Optimal and Asymptotically Optimal Control for Some Inventory Models

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Abstract A multi-supplier discrete-time inventory model is considered as illustration of problems arising in applied probability. Optimal and asymptotically optimal control is established for all values of parameters involved. The model stability is also investigated.

Keywords Optimal and asymptotically optimal policies • Discrete-time inventory models • Stability

Mathematics Subject Classification (2010): Primary 90B05, Secondary 90C31

1 Introduction

It was my scientific adviser Professor Yu.V. Prokhorov who proposed optimal control of some inventory systems as a topic of my Phd thesis. At the time it was a new research direction. The subject of my habilitation thesis was stochastic inventory models. So I decided to return to these problems in the paper devoted to jubilee of academician of Russian Academy of Sciences Yu.V. Prokhorov.

Optimal control of inventory systems is a particular case of decision making under uncertainty (see, e.g., [5]). It is well known that construction of a mathematical model is useful to investigate a real life process or system and make a correct decision.

There always exist a lot of models describing the process under consideration more or less precisely. Therefore it is necessary to choose an appropriate model.

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Usually the model depends on some parameters not known exactly. So they are estimated on the base of previous observations. The same is true of underlying processes distributions. Hence, the model must be stable with respect to small parameters fluctuations and processes perturbations (see, e.g., [6]).

To illustrate the problems arising and the methods useful for their solution, a multi-supplier inventory model is considered.

2 Main Results

The aim of investigation is to establish optimal and asymptotically optimal control. It is reasonable to begin by some definitions.

2.1 *Definitions*

To describe any applied probability model one needs to know the following elements: the planning horizon T, input process $Z = \{Z(t), t \in [0, T]\}$, output process $Y = \{Y(t), t \in [0, T]\}$ and control $U = \{U(t), t \in [0, T]\}$. The system state is given by $X = \Psi(Z, Y, U)$ where functional Ψ represents the system configuration and operation mode. Obviously, one has also $X = \{X(t), t \in [0, T]\}$. Moreover, processes Z, Y, U and X can be multi-dimensional and their dimensions may differ. For evaluation of the system performance quality it is necessary to introduce an objective function $\mathscr{L}(Z, Y, U, X, T)$. For brevity it will be denoted by $\mathscr{L}_T(U)$. So, a typical applied probability model is described by a six-tuple $(Z, Y, U, \Psi, \mathscr{L}, T)$.

Such description is useful for models classification. It also demonstrates the similarity of models arising in different applied probability domains such as inventory and dams theory, insurance and finance, queueing and reliability theory, as well as population growth and many others (see, e.g., [7]). One only gives another interpretation to processes Z, Y, X in order to switch from one research domain to another. Thus, input to inventory system is replenishment delivery (or production) and output is demand, whereas for a queueing system it is arrival and departure of customers respectively (for details see, e.g., [6]).

Definition 1. A control $U_T^* = \{U^*(t), t \in [0, T]\}$ is called *optimal* if

$$\mathscr{L}_T(U_T^*) = \inf_{U_T \in \mathscr{U}_T} \mathscr{L}_T(U_T) \quad (\text{or} \quad \mathscr{L}_T(U_T^*) = \sup_{U_T \in \mathscr{U}_T} \mathscr{L}_T(U_T)), \tag{1}$$

where $\mathscr{U}_T$ is a class of all feasible controls. Furthermore, $U^* = \{U_T^*, T \geq 0\}$ is called an *optimal policy*.

The choice of inf or sup in (1) is determined by the problem we want to solve. Namely, if we are interested in minimization of losses (or ruin probability) we use the first expression, whereas for profit (or system life-time) maximization we use the second one in (1).

