definitions of V and $\Phi = \psi_1 \oplus \psi_2$ we get

$$\Phi(a)V\xi = \lambda^{1/2}\phi_1(1)^{-1/2}\phi_1(a)\xi \oplus \mu^{1/2}\phi_2(1)^{-1/2}\phi_2(a)\xi.$$

This is equal to

$$V\phi(a)\xi = \lambda^{1/2}\phi_1(1)^{1/2}\phi(a)\xi \oplus \mu^{1/2}\phi_2(a)^{1/2}\phi(a)\xi,$$

so that

$$\phi_i(1)\phi(a)\xi = \phi_i(a)\xi, \quad \text{for all } \xi \in H,$$

hence $\phi_i = \phi_i(1)\phi$.

In case (i) in the theorem $\phi_i(1) = 1$, so $\phi_i = \phi$, and the conclusion in (i) follows.

In case (ii) $\phi_i(a) = \phi_i(1)\phi(a)$ for all a. Taking adjoints for a self-adjoint we see that $\phi_i(1)$ commutes with the self-adjoint operator $\phi(a)$, and therefore $\phi_i(1) \in \phi(A)'$, which we assumed is the scalar operators. Thus ϕ_i is a scalar multiple of ϕ, and thus ϕ is extremal. □

It is a quite special property to be a nonextendible map. Our next result is an example of a nonextendible map. It is an extension of Proposition 3.1.5, where it was shown that Jordan homomorphisms were extremal in the set of positive unital maps.

Theorem 3.4.4 *Let A be a C^*-algebra and $\phi \in B(A, H)$ a unital Jordan homomorphism. Then ϕ is nonextendible.*

Proof Since $\phi(1)$ is always a projection the assumption that ϕ is unital is just made for convenience. Let $\Phi \supset \phi$ be an extension, so $\phi(a) = P\Phi(a)P$, where P is the projection of K onto H, $\Phi : A \to B(K)$ positive and unital. If $a \in A$ is self-adjoint then the Kadison-Schwarz inequality, Theorem 1.3.1, applied to Φ, implies with 1 the identity in $B(K)$,

$$\begin{aligned} 0 &\le P\Phi(a)(1-P)\Phi(a)P \\ &= P\Phi(a)^2P - \phi(a)^2 \\ &= P\Phi(a)^2P - \phi\big(a^2\big) \\ &= P\big(\Phi(a)^2 - \Phi\big(a^2\big)\big)P \le 0. \end{aligned}$$

It follows that $(1-P)\Phi(a)P = 0$, hence $\Phi(a)\xi \in H$ for all $\xi \in H$. Thus Φ is a trivial extension of ϕ. □

In the converse direction we see that if ϕ is a nonextendible unital completely positive map, then the Stinespring Theorem, 1.2.7, shows that ϕ has an extension which is a representation, hence by nonextendibility ϕ, is itself a homomorphism. It is interesting that this conclusion holds in much more generality. Recall from Definition 1.2.1 that a map $\phi \in B(A,H)$ is 2-positive if $\phi \otimes \iota$ is positive, where ι is the identity map of M_2 onto itself. This means that

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in M_2(A)^+ \Rightarrow \begin{pmatrix} \phi(a) & \phi(b) \\ \phi(b)^* & \phi(c) \end{pmatrix} \in M_2\big(B(H)\big)^+.$$

Theorem 3.4.5 *Let A be a C^*-algebra and $\phi \in B(A,H)$ a unital 2-positive nonextendible map. Then ϕ is a homomorphism.*

Proof Let $a, b \in A$ with $a \geq 0$. Then

$$\begin{pmatrix} a & ab^* \\ ba & bab^* \end{pmatrix} = \begin{pmatrix} a^{1/2} & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & a^{1/2} \\ a^{1/2} & a \end{pmatrix} \begin{pmatrix} a^{1/2} & 0 \\ 0 & b^* \end{pmatrix} \geq 0.$$

Let b be fixed, and, then since ϕ is 2-positive,

$$\psi(a) = \begin{pmatrix} \phi(a) & \phi(ab^*) \\ \phi(ba) & \phi(bab^*) \end{pmatrix}$$

defines a positive map of A into $B(H \oplus H)$. Then $\psi(1)$ is invertible on $\psi(1)H \oplus H$. Let K denote the closure of $\psi(1)H \oplus H$. Define a map $\Phi : A \to B(H \oplus H)$ by

$$\Phi(a) = \psi(1)^{-1/2}\psi(a)\psi(1)^{-1/2}.$$

Then Φ is a positive unital map of A into $B(K)$. Let $V : H \to K$ be the linear operator defined by

$$V\xi = \psi(1)^{1/2}(\xi \oplus 0).$$

Thus for $\xi, \eta \in H$ we immediately get

$$\big(\phi(a)\xi, \eta\big) = \big(\Phi(a)V\xi, V\eta\big).$$

In particular, if $a = 1$, so $\phi(a) = 1$, we see that $V : H \to K$ is an isometric imbedding, and so

$$\phi(a) = V^*\Phi(a)V.$$

Thus Φ is an extension of ϕ, and since ϕ is nonextendible, $\Phi \supset \phi$ is a trivial extension. Therefore

$$\Phi(a)V\xi = V\phi(a)\xi.$$

Using the defining formulas for Φ and V we then get

$$\psi(1)^{-1/2}\begin{pmatrix}\phi(a) & \phi(ab^*)\\ \phi(ba) & \phi(bab^*)\end{pmatrix}\begin{pmatrix}\xi\\ 0\end{pmatrix} = \Phi(a)V\xi = V\phi(a)\xi = \psi(1)^{1/2}\begin{pmatrix}\phi(a)\xi\\ 0\end{pmatrix}.$$

If we multiply on the left by $\psi(1)^{1/2}$, we get

$$\begin{pmatrix}\phi(a) & \phi(ab^*)\\ \phi(ba) & \phi(bab^*)\end{pmatrix}\begin{pmatrix}\xi\\ 0\end{pmatrix} = \begin{pmatrix}1 & \phi(b)^*\\ \phi(b) & \phi(bb^*)\end{pmatrix}\begin{pmatrix}\phi(a)\xi\\ 0\end{pmatrix},$$

hence $\phi(ba)\xi = \phi(b)\phi(a)\xi$ for all $\xi \in H$, proving that ϕ is a homomorphism. □

3.5 A Radon-Nikodym Theorem

One version of the classical Radon-Nikodym theorem for measures states that if μ and η are finite measures on a measure space, and $\eta \leq \mu$, then there exists a measurable function $0 \leq f \leq 1$ such that

$$\int g\, d\eta = \int fg\, d\mu$$

for all integrable functions g. We shall in the present section prove an analogous result for completely positive maps and then apply this to characterize maps which are extremal among the completely positive ones. We first show a sharpening of the Stinespring Theorem 1.2.7.

Lemma 3.5.1 *Let A be a C^*-algebra and $\phi : A \to B(H)$ a completely positive map. Then there exist a Hilbert space K, a representation π of A on K, a bounded operator $V : H \to K$ with the property that the closed subspace*

$$\left[\pi(A)VH\right] = \left\{\pi(a)V\xi\ :\ a \in A, \xi \in H\right\}^-$$

equals K, and such that $\phi = V^*\pi V$.

Proof Let $W^*\pi_0 W$ be a Stinespring decomposition of ϕ as in Theorem 1.2.7 with π_0 a representation of A on a Hilbert space K_0, and $W : H \to K_0$ a bounded operator. Let e be the projection onto $[\pi_0(A)WH]$. Then e belongs to the commutant $\pi_0(A)'$ of $\pi_0(A)$, because if $a, b \in A$ then

$$\pi_0(b)\bigl(\pi_0(a)W\xi\bigr) = \pi_0(ba)W\xi \in \left[\pi_0(A)WH\right].$$

Let $K = eK_0$, $\pi = e\pi_0$ and $V = eW$, then

$$V^*\pi(a)V = V^*e\pi_0(a)eV = W\pi_0(a)W = \phi(a),$$

and

$$\left[\pi(A)VH\right] = \left[e\pi_0(A)WH\right] = eK_0 = K.$$ □

Lemma 3.5.2 *Let ϕ_1 and ϕ_2 be completely positive maps of A into $B(H)$ such that $\phi_2 - \phi_1$ is completely positive. Let $\phi_i(a) = V_i^*\pi_i(a)V_i$ be the Stinespring decompositions such that $[\pi_i(A)V_iH] = K_i, i = 1, 2$. Then there exists an operator $T : K_2 \to K_1$ with $\|T\| \leq 1$ such that*

(i) $TV_2 = V_1$.
(ii) $T\pi_2(a) = \pi_1(a)T, a \in A$.

Proof Let $\xi_1, \dots, \xi_n \in H$, $a_1, \dots, a_n \in A$. Then

$$\begin{aligned}\Big\|\sum_j \pi_1(a_j)V_1\xi_j\Big\|^2 &= \sum_{ij}\big(V_1^*\pi_1\big(a_i^*a_j\big)V_1\xi_j, \xi_i\big)\\ &= \sum_{ij}\big(\phi_1\big(a_i^*a_j\big)\xi_j, \xi_i\big)\\ &\leq \sum_{ij}\big(\phi_2\big(a_i^*a_j\big)\xi_j, \xi_i\big)\\ &= \Big\|\sum \pi_2(a_j)V_2\xi_j\Big\|^2,\end{aligned}$$

since $\phi_2 - \phi_1$ is completely positive and $(a_i^*a_j) \in (A \otimes M_n)^+$. Therefore there exists a unique contraction T defined on $[\pi_2(A)VH] = K_2$ which satisfies $T\pi_2(a)V_2\xi = \pi_1(a)V_1\xi$ for all $a \in A, \xi \in H$. Taking $a = 1$, we have $TV_2 = V_1$. If $a, b \in A$ then

$$T\pi_2(a)\pi_2(b)V_2\xi = T\pi_2(ab)V_2\xi = \pi_1(ab)V_1\xi = \pi_1(a)T\pi_2(b)V_2\xi,$$

so that $T\pi_2(a) = \pi_1(a)T$, using that $[\pi_2(A)V_2H] = K_2$. □

Let ϕ be a completely positive map of A into $B(H)$ with Stinespring decomposition $\phi = V^*\pi V$. If $0 \leq T \leq 1$ is an operator in $\pi(A)'$ then the map $\phi_T(a) = V^*T\pi(a)V$ is a completely positive map of A into $B(H)$, because if $W = T^{1/2}V$, then $\phi_T(a) = W^*\pi(a)W$, so is completely positive by the Stinespring theorem, 1.2.7. If we apply this to $1 - T$, we see that $\phi - \phi_T = \phi_{1-T}$ is also completely positive.

Theorem 3.5.3 *Let A be a C^*-algebra and ϕ and ψ completely positive maps of A into $B(H)$ such that $\phi - \psi$ is completely positive. Let $\phi = V^*\pi V$ be the Stinespring decomposition of ϕ with $[\pi(A)VH] = K$. Then there is a unique operator $T \in \pi(A)'$ with $0 \leq T \leq 1$ such that $\psi(a) = \phi_T(a) = V^*T\pi(a)V$.*

Proof The map $T \to \phi_T$ is clearly linear, and if $\phi_T = 0$ then for all $a, b \in A$ and $\xi, \eta \in H$ we have

$$\big(T\pi(a)V\xi, \pi(b)V\eta\big) = \big(V^*T\pi\big(b^*a\big)V\xi, \eta\big) = \big(\phi_T\big(b^*a\big)\xi, \eta\big) = 0.$$

Since $[\pi(A)VH] = K$, $T = 0$, so we have uniqueness in the theorem.

It remains to show that $\psi = \phi_T$ for $0 \le T \le 1, T \in \pi(A)'$. By Lemma 3.5.1 ψ has a Stinespring decomposition, $\psi = W^*\sigma W$, where $W : H \to K_1$ and $K_1 = [\sigma(A)WH]$. By Lemma 3.5.2 there is a contraction $X : K \to K_1$ such that $XV = W$ and $X\pi(a) = \sigma(a)X$ for all $a \in A$, and taking adjoints, $\pi(a)X^* = X^*\sigma(a)$ for $a \in A$. Let $T = X^*X$. Then clearly $0 \le T \le 1$, and $T\pi(a) = X^*\sigma(a)X = \pi(a)T$, so that $T \in \pi(A)'$. Finally, we have for $\xi, \eta \in H$,

$$\begin{aligned}
\big(\phi_T(a)\xi, \eta\big) &= \big(X^*X\pi(a)V\xi, V\eta\big) \\
&= \big(X\pi(a)V\xi, XV\eta\big) \\
&= \big(\sigma(a)XV\xi, XV\eta\big) \\
&= \big(\sigma(a)W\xi, W\eta\big) \\
&= \big(\psi(a)\xi, \xi\big),
\end{aligned}$$

completing the proof of the theorem. □

We can now show the promised characterization of maps extremal in the cone of completely positive maps. For this we make the following,

Definition 3.5.4 Let $\phi : A \to B(H)$ be completely positive. We say ϕ is *pure* if every completely positive map $\psi : A \to B(H)$ with $\phi - \psi$ completely positive is a scalar multiple of ϕ.

It is well known that a state is pure if and only if its GNS-representation is irreducible. This extends to completely positive maps as follows.

Corollary 3.5.5 *Let $\phi : A \to B(H)$ be completely positive with Stinespring decomposition $\phi = V^*\pi V$, such that $V : H \to K$ and $[\pi(A)VH] = K$. Then ϕ is pure if and only if π is irreducible.*

Proof Let ϕ be pure. By the comments before Theorem 3.5.3 the set $\{T \in \pi(A)' : 0 \le T \le 1\}$ consists of scalar multiple of the identity, which implies that $\pi(A)$ is irreducible.

Conversely, if π is irreducible and $\psi : A \to B(H)$ is a map such that ψ and $\phi - \psi$ are completely positive, then by Theorem 3.5.3 $\psi = \phi_T$ for some $T \in \pi(A)', 0 \le T \le 1$. Since $\pi(A)'$ consists of scalar operators, $T = \lambda 1$ for some $0 \le \lambda \le 1$, so ψ is a scalar multiple of ϕ, hence ϕ is pure. □

In the finite dimensional case we get a stronger extremality result for pure maps. The result can easily be extended to maps $\phi : A \to B(H)$, where A is a C^*-algebra all of whose irreducible representations are finite dimensional.

Corollary 3.5.6 *Let K_0 be a finite dimensional Hilbert space and $\phi : B(K_0) \to B(H)$ completely positive. Then ϕ is pure if and only if it is an extremal positive map in $B(B(K_0), H)^+$.*

Proof It is clear that if ϕ is extremal then it is in particular pure. Conversely, assume ϕ is pure with Stinespring decomposition $\phi = V^*\pi V$, where by Corollary 3.5.5 π is irreducible. Since K_0 is finite dimensional, $\pi(B(K_0)) = B(K)$, K as in Corollary 3.5.5, and by finiteness π is an isomorphism. By Proposition 3.1.3 $AdV : B(K) \to B(H)$ is extremal. Let $\psi \in B(B(K_0), H)^+$, with $\psi \le \phi$. Then $\psi \circ \pi^{-1} \le AdV$, so by extremality of AdV, $\psi \circ \pi^{-1} = \lambda AdV$ for $0 \le \lambda \le 1$. Thus $\psi = \lambda AdV \circ \pi = \lambda\phi$, so ϕ is extremal. □

3.6 Notes

Extreme points of the convex set of unital positive maps were studied in [71]. The results in Sect. 3.1, except Proposition 3.1.7, are mostly variations of results in [71]. Proposition 3.1.7 is a special case of well known results on automorphisms of von Neumann algebras.

As mentioned in the introduction to Sect. 3.2 Jacobson and Rickart [29] showed that Jordan homomorphisms of matrix algebras over certain rings are sums of homomorphisms and anti-homomorphisms. Their result was used by Kadison [35] to show that surjective Jordan homomorphisms between C^*-algebras were sums of homomorphisms and anti-homomorphims, and finally the author [72] showed the same result for Jordan homomorphisms of a C^*-algebra into another C^*-algebra. Theorem 3.2.4 is a special case of Kadison's result, but the proof is quite different from the proofs in the papers referred to above. In [9] surjective Jordan homomorphisms were characterized as those positive maps which map invertible operators onto invertible operators.

Theorem 3.3.2 and its proof is taken from [71], but its followup, Theorem 3.3.4 is, with a different proof, due to Marciniak [50]. For a closely related result for maps which are not necessarily positive, see [31, 46–48]. Theorem 3.3.7 is also due to Marciniak [50]. For further work on nonextendible maps see [95, 96].

The contents of Sect. 3.4 on nonextendible maps are all due to Woronowicz [99], see also [42].

The Radon-Nikodym type theorem, Theorem 3.5.3 is due to Arveson [1].

If K and H are finite dimensional the facial structure of the cone $B(B(K), H)^+$ has been studied by several authors; see [45] for a survey. In this context maps which generate exposed rays in $B(B(K), H)^+$, called exposed maps have attracted much attention as they form a dense subset of the extremal maps, see e.g. [13, 19].

Chapter 4
Choi Matrices and Dual Functionals

In the theory of positive maps from the $n \times n$ matrices M_n ($=B(K)$ with $K = \mathbb{C}^n$) into $B(H)$, the Choi matrix corresponding to a map is very important. The present chapter is devoted to the close relationship between maps and their Choi matrices. In Sect. 4.1 we present the basic definitions and results. Then in Sect. 4.2 we introduce the dual functional to a map and show how its properties reflect the positivity properties of the map.

4.1 The Choi Matrix

In this section K is a finite dimensional Hilbert space. The vector space of linear maps of $B(K)$ into $B(H)$ can be identified with $B(K) \otimes B(H)$. In our treatment this identification will be done via the Choi matrix for a map.

Definition 4.1.1 Let $K = \mathbb{C}^n$ and let $\phi : B(K) \to B(H)$ be a linear map. Let (e_{ij}), $i, j = 1, \dots, n$ be a complete set of matrix units for $B(K)$. Then the *Choi matrix* for ϕ is the operator

$$C_\phi = \sum_{i,j=1}^{n} e_{ij} \otimes \phi(e_{ij}) \in B(K) \otimes B(H).$$

The map $\phi \to C_\phi$ is clearly linear and injective, and given an operator $\sum e_{ij} \otimes a_{ij} \in B(K) \otimes B(H)$, then we can define a linear map ϕ by $\phi(e_{ij}) = a_{ij}$. Thus the map $\phi \to C_\phi$ is surjective. This map is often called the *Jamiolkowski isomorphism*.

As defined the Choi matrix depends on the choice of matrix units (e_{ij}). The next lemma describes it with respect to another set of matrix units. Recall the notation $B(B(K), H)$ is the linear space of all linear maps from $B(K)$ into $B(H)$.

Lemma 4.1.2 *Let $\phi \in B(B(K), H)$ have Choi matrix C_ϕ with respect to a complete set of matrix units (e_{ij}). Let (f_{ij}) be another complete set of matrix units and w*

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a unitary operator such that $w^* e_{ij} w = f_{ij}$. *Then the Choi matrix* C_ϕ^f *with respect to* (f_{ij}) *is given by*

$$C_\phi^f = Ad(w \otimes 1)(C_{\phi \circ Adw}).$$

Proof

$$\begin{aligned} C_{\phi \circ Adw} &= \sum e_{ij} \otimes \phi\big(w^* e_{ij} w\big) \\ &= \sum e_{ij} \otimes \phi(f_{ij}) \\ &= (w \otimes 1)\Big(\sum_{i,j} f_{ij} \otimes \phi(f_{ij})\big(w^* \otimes 1\big)\Big). \end{aligned}$$

Hence $C_\phi^f = (w^* \otimes 1) C_{\phi \circ Adw}(w \otimes 1)$. □

Two special cases are important.

Proposition 4.1.3 *Let* ω *be a linear functional on* $B(K)$ *with density operator* h, *viz.* $\omega(a) = Tr(ha)$, $a \in B(K)$. *Let* $a \in B(H)^+$, *and identify* $b\omega$ *with the map* $a \to \omega(a)b$ *of* $B(K)$ *into* $B(H)$. *Then*

$$C_{b\omega} = h^t \otimes b.$$

Proof

$$\begin{aligned} C_{b\omega} &= \sum e_{ij} \otimes \omega(e_{ij})b \\ &= \sum \omega(e_{ij}) e_{ij} \otimes b \\ &= \sum Tr(he_{ij}) e_{ij} \otimes b \\ &= \sum h_{ji} e_{ij} \otimes b \\ &= h^t \otimes b. \end{aligned}$$

□

Proposition 4.1.4 *Suppose* $\dim H = m < \infty$. *Let* $\xi_1, \dots, \xi_n$ *(resp.* $\eta_1, \dots, \eta_m$*) be an orthonormal basis for* K *(resp.* H*), and* (e_{ij}) *(resp.* (f_{kl})*) be the corresponding complete set of matrix units, so* $e_{ij}\xi_k = \delta_{jk}\xi_i$, *and similarly for* (f_{kl}). *Let* $V : H \to K$ *be defined by* $V\eta_k = \sum_i v_{ik}\xi_i$. *Let*

$$g_{(i,k),(j,l)} = e_{ij} \otimes f_{kl}.$$

Then the set $(g_{(i,k),(j,l)})$ *is a complete set of matrix units for* $B(K \otimes H)$, *and*

$$C_{AdV} = \sum v_{jl} \overline{v}_{ik} g_{(i,k),(j,l)}$$

is a positive scalar multiple of the projection onto $\omega = \sum \overline{v}_{ik} \xi_i \otimes \eta_k$.

Proof It is obvious that $(g_{(i,k),(j,l)})$ is a complete set of matrix units for $B(K \otimes H)$. Let $\xi = \sum a_k \eta_k \in H$. Then

$$\begin{aligned}(\xi, V^*\xi_i) &= (V\xi, \xi_i) = \sum_k a_k (V\eta_k, \xi_i) \\ &= \sum_k a_k v_{ik} = \sum_k a_k (\eta_k, \overline{v}_{ik}\eta_k) = \sum_k (\xi, \overline{v}_{ik}\eta_k).\end{aligned}$$

Thus

$$V^*\xi_i = \sum_k \overline{v}_{ik}\eta_k, \quad \text{for all } i. \tag{4.1}$$

It follows that

$$V^* e_{ij} V \eta_k = V^* e_{ij} \sum_s v_{sk}\xi_s = V^* v_{jk}\xi_i = \sum_l v_{jk}\overline{v}_{il}\eta_l.$$

Therefore we get

$$\begin{aligned}C_{AdV}(\xi_s \otimes \eta_t) &= \left(\sum_{ij} e_{ij} \otimes V^* e_{ij} V\right)(\xi_s \otimes \eta_t) \\ &= \sum \xi_i \otimes v_{st}\overline{v}_{ik}\eta_k \\ &= \left(\sum_{ik} e_{is} \otimes v_{st}\overline{v}_{ik} f_{kt}\right)(\xi_s \otimes \eta_t) \\ &= \left(\sum_{ik} v_{st}\overline{v}_{ik} g_{(i,k)(s,t)}\right)(\xi_s \otimes \eta_t).\end{aligned}$$

Thus

$$C_{AdV} = \sum_{i,j,k,l} v_{jl}\overline{v}_{ik} g_{(i,k)(j,l)}. \qquad \square$$

In the above proposition the rank of V is reflected in how ω is written as a tensor product of vectors.

Definition 4.1.5 Let $\xi \in K \otimes H$. Then ξ has *Schmidt rank* r denoted by $SR\xi$, if r is the smallest number m such that ξ can be written as $\xi = \sum_{i=1}^m \xi_i \otimes \eta_i$ with $\xi_i \in K$, $\eta_i \in H$.

Then we can find an orthonormal family $\omega_1, \dots, \omega_r \in H$ and vectors $\rho_i \in K$ such that $\xi = \sum_{i=1}^r \rho_i \otimes \omega_i$. To show this, note that the span of the η_i's must be r-dimensional by minimality of r, so we can write the η_i's as linear combinations of r orthonormal vectors $\omega_1, \dots, \omega_r$ in H. Using this we can give more specific information on V and ω in the last proposition.

Proposition 4.1.6 *Let $V : H \to K$ and ω be as in Proposition 4.1.4. Then $C_{AdV} = \lambda[\omega]$ for some $\lambda \geq 0$. ω has Schmidt rank r if and only if* $\operatorname{rank} V = r$.

Proof Suppose $\operatorname{rank} V = r$. Choose an orthonormal basis $\eta_1, \ldots, \eta_m$ for H such that $V^*V\eta_k = \lambda_k\eta_k$ with $\lambda_1, \ldots, \lambda_r > 0$ and $\lambda_k = 0$ for $k > r$. Let $\xi_1, \ldots, \xi_n$ be an orthonormal basis for K. By Proposition 4.1.4

$$C_{AdV} = \lambda[\omega], \quad \omega = \sum_k \left(\sum_i \overline{v}_{ik}\xi_i \right) \otimes \eta_k$$

and

$$V\eta_k = \sum_i v_{ik}\xi_i .$$

Thus by (4.1)

$$\lambda_k\eta_k = V^*V\eta_k = \sum_i V^* v_{ik}\xi_i = \sum_{i,l} v_{ik}\overline{v}_{il}\eta_l, \tag{4.2}$$

hence $\overline{v}_{il} = 0$ for $l \neq k$, and $\sum_i |v_{ik}|^2 = \lambda_k$. Thus $v_{ik} \neq 0$ for some i when $k \leq r$, so that ω has Schmidt rank r.

Conversely, if $SR\omega = r$, choose and orthonormal basis $\eta_1, \ldots, \eta_m$ in H such that

$$\omega = \sum_{k=1}^{r} \left(\sum_i \overline{v_{ik}}\xi_i \right) \otimes \eta_k = \sum_{ik} \overline{v}_{ik}\xi_i \otimes \eta_k .$$

If we define $V : H \to K$ by $V\eta_k = \sum_i v_{ik}\xi_i \neq 0$ if $k \leq r$, and $V\eta_k = 0$ for $k > r$, Proposition 4.1.4 shows us that C_{AdV} is a scalar multiplum of $[\omega]$. By construction V has rank r. Since $\phi \to C_\phi$ is an isomorphism, and $AdV = AdW$ if and only if $W = zV$, $|z| = 1$, the rank of V is uniquely defined whenever $C_{AdV} = \lambda[\omega]$ with $\lambda > 0$. Thus $\operatorname{rank} V = r$. □

Remark 4.1.7 If $\dim K = n$, and ι denotes the identity map of $B(K)$ into itself, then for $V = 1$ we get

$$C_\iota = C_{Ad1} = \sum e_{ij} \otimes e_{ij}$$

is n times the projection onto $\frac{1}{\sqrt{n}}\xi_i \otimes \xi_i$, called the *maximally entangled state*. For more on entanglement see the discussion after Proposition 4.1.11 and Sect. 7.4.

Note that by Proposition 4.1.6 $\operatorname{rank} V = 1$ if and only if $C_{AdV} = \lambda[\xi] \otimes [\eta]$ if and only if $\omega = \xi \otimes \eta$ is a product vector.

As an immediate consequence of Proposition 4.1.4 we have

Theorem 4.1.8 *Let K and H be finite dimensional and $\phi \in B(B(K), H)$. Then the following conditions are equivalent*:

(i) *ϕ is completely positive.*
(ii) $C_\phi \geq 0$.
(iii) $\phi = \sum_{i=1}^{m} AdV_i$ *with* $V_i : H \to K$ *linear, and* $m \leq \dim K \cdot \dim H$.
(iv) $\phi = \sum_{i=1}^{k} AdW_i$, *with* $W_i : H \to K$ *linear and* $k \in \mathbb{N}$.

Proof (i) ⇒ (ii). Let $n = \dim K$, $m = \dim H$. If ϕ is completely positive then $\iota_n \otimes \phi : M_n \otimes B(K) \to M_n \otimes B(H)$ is positive, where $\iota_n$is the identity map on M_n. Hence

$$C_\phi = \sum_{ij} e_{ij} \otimes \phi(e_{ij}) = \iota_n \otimes \phi\left(\sum_{ij} e_{ij} \otimes e_{ij}\right) \geq 0.$$

(ii) ⇒ (iii). If $C_\phi \geq 0$ then $C_\phi = \sum_{i=1}^{mn} \lambda_i [\omega_i]$ with ω_i an orthonormal basis for $K \otimes H$, $1 \leq i \leq mn$, $\lambda_i \geq 0$. By Proposition 4.1.4 $[\omega_i] = C_{AdV_i}$ for an operator $V_i : H \to K$. Thus $\phi = \sum_1^{mn} \lambda_i AdV_i$. If we replace V_i by $\lambda_i^{-1/2} V_i$ whenever $\lambda_i \neq 0$, we have (iii).

(iii) ⇒ (iv) is trivial.

(iv) ⇒ (i). This follows since $\iota_n \otimes AdV = Ad(\iota_n \otimes V)$ is positive, so AdV is completely positive (see also Lemma 1.2.2). □

The decomposition (iii) in the above theorem is usually called the *Kraus decomposition* for ϕ.

Corollary 4.1.9 *Let* $\phi : B(K) \to B(H)$, *with* $K = \mathbb{C}^n$, $H = \mathbb{C}^m$, *and let* $k = \min(m, n)$. *Then ϕ is completely positive if and only if ϕ is k-positive.*

Proof Suppose ϕ is k-positive. Assume first $k = n$. Then $\iota_n \otimes \phi$ is positive, so $C_\phi = \iota_n \otimes \phi(\sum_{ij} e_{ij} \otimes e_{ij}) \geq 0$. Thus by Theorem 4.1.8 ϕ is completely positive. If $k = m$ then $\phi^* : B(H) \to B(K)$ is k-positive from Proposition 1.4.3, hence by the first part ϕ^* is completely positive. Then by the same proposition ϕ is completely positive. The converse is obvious. □

The above corollary can be extended to maps of C^*-algebras. Then it states that every k-positive map of a C^*-algebra A into another B is completely positive if and only if either A or B has all its irreducible representations on Hilbert spaces of dimension less than or equal to k, see [93].

