Generalizations of Integral Inequalities for Integrals Based on Nonadditive Measures

Endre Pap and Mirjana Štrboja

Abstract. There is given an overview of generalizations of the integral inequalities for integrals based on nonadditive measures. The Hölder, Minkowski, Jensen, Chebishev and Berwald inequalities are generalized to the Choquet and Sugeno integrals. A general inequality which cover Hölder and Minkowski type inequalities is considered for the universal integral. The corresponding inequalities for important cases of the pseudo-integral and applications of these inequalities in pseudo-probability are also given.

Keywords: Nonadditive measure, Choquet integral, Sugeno integral, pseudo-integral, Jensen inequality, Hölder inequality, Minkowski inequality, Chebishev inequality, Berwald inequality.

1 Introduction

The Hölder, Minkowski, Jensen and Chebyshev inequalities for Lebesgue integral, see [12], play an important role in mathematical analysis and in other areas of mathematics, especially in theory of probability, differential equations, geometry, and wider, e.g., information sciences, economics, engineering.

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The classical measure theory is based on countable additive measures and the Lebesgue integral. However, additivity measures do not allow modeling many phenomena. For this reason, nonadditive measure, called also fuzzy measure or capacity, and the corresponding integrals, e.g., Choquet, Sugeno, are introduced, see [15, 18, 36, 45]. The Choquet and Sugeno integrals have important applications as aggregation functions in decision theory (multiple criteria, multiple attributes, multiperson decision making, multiobjective optimization), information or data fusion, artificial intelligence and fuzzy logic, see [17, 44, 47, 48].

The pseudo-analysis as a generalization of the classical analysis is based on a special nonadditive measures, called pseudo-additive measures, and one of its tools is the pseudo-integral. There we consider instead of the field of real numbers a semiring, i.e., a real interval $[a,b] \subseteq [-\infty,\infty]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$, see [34, 35, 36, 37, 39, 40, 46]. On this structure the notions of $\oplus$-measure (pseudo-additive measure) and corresponding integral (pseudo-integral) are introduced. Methods of the pseudo-analysis can be applied for solving problems in many different fields such as system theory, optimization, decision making, control theory, differential equations, difference equations, etc. Similar approach was introduced independently by Maslov and his collaborators in the framework of idempotent measures theory, see [20, 22, 25]. The theory of cost measures based on idempotent measures and integrals of Maslov was delevoped, see [7, 8, 9].

Considering the wide applications of integrals based on nonadditive measure, there is a need for the study of inequalities for those integrals. The study of inequalities for Choquet and Sugeno integral, were given in [1, 2, 16, 27, 28, 30, 31, 43]. The first of all Jensen type inequality for Sugeno integral was obtained by Román-Flores *et al.* [43]. A fuzzy Chebyshev type inequalty has been considered by a several authors, see [2, 16, 28, 30, 32]. Inequalities with respect to the Choquet integral is observed by Wang [31] and Mesiar, Li, and Pap [27]. The generalizations of the classical integral inequalities for the universal integral (introduced in [19]) were investigated in [6]. In [1, 5, 41, 42] inequalities with respect to pseudo-integrals were obtained.

In the Section 2 an overview on generalizations of the Jensen, Hölder, Minkowski, Chebishev and Berwald type inequalities for Choquet and Sugeno integrals are given. In Section 3, we review results related to the universal integral, as generalization of Choquet and Sugeno integral. There are given a general inequality which cover Hölder and Minkowski type inequalities. Generalizations of the Hölder, Minkowski, Jensen and Chebyshev type inequalities for important cases of the pseudo-integral are presented in Section 4. Inspired by applications integral inequalities in the probability theory, the pseudo-probability is introduced and the inequalities valid for the pseudo-integral is applied in this theory in Section 4.6.1. Finally, using the notions of the cost measure we review these inequalities related to the value of a cost variables.

2 General Nonadditive Measures and Integrals Based on Them

Let X be a non-empty set and $\mathscr{A}$ a σ-algebra of subsets of X. Then $(X,\mathscr{A})$ is measurable space and a function $f : X \to [0,\infty]$ is called $\mathscr{A}$-measurable if, for each $B \in \mathscr{B}([0,\infty])$, the σ -algebra of Borel subsets of $[0,\infty]$, the preimage $f^{-1}(B)$ is an element of $\mathscr{A}$.

Definition 1. ([36, 48]) A monotone measure m on a measurable space $(X,\mathscr{A})$ is a function $m : \mathscr{A} \to [0,\infty]$ satisfying
(i) $m(\varnothing) = 0$,
(ii) $m(X) > 0$,
(iii) $m(A) \leqslant m(B)$ whenever $A \subseteq B$.

Normed monotone measures on $(X,\mathscr{A})$, i.e., monotone measures satisfying $m(X) = 1$, are also called fuzzy measures (see [36, 48]).

The Choquet, Sugeno and Shilkret integrals (see [10, 36, 38, 48]) are based on monotone measure and they are defined, respectively, for any measurable space $(X,\mathscr{A})$, for any measurable function f and for any monotone measure m, by

$$\begin{aligned}
\mathbf{Ch}(m,f) &= \int_0^\infty m(\{f \geqslant t\})\,dt,\\
\mathbf{Su}(m,f) &= \sup\{\min(t, m(\{f \geqslant t\})) \mid t \in]0,\infty]\},\\
\mathbf{Sh}(m,f) &= \sup\{t \cdot m(\{f \geqslant t\}) \mid t \in]0,\infty]\},
\end{aligned}$$

where the convention $0 \cdot \infty = 0$ is used.

Now we give a short overview on results related to the generalizations of the classical integral inequalities for Choquet and Sugeno integrals.

Jensen inequality and reverse Jensen inequality for Sugeno integral is obtained in [43]. Jensen, Chebyshev, Hölder and Minkowski type inequalities for Choquet integral and several convergence concepts as applications of these inequalities are observed in [31]. An approach to the Choquet integral as Lebesgue integral is given in [27] and in this way there are obtained the related inequalities.