Since extremum in (1) may be not attained we introduce the following

Definition 2. A control U_T^ε is *ε-optimal* if

$$\mathscr{L}_T(U_T^\varepsilon) < \inf_{U_T \in \mathscr{U}_T} \mathscr{L}_T(U_T) + \varepsilon \qquad (\text{or} \quad \mathscr{L}_T(U_T^\varepsilon) > \sup_{U_T \in \mathscr{U}_T} \mathscr{L}_T(U_T) - \varepsilon).$$

Definition 3. A policy $\tilde{U} = \{\tilde{U}_T, T \geq 0\}$ is *stationary* if for any $T, S \geq 0$

$$\tilde{U}_T(t) = \tilde{U}_S(t), \quad t \leq \min(T, S).$$

Definition 4. A policy $\widehat{U} = (\widehat{U}_T, T \geq 0)$ is *asymptotically optimal* if

$$\lim_{T \to \infty} T^{-1} \mathscr{L}_T(\widehat{U}_T) = \lim_{T \to \infty} T^{-1} \mathscr{L}_T(U_T^*).$$

The changes necessary for discrete-time models are obvious.

2.2 *Model Description*

Below we consider a discrete-time multi-supplier one-product inventory system. It is supposed that a store created to satisfy the customers demand can be replenished periodically. Namely, at the end of each period (e.g., year, month, week, day etc.) an order for replenishment of inventory stored can be sent to one of m suppliers or to any subset of them. The i-th supplier delivers an order with $(i-1)$-period delay, $i = \overline{1, m}$. Let a_i be the maximal order possible at the i-th supplier, and the ordering price is c_i per unit, $i = \overline{1, m}$. For simplicity, the constant delivery cost associated with each order is ignored. However we take into account holding cost h per unit stored per period and penalty p for deficit of unit per period.

Let the planning horizon be equal to n periods. The demand is described by a sequence of independent identically distributed nonnegative random variables $\{\xi_k\}_{k=1}^n$. Here ξ_k is amount demanded during the k-th period. Assume $F(x)$ to be the distribution function of ξ_k having a density $\varphi(s) > 0$ for $s \in [\underline{\kappa}, \overline{\kappa}] \subset [0, \infty)$. It is also supposed that there exists $\mathsf{E}\xi_k = \mu$, $k = \overline{1, n}$.

Unsatisfied demand is backlogged. That means, the inventory level x_k at the end of the k-th period can be negative. In this case $|x_k|$ is the deficit amount.

Expected discounted n-period costs are chosen as objective function. We denote by $f_n(x, y_1, \ldots, y_{m-2})$ the minimal value of objective function if inventory on hand (or initial inventory level) is x and y_i is already ordered (during previous periods) quantity to be delivered i periods later, $i = \overline{1, m-2}$.

2.2.1 Notation and Preliminary Results

It is supposed that the order amounts at the end of each period depend on the inventory level x and yet undelivered quantities $y_1, \ldots, y_{m-2}$. Using the Bellman optimality principle (see, e.g., [2]) it is possible to obtain, for $n \geq 1$, the following functional equation

$$f_n(x, y_1, \ldots, y_{m-2}) = \min_{0 \leq z_i \leq a_i, i=\overline{1,m}} [\sum_{i=1}^{m} c_i z_i + L(x + z_1) + \tag{2}$$

$$+\alpha \mathsf{E} f_{n-1}(x + y_1 + z_1 + z_2 - \xi_1, y_2 + z_3, \ldots, y_{m-2} + z_{m-1}, z_m)].$$

Here α is the discount factor, E stands for mathematical expectation and z_i is the order size at the first step of n-step process from the i-th supplier, $i = \overline{1,m}$. Furthermore, the one-period mean holding and penalty costs are represented by

$$L(v) = \mathsf{E}[h(v - \xi_1)^+ + p(\xi_1 - v)^+], \quad \text{with} \quad a^+ = \max(a, 0),$$

if inventory level available to satisfy demand is equal to v.