We shall need to know the Choi matrix for ϕ^* when $\phi \in P(H)$, the cone of positive maps of $B(H)$ into itself.

Lemma 4.1.10 *Let* $\dim H = n$ *and* $\xi_1, \ldots, \xi_n$ *be an orthonormal basis for* H. *Let* J *be the conjugation of* $H \otimes H$ *defined by*

$$J z\xi_i \otimes \xi_j = \bar{z}\xi_j \otimes \xi_i$$

with $z \in \mathbb{C}$. *Let* $\phi \in P(H)$. *Then* $C_{\phi^*} = J C_\phi J$.

Proof Let $V = (v_{ij})_{i,j \leq n} \in B(H)$, and let e_{ij} denote the matrix units such that $e_{ij}\xi_k = \delta_{jk}\xi_i$. Then a straightforward computation yields

$$AdV(e_{kl}) = V^* e_{kl} V = (\overline{v}_{ki} v_{lj})_{ij}.$$

Since $V^* = (\overline{v}_{ji})$ it follows that

$$AdV^*(e_{kl}) = V e_{kl} V^* = (v_{ik}\overline{v}_{jl})_{ij}.$$

From the definition of J it thus follows that

$$\begin{aligned}
JC_{AdV}J(z\xi_p \otimes \xi_q) &= J\Bigg(\sum_{ijkl} e_{kl} \otimes \overline{v}_{ki} v_{lj} e_{ij}\Bigg)\overline{z}\xi_q \otimes \xi_p \\
&= \sum v_{ki}\overline{v}_{lj} e_{ij}\xi_p \otimes z e_{kl}\xi_q \\
&= \Bigg(\sum_{ik} v_{ik}\overline{v}_{jl} e_{kl} \otimes e_{ij}\Bigg)(z\xi_p \otimes \xi_q) \\
&= \Big(\sum e_{kl} \otimes V e_{kl} V^*\Big)(z\xi_p \otimes \xi_q) \\
&= C_{AdV^*}(z\xi_p \otimes \xi_q),
\end{aligned}$$

where we at the third equality sign exchanged (i, j) with (k, l). Since the vectors $\xi_p \otimes \xi_q$ form a basis for $H \otimes H$, $JC_{AdV}J = C_{AdV^*}$. Now, if ϕ is a positive map then C_ϕ is self-adjoint, hence the difference between two positive operators, which both are Choi matrices for completely positive maps by Theorem 4.1.8. Hence by Theorem 4.1.8 again ϕ is a real linear sum of maps AdV. By Proposition 1.4.2 the adjoint map of AdV is AdV^*. Applying this to each summand AdV, we thus get $JC_\phi J = C_{\phi^*}$. □

Proposition 4.1.11 *Let H be a Hilbert space of arbitrary dimension. Let $\phi \in B(B(K), H)$. Then ϕ is positive if and only if $Tr(C_\phi\, a \otimes b) \geq 0$ for all $a \in B(K)^+$ and b a positive trace class operator on H.*

Proof Computing we get

$$\begin{aligned}
Tr(C_\phi a \otimes b) &= \sum_{ij} Tr\big(\big(e_{ij} \otimes \phi(e_{ij})\big)(a \otimes b)\big) \\
&= \sum_{ij} Tr(e_{ij}a)Tr\big(\phi(e_{ij})b\big) \\
&= \sum a_{ji} Tr\big(\phi(e_{ij})b\big) \\
&= Tr\big(\phi\big(a^t\big)b\big).
\end{aligned}$$

Since this holds for all positive trace class operators b, and $a \geq 0$ if and only if $a^t \geq 0$, $\phi(a) \geq 0$ if and only if $Tr(C_\phi a \otimes b) \geq 0$ for all positive a and b. □

In quantum information theory C_ϕ is often called an *entanglement witness* when ϕ is not completely positive, because the proposition shows that if $h = \sum a_i \otimes b_i \geq 0$ is the density operator for a state ω on $B(K \otimes H)$, then ω is *entangled*, i.e. h cannot be written in the above form with all $a_i, b_i \geq 0$, if there exists a positive map $\phi : B(K) \to B(H)$ such that $Tr(C_\phi h) < 0$.

Let $\phi \in B(B(K), H)$ be a self-adjoint linear map, so $\phi(a)$ is self-adjoint when a is self-adjoint. Then it is easily seen that C_ϕ is a self-adjoint operator, hence is the difference of two positive operators C_ϕ^+ and C_ϕ^- such that $C_\phi^+ C_\phi^- = 0$.

We shall see later, Theorem 7.4.3, that C_ϕ^- contains much information. Presently we concentrate on C_ϕ^+. Let $c \geq 0$ be the smallest positive number such that $c1 \geq C_\phi$. Then $c = \|C_\phi^+\|$. Hence, if $c \neq 0$ there exists a map $\phi_{cp} : B(K) \to B(H)$ such that its Choi matrix $C_{\phi_{cp}} = 1 - \frac{1}{c} C_\phi$ is a positive operator. Thus if Tr is identified with the positive map $a \to Tr(a)1$, it is straightforward to show that $C_{Tr} = 1$, so $\frac{1}{c}\phi = Tr - \phi_{cp}$. By Theorem 4.1.8, ϕ_{cp} is completely positive. We have

Theorem 4.1.12 *Let $\phi \in B(B(K), H)$ be a self-adjoint linear map such that $-\phi$ is not completely positive. Then there exists a completely positive map $\phi_{cp} : B(K) \to B(H)$ such that*

$$\left\|C_\phi^+\right\|^{-1} \phi = Tr - \phi_{cp}.$$

Furthermore, ϕ is positive if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all product states $\rho = \omega_1 \otimes \omega_2$ on $B(K) \otimes B(H)$.

Proof The existence of ϕ_{cp} was shown above. To show the second part let $\rho(x) = Tr \otimes Tr((a \otimes b)x)$ be a product state on $B(K) \otimes B(H)$ with density operator $a \otimes b$. Then

$$\rho(C_{\phi_{cp}}) = Tr \otimes Tr(C_{\phi_{cp}} a \otimes b),$$

so that $Tr(C_\phi a \otimes b) \geq 0$ if and only if $\rho(C_{\phi_{cp}}) \leq 1$. Hence the theorem follows from Proposition 4.1.11. □

Recall from Definition 1.2.1 that a map ϕ is k-positive if $\iota_k \otimes \phi$ is positive, where ι_k denotes the identity map on M_k. We now give several characterizations of k-positive maps, one of them in terms of the Choi matrix.

Definition 4.1.13 An operator C on $K \otimes H$ is called *k-block positive* if $(C \sum_{i=1}^k \xi_i \otimes \eta_i, \sum_{i=1}^k \xi_i \otimes \eta_i) \geq 0$ for all choices of vectors $\xi_1, \ldots, \xi_k \in K$, and $\eta_1, \ldots, \eta_k \in H$.

Remark 4.1.14 Note that a vector $\xi \in K \otimes H$ is of the form $\sum_{i=1}^k \xi_i \otimes \eta_i$ if and only if $\xi = (1 \otimes q)\psi$ for a vector $\psi \in K \otimes H$ and projection $q \in B(H)$ of dimension k.

Indeed, if $\xi = \sum_{i=1}^{k} \xi_i \otimes \eta_i$ let q denote the projection onto the span of $\eta_1, \dots, \eta_k$, then $\xi = (1 \otimes q)\xi$. Conversely, if $\xi = (1 \otimes q)\psi$ with $\psi = \sum_{i=1}^{n} \xi_i \otimes \eta_i$, q as above, we can choose a basis $\gamma_1, \dots, \gamma_k$ for qH such that $q\eta_i = \sum \alpha_{ij}\gamma_j$. Then

$$1 \otimes q(\psi) = \sum \xi_i \otimes q\eta_i = \sum \alpha_{ij}\xi_i \otimes \gamma_j = \sum_{j=1}^{k} \Big(\sum_i \alpha_{ij}\xi_i \Big) \otimes \gamma_j .$$

The same argument also yields that $\xi = \sum_1^k \xi_i \otimes \eta_i$ if and only if $\xi = p \otimes q(\psi)$ for $\psi \in K \otimes H$, and p and q k-dimensional projections in $B(K)$ and $B(H)$ respectively.

Theorem 4.1.15 *Let $\phi \in B(B(K), H)$ and $k \le \min(\dim K, \dim H)$. Then the following conditions are equivalent.*

(i) *ϕ is k-positive.*
(ii) *$\phi \circ AdV$ is completely positive for all $V \in B(K)$ with* $\operatorname{rank} V \le k$.
(iii) *$AdW \circ \phi$ is completely positive for all $W \in B(H)$ with* $\operatorname{rank} W \le k$.
(iv) *C_ϕ is k-block positive.*

Proof The proof goes as follows. (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Let ϕ be k-positive and $V \in B(K)$ with rank $V \le k$. Let $e = $ support V. Then $\dim e \le k$. Thus

$$\phi \circ AdV = \phi \circ AdV \circ Ade : eB(K)e \to B(H).$$

Since $eB(K)e \cong M_l$ with $l \le k$, and ϕ is k-positive, $\phi \circ AdV$ is completely positive by Corollary 4.1.9.

(ii) $\Rightarrow$ (i). Let $(e_{ij})_{i,j\le k}$ be a complete set of matrix units for M_k. Let $a = \sum_{i,j\le k} e_{ij} \otimes a_{ij} \in (M_k \otimes B(K))^+$. Again by Corollary 4.1.9 $a = C_\psi$ for a completely positive map $\psi : M_k \to B(K)$. By Theorem 4.1.8 $\psi = \sum AdV_i$ with $V_i : K \to \mathbb{C}^k$. Since $k \le \dim K$ we may assume $\mathbb{C}^k \subset K$, hence $V_i \in B(K)$ with rank $V_i \le k$ for all i. Thus by (ii) $\phi \circ \psi$ is completely positive, hence by Theorem 4.1.8

$$\iota_k \otimes \phi(a) = \iota_k \otimes \phi(C_\psi) = C_{\phi\circ\psi} \ge 0,$$

so that ϕ is k-positive.

(ii) $\Rightarrow$ (iv). Let $\xi = \sum_1^k \xi_i \otimes \eta_i \in K \otimes H$ have Schmidt rank k. Let q be a k-dimensional projection in $B(H)$ such that $q\eta_i = \eta_i$ for all i. Let (e_{ij}) be a complete set of matrix units in $B(K)$ such that $C_\phi = \sum e_{ij} \otimes \phi(e_{ij})$. Then we have

$$C_{Adq\circ\phi} = \sum e_{ij} \otimes Adq\big(\phi(e_{ij})\big) = Ad(1 \otimes q)(C_\phi).$$

Thus by (ii) and Theorem 4.1.8 $Ad(1 \otimes q)(C_\phi) \ge 0$. It follows that

$$(C_\phi\xi, \xi) = \big(C_\phi(1 \otimes q)\xi, (1 \otimes q)\xi\big) = \big(Ad(1 \otimes q)(C_\phi)\xi, \xi\big) \ge 0.$$

Thus C_ϕ is k-block positive.

(iv) $\Rightarrow$ (iii). Let $W \in B(H)$ with rank $W \leq k$. Let $\xi = \sum \xi_i \otimes \eta_i \in K \otimes H$. Let e support W, so $\dim e \leq k$. Then there exist k vectors $\alpha_1, \ldots, \alpha_k \in H$ such that $e\eta_i = \sum_1^n c_{ij}\alpha_j$, $c_{ij} \in \mathbb{C}$. We can therefore write $1 \otimes W\xi = \sum_1^k \xi'_j \otimes \beta_j$ with $\xi'_j \in K$, $\beta_j \in H$. Thus $1 \otimes W\xi$ has Schmidt rank$\leq k$, hence by the assumption that C_ϕ is k-block positive $(C_\phi(1 \otimes W)\xi, (1 \otimes W)\xi) \geq 0$. Thus

$$C_{Adw\circ\phi} = (1 \otimes AdW)(C_\phi) \geq 0,$$

so that $AdW \circ \phi$ is completely positive.

(iii) $\Rightarrow$ (i). Let $V \in B(H)$ with rank $V \leq k$. Then $(AdV)^* = AdV^*$. Hence

$$\phi^* \circ AdV^* = (AdV \circ \phi)^* : B(H) \to B(K).$$

Since by assumption $AdV \circ \phi$ is completely positive, so is $\phi^* \circ AdV^*$. We have therefore shown that $\phi^* \circ AdW$ is completely positive for all $W \in B(H)$ with rank $W \leq k$. Therefore by the equivalence (i) $\Leftrightarrow$ (ii) applied to $\phi^* : B(H) \to B(K)$, ϕ^* is k-positive, hence so is ϕ. $\square$

4.2 The Dual Functional of a Map

In the previous section we studied the duality between positive maps of $B(K)$ into $B(H)$ as matrices via the Jamiolkowski isomorphism $\phi \to C_\phi \in B(K \otimes H)$. In this section we consider the duality between maps and linear functionals on $B(K) \otimes \mathscr{T}(H)$, or more generally $A \otimes \mathscr{T}(H)$, where $\mathscr{T}(H)$ denotes the trace class operators on $B(H)$, and A is an *operator system*, i.e. a unital linear subspace of $B(K)$ such that $a \in A$ implies $a^* \in A$.

Definition 4.2.1 Let A be an operator system and $\phi : A \to B(H)$ a bounded linear map. Then its *dual functional* $\widetilde{\phi}$ on $A \otimes \mathscr{T}(H)$ is the functional defined by

$$\widetilde{\phi}(a \otimes b) = Tr\big(\phi(a)b^t\big),$$

where t is the transpose on $B(H)$ defined by a fixed orthonormal basis.

$\widetilde{\phi}$ is well defined because $\phi(a)$ is a bounded operator in $B(H)$, and b is a trace class operator. Let the *projective norm* on the algebraic tensor product of A and $\mathscr{T}(H)$ be defined by

$$\|x\|_\wedge = \inf\left\{\sum \|a_i\|\|b_i\|_1 \,:\, x = \sum_{i=1}^n a_i \otimes b_i, a_i \in A, b_i \in \mathscr{T}(H)\right\}$$

where $\|b\|_1$ is the trace norm $\|b\|_1 = Tr(|b|)$. We denote by $A\widehat{\otimes}\mathscr{T}(H)$ the completion of the algebraic tensor product with respect to the projective norm, and by $A^+\widehat{\otimes}\mathscr{T}(H)^+$ the closed cone generated by operators $\sum_i a_i \otimes b_i$ with $a_i \in A^+$, $b_i \in \mathscr{T}(H)^+$. $A\widehat{\otimes}\mathscr{T}(H)$ is called the *projective tensor product* of A and $\mathscr{T}$.

Lemma 4.2.2 *Let A be an operator system. Then the map $\phi \to \widetilde{\phi}$ is an isometric isomorphism of the space of bounded linear maps of A into $B(H)$ and $(A\widehat{\otimes}\mathscr{T}(H))^*$. Furthermore ϕ is positive if and only if $\widetilde{\phi}$ is positive on $A^+\widehat{\otimes}\mathscr{T}(H)^+$.*

Proof Let $x = \sum_1^n a_i \otimes b_i \in A\widehat{\otimes}\mathscr{T}(H)$ be a finite tensor. Then

$$\begin{aligned}\big|\widetilde{\phi}(x)\big| &= \Big|\sum_i Tr\big(\phi(a_i)b_i^t\big)\Big| \\ &\le \sum\big|Tr\big(\phi(a_i)b_i^t\big)\big| \\ &\le \sum\big\|\phi(a_i)\big\|\,\|b_i\|_1 \\ &\le \|\phi\|\sum\|a_i\|\,\|b_i\|_1.\end{aligned}$$

Thus $\|\widetilde{\phi}\| \le \|\phi\|$.

Conversely, since $\|\ \|_\wedge$ is a cross norm,

$$\begin{aligned}\|\phi\| = \sup_{\|a\|=1}\big\|\phi(a)\big\| &= \sup_{\|a\|=1,\,\|b\|_1=1}\big|Tr\big(\phi(a)b^t\big)\big| \\ &= \sup\big|\widetilde{\phi}(a\otimes b)\big| \\ &= \sup\|\widetilde{\phi}\|\,\|a\otimes b\| \\ &\le \sup\|\widetilde{\phi}\|\,\|a\|\,\|b\|_1 \\ &\le \|\widetilde{\phi}\|.\end{aligned}$$

Thus the map $\phi \to \widetilde{\phi}$ is an isometry. The last part of the lemma follows from the proof of Proposition 4.1.11. □

The connection between the Choi matrix C_ϕ and $\widetilde{\phi}$ is given by the following result.

Lemma 4.2.3 *Let K be finite dimensional and $\phi \in B(B(K), H)$. Then C_ϕ^t is the density operator for $\widetilde{\phi}$.*

Proof Since the transpose is Tr-invariant, if $a \otimes b \in B(K) \otimes \mathscr{T}(H)$,

$$\begin{aligned}Tr\big(C_\phi^t a\otimes b\big) &= Tr\big(C_\phi a^t \otimes b^t\big) \\ &= \sum_{ij} Tr\big(e_{ij}a^t \otimes \phi(e_{ij})b^t\big) \\ &= \sum_{ij} Tr\big(e_{ij}a^t\big)Tr\big(\phi(e_{ij})b^t\big)\end{aligned}$$

$$= \sum a_{ij} Tr\big(e_{ij}\phi^*\big(b^t\big)\big)$$
$$= Tr\big(a\phi^*\big(b^t\big)\big)$$
$$= \widetilde{\phi}(a \otimes b),$$

proving the lemma. □

We shall often encounter the situation when we compose a map by the transpose map both in the domain and the range of ϕ. Let as before t denote the transpose both of $B(K)$ and $B(H)$.

Definition 4.2.4 Let $\phi \in B(B(K), H)$. Then we denote by

$$\phi^t = t \circ \phi \circ t.$$

The basic properties are given in

Lemma 4.2.5 *Let $\phi \in B(B(K), H)$. Then we have*

(i) *If ϕ is k-positive (resp. completely positive), so is ϕ^t.*
(ii) *If $\phi = AdV$ then $\phi^t = AdV^{t*}$.*
(iii) *If* $\dim K < \infty$ *then $C_{\phi^t} = C_\phi^t$, where C_ϕ^t is the transpose on $B(K \otimes H)$.*

Proof (i) Let $\iota = \iota_k$ be the identity map on M_k. Then

$$\iota \otimes \phi^t = (\iota \otimes t) \circ (\iota \otimes \phi) \circ (\iota \otimes t) = (t \otimes t) \circ (\iota \otimes \phi) \circ (t \otimes t),$$

is positive, since $t \otimes t$ is the transpose on $B(K \otimes H)$, so is a positive map, and $\iota \otimes \phi$ is a positive map when ϕ is k-positive.

(ii) $(AdV)^t(a) = (AdV(a^t))^t = (V^* a^t V)^t = V^t a V^{*t}$.

(iii) $C_{\phi^t} = \sum e_{ij} \otimes \phi^t(e_{ij}) = \sum e_{ij} \otimes \phi(e_{ji})^t = (\sum e_{ji} \otimes \phi(e_{ji}))^t = C_\phi^t$. □

The relationship between $\widetilde{\phi}$ and ϕ^t is given in the next result.

Lemma 4.2.6 *Let K and H be finite dimensional. Let $\pi : B(K) \otimes B(K) \to B(K)$ be defined by $\pi(a \otimes b) = b^t a$. Then $Tr \circ \pi$ is positive and linear. Let $\phi \in B(B(K), H)$. Then*

$$\widetilde{\phi} = Tr \circ \pi \circ \big(\iota \otimes \phi^{*t}\big).$$

Proof Linearity of $Tr \circ \pi$ is clear. To show positivity let $x = \sum a_i \otimes b_i \in B(K) \otimes B(K)$. Then

$$Tr \circ \pi\big(xx^*\big) = \sum_{ij} Tr \circ \pi\big(a_i a_j^* \otimes b_i b_j^*\big)$$
$$= \sum Tr\big(b_j^{*t} b_i^t a_i a_j^*\big)$$

$$= \sum_{ij} Tr\big((b_j^t a_j)^* (b_i^t a_i)\big)$$

$$= Tr\Big(\Big(\sum b_j^t a_j\Big)^* \Big(\sum b_i^t a_i\Big)\Big) \geq 0,$$

so $Tr \circ \pi$ is positive. The formula in the lemma follows from the computation

$$\widetilde{\phi}(a \otimes b) = Tr\big(\phi(a) b^t\big) = Tr\big(a \phi^*(b^t)\big) = Tr\big(a \phi^{*t}(b)^t\big) = Tr \circ \pi\big(\iota \otimes \phi^{*t}(a \otimes b)\big).$$

□

In the finite dimensional case we showed in Theorem 4.1.8 that $\phi \in B(B(K), H)$ is completely positive if and only if $C_\phi \geq 0$, hence by Lemma 4.2.3 if and only if $\widetilde{\phi}$ is positive. We now show a generalization of this. When H is infinite dimensional we define the positive cone $(A \widehat{\otimes} \mathscr{T}(H))^+$ in $A \widehat{\otimes} \mathscr{T}(H)$ for A an operator system, to be the closure of the positive cone in the algebraic tensor product $A \otimes \mathscr{T}(H)$.

Theorem 4.2.7 *Let A be an operator system and $\phi : A \to B(H)$. Then ϕ is completely positive if and only if $\widetilde{\phi}$ is a positive linear functional on $A \widehat{\otimes} \mathscr{T}(H)$.*

Proof We first assume H is finite dimensional. Then we have

$$\widetilde{\phi}^t(a \otimes b) = \widetilde{\phi}^*(b \otimes a), \quad a \in A, b \in B(H). \tag{4.3}$$

This follows from the computation

$$\widetilde{\phi}^t(a \otimes b) = Tr\big(\phi(a^t)^t b^t\big) = Tr\big(a^t \phi^*(b)\big) = \widetilde{\phi}^*(b \otimes a).$$

Assume $\widetilde{\phi}$ is a positive linear functional on $A \otimes B(H)$. Since $1 \otimes 1$ is an interior point of the positive cone $(A \otimes B(H))^+$ in the algebraic tensor product $A \otimes B(H)$, and $\widetilde{\phi}$ is positive on $(A \otimes B(H))^+$, it follows from Appendix A.3.1 that $\widetilde{\phi}$ has an extension to a positive linear functional ρ on $B(K) \otimes B(H)$. Since $\rho(1 \otimes 1) = \widetilde{\phi}(1 \otimes 1)$, ρ is bounded, and by the definition of the dual functional and Lemma 4.2.2, ρ is of the form $\rho = \widetilde{\psi}$ for a positive map $\psi \in B(B(K), H)$.

Let $\sum_i a_i \otimes b_i \geq 0$ in $B(K) \otimes B(H)$. Then $\sum a_i^t \otimes b_i^t = (\sum a_i \otimes b_i)^t \geq 0$, hence $\sum b_i^t \otimes a_i^t \geq 0$. Thus by (4.3)

$$\widetilde{\psi}^*\Big(\sum b_i \otimes a_i\Big) = \widetilde{\psi}^t\Big(\sum a_i \otimes b_i\Big) = \widetilde{\psi}\Big(\sum a_i^t \otimes b_i^t\Big) \geq 0,$$

so $\widetilde{\psi}^*$ is positive.

To continue the proof assume first K finite dimensional. Then, $\widetilde{\psi}^* \geq 0$ implies $C_{\psi^{*t}} = C_{\psi^*}^t \geq 0$ by Lemma 4.2.3, hence ψ^* is completely positive by Theorem 4.1.8. In the general case let e be a finite dimensional projection in $B(K)$ such that $e^t = e$. Then

$$\big(Ade \circ \psi^*\big)^{\widetilde{}} = \widetilde{\psi^{*t}} \circ Ad(1 \otimes e), \tag{4.4}$$

which is positive, so $\psi^* : B(H) \to eB(K)e$ is completely positive by the finite dimensional case. Since this holds for all e as above, ψ^* is completely positive. But then ψ is completely positive by Proposition 1.4.3. Since $\widetilde{\psi}$ is an extension of $\widetilde{\phi}$, ψ is an extension of ϕ. Thus ϕ is completely positive.

If $\dim H = \infty$, we use the same argument, and let (e_γ) be a net of finite dimensional projections in $B(H)$, such that $e_\gamma = e_\gamma^t$, and $e_\gamma \to 1$. Then as in (4.4)

$$(Ade_\gamma \circ \phi)^{\sim} = \widetilde{\phi}^t \circ Ad(1 \otimes e_\gamma) \tag{4.5}$$

is positive, so by the first part of the proof $Ade_\gamma \circ \phi$ is completely positive, and finally by taking limits ϕ is completely positive.

Conversely suppose ϕ is completely positive. Assume first that $\dim H = n < \infty$, so $B(H) = M_n$. Let $\phi_n = \phi \otimes \iota_n$. Then ϕ_n is a positive map $A \otimes M_n \to M_n \otimes M_n$. Let $\pi : M_n \otimes M_n \to M_n$ be defined by $\phi(a \otimes b) = b^t a$. By Lemma 4.2.6 $Tr \circ \pi$ is positive. Let $\sum_i a_i \otimes b_i \in (A \otimes M_n)^+$. Then we have

$$\begin{aligned}
\widetilde{\phi}\Big(\sum_i a_i \otimes b_i\Big) &= \sum_i Tr\big(\phi(a_i) b_i^t\big) \\
&= \sum_i Tr \circ \pi\big(\phi(a_i) \otimes b_i\big) \\
&= Tr \circ \pi\Big(\phi_n\Big(\sum_i a_i \otimes b_i\Big)\Big) \geq 0,
\end{aligned}$$

so $\widetilde{\phi}$ is positive. In the general case let (e_γ) be an increasing net in $B(H)$ as in the previous paragraph. Then $Ade_\gamma \circ \phi : A \to e_\gamma B(H) e_\gamma$ is completely positive, so by the above $(Ade_\gamma \circ \phi)^{\sim}$ is positive.

For each $a \in B(H)$, $e_\gamma a e_\gamma \to a$ strongly. Thus for each trace class operator b,

$$Tr(a e_\gamma b e_\gamma) = Tr(e_\gamma a e_\gamma b) \to Tr(ab).$$

Hence $e_\gamma b e_\gamma \to b$ as trace class operators. Thus if $\sum a_i \otimes b_i \in (A \otimes \mathcal{T}(H))^+$ we get

$$\begin{aligned}
\widetilde{\phi}\Big(\sum_i a_i \otimes b_i\Big) &= \sum_i Tr\big(\phi(a_i) b_i^t\big) \\
&= \lim \sum_i Tr\big(\phi(a_i) e_\gamma b_i^t e_\gamma\big)
\end{aligned}$$

is positive, since $\sum_i a_i \otimes e_\gamma b_i e_\gamma = Ad(1 \otimes e_\gamma)(\sum a_i \otimes b_i) \geq 0$. Thus $\widetilde{\phi} \geq 0$. □

4.3 Notes

The results in Sect. 4.1 are due to several authors. The Kraus decomposition was noted by Kraus [41] and the Jamiolkowski isomorphism by Jamiolkowski [30] a year later. Then Choi introduced the Choi matrix [7] and showed Theorem 4.1.8. Propositions 4.1.4, 4.1.6, and Theorem 4.1.15 can be found in [67–69], but some of these results were previously known in the literature in one form or the other, see [2, Sect. 10.3].

The results in Sect. 4.2 can be found in [78], except for Lemma 4.2.6, which is taken from [80].

Chapter 5
Mapping Cones

In the theory of positive maps the completely positive ones have by far attracted most attention. We shall in the present chapter see that if we consider cones of positive maps with selected properties then we can prove results similar to those for completely positive maps. In Sect. 5.1 we introduce the main concepts and prove the basic results, and in Sect. 5.2 we show a Hahn-Banach like extension theorem for maps positive with respect to cones.

5.1 Basic Properties

The problems on positive maps $\phi : A \to B(H)$ encountered in the present chapter are to a great extent independent of the C^*-algebra structure of A. We shall therefore concentrate on the more general situation when A is an operator system, i.e. a complex linear subspace of $B(K)$ such that $a^* \in A$ whenever $a \in A$ with $1 \in A$. Let as before $B(A, H)$ denote the linear space of bounded linear maps of A into $B(H)$, and let $P(H)$ denote the positive linear maps of $B(H)$ into itself. The *BW-topology* (see Appendix A.1.1) on $B(A, H)$ is the topology where a bounded net (ϕ_α) in $B(A, H)$ converges to $\phi \in B(A, H)$ whenever $\phi_\alpha(a) \to \phi(a)$ in the weak topology for all $a \in A$. With the duality of $B(A, H)$ and $(A \widehat{\otimes} \mathscr{T}(H))^*$ given by $\phi \to \widetilde{\phi}$ in Definition 4.2.1 we have that $\phi_\alpha \to \phi$ in the BW-topology if and only if $\widetilde{\phi}_\alpha \to \widetilde{\phi}$ in the w^*-topology on bounded functionals in $(A \widehat{\otimes} \mathscr{T}(H))^*$. This is easily seen, since $\phi_\alpha(a) \to \phi(a)$ weakly if and only if $\widetilde{\phi}_\alpha(a \otimes b) = Tr(\phi_\alpha(a)b^t) \to Tr(\phi(a)b^t) = \widetilde{\phi}(a \otimes b)$ for all $a \otimes b \in A \otimes \mathscr{T}(H)$.

It should be remarked that if A and H are finite dimensional then the BW-topology reduces to the norm topology on $B(A, H)$.