Chebyshev type inequality for Sugeno integral based on Lebesgue measure are obtained in [16]. Previous results from [16] are generalised in [30]. Namely, there is presented a Chebyshev type inequality for Sugeno integral based on an arbitrary fuzzy measure. This inequality for comonotone functions and arbitrary fuzzy measure-base Sugeno integral were given in [28]

A general Minkowski type inequality for Sugeno integral is obtained in [1].

Berwald inequality for Sugeno integral is studied in [4]. This inequality holds in the following form:

Theorem 1. *Let* $0 < r < s < \infty$, $f : [a,b] \to [0,\infty[$ *be a concave function and m be the Lebesgue measure on* $\mathbb{R}$. *Then*
(a) if $f(a) < f(b)$, *then*

$$(\mathbf{Su}(m,f^r))^{\frac{1}{r}} \geqslant \min\left\{\begin{array}{c} \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{Su}(m,f^s)}{b-a}\right)^{\frac{1}{s}}, \\ \left(b-\frac{\frac{(b-a)^{\frac{1+r}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{Su}(m,f^s)}{b-a}\right)^{\frac{1}{s}}+af(b)-bf(a)}{f(b)-f(a)}\right)^{\frac{1}{r}} \end{array}\right\},$$

(b) if $f(a)=f(b)$*, then*

$$(\mathbf{Su}(m,f^r))^{\frac{1}{r}} \geqslant \min\left\{f(a),(b-a)^{\frac{1}{r}}\right\},$$

(c) if $f(a)>f(b)$*, then*

$$(\mathbf{Su}(m,f^r))^{\frac{1}{r}} \geqslant \min\left\{\begin{array}{c} \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{Su}(m,f^s)}{b-a}\right)^{\frac{1}{s}}, \\ \left(\frac{\frac{(b-a)^{\frac{1+r}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\mathbf{Su}(m,f^s)}{b-a}\right)^{\frac{1}{s}}+af(b)-bf(a)}{f(b)-f(a)}-a\right)^{\frac{1}{r}} \end{array}\right\}.$$

3 Universal Integral

In order to define the notion of the universal integral the following notions are needed.

Definition 2. ([19]) Let $(X,\mathscr{A})$ be a measurable space.
(i) $\mathscr{F}^{(X,\mathscr{A})}$ is the set of all $\mathscr{A}$ -measurable functions $f: X \to [0,\infty]$;
(ii) For each number $a \in]0,\infty]$, $\mathscr{M}_a^{(X,\mathscr{A})}$ is the set of all monotone measures (in the sense of Definition 1) satisfying $m(X)=a$; and we take

$$\mathscr{M}^{(X,\mathscr{A})} = \bigcup_{a\in]0,\infty]} \mathscr{M}_a^{(X,\mathscr{A})}.$$

An equivalence relation between pairs of measures and functions was introduced in [19].

Definition 3. Two pairs $(m_1,f_1) \in \mathscr{M}^{(X_1,\mathscr{A}_1)} \times \mathscr{F}^{(X_1,\mathscr{A}_1)}$ and $(m_2,f_2) \in \mathscr{M}^{(X_2,\mathscr{A}_2)} \times \mathscr{F}^{(X_2,\mathscr{A}_2)}$ satisfying

$$m_1(\{f_1 \geqslant t\}) = m_2(\{f_2 \geqslant t\}) \text{ for all } t \in]0,\infty],$$

will be called integral equivalent, in symbols

$$(m_1, f_1) \sim (m_2, f_2)\,.$$

Notion of the pseudo-multiplication is necessary to introduce the universal integral.

Definition 4. ([36, 46]) A function $\otimes\colon [0,\infty]^2 \to [0,\infty]$ is called a pseudo-multiplication if it satisfies the following properties:

(i) it is non-decreasing in each component, i.e., for all $a_1, a_2, b_1, b_2 \in [0,\infty]$ with $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$ we have $a_1 \otimes b_1 \leqslant a_2 \otimes b_2$;
(ii) 0 is an annihilator of , i.e., for all $a \in [0,\infty]$ we have $a \otimes 0 = 0 \otimes a = 0$;
(iii) has a neutral element different from 0, i.e., there exists an $e \in]0,\infty]$ such that, for all $a \in [0,\infty]$, we have $a \otimes e = e \otimes a = a$.

Let $\mathscr{S}$ be the class of all measurable spaces, and take

$$\mathscr{D}_{[0,\infty]} = \bigcup_{(X,\mathscr{A}) \in \mathscr{S}} \mathscr{M}^{(X,\mathscr{A})} \times \mathscr{F}^{(X,\mathscr{A})}.$$

The Choquet, Sugeno and Shilkret integrals are particular cases of the following integral given in [19].

Definition 5. A function $\mathbf{I}\colon \mathscr{D}_{[0,\infty]} \to [0,\infty]$ is called a universal integral if the following axioms hold:

(I1) For any measurable space $(X,\mathscr{A})$, the restriction of the function $\mathbf{I}$ to $\mathscr{M}^{(X,\mathscr{A})} \times \mathscr{F}^{(X,\mathscr{A})}$ is non-decreasing in each coordinate;
(I2) there exists a pseudo-multiplication $\otimes\colon [0,\infty]^2 \to [0,\infty]$ such that for all pairs $(m, c \cdot \mathbf{1}_A) \in \mathscr{D}_{[0,\infty]}$ (where $\mathbf{1}_A$ is the characteristic function of the set A)

$$\mathbf{I}(m, c \cdot \mathbf{1}_A) = c \otimes m(A);$$

(I3) for all integral equivalent pairs $(m_1, f_1), (m_2, f_2) \in \mathscr{D}_{[0,\infty]}$ we have

$$\mathbf{I}(m_1, f_1) = \mathbf{I}(m_2, f_2)\,.$$

By Proposition 3.1 from [19] we have the following important characterization.

Theorem 2. *Let $\otimes\colon [0,\infty]^2 \to [0,\infty]$ be a pseudo-multiplication on $[0,\infty]$. Then the smallest universal integral $\mathbf{I}$ and the greatest universal integral $\mathbf{I}$ based on $\otimes$ are given by*

$$\mathbf{I}_\otimes(m,f) = \sup\{t \otimes m(\{f \geqslant t\}) \mid t \in]0,\infty]\}\,,$$
$$\mathbf{I}^\otimes(m,f) = \mathrm{essup}_m f \otimes \sup\{m(\{f \geqslant t\}) \mid t \in]0,\infty]\}\,,$$

where $\mathrm{essup}_m f = \sup\{t \in [0,\infty] \mid m(\{f \geqslant t\}) > 0\}\,.$

Notice that $\mathbf{Su} = \mathbf{I}_{Min}$ and $\mathbf{Sh} = \mathbf{I}_{Prod}$, where the pseudo-multiplications Min and $Prod$ are given by $Min(a,b) = \min(a,b)$ and $Prod(a,b) = a \cdot b$.