The calculations for arbitrary m being too cumbersome, we treat below in detail the case $m = 2$. Then we need to know only the initial level x and Eq. (2) takes the form

$$f_n(x) = \min_{0 \leq z_i \leq a_i, i=1,2} [c_1 z_1 + c_2 z_2 + L(x + z_1) + \alpha \mathsf{E} f_{n-1}(x + z_1 + z_2 - \xi_1)] \tag{3}$$

with $f_0(x) \equiv 0$. Let us introduce the following notation $v = x + z_1$, $u = v + z_2$ and

$$G_n(v, u) = (c_1 - c_2)v + c_2 u + L(v) + \alpha \mathsf{E} f_{n-1}(u - \xi_1).$$

Then Eq. (3) can be rewritten as follows

$$f_n(x) = -c_1 x + \min_{(v,u) \in D_x} G_n(v, u) \tag{4}$$

where $D_x = \{x \leq v \leq x + a_1, v \leq u \leq v + a_2\}$.

The minimum in (4) can be attained either inside of D_x or at its boundary.

To formulate the main results we need the following functions

$$\frac{\partial G_n}{\partial v}(v, u) = c_1 - c_2 + L'(v) := K(v),$$

$$\frac{\partial G_n}{\partial u}(v, u) = c_2 + \alpha \int_0^\infty f'_{n-1}(u - s)\varphi(s)\, ds := S_n(u).$$

Moreover, $T_n(v) = S_n(v) + K(v)$ and $B_n(v) = S_n(v + a_2) + K(v)$ represent $\dfrac{dG_n(v,v)}{dv}$ and $\dfrac{dG_n(v,v+a_2)}{dv}$ respectively, whereas

$$R^a(u) = c_2 - \alpha c_1 + \alpha \int_0^{u-\bar{v}} K(u-s)\varphi(s)\,ds + \alpha \int_{u+a-\bar{v}}^{\infty} K(u+a-s)\varphi(s)\,ds. \quad (5)$$

Let $\bar{v}$, u_n, v_n, w_n and u^a be the roots of the following equations

$$K(\bar{v}) = 0, \quad S_n(u_n) = 0, \quad T_n(v_n) = 0, \quad B_n(w_n) = 0, \quad R^a(u^a) = 0, \quad (6)$$

provided the solutions exist for a given set of cost parameters. In particular, $\bar{v} \in [\underline{\kappa}, \overline{\kappa}]$ is given by

$$F(\bar{v}) = \frac{p - c_1 + c_2}{p + h}$$

if $(c_1, c_2) \in \Gamma = \{(c_1, c_2) : (c_1 - p)^+ \le c_2 \le c_1 + h\}$. Otherwise, we set $\bar{v} = -\infty$, if $K(v) > 0$ for all v, that is, $(c_1, c_2) \in \Gamma^- = \{(c_1, c_2) : c_2 < (c_1 - p)^+\}$, and $\bar{v} = +\infty$, if $K(v) < 0$ for all v, that is, $(c_1, c_2) \in \Gamma^+ = \{(c_1, c_2) : c_2 > c_1 + h\}$. A similar assumption holds for $S_n(u)$, $T_n(v)$, $B_n(w)$ and u_n, v_n, w_n, $n \ge 1$, as well as $R^a(u)$ and u^a. Below we are going to use also the following notation. For $k \ge 0$ set

$$\Delta_k = \{(c_1, c_2) : p\sum_{i=0}^{k-1}\alpha^i < c_1 \le p\sum_{i=0}^{k}\alpha^i\}, \quad \Delta^k = \{(c_1, c_2) : p\sum_{i=1}^{k}\alpha^i < c_2 \le p\sum_{i=1}^{k+1}\alpha^i\},$$

where as always the sum over empty set is equal to 0,

$$\Delta_k^l = \Delta_k \cap \Delta^l,\ A_k = \cup_{l \ge k}\Delta_l,\ A^k = \cup_{l \ge k}\Delta^l,\ \Gamma^\alpha = \{(c_1, c_2) : (c_1 - p)^+ \le c_2 \le \alpha c_1\},$$

$$\Gamma_n^- = \{(c_1, c_2) \in \Gamma : S_n(\bar{v}) < 0\},\ \Gamma_n^+ = \{(c_1, c_2) \in \Gamma : S_n(\bar{v}) > 0\},$$

whereas $\Gamma_n^0 = \{(c_1, c_2) \in \Gamma : S_n(\bar{v}) = 0\}$. As usual dealing with dynamic programming all the proofs are carried out by induction.