Definition 5.1.1 A *mapping cone* is a BW-closed convex subcone $\mathscr{C}$ of the positive maps $P(H)$ of $B(H)$ into itself such that

(i) if $0 \neq a \in B(H)^+$ then there is $\phi \in \mathscr{C}$ such that $\phi(a) \neq 0$,
(ii) $\mathscr{C}$ is invariant in the sense that if $\phi \in \mathscr{C}$ and $a, b \in B(H)$, then the map

E. Størmer, *Positive Linear Maps of Operator Algebras*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-34369-8_5,

$$x \to a^*\phi\big(b^*xb\big)a = Ada \circ \phi \circ Adb(x) \in \mathscr{C}.$$

Note that if H is finite dimensional condition (ii) is by Theorem 4.1.8 equivalent to

(iii) If $\phi \in \mathscr{C}$ and $\alpha, \beta \in CP(H)$, the cone of completely positive maps, then $\alpha \circ \phi \circ \beta \in \mathscr{C}$.

Many well known cones are mapping cones. Clearly $P(H)$ and $CP(H)$ are mapping cones.

The cone $P_k(H)$ consisting of all k-positive maps in $P(H)$ is a mapping cone, since if $\phi \in P_k(H)$ and $a, b \in B(H)$ then

$$\iota_k \otimes Ada \circ \phi \circ Adb = Ad(1 \otimes a) \circ (\iota_k \otimes \phi) \circ Ad(1 \otimes b)$$

is positive, so $Ada \circ \phi \circ Adb \in P_k(H)$.

Some other classes are defined as follows.

Definition 5.1.2 For each $k \in \mathbb{N}$ let $SP_k(H)$ denote the closed convex cone generated by maps $AdV \in P(H)$ with $V \in B(H)$ having rank less than or equal to k.

A map $\phi \in SP_1(H)$ is called *super-positive* and a map $\phi \in SP_k(H)$, $k \geq 2$ is *k-super-positive*. Super-positive maps are also called *entanglement breaking* in the literature.

Lemma 5.1.3 *$SP_1(H)$ is generated by maps $a \to \omega(a)x$ with ω a normal state on $B(H)$ and $x \in B(H)^+$. In particular, if $\phi \in SP_1(H)$ and $\alpha, \beta \in P(H)$ then both $\alpha \circ \phi, \phi \circ \beta \in SP_1(H)$.*

Proof If $\phi = AdV$ with rank $V = 1$, let q be the range projection of V. Then q is the projection onto the 1-dimensional subspace spanned by a unit vector η. Thus

$$AdV(a) = V^*qaqV = V^*(a\eta, \eta)V = \omega_\eta(a)V^*V,$$

is of the form described in the lemma. Here ω_η is the vector state $\omega_\eta(a) = (a\eta, \eta)$.

Conversely, if $\phi(a) = \omega(a)x$ with ω a normal state on $B(H)$, then ω is a convex sum of vector states. We may therefore assume $\omega = \omega_\eta$ with η as above, and $q = [\eta]$. We have to approximate ϕ in the BW-topology by maps of the form $\sum AdV_i$ with V_i of rank 1. Let $a_1, \dots, a_n \in B(H)$. By weak approximation we may assume there is a finite dimensional projection $e \in B(H)$ such that $q, a_k, x \in eB(H)e$ for all k. Let (e_{ij}) be a complete set of matrix units for $eB(H)e$ with $e_{11} = q$. Then

$$\begin{aligned}\phi(a_k) = \omega_\eta(a_k)x &= x^{1/2}Tr(e_{11}a_ke_{11})x^{1/2}\\ &= x^{1/2}\sum_i e_{i1}a_ke_{1i}x^{1/2}\\ &= \sum\big(e_{1i}x^{1/2}\big)^*a_k\big(e_{1i}x^{1/2}\big),\end{aligned}$$

is of the form $\sum AdV_i$ with $V_i = e_{1i}x^{1/2}$ of rank 1. Thus $\phi \in SP_1(H)$.

Note that since the normal states are w^*-dense in the state space of $B(H)$ it suffices to consider normal states in the lemma. □

Since for each $a \neq 0$ in $B(H)^+$ there is a normal state ω such that $\omega(a) \neq 0$, it follows that $SP_1(H)$ is a mapping cone. Since rank $V \leq k$ implies rank $aVb \leq k$ for all $a, b \in B(H)$, and $SP_k(H) \supset SP_1(H)$, it is clear that $SP_k(H)$ is also a mapping cone. The name "entanglement braking" comes from the last statement of Lemma 5.1.3.

Another characterization of the super-positive maps is given in the next proposition. Recall that a linear functional ρ on $A \otimes B$ is said to be *separable* if it is of the form $\rho = \sum_i \omega_i \otimes \rho_i$ with ω_i and ρ_i positive linear functionals on A and B respectively.

Proposition 5.1.4 *Let $\phi \in P(H)$. Then ϕ is super-positive if and only if its dual functional $\widetilde{\phi}$ is a w^*-limit of separable functionals.*

Proof Since $\mathscr{T}(H)$ is weakly dense in $B(H)$, by Lemma 5.1.3 $SP_1(H)$ is the mapping cone generated by maps of the form $a \to \omega(a)x$ with ω a state on $B(H)$ and x a positive operator in $\mathscr{T}(H)$. Let ρ denote the positive functional, $\rho(b) = Tr(xb^t)$ on $B(H)$. Thus, if $\phi(a) = \omega(a)x$ then

$$\widetilde{\phi}(a \otimes b) = Tr\big(\omega(a)xb^t\big) = \omega(a)Tr\big(xb^t\big) = \omega \otimes \rho(a \otimes b),$$

and the proposition follows easily. □

Lemma 5.1.5 *If $\mathscr{C}$ is a mapping cone in $P(H)$ then $SP_1(H) \subset \mathscr{C} \subset P(H)$.*

Proof By definition $\mathscr{C} \subset P(H)$. By condition (i) in Definition 5.1.1 if e is a 1-dimensional projection in $B(H)$ there is $\phi \in \mathscr{C}$ such that $\phi(e) \neq 0$. Let ω be the pure state on $B(H)$ defined by $eae = \omega(a)e$. Then the map

$$a \to \phi(eae) = \omega(a)\phi(e)$$

belongs to $SP_1(H) \cap \mathscr{C}$. Since every vector state is of the form $\omega \circ AdU$ for a unitary operator U, and each normal state is a norm limit of convex combinations of vector states, and each positive operator is approximated in the weak topology by finite sums $\sum \lambda_i e_i$ with $\lambda_i > 0$ and e_i 1-dimensional projections, it follows that $SP_1(H) \subset \mathscr{C}$. □

In order to study positivity properties of maps relative to a mapping cone we need the following cones.

Definition 5.1.6 Let $\mathscr{C}$ be a mapping cone in $P(H)$, and let A be an operator system. Then $P(A, \mathscr{C})$ is defined by

$$P(A, \mathscr{C}) = \big\{x \in \big(A \widehat{\otimes} \mathscr{T}(H)\big)_{sa} : \iota \otimes \alpha(x) \geq 0 \text{ for all } \alpha \in \mathscr{C}\big\},$$

where ι is the identity map on A.

Lemma 5.1.7 *In the above notation $P(A,\mathscr{C})$ is a proper norm closed convex cone in $A\widehat{\otimes}\mathscr{T}(H)$ containing the cone $A^+\otimes\mathscr{T}(H)^+$.*

Proof Since $\|b\|\le\|b\|_1$ for all $b\in\mathscr{T}(H)$, if $\alpha\in P(H)$ and $\sum_i a_i\otimes b_i\in A\otimes\mathscr{T}(H)$, we have

$$\left\|\iota\otimes\alpha\left(\sum_i a_i\otimes b_i\right)\right\|=\left\|\sum a_i\otimes\alpha(b_i)\right\|\le\|\alpha\|\sum\|a_i\|\|b_i\|$$
$$\le\|\alpha\|\sum\|a_i\|\|b_i\|_1.$$

It follows that $\|\iota\otimes\alpha(x)\|\le\|\alpha\|\|x\|_\wedge$ for all $x\in A\widehat{\otimes}\mathscr{T}(H)$ where as before $\|x\|_\wedge$ is the projective norm of $x\in A\widehat{\otimes}\mathscr{T}(H)$. In particular $\iota\otimes\alpha$ is a bounded map of $A\widehat{\otimes}\mathscr{T}(H)$ into $A\otimes B(H)$, and so $P(A,\mathscr{C})$ is well defined and closed. Since it is trivially convex, it remains to show that it is proper. For this let $x\in P(A,\mathscr{C})$ be such that $\iota\otimes\alpha(x)=0$ for all $\alpha\in\mathscr{C}$. If $\omega\in A^*$ we have $\omega\otimes\alpha(x)=0$ for all $\alpha\in\mathscr{C}$. Since $SP_1(H)\subset\mathscr{C}$ by Lemma 5.1.5, $\omega\otimes\rho(x)=0$ for all states ρ on $B(H)$ and $\omega\in A^*$. Since these functionals span a w^*-dense subspace of $(A\widehat{\otimes}\mathscr{T}(H))^*$, $x=0$. In particular, if $x\in P(A,\mathscr{C})\cap(-P(A,\mathscr{C}))$ then

$$\iota\otimes\alpha(x)\in\big(A\otimes B(H)\big)^+\cap\big(-\big(A\otimes B(H)\big)^+\big)=\{0\}$$

for all $\alpha\in\mathscr{C}$. Thus $x=0$, and $P(A,\mathscr{C})$ is a proper cone. Since it is trivial that $(A\widehat{\otimes}\mathscr{T}(H))^+\supset A^+\otimes\mathscr{T}(H)^+$, the proof is complete. □

Lemma 5.1.8 *Let H be finite dimensional and $\mathscr{C}$ a mapping cone in $P(H)$. Then the linear isomorphisms of $B(H)$ onto itself belonging to $\mathscr{C}$ are norm dense in $\mathscr{C}$.*

Proof Let $n=\dim H$ and identify $B(H)$ with M_n. We first show there exists a linear isomorphism of $B(H)$ onto itself belonging to $SP_1(H)$. Since $\dim M_n=n^2$ each set of n^2+1 positive matrices in M_n is linearly dependent. Since span $M_n^+=M_n$ there exists a basis $\{a_{ij}:i,j=1,\dots,n\}$ for M_n with $a_{ij}\in M_n^+$. Similarly M_n^* has a basis consisting of n^2 states ω_{ij}, $i,j=1,\dots,n$. Then the linear map of M_n into itself defined by $a\to(\omega_{ij}(a))$ is a linear isomorphism, hence is in particular surjective. But then the map

$$\beta(a)=\sum_{ij}\omega_{ij}(a)a_{ij}$$

is a linear isomorphism of M_n onto itself belonging to $SP_1(H)$, see Lemma 5.1.3.

To complete the proof let $\varepsilon>0$ and $\alpha\in\mathscr{C}$, and let β be as above. Scaling β we may assume $\|\beta\|<\varepsilon/2$ and $\alpha+\beta\ne0$. If $\alpha(a)+\beta(a)=0$ for some $a\in M_n$ then $-1\in\mathrm{Spec}(\beta^{-1}\circ\alpha)$—the spectrum of $\beta^{-1}\circ\alpha$. Since $\mathrm{Spec}(\beta^{-1}\circ\alpha)$ is finite there exists $\lambda\in[\frac{1}{2},\frac{3}{2}]$ such that $-1\notin\lambda\,\mathrm{Spec}(\beta^{-1}\circ\alpha)=\mathrm{Spec}(\lambda\beta^{-1}\circ\alpha)$. Thus $\lambda\beta^{-1}\circ\alpha(a)\ne-a$ for all a, so that $\gamma=\alpha+\lambda^{-1}\beta$ is a linear isomorphism of M_n into itself, so onto by finite dimensionality, satisfying $\|\alpha-\gamma\|\le\lambda^{-1}\|\beta\|<\varepsilon$. Since $\beta\in SP_1(H)$, $\gamma\in\mathscr{C}$ by Lemma 5.1.5, completing the proof. □

We can now state the crucial positivity condition for maps with respect to mapping cones.

Definition 5.1.9 Let $\mathscr{C}$ be a mapping cone in $P(H)$ and A an operator system. Then a map $\phi \in B(A, H)$ is said to be *$\mathscr{C}$-positive* if its dual functional $\widetilde{\phi}$ is positive on the cone $P(A, \mathscr{C})$, or equivalently

$$\sum_i Tr\big(\phi(a_i)b_i^t\big) \geq 0 \quad \text{for all } \sum a_i \otimes b_i \in P(A, \mathscr{C}).$$

Since $P(A, \mathscr{C}) \supset A^+ \otimes \mathscr{T}(H)^+$ by Lemma 5.1.7 it is immediate from Lemma 4.2.2 that a $\mathscr{C}$-positive map is positive.

Proposition 5.1.10 *Let $\mathscr{C}$ be a mapping cone in $P(H)$ and A an operator system. Let $\phi \in B(A, H)$ be $\mathscr{C}$-positive, and let $V \in B(H)$, $\beta : A \to A$ be completely positive. Then $AdV \circ \phi \circ \beta$ is $\mathscr{C}$-positive.*

Proof For simplicity of notation let $\alpha = AdV$. Recall that $\alpha^t = t \circ \alpha \circ t$ with t the transpose map on $B(H)$, and α^* is the adjoint map defined by $Tr(\alpha(a)b) = Tr(a\alpha^*(b))$, $a \otimes b \in A \otimes \mathscr{T}(H)$. We first show $\alpha \circ \phi$ is $\mathscr{C}$-positive. Let $a \otimes b \in A \otimes \mathscr{T}(H)$. Then

$$\begin{aligned}\widetilde{\alpha \circ \phi}(a \otimes b) &= Tr\big(\alpha \circ \phi(a)b^t\big) = Tr\big(\phi(a)\alpha^*\big(b^t\big)\big)\\ &= Tr\big(\phi(a)\alpha^{*t}(b)^t\big)\\ &= \widetilde{\phi}\big(\iota \otimes \alpha^{*t}(a \otimes b)\big),\end{aligned}$$

so $\widetilde{\alpha \circ \phi} = \widetilde{\phi} \circ (\iota \otimes \alpha^{*t})$.

For each matrix a, $a^{*t*} = a^t$. Thus with $\alpha = AdV$, $\alpha^{*t} = AdV^{*t*} = AdV^t$ by Lemma 4.2.5 and Proposition 1.4.2. If $\psi \in \mathscr{C}$, then $\psi \circ \alpha^{*t} \in \mathscr{C}$ by definition of mapping cone. Thus if $x \in P(A, \mathscr{C})$ den

$$\iota \otimes \psi\big(\iota \otimes \alpha^{*t}\big)(x) = \iota \otimes \psi \circ \alpha^{*t}(x) \geq 0,$$

hence $\iota \otimes \alpha^{*t}(x) \in P(A, \mathscr{C})$, so by the above $\widetilde{\alpha \circ \phi}(x) \geq 0$. Thus $AdV \circ \phi$ is $\mathscr{C}$-positive.

We next show $\phi \circ \beta$ is $\mathscr{C}$-positive, where β is a completely positive map of A into itself. We then have

$$\widetilde{\phi \circ \beta}(a \otimes b) = \widetilde{\phi}\big(\beta(a) \otimes b\big) = \widetilde{\phi} \circ (\beta \otimes \iota)(a \otimes b),$$

so $\widetilde{\phi \circ \beta} = \widetilde{\phi} \circ (\beta \otimes \iota)$.

If $x \in P(A, \mathscr{C})$ and $\psi \in \mathscr{C}$ then

$$\iota \otimes \psi(\beta \otimes \iota)(x) = (\beta \otimes \iota) \circ (\iota \otimes \psi)(x) \geq 0,$$

since $\beta \otimes \iota$ is positive when β is completely positive. Thus $\beta \otimes \iota(x) \in P(A, \mathscr{C})$, so $\widetilde{\phi \circ \beta} = \widetilde{\phi}(\beta \otimes \iota(x)) \geq 0$, proving that $\phi \circ \beta$ is $\mathscr{C}$-positive. □

If dim $H < \infty$ then by Theorem 4.1.8 each completely positive map is a sum of maps of the form AdV. Thus we get

Corollary 5.1.11 *Let H be finite dimensional and $\mathscr{C}$ a mapping cone in $P(H)$. If $\phi \in B(A, H)$ is $\mathscr{C}$-positive then $\alpha \circ \phi \circ \beta$ is $\mathscr{C}$-positive for all completely positive maps $\alpha \in CP(H)$ and $\beta : A \to A$.*

In Proposition 5.1.10 we had to restrict attention to maps AdV, or finite sums of such in order to conclude that $AdV \circ \phi \circ \beta$ was $\mathscr{C}$-positive. As can be seen from the proof, the reason for this is that if $\alpha \in CP(H)$ we cannot conclude that α^* is well defined on $B(H)$, as it is only defined on $\mathscr{T}(H)$. To avoid technicalities we therefore state the following definition for finite dimensional Hilbert spaces.

Definition 5.1.12 Let H be finite dimensional. A mapping cone $\mathscr{C}$ in $P(H)$ is said to be *symmetric* if $\phi \in \mathscr{C}$ implies both ϕ^* and ϕ^t belong to $\mathscr{C}$.

It is clear that the cones $P_k(H)$, $SP_k(H)$, $CP(H)$, $P(H)$ are all symmetric mapping cones. We next show that if $\mathscr{C}$ is symmetric the $\mathscr{C}$-positive maps have a more intuitive interpretation than in Definition 5.1.9.

Theorem 5.1.13 *Let A be an operator system and H a finite dimensional Hilbert space. Let $\mathscr{C}$ be a symmetric mapping cone in $P(H)$ and denote by $C_{\mathscr{C}}$ the BW-closed cone in $B(A, H)$ generated by all maps of the form $\alpha \circ \psi$ with $\alpha \in \mathscr{C}$ and $\psi : A \to B(H)$ completely positive. Then a map $\phi \in B(A, H)$ is $\mathscr{C}$-positive if and only if $\phi \in C_{\mathscr{C}}$.*

We first prove a lemma.

Lemma 5.1.14 *Let $\mathscr{C}$ be a symmetric mapping cone in $P(H)$. Suppose $\alpha \in \mathscr{C}$ is a linear isomorphism of $B(H)$ onto itself. Let A be an operator system and let*

$$P_\alpha = \left\{x \in A \otimes B(H) : \iota \otimes \alpha^{*t}(x) \geq 0\right\}.$$

If $\phi \in B(A, H)$ is such that $\widetilde{\phi}$ is positive on P_α, then there exists $\psi \in B(A, H)$ which is completely positive such that $\phi = \alpha \circ \psi$.

Proof Let $\psi = \alpha^{-1} \circ \phi$. Then $\psi \in B(A, H)$. The proof is complete as soon as we can show ψ is completely positive. For this let $x = \sum a_i \otimes b_i \in (A \otimes B(H))^+$. Then, since α^{*t} is also invertible,

$$\iota \otimes \alpha^{*t}\left(\sum a_i \otimes \left(\alpha^{*t}\right)^{-1}(b_i)\right) = x \geq 0, \tag{5.1}$$

so that $\sum a_i \otimes (\alpha^{*t})^{-1}(b_i) \in P_\alpha$. Since $\widetilde{\phi}$ is positive on P_α we have

$$\begin{aligned}\widetilde{\psi}(x) &= \sum_i Tr\big(\psi(a_i)b_i^t\big)\\ &= \sum Tr\big(\phi(a_i)\big(\alpha^{-1}\big)^*\big(b_i^t\big)\big)\\ &= \sum Tr\big(\phi(a_i)\big(\big(\alpha^{-1}\big)^{*t}(b_i)\big)^t\big)\\ &= \widetilde{\phi}\Big(\sum a_i \otimes \big(\alpha^{-1}\big)^{*t}(b_i)\Big).\end{aligned}$$

For a map α we have $(\alpha^{-1})^* = (\alpha^*)^{-1}$, because this holds for invertible operators on a Hilbert space. Since $t^{-1} = t$ we get

$$\big(\alpha^{-1}\big)^{*t} = \big(\big(\alpha^*\big)^{-1}\big)^t = t \circ \big(\alpha^*\big)^{-1} \circ t = \big(t \circ \alpha^* \circ t\big)^{-1} = \big(\alpha^{*t}\big)^{-1}.$$

Therefore by (5.1) $\widetilde{\psi}$ is positive, hence by Theorem 4.2.7, ψ is completely positive. Since $\phi = \alpha \circ \psi$ the proof is complete. □

Proof of Theorem 5.1.13 Suppose $\phi \in C_{\mathscr{C}}$. Now, sums of $\mathscr{C}$-positive maps are $\mathscr{C}$-positive, and if (ψ_α) is a net of $\mathscr{C}$-positive maps converging to $\psi \in P(H)$ in the BW-topology, then as remarked in the beginning of Sect. 5.1, $\widetilde{\psi_\alpha} \to \widetilde{\psi}$ in the w^*-topology. Thus we may, in order to show ϕ is $\mathscr{C}$-positive, assume $\phi = \alpha \circ \psi$ with $\alpha \in \mathscr{C}$, ψ is completely positive, $\psi \in B(A, H)$. Let $x = \sum a_i \otimes b_i \in P(A, \mathscr{C})$. Since $\mathscr{C}$ is symmetric $\sum a_i \otimes \alpha^{*t}(b_i) \in (A \otimes B(H))^+$, hence

$$\begin{aligned}\widetilde{\phi(x)} &= \sum Tr\big(\alpha \circ \psi(a_i)b_i^t\big) = \sum Tr\big(\psi(a_i)\alpha^{*t}(b_i)^t\big)\\ &= \widetilde{\psi}\Big(\sum a_i \otimes \alpha^{*t}(b_i)\Big) \geq 0,\end{aligned}$$

because $\widetilde{\psi}$ is positive by Theorem 4.2.7. Thus ϕ is $\mathscr{C}$-positive.

Assume ϕ is $\mathscr{C}$-positive. By Lemma 5.1.8 $C_{\mathscr{C}}$ is generated by maps α with α a linear isomorphism of $B(H)$ onto itself belonging to $\mathscr{C}$. Let $\alpha \in \mathscr{C}$ be a linear isomorphism, and let P_α be as in Lemma 5.1.14. Then

$$P(A, \mathscr{C}) = \bigcap_\alpha P_\alpha,$$

the intersection being taken over all linear isomorphisms in $\mathscr{C}$. The dual cone of $P(A, \mathscr{C})$ in $(A \otimes B(H))^*$ is the cone spanned by all dual cones of P_α's. Hence it is the w^*-closure of all finite sums $\sum \widetilde{\phi}_\alpha$ with $\widetilde{\phi}_\alpha$ positive on P_α.

By Lemma 5.1.14, $\phi_\alpha = \alpha \circ \psi_\alpha$, ψ_α completely positive map of A into $B(H)$. For such a sum we have

$$\sum \widetilde{\phi}_\alpha = \sum (\alpha \circ \psi_\alpha)\widetilde{\ } = \Big(\sum \alpha \circ \psi_\alpha\Big)\widetilde{\ }.$$

Since our given map is $\mathscr{C}$-positive there exists a bounded net (ϕ_γ) of the form $\widetilde{\phi}_\gamma = \sum \widetilde{\phi}_\alpha$ as above such that $\widetilde{\phi}_\gamma \to \widetilde{\phi}$ in the w^*-topology. Hence $\widetilde{\phi}$ is a w^*-limit of maps $(\sum \alpha \circ \psi_\alpha)\widetilde{}$, hence ϕ is a BW-limit of maps $\sum \alpha \circ \psi_\alpha$ with α a linear isomorphism in $\mathscr{C}$ and $\psi_\alpha \in B(A, H)$ completely positive. But that means $\phi \in C_{\mathscr{C}}$. □

Remark 5.1.15 Since the identity map is completely positive it is obvious that $P(A, CP(H)) = (A \widehat{\otimes} \mathscr{T}(H))^+$ for an operator system A. Also, it is immediate by Lemma 5.1.3 that

$$P\big(A, SP_1(H)\big) \\ = \big\{x \in A \widehat{\otimes} \mathscr{T}(H) : \rho \otimes \omega(x) \geq 0 \text{ for all states } \rho \text{ of } A \text{ and } \omega \text{ of } B(H)\big\}$$

because $\iota \otimes \omega(x) \geq 0$ for all states ω of $B(H)$ if and only if $\rho \otimes \omega(x) \geq 0$ for all states ρ of A.

We shall see later, Remark 7.1.4, that if H is finite-dimensional, then $P(B(K), P(H)) = B(K)^+ \otimes B(H)^+$.

Proposition 5.1.16 *Let H be finite dimensional. Then $P(B(H), P(H)) = B(H)^+ \otimes B(H)^+$. In particular, if $h \in B(H)^+$ is the density operator for a state ρ, then $\iota \otimes \alpha(h) \geq 0$ for all $\alpha \in P(H)$ if and only if $h \in B(H)^+ \otimes B(H)^+$, i.e. if and only if ρ is a separable state.*

Proof By Theorem 5.1.13 a map $\phi : B(H) \to B(H)$ is $P(H)$-positive if and only if $\phi \in P(H)$, which by Lemma 4.2.2 is equivalent to $\widetilde{\phi}$ being positive on $B(H)^+ \otimes B(H)^+$. Since $P(B(H), P(H)) \supset B(H)^+ \otimes B(H)^+$, and a linear functional is positive on the smallest cone if and only if it is positive on the largest, it follows from the Hahn-Banach theorem for cones, that $P(B(H), P(H)) = B(H)^+ \otimes B(H)^+$. The last statement is obvious. □

5.2 The Extension Theorem

In this section we prove the analogue of the Hahn-Banach theorem for $\mathscr{C}$-positive maps. For this we need two lemmas.

Lemma 5.2.1 *Let H be a Hilbert space and $\mathscr{C}$ a mapping cone in $P(H)$. Let A be an operator system and e a finite dimensional projection in $B(H)$. Then $1 \otimes e$ is an interior point of $(1 \otimes e)P(A, \mathscr{C})(1 \otimes e)$.*

Proof Since each map in $\mathscr{C}$ is positive, it is clear that $1 \otimes e \in P(A, \mathscr{C})$. Let $\alpha \in \mathscr{C}$. By Lemma 5.1.8 we can add a linear isometry in $\mathscr{C}$ of $eB(H)e$ onto itself of small norm to α, so we may assume the range projection of $Ade \circ \alpha(1)$ equals e. Since e is finite dimensional, there is $a \in (eB(H)e)^+$ such that $a(Ade \circ \alpha(1))a = a\alpha(1)a = e$. Again, since e is finite dimensional,

$$M = \sup\bigl\{\|\iota \otimes \gamma\| : \gamma = Ade \circ \gamma',\ \gamma' \in \mathscr{C},\ \gamma'(1) = e\bigr\}$$

is finite. Let $\beta = Ada \circ \alpha$. Then $\beta \in \mathscr{C}$, and $\beta(1) = e$, so $\|\iota \otimes \beta\| \leq M$. Let $x \in A \otimes \mathscr{T}(H)$ be self-adjoint and $\|x\| < 1/M$. Then

$$1 \otimes e + \iota \otimes \beta(x) \geq 0.$$

Hence

$$\begin{aligned}\iota \otimes \alpha(1 \otimes 1 + x) &= \bigl(\iota \otimes Ada^{-1}\bigr) \circ (\iota \otimes \beta)(1 \otimes 1 + x)\\ &= \bigl(\iota \otimes Ada^{-1}\bigr)\bigl(1 \otimes e + \iota \otimes \beta(x)\bigr) \geq 0.\end{aligned}$$

Since α was an arbitrary map in $\mathscr{C}$, $1 \otimes 1 + x \in P(A, \mathscr{C})$. But then

$$1 \otimes e + (1 \otimes e)x(1 \otimes e) \in (1 \otimes e)P(A, \mathscr{C})(1 \otimes e),$$

so $1 \otimes e$ is an interior point of $(1 \otimes e)P(A, \mathscr{C})(1 \otimes e)$. □

Lemma 5.2.2 *Let $A \subset B$ be operator systems on the same Hilbert space. Let $\mathscr{C}$ be a mapping cone in $P(H)$ and e a finite dimensional projection in $B(H)$. Then*

$$(1 \otimes e)P(A, \mathscr{C})(1 \otimes e) = (1 \otimes e)P(B, \mathscr{C})(1 \otimes e) \cap A\widehat{\otimes}\mathscr{T}(H).$$

Proof If $x \in P(A, \mathscr{C})$ then for all $\alpha \in \mathscr{C}$,

$$\iota \otimes \alpha(x) \in \bigl(A \otimes B(H)\bigr)^+ \subset \bigl(B \otimes B(H)\bigr)^+,$$

hence $x \in P(B, \mathscr{C})$. Thus $P(A, \mathscr{C}) \subset P(B, \mathscr{C}) \cap A\widehat{\otimes}\mathscr{T}(H)$, and therefore $(1 \otimes e)P(A, \mathscr{C})(1 \otimes e) \subset (1 \otimes e)P(B, \mathscr{C})(1 \otimes e) \cap A\widehat{\otimes}\mathscr{T}(H)$.

Conversely, if $x \in P(B, \mathscr{C}) \cap A\widehat{\otimes}\mathscr{T}(H)$, then $\iota \otimes \alpha(x) \geq 0$ for all $\alpha \in \mathscr{C}$. Since $(1 \otimes e)x(1 \otimes e) \in (1 \otimes e)P(B, \mathscr{C})(1 \otimes e) \cap A\widehat{\otimes}\mathscr{T}(H)$, it follows that

$$(1 \otimes e)x(1 \otimes e) \in (1 \otimes e)P(A, \mathscr{C})(1 \otimes e).$$ □

We are now in position to prove the extension theorem for $\mathscr{C}$-positive maps.