There is neither a smallest nor a greatest pseudo-multiplication on $[0,\infty]$. But, if we fix the neutral element $e \in]0,\infty]$, then the smallest pseudo-multiplication $\otimes_e$ with neutral element e is given by

$$a \otimes_e b = \begin{cases} 0 & \text{if } (a,b) \in [0,e[^2, \\ \max(a,b) & \text{if } (a,b) \in [e,\infty]^2, \\ \min(a,b) & \text{otherwise.} \end{cases}$$

Then by Proposition 3.2 from [19] there exists the smallest universal integral $\mathbf{I}_{\otimes_e}$ among all universal integrals given by

$$\mathbf{I}_{\otimes_e}(m,f) = \max(m(\{f \geqslant e\}), \mathrm{essinf}_m f),$$

where $\mathrm{essinf}_m f = \sup\{t \in [0,\infty] \mid m(\{f \geqslant t\}) = e\}$.

3.1 A General Inequality for the Universal Integral

We will give first a main inequality, see [6], and then the Minkowski and Chebyshev type inequalities appear as special cases.

The following important property of a pair of functions is needed, see [15, 36]. Functions $f,g\colon X \to \mathbb{R}$ are said to be comonotone if for all $x,y \in X$,

$$(f(x) - f(y))(g(x) - g(y)) \geqslant 0.$$

The comonotonicity of functions f and g is equivalent to the nonexistence of points $x,y \in X$ such that $f(x) < f(y)$ and $g(x) > g(y)$.

Theorem 3. *Let* $\star\colon [0,\infty[^2 \to [0,\infty[$ *be continuous and nondecreasing in both arguments and* $\varphi\colon [0,\infty[\to [0,\infty[$ *be continuous and strictly increasing function. Let* $f,g \in \mathscr{F}^{(X,\mathscr{A})}$ *be two comonotone measurable functions and* $\otimes_e\colon [0,\infty]^2 \to [0,\infty]$ *be a smallest pseudo-multiplication on* $[0,\infty]$ *with neutral element* $e \in]0,\infty]$ *and* $m \in \mathscr{M}^{(X,\mathscr{A})}$ *be a monotone measure such that* $\mathbf{I}_{\otimes_e}(m,\varphi(f))$ *and* $\mathbf{I}_{\otimes_e}(m,\varphi(g))$ *are finite. If*

$$\varphi^{-1}((\varphi(a \star b) \otimes_e c)) \geqslant \left(\varphi^{-1}((\varphi(a) \otimes_e c)) \star b\right) \vee \left(a \star \varphi^{-1}((\varphi(b) \otimes_e c))\right),$$

then the inequality

$$\varphi^{-1}(\mathbf{I}_{\otimes_e}(m,\varphi(f \star g))) \geqslant \varphi^{-1}(\mathbf{I}_{\otimes_e}(m,\varphi(f))) \star \varphi^{-1}(\mathbf{I}_{\otimes_e}(m,\varphi(g)))$$

holds.

3.2 Minkowski's Inequality for Universal Integral

As a corollary of Theorem 3 we obtain an inequality related to Minkowski type for universal integral. Hence, if $\varphi(x) = x^s$ for all $s > 0$, the following holds:

Corollary 1. *Let $f,g \in \mathscr{F}^{(X,\mathscr{A})}$ be two comonotone measurable functions and $\otimes_e\colon [0,\infty]^2 \to [0,\infty]$ be a smallest pseudo-multiplication on $[0,\infty]$ with neutral element $e \in\,]0,\infty]$ and $m \in \mathscr{M}^{(X,\mathscr{A})}$ be a monotone measure such that $\mathbf{I}_{\otimes_e}(m, f^s)$ and $\mathbf{I}_{\otimes_e}(m, g^s)$ are finite. Let $\star\colon [0,\infty[^2 \to [0,\infty[$ be continuous and nondecreasing in both arguments. If*

$$((a\star b)^s \otimes_e c)^{\frac{1}{s}} \geqslant \left((a^s \otimes_e c)^{\frac{1}{s}} \star b\right) \vee \left(a \star (b^s \otimes_e c)^{\frac{1}{s}}\right),$$

then the inequality

$$(\mathbf{I}_{\otimes_e}(m, (f\star g)^s))^{\frac{1}{s}} \geqslant (\mathbf{I}_{\otimes_e}(m, f^s))^{\frac{1}{s}} \star (\mathbf{I}_{\otimes_e}(m, g^s))^{\frac{1}{s}}$$

holds for all $s > 0$.

3.3 Chebyshev's Inequality for Universal Integral

Due to Theorem 3, if $s = 1$ we have the Chebyshev type inequality.

Corollary 2. *Let $f,g \in \mathscr{F}^{(X,\mathscr{A})}$ be two comonotone measurable functions and $\otimes_e\colon [0,\infty]^2 \to [0,\infty]$ be a smallest pseudo-multiplication on $[0,\infty]$ with neutral element $e \in\,]0,\infty]$ and $m \in \mathscr{M}^{(X,\mathscr{A})}$ be a monotone measure such that $\mathbf{I}_{\otimes_e}(m, f)$ and $\mathbf{I}_{\otimes_e}(m, g)$ are finite. Let $\star\colon [0,\infty[^2 \to [0,\infty[$ be continuous and nondecreasing in both arguments. If*

$$(a\star b) \otimes_e c) \geqslant [(a \otimes_e c) \star b] \vee [a \star (b \otimes_e c)],$$

then the inequality

$$\mathbf{I}_{\otimes_e}(m, (f\star g)) \geqslant \mathbf{I}_{\otimes_e}(m, f) \star \mathbf{I}_{\otimes_e}(m, g)$$

holds.