Thus, it will be proved that functions $f_n'(x)$ are non-decreasing as well as $K(v)$, $S_n(v)$, $T_n(v)$ and $B_n(v)$. Moreover, to establish that sequences $\{u_n\}$, $\{v_n\}$, $n \ge 1$, are non-decreasing it is enough to check that $f_n'(x) - f_{n-1}'(x) \le 0$ for $x \le \max(u_n, v_n)$, since $S_{n+1}(u) = S_n(u) + \alpha H_n(u)$ and $T_{n+1}(v) = T_n(v) + \alpha H_n(v)$ where $H_n(u) = (f_n' - f_{n-1}') * F(u)$, here and further on $*$ denotes the convolution.

The crucial role for classification of possible variants of optimal behaviour plays the following

Lemma 1. *If* $(c_1, c_2) \in \Gamma_n^-$, *then* $\bar{v} < v_n < u_n$; *if* $(c_1, c_2) \in \Gamma_n^+$, *then* $\bar{v} > v_n > u_n$, *whereas* $\bar{v} = v_n = u_n$ *if* $(c_1, c_2) \in \Gamma_n^0$, *and* $u_n, v_n, \bar{v}$ *are defined by (6). Moreover, if* $(c_1, c_2) \in \Gamma^-$, *then* $v_n < u_n$ *and* $v_n > u_n$, *if* $(c_1, c_2) \in \Gamma^+$, *for all* n.

Proof. The statement is obvious, since functions $K(v)$, $S_n(v)$ and $T_n(v)$ are non-decreasing in v, $T_n(v) = S_n(v) + K(v)$ and $K(v) < 0$ for $v < \bar{v}$, while $K(v) > 0$ for $v > \bar{v}$. □

2.3 *Optimal Control*

We begin by treating the case without constraints on order sizes. Although Corollary 1 was already formulated in [4] (under assumption $\alpha = 1$) a more thorough investigation undertaken here lets clarify the situation and provides useful tools for the case with order constraints.

2.3.1 Unrestricted Order Sizes

At first we suppose that the order size at both suppliers may assume any value, that is, $a_i = \infty$, $i = 1, 2$.

Theorem 1. *If* $c_2 > \alpha c_1$, *the optimal behaviour at the first step of n-step process has the form* $u_n(x) = v_n(x) = \max(x, v_n)$. *The sequence* $\{v_n\}$ *of critical levels given by (6) is non-decreasing and there exists* $\lim_{n\to\infty} v_n = \widehat{v}$ *satisfying the following relation*

$$F(\widehat{v}) = \frac{p - c_1(1-\alpha)}{p+h}. \tag{7}$$

Moreover, for $(c_1, c_2) \in \Delta_k$, $k = 0, 1, \ldots,$ *one has* $v_n = -\infty$, $n \le k$ *and* v_{k+1} *is a solution of the equation*

$$\sum_{i=1}^{k+1} \alpha^{i-1} F^{i*}(v_{k+1}) = \frac{p\sum_{i=0}^{k} \alpha^i - c_1}{p+h}. \tag{8}$$

Proof. For $n = 1$ it is optimal to take $u = v$, since $S_1(u) = c_2 > 0$ for all u, that means $u_1 = -\infty$. On the other hand, $T_1(v) = c_1 - p + (p+h)F(v)$, therefore $v_1 = -\infty$ in A_1, whereas in Δ_0 there exists $v_1 \in [0, \bar{v}]$ such that $F(v_1) = (p - c_1)/(p+h)$. Thus, the optimal decision has the form $u_1(x) = v_1(x) = \max(x, v_1)$.