Theorem 5.2.3 *Let $A \subset B$ be operator systems on the same Hilbert space, and let $\mathscr{C}$ be a mapping cone in $P(H)$. Then each $\mathscr{C}$-positive map $\phi \in B(A, H)$ has a $\mathscr{C}$-positive extension $\psi \in B(B, H)$.*

Proof Let e be a finite dimensional projection in $B(H)$ such that $e = e^t$. Let ϕ_e denote the map $Ade \circ \phi \in B(A, eH)$, where $B(eH)$ is identified with $eB(H)e$. By Lemma 5.2.1 $1 \otimes e$ is an interior point of $(1 \otimes e)P(A, \mathscr{C})(1 \otimes e)$. Since the dual functional $\widetilde{\phi}$ of ϕ is positive on $P(A, \mathscr{C})$, $\widetilde{\phi}$ is positive on $(1 \otimes e)P(A, \mathscr{C})(1 \otimes e)$ hence so is $\widetilde{\phi}_e = \widetilde{\phi} \circ (\iota \otimes Ade)$, as is easily shown. By Lemma 5.2.2 it follows from a theorem of Krein, see Appendix A.3.1 that $\widetilde{\phi}_e$ has an extension $\widetilde{\psi_e}$ in $B(B, \mathscr{T}(H))^*$ which is positive on $(1 \otimes e)P(B, \mathscr{C})(1 \otimes e)$, hence $\widetilde{\psi}_e$ is the dual of a map ψ_e in $B(B, eH)$. Since $a \otimes ebe \in (1 \otimes e)P(B, \mathscr{C})(1 \otimes e)$ for $a \in A^+$, $ebe \in B(eH)$, ψ_e is a positive map by Proposition 4.1.11. Note that if H is finite dimensional and we let $e = 1$, then ψ_e is the desired extension of ϕ.

Since $1 \in A$, A being an operator system, and ψ_e is positive, by Theorem 1.3.3

$$\left\|\psi_e\right\| = \left\|\psi_e(1)\right\| = \left\|\phi_e(1)\right\| \leq \left\|\phi(1)\right\| = \|\phi\|. \tag{5.2}$$

To complete the proof let (e_γ) be an increasing net of finite dimensional projections in $B(H)$ converging strongly to 1, and $e_\gamma = e_\gamma^t$. For each γ let by the above ψ_γ be an extension of ϕ_{e_γ} to $B(B, H)$ such that $\widetilde{\psi}_\gamma$ is positive on $(1 \otimes e_\gamma)P(B, \mathscr{C})(1 \otimes e_\gamma)$. By (5.2) the net (ψ_γ) is uniformly bounded, so by compactness of the unit ball in $B(B, H)$, (ψ_γ) has a BW-limit point $\psi \in B(B, H)$. Let a subnet (ψ_α) converge to ψ. Since each ψ_α is an extension of ϕ_{e_γ}, so is ψ. Hence ψ is an extension of ϕ. To show ψ is $\mathscr{C}$-positive let $x = \sum_{i=1}^n a_i \otimes b_i \in P(B, \mathscr{C})$. Then

$$(1 \otimes e_\alpha)x(1 \otimes e_\alpha) \in (1 \otimes e_\alpha)P(B, \mathscr{C})(1 \otimes e_\alpha) \quad \text{for all } \alpha,$$

hence, since $e_\alpha = e_\alpha^t$, and $b_i \in \mathscr{T}(H)$, so $e_\alpha b_i^t e_\alpha$ is close to b_i^t in norm for large α,

$$\begin{aligned}
\widetilde{\psi}(x) = \widetilde{\psi}\Big(\sum a_i \otimes b_i\Big) &= \sum_i Tr\big(\psi(a_i)b_i^t\big) \\
&= \lim_\alpha \sum_i Tr\big(\psi_\alpha(a_i)b_i^t\big) \\
&= \lim_\alpha \sum_i Tr\big(\psi_\alpha(a_i)e_\alpha b_i^t e_\alpha\big) \\
&= \lim_\alpha \widetilde{\psi_\alpha}\big((1 \otimes e_\alpha)x(1 \otimes e_\alpha)\big) \\
&\geq 0.
\end{aligned}$$

Since operators like x are norm dense in $P(B, \mathscr{C})$, $\widetilde{\psi}$ is positive on $P(B, \mathscr{C})$, hence ψ is $\mathscr{C}$-positive. □

As a consequence of Theorem 5.2.3 it suffices in many cases to study maps from $B(K)$ into $B(H)$ rather than maps from operator systems into $B(H)$. The proof we shall now give of Arveson's extension theorem for completely positive maps is an example of this.

Corollary 5.2.4 *Let $A \subset B(K)$ be an operator system. Let $\phi \in B(A, H)$ be a completely positive map. Then ϕ has a completely positive extension $\psi \in B(B(K), H)$.*

Proof By Theorem 4.2.7 ϕ is completely positive if and only if $\widetilde{\phi}$ is positive on $A\widehat{\otimes}\mathscr{T}(H)$, hence on $P(A, CP(H))$, by Remark 5.1.15. Thus ϕ is completely positive if and only if ϕ is $CP(H)$-positive, hence the corollary follows from Theorem 5.2.3. □

If $A \subset B(H)$, the $\mathscr{C}$-positive maps of A into $B(H)$ with H finite dimensional have a very nice form when $\mathscr{C}$ is symmetric.

Theorem 5.2.5 *Let A be an operator system contained in $B(H)$ with H finite dimensional, and let $\mathscr{C}$ be a symmetric mapping cone in $P(H)$. Then a map of A into $B(H)$ is $\mathscr{C}$-positive if and only if it is the restriction of a map in $\mathscr{C}$ to A.*

Proof Let $\phi \in B(A, H)$ be $\mathscr{C}$-positive. By Theorem 5.2.3 ϕ has a $\mathscr{C}$-positive extension to a map in $P(H)$. We may thus replace A by $B(H)$. Let $C_{\mathscr{C}}$ denote the BW-closed cone in $P(H)$ generated by maps of the form $\alpha \circ \psi$ with $\alpha \in \mathscr{C}$, $\psi \in CP(H)$. By Theorem 5.1.13 $\phi \in C_{\mathscr{C}}$. By Proposition 5.1.10, each $\alpha \circ \psi \in \mathscr{C}$, hence ϕ, being a limit of such maps, belongs to $\mathscr{C}$. This shows that the ϕ we started with is the restriction to A of a map in $\mathscr{C}$.

Conversely, if $\alpha \in \mathscr{C}$ then $\alpha \in C_{\mathscr{C}}$, hence is $\mathscr{C}$-positive by Theorem 5.1.13. Thus the restriction to A is $\mathscr{C}$-positive. □

5.3 Notes

Most of the results in Chap. 5 have been taken from [78], but not all. Proposition 5.1.4 is due to P. Horodecki, P.W. Shor, and M.B. Ruskai [26], with a different proof, see also [79]. Proposition 5.1.16 is due to M., P., and R. Horodecki [25]. The Arveson Extension Theorem, Corollary 5.2.4 was shown by Arveson in [1] and has been very important in the study of completely positive maps.

Chapter 6
Dual Cones

If C is a closed convex cone in a Hilbert space H, its dual cone is defined as the cone

$$C^\circ = \{\xi \in H : \langle \xi, \eta \rangle \geq 0 \text{ for all } \eta \in \mathbb{C}\}$$

If K and H are finite dimensional Hilbert spaces we shall study dual cones in $B(B(K), H)$ with respect to the Hilbert-Schmidt structure. Because of the Extension Theorem 5.2.3 we shall concentrate on $B(B(K), H)$ rather than $B(A, H)$ as we did previously.

The chapter is divided into three sections. In Sect. 6.1 we develop the basic theory for the dual cone of the cone of $\mathscr{C}$-positive maps. In Sect. 6.2 we describe the dual cone for the main mapping cones. These results are used in Sect. 6.3 to show that all positive maps of M_2 into itself are decomposable. Finally, in Sect. 6.4 we consider tensor products of positive maps.

6.1 Basic Results

Throughout this section K and H are finite dimensional. $B(B(K), H)$ denotes the linear maps of $B(K)$ into $B(H)$.

Definition 6.1.1 Let $S \subset B(B(K), H)$ be a closed convex cone. Then its *dual cone* S° is defined as

$$S^\circ = \{\phi \in B\big(B(K), H\big) : Tr(C_\phi C_\psi) \geq 0 \text{ for all } \psi \in S\},$$

where as before, C_ϕ and C_ψ are the Choi matrices for ϕ and ψ. Here Tr denotes the usual trace on $B(K \otimes H)$.

We shall mainly study dual cones for mapping cones and $\mathscr{C}$-positive maps. Note that in the Hilbert space case considered above, it is well known that $C^{\circ\circ} = C$. Thus we get the same result for S as above.

E. Størmer, *Positive Linear Maps of Operator Algebras*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-34369-8_6,

Lemma 6.1.2 *Let $S \subset B(B(K), H)^+$, the positive maps of $B(K)$ into $B(H)$, be a closed convex cone. Then $S^{\circ\circ} = S$.*

Our first result on dual cones shows that the dual cone of a mapping cone has similar properties. In this case $K = H$.

Theorem 6.1.3 *Let $\mathscr{C}$ be a mapping cone in $P(H)$. Then its dual cone $\mathscr{C}^\circ$ is a mapping cone. Furthermore, if $\mathscr{C}$ is symmetric, so is $\mathscr{C}^\circ$.*

Proof We first show $\mathscr{C}^\circ$ is a mapping cone. By Lemma 5.1.5 $\mathscr{C} \supset SP_1(H)$, the super-positive maps in $P(H)$. By Proposition 4.1.3 the Choi matrix for a super-positive map is a sum $\sum_i a_i \otimes b_i \in B(H)^+ \otimes B(H)^+$. Thus by Lemma 4.1.10 a map ϕ is positive if and only if it is in the dual cone of $SP_1(H)$. Since $\mathscr{C}^\circ \subset SP_1(H)^\circ$ it follows that every map in $\mathscr{C}^\circ$ is positive.

Now let $\alpha \in CP(H)$, the completely positive maps in $P(H)$, $\psi \in \mathscr{C}^\circ$, and $\phi \in \mathscr{C}$. Then by Proposition 1.4.3 $\alpha^* \in CP(H)$, so $\alpha^* \circ \phi \in \mathscr{C}$. Hence

$$Tr(C_\phi C_{\alpha\circ\psi}) = Tr\big(C_\phi(\iota \otimes \alpha)(C_\psi)\big) = Tr\big(\iota \otimes \alpha^*(C_\phi)C_\psi\big) = Tr(C_{\alpha^*\circ\phi}C_\psi) \geq 0.$$

It follows that $\alpha \circ \psi \in \mathscr{C}^\circ$.

By Lemma 4.1.10, if $\xi_1, \ldots, \xi_n$ is an orthonormal basis for H and J the conjugation on $H \otimes H$ given by $Jz\xi_i \otimes \xi_j = \sum \bar{z}\xi_j \otimes \xi_i$, then $C_{\alpha^*} = JC_\alpha J$ for $\alpha \in P(H)$. The map $a \to Ja^*J$ is an anti-automorphism of order 2 of $B(H \otimes H)$, so by uniqueness of the trace, $Tr(Ja^*J) = Tr(a)$ for all $a \in B(H \otimes H)$. Thus if $\phi \in \mathscr{C}, \psi \in \mathscr{C}^\circ, \alpha \in CP(H)$, we have

$$\begin{aligned}
Tr(C_\phi C_{\psi\circ\alpha}) &= Tr\big(C_\phi(\iota \otimes \psi)(C_\alpha)\big)\\
&= Tr(C_{\psi^*\circ\phi}C_\alpha)\\
&= Tr(JC_{\phi^*\circ\psi}JC_\alpha)\\
&= Tr(C_{\phi^*\circ\psi}JC_\alpha J)\\
&= Tr\big(\iota \otimes \phi^*(C_\psi)C_{\alpha^*}\big)\\
&= Tr(C_\psi C_{\phi\circ\alpha^*})\\
&\geq 0,
\end{aligned}$$

since $\phi \circ \alpha^* \in \mathscr{C}$. Thus $\psi \circ \alpha \in \mathscr{C}^\circ$, so $\mathscr{C}^\circ$ is a mapping cone.

Assume $\mathscr{C}$ is a symmetric mapping cone. Let $\psi \in \mathscr{C}^\circ$. We have to show ψ^t and $\psi^* \in \mathscr{C}^\circ$. Let $\phi \in \mathscr{C}$. Then, since $C_{\phi^t} = C_\phi^t$ by Lemma 4.2.5,

$$0 \leq Tr(C_\psi C_\phi) = Tr(C_{\psi^{tt}}C_\phi) = Tr\big(C_{\psi^t}^t C_\phi\big) = Tr\big(C_{\psi^t}C_\phi^t\big) = Tr(C_{\psi^t}C_{\phi^t}).$$

Since $\phi \in \mathscr{C}$ if and only if $\phi^t \in \mathscr{C}$, it follows that $\psi^t \in \mathscr{C}^\circ$. Similarly we have by Lemma 4.1.10

$$Tr(C_{\psi^*}C_\phi) = Tr(JC_\psi JC_\phi) = Tr(C_\psi JC_\phi J) = Tr(C_\psi C_{\phi^*}) \geq 0.$$

So $\psi^* \in \mathscr{C}^\circ$, since $\phi^* \in \mathscr{C}$ if and only if $\phi \in \mathscr{C}$. □

Notation 6.1.4 Let $\mathscr{C}$ be a mapping cone in $P(H)$. We denote by $P_{\mathscr{C}}(K)$ the closed cone in $B(B(K), H)^+$ of $\mathscr{C}$-positive maps of $B(K)$ into $B(H)$.

Remark 6.1.5 Note that if $K = H$ then by Theorem 5.2.5 if $\mathscr{C}$ is symmetric, then $P_{\mathscr{C}}(H) = \mathscr{C}$. Thus by Theorem 6.1.3, $P_{\mathscr{C}}(H)^\circ = \mathscr{C}^\circ = P_{\mathscr{C}^\circ}(H)$.

If $K \neq H$ the situation is more complicated.

Theorem 6.1.6 *Let $\mathscr{C}$ be a symmetric mapping cone in $P(H)$ and $\phi \in B(B(K), H)$ a positive map. Then the following conditions are equivalent.*

(i) $\phi \in P_{\mathscr{C}}(K)^\circ$.
(ii) $C_\phi \in P(B(K), \mathscr{C})$, *i.e.* $\iota \otimes \alpha(C_\phi) \geq 0$ *for all* $a \in \mathscr{C}$.
(iii) $\widetilde{\phi} \circ (\iota \otimes \alpha) \geq 0$ *for all* $\alpha \in \mathscr{C}$.
(iv) $\alpha \circ \phi$ *is completely positive for all* $\alpha \in \mathscr{C}$.

Proof We shall prove the equivalences (i) ⇔ (ii) ⇒ (iii) ⇒ (i), and (iii) ⇔ (iv).

(i) ⇔ (ii). Let (e_{ij}) be a complete set of matrix units for $B(K)$, and let $p = \sum_{ij} e_{ij} \otimes e_{ij}$, so by definition $C_\phi = \sum e_{ij} \otimes \phi(e_{ij}) = \iota \otimes \phi(p)$. By Theorem 5.1.13 $P_{\mathscr{C}}(K)$ is generated by maps of the form $\alpha \circ \psi$ with $\alpha \in \mathscr{C}$, $\psi \in B(B(K), H)$ completely positive. We thus have

$$\begin{aligned} \phi \in P_{\mathscr{C}}(K)^\circ &\Leftrightarrow Tr(C_\phi C_{\alpha\circ\psi}) \geq 0 \quad \text{for all } \alpha, \psi \text{ as above} \\ &\Leftrightarrow Tr\big(\iota \otimes \alpha^*(C_\phi)C_\psi\big) \geq 0 \quad \text{for all } \alpha, \psi \\ &\Leftrightarrow \iota \otimes \alpha^*(C_\phi) \geq 0, \end{aligned}$$

because by Theorem 4.1.8 $B(K \otimes H)^+ = \{C_\psi : \psi \in B(B(K), H) \text{ completely positive}\}$. Thus $\phi \in P_{\mathscr{C}}(K)^\circ$ if and only if $\iota \otimes \alpha(C_\phi) \geq 0$ for all $\alpha \in \mathscr{C}$, because $\mathscr{C}$ is symmetric, hence if and only if $C_\phi \in P(B(K), \mathscr{C})$.

(ii) ⇒ (iii). We have for $\alpha \in \mathscr{C}$

$$\begin{aligned} \iota \otimes \alpha^t(C_\phi) &= (t \otimes t) \circ (\iota \otimes \alpha) \circ (t \otimes t)(C_\phi) \\ &= (t \otimes t) \circ (\iota \otimes \alpha)\big(C_\phi^t\big). \end{aligned}$$

Since $\alpha^t \in \mathscr{C}$ and $t \otimes t$ is an anti-automorphism of $B(K \otimes H)$, $\iota \otimes \alpha(C_\phi^t) \geq 0$ for all $\alpha \in \mathscr{C}$, by (ii). By Lemma 4.2.3, if $x \in B(K \otimes H)^+$,

$$\widetilde{\phi} \circ (\iota \otimes \alpha)(x) = Tr\big(C_\phi^t(\iota \otimes \alpha)(x)\big) = Tr\big(\iota \otimes \alpha^*\big(C_\phi^t\big)x\big) \geq 0,$$

since $\alpha^* \in \mathscr{C}$. Thus (iii) follows.

(iii) $\Rightarrow$ (i). If $\widetilde{\phi} \circ (\iota \otimes \alpha)$ is positive, and p is as in the first paragraph of the proof,

$$0 \leq \widetilde{\phi} \circ (\iota \otimes \alpha)(p) = Tr\big(C_\phi^t C_\alpha\big) = Tr(C_\phi C_{\alpha^t})$$

for all $\alpha \in \mathscr{C}$. Since $\alpha \in \mathscr{C}$ if and only if $\alpha^t \in \mathscr{C}$, since $\mathscr{C}$ is symmetric, $\phi \in P_{\mathscr{C}}(K)^\circ$.

(iii) $\Leftrightarrow$ (iv). By the computations in the proof of (ii) $\Leftrightarrow$ (iii) $\widetilde{\phi} \circ (\iota \otimes \alpha)$ is positive for all $\alpha \in \mathscr{C}$ if and only if $C_{\alpha\circ\phi} = \iota \otimes \alpha(C_\phi) \geq 0$ for all $\alpha \in \mathscr{C}$ if and only if $\alpha \circ \phi$ is completely positive by Theorem 4.1.8, for all $\alpha \in \mathscr{C}$. □

Theorem 6.1.6 gave conditions for a map to belong to $P_{\mathscr{C}}(K)^\circ$ in terms of properties of its Choi matrix, its dual functional and composition with maps in $\mathscr{C}$. We now show that a map ϕ belongs to $P_{\mathscr{C}}(K)^\circ$ if and only if ϕ is $\mathscr{C}^\circ$-positive.

Theorem 6.1.7 *Let $\mathscr{C}$ be a symmetric mapping cone in $P(H)$. Then*

$$P_{\mathscr{C}}(K)^\circ = P_{\mathscr{C}^\circ}(K).$$

We divide the proof into two lemmas.

Lemma 6.1.8 *Let $K^{\mathscr{C}}$ denote the closed convex cone generated by the cones*

$$\iota \otimes \alpha\big(B(K \otimes H)^+\big), \quad \alpha \in \mathscr{C}.$$

Then

$$\begin{aligned}\mathscr{C}^\circ &= \big\{\beta \in P(H) : \iota \otimes \beta(x) \geq 0 \textit{ for all } x \in K^{\mathscr{C}}\big\}\\ &= \big\{\beta \in P(H) : \beta \circ \alpha \in CP(H) \textit{ for all } \alpha \in \mathscr{C}\big\}.\end{aligned}$$

Proof Let $\beta \in P(H)$. Then we have

$$\begin{aligned}&\iota \otimes \beta(x) \geq 0 \quad \text{for all } x \in K^{\mathscr{C}}\\ &\Leftrightarrow \quad (\iota \otimes \beta) \circ (\iota \otimes \alpha) \geq 0 \quad \text{for all } \alpha \in \mathscr{C}\\ &\Leftrightarrow \quad \iota \otimes \beta \circ \alpha \geq 0 \quad \text{for all } \alpha \in \mathscr{C}\\ &\Leftrightarrow \quad \beta \circ \alpha \in CP(H) \quad \text{for all } \alpha \in \mathscr{C}\\ &\Leftrightarrow \quad \alpha^* \circ \beta^* = (\beta \circ \alpha)^* \in CP(H) \quad \text{for all } \alpha \in \mathscr{C}\\ &\Leftrightarrow \quad \beta^* \in \mathscr{C}^\circ \quad \text{by Theorem 6.1.6 since } \mathscr{C} \text{ is symmetric.}\\ &\Leftrightarrow \quad \beta \in \mathscr{C}^\circ \quad \text{by Theorem 6.1.3,}\end{aligned}$$

proving the lemma. □

Lemma 6.1.9 $K^{\mathscr{C}} = P(B(K), \mathscr{C}^\circ)$.

Proof If $x \in K^{\mathscr{C}}$ then by Lemma 6.1.8 $\iota \otimes \beta(x) \geq 0$ for all $\beta \in \mathscr{C}^\circ$, hence $K^{\mathscr{C}} \subset P(B(K), \mathscr{C}^\circ)$. By the proof of Lemma 5.2.1, $1 \otimes 1$ is an interior point of $K^{\mathscr{C}}$. If the inclusion is strict there exists $y_0 \in P(B(K), \mathscr{C}^\circ)$ such that $y_0 \notin K^{\mathscr{C}}$. Thus by the Krein theorem, see Appendix A.3.1 there exists a linear functional $\widetilde{\phi}$ on $B(K \otimes H)$ with $\phi \in B(B(K), H)$ such that $\widetilde{\phi}$ is positive on $K^{\mathscr{C}}$, while $\widetilde{\phi}(y_0) < 0$. By Theorem 6.1.6(iii), since $\widetilde{\phi}$ is positive on $K^{\mathscr{C}}$, $\phi \in P_{\mathscr{C}}(K)^\circ$. Write y_0 in the form $y_0 = \sum_i a_i \otimes b_i$, $a_i \in B(K)$, $b_i \in B(H)$, and let $\pi : B(K) \otimes B(K) \to B(K)$ be given by $\pi(a \otimes b) = b^t a$. By Lemma 4.2.6

$$Tr \circ \pi\big(\iota \otimes \phi^{*t}(y_0)\big) = \widetilde{\phi}(y_0) < 0. \tag{6.1}$$

Since by Lemma 4.2.6, $Tr \circ \pi$ is positive, $\iota \otimes \phi^{*t}(y_0)$ is not positive. We shall show that $\iota \otimes \phi^{*t}(y_0) \geq 0$, so we obtain a contradiction, and thus completing the proof of the lemma.

First note that $\phi^t \in P_{\mathscr{C}}(K)^\circ$. Indeed, by Theorem 6.1.6 $\alpha \circ \phi$ is completely positive for all $\alpha \in \mathscr{C}$, hence $\alpha^t \circ \phi^t = (\alpha \circ \phi)^t$ is completely positive, so $\alpha \circ \phi^t$ is completely positive since $\mathscr{C}$ is symmetric, and therefore $\phi^t \in P_{\mathscr{C}}(K)^\circ$.

We have $y_0 \in P(B(K), \mathscr{C}^\circ)$. Let $\psi \in B(B(K), H)$ be a map such that $y_0 = C_\psi$. Then $C_{\alpha\circ\psi} = \iota \otimes \alpha(C_\psi) \geq 0$ for all $\alpha \in \mathscr{C}^\circ$, hence $\alpha \circ \psi$ is completely positive for all $\alpha \in \mathscr{C}^\circ$, hence by Theorem 6.1.6, $\psi \in P_{\mathscr{C}^\circ}(K)^\circ$.

Let $\gamma \in B(B(H), K)$ be completely positive, and let $\alpha \in \mathscr{C}$. Then $\alpha \circ (\phi^t \circ \gamma) = (\alpha \circ \phi^t) \circ \gamma$ is completely positive since $\alpha \circ \phi^t$ is completely positive. Since this holds for all $\alpha \in \mathscr{C}$, $\phi^t \circ \gamma \in P_{\mathscr{C}}(H)^\circ$, which by Remark 6.1.5 equals $\mathscr{C}^\circ$. Thus $\phi^t \circ \gamma \in \mathscr{C}^\circ$, hence $\gamma^* \circ \phi^{t*} = (\phi^t \circ \gamma)^* \in \mathscr{C}^\circ$ since $\mathscr{C}^\circ$ is symmetric. Again by Theorem 6.1.6, $\gamma^* \circ \phi^{t*} \circ \psi$ is completely positive, hence for all $x \in B(K \otimes H)^+$

$$0 \leq Tr(C_{\gamma^*\circ\phi^{t*}\circ\psi}\, x) = Tr\big(C_{\phi^{t*}\circ\psi}(\iota \otimes \gamma)(x)\big).$$

Now γ was an arbitrary completely positive map of $B(H)$ into $B(K)$. Hence it follows that $B(K \otimes K)^+$ is the closed convex cone generated by the set

$$\big\{\iota \otimes \gamma(x) : x \in B(K \otimes H)^+,\ \gamma \in B\big(B(H), K\big) \text{ completely positive}\big\}.$$

It follows that $\iota \otimes \phi^{t*}(C_\psi) = C_{\phi^{t*}\circ\psi} \geq 0$. Now $\phi^{t*} = \phi^{*t}$. Indeed, if $a \in B(K)$, $b \in B(H)$,

$$\begin{aligned} Tr\big(\phi^{*t}(a)b\big) &= Tr\big(\phi^*\big(a^t\big)^t b\big) = Tr\big(\phi^*\big(a^t\big)b^t\big) \\ &= Tr\big(a^t\phi\big(b^t\big)\big) = Tr\big(a\phi^t(b)\big) = Tr\big(\phi^{t*}(a)b\big). \end{aligned}$$

Thus we have shown $\iota \otimes \phi^{*t}(y_0) \geq 0$, contradicting (6.1). □

Proof of Theorem 6.1.7 By Theorem 6.1.6, if $\phi \in B(B(K), H)$, then $\phi \in P_{\mathscr{C}}(K)^\circ$ if and only if $\widetilde{\phi} \circ (\iota \otimes \alpha)$ is positive for all $\alpha \in \mathscr{C}$, if and only if $\widetilde{\phi}$ is positive on $K^{\mathscr{C}}$, so by Lemma 6.1.9 if and only if ϕ is $\mathscr{C}^\circ$-positive, i.e. $\phi \in P_{\mathscr{C}^\circ}(K)$. □

6.2 Examples of Dual Cones

In this section we describe the dual cones of the main mapping cones. H is, except in Theorem 6.2.6, a finite dimensional Hilbert space.

Proposition 6.2.1 $P(H)^\circ = SP_1(H)$*—the super-positive maps in* $P(H)$.

Proof By Proposition 4.1.11, a map $\phi \in P(H)$ if and only if $Tr(C_\phi a \otimes b) \geq 0$ for all $a, b \in B(H)^+$, so by Proposition 4.1.3 if and only if $Tr(C_\phi C_\psi) \geq 0$ for all $\psi \in SP_1(H)$, hence if and only if $\phi \in SP_1(H)^\circ$. Thus the proposition follows from Lemma 6.1.2. □

Proposition 6.2.2 $CP(H)^\circ = CP(H)$.

Proof Since each self-adjoint operator is the Choi matrix for a self-adjoint map, it follows by Theorem 4.1.8 that an operator is positive if and only if it is the Choi matrix for a completely positive map. Thus a map $\phi \in P(H)$ belongs to $CP(H)^\circ$ if and only if $Tr(C_\phi C_\psi) \geq 0$ for all $\psi \in CP(H)$ if and only if $Tr(C_\phi x) \geq 0$ for $x \in B(H \otimes H)^+$, if and only if $C_\phi \geq 0$, if and only if $\phi \in CP(H)$. □

Recall from Definition 5.1.2 that a map ϕ in $P(H)$ belongs to the cone $SP_k(H)$ of k-super-positive maps if and only if $\phi = \sum_i AdV_i$, where $V_i \in B(H)$ has rank $V_i \leq k$. Since the rank of the product of two operators is smaller than or equal to the minimum of the ranks of the two operators, it is clear that $SP_k(H)$ is a mapping cone. Since rank $V^* =$ rank $V^t =$ rank V for $V \in B(H)$, it follows from Lemma 4.2.5 and Proposition 1.4.2 that $SP_k(H)$ is a symmetric mapping cone. Recall that $P_k(H)$ denotes the mapping cone of k-positive maps. It is also easily seen to be symmetric, see e.g. Lemma 4.2.5.

Proposition 6.2.3 $P_k(H)^\circ = SP_k(H)$.

Proof By Theorem 4.1.15 a map ϕ belongs to $P_k(H)$ if and only if $AdV \circ \phi$ is completely positive for all $V \in B(H)$ with rank $V \leq k$, which holds if and only if for all $\psi \in CP(H)$,

$$0 \leq Tr(C_{AdV\circ\phi} C_\psi) = Tr(C_\phi C_{AdV^*\circ\psi}),$$

if and only if $0 \leq Tr(C_\phi C_\psi)$ for all $\psi \in SP_k(H)$, using Theorem 4.1.8, hence if and only if $\phi \in SP_k(H)^\circ$. By Lemma 6.1.2, $P_k(H)^\circ = SP_k(H)$. □

In Definition 1.2.8 we defined a map $\phi \in P(H)$ to be copositive if $t \circ \phi \in CP(H)$, and ϕ is decomposable if $\phi = \phi_1 + \phi_2$ with $\phi_1 \in CP(H)$ and ϕ_2 copositive. We can do the same for maps in the cones $P_k(H)$ and $SP_k(H)$ and call a map ϕ *co-k-positive* if $t \circ \phi \in P_k$, and similarly *co-k-super-positive* if $t \circ \phi \in SP_k(H)$.