4 Pseudo-Integral

Let $[a,b]$ be a closed (in some cases semiclosed) subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$. This can be the usual order of the real line, but it can be another order. The operation $\oplus$ (pseudo-addition) is a commutative, non-decreasing (with respect to $\preceq$), associative function $\oplus : [a,b] \times [a,b] \to [a,b]$ with a zero (neutral) element denoted by $\mathbf{0}$. Denote $[a,b]_+ = \{x : x \in [a,b], \mathbf{0} \preceq x\}$. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a,b] \times [a,b] \to [a,b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$, $z \in [a,b]_+$, associative and for which there exist a unit element $\mathbf{1} \in [a,b]$, i.e., for each $x \in [a,b]$, $1 \odot x = x$. We assume $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

The structure $([a,b], \oplus, \odot)$ is called a *semiring* (see [23, 36]). We will consider only semirings with the following continuous operations:

Case I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = \sup(x,y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a,b]$ (or $[a,b)$). We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $\sup(x,y) = y$.

(b) $x \oplus y = \sup(x,y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a,b]$ (or $(a,b]$). We have $\mathbf{0} = b$ and the idempotent operation inf induces a full order in the following way: $x \preceq y$ if and only if $\inf(x,y) = y$.

Case II: The pseudo-operations are defined by a monotone and continuous function $g : [a,b] \to [0,\infty]$, i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \text{ and } x \odot y = g^{-1}(g(x) \cdot g(y)).$$

If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$.

If the generator g is increasing (decreasing), the operation $\oplus$ induces the usual order (opposite to the usual order) on the interval $[a,b]$ in the following way: $x \preceq y$ if and only if $g(x) \le g(y)$.

Case III: Both operations are idempotent. We have

(a) $x \oplus y = \sup(x,y)$, $x \odot y = \inf(x,y)$, on the interval $[a,b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation sup induces the usual order ($x \preceq y$ if and only if $\sup(x,y) = y$).

(b) $x \oplus y = \inf(x,y)$, $x \odot y = \sup(x,y)$, on the interval $[a,b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation inf induces an order opposite to the usual order ($x \preceq y$ if and only if $\inf(x,y) = y$).

In order to present the Hölder and Minkowski integral inequalities for the pseudo-integral, it is necessary to introduce the pseudo-power. For $x \in [a,b]_+$ and $p \in]0,\infty[$, the pseudo-power $x_\odot^{(p)}$ is defined in the following way. If $p = n$ is an integer then $x_\odot^{(n)} = \underbrace{x \odot x \odot \cdots \odot x}_{n}$. Moreover, $x_\odot^{(\frac{1}{n})} = \sup\left\{y \mid y_\odot^{(n)} \le x\right\}$. Then $x_\odot^{(\frac{m}{n})} = x_\odot^{(r)}$ is well defined for any rational $r \in]0,\infty[$, independently of representation $r = \frac{m}{n} = \frac{m_1}{n_1}$, m, n, m_1, n_1 being positive integers (the result follows from the continuity and monotonicity of $\odot$). Due to continuity of $\odot$, if $p \in]0,\infty[$ is not rational, then

$$x_\odot^{(p)} = \sup\left\{x_\odot^{(r)} \mid r \in]0,p[,\ r \text{ is rational}\right\}.$$

Evidently, if $x \odot y = g^{-1}(g(x) \cdot g(y))$, then $x_{\odot}^{(p)} = g^{-1}(g^{p}(x))$. On the other hand, if $\odot$ is idempotent, then $x_{\odot}^{(p)} = x$ for any $x \in [a,b]$ and $p \in]0,\infty[$.

Let $(X, \mathscr{A})$ be a measurable space. A set function $m : \mathscr{A} \to [a,b]$ is a σ-$\oplus$-measure if there hold

(i) $m(\varnothing) = \mathbf{0}$ (if $\oplus$ is not idempotent),
(ii)$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} m(A_i)$ holds for any sequence $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathscr{A}$.

We suppose that $([a,b],\oplus)$ and $([a,b],\odot)$ are complete lattice ordered semigrups. We suppose that $[a,b]$ is endowed with a metric d compatible with sup and inf, i.e. $\limsup x_n = x$ and $\liminf x_n = x$, imply $\lim_{n\to\infty} d(x_n,x) = 0$, and which satisfies at least one of the following conditions:

(a) $d(x \oplus y, x' \oplus y') \leq d(x,x') + d(y,y')$
(b) $d(x \oplus y, x' \oplus y') \leq \max\{d(x,x'), d(y,y')\}$.

Let f and h be two functions defined on X and with values in $[a,b]$. We define for any $x \in X$ for functions f and g that $(f \oplus g)(x) = f(x) \oplus g(x)$ and $(f \odot g)(x) = f(x) \odot g(x)$, and for any $\lambda \in [a,b]$ $(\lambda \odot f)(x) = \lambda \odot f(x)$.

The pseudo-characteristic function with values in a semiring is defined with

$$\chi_A(x) = \begin{cases} \mathbf{0}, & x \notin A \\ \mathbf{1}, & x \in A \end{cases}.$$

A step (measurable) function is a mapping $e : X \to [a,b]$ that has the following representation $e = \bigoplus_{i=1}^{n} a_i \odot \chi_{A_i}$ for $a_i \in [a,b]$ and sets $A_i \in \mathscr{A}$ are pairwise disjoint if $\oplus$ is nonidempotent.

Let ε be a positive real number, and $B \subset [a,b]$. A subset $\{l_i^{\varepsilon}\}_{n\in\mathbb{N}}$ of B is a ε-net in B if for each $x \in B$ there exists l_i^{ε} such that $d(l_i^{\varepsilon},x) \leqslant \varepsilon$. If we have $l_i^{\varepsilon} \preceq x$, than we call $\{l_i^{\varepsilon}\}$ a lower ε-net. If $l_i^{\varepsilon} \preceq l_{i+1}^{\varepsilon}$ holds, than $\{l_i^{\varepsilon}\}$ is monotone, for more details see [33, 36].

Let $m : \mathscr{A} \to [a,b]$ be a $\oplus$-measure.