For further investigation we need only to know

$$f_1'(x) = -c_1 + \begin{cases} 0, & x < v_1, \\ T_1(x), & x \ge v_1, \end{cases} = \begin{cases} -c_1, & x < v_1, \\ L'(x), & x \ge v_1. \end{cases} \tag{9}$$

It is obvious that $f_1'(x)$ is non-decreasing, the same being true of

$$S_2(u) = c_2 - \alpha c_1 + \alpha \int_0^{u-v_1} T_1(u-s)\varphi(s)\,ds \tag{10}$$

and

$$T_2(v) = Q(v) + \alpha \int_0^{v-v_1} T_1(v-s)\varphi(s)\,ds \tag{11}$$

with $Q(v) = c_1(1-\alpha) + L'(v)$. Note that in the case $v_1 = -\infty$ the meaning of $\int_0^{u-v_1}$ in (10) and $\int_0^{v-v_1}$ in (11) is $\int_0^{\infty}$. The same agreement will be used further on.

Thus, $S_2(u) > 0$ for all u under assumption $c_2 > \alpha c_1$, that is, $u_2 = -\infty$. Since $T_2(v) \geq Q(v)$, it follows immediately that $v_2 \leq \widehat{v}$ and $\widehat{v}$ is given by (7), hence $\widehat{v} < \bar{v}$. Moreover, $\widehat{v} = -\infty$ for $c_1 > p(1-\alpha)^{-1}$. It is also clear that $v_2 > v_1$ in Δ_0 because $T_2(v_1) = -\alpha c_1$. Recalling that in A_1

$$T_2(v) = c_1 + L'(v) + \alpha \int_0^{\infty} L'(v-s)\varphi(s)\,ds$$

we get

$$F(v_2) + \alpha F^{2*}(v_2) = \frac{p(1+\alpha) - c_1}{p+h} \quad \text{in} \quad \Delta_1,$$

whereas $v_2 = -\infty$ in A_2. Hence, $u_2(x) = v_2(x) = \max(x, v_2)$.

Assuming now the statement of the theorem to be valid for $k \leq m$, one has

$$f_k'(x) = -c_1 + \begin{cases} 0, & x < v_k, \\ T_k(x), & x \geq v_k, \end{cases} \tag{12}$$

and

$$f_m'(x) - f_{m-1}'(x) = \begin{cases} 0, & x < v_{m-1}, \\ -T_{m-1}(x), & v_{m-1} \leq x < v_m, \\ T_m(x) - T_{m-1}(x), & x \geq v_m. \end{cases} \tag{13}$$

Thus, $S_{m+1}(u) > 0$ for all u, that entails $u_{m+1} = -\infty$. Moreover, $T_{m+1}(v) \geq Q(v)$ and $H_m(v_m) \leq 0$. Hence, $v_m < v_{m+1} \leq \widehat{v}$ in $\cup_{k=0}^{m-1} \Delta_k$ and v_{m+1} satisfies (8) with $k = m$ in Δ_m, whereas $v_{m+1} = -\infty$ in A_{m+1}. That means, the theorem statement is valid for $m+1$.

The sequence $\{v_n\}$ is non-decreasing and bounded. Consequently there exists $\lim_{n\to\infty} v_n = \check{v} \leq \widehat{v}$. It remains to prove that $\check{v} = \widehat{v}$. In fact, for $n > k+1$

$$T_n(v) = Q(v) + \alpha \int_0^{v-v_{n-1}} T_{n-1}(v-s)\varphi(s)\,ds \quad \text{in} \quad \Delta_k, \quad k \geq 0,$$

so

$$|Q(v_n)| = \alpha \int_0^{v_n - v_{n-1}} T_{n-1}(v_n - s)\varphi(s)\,ds$$

$$\le T_{n-1}(\widehat{v})\alpha \int_0^{v_n - v_{n-1}} \varphi(s)\,ds \le \ldots \le T_k(\widehat{v})\alpha^{n-k} \int_0^{v_n - v_{n-1}} \varphi(s)\,ds$$

where $T_k(\widehat{v}) \le c_1 + h\sum_{i=0}^{k}\alpha^i \le c_1 + h(1-\alpha)^{-1}$.