We denote the corresponding cones by $coP_k(H)$ and $coSP_k(H)$. For two mapping cones $\mathscr{C}_1$ and $\mathscr{C}_2$ their intersection $\mathscr{C}_1 \cap \mathscr{C}_2$ is a mapping cone, as is the closed cone $\mathscr{C}_1 \vee \mathscr{C}_2$ they generate. By standard results from Hilbert space

$$(\mathscr{C}_1 \cap \mathscr{C}_2)^\circ = \mathscr{C}_1^\circ \vee \mathscr{C}_2^\circ, \qquad (\mathscr{C}_1 \vee \mathscr{C}_2)^\circ = \mathscr{C}_1^\circ \cap \mathscr{C}_2^\circ.$$

We thus get from Proposition 6.2.3,

$$\big(P_k(H) \cap coP_l(H)\big)^\circ = SP_k(H) \vee coSP_l(H) \quad \text{when } k, l \le \dim H,$$
$$\big(P_k(H) \vee coP_l(H)\big)^\circ = SP_k(H) \cap coSP_l(H).$$

Recall from Remark 1.2.9 that a map $\phi \in P(H)$ is *atomic* if it is not the sum of a 2-positive and a co-2-positive map, hence if $\phi \notin P_2(H) \vee coP_2(H)$, or by the above, if $\phi \notin (SP_2(H) \cap coSP_2(H))^\circ$. This yields a technique for showing that a map is atomic. One example of this will be shown in Chap. 7.

Proposition 6.2.4 *Let $\phi \in B(B(K), H)$. Then ϕ is P_k-positive if and only if ϕ is k-positive. In particular ϕ is $P(H)$-positive if and only if ϕ is positive.*

Proof It follows from Theorem 4.1.15 that a map $\phi \in B(B(K), H)$ is k-positive if and only if $AdV \circ \phi$ is completely positive for all $V \in B(K)$ with rank $V \le k$. But these maps AdV generate $SP_k(H)$, hence by Theorem 6.1.6, ϕ is k-positive if and only if $\phi \in P_{SP_k}(K)^\circ$, which equals $P_{P_k}(K)$ by Proposition 6.2.3 and Theorem 6.1.7.

The last part follows since $P(H) = P_1(H)$. □

Remark 6.2.5 Recall that a map ϕ is decomposable if it is the sum of a completely positive and a copositive map. Thus $\phi \in P(H)$ is decomposable if and only if

$$\phi \in CP(H) \vee coCP(H) = \big(CP(H) \cap coCP(H)\big)^\circ.$$

If as above $\dim H < \infty$ then by Lemmas 6.1.8 and 6.1.9 ϕ is decomposable if and only if $\iota \otimes \phi(x) \ge 0$ whenever $x = C_\psi$ with $\psi \in CP(H) \cap coCP(H)$, i.e. whenever x and $t \otimes \iota(x)$ are positive. If $\dim H = n$ we can identify $B(H) \otimes B(H)$ with $M_n(B(H))$ and reformulate the above as follows: ϕ is decomposable if and only if $\phi(x_{ij}) \in M_n(B(H))^+$ whenever (x_{ij}) and (x_{ji}) are in $M_n(B(H))^+$.

We shall now generalize this result to maps on C^*-algebras.

Theorem 6.2.6 *Let A be a C^*-algebra and ϕ a unital positive map of A into $B(H)$, where H is an arbitrary Hilbert space. Then ϕ is decomposable if and only if for all $n \in \mathbb{N}$ whenever (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$.*

Proof Suppose ϕ is decomposable. By Theorem 1.2.11 and its proof $\phi = v^*\pi v$, where π is a Jordan homomorphism π of A into $B(K)$ for some Hilbert space K, such that π is the sum of a homomorphism and an anti-homomorphism, and $v : H \to K$ a bounded linear operator. Thus if (x_{ij}) and $(x_{ji}) \in M_n(A)^+$ it is immediate that $(\phi(x_{ij})) \in M_n(B(H))^+$.

Conversely suppose (x_{ij}) and $(x_{ji}) \in M_n(A)^+$ implies $(\phi(x_{ij})) \in M_n(B(H))^+$. We may assume $A \subset B(K)$ for a Hilbert space K. Let t denote the transpose map on $B(K)$ with respect to some orthonormal basis. Let

$$V = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^t \end{pmatrix} \in M_2\big(B(K)\big) : x \in A \right\}.$$

Then V is a self-adjoint subspace of $M_2(B(K))$ containing the identity. Let $n \in N$ and let θ_n on $M_n(B(K))$ be defined by $\theta_n((x_{ij})) = (x^t_{ji})$. Hence if we write $M_n(B(K))$ in tensor form $B(K) \otimes M_n$, then $\theta_n = t \otimes t$. Then θ_n is an anti-isomorphism of order 2. Hence if $(x_{ij}) \in M_n(A)$ then $(x_{ji}) \in M_n(A)^+$ if and only if $(x^t_{ij}) = \theta_n((x_{ji})) \in M_n(B(K))^+$. Therefore both (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ if and only if

$$\left(\begin{pmatrix} x_{ij} & 0 \\ 0 & x^t_{ij} \end{pmatrix} \right) \in M_n(V)^+.$$

Let $\overline{\phi} : V \to B(H)$ be defined by

$$\overline{\phi}\left(\begin{pmatrix} x & \\ & x^t \end{pmatrix} \right) = \phi(x).$$

Then $\overline{\phi}$ is completely positive by our hypothesis on ϕ and the above equivalence. By Corollary 5.2.4 $\overline{\phi}$ has a completely positive extension $\overline{\overline{\phi}} : M_2(B(K)) \to B(H)$. Thus by Stinespring's theorem, 1.2.7, there are a Hilbert space L, a bounded linear map $v : H \to L$ and a representation $\pi_1 : M_2(B(K)) \to B(L)$ such that $\overline{\overline{\phi}} = v^*\pi_1 v$. Let π_2 be the Jordan homomorphism of A into $M_2(B(K))$ defined by

$$\pi_2(x) = \begin{pmatrix} x & 0 \\ 0 & x^t \end{pmatrix}, \quad x \in A.$$

Then π_2 is the sum of a homomorphism and an anti-homomorphism, and so is $\pi = \pi_1 \circ \pi_2$. Thus $\phi(x) = v^*\pi(x)v$ is decomposable. □

In the next section we shall show that all maps in $P(H)$ with $H = \mathbb{C}^2$, are decomposable. For this we shall need our next proposition. Recall that if $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ then $\overline{\xi} = (\overline{\xi_1}, \dots, \overline{\xi_n})$. Then if ξ is a unit vector, $[\xi] = (\xi_i \overline{\xi_j})$, so that $[\overline{\xi}] = [\xi]^t$. Recall also that $SP_1(H)$ denotes the super-positive maps in $P(H)$.

Proposition 6.2.7 *Let H be finite dimensional. Then $SP_1(H) = CP(H) \cap coCP(H)$ if and only if for all operators $a \in B(H \otimes H)^+$ such that $(t \otimes \iota)(a) \geq 0$ there exists a nonzero product vector $\xi \otimes \eta \in$ range a such that $\overline{\xi} \otimes \eta \in$ range$(t \otimes \iota)(a)$. In particular, if these conditions hold, then every map $\phi \in P(H)$ is decomposable.*

Proof Suppose $SP_1(H) = CP(H) \cap coCP(H)$, and let $a \geq 0$ with $t \otimes \iota(a) \geq 0$. Then $a = C_\phi$ with $\phi \in CP(H) \cap coCP(H)$, so $\phi \in SP_1$. By Proposition 5.1.4 its dual functional $\widetilde{\phi}$ is separable, hence C_ϕ being the transpose of the density matrix for $\widetilde{\phi}$ by Lemma 4.2.3, is a sum $\sum_i a_i \otimes b_i$ with $a_i, b_i \in B(H)^+$. If $\xi_i \in \text{range}\, a_i$ and $\eta_i \in \text{range}\, b_i$ then $\overline{\xi_i} \otimes \eta_i \in \text{range}\, a_i^t \otimes b_i$, hence in range$(t \otimes \iota)a$, proving necessity in the proposition.

Conversely let $a = C_\phi$ with ϕ extremal in $CP(H) \cap coCP(H)$. Let $\xi \otimes \eta \in \text{range}\, a$, $\overline{\xi} \otimes \eta \in \text{range}\, t \otimes \iota(a)$. Since H is finite dimensional and both a and $t \otimes \iota(a)$ are positive, there exists $\varepsilon > 0$ such that $a \geq \varepsilon[\xi] \otimes [\eta]$, and $t \otimes \iota(a) \geq \varepsilon[\xi]^t \otimes [\eta]$. Thus the map ψ with $C_\psi = a - \varepsilon[\xi] \otimes [\eta]$ belongs to $CP(H) \cap coCP(H)$, and is majorized by ϕ. Since ϕ is extremal in $CP(H) \cap coCP(H)$, there exists $\lambda > 0$ such that $a = \lambda[\xi] \otimes [\eta]$, and $\phi \in SP_1(H)$, so $SP_1(H) = CP(H) \cap coCP(H)$.

Finally, from the last statement, we have, using Propositions 6.2.1, 6.2.2 and Remark 6.2.5,

$$P(H) = \big(SP_1(H)\big)^\circ = \big(CP(H) \cap coCP(H)\big)^\circ = CP(H) \vee coCP(H),$$

proving that each map in $P(H)$ is decomposable. □

6.3 Maps on the 2 × 2 Matrices

The only cases where the positive maps from $B(K)$ into $B(H)$ are fully understood are when $\dim K = 2$ and $\dim H \leq 3$, or when $\dim K = 3$ and $\dim H = 2$. In this section we consider the case when $\dim K = \dim H = 2$. Our proof follows from that of Woronowicz [98] and can without much work be extended to the case when one of K and H is three dimensional.

Theorem 6.3.1 *Every positive map of M_2 into itself is decomposable.*

Since each completely positive map is a sum of maps of the form AdV by Theorem 4.1.8, and each map AdV is extremal by Proposition 3.1.3. It follows by composing such a map by the transpose and using Lemma 3.1.2, that we have as an immediate consequence of Theorem 6.3.1,

Corollary 6.3.2 *A map in $P(\mathbb{C}^2)$ is extremal if and only if it is of the form AdV or $t \circ AdV$.*

In order to prove Theorem 6.3.1 we shall need a result on anti-automorphisms of $B(H)$ with H finite dimensional. Recall that a *conjugation* of H is a conjugate linear isometry J on H such that $J^2 = 1$. Then the map $a \to Ja^*J$ is an anti-automorphism of order 2. In [73] it was shown that each anti-automorphism of order 2 of a factor, i.e. a von Neumann algebra with center the scalars, is either of the above form or of the form $a \to -J_0 a^* J_0$ where J_0 is a conjugate linear isometry such that $J_0^2 = -1$. We shall need the following rather special result on the existence of an anti-automorphism implemented by a conjugation.

Proposition 6.3.3 *Let H be finite dimensional and $b \in B(H)$. Suppose there exist $\lambda > 0$ and unit vectors $\xi, \eta \in H$ such that*

$$b^*b - bb^* = \lambda[\eta] - \lambda[\xi].$$

Then there exists a conjugation J on H such that $Jb^*J = b$, and $J\xi = \eta$.

Proof Note that if $\lambda = 0$, b is a normal operator, and the existence of J such that $Jb^*J = b$, is an easy consequence of the spectral theorem. We have $\lambda > 0$. Multiply b by $\lambda^{-1/2}$ and assume

$$b^*b - bb^* = [\eta] - [\xi].$$

In the proof we shall consider products where each factor is either b or b^*. It will therefore be convenient to write b_i or b'_i, $i \geq 1$, for $b_i, b'_i \in \{b, b^*\}$. Similarly we shall denote by ξ^* the vector η, and $\eta^* = \xi$. In some cases we shall write ψ and ψ_1 for ξ or η. In that case $\psi^* = \xi$ if and only if $\psi = \eta$, $\psi^* = \eta$ if $\psi = \xi$, and similarly for ψ_1. For $s \in \mathbb{R}$ let

$$A(s) = b + sb^*.$$

By direct computation we have

$$\frac{1}{1 - s^2}\big(A(s)^*A(s) - A(s)A(s)^*\big) = [\eta] - [\xi].$$

For an integer $n \geq 1$ we therefore have

$$\begin{aligned}\big(A(s)^n\eta, \eta\big) - \big(A(s)^n\xi, \xi\big) &= Tr\big(A(s)^n\big([\eta] - [\xi]\big)\big)\\ &= \frac{1}{1 - s^2}Tr\big(A(s)^n\big(A(s)^*A(s) - A(s)A(s)^*\big)\big)\\ &= \frac{1}{1 - s^2}Tr\big(A(s)^{n+1}A(s)^* - A(s)^{n+1}A(s)^*\big)\\ &= 0.\end{aligned}$$

With the notation introduced above this reduces to

$$\big(A(s)^n\psi, \psi_1\big) = \big(A(s)^n\psi_1^*, \psi^*\big), \tag{6.2}$$

because when $\psi = \psi_1$ this follows from the above, and when $\psi = \psi_1^*$, then $\psi = \psi_1^*$, and $\psi_1 = \psi^*$, so (6.2) is trivial.

Both sides of (6.2) are polynomials of order n in s. We shall compare the coefficients of s^k for all k. To see the pattern most easily consider as an example

$$A(s)^3 = b^3 + \big(b^2b^* + bb^*b + b^*b^2\big)s + \big(bb^{*2} + b^*bb^* + b^{*2}b\big)s^2 + b^{*3}s^3.$$

For $0 \le k \le n$ let σ_k consist of all products $b_{1k}b_{2k}\cdots b_{nk}$ with $n-k$ b's and k b^*'s. Then, as is easily seen by induction on n,

$$A(s)^n = \sum_{k=0}^{n}\Bigl(\sum_{\sigma_k} b_{1k}\cdots b_{nk}\Bigr)s^k.$$

Since each coefficient of s^k is symmetric in the indices we have

$$\sum_{\sigma_k} b_{1k}\cdots b_{nk} = \sum_{\sigma_k} b_{nk}\cdots b_{1k}.$$

By (6.2) we thus get from the uniqueness of the coefficients for each s^k,

$$\sum_{\sigma_k}(b_{1k}\cdots b_{nk}\psi,\psi_1) = \sum_{\sigma_k}\bigl(b_{nk}\cdots b_{1k}\psi_1^*,\psi^*\bigr),$$

or rather

$$\sum_{\sigma_k}\bigl\{(b_{1k}\cdots b_{nk}\psi,\psi_1) - \bigl(b_{nk}\cdots b_{1k}\psi_1^*,\psi^*\bigr)\bigr\} = 0. \tag{6.3}$$

As will be seen later, the existence of the conjugation J satisfying the conditions of the proposition is equivalent to the following identity.

$$(b_1 b_2\cdots b_m\psi,\psi_1) = \bigl(b_m\cdots b_2 b_1\psi_1^*,\psi^*\bigr) \tag{6.4}$$

for all products of b_i's. For $m=0$ this relation was shown in (6.2) with $n=0$. Use induction on n, and assume (6.4) holds for all $m \le n-1$. Then using that $b^*b - bb^* = [\eta] - [\xi]$ and remembering our conventions on ψ and ψ_1, and the fact that for operators x and y,

$$\bigl(x[\eta]y\psi,\psi_1\bigr) = \bigl([\eta]y\psi,[\eta]x^*\psi_1\bigr) = (y\psi,\eta)\overline{\bigl(x^*\psi_1,\eta\bigr)} = (y\psi,\eta)(x\eta,\psi_1)$$

and using the induction hypothesis we have

$$\begin{aligned}
&\bigl(b_1\cdots b_k b^*b b_{k+3}\cdots b_n\psi,\psi_1\bigr) - \bigl(b_1\cdots b_k bb^* b_{k+3}\cdots b_n\psi,\psi_1\bigr)\\
&\quad= \bigl(b_1\cdots b_k[\eta]b_{k+3}\cdots b_n\psi,\psi_1\bigr) - \bigl(b_1\cdots b_k[\xi]b_{k+3}\cdots b_n\psi,\psi_1\bigr)\\
&\quad= (b_1\cdots b_k\eta,\psi_1)(b_{k+3}\cdots b_n\psi,\eta) - (b_1\cdots b_k\xi,\psi_1)(b_{k+3}\cdots b_n\psi,\xi)\\
&\quad= \bigl(b_k\cdots b_1\psi_1^*,\eta^*\bigr)\bigl(b_n\cdots b_{k+3}\eta^*,\psi^*\bigr) - \bigl(b_k\cdots b_1\psi_1^*,\xi^*\bigr)\bigl(b_n\cdots b_{k+3}\xi^*,\psi^*\bigr)\\
&\quad= \bigl(b_n\cdots b_{k+3}\xi,\psi^*\bigr)\bigl(b_k\cdots b_1\psi_1^*,\xi\bigr) - \bigl(b_n\cdots b_{k+3}\eta,\psi^*\bigr)\bigl(b_k\cdots b_1\psi^*,\eta\bigr)\\
&\quad= \bigl(b_n\cdots b_{k+3}[\xi]b_k\cdots b_1\psi_1^*,\psi^*\bigr) - \bigl(b_n\cdots b_{k+3}[\eta]b_k\cdots b_1\psi_1^*,\psi^*\bigr)\\
&\quad= \bigl(b_n\cdots b_{k+3}bb^*b_k\cdots b_2\psi_1^*,\psi^*\bigr) - \bigl(b_n\cdots b_{k+3}b^*bb_k\cdots b_1\psi_1^*,\psi^*\bigr).
\end{aligned}$$

By the equality of the first and the last expression of this computation we see that the difference

$$(b_1 \cdots b_n \psi, \psi_1) - \left(b_n \cdots b_1 \psi_1^*, \psi^*\right) = \alpha(k)$$

is independent of the order of the sequence $b_1, \ldots, b_n$ as long as the sequence contains $n-k$ entries of b and k entries of b^*. Thus all summands of (6.3) are equal to $\alpha(k)$. Since the sum is 0, $\alpha(k) = 0$, and therefore (6.4) is verified for $m = n$. Thus by induction we have shown that (6.4) holds for all non-negative integers m. Let

$$u = b_1 \cdots b_n \psi, \qquad v = b_1' \cdots b_m' \psi,$$
$$u^* = b_1{}^* \cdots b_n{}^* \psi^*, \qquad v^* = b'{}_1{}^* \cdots b'{}_m{}^* \psi^*.$$

Then it follows from (6.4) that

$$(u, v) = \left(v^*, u^*\right). \tag{6.5}$$

Let H_0 be the subspace of H generated by all vectors of the form $b_1 \cdots b_n \psi$. By (6.5) there exists a conjugation J_0 acting on H_0 such that

$$J_0 b_1 \cdots b_n \psi = b_1^* \cdots b_n^* \psi^*.$$

If $H_0 = H$ this conjugation solves our problem. In the general case $H = H_0 \oplus H_1$. Then H_0 is invariant under b and b^*, so $b = c_0 \oplus c_1$, where c_i is the restriction of b to H_i. Since $\xi, \eta \in H_0$ the operator c_1 is normal. As remarked at the beginning of the proof there exists a conjugation J_1 on H_1 such that $J_1 c_1^* J_1 = c_1$. Thus $J = J_0 \oplus J_1$ satisfies our requirements. □

We are now in position to prove Theorem 6.3.1. The proof will be divided into some lemmas. Recall from the Appendix that we identify $M_2 \otimes M_2$ with $M_2(B(\mathbb{C}^2)) = M_2(M_2)$.

Lemma 6.3.4 *Let $H = \mathbb{C}^2$, and let*

$$a = \begin{pmatrix} 1 & b \\ b^* & c \end{pmatrix} \in M_2\big(B\big(\mathbb{C}^2\big)\big)$$

satisfy $a \geq 0$ and $t \otimes \iota(a) \geq 0$. Let $s \in \mathbb{C}$ and let H_s be the subspace of H spanned by $(b - s1)\ker(c - b^*b)$ and $(b - s1)^* \ker(c - bb^*)$. If $0 \neq \eta \in H$, $\eta \perp H_s$ and $\xi = (1, \overline{s}) \in \mathbb{C}^2$, then $\xi \otimes \eta \in \operatorname{range} a$ and $\overline{\xi} \otimes \eta \in \operatorname{range}(t \otimes 1)a$.

Proof Since $\eta \perp (b - s1)\ker(c - b^*b)$, $(b - s1)^*\eta \perp \ker(c - b^*b)$. For a self-adjoint operator the kernel coincides with the orthogonal complement of the image. Therefore there exists $\psi \in H$ such that

$$(b - s1)^*\eta = \big(c - b^*b\big)\psi.$$

Using this relation we easily find that

$$\begin{pmatrix} \eta \\ \bar{s}\eta \end{pmatrix} = \begin{pmatrix} 1 & b \\ b^* & c \end{pmatrix} \begin{pmatrix} \eta + b\psi \\ -\psi \end{pmatrix},$$

so that $(1, \bar{s}) \otimes \eta \in \operatorname{range} a$. In the same way we show $(1, s) \otimes \eta \in \operatorname{range}(t \otimes \iota)a$. □

In the case when b in Lemma 6.3.4 is normal then the conclusion of the lemma is immediate with the assumption on the vector η. Indeed we have

Lemma 6.3.5 *If in Lemma 6.3.4 b is a normal operator then there exist ξ and $\eta \in H$ such that $\xi \otimes \eta \in \operatorname{range} a$, $\bar{\xi} \otimes \eta \in \operatorname{range}(t \otimes \iota)(a)$.*

Proof Let s be an eigenvalue of b and η the corresponding eigenvector, so $b\eta = s\eta$, $b^*\eta = \bar{s}\eta$. Then

$$\begin{pmatrix} \eta \\ \bar{s}\eta \end{pmatrix} = \begin{pmatrix} 1 & b \\ b^* & c \end{pmatrix} \begin{pmatrix} \eta \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \eta \\ s\eta \end{pmatrix} = \begin{pmatrix} 1 & b^* \\ b & c \end{pmatrix} \begin{pmatrix} \eta \\ 0 \end{pmatrix},$$

so the lemma follows with $\xi = (1, \bar{s})$ since we can identify $\xi \otimes \eta$ with $\left(\begin{smallmatrix} \eta \\ \bar{s}\eta \end{smallmatrix}\right)$, see the Appendix. □

We next show that with the notation and assumptions as in Lemma 6.3.4 the vector $\eta \perp H_s$ exists, or b is normal.

Lemma 6.3.6 *Let a be as in Lemma 6.3.4, and assume b is not normal. Then there exist $z \in \mathbb{C}$ and a non-zero vector $\psi \perp H_z$.*

Proof We first note that $c - b^*b \geq 0$ and $c - bb^* \geq 0$. The first inequality follows, since if $\alpha, \beta \in H$ then

$$\left(a \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) = \|\alpha + b\beta\|^2 + \big((c - b^*b)\beta, \beta\big),$$

so that $a \geq 0$ if and only if $c - b^*b \geq 0$. Similarly $t \otimes \iota(a) \geq 0$ if and only if $c - bb^* \geq 0$.

Let $n_+ = \dim\ker(c - b^*b)$, $n_- = \dim\ker(c - bb^*)$. To prove the lemma we must consider the following cases.

If $n_+ = 2$ then $c = b^*b$, so that

$$0 = Tr\big(c - b^*b\big) = Tr\big(c - bb^*\big) = 0,$$

hence $b^*b = c = bb^*$, and b is normal, a case which is ruled out by assumption. Similarly $n_- \neq 2$.

If $n_+ + n_- \leq 1$, then $\dim H_s \leq 1$, so the existence of $\eta \perp H_s$ is obvious.

We are therefore left with the case $n_+ + n_- = 2$, and so by the above, $n_+ = n_- = 1$. Then the operators $c - b^*b$ and $c - bb^*$ have rank 1. Since they have the same trace there exist $\lambda > 0$ and unit vectors ξ and η such that

$$c - b^*b = \lambda[\xi], \qquad c - bb^* = \lambda[\eta]. \tag{6.6}$$

Furthermore $[\xi] \neq [\eta]$, so ξ and η are not proportional.

Consider the vectors $\xi, \eta, b\xi, b^*\eta$. Since $\dim H = 2$, they are linearly dependent. Therefore there are complex numbers $\alpha, \beta, \gamma, \delta$ such that

$$\alpha b^*\eta + \beta\eta = \gamma b\xi + \delta\xi. \tag{6.7}$$

We may assume α and γ are real and non-negative, since possible phase factors can be absorbed in ξ and η. Also we may assume $\alpha + \gamma > 0$, since otherwise η and ξ would be proportional.

By Proposition 6.3.3 there exists a conjugation J on H such that $JbJ = b^*$ and $J\xi = \eta$. Applying J to (6.7) we get

$$\alpha b\xi + \overline{\beta}\xi = \gamma b^*\eta + \overline{\delta}\eta.$$

Combining this with (6.7) we obtain

$$(b - s1)\xi = (b - s1)^*\eta, \tag{6.8}$$

where $s = -\frac{\overline{\beta}+\delta}{\alpha+\gamma}$.

Let $\psi \in H$, $z, w \in \mathbb{C}$. Since $b^*b - bb^* = \lambda[\xi] - \lambda[\eta]$ with ξ and η unit vectors, and using (6.4) it follows from a straightforward computation that

$$\begin{aligned} &\left\|(b - z1)\psi + w\eta\right\|^2 + \left|(\xi, \psi) + (s - z)w\right|^2 \\ &\quad = \left\|(b - z1)^*\psi + w\xi\right\|^2 + \left|(\eta, \psi) + (\overline{s} - \overline{z})w\right|^2. \end{aligned} \tag{6.9}$$

For each $z \in \mathbb{C}$ let

$$D_z = (b - z1) + \frac{1}{z - s}v, \quad z \neq s,$$

where v is a partial isometry such that $v^*v = [\xi]$, $vv^* = [\eta]$. The determinant $\det D_z$ is a rational function of z and tends to infinity as $z \to \infty$. Since any rational function defined on the one-point compactification of the complex plane takes any complex value, there exists $z \in \mathbb{C}$ such that $\det D_z = 0$. Thus there exists a nonzero vector $\psi \in H$ such that $D_z\psi = 0$, or more explicitly

$$(b - z1)\psi + \frac{(\psi, \xi)}{z - s}\eta = 0.$$

This shows that $(b - z1)\psi$ is proportional to η. Since $c - bb^* = \lambda[\eta]$, $(b - z1)\psi$ is orthogonal to $\ker(c - bb^*)$, and consequently

$$\psi \perp (b - z1)^* \ker\big(c - bb^*\big).$$

Let $w = \frac{(\psi,\xi)}{s-z}$. We see that the left side of (6.9) is zero, hence the summands on the right side are zero, in particular $(b - z1)^*\psi$ is proportional to ξ, so $(b - z1)^*\psi$ is orthogonal to $\ker(c - b^*b)$, and so

$$\psi \perp (b - z1)\ker\bigl(c - b^*b\bigr).$$

We have thus found the desired vector orthogonal to H_z for some $z \in \mathbb{C}$. □

In the above lemmas we considered operators a of the form

$$a = \begin{pmatrix} 1 & b \\ b^* & c \end{pmatrix}.$$

We must now extend the results to the general case.

Lemma 6.3.7 *Let* $a = \left(\begin{smallmatrix} x & b \\ b^* & c \end{smallmatrix}\right) \in (M_2 \otimes M_2)^+$ *satisfy* $(t \otimes \iota)(a) \geq 0$. *Then there exists a product vector* $\xi \otimes \eta \in \operatorname{range} a$ *such that* $\overline{\xi} \otimes \eta \in \operatorname{range}(t \otimes \iota)(a)$.

Proof Clearly $x, c \geq 0$. There are two cases.

Case 1 x is invertible. Let

$$a_1 = \begin{pmatrix} x^{-1/2} & 0 \\ 0 & x^{-1/2} \end{pmatrix} a \begin{pmatrix} x^{-1/2} & 0 \\ 0 & x^{-1/2} \end{pmatrix}.$$

Then a_1 has the form in Lemmas 6.3.4 and 6.3.5. Thus by the lemmas there exists a product vector $\xi \otimes \eta \in \operatorname{range} a_1$ such that $\overline{\xi} \otimes \eta \in \operatorname{range}(t \otimes \iota)(a_1)$. But then $(1 \otimes x^{1/2})\,\xi \otimes \eta \in \operatorname{range} a$, and $(1 \otimes x^{1/2})\,\overline{\xi} \otimes \eta \in \operatorname{range} t \otimes \iota(a)$.

Case 2 x is non-invertible. Since $\dim H = 2$, $x = \lambda p$ with p a 1-dimensional projection. Let $q = 1 - p$. Then

$$\begin{pmatrix} 0 & qb \\ b^*q & c \end{pmatrix} = \begin{pmatrix} q & \\ & 1 \end{pmatrix} a \begin{pmatrix} q & \\ & 1 \end{pmatrix} \geq 0$$

and

$$\begin{pmatrix} 0 & qb^* \\ bq & c \end{pmatrix} = \begin{pmatrix} q & \\ & 1 \end{pmatrix} t \otimes \iota(a) \begin{pmatrix} q & \\ & 1 \end{pmatrix} \geq 0.$$

Hence

$$qb = b^*q = qb^* = bq = 0. \tag{6.10}$$

Suppose $cq \neq 0$. If $q = [\eta]$ then $c\eta \neq 0$, so by (6.10)

$$\begin{pmatrix} 0 \\ c\eta \end{pmatrix} = a \begin{pmatrix} 0 \\ \eta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ c\eta \end{pmatrix} = (t \otimes \iota)(a) \begin{pmatrix} 0 \\ \eta \end{pmatrix}. \tag{6.11}$$

Thus $\binom{0}{c\eta} = \binom{0}{1} \otimes c\eta$ is the desired product vector.