(i) The pseudo-integral of a step function $e : X \to [a,b]$ is defined by

$$\int_X^{\oplus} e \odot dm = \bigoplus_{i=1}^{n} a_i \odot m(A_i).$$

(ii)The pseudo-integral of a measurable function $f : X \to [a,b]$, (if $\oplus$ is not idempotent we suppose that for each $\varepsilon > 0$ there exists a monotone ε-net in $f(X)$) is defined by

$$\int_X^{\oplus} f \odot dm = \lim_{n\to\infty} \int_X^{\oplus} e_n(x) \odot dm,$$

where $(e_n)_{n\in\mathbb{N}}$ is a sequence of step functions such that $d(e_n(x), f(x)) \to 0$ uniformly as $n \to \infty$.

4.1 Two Important Special Cases

If the pseudo-operations are defined by a monotone and continuous function $g:[a,b]\to[0,\infty]$, the pseudo-integral for a measurable function $f:X\to[a,b]$ is given by,

$$\int_X^{\oplus} f\odot dm = g^{-1}\left(\int_X (g\circ f)\, d\,(g\circ m)\right), \tag{1}$$

where the integral applied on the right side is the standard Lebesgue integral. In special case, when $X=[c,d]$, $\mathscr{A}=\mathscr{B}(X)$ and $m=g^{-1}\circ\lambda$, λ the standard Lebesgue measure on $[c,d]$, then the pseudo-integral reduces on g -integral. Therefore, due to (1) we have

$$\int_{[c,d]}^{\oplus} f dx = g^{-1}\left(\int_c^d g\,(f\,(x))\,dx\right).$$

When the semiring is of the form $([a,b],\sup,\odot)$, cases I(a) and III(a) from section 4 we shall consider complete sup-measure m only and $\mathscr{A}=2^X$, i.e., for any system $(A_i)_{i\in I}$ of measurable sets, $m\left(\bigcup_{i\in I} A\right)=\sup_{i\in I} m\,(A_i)$. Recall that if X is countable (especially, if X is finite) then any σ-sup-measure m is complete and, moreover, $m\,(A)=\sup_{x\in A}\psi\,(x)$, where $\psi:X\to[a,b]$ is a density function given by $\psi\,(x)=m\,(\{x\})$. Then the pseudo-integral for a function $f:X\to[a,b]$ is given by

$$\int_X^{\oplus} f\odot dm = \sup_{x\in X}\left(f\,(x)\odot\psi\,(x)\right),$$

where function ψ defines sup-measure m.

In [29] is shown that any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition.

Theorem 4. *Let m be a* sup-*measure on* $([0,\infty],\mathscr{B}([0,\infty]))$, *where* $\mathscr{B}([0,\infty])$ *is the Borel* σ*-algebra on* $[0,\infty]$,

$$m\,(A)=\mathrm{essup}_{\mu}\left(\psi\,(x):x\in A\right), \tag{2}$$

and $\psi:[0,\infty]\to[0,\infty]$ *is a continuous density. Then for any pseudo-addition* $\oplus$ *with a generator g there exists a family* $\{m_\lambda\}$ *of* $\oplus_\lambda$*-measure on* $([0,\infty[,\mathscr{B})$, *where* $\oplus_\lambda$ *is generated by* g^λ *(the function g of the power* λ*),* $\lambda\in\,]0,\infty[$ *, such that*

$$\lim_{\lambda\to\infty} m_\lambda = m.$$

For any continuous function $f:[0,\infty]\to[0,\infty]$ the integral $\int^{\oplus} f\odot dm$ can be obtained as a limit of g-integrals, ([29]).

Theorem 5. *Let* $([0,\infty], \sup, \odot)$ *be a semiring with* $\odot$ *generated by the generator* g*. Let* m *be the same as in Theorem 4. Then exists a family* $\{m_\lambda\}$ *of* $\oplus_\lambda$*-measures, where* $\oplus_\lambda$ *is generated by* g^λ*,* $\lambda \in]0,\infty[$*, such that for every continuous function* $f : [0,\infty] \to [0,\infty]$

$$\int^{\sup} f \odot dm = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot_\lambda dm_\lambda .$$

In the following we will give the Hölder, Minkowski, Jensen and Chebyshev type inequaities for important cases of the pseudo-integral.

Since the cases I(b) and III(b) are linked to the cases I(a) and III(a) by duality, all inequaities for the pseudo-integral related to the cases I(a) and III(a) can be transformed into inequaities for pseudo-integral related to the cases I(b) and III(b).

4.2 *Hölder's Inequality for Pseudo-integral*

Now we shall give a generalization of the classical Hölder inequality on the semiring with generated pseudo-operations based on [5].

Recall that if p and q are positive real number such that $\frac{1}{p} + \frac{1}{q} = 1$, then p and q is a pair of conjugate exponents.

Theorem 6. *Let* p *and* q *be conjugate exponents,* $1 < p < \infty$*. For a given measurable space* $(X, \mathscr{A})$ *let* $u, v : X \to [a,b]$ *be two measurable functions and let a generator* $g : [a,b] \to [0,\infty]$ *of the pseudo-addition* $\oplus$ *and the pseudo-multiplication* $\odot$ *be an increasing function. Then for any* σ*-*$\oplus$*-measure* m *it holds:*

$$\int_X^{\oplus} (u \odot v) \odot dm \leqslant \left(\int_X^{\oplus} u_{\odot}^{(p)} \odot dm \right)_{\odot}^{\left(\frac{1}{p}\right)} \odot \left(\int_X^{\oplus} v_{\odot}^{(q)} \odot dm \right)_{\odot}^{\left(\frac{1}{q}\right)} .$$

Example 1. (i) Let $[a,b] = [0,\infty]$ and $g(x) = x^\alpha$ for some $\alpha \in [1,\infty[$. The corresponding pseudo-operations are $x \oplus y = \sqrt[\alpha]{x^\alpha + y^\alpha}$ and $x \odot y = xy$. Then the inequality from *Theorem* 6 reduces on the following inequality

$$\sqrt[\alpha]{\int_{[c,d]} u(x)^\alpha v(x)^\alpha \, dx} \leqslant \sqrt[p\alpha]{\int_{[c,d]} u(x)^{p\alpha} \, dx} \sqrt[q\alpha]{\int_{[c,d]} v(x)^{q\alpha} \, dx}.$$