Hence, $Q(v_n) \to 0 = Q(\widehat{v})$, as $n \to \infty$. On the other hand, $Q(v_n) \to Q(\breve{v})$, therefore $\breve{v} = \widehat{v}$. It is clear that this result is true for any $0 < \alpha \le 1$. □

Remark 1. The main result of Theorem 1 can be reformulated in the following way:

$$z_n^{(1)}(x) = z_n^{(2)}(x) = 0 \quad \text{for} \quad n \le k$$

and

$$z_n^{(1)}(x) = (v_n - x)^+, \quad z_n^{(2)}(x) = 0 \quad \text{for} \quad n > k,$$

if $(c_1, c_2) \in \Delta_k$, $k = 0, 1, \ldots$.

That means, for $c_2 > \alpha c_1$ it is optimal to use only the first supplier. The inventory level is raised up to a prescribed critical value v_n if the initial level x at the first step of n-step process is less than v_n. Nothing is ordered for $x \ge v_n$. Furthermore, if $c_1 > p\sum_{i=0}^{k-1}\alpha^i$ then for $n \le k$ nothing is ordered for all initial inventory levels x at the first step of n-step process. If $c_1 > p(1-\alpha)^{-1}$, it is optimal never to order for any initial level.

Theorem 2. *If $c_2 < (c_1 - p)^+$, the optimal behaviour at the first step of n-step process has the form $v_n(x) = x$, $u_n(x) = \max(x, u_n)$. The sequence $\{u_n\}$ defined by (6) is non-decreasing and there exists $\lim_{n\to\infty} u_n = u^0$, where u^0 is given by*

$$F^{2*}(u^0) = \frac{\alpha p - c_2(1-\alpha)}{\alpha(p+h)}. \tag{14}$$

Moreover, for $(c_1, c_2) \in \Delta^{k-1}$, $k = 1, 2, \ldots$, one has $u_n = -\infty$, $n \le k$, and

$$\sum_{i=2}^{k+1} \alpha^{i-2} F^{i*}(u_{k+1}) = \frac{p\sum_{i=1}^{k}\alpha^i - c_2}{\alpha(p+h)}. \tag{15}$$

Proof. Recall that $\bar{v} = -\infty$ in Γ^- and $\Gamma^- \subset A_1$. It follows immediately from here that $u_1(x) = v_1(x) = x$ and $f_1'(x) = L'(x)$. Now turn to $n = 2$. Since

$$S_2(u) = c_2 - \alpha p + \alpha(p+h)F^{2*}(u),$$

it is obvious that $S_2(u) > 0$ for all u (that is, $u_2 = -\infty$) in A^1 and there exists $u_2 \ge 0$ satisfying (15) with $k = 1$ in Δ^0. According to Lemma 1 one has $v_2 < u_2$, therefore it is optimal to have $v_2(x) = x$ and $u_2(x) = \max(x, u_2)$. Thus,

$$f_2'(x) = -c_1 + \begin{cases} K(x), & x < u_2, \\ T_2(x), & x \geq u_2, \end{cases} = -c_2 + L'(x) + \begin{cases} 0, & x < u_2, \\ S_2(x), & x \geq u_2, \end{cases}$$

and

$$S_3(u) = R^0(u) + \alpha \int_0^{u-u_2} S_2(u-s)\varphi(s)\,ds$$

where $R^0(u)$ given by (5) with $a = 0$ has the form

$$c_2(1-\alpha) + \alpha \int_0^{\infty} L'(u-s)\varphi(s)\,ds = c_2(1-\alpha) - \alpha p + \alpha(p+h)F^{2*}(u). \quad (16)$$

It is clear that there exists u^0 satisfying $R^0(u^0) = 0$. For $c_2 \leq \alpha p(1-\alpha)^{-1}$ it is given by (14), otherwise $u^0 = -\infty$. Since $S_3(u) \geq R^0(u)$ and $S_3(u_2) = -\alpha c_2$ in Δ^0, one has $u_2 < u_3 \leq u^0$. Moreover, in Δ^1 there exists u_3 satisfying (15) with $k = 2$, whereas $u_3 = -\infty$ in A^2.