If $cq = 0$ then all operators x, b, c act on the 1-dimensional Hilbert space pH, in which case the lemma is trivial, or can be deduced from Lemma 6.3.5 if desired. □

Proof of Theorem 6.3.1 By Lemma 6.3.6 for each operator $a \in (M_2 \otimes M_2)^+$ such that $(t \otimes \iota)(a) \geq 0$, there exists a product vector $\xi \otimes \eta \in \text{range}\, a$ such that $\overline{\xi} \otimes \eta \in \text{range}(t \otimes 1)(a)$. Then by Proposition 6.2.7 each map in $P(H)$ is decomposable. □

6.4 Tensor Products

A major problem with positive maps is that their tensor products are usually not positive. It follows from the Stinespring Theorem 1.2.7, that the tensor product of two completely positive maps is positive, indeed it is completely positive. As a consequence the tensor product of two copositive maps is copositive. We shall see that this result follows from the fact that $CP(H)^\circ = CP(H)$, see Proposition 6.2.2, and similarly for copositive maps.

We assume in this section that H is a finite dimensional Hilbert space, and (e_{ij}) is a complete set of matrix units for $B(H)$. We put $p = \sum_{ij} e_{ij} \otimes e_{ij}$. Then we have

Theorem 6.4.1 *Let $\mathscr{C}$ be a symmetric mapping cone in $P(H)$, and let $\phi \in P(H)$. Then the following conditions are equivalent*:

(i) *$\phi \in \mathscr{C}^\circ$—the dual cone of $\mathscr{C}$.*
(ii) *$\phi \circ \psi$ is completely positive for all $\psi \in \mathscr{C}$.*
(iii) *$\psi \otimes \phi$ is positive for all $\psi \in \mathscr{C}$.*
(iv) *$\psi \otimes \phi(p) \geq 0$ for all $\psi \in \mathscr{C}$.*

We first do some preliminaries. Let $\pi : B(H) \otimes B(H) \to B(H)$ be the map $\pi(a \otimes b) = b^t a$. By Lemma 4.2.6 if $\phi \in P(H)$ then

$$\widetilde{\phi} = Tr \circ \pi \circ (\iota \otimes \phi^{*t}), \tag{6.12}$$

where ι is the identity map on $B(H)$. Note also that since $C_\iota = p$ is the Choi matrix for ι, we have by Lemma 4.2.3,

$$\widetilde{\iota}(x) = Tr(C_\iota^t x) = Tr(px), \quad x \in B(H \otimes H).$$

Thus by (6.12) applied to ι we obtain, since $\widetilde{\iota} = Tr \circ \pi$, and the fact that $\phi^{*t} = \phi^{t*}$, see the proof of Lemma 6.1.9,

$$\widetilde{\phi}(x) = Tr \circ \pi(\iota \otimes \phi^{*t}(x)) = Tr(p(\iota \otimes \phi^{*t}(x))) = Tr(\iota \otimes \phi^t(p)x). \tag{6.13}$$

Lemma 6.4.2 *Let $\phi, \psi \in P(H)$. Then we have*

(i) $(\phi \circ \psi)\widetilde{\ }(x) = Tr((\psi^* \otimes \phi^t)(p)x), \ x \in B(H \otimes H)$.
(ii) $\psi^{*t} \otimes \phi(p) = \iota \otimes (\phi \circ \psi)(p)$.

(iii) *Furthermore if* $\gamma \in B(B(H), K)$ *for another Hilbert space* K, *then* $C_{\gamma\circ\phi\circ\psi} = \psi^{*t} \otimes \gamma(C_\phi)$.

Proof Using the above formulas we get for $a, b \in B(H)$,

$$\begin{aligned}
(\phi \circ \psi)\tilde{}(a \otimes b) &= Tr \circ \pi\big(\iota \otimes (\phi \circ \psi)^{*t}(a \otimes b)\big) \\
&= Tr \circ \pi\big(a \otimes \big(\psi^* \circ \phi^*\big)\big(b^t\big)^t\big) \\
&= Tr\big(a\big(\psi^* \circ \phi^*\big)\big(b^t\big)\big) \\
&= Tr\big(\psi(a)\phi^*\big(b^t\big)\big) \\
&= Tr \circ \pi\big(\psi(a) \otimes \phi^{*t}(b)\big) \\
&= Tr\big(p\big(\psi(a) \otimes \phi^{*t}(b)\big)\big) \\
&= Tr\big(\big(\psi^* \otimes \phi^t\big)(p)(a \otimes b)\big),
\end{aligned}$$

proving (i).

We also have by (6.13) that

$$(\phi \circ \psi)\tilde{}(x) = Tr\big(\iota \otimes (\phi \circ \psi)^t(p)x\big). \tag{6.14}$$

It is straightforward to show $(\phi \circ \psi)^t = \phi^t \circ \psi^t$. We therefore get from (6.14) and (i)

$$\psi^* \otimes \phi^t(p) = \iota \otimes (\phi \circ \psi)^t(p) = \iota \otimes \phi^t \circ \psi^t(p).$$

Since $P(H)$ is symmetric and the last equation holds for all ϕ^t and ψ^t in $P(H)$, (ii) follows.

To show (iii) notice that it is immediate from the definition of the Choi matrix that

$$C_{\gamma\circ(\phi\circ\psi)} = \iota \otimes \gamma(C_{\phi\circ\psi}).$$

Thus by (ii)

$$\begin{aligned}
C_{\gamma\circ\phi\circ\psi} &= \iota \otimes \gamma\big(\psi^{*t} \otimes \phi\big)(p) \\
&= \big(\psi^{*t} \otimes \gamma\big)(\iota \otimes \phi)(p) \\
&= \psi^{*t} \otimes \gamma(C_\phi).
\end{aligned}$$

□

Proof of Theorem 6.4.1 The pattern of the proof is (i) ⇔ (ii) ⇔ (iv) and (i) ⇒ (iii) ⇒ (iv).

(i) ⇔ (ii). By Theorem 5.2.5 $\mathscr{C} = P_{\mathscr{C}}(H)$—the $\mathscr{C}$-positive maps in $P(H)$. By Theorem 6.1.3 $\mathscr{C}^\circ$ is symmetric. Thus by Theorem 6.1.6 $\phi \in \mathscr{C}^\circ$ if and only if $\phi^* \in \mathscr{C}^\circ$ if and only if $\psi \circ \phi^*$ is completely positive if and only if $\phi \circ \psi^* = (\psi \circ \phi^*)^*$ is completely positive if and only if $\phi \circ \psi$ is completely positive for all $\psi \in \mathscr{C}$, proving (i) ⇔ (ii).

(ii) ⇔ (iv). By Theorem 4.1.8 and Lemma 6.4.2 $\phi \circ \psi$ is completely positive if and only if

$$0 \le C_{\phi\circ\psi} = \iota \otimes \phi \circ \psi(p) = \psi^{*t} \otimes \phi(p).$$

Since $\mathscr{C}$ is symmetric the equivalence (ii) $\Leftrightarrow$ (iv) follows.

Clearly (iii) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (iii). With the chosen complete set of matrix units (e_{ij}) we have $p = \sum e_{ij} \otimes e_{ij}$, which is a positive rank 1 operator with range the vector $\sum_i \xi_i \otimes \xi_i$, where $\xi_1, \ldots, \xi_n, n = \dim H$, is an orthonormal basis for H such that $e_{ij}\xi_k = \delta_{jk}\xi_i$. Let $\xi = \sum \xi_i \otimes \eta_i$ be a vector in $H \otimes H$. Let $v \in B(H)$ be defined by $v\xi_i = \eta_i$, so

$$\xi = 1 \otimes v\left(\sum_i \xi_i \otimes \xi_i\right).$$

Let q be the 1-dimensional projection $[\xi]$ onto $\mathbb{C}\xi$. Then it follows that

$$Ad(1 \otimes v)(p) = \lambda q \quad \text{for some } \lambda > 0.$$

We have thus shown that given a 1-dimensional projection $q \in B(H)$ then there exists $v \in B(H)$ such that

$$1 \otimes Adv(p) = q.$$

Since $\mathscr{C}^\circ$ is a mapping cone by Theorem 6.1.3, and $\phi \in \mathscr{C}^\circ$, $\phi \circ Adv \in \mathscr{C}^\circ$. Thus by Theorem 6.1.6 $\phi \circ Adv \circ \psi$ is completely positive for all $\psi \in \mathscr{C}$, hence by Lemma 6.4.2

$$\psi^{*t} \otimes \phi(q) = \left(\psi^{*t} \otimes \phi \circ Adv\right)(p) = \iota \otimes (\phi \circ Adv \circ \psi)(p) \ge 0.$$

Since $\mathscr{C}$ is symmetric, $\psi \otimes \phi(q) \ge 0$ for all $\psi \in \mathscr{C}$ and 1-dimensional projections q. It follows that $\psi \otimes \phi$ is positive for all $\psi \in \mathscr{C}$. Thus (i) $\Rightarrow$ (iii), and the proof is complete. □

The above theorem is about maps in $P(H)$. We next apply the theorem to maps from different $B(K)$'s into $B(H)$.

Corollary 6.4.3 *Let H, K, L be finite dimensional Hilbert spaces. Let $\mathscr{C}$ be a symmetric mapping cone in $P(H)$. Suppose $\psi \in B(B(K), H)$ is $\mathscr{C}$-positive and $\phi \in B(B(L), H)$ is $\mathscr{C}^\circ$-positive. Then $\psi \otimes \phi : B(K \otimes L) \to B(H \otimes H)$ is positive.*

Proof By Theorem 5.1.13 it suffices to consider maps of the form $\psi = \alpha \circ \beta$ with $\alpha \in \mathscr{C}$, $\beta : B(K) \to B(H)$ completely positive, and $\phi = \gamma \circ \delta$ with $\gamma \in \mathscr{C}^\circ$, $\delta : B(L) \to B(H)$ completely positive.

Thus

$$\psi \otimes \phi = (\alpha \otimes \gamma) \circ (\beta \otimes \delta)$$

is positive, since $\beta \otimes \delta$ is completely positive, and $\alpha \otimes \gamma$ is positive by Theorem 6.4.1. □

6.5 Notes

Duality of cones of positive maps has been studied for some time, see e.g [17, 21, 27] and [2].

The results in Sects. 6.1, 6.2 and 6.4, except for the examples 6.2.1 and 6.2.3 in Sect. 6.2, which are taken from [69], are to a great extent taken from papers by the author. However, Proposition 6.2.4 was shown by Itoh [28]. For Theorem 6.1.3 see [82], for Theorem 6.1.6 [80], and for Theorem 6.2.6 see [77]. The results in Sect. 6.4 are taken from [84].

For an extension of Remark 6.2.5 to maps which are sums of k-positive and l-copositive maps, see [16].

Section 6.3 on maps on M_2 is due to Woronowicz [98]. Related results on maps on M_2 can be found in [71].

Chapter 7
States and Positive Maps

The duality $\phi \to \widetilde{\phi}$ between the bounded maps of $B(K)$ into $B(H)$, $B(B(K), H)$, and the dual $(B(K)\widehat{\otimes}\mathscr{T}(H))^*$ of the projective tensor product of $B(K)$ and $\mathscr{T}(H)$, see 4.2, shows a close relationship between positive maps and linear functionals. In this chapter we shall elaborate on this relationship. In Sect. 7.1 we shall translate the duality theorem, 6.1.6, to a theorem on linear functionals and show some consequences. In Sect. 7.2 we consider PPT-states on tensor products $B(K) \otimes B(H)$ and show their relationship to decomposable maps. Section 7.3 is devoted to entanglement. It turns out that the negative part C_ϕ^- of the Choi matrix C_ϕ for a map ϕ contains much information related to entanglement. Finally in Sect. 7.4 we shall relate positive maps to super-positive maps.

7.1 Positivity Properties of Linear Functionals

The main result in the present section is the following theorem, which is essentially a translation of Theorem 6.1.6 to linear functionals. We denote by $P_{\mathscr{C}}(K)$ the $\mathscr{C}$-positive maps from $B(K)$ into $B(H)$.

Theorem 7.1.1 *Let K and H be finite dimensional Hilbert spaces and $\mathscr{C}$ a symmetric mapping cone in $P(H)$. Let ρ be a linear functional on $B(K) \otimes B(H)$ with density operator h, so $\rho(x) = Tr(hx)$. Then the following conditions are equivalent.*

(i) $\rho = \widetilde{\phi}$ *with* $\phi \in P_{\mathscr{C}}(K)^\circ$.
(ii) $\rho(C_\alpha) \geq 0$ *for all* $\alpha \in \mathscr{C}$.
(iii) $\iota \otimes \alpha(h) \geq 0$ *for all* $\alpha \in \mathscr{C}$, *i.e.* $h \in P(B(K), \mathscr{C})$.
(iv) $\rho \circ (\iota \otimes \alpha) \geq 0$ *for all* $\alpha \in \mathscr{C}$.
(v) ρ *is positive on the cone* $P(B(K), \mathscr{C}^\circ)$.

Proof (i) $\Leftrightarrow$ (ii). By Lemma 4.2.2 $\rho = \widetilde{\phi}$ for some $\phi \in B(B(K), H)$. By Lemma 4.2.3 C_ϕ^t is the density operator for $\widetilde{\phi}$. We thus have for $\alpha \in \mathscr{C}$, using Lemma 4.2.5,

E. Størmer, *Positive Linear Maps of Operator Algebras*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-34369-8_7,

$$\rho(C_\alpha) = Tr\big(C_\phi^t C_\alpha\big) = Tr\big(C_\phi C_\alpha^t\big) = Tr(C_\phi C_{\alpha^t}).$$

Since $\mathscr{C}$ is symmetric it follows that $\phi \in P_{\mathscr{C}}(K)^\circ$ if and only if $\rho(C_\alpha) \geq 0$ for all $\alpha \in \mathscr{C}$, proving (i) $\Leftrightarrow$ (ii).

(iii) $\Leftrightarrow$ (iv). We let (e_{ij}) be a complete set of matrix units for $B(K)$ and $p = \sum e_{ij} \otimes e_{ij}$. Then since $C_\phi^t = C_{\phi^t}$,

$$\iota \otimes \alpha(h) = \iota \otimes \alpha\big(C_\phi^t\big) = (\iota \otimes \alpha) \circ \big(\iota \otimes \phi^t\big)(p). \tag{7.1}$$

Hence

$$\rho \circ (\iota \otimes \alpha)(x) = Tr\big(C_{\phi^t}\, \iota \otimes \alpha(x)\big) = Tr\big(\iota \otimes \big(\alpha^* \circ \phi^t\big)(p)x\big).$$

Thus by (7.1) $\rho \circ (\iota \otimes \alpha) \geq 0$ for all $\alpha \in \mathscr{C}$ if and only if $\iota \otimes (\alpha \circ \phi^t)(p) \geq 0$ for all $\alpha \in \mathscr{C}$, if and only if $\iota \otimes \alpha(h) \geq 0$ for all $a \in \mathscr{C}$, proving (iii) $\Leftrightarrow$ (iv).

(i) $\Leftrightarrow$ (iii). Since $p = p^t = t \otimes t(p)$ we have

$$\begin{aligned}(t \otimes t) \circ \big(\iota \otimes \alpha \circ \phi^t\big)(p) &= \iota \otimes (t \circ \alpha \circ t \circ \phi)\big(t \otimes t(p)\big)\\ &= \iota \otimes \alpha^t \circ \phi\big(p^t\big)\\ &= \iota \otimes \alpha^t \circ \phi(p).\end{aligned}$$

Since $\mathscr{C}$ is symmetric, and $t \otimes t$ is an anti-isomorphism, it follows from (7.1) that $\alpha \circ \phi$ is completely positive if and only if $\iota \otimes \alpha(h) \geq 0$. Hence by Theorem 6.1.6 $\phi \in P_{\mathscr{C}}(K)^\circ$ if and only if $\iota \otimes \alpha(h) \geq 0$, i.e. (i) $\Leftrightarrow$ (iii).

(i) $\Leftrightarrow$ (v). By Theorem 6.1.7 $P_{\mathscr{C}}(K)^\circ = P_{\mathscr{C}^\circ}(K)$. Thus $\phi \in P_{\mathscr{C}}(K)^\circ$ if and only if ϕ is $\mathscr{C}^\circ$-positive if and only if $\rho = \widetilde{\phi}$ is positive on $P(B(K), \mathscr{C}^\circ)$, proving (i) $\Leftrightarrow$ (v). Thus all conditions (i), ..., (v) are equivalent. □

Corollary 7.1.2 *Let K and H be finite dimensional Hilbert spaces and ρ a state on $B(K) \otimes B(H)$ with density operator h. Then ρ is separable if and only if $\iota \otimes \alpha(h) \geq 0$ for all $\alpha \in P(H)$.*

Proof By Proposition 5.1.4 the mapping cone $SP_1(H)$ of super-positive maps in $P(H)$ consists of maps ϕ with $\widetilde{\phi}$ a separable positive linear functional. By Proposition 6.2.1 $SP_1(H) = P(H)^\circ$. Thus by the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 7.1.1 $\rho\ (= \widetilde{\phi})$ is separable if and only if $\iota \otimes \alpha(h) \geq 0$ for all $\alpha \in P(H)$. □

Remark 7.1.3 The above corollary can easily be extended to the infinite dimensional case if we assume ρ is a normal state and the maps α are normal. The proof is then obtained by reduction to the finite dimensional case by considering $e \otimes f\ h\ e \otimes f$ for e and f finite dimensional projections in $B(K)$ and $B(H)$ respectively, and then taking limits. Considering adjoint maps we can also show the analogue result when the α's map $B(H)$ into $B(K)$.

Remark 7.1.4 An equivalent formulation of Corollary 7.1.2 is the identity

$$P\big(B(K), P(H)\big) = B(K)^+ \otimes B(H)^+. \tag{7.2}$$

Indeed, by definition of $P(B(K), P(H))$, (Definition 5.1.6), a positive operator $h \in B(K \otimes H)$ belongs to $P(B(K), P(H))$ if and only if $\iota \otimes \alpha(h) \geq 0$ for all $\alpha \in P(H)$, so by Corollary 7.1.2, if and only if $Tr(h\,\cdot)$ is a separable positive linear functional, i.e., if and only if $h \in B(K)^+ \otimes B(H)^+$.

Thus by Lemma 5.2.1, $1 \otimes 1$ is an interior point of $B(K)^+ \oplus B(H)^+$. Since $C_{Tr} = 1 \otimes 1$, it follows that for each positive linear functional ρ of small enough norm, $Tr + \rho$ is separable. In Sect. 7.5 we shall prove a strengthening of this result.

7.2 PPT-States

PPT-states, i.e. states with positive partial transpose, are rough approximations to separable states, and have attracted much attention in the literature. We show in this section how they relate to positive maps and in particular to decomposable maps. They are defined as follows.

Definition 7.2.1 Let A be an operator system and H a Hilbert space. A state ρ on $A \widehat{\otimes} \mathcal{T}(H)$ is said to be a *PPT-state* if $\rho \circ (\iota \otimes t)$ is a state on $A \widehat{\otimes} \mathcal{T}(H)$ as well.

Theorem 7.2.2 *Let A and H be as above and ρ a state on $A \widehat{\otimes} \mathcal{T}(H)$. Let $\phi \in B(A, H)$ be the map such that $\rho = \widetilde{\phi}$. Then ρ is a PPT-state if and only if ϕ is both completely positive and copositive.*

Proof Since ρ is a state ϕ, is completely positive by Theorem 4.2.7. Let $a \in A$ and b be a trace class operator on H. Since the trace is invariant under transposition,

$$\begin{aligned}\widetilde{\phi} \circ (\iota \otimes t)(a \otimes b) &= \widetilde{\phi}\big(a \otimes b^t\big) = Tr\big(\phi(a)b\big)\\ &= Tr\big(t \circ \phi(a)b^t\big) = (t \circ \phi)\widetilde{\ }(a \otimes b).\end{aligned}$$

Thus $\rho = \widetilde{\phi}$ is PPT if and only if both ϕ and $t \circ \phi$ are completely positive, hence if and only if ϕ is both completely positive and copositive. □

Corollary 7.2.3 *Let $H = \mathbb{C}^2$. Then a state ρ on $M_2 \otimes M_2$ is separable if and only if it is PPT.*

Proof By Theorem 6.3.1 each positive map in $P(\mathbb{C}^2)$ is decomposable. Hence

$$CP\big(\mathbb{C}^2\big) \cap coCP\big(\mathbb{C}^2\big) = P\big(\mathbb{C}^2\big)^\circ = SP_1\big(\mathbb{C}^2\big).$$

It follows that a map is both completely positive and copositive if and only if it is super-positive. Hence the corollary follows from Proposition 5.1.4. □

If $\phi \in B(B(K), H)$ for K and H finite dimensional we have the following application of Theorem 7.1.1 to PPT-states.

Corollary 7.2.4 *Let K and H be finite dimensional and ρ a state on $B(K) \otimes B(H)$ with density operator h. Let $\mathscr{C} = CP(H) \vee coCP(H)$ be the mapping cone generated by $CP(H)$ and $coCP(H)$. Then the following conditions are equivalent.*

(i) *ρ is a PPT-state.*
(ii) *$\alpha \circ \phi$ is completely positive for all $\alpha \in \mathscr{C}$, where $\rho = \widetilde{\phi}$.*
(iii) *$\iota \otimes \alpha(h) \geq 0$ for all $\alpha \in \mathscr{C}$.*

Proof Since by Proposition 6.2.2 $CP(H)^\circ = CP(H)$ and similarly for $coCP(H)$, $\mathscr{C}^\circ = CP(H) \cap coCP(H)$. Therefore by Theorem 7.2.2 $\widetilde{\phi}$ is PPT if and only if $\phi \in \mathscr{C}^\circ$, hence by Theorem 7.1.1 if and only if $\iota \otimes \alpha(h) \geq 0$ for all $\alpha \in \mathscr{C}$, hence (i) $\Leftrightarrow$ (iii).

(ii) $\Leftrightarrow$ (iii). By Theorem 7.1.1 $\iota \otimes \alpha(h) \geq 0$ for all $\alpha \in \mathscr{C}$ if and only if $\phi \in P_{\mathscr{C}}(K)^\circ$, which by Theorem 6.1.6 is equivalent to $\alpha \circ \phi$ being completely positive for all $\alpha \in \mathscr{C}$. □

The relationship between PPT-states and decomposable maps is clear from the next result.

Corollary 7.2.5 *Let $\phi \in P(H)$. Then ϕ is decomposable if and only if $\rho(C_\phi) \geq 0$ for all PPT-states ρ on $B(H) \otimes B(H)$.*

Proof ϕ is decomposable if and only if $\phi \in CP(H) \vee coCP(H) = (CP(H) \cap coCP(H))^\circ$, hence by Theorem 7.2.2, if and only if $Tr(C_\phi C_\psi) \geq 0$ for all ψ with $\widetilde{\psi}$ a PPT-state. Since $\widetilde{\psi}$ is PPT if and only if $\widetilde{\psi}^t$ is PPT it follows that ϕ is decomposable if and only if $\rho(C_\phi) \geq 0$ for all PPT-states ρ. □

7.3 The Choi Map

For some time it was a problem whether all PPT-states were separable. By Corollary 7.2.5 and the proof of Corollary 7.2.3 this would via Proposition 4.1.11 be the same as saying that all positive maps are decomposable. But we saw in Proposition 2.3.3 that the positive projection of M_n onto the spin factor V_k, $k = 4$ or $k \geq 6$, is indecomposable, so there exist PPT-states which are not separable.

A celebrated example of an indecomposable positive map is the Choi map in $P(\mathbb{C}^3)$. It was the first example known of an indecomposable map and has been generalized to higher dimensions. We shall for simplicity of the argument only study the simplest case, namely

Definition 7.3.1 The *Choi map* is the map $\phi \in P(\mathbb{C}^3)$ defined as follows: If $x = (x_{ij}) \in M_3$ then

$$\phi(x) = \begin{pmatrix} x_{11} + x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} + x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} + x_{22} \end{pmatrix} = \Delta(x) - x,$$

where

$$\Delta(x) = \begin{pmatrix} 2x_{11} + x_{33} & 0 & 0 \\ 0 & 2x_{22} + x_{11} & 0 \\ 0 & 0 & 2x_{33} + x_{22} \end{pmatrix}.$$

It is not immediate that ϕ is positive. For this we shall need two lemmas.

Lemma 7.3.2 *Let ξ_0 be a unit vector in the finite dimensional Hilbert space H, let $p = [\xi_0]$ be the projection onto $\mathbb{C}\xi_0$, and let $a \in B(H)^+$ be invertible. Then $a \geq p$ if and only if $(a^{-1}\xi_0, \xi_0) \leq 1$.*

Proof Suppose $(a^{-1}\xi_0, \xi_0) \leq 1$. By the Cauchy-Schwarz inequality for states

$$1 = (\xi_0, \xi_0)^2 = \left(a^{1/2}\xi_0, a^{-1/2}\xi_0\right)^2 \leq (a\xi_0, \xi_0)\left(a^{-1}\xi_0, \xi_0\right).$$

If it is not the case that $a \geq p$, then $(a\xi_0, \xi_0) < 1$, hence $(a^{-1}\xi_0, \xi_0) > 1$, contrary to assumption. Thus $a \geq p$. Conversely, if $a \geq p$ then, since H is finite dimensional and a is invertible, there is $\varepsilon > 0$ such that $a \geq p + \varepsilon(1 - p)$. Thus $a^{-1} \leq p + \frac{1}{\varepsilon}(1 - p)$, so that $pa^{-1}p \leq p$. Thus $(a^{-1}\xi_0, \xi_0) = (pa^{-1}p\xi_0, \xi_0) \leq 1$. □

Lemma 7.3.3 *Let $\alpha, \beta, \gamma \geq 0$. Then*

$$\frac{\alpha}{2\alpha + \gamma} + \frac{\beta}{2\beta + \alpha} + \frac{\gamma}{2\gamma + \beta} \leq 1.$$

Proof Put

$$x = \frac{\gamma}{\alpha}, \qquad y = \frac{\alpha}{\beta}, \qquad z = \frac{\beta}{\gamma}.$$

Then $xyz = 1$, so the inequality in the lemma becomes

$$\frac{1}{2 + x} + \frac{1}{2 + y} + \frac{1}{2 + z} \leq 1.$$

If we multiply out this reduces to showing

$$xy + xz + yz \geq 3,$$

or since $z = \frac{1}{xy}$, to showing

$$f(x, y) = x^2y^2 + x + y - 3xy \geq 0 \quad \text{for } x, y \geq 0.$$

Then $f(0, 0) = 0$, $f(x, y) \to +\infty$ if either $x \to +\infty$ or $y \to +\infty$. Straightforward calculus shows that the only minimum point for f in $(0, \infty) \times (0, \infty)$ is $(1, 1)$, with value $f(1, 1) = 0$. Thus $f(x, y) \geq 0$, and the lemma is proved. □

Proposition 7.3.4 *The Choi map is positive.*

Proof It suffices to show $\phi(p) \geq 0$ for all 1-dimensional projections p. Let $p = [\xi_0]$ for a unit vector $\xi_0 = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$. Then

$$\Delta(p) = \begin{pmatrix} 2|\alpha_1|^2 + |\alpha_3|^2 & 0 & \\ 0 & 2|\alpha_2|^2 + |\alpha_1|^2 & 0 \\ 0 & 0 & 2|\alpha_3|^2 + |\alpha_2|^2 \end{pmatrix}.$$

By Lemma 7.3.3

$$\left(\Delta(p)^{-1}\xi_0, \xi_0\right) = \frac{|\alpha_1|^2}{2|\alpha_1|^2 + |\alpha_3|^2} + \frac{|\alpha_2|^2}{2|\alpha_2|^2 + |\alpha_1|^2} + \frac{|\alpha_3|^2}{2|\alpha_3|^2 + |\alpha_2|^2} \leq 1.$$

Thus by Lemma 7.3.2

$$\phi(p) = \Delta(p) - p \geq 0,$$

so ϕ is positive. □

One can show that ϕ is atomic and extremal. The proofs are rather involved, so we shall only prove the following result.

Proposition 7.3.5 *The Choi map ϕ is not 2-positive.*

Proof Let $\xi_0 = (0, 1, 1, 1, 1, 0) \in \mathbb{C}^6 = \mathbb{C}^3 \otimes \mathbb{C}^2$. Let $p = [\xi_0]$. Then

$$p = \frac{1}{4}\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = \frac{1}{4}\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

with $p_{ij} \in M_3$ as indicated. Then

$$\iota_2 \otimes \phi(p) = \frac{1}{4}\left(\phi(p_{ij})\right) \in M_2 \otimes M_3.$$

A straightforward calculation shows that

$$\left(\iota_2 \otimes \phi(p)\xi_0, \xi_0\right) = -\frac{1}{4} < 0.$$

Thus ϕ is not 2-positive. □

In addition to the negative result that ϕ is not 2-positive we next show that ϕ is indecomposable.

Proposition 7.3.6 *There exists a PPT-state ρ such that $\rho(C_\phi) < 0$. Hence ϕ is indecomposable.*

Proof We have

$$C_\phi = \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Let $S \in M_3 \otimes M_3$ be the matrix

$$S = \left(\begin{array}{ccc|ccc|ccc} 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{array}\right).$$

Then $S \geq 0$ and $\iota \otimes t(S) \geq 0$, as is easily checked. Let $\rho = \frac{1}{21} Tr(S\,\cdot)$. Then ρ is a PPT-state, and $\rho(C_\phi) < 0$, so by Corollary 7.2.5 ϕ is indecomposable. □

We note that the state ρ is entangled, because ϕ is positive, so $Tr(C_\phi a \otimes b) \geq 0$ for all $a, b \in M_3^+$, using Proposition 4.1.11.