(ii) Let $[a,b] = [0,\infty]$ and $g(x) = x^\alpha$ for some $\alpha \in [1,\infty[$. The corresponding pseudo-operations are $x \oplus y = \sqrt[\alpha]{x^\alpha + y^\alpha}$ and $x \odot y = xy$. Then the inequality from *Theorem* 6 reduces on the following inequality

$$\sqrt[\alpha]{\int_{[c,d]} u(x)^\alpha v(x)^\alpha \, dx} \leqslant \sqrt[p\alpha]{\int_{[c,d]} u(x)^{p\alpha} \, dx} \sqrt[q\alpha]{\int_{[c,d]} v(x)^{q\alpha} \, dx}.$$

Let $x \oplus y = \sup(x,y)$ and $x \odot y = g^{-1}(g(x)g(y))$. As a consequence of the Hölder type theorem for a complete sup-measure the following result holds for general σ-sup-measure by [5].

Theorem 7. *Let $\odot$ be represented by an increasing generator g and m be σ-*sup-*measure. Let p and q be conjugate exponents with $1 < p < \infty$. Then for any measurable functions $u, v : X \to [a,b]$, it holds:*

$$\int_X^{\sup} (u \odot v) \odot dm \leq \left[\int_X^{\sup} u_{\odot}^{(p)} \odot dm\right]_{\odot}^{\left(\frac{1}{p}\right)} \odot \left[\int_X^{\sup} v_{\odot}^{(q)} \odot dm\right]_{\odot}^{\left(\frac{1}{q}\right)}.$$

In the case III(a) and $p > 0$, $x_{\odot}^{(p)} = x$, the Hölder inequality reduces to the inequality

$$\int_X^{\sup} (u \odot v) \odot dm \leq \left(\int_X^{\sup} u \odot dm\right) \odot \left(\int_X^{\sup} v \odot dm\right),$$

which trivially holds because of distributivity of sup and inf.

Example 2. Let $[a,b] = [-\infty, \infty]$ and g generating $\odot$ be given by $g(x) = e^x$. Then

$$x \odot y = x + y,$$

and Hölder type inequality from *Theorem* 7 reduces on

$$\sup_{x \in X} (u(x) + v(x) + \psi(x))$$

$$\leqslant \frac{1}{p} \sup_{x \in X} (p \cdot u(x) + \psi(x)) + \frac{1}{q} \sup_{x \in X} (q \cdot v(x) + \psi(x))$$

where u, v, ψ are arbitrary real function on X.

4.3 Minkowski's Inequality for Pseudo-integral

In [5] is given Minkowski's inequality for three cases of the pseudo-integrals. The following inequality holds for the case II and corresponding pseudo-integral.

Theorem 8. *Let $u, v : X \to [a,b]$ be two measurable functions and $p \in [1, \infty[$. If an additive generator $g : [a,b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ are increasing. Then for any σ-$\oplus$-measure m it holds:*

$$\left(\int_X^{\oplus} (u \oplus v)_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)}$$

$$\leqslant \left(\int_X^{\oplus} u_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)} \oplus \left(\int_X^{\oplus} v_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)}.$$

If we observe semiring $([a,b], \sup, \inf)$, then the corresponding inequality means (recall that $x_{\odot}^{(p)} = x$ for each $x \in [a,b]$, $p > 0$)

$$\int_X^{\oplus} (u \oplus v) \odot dm \leqslant \sup\left(\int_X^{\oplus} u \odot dm, \int_X^{\oplus} v \odot dm\right).$$

In the case I(a) also Minkowski type inequality holds.

Theorem 9. *Let $\odot$ be represented by an increasing generator g, m be a complete* sup*-measure and $p \in]0,\infty[$. Then for any functions $u, v : X \to [a,b]$, it holds:*

$$\left(\int_X^{\sup} (\sup(u,v))_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)}$$

$$= \sup\left(\left(\int_X^{\sup} u_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)}, \left(\int_X^{\sup} v_{\odot}^{(p)} \odot dm\right)_{\odot}^{\left(\frac{1}{p}\right)}\right).$$

4.4 Jensen Inequality for Pseudo-integral

Due to [42] we have the following generalization Jensen inequality for two cases of the pseudo-integral.

Theorem 10. *Let $(X, \mathscr{A})$ be a measurable space, m be a σ-$\oplus$-measure and $m(X) = \mathbf{1}$. Let a generator g of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function. If $f : X \to [a,b]$ is a measurable function such that $\int_X^{\oplus} f \odot dm < b$ and Φ is a convex and nonincreasing function on a subinterval of $[a,b]$ containing the range of f, with values in $[a,b]$, then we have*

$$\Phi\left(\int_X^{\oplus} f \odot dm\right) \leqslant \int_X^{\oplus} (\Phi \circ f) \odot dm.$$

Example 3. (i) Let $g(x) = x^{\alpha}$ for some $\alpha \in [1,\infty[$. The corresponding pseudo-operations are $x \oplus y = \sqrt[\alpha]{x^{\alpha} + y^{\alpha}}$ and $x \odot y = xy$. Then the inequality from *Theorem* 10 has the following form

$$\Phi\left(\sqrt[\alpha]{\int_{[0,1]} f(x)^{\alpha}\, dx}\right) \leqslant \sqrt[\alpha]{\int_{[0,1]} \Phi(f(x))^{\alpha}\, dx}.$$

(ii)Let $g(x) = e^x$. The corresponding pseudo-operations are $x \oplus y = \ln(e^x + e^y)$ and $x \odot y = x + y$. Then we have the following inequality

$$\Phi\left(\ln \int_{[0,1]} e^{f(x)}\, dx\right) \leqslant \ln\left(\int_{[0,1]} e^{\Phi(f(x))}\, dx\right).$$

Now we consider the case when $\oplus = \max$, and $\odot = g^{-1}(g(x)g(y))$. From Theorem 5 and the previous theorem it follows the next theorem.