Assuming the statement of the theorem to be valid for $k \leq m$ one has

$$f_k'(x) = -c_1 + K(x) + \begin{cases} 0, & x < u_k, \\ S_k(x), & x \geq u_k, \end{cases} \quad (17)$$

and

$$f_m'(x) - f_{m-1}'(x) = \begin{cases} 0, & x < u_{m-1}, \\ -S_{m-1}(x), & u_{m-1} \leq x < u_m, \\ S_m(x) - S_{m-1}(x), & x \geq u_m. \end{cases}$$

That means $S_{m+1}(u) \geq R^0(u)$ for all u and $H_m(u_m) < 0$. Thus, $u_m < u_{m+1} \leq u^0$ in $\cup_{k=0}^{m-2}\Delta^k$ and $u_{m+1} \leq u^0$ satisfies (15) with $k = m$ in Δ^{m-1}, whereas $u_{m+1} = -\infty$ in A^m. It follows immediately that $v_{m+1}(x) = x$ and $u_{m+1}(x) = \max(x, u_{m+1})$. Clearly, the theorem statement is valid for $m+1$.

The sequence $\{u_n\}$ is non-decreasing and bounded, consequently there exists $\breve{u} = \lim_{n\to\infty} u_n$. It remains to prove that $\breve{u} = u^0$. In fact, for $n > k+2$,

$$S_n(u) = R^0(u) + \alpha \int_0^{u-u_{n-1}} S_{n-1}(u-s)\varphi(s)\,ds \quad \text{in} \quad \Delta^k, \quad k = 0, 1, \ldots,$$

and

$$|R^0(u_n)| = \alpha \int_0^{u_n - u_{n-1}} S_{n-1}(u_n - s)\varphi(s)\,ds$$

$$\leq S_{n-1}(u^0)\alpha \int_0^{u_n-u_{n-1}} \varphi(s)\,ds \leq \ldots \leq S_k(u^0)\alpha^{n-k} \int_0^{u_n-u_{n-1}} \varphi(s)\,ds,$$

where $S_k(u^0) \leq c_2 + h\sum_{i=1}^{k+1} \alpha^i \leq c_2 + h(1-\alpha)^{-1}$.

It is clear that $R^0(u_n) \to 0 = R^0(u^0)$, as $n \to \infty$, hence, $\breve{u} = u^0$ for $0 < \alpha \leq 1$.

□

Remark 2. In other words, Theorem 2 states that for $c_2 < c_1 - p$ one has to use only the second supplier, the order sizes being

$$z_n^{(1)}(x) = 0, \quad z_n^{(2)}(x) = 0, \quad n \le k+1,$$

and

$$z_n^{(1)}(x) = 0, \quad z_n^{(2)}(x) = (u_n - x)^+, \quad n > k+1,$$

if $(c_1, c_2) \in \Delta^k$, $k = 0, 1, \ldots$.

Now let us turn to the last and most complicated case.

Theorem 3. *If $(c_1, c_2) \in \Gamma^\alpha$, the optimal behaviour at the first step of n-step process has the form $v_n(x) = \max(x, \min(v_n, \bar{v}))$, $u_n(x) = max(v_n(x), u_n)$. The sequence $\{u_n\}$ is non-decreasing and there exists $\lim_{n\to\infty} u_n = u^\infty$ where u^∞ is given by* (5) and (6) *with $a = \infty$.*

Proof. It is obvious that $\Gamma^\alpha \subset \cup_{i=0}^{\infty}(\Delta_i^i \cup \Delta_{i+1}^i)$ and $\bar{v} \ge \underline{\kappa}$ in Γ^α. As in Theorem 1, for $n = 1$ one has $u_1(x) = v_1(x) = \max(x, v_1)$ where v_1 is given by (8) with $k = 0$ in Δ_0 and $v_1 = -\infty$ in A_1. Thus $f_1'(x)$ has the form (9). Note also that

$$\Gamma_1^0 = \{(c_1, c_2) : 0 \le c_1 \le p, c_2 = 0\}.$$

Moreover, $S_2(u)$ is given by (10) and $u_2 \ge v_1$. It is also clear that $\{(c_1, c_2) : c_2 = \alpha c_1\} \subset \Gamma_2^+$. On the other hand, $\Gamma_1^0 \subset \Gamma_2^-$, since $S_2(\bar{v}) = -\alpha c_1$ in Γ_1^0. Furthermore, in Δ_0^0 the function $c_2 = g_2(c_1)$ is defined implicitly by