Remark 7.3.7 The Choi map has a natural extension to M_n. Let P be the positive projection of M_n onto the diagonal. Let s be the shift unitary in M_n, so $s = (\delta_{i,i+1})$, where $\delta_{i,j}$ is the Kronecker symbol, and where the indices are understood modulo n. Then we have an extension of the Choi map to $P(\mathbb{C}^n)$ defined by

$$\phi(a) = (n-1)P(a) + \sum_{i=1}^{k} P\left(s^i a s^{i*}\right) - x.$$

It was shown by Tanahashi and Tomiyama [88] and Ha [17] that for $n \geq 3$ and $1 \leq k \leq n-2$, ϕ is atomic, hence in particular indecomposable.

7.4 Entanglement

A state on a tensor product $B(K) \otimes B(H)$ is called *entangled* if it is not separable. We shall see in this section how entanglement is related to positive maps. We first give a definition of entanglement related to a given mapping cone.

Definition 7.4.1 Let $\mathscr{C}$ be a mapping cone in $P(H)$, $\mathscr{C} \supset CP(H)$ with $\dim H < \infty$, and let K be another finite dimensional Hilbert space. Let

$$S_{\mathscr{C}} = \left\{\rho \in B(K \otimes H)^* : \rho = Tr(C_\psi \cdot) \text{ is a state with } \psi \in P_{\mathscr{C}}(K)^\circ\right\},$$

where as before $P_{\mathscr{C}}(K)$ denotes the cone of $\mathscr{C}$ positive maps of $B(K)$ into $B(H)$. We say a state ω on $B(K \otimes H)$ is $\mathscr{C}$*-entangled* if $\omega \notin S_{\mathscr{C}}$.

Note that if $\mathscr{C} = P(H)$, then $\mathscr{C}^\circ = SP_1(H)$, so by Theorem 6.1.7 $P_{\mathscr{C}}(K)^\circ = P_{\mathscr{C}^\circ}(K)$ is the cone of $SP_1(H)$-positive maps, so the corresponding states are separable by Proposition 5.1.4. Thus in this case a state is $\mathscr{C}$-entangled if and only if it is entangled.

We shall need the following lemma. If $\mathscr{C}$ is a symmetric mapping cone it follows from Theorem 6.1.6.

Lemma 7.4.2 *Let $\mathscr{C} \supset CP(H)$ be a mapping cone in $P(H)$ and K finite dimensional. Then each map in $P_{\mathscr{C}}(K)^\circ$ is completely positive.*

Proof Let $\phi \in B(B(K), H)$ belong to $P_{\mathscr{C}}(K)^\circ$. Then $Tr(C_\phi C_\psi) \geq 0$ for all $\psi \in P_{\mathscr{C}}(K)$. Since $\mathscr{C} \supset CP(H)$, $P_{\mathscr{C}}(K) \supset P_{CP}(K)$. By, for example Theorem 5.1.13, $P_{CP}(K)$ consists of the completely positive maps of $B(K)$ into $B(H)$. By Theorem 4.1.8 a map ψ in $B(B(K), H)$ is completely positive if and only if $C_\psi \geq 0$. Thus $Tr(C_\phi x) \geq 0$ for all $x \in B(K \otimes H)^+$, hence $C_\phi \geq 0$, and therefore ϕ is completely positive by Theorem 4.1.8. □

If $\phi \in B(B(K), H)^+$, we let as before, see Sect. 4.1, C_ϕ^+ and C_ϕ^- denote the positive and negative parts of the Choi matrix C_ϕ, so $C_\phi = C_\phi^+ - C_\phi^-$ with $C_\phi^+ C_\phi^- = 0$.

Theorem 7.4.3 *Let e be a projection in $B(K \otimes H)$ and $\mathscr{C}$ be a mapping cone in $P(H)$ such that $\mathscr{C}$ strictly contains $CP(H)$. Then each state ω on $B(K \otimes H)$ with support in e is $\mathscr{C}$-entangled if and only if there exists a $\mathscr{C}$-positive map $\phi \in B(B(K), H)$ with support $C_\phi^- = e$.*

Proof Suppose ϕ is a $\mathscr{C}$-positive map as in the theorem. Let ω be a state with support $\omega \leq e$. Then

$$\omega(C_\phi) = \omega(eC_\phi e) = -\omega\left(C_\phi^-\right) < 0.$$

Thus if ψ is a map in $B(B(K), H)$ such that C_ψ is the density matrix for ω, then $Tr(C_\psi C_\phi) < 0$, so $\psi \notin P_{\mathscr{C}}(K)^\circ$, and therefore $\omega \notin S_{\mathscr{C}}$.

Conversely let $\mu = \sup_{\rho \in S_{\mathscr{C}}} \rho(e)$. We claim that $\mu < 1$. We have

$$1 = \|e\| = \sup\{Tr(eh) : 0 \leq h \leq 1,\ Tr(h) = 1\}.$$

Now $Tr(eh) = Tr(h)$ if and only if $h \leq e$. For such an h the state $Tr(h\,\cdot)$ is by assumption $\mathscr{C}$-entangled, hence $Tr(h\,\cdot) \notin S_{\mathscr{C}}$. It follows that $Tr(eC_\psi) < 1$ for all states $Tr(C_\psi\,\cdot)$ in $S_{\mathscr{C}}$. By compactness of $S_{\mathscr{C}}$ and continuity of the maps $\psi \to Tr(eC_\psi)$, $\mu < 1$ as claimed.

Let $\lambda = 1/\mu$, and let ϕ be the map defined by $C_\phi = 1 - \lambda e$. If $Tr(C_\psi\,\cdot) \in S_{\mathscr{C}}$, so in particular $\psi \in P_{\mathscr{C}}(K)^\circ$, then by definition of μ

$$Tr(C_\phi C_\psi) = 1 - \lambda Tr(eC_\psi) \geq 1 - \lambda\mu = 0. \tag{7.3}$$

If $\psi \in P_{\mathscr{C}}(K)^\circ$ then ψ is completely positive by Lemma 7.4.2, so $Tr(C_\psi\,\cdot)$ is automatically positive and therefore a positive multiple of a state in $S_{\mathscr{C}^\circ}$. Thus (7.3) implies that $\phi \in P_{\mathscr{C}}(K)^{\circ\circ} = P_{\mathscr{C}}(K)$, using Lemma 6.1.2. Thus ϕ is $\mathscr{C}$-positive, with $C_\phi^+ = 1 - e$, and $C_\phi^- = (\lambda - 1)e$ with support e. □

As an immediate corollary we have

Corollary 7.4.4 *Let $\phi \in B(B(K), H)$ be $\mathscr{C}$-positive with $\mathscr{C}$ as in Theorem 7.4.3. Then every state with support in the support of C_ϕ^- is entangled. In particular, this holds for all positive maps $\phi : B(K) \to B(H)$.*

Remark 7.4.5 If $\phi : B(K) \to B(H)$ is unital and positive, and f is the projection onto the eigenspace of C_ϕ corresponding to the eigenvalues $\lambda > 1$, then each state with support majorized by f is entangled. Indeed, since $Tr(a)1 - a \geq 0$ for all $a \geq 0$, the map $\psi = Tr - \phi = \phi \circ (Tr - \iota)$ is positive, $C_\psi = 1 - C_\phi$, so $C_\psi^- = f$, hence our assertion follows from Corollary 7.4.4.

Corollary 7.4.4 has a natural extension to k-positive maps. Recall that a vector $\xi \in K \otimes H$ has Schmidt rank k if k is the smallest number m such that $\xi = \sum_{i=1}^m \xi_i \otimes \eta_i$. For simplicity we state the next result for the case $K = H$.

Corollary 7.4.6 *Let $\phi \in P(H)$ be k-positive and not completely positive. Let ξ be a unit vector in support C_ϕ^-. Then the Schmidt rank of ξ is greater than k.*

Proof Let $\rho = \omega_\xi$ be the vector state defined by ξ. Then $[\xi] = \text{support}\,\rho \leq \text{support}\,C_\phi^-$, hence by Theorem 7.4.3 ρ is P_k-entangled. Thus $\rho = Tr([\xi]\cdot) = Tr(C_\psi\,\cdot)$ with $\psi \notin P_k(H)^\circ$. By Proposition 6.2.3 $P_k(H)^\circ = SP_k(H)$ is the cone of k-super-positive maps. By Proposition 4.1.4 $\psi = AdV$. Since $AdV \in SP_k(H)$ if and only if rank $V \leq k$, it follows that rank $V > k$. Since $[\xi] = C_{AdV}$ it follows from Proposition 4.1.6 that ξ has Schmidt rank $SR(\xi) > k$. □

Remark 7.4.7 The above corollary can easily be extended to the case when $\dim K = m \neq n = \dim H$. See [43] and also [32]. In the last reference. it was also shown that it follows from [91] that

$$\dim \operatorname{supp} C_\phi^- \leq (m-k)(n-k).$$

Using Theorem 7.4.3 we can obtain a large class of indecomposable maps. Let as before K and H be finite dimensional Hilbert spaces. An orthogonal family of product vectors $\xi_i \otimes \eta_i$ in $K \otimes H$ is called an *unextendible product basis* if the orthogonal complement of the span of $\{\xi_i \otimes \eta_i\}$ contains no product vector.

Theorem 7.4.8 *Let $\{\xi_i \otimes \eta_i\}$ be an unextendible product basis for $K \otimes H$, and let X denote the linear span of $\{\xi_i \otimes \eta_i\}$ in $K \otimes H$. Let e denote the orthogonal projection onto the orthogonal complement $X^\perp$ of X. Then there exists $\lambda > 1$ such that the map $\phi : B(K) \to B(H)$ with $C_\phi = 1 - \lambda e$ is indecomposable.*

Proof Applying Theorem 7.4.3 and its proof to the symmetric mapping cone $P(H)$ we see that there exists $\lambda > 1$ such that the map $\phi \in B(B(K), H)$ with $C_\phi = 1 - \lambda e$ is positive. Let $[\xi_i]$ (resp $[\eta_i]$) denote the one dimensional projection onto $\mathbb{C}\xi_i$ (resp. $\mathbb{C}\eta_i$). Then

$$e = 1 \otimes 1 - \sum_i [\xi_i] \otimes [\eta_i].$$

We assert that $\iota \otimes t(e)$ is a projection. Indeed, suppose $f, g \in B(K)$, $p, q \in B(H)$ are projections such that $f \otimes p \perp g \otimes q$, then $f \otimes p^t \perp g \otimes q^t$. This follows, since

$$\begin{aligned} Tr \otimes Tr\big((f \otimes p^t)(g \otimes q^t)\big) &= Tr \otimes Tr\big(fg \otimes p^t q^t\big) \\ &= Tr(fg) Tr\big(p^t q^t\big) \\ &= Tr(fg) Tr(pq) \\ &= Tr \otimes Tr\big((f \otimes p)(g \otimes q)\big) = 0. \end{aligned}$$

It follows that $\iota \otimes t(e)$ being the sum of the orthogonal projections $[\xi_i] \otimes [\eta_i]^t$ is a projection as claimed. But then $\iota \otimes \psi(e) \geq 0$ for all completely positive and copositive maps in $B(B(H), K)$ and thus for all decomposable maps of $B(H)$ into $B(K)$.

Let (e_{ij}) be a complete set of matrix units in $B(K)$ and $p = \sum e_{ij} \otimes e_{ij}$. Then $C_\phi = \iota \otimes \phi(p)$. We thus have for the trace Tr on $B(K \otimes H)$,

$$\begin{aligned} Tr\big(p\big(\iota \otimes \phi^*\big)(e)\big) &= Tr\big(\iota \otimes \phi(p) e\big) \\ &= Tr(C_\phi e) \\ &= Tr(e - \lambda e) \\ &= (1 - \lambda) Tr(e) < 0. \end{aligned}$$

Thus $\iota \otimes \phi^*(e)$ is not positive, hence ϕ^* is indecomposable by the above paragraph. But then ϕ is also indecomposable. □

7.5 Super-positive Maps

One of the main problems in the study of states and positive maps in quantum information theory is to find criteria for when a state is separable, or equivalently for a positive map to be super-positive. We have already seen two approaches, that of PPT-states and the Horodecki theorem, Corollary 7.1.2. A third approach is that of optimal maps, i.e. maps which do not majorize any completely positive maps. For those maps one can construct their SPA, physical structural approximation, which for a unital map ϕ is defined by having a Choi matrix of the form $\frac{1-p}{d^2}1 + pC_\phi$, where $d = \dim H$, and $p \in [0, 1]$ is maximal such that the above Choi matrix is positive. It was conjectured that the corresponding state is separable, see e.g. [12]. This has recently been shown to be false, as shown in [20] and [83].

In the present section we shall show a similar result for any positive map, namely that $\phi(1)Tr + \phi$ is always super-positive. Our approach is close to that used in the study of the SPA above. For this we need some preliminary results.

Lemma 7.5.1 *Let H have an orthonormal basis $\xi_1, \dots, \xi_d$, and let (e_{ij}) be the matrix units such that $e_{ij}\xi_k = \delta_{jk}\xi_i$. Let V denote the flip $V\xi \otimes \eta = \eta \otimes \xi$ on $H \otimes H$. Then*

$$V = \sum e_{ij} \otimes e_{ji},$$

and $Ad(U \otimes U)(V) = V$ for all unitary operators U in $B(H)$.

Proof $\sum e_{ij} \otimes e_{ji}(\xi_k \otimes \xi_l) = \sum \delta_{jk}\xi_i \otimes \delta_{il}\xi_j = \xi_l \otimes \xi_k$, so V is given by the formula in the lemma. To show V is invariant let U be a unitary operator, and $\xi \otimes \eta \in H \otimes H$. Then

$$\begin{aligned} Ad(U \otimes U)(V)\xi \otimes \eta &= \big(U^* \otimes U^*\big)V(U\xi \otimes U\eta) \\ &= \big(U^* \otimes U^*\big)(U\eta \otimes U\xi) \\ &= \eta \otimes \xi \\ &= V(\xi \otimes \eta). \end{aligned}$$

□

Lemma 7.5.2 *Let $(e_{ij})_{i,j=1,2}$ be the usual matrix units in M_2. Let $A \in M_2 \otimes M_2$ be invariant under all automorphisms of $M_2 \otimes M_2$ of the form $Ad\, U \otimes U$ with U unitary in M_2. Then A is of the form*

$$A = a1 + bV,$$

with V as in Lemma 7.5.1.

Proof The proof goes in three steps.

Step 1 Let $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then it follows easily that

$$A = \begin{pmatrix} a_1 & 0 & 0 & b_4 \\ 0 & a_2 & b_3 & 0 \\ 0 & b_2 & a_3 & 0 \\ b_1 & 0 & 0 & a_4 \end{pmatrix}.$$

Step 2 Let $U = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$ with $|z| = 1$. Then with A as above

$$A = Ad\, U \otimes U(A) = \begin{pmatrix} a_4 & 0 & 0 & z^2 b_1 \\ 0 & a_3 & b_2 & 0 \\ 0 & b_3 & a_2 & 0 \\ \overline{z}^2 b_4 & 0 & 0 & a_1 \end{pmatrix}.$$

Thus $a_1 = a_4$, $a_2 = a_3$, $z^2 b_1 = b_4$, $b_2 = b_3$. Since this holds for all $z \in \mathbb{C}$ with $|z| = 1$, $b_1 = b_4 = 0$. Therefore

$$A = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_2 & a_2 & 0 \\ 0 & 0 & 0 & a_1 \end{pmatrix}.$$

If we subtract $b = b_2$ from a_1 we see that A has the form

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} + \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}.$$

The right summand is $b \sum e_{ij} \otimes e_{ji}$, which is invariant under all automorphisms $Ad\, U \otimes U$ by Lemma 7.5.1. Therefore the right summand is invariant, so it remains to show

Step 3 $a = c$. Rewriting we have

$$\begin{aligned} \begin{pmatrix} a & & & \\ & c & & \\ & & c & \\ & & & a \end{pmatrix} &= (a-c) \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} + c 1 \otimes 1 \\ &= (a-c) e_{11} \otimes e_{11} + (a-c) e_{22} \otimes e_{22} + c 1 \otimes 1. \end{aligned}$$

Since $1 \otimes 1$ is invariant we have $(a-c)[e_{11} \otimes e_{11} + e_{22} \otimes e_{22}]$ is invariant. But if we choose U such that

$$Ad\,U(e_{11}) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad Ad\,U(e_{22})\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

then it is easily seen that

$$e_{11} \otimes e_{11} + e_{22} \otimes e_{22} \neq Ad(U \otimes U)(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}).$$

Therefore $a - c = 0$, completing the proof of Step 3, and therefore of the lemma. □

We next extend Lemma 7.5.2 to arbitrary dimensions.

Lemma 7.5.3 *Let $A \in M_d \otimes M_d$ be invariant under all automorphisms $Ad\,U \otimes U$ with U unitary in M_d. Then A is of the form $A = a\,1 \otimes 1 + bV$ with V the flip on $\mathbb{C}^d \otimes \mathbb{C}^d$.*

Proof Let $\xi_1, \ldots, \xi_d$ be an orthonormal basis for $H = \mathbb{C}^d$ and (e_{ij}) the corresponding complete set of matrix units. Let $n \neq m$ belong to $\{1, \ldots, d\}$, and let F_{mn} be the orthogonal projection onto $\mathrm{span}\{\xi_i \otimes \xi_j : i, j \in \{m, n\}\}$. If $\{m, n\} \neq \{p, q\}$ with $p \neq q$, $r \in \{m, n, p, q\}$ and equal to only one of them, and U is the unitary operator such that $U\xi_r = -\xi_r$, and $U\xi_j = \xi_j$ for $j \neq r$, then

$$(A\xi_m \otimes \xi_n, \xi_p \otimes \xi_q) = (AU\xi_m \otimes U\xi_n, U\xi_p \otimes U\xi_q) = -(A\xi_m \otimes \xi_n, \xi_p \otimes \xi_q),$$

hence $(A\xi_m \otimes \xi_n, \xi_p \otimes \xi_q) = 0$ whenever $\{m, n\} \neq \{p, q\}$. Since any permutation of the basis elements is implemented by a unitary operator, A depends only on the matrix elements m, n, p, q, such that $\{m, n\} = \{p, q\}$, i.e.

$$A = \sum a_{(m,p)(n,q)} e_{mp} \otimes e_{nq} \tag{7.4}$$

with $a_{(m,p)(n,q)} = 0$ unless $m = n = p = q$, or $m = p \neq q = n$, or $m = q \neq n = p$.

Considering the unitaries U in M_d such that $U\xi_i = \xi_i$ for $i \neq m, n$ and $U \otimes U F_{mn}\mathbb{C}^d \otimes \mathbb{C} = F_{mn}\mathbb{C}^d \otimes \mathbb{C}$ it follows that $Ad\,U \otimes U(F_{mn}M_d \otimes M_d F_{mn}) = F_{mn}M_d \otimes M_d F_{mn}$. Furthermore $F_{mn}AF_{mn}$ is fixed under the restrictions of $Ad\,U \otimes U$. Thus by Lemma 7.5.2

$$F_{mn}AF_{mn} = aF_{mn} + bV_{mn}, \tag{7.5}$$

where V_{mn} is the restriction of the flip V to $F_{mn}\mathbb{C}^d \otimes \mathbb{C}^d$. If we take U self-adjoint such that $U\xi_m = \xi_p$, $U\xi_n = \xi_q$ and $U\xi_j = \xi_j$ for $j \notin \{m, n, p, q\}$ then

$$Ad\,U \otimes U(F_{mn}AF_{mn}) = F_{pq}AF_{pq}.$$

Thus the coefficients a and b in (7.5) remain the same for $F_{mn}AF_{mn}$ and $F_{pq}AF_{pq}$. It follows that the coefficients $a_{(m,p)(n,q)}$ in (7.4) are given by the formula

$$a_{(m,p)(n,q)} = \begin{cases} a+b & \text{when } m=p, n=q, \\ b & \text{when } m=q\neq p=n, \\ 0 & \text{otherwise.} \end{cases}$$

But this means that $A = a\,1\otimes 1 + bV$ with V as in Lemma 7.5.1. □

Theorem 7.5.4 *Let H be a finite dimensional Hilbert space and $\phi \in P(H)$. Then the map $\phi(1)Tr + \phi$ is super-positive.*

Proof We first show that $Tr + t$, t being the transpose, is super-positive. Let G denote the compact group $G = \{Ad\,U \otimes U : U \text{ unitary in } B(H)\}$. Let dU denote the normalized Haar measure on G. Then

$$P(A) = \int_G Ad\,U \otimes U(A)\,dU$$

is a unital positive projection of $B(H \otimes H)$ onto the fixed point algebra of G, which by Lemma 7.5.3 equals the span $\{a1 \otimes 1 + bV : a, b \in \mathbb{C}\}$. Clearly P is trace invariant, so if h is the density operator for a state ρ on $B(H \otimes H)$, then $Tr(P(h)) = Tr(h) = 1$. Thus

$$P(h) = a1 \otimes 1 + bV, \quad a, b \in \mathbb{R}, \tag{7.6}$$

has trace 1. By Lemma 7.5.1 $Tr(V) = d$ where $d = \dim H$, so that

$$1 = Tr\big(P(h)\big) = ad^2 + bd. \tag{7.7}$$

We have

$$\begin{aligned} Tr\big(P(h)V\big) &= \int Tr\big(Ad\,U \otimes U(h)V\big)\,dU \\ &= \int Tr\big(hAd\,U^* \otimes U^*(V)\big)\,dU \\ &= Tr(hV). \end{aligned} \tag{7.8}$$

Let $e = (h_i\overline{h}_j)$ be a 1-dimensional projection in $B(H)$, and let $h = e \otimes e$, so $\rho = Tr(h\,\cdot)$ is a product state. Then by Lemma 7.5.1

$$\begin{aligned} Tr(hV) &= Tr\left(e \otimes e \sum_{ij} e_{ij} \otimes e_{ji}\right) \\ &= \sum_{ij} Tr(ee_{ij})Tr(ee_{ji}) \\ &= \sum_{ij} (h_i\overline{h}_j)(h_j\overline{h}_i) \\ &= \sum_{ij} |h_i|^2|h_j|^2 = 1. \end{aligned}$$

By (7.6) $P(h)V = aV + b1 \otimes 1$. Thus by (7.7) and (7.8)

$$1 = Tr\big(P(h)V\big) = Tr(aV + b1 \otimes 1) = ad + bd^2.$$

Combining this with (7.7) we get

$$a = b = 1/d(d+1).$$

Hence

$$P(h) = \frac{1}{d(d+1)}(1 \otimes 1 + V),$$

which is the Choi matrix for $\frac{1}{d(d+1)}(Tr + t)$.

Let $e(U) = Ad\,U(e)$, and put $\psi = \frac{1}{d(d+1)}(Tr + t)$. Then $C_\psi = P(h)$, so if $\phi \in P(H)$ we have

$$Tr(C_\psi C_\phi) = Tr\bigg(\bigg(\int e(U) \otimes e(U)\,dU\bigg)C_\phi\bigg) = \int Tr\big(e(U) \otimes e(U)C_\phi\big)\,dU \geq 0,$$

since the integrand is positive by Proposition 4.1.11. Since this holds for all $\phi \in P(H)$, by Proposition 6.2.1 $\psi \in P(H)^\circ = SP_1(H)$. Thus $Tr + t$ is super-positive.

The super-positive maps have the property that their compositions with positive maps remain super-positive, hence $Tr + \iota = t \circ (Tr + t)$ is super-positive. Thus if ϕ is a positive map then

$$\phi(1)Tr(a) + \phi(a) = \phi\big(Tr(a)1\big) + \phi(a) = \phi \circ (Tr + \iota)(a)$$

is super-positive, completing the proof of the theorem. □

If $\|\phi\| \leq 1$ then, since Tr as a map in $P(H)$ is super-positive, $(\|\phi\| - \phi(1))Tr$ is super-positive, hence we have, since $\|\phi\|Tr + \phi = (\|\phi\| - \phi(1))Tr + \phi(1)Tr + \phi$.

Corollary 7.5.5 *If $\phi \in P(H)$ has norm $\|\phi\| \leq 1$, then $Tr + \phi$ is super-positive.*

If we translate the theorem to states we get the following

Corollary 7.5.6 *Let ρ be a state on $B(H \otimes H)$ with $d = \dim H$. Let ρ_2 be the state on the second factor defined by $\rho_2(b) = \rho(1 \otimes b)$. Then the state $\frac{1}{d+1}(Tr \otimes \rho_2 + \rho)$ is separable.*

Proof Since ρ is a state, $\rho = \widetilde{\phi}$ for a completely positive map ϕ, see Theorem 4.2.7. By Theorem 7.5.4 the map $\psi = \phi(1)Tr + \phi$ is super-positive, hence $\widetilde{\psi}$ is separable by Proposition 5.1.4. We have

$$\begin{aligned}\widetilde{\psi}(a \otimes b) &= Tr\big(\big(\phi(1)Tr(a) + \phi(a)\big)b^t\big)\\ &= Tr(a)Tr\big(\phi(1)b^t\big) + Tr\big(\phi(a)b^t\big)\end{aligned}$$

$$= Tr(a)\rho(1 \otimes b) + \rho(a \otimes b)$$
$$= (Tr \otimes \rho_2 + \rho)(a \otimes b),$$

proving the corollary. □

If a map ϕ is not completely positive and $C_\phi = C_\phi^+ - C_\phi^-$, then C_ϕ^- is a nonzero positive operator. We next give an estimate for the norm of C_ϕ^-.

Corollary 7.5.7 *Let* $\phi \in P(H)$ *with* $\|\phi\| \leq 1$. *Then* $C_\phi \geq -1 \otimes 1$, *hence* $\|C_\phi^-\| \leq 1$.

Proof By Corollary 7.5.5 $Tr + \phi$ is super-positive so in particular completely positive. Hence

$$0 \leq C_{Tr+\phi} = C_{Tr} + C_\phi = 1 + C_\phi = 1 + C_\phi^+ - C_\phi^-.$$

Since $C_\phi^+ C_\phi^- = 0$, it follows that $C_\phi^- \leq 1$. □

It follows that for all $\phi \in P(H)$, $\|C_\phi^-\| \leq \|\phi\|$.

7.6 Notes

Behind much of the theory in Chap. 7 lies the duality $\phi \to \widetilde{\phi}$. Theorem 7.1.1 is essentially a translation of Theorem 6.1.6 to states and is taken from [82]. Corollary 7.1.2 is in the form given, a celebrated result of Horodecki [25], and is often referred to as one of the main results which show the importance of general positive maps rather than completely positive maps. The identity (7.2) in Remark 7.1.4 can be found in [78] in the case $K = H$.

PPT-states were introduced by Peres [60]. Theorem 7.2.2 and its corollaries have been observed independently by several authors. For a discussion on PPT-states, see [2]. By work of Woronowicz [98] Corollary 7.2.3 is also true for states on $M_2 \otimes M_3$ and $M_3 \otimes M_2$.

The Choi map described in Sect. 7.3 was introduced by Choi [8]. It has in generalized form been studied by others, see [4, 44] and [55], because it is an indecomposable map in the least possible dimension, 3×3 matrices. It and its extension to higher dimensions as in Remark 7.3.7 have been shown to be both atomic and extremal see [17, 55–57, 88]. Related results on extremal and indecomposable maps were obtained in [39]. The example in 7.3.6 was exhibited in [77]. This was before the introduction of PPT-states by Peres, see also [21]. An example of a PPT-state which was not separable was later exhibited by P. Horodecki [23].

Theorem 7.4.3 is due to Skowronek and the author, [68], but a related result for $P(H)$ is due to Parthasarathy [58]. For Corollary 7.4.6 see the paper by Kuah and Sudarshan [43] and Sarbicki [66]. Theorem 7.4.8 is due to Terhal [89].

Section 7.5 is based on work of Werner [97], Horodecki [24] and Chruściński, Pytel [11]. They were mainly interested in optimal maps and whether they were super-positive. Theorem 7.5.4 is different and new, being a result on all possible positive maps, but it can easily be deduced from [24].

Chapter 8
Norms of Positive Maps

As we saw in Chap. 1 the uniform norm of a positive map ϕ from an operator system A into $B(H)$ is defined by

$$\|\phi\| = \sup_{\|a\|\le 1} \left\|\phi(a)\right\|,$$

which equals $\|\phi(1)\|$ if A is a unital C^*-algebra. However, there are many other norms that could be used. In this chapter we shall consider some of these norms, first for general positive maps in $B(B(K), H)$ and then for the so-called Werner maps of the form $Tr - AdV$, $V : H \to K$.

8.1 Norms of Maps

Let K and H be finite dimensional Hilbert spaces, $\mathscr{C}$ a mapping cone in $P(H)$ and $P_{\mathscr{C}}(K)$ the $\mathscr{C}$-positive maps in $B(B(K), H)$, and $P_{\mathscr{C}}(K)^\circ$ the dual cone of $P_{\mathscr{C}}(K)$. Let in analogy with Definition 7.4.1

$$S_{\mathscr{C}} = \left\{\rho \in B(K \otimes H)^* : \rho = Tr(C_\psi \,\cdot), Tr(C_\psi) = 1, \psi \in P_{\mathscr{C}}(K)^\circ\right\}.$$

Thus $S_{\mathscr{C}}$ is a convex set of linear functionals. Note that if $\mathscr{C} \supset CP(H)$, then as pointed out in the proof of Lemma 7.4.2, every map $\psi \in P_{\mathscr{C}}(K)^\circ$ is completely positive, hence the definitions of $S_{\mathscr{C}}$ given in Definition 7.4.1 and above, coincide. If $a \in B(K \otimes H)$ let

$$\|a\|_{S_{\mathscr{C}}} = \sup\left\{\left|\rho(a)\right| : \rho \in S_{\mathscr{C}}\right\},$$

and if $\phi \in B(B(K), H)$, let

$$\|\phi\|_{\mathscr{C}} = \sup\left\{\left|Tr(C_\phi C_\psi)\right| : \rho = Tr(C_\psi \,\cdot) \in S_{\mathscr{C}}\right\}.$$

Then $\|\phi\|_{\mathscr{C}} = \|C_\phi\|_{S_{\mathscr{C}}}$.