Theorem 11. *Let the pseudo-multiplication $\odot$ is represented by a convex and increasing generator g, μ be the usual Lebesgue measure on $\mathbb{R}$ and m be a* sup-*measure on $([c,d], \mathscr{B}([c,d]))$ defined by (2) where $\psi : [c,d] \to [0,\infty]$ is a continuous density and $m([c,d]) = \mathbf{1}$. If $f : [c,d] \to [0,\infty]$ is continuous function such that pseudo-integral $\int_{[c,d]}^{\sup} f \odot dm$ is finite and Φ is a convex and non-increasing function on the range of f, then it holds:*

$$\Phi\left(\int_{[c,d]}^{\sup} f \odot dm\right) \leqslant \int_{[c,d]}^{\sup} (\Phi \circ f) \odot dm.$$

Example 4. Using Example 3(ii) we have that $g^{\lambda}(x) = e^{\lambda x}$. Then

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \ln\left(e^{\lambda x} + e^{\lambda y}\right) = \max(x,y),$$

and

$$x \odot_{\lambda} y = x + y.$$

Therefore Jensen type inequality from *Theorem* 11 reduces on

$$\Phi\left(\sup(f(x) + \psi(x))\right) \leqslant \sup\left(\Phi(f(x)) + \psi(x)\right),$$

where ψ is a density function related to m..

4.5 Chebyshev's Inequality for Pseudo-integral

The Chebyshev type inequality for pseudo-integral is studied in [1, 41]. Let in the following theorems $([c,d], \mathscr{A})$ be a measure space and m be σ-$\oplus$ -measure such that $m([c,d]) = \mathbf{1}$.

Theorem 12. *Let $f_1, f_2 : [c,d] \to [a,b]$ be measurable functions. If an additive generator g of the pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ is an increasing function and f_1, f_2 are either both increasing or both decreasing, then*

$$\int_{[c,d]}^{\oplus} f_1 \odot dm \odot \int_{[c,d]}^{\oplus} f_2 \odot dm \leqslant \int_{[c,d]}^{\oplus} (f_1 \odot f_2) \odot dm.$$

In special case, when $[c,d] = [0,1]$, $\mathscr{A} = \mathscr{B}([0,1])$ and $m = g^{-1} \circ \lambda$, λ the standard Lebesgue measure on $[0,1]$, we have the inequality from [41].

Example 5. (i) Let $g(x) = x^{\alpha}$ for some $\alpha \in [1,\infty)$. The corresponding pseudo-operations are $x \oplus y = \sqrt[\alpha]{x^{\alpha} + y^{\alpha}}$ and $x \odot y = xy$. Then by the previous theorem we obtain the following inequality

$$\sqrt[\alpha]{\int_{[0,1]} f_1(x)^{\alpha}\, dx}\, \sqrt[\alpha]{\int_{[0,1]} f_2(x)^{\alpha}\, dx} \leqslant \sqrt[\alpha]{\int_{[0,1]} f_1(x)^{\alpha} f_2(x)^{\alpha} dx}.$$

(ii) Let $g(x) = e^x$. The corresponding pseudo-operations are $x \oplus y = \ln(e^x + e^y)$ and $x \odot y = x + y$. Then by the previous theorem we obtain the following inequality

$$\ln \int_{[0,1]} e^{f_1(x)}\, dx + \ln \int_{[0,1]} e^{f_2(x)}\, dx \leqslant \ln\left(\int_{[0,1]} e^{f_1(x)+f_2(x)} dx\right).$$

Now, we consider the second case, when $\oplus = \max$, and $\odot = g^{-1}(g(x)g(y))$. Due to Theorem 5 and the previous theorem the following holds:

Theorem 13. *Let $\odot$ is represented by an increasing multiplicative generator g and m be the same as in Theorem 11. Then for any continuous functions $f_1, f_2 : [c,d] \to [a,b]$, which are either both increasing or both decreasing, holds:*

$$\int_{[c,d]}^{\sup} f_1 \odot dm \odot \int_{[c,d]}^{\sup} f_2 \odot dm \leqslant \int_{[c,d]}^{\sup} (f_1 \odot f_2) \odot dm.$$

Example 6. Using Example 5(ii) we have that $g^{\lambda}(x) = e^{\lambda x}$. Then

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \ln\left(e^{\lambda x} + e^{\lambda y}\right) = \max(x,y),$$

and

$$x \odot_{\lambda} y = x + y.$$

Therefore, Chebyshev type inequality from *Theorem* 13 reduces on

$$\sup(f_1(x) + \psi(x)) + \sup(f_2(x) + \psi(x)) \leqslant \sup(f_1(x) + f_2(x) + \psi(x)),$$

where ψ is a density function related to m.

4.6 Applications

4.6.1 Pseudo-probability

The pseudo-probability is a generalization of the classical probability. In an analogous way as in the probability theory, see [11], we will introduced the corresponding notions in the framework of the σ-$\oplus$-measure and the pseudo-integral.

Let $([a,b], \oplus, \odot)$ be a semiring and $(X, \mathscr{A})$ be a measurable space.

The pseudo-probability $\mathbb{P}$ is σ-$\oplus$-measure on $(X, \mathscr{A})$ satisfying $\mathbb{P}(X) = \mathbf{1}$.

Specially, if we observe case II and P is a classical probability then $\mathbb{P} = g^{-1} \circ P$ is distorted probability, see [13].

The function $Y : X \to [a,b]$ is a pseudo-random variable if for any $y \in [a,b]$ it holds

$$\{\omega \in X \mid Y(\omega) \prec y\} = \{Y \prec y\} \in \mathscr{A}.$$

The pseudo-expectation of the pseudo-variable Y is introduced by

$$\mathbb{E}(Y) = \int_X^{\oplus} Y \odot d\mathbb{P}.$$

If the pseudo-expectation of the pseudo-random variable Y has a finite value in the sense of a given semiring, i.e., if the operation $\oplus$ induces the usual order (opposite to the usual order) on the interval $[a,b]$ it means that $\mathbb{E}(Y) < b$, ($\mathbb{E}(Y) > a$), then Y is integrable.

Due to definition of the pseudo-expectation, we have

$$\mathbb{E}(f(Y)) = \int_X^{\oplus} f(Y) \odot d\mathbb{P} \tag{3}$$

for any measurable function $f : [a,b] \to [a,b]$.