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Lemma 8.1.1 *$\| \ \|_{S_{\mathscr{C}}}$ and $\| \ \|_{\mathscr{C}}$ are norms on $B(K \otimes H)$ and $B(B(K), H)$ respectively.*

Proof The norm properties $\|\lambda a\|_{S_{\mathscr{C}}} = |\lambda| \|a\|_{S_{\mathscr{C}}}$ and $\|\lambda\phi\|_{\mathscr{C}} = |\lambda| \|\phi\|_{\mathscr{C}}$ are clear, and the same is subadditivity, i.e. $\|\phi + \psi\|_{\mathscr{C}} \leq \|\phi\|_{\mathscr{C}} + \|\psi\|_{\mathscr{C}}$. Since the composition of a super-positive map with a positive map is super-positive by Lemma 5.1.3, the super-positive maps in $B(B(K), H)$ belong to the dual cone $P_{\mathscr{C}}(K)^\circ$ by Theorem 6.1.6. Thus $S_{\mathscr{C}}$ contains all states with density operator corresponding to super-positive maps, hence all separable states by Proposition 5.1.4. By Lemma 5.1.7 and its proof, if $\omega(a) = 0$ for all separable states then $\rho(a) = 0$ for all states ρ, hence $a = 0$. Thus $\| \ \|_{S_{\mathscr{C}}}$ and $\| \ \|_{\mathscr{C}}$ are norms. □

Recall that if $\phi \in B(B(K), H)$ is positive then C_ϕ is a self-adjoint operator with positive and negative parts C_ϕ^+ and C_ϕ^-, so $C_\phi = C_\phi^+ - C_\phi^-$, and $C_\phi^+ C_\phi^- = 0$. Let ϕ^+ and ϕ^- be the completely positive maps such that $C_{\phi^+} = C_\phi^+$, $C_{\phi^-} = C_\phi^-$. Then we have

Proposition 8.1.2 *Let $\mathscr{C}$ be a mapping cone in $P(H)$ containing $CP(H)$. Let $\phi \in B(B(K), H)$ be $\mathscr{C}$-positive. Then*

$$\left\|\phi^+\right\|_{\mathscr{C}} \geq \left\|\phi^-\right\|_{\mathscr{C}},$$

or equivalently, $\|C_\phi^+\|_{S_{\mathscr{C}}} \geq \|C_\phi^-\|_{S_{\mathscr{C}}}$.

Proof As noted in Lemma 7.4.2, since $\mathscr{C} \supset CP(H)$, $P_{\mathscr{C}}(K)^\circ$ is contained in the cone of completely positive maps of $B(K)$ into $B(H)$. Therefore if $\rho = Tr(C_\psi \cdot) \in S_{\mathscr{C}}$, then ρ is a state. Since $\phi \in P_{\mathscr{C}}(K)$,

$$0 \leq Tr(C_\phi C_\psi) = Tr\left(C_\phi^+ C_\psi\right) - Tr\left(C_\phi^- C_\psi\right).$$

Thus, since $C_\psi \geq 0$ by Theorem 4.1.8,

$$\left\|\phi^+\right\|_{\mathscr{C}} \geq \sup_\psi Tr\left(C_\phi^- C_\psi\right) = \left\|\phi^-\right\|_{\mathscr{C}}.$$

Since $\|C_\phi^+\|_{S_{\mathscr{C}}} = \|\phi^+\|_{\mathscr{C}}$, and the same for ϕ^-, the proof is complete. □

The reader should note the related result, Corollary 7.5.7, that if ϕ is unital then $\|C_\phi^-\| \leq 1 = \|\phi\|$.

We saw in Theorem 4.1.12 that each positive map in $B(B(K), H)$ can be written in the form

$$\left\|C_\phi^+\right\|^{-1} \phi = Tr - \phi_{cp},$$

where Tr is the usual trace on $B(K)$ identified with the map $a \to Tr(a)1$, and $\phi_{cp} \in B(B(K), H)$ is completely positive. In the next proposition we just consider $Tr - \phi_{cp}$.

Proposition 8.1.3 *Let $\mathscr{C}$ be a mapping cone in $P(H)$ containing $CP(H)$. Let $\phi = Tr - \phi_{cp}$ as above. Then ϕ is $\mathscr{C}$-positive if and only if*

$$\|\phi_{cp}\|_{\mathscr{C}} \leq 1.$$

Proof By Lemma 6.1.2 $P_{\mathscr{C}}(K) = P_{\mathscr{C}}(K)^{\circ\circ}$, so ϕ is $\mathscr{C}$-positive if and only if $Tr(C_{\phi}C_{\psi}) \geq 0$ for all $\psi \in P_{\mathscr{C}}(K)^{\circ}$, if and only if $Tr(C_{\phi}C_{\psi^t}) \geq 0$ for all ψ such that $\tilde{\psi} \in S_{\mathscr{C}}$ (recall that C_{ψ^t} is the density operator for $\tilde{\psi}$ by Lemma 4.2.3). Now $C_{Tr} = 1$. Thus ϕ is $\mathscr{C}$-positive if and only if

$$\begin{aligned}
0 &\leq \inf_{\tilde{\psi} \in S_{\mathscr{C}}} Tr(C_{\phi}C_{\psi^t}) \\
&= \inf_{\tilde{\psi}} Tr\big((1 - C_{\phi_{cp}})C_{\psi^t}\big) \\
&= 1 - \sup_{\tilde{\psi}} Tr(C_{\phi_{cp}}C_{\psi^t}) \\
&= 1 - \|\phi_{cp}\|_{\mathscr{C}},
\end{aligned}$$

if and only if $\|\phi_{cp}\| \leq 1$, where we used that $\mathscr{C} \supset CP(H)$ to conclude that $\sup Tr(C_{\phi_{cp}}C_{\psi^t}) = \|\phi_{cp}\|_{\mathscr{C}}$. □

We next specialize to the cone of k-positive maps P_k. Recall from Proposition 6.2.3 that $P_k(H)^{\circ} = SP_k(H)$, where $SP_k(H)$ is the cone of maps of the form $\sum_i AdV_i$, where each $V_i \in B(H)$ has rank $V_i \leq k$. For a vector $\xi \in K \otimes H$ its Schmidt rank is by Definition 4.1.5 the smallest m such that $\sum_{i=1}^{m} \xi_i \otimes \eta_i \in K \otimes H$ represents ξ, i.e. $SR(\xi) = \min m$, with m as above. For a self-adjoint operator $a \in B(K \otimes H)$ we define a norm

$$\|a\|_{S(k)} = \sup\big\{\big|(a\xi, \xi)\big| : \xi \in K \otimes H, \|\xi\| = 1, SR(\xi) \leq k\big\}.$$

Lemma 8.1.4 *Let ϕ be a positive map in $B(B(K), H)$. Then*

$$\|\phi\|_{P_k(H)} = \|C_{\phi}\|_{S(k)}.$$

Proof Recall from Theorem 5.1.13 that the cone $P_{SP_k}(K)$ of k-super-positive maps is generated as a cone by maps $\alpha \circ \beta$ with $\alpha \in SP_k(H)$ and $\beta \in B(B(K), H)$ completely positive. Since $SP_k(H)$ is generated by maps AdV with $V \in B(H)$ with rank $V \leq k$ and β is a sum of maps AdW with $W : H \to K$, it follows that $P_{SP_k}(K)$ is generated by maps $V : H \to K$ with rank $V \leq k$. Recall also from Proposition 4.1.6 that if $[\xi]$ is the 1-dimensional projection on the space $\mathbb{C}\xi$ then $[\xi] = C_{AdV}$ with $V : H \to K$, and rank $V = SR(\xi)$. Using this we have for ϕ in the lemma,

$$\begin{aligned}
\|C_{\phi}\|_{S(k)} &= \sup\big\{\big|(C_{\phi}\xi, \xi)\big| : \|\xi\| = 1,\ SR(\xi) \leq k\big\} \\
&= \sup\big\{\big|Tr\big(C_{\phi}[\xi]\big)\big| : SR(\xi) \leq k\big\}
\end{aligned}$$

$$= \sup\{|Tr(C_\phi C_{AdV})| : Tr(C_{AdV}) = 1,\ \text{rank}\, V \le k\}$$
$$= \sup\{|Tr(C_\phi C_\psi)| : \psi \in SP_k(K),\ Tr(C_\psi) = 1\}$$
$$= \|\phi\|_{P_k(H)}. \qquad \square$$

From this result we get immediately from Proposition 8.1.3

Proposition 8.1.5 *Let $\phi \in B(B(K), H)$ be of the form $\phi = Tr - \phi_{cp}$ with ϕ_{cp} completely positive. Then ϕ is k-positive if and only if $\|C_{\phi_{cp}}\|_{S(k)} \le 1$.*

In particular, if $\phi_{cp} = AdV$ we get

Corollary 8.1.6 *Let $V : H \to K$ be a linear operator. Then the map $\phi = Tr - AdV$ is k-positive if and only if*

$$\|C_{AdV}\|_{S(k)} \le 1.$$

It turns out that the norms $\|AdV\|_{P_k(H)}$ and $\|C_{AdV}\|_{S(k)}$ are closely related to the Ky Fan norms defined as follows.

Definition 8.1.7 Let $a \in B(H)^+$, and $\dim H = d$. For $k \in \{1, \dots, d\}$ define the *Ky Fan norm* of a to be

$$\|a\|_{(k)} = \sum_{i=1}^{k} s_i,$$

where $s_1 \ge s_2 \ge \cdots \ge s_d$ are the eigenvalues of a in decreasing order.

A useful characterization of the Ky Fan norm of a positive operator is given by the *Ky Fan maximum principle*.

Lemma 8.1.8 *Let $a \in B(H)^+$. Then*

$$\|a\|_{(k)} = \max\left\{\sum_{i=1}^{k}(a\xi_i, \xi_i) : (\xi_i)_{i=1}^{k} \text{ is an orthonormal set in } H\right\}$$
$$= \max\{Tr(ae) : e\ k\text{-dimensional projection in } B(H)\}.$$

Proof If e is a k-dimensional projection, then $e = \sum_1^k [\xi_i]$ with $\{\xi_, \dots, \xi_k\}$ an orthonormal set of vectors in $e(H)$. Since

$$Tr(ae) = \sum Tr(a[\xi_i]) = \sum (a\xi_i, \xi_i),$$

the last equality in the lemma is obvious.

Let $s_1 \geq s_2 \geq \cdots \geq s_d$ be the eigenvalues of a in decreasing order. Let η_i be an eigenvector for the eigenvalue s_i. Then $(a\eta_i, \eta_i) = s_i$, so that

$$\|a\|_{(k)} = \sum_1^k (a\eta_i, \eta_i) \leq \max \sum_1^k (a\xi_i, \xi_i) = Tr(ea),$$

with e and (ξ_i) as above. Thus $\|a\|_{(k)} \leq \max_e Tr(ea)$.

We must next show the opposite inequality. By a small perturbation of the s_i we may assume they are distinct. Let $\{\xi_1, \ldots, \xi_k\}$ be an orthonormal set in H, and arrange the indices so that

$$(a\xi_1, \xi_1) \geq (a\xi_2, \xi_2) \geq \cdots \geq (a\xi_k, \xi_k).$$

We use induction to conclude that $(a\xi_i, \xi_i) \leq s_i$. Since $s_1 = \|a\|$, clearly $(a\xi_1, \xi_1) \leq s_1$. Assume we have shown $(a\xi_i, \xi_i) \leq s_i$ for $i = 1, \ldots, m-1$. There are two cases.

Case (1). $(a\xi_{m-1}, \xi_{m-1}) \geq s_m$.

In this case $(a\xi_m, \xi_m) \leq s_m$, because the dimension of the eigenspace corresponding to the eigenvalues greater than s_m is $m-1$, using that the s_i are assumed to be distinct.

Case (2). $(a\xi_{m-1}, \xi_{m-1}) < s_m$.

Then $(a\xi_m, \xi_m) < s_m$, so in either case $(a\xi_m, \xi_m) \leq s_m$, completing the induction argument. It follows that $\sum_1^k (a_i\xi_i, \xi_i) \leq \sum_1^k s_i = \|a\|_{(k)}$, completing the proof. □

In order to relate the norms for AdV discussed above to the Ky Fan norm we have

Theorem 8.1.9 *Let $V \in B(H)$. Then*

$$\begin{aligned}\left\|VV^*\right\|_{(k)} &= \sup\left\{Tr(C_{AdV}C_{AdW}) \,:\, \text{rank}\, W \leq k, Tr(C_{AdW}) = 1\right\} \\ &= \|AdV\|_{P_k(H)}.\end{aligned}$$

Proof The last equality follows from the definition of $\|AdV\|_{P_k(H)}$ and the fact that $P_k(H)^\circ = SP_k(H)$, see Proposition 6.2.3.

We first show that $\|VV^*\|_{(k)}$ majorizes the right side of the equality in the theorem. Let $W \in B(H)$ have rank $W \leq k$ and $Tr(C_{AdW}) = 1$. Then the range projection e of W has dimension $\leq k$, and $W = eW$. Let $\xi_1, \ldots, \xi_d$ be an orthonormal basis for H, and e_{ij} matrix units such that $e_{ij}\xi_k = \delta_{jk}\xi_i$. Suppose $V\xi_k = \sum_i v_{ik}\xi_i$. Then by Proposition 4.1.4

$$C_{AdV} = \sum v_{jl}\overline{v_{ik}} e_{ij} \otimes e_{kl}$$

is a scalar multiple of the projection onto $\mathbb{C}\xi$ with $\xi = \sum \overline{v_{ik}}\xi_i \otimes \xi_k$. Similarly by our assumption on W, C_{AdW} is the projection onto $\mathbb{C}\eta$ with $\eta = \overline{\sum w_{ik}}\xi_i \otimes \xi_k$. We thus have

$$\begin{aligned}
Tr(C_{AdV}C_{AdW}) &= Tr\Big(\sum v_{jl}\overline{v_{ik}}e_{ij}\otimes e_{kl}\sum w_{st}\overline{w_{uv}}e_{us}\otimes e_{vt}\Big)\\
&= \sum Tr(\delta_{ju}\delta_{lv}v_{jl}\overline{v_{ik}}w_{st}\overline{w_{jl}}e_{is}\otimes e_{kt})\\
&= \sum Tr\Big(\sum v_{jl}\overline{v_{ik}}w_{ik}\overline{w_{jl}}e_{ii}\otimes e_{kk}\Big)\\
&= \Big(\sum_{jl} v_{jl}\overline{w_{jl}}\Big)\Big(\sum \overline{v_{ik}}w_{ik}\Big)\\
&= Tr\big(VW^*\big)Tr\big(V^*W\big)\\
&= \big|Tr\big(VW^*\big)\big|^2. \qquad (8.1)
\end{aligned}$$

Now C_{AdW} is a 1-dimensional projection, so applying the above to $V = W$, we get

$$1 = Tr(C_{AdW}) = Tr\big(C_{AdW}^2\big) = Tr\big(WW^*\big). \qquad (8.2)$$

Since $W = eW$ we thus have from the above and Lemma 8.1.8,

$$\begin{aligned}
Tr(C_{AdV}C_{AdW}) &= \big|Tr\big(VW^*e\big)\big|^2\\
&\le Tr\big(eV(eV)^*\big)Tr\big(WW^*\big)\\
&= Tr\big(eVV^*\big)\\
&\le \sup_{\operatorname{rank} f\le k} Tr\big(fVV^*\big)\\
&= \big\|VV^*\big\|_{(k)}^2. \qquad (8.3)
\end{aligned}$$

It remains to show the opposite inequality. Since H is finite dimensional, we can by compactness find a projection e with $\operatorname{rank} e \le k$ such that by Lemma 8.1.8

$$\big\|VV^*\big\|_{(k)} = \sup_{\operatorname{rank} f\le k} Tr\big(fVV^*\big) = Tr\big(eVV^*\big).$$

Let $W = \|VV^*\|_{(k)}^{-1/2}eV$. Then $\operatorname{rank} W \le k$, and

$$\|W\|_{HS}^2 = \big\|VV^*\big\|_{(k)}^{-1}Tr\big((eV)(eV)^*\big) = \big\|VV^*\big\|_{(k)}^{-1}Tr\big(eVV^*\big) = 1.$$

In particular, $1 = \|VV^*\|_{(k)}^{-1/2}Tr(WV^*) = \|VV^*\|_{(k)}^{-1/2}Tr(VW^*)$. Since by (8.2), $Tr(C_{AdW}) = 1$, we thus have by (8.1)

$$\begin{aligned}
\big\|VV^*\big\|_{(k)} &= \big\|VV^*\big\|_{(k)}^{1/2}Tr\big(VW^*\big)\cdot 1\\
&= \big\|VV^*\big\|_{(k)}^{1/2}Tr\big(VW^*\big)\big\|VV^*\big\|_{(k)}^{-1/2}Tr\big(WV^*\big)\\
&= \big|Tr\big(VW^*\big)\big|^2\\
&= Tr(C_{AdV}C_{AdW}).
\end{aligned}$$

Thus the sup on the right side of the equation in the theorem is attained and equal to $\|VV^*\|_{(k)}$, and we have the asserted equality. □

Corollary 8.1.10 *Let $V \in B(H)$. Then the map $Tr - AdV$ is k-positive if and only if*

$$\left\|VV^*\right\|_{(k)} \leq 1.$$

Proof By Theorem 8.1.9 $\|VV^*\|_{(k)} = \|AdV\|_{P_k(H)}$, which equals $\|C_{AdV}\|_{S(k)}$ by Lemma 8.1.4, so the corollary follows from Corollary 8.1.6. □

As a consequence it is easy to exhibit maps of the form $Tr - AdV$ which are k-positive but not $(k+1)$-positive. Just take V with $\|VV^*\|_{(k)} = 1 < \|VV^*\|_{(k+1)}$. For more results along these lines see [5, 85, 94].

8.2 Notes

The treatment in the last chapter follows closely the paper [68]. For Proposition 8.1.3 see [81]. However, for k-positive maps of the form $Tr - AdV$ the results had been obtained earlier by Chruscinski and Kossakowski [10]. Corollary 8.1.6 is due to Johnston and Kribs [32], where one also can find a proof of the Ky Fan maximum principle, Lemma 8.1.8. For further study of norms and the relation to operator spaces see [33].

Appendix

In this appendix we collect a few basic results which are needed in the text. They are on topologies on $B(H)$, tensor products, and an extension theorem for linear functionals which are positive on a cone.

A.1 Topologies on $B(H)$

In addition to the norm topology we shall come across two topologies on $B(H)$, the strong and weak topologies. Their definitions are as follows. The *strong topology* has neighborhood basis around $a \in B(H)$ given by the sets

$$\{b \in B(H) : \|b\xi_i - a\xi_i\| < \varepsilon,\ \xi_1, \dots, \xi_n \in H\}.$$

The *weak topology* has neighborhood basis around $a \in B(H)$ given by the sets

$$\{b \in B(H) : |(b\xi_i, \eta_i) - (a\xi_i, \eta_i)| < \varepsilon,\ \xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in H\}.$$

We refer the reader to Chap. 5 in [38] for the properties of the strong and weak topologies. We now list with references to [38] in brackets some of the main results we shall need.

A1.1 (5.1.2) The weak and strong closures of a convex subset of $B(H)$ coincide. If particular, a unital C^*-algebra is a von Neumann algebra if it is weakly closed, or equivalently if it is strongly closed.

A1.2 (5.1.3) The unit ball in $B(H)$ is weakly compact.

A1.3 (5.1.4) If (a_α) is a monotone increasing net of self-adjoint operators in $B(H)$ with $a_\alpha \le k1$ for some $k > 0$ for all α, then (a_α) is strongly convergent to a self-adjoint operator a, and a is the least upper bound for (a_α).

For us the last result is very important for going from the finite dimensional case to infinite dimensions. Many properties of positive maps can be extended to infinite

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dimensions from the finite dimensional case, by considering $e_\alpha \phi e_\alpha$ for $\phi : B(K) \to B(H)$, or similarly by considering $\phi(e_\alpha \cdot e_\alpha)$ if $\phi : B(K) \to B(H)$ where (e_α) is an increasing net of finite dimensional projections converging strongly to 1.

A positive map $\phi : A \to B(H)$ with A a von Neumann algebra is said to be *normal* if for each net (a_α) in A as in Sect. A.1 we have $\phi(a_\alpha) \to \phi(a)$ strongly. In particular if $\omega : A \to \mathbb{C}$ is a state we have the following theorem.

A1.4 (7.1.2) The following conditions on a state ω on a von Neumann algebra A acting on a Hilbert space H are equivalent:

(i) $\omega = \sum_{i=1}^{\infty} \omega_{\xi_i}$ with $\sum_i \|\xi_i\|^2 = 1$ an orthogonal family of vectors in H.
(ii) $\omega = \sum_1^{\infty} \omega_{\eta_i}$ with $\eta_i \in H$, $\sum_i \|\eta_i\|^2 = 1$.
(iii) ω is weakly continuous on the unit ball of A.
(iv) ω is strongly continuous on the unit ball of A.
(v) ω is normal.

It follows that a positive map $\phi : A \to B(K)$ is normal if and only if ϕ is weakly continuous on the unit ball of A. Concerning the norm topology we have the following result on the convex hull of the unitary operators in a C^*-algebra—the Russo-Dye theorem, see [65] or (10.5.4).

A1.5 Let A be a unital C^*-algebra. Then the convex hull of the unitary operators in A is norm dense in the unit ball of A.

Let A be an operator system and as before, $B(A, H)$ the bounded linear maps of A into $B(H)$. Then the *BW-topology* (BW stands for bounded-weak) on $B(A, H)$ is the topology, where a bounded net (ϕ_α) in $B(A, H)$ converges to $\phi \in B(A, H)$ if $\phi_\alpha(a) \to \phi(a)$ weakly for each $a \in A$.

Theorem A.1.1 *With the above notation let A_1 denote (resp. $B(H)_1$) the unit ball of A (resp. $B(H)$). Let*

$$S = \{\phi \in B(A, H) : \|\phi\| \le 1\}.$$

Then S is compact in the BW-topology.

Proof Let $X = \prod_{a \in A_1} B(H)_{1a}$, where $B(H)_{1a} = B(H)_1$, be the product space of $B(H)_1$ indexed by A_1. By the Tychonoff theorem X is compact in the product topology when $B(H)_1$ is given the weak topology, so is weakly compact. Consider the map $S \to X$ defined by

$$\phi \to \phi' = (\phi(a)) \in X,$$

where $\phi(a)$ is the ath coordinate of ϕ'. By definition of the BW-topology and the product topology the map $\phi \to \phi'$ is a homeomorphism of S onto its image $S' \subset X$. To show S is compact it follows from the compactness of X that it remains to show S' is closed in X. So let ψ' be a limit point of S' in X. We must show there exists $\phi \in S$ such that $\phi' = \psi'$. Let ψ be the map of A_1 into X such that $\psi(a) = \psi'(a)$

for $a \in A_1$. Since A_1 spans A, ψ can be extended linearly to a map $\phi : A \to B(H)$. To show $\phi \in B(A, H)$ we must show that ϕ is single valued and linear, hence to show that if $a_i \in A_1, \alpha_i \in \mathbb{C}, i = 1, \ldots, n$, and $\sum_i \alpha_i a_i = 0$, then $\sum \alpha_i \phi(a_i) = 0$. For this let ρ be a normal state on $B(H)$ and $\varepsilon > 0$. Let $c = 1 + \sum_i |\alpha_i|$. Since ψ' was a limit point of S', and $\psi'(a) = \phi(a)$ for $a \in A_1$, there exists $\tau \in S$ such that $|\rho(\tau(a_i) - \phi(a_i))| < \varepsilon/c$ for all i. Then since $\sum \alpha_i a_i = 0$, and τ is linear,

$$\left|\rho\left(\sum_i \alpha_i \phi(a_i)\right)\right| = \left|\rho\left(\sum_i \alpha_i \phi(a_i)\right) - \sum \alpha_i \tau(a_i)\right| < \varepsilon.$$

Since the normal states separate $B(H)$, and ε is arbitrary $\sum_i \alpha_i \phi(a_i) = 0$. Thus $\phi \in S$, and $\psi' = \phi' \in S'$, so S' is closed, proving the theorem. □

This proof is based on a more general result in [37]. Note that with $H = \mathbb{C}$, the above theorem reduces to the well known result that the state space of A is w^*-compact.

A.2 Tensor Products

Let K and H be Hilbert spaces, and let $(\xi_i)_{i \in I}$, I an index set, be an orthonormal basis for K. Let $H_i = H$ for $i \in I$, and let $\widetilde{H} = \oplus_{i \in I} H_i$ be the Hilbert space direct sum of the H_i. Let $\xi = \sum_{i \in I} \alpha_i \xi_i \in K$ and $\eta \in H$. Define the *product vector* $\xi \otimes \eta \in \widetilde{H}$ by

$$\xi \otimes \eta = (\alpha_i \eta)_{i \in I}.$$

Then

$$\|\xi \otimes \eta\|^2 = \sum |\alpha_i|^2 \|\eta\|^2 = \|\xi\|^2 \|\eta\|^2,$$

so $\xi \otimes \eta$ is well defined. We define the algebraic tensor product of K and H as the linear span of all product vectors as above, and denote by $K \otimes H$ the completion in $\widetilde{H}$. We define an inner product on product vectors by

$$(\xi \otimes \eta, \psi \otimes \mu) = (\xi, \psi)(\eta, \mu), \quad \xi, \psi \in K,\ \eta, \mu \in H,$$

and extend it bilinearly to $K \otimes H$. Thus $K \otimes H$ is a Hilbert space. If $a \in B(K), b \in B(H)$ we let $a \otimes b$ be the operator on $K \otimes H$ defined by

$$a \otimes b(\xi \otimes \eta) = a\xi \otimes b\eta.$$

We let products and adjoints be given by coordinate action, i.e.

$$(a \otimes b)(c \otimes d) = ac \otimes bd, \quad a, c \in B(K),\ b, d \in B(H),$$
$$(a \otimes b)^* = a^* \otimes b^*.$$

It follows that the linear span of all operators $a \otimes b$ is a $*$-subalgebra of $B(K \otimes H)$. Its weak closure is the von Neumann algebra denoted by $B(K) \otimes B(H)$.

If $(\eta_j)_{j \in J}$ is an orthonormal basis for H, then $(\xi_i \otimes \eta_j)_{(i,j) \in I \times J}$ is an orthonormal basis for $K \otimes H$. One can then use this to show $B(K) \otimes B(H) = B(K \otimes H)$.

Assume now K is finite dimensional; let $n = \dim K$. Then $B(K) \cong M_n$—the complex $n \times n$-matrices. Let (e_{ij}) be a complete set of matrix units for $B(K)$, $1 \leq i$, $j \leq n$, so $\sum e_{ii} = 1$, $e_{ij} e_{kl} = \delta_{jk} e_{il}$. Let $a = (a_{ij}) = \sum a_{ij} e_{ij} \in B(K)$. Then

$$a \otimes b = \sum a_{ij} e_{ij} \otimes b.$$

If $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ is the unit vector with 1 in the ith coordinate, then $(e_1, \dots, e_n)$ is an orthonormal basis for K with $e_{ij} e_k = \delta_{jk} e_i$. Thus if $\xi = \sum \xi_i e_i \in K$ and $\eta \in H$ we get

$$\begin{aligned} a \otimes b(\xi \otimes \eta) &= \left((a_{ij}) \sum_{k=1}^{n} \xi_k e_k \right) \otimes b\eta \\ &= \sum a_{ij} \xi_j e_j \otimes b\eta \\ &= \Big(\sum a_{1j} \xi_j e_1 \otimes b\eta, \dots, \sum a_{nj} \xi_j e_n \otimes \eta \Big). \end{aligned}$$

This can be written as matrices, where we now write the vectors as column vectors. We then get

$$a \otimes b(\xi \otimes \eta) = \begin{pmatrix} a_{11} b & \cdots & a_{1n} b \\ \vdots & & \vdots \\ a_{n1} b & \cdots & a_{nn} b \end{pmatrix} \begin{pmatrix} \xi_1 b\eta \\ \vdots \\ \xi_n b\eta \end{pmatrix}.$$

Hence $a \otimes b$ is the $n \times n$ block matrix $(a_{ij} b)$ over $B(K)$, so that

$$B(K) \otimes B(H) = M_n\big(B(H)\big).$$

Since the flip $a \otimes b \to b \otimes a$ defines an isomorphism of $B(K) \otimes B(H)$ onto $B(H) \otimes B(K)$ we can, if $\dim H = m < \infty$, also consider $a \otimes b$ with $b = (b_{kl}) \in B(H)$ as the block matrix $(ab_{kl}) \in M_m(B(K))$. This will be done on some occasions.

A.3 An Extension Theorem

Results of the Hahn-Banach type, where one extends a linear functional or map from a subspace to a larger space, are of crucial importance in functional analysis. We shall need the following result of Krein, see [53, Ch. 1, §3, Theorem 2].

Theorem A.3.1 *Suppose K is a convex cone in a real locally convex space X. Assume K contains interior points, and let M be a subspace of X which contains* *at least one interior point of* K. *Then every linear functional* $f(x)$ *on* M *which is positive on* $K \cap M$ *can be extended to a linear functional* $F(x)$ *on* X *which is positive on* K.

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