As consequences of (3) and the inequalities valid for pseudo-integral, with the same assumptions as in the corresponding inequalities type theorems the following hold:

(i) By Hölder's inequality we have

$$E(Y \odot Z) \preceq \left(E\left((Y)_{\odot}^{(p)}\right)\right)_{\odot}^{\left(\frac{1}{p}\right)} \odot \left(E(Z)_{\odot}^{(q)}\right)_{\odot}^{\left(\frac{1}{q}\right)}, \tag{4}$$

(ii) By Minkowski's inequality we have

$$\left(E\left((Y \oplus Z)_{\odot}^{(p)}\right)\right)_{\odot}^{\left(\frac{1}{p}\right)} \preceq \left(E\left((Y)_{\odot}^{(p)}\right)\right)_{\odot}^{\left(\frac{1}{p}\right)} \oplus \left(E\left((Z)_{\odot}^{(p)}\right)\right)_{\odot}^{\left(\frac{1}{p}\right)}, \tag{5}$$

(iii) By Jensen's inequality we have

$$\boldsymbol{\Phi}(\mathbf{E}(Y)) \preceq \mathbf{E}(\boldsymbol{\Phi}(Y)),$$

(iv) By Chebyshev's inequality we have

$$\mathbf{E}(f_1(Y)) \odot \mathbf{E}(f_2(Y)) \preceq \mathbf{E}(f_1(Y) \odot f_2(Y)).$$

4.6.2 Cost Measure and Decision Variable

The duality between probability and optimization was considered in [9]. Hence, there $(\mathbb{R}^+, +, \times)$ is replaced by the semiring $(]-\infty, \infty], \min, +)$. By analogy with probability theory there cost mesure, decision variable and related notions were introduced. Let the semiring $(]-\infty,\infty], \min, +)$ is denoted by $\mathbb{R}_{\min}$.

Let U be a topological spaces, $\mathscr{U}$ be the set of open sets of U. A set function $\mathbb{K} : \mathscr{U} \to \mathbb{R}_{\min}$ is a cost measure if there hold

(i) $\mathbb{K}(\emptyset) = \infty$,
(ii) $\mathbb{K}(U) = 0$,
(iii) $\mathbb{K}(\bigcup_n A_n) = \inf_n \mathbb{K}(A_n)$ za sve A_n iz $\mathscr{U}$.

The triplet $(U, \mathscr{U}, \mathbb{K})$ is called a decision space.

A function $c : U \to \mathbb{R}_{\min}$ such that $\mathbb{K}(A) = \inf_{u\in A} c(u)$ for any $A \in \mathscr{U}$ is a cost density of the cost measure $\mathbb{K}$.

A l.s.c. function $c : U \to \mathbb{R}_{\min}$ such that $\inf_u c(u) = 0$ defines a cost measure on $(U, \mathscr{U})$ by $\mathbb{K}(A) = \inf_{u\in A} c(u)$ ([8]). Also, for any cost measure $\mathbb{K}$ defined on open sets of a topological space with a countable basis of open sets there exists a unique minimal extension $\mathbb{K}_*$ to $\mathscr{P}(U)$ ([21, 26]).

Now we recall the definitions of the decision variables and related notions [9].

The mapping $Y : U \to E$ is a decision variables on $(U, \mathscr{U}, \mathbb{K})$, where E is a topological space with a countable basis of open set. It induces a cost measure $\mathbb{K}_Y$ on $(E, \mathscr{B})$ (where $\mathscr{B}$ denotes the set of open sets of E) defined by $\mathbb{K}_Y(A) = \mathbb{K}_*\left(Y^{-1}(A)\right)$ for all $A \in \mathscr{B}$. The cost measure $\mathbb{K}_Y$ has a l.s.c. density denoted c_Y. When $E = \mathbb{R}_{\min}$ a decision variable Y is called a cost variable.

The value of a cost variable Y is defined by

$$\mathbb{V}(Y) = \inf_x (x + c_Y(x)).$$

The convergence of decision variables, law of large numbers, Bellman chains and processes are also considered in [9].

Remark 1. The cost measure, decision variables and related notions can also be defined in a general idempotent semiring, see [8].

Notice that the value of a cost variable Y is defined by the pseudo-integral with respect to σ-$\oplus$-measure $\mathbb{K}_*$, i.e.,

$$\mathbb{V}(Y) = \int_U^{\inf} Y \odot d\mathbb{K}_*.$$

Let Y and Z be decision variables on $(U, \mathscr{U}, \mathbb{K})$. Due to the previous notations the inequalities (4) and (5) have the following forms:

(i) For Hölder's inequality

$$\mathbb{V}(Y+Z) \geq \frac{1}{p}\mathbb{V}(pY) + \frac{1}{q}\mathbb{V}(qZ),$$

where p and q are conjugate exponents, $1 < p < \infty$,

(ii) For Minkowski's inequality

$$\frac{1}{p}\mathbb{V}\left(p\inf\left(Y,Z\right)\right)=\inf\left(\frac{1}{p}\mathbb{V}\left(pY\right),\frac{1}{p}\mathbb{V}\left(pZ\right)\right),$$

where $p\in\,]0,\infty[$.

In [14] the semiring $([-\infty,\infty),\sup,+)$ are considered and there are given the corresponding inequalities related to idempotent integral introduced in [24]. These inequalities also have applications in decision theory. Hence, in this case Hölder's and Minkowski's inequalities have the following forms:

$$\mathbb{E}\left(Y+Z\right)\leq\frac{1}{p}\mathbb{E}\left(pY\right)+\frac{1}{q}\mathbb{E}\left(qZ\right),$$

where p and q are conjugate exponents, $0<p\leq q<\infty$ and

$$\frac{1}{p}\mathbb{E}\left(p\sup\left(Y,Z\right)\right)=\sup\left(\frac{1}{p}\mathbb{E}\left(pY\right),\frac{1}{p}\mathbb{E}\left(pZ\right)\right),$$

where $p\in(0,\infty)$ and $\mathbb{E}\left(Y\right)$ is the value of a decision variables (in the sense of a given semiring) defined by sup-integral (see [24]).

5 Conclusion

We have given the generalizations of the mostly used integral inequalities for non-additive integrals (Choquet integral, Sugeno integral, universal integral, pseudo integral). The future work will be the investigation of new applications of obtained results in many fields.

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