$$S_2(\bar{v}) = c_2 - \alpha c_1 + \alpha \int_0^{\bar{v}-v_1} T_1(\bar{v} - s)\varphi(s)\,ds = 0,$$

whence it follows $g_2(0) = 0$ and

$$g_2'(c_1) = \alpha \frac{\varphi(\bar{v}) \int_{\bar{v}-v_1}^{\infty} \varphi(s)\,ds + \int_0^{\bar{v}-v_1} \varphi(\bar{v}-s)\varphi(s)\,ds}{\varphi(\bar{v}) + \alpha \int_0^{\bar{v}-v_1} \varphi(\bar{v}-s)\varphi(s)\,ds}.$$

Thus, it is clear that $1 \ge g_2'(c_1) \ge 0$ and $g_2'(0) = \alpha$, since $\bar{v} = v_1$ for $c_1 = c_2 = 0$.

For $c_1 = p$ two expressions for $S_2(\bar{v})$ coincide because

$$S_2(\bar{v}) \to c_2 - \alpha p + \alpha(p+h)F^{2*}(\bar{v}), \quad \text{as} \quad c_1 \uparrow p,$$

and in Δ_1^0 one has $S_2(u) = c_2 - \alpha p + \alpha(p+h)F^{2*}(u)$. It is easy to get that u_2 is determined by (15) with $k = 1$ in Δ_1^0 and $u_2 = -\infty$ in Δ^1. We have also

$$g_2'(c_1) = \frac{\alpha\varphi^{2*}(\bar{v})}{\varphi(\bar{v}) + \alpha\varphi^{2*}(\bar{v})} \quad \text{in} \quad \Delta_1^0$$

and $\{(c_1, c_2) \in \Delta_1^0 : c_2 = c_1 - p\} \subset \Gamma_2^- \cup \Gamma_2^0$, more precisely, $g_2(p(1+\alpha)) = \alpha p$.

Hence, $\Gamma_2^- \subset \Delta_0^0 \cup \Delta_1^0$, moreover, we are going to establish that $\Gamma_2^- \subset \Gamma_3^-$, whereas $\Gamma_3^+ \subset \Gamma_2^+$. In fact, due to Lemma 1 one has $\bar{v} < v_2 < u_2$ in Γ_2^-. It follows immediately that $v_2(x) = \max(x, \bar{v})$ and $u_2(x) = \max(v_2(x), u_2)$. That means,

$$f_2'(x) = -c_1 + \begin{cases} 0, & x < \bar{v}, \\ K(x), & \bar{v} \le x < u_2, \\ T_2(x), & x \ge u_2, \end{cases} \tag{18}$$

and

$$S_3(u) = R^\infty(u) + \alpha \int_0^{u-u_2} S_2(u-s)\varphi(s)\, ds$$

with $R^\infty(u)$ given by (5) with $a = \infty$. In other words, we have

$$R^\infty(u) = c_2 - \alpha c_1 + \alpha \int_0^{u-\bar{v}} K(u-s)\varphi(s)\, ds.$$

Since $S_3(\bar{v}) = c_2 - \alpha c_1 < 0$ in Γ_2^-, it is clear that $\Gamma_2^- \subset \Gamma_3^-$. From (9) and (18) one gets

$$f_2'(x) - f_1'(x) = \begin{cases} 0, & x < v_1, \\ -T_1(x), & v_1 \le x < \bar{v}, \\ -c_2, & \bar{v} \le x < u_2, \\ T_2(x) - T_1(x), & x \ge u_2. \end{cases}$$

Thus, $f_2'(x) - f_1'(x) \le 0$ for $x \le u_2$, that is, $H_2(u_2) < 0$ and $u_2 < u_3$. As soon as $S_3(u) \ge R^\infty(u)$, it is obvious that $u_3 \le u^\infty$. Hence, $f_3'(x)$ has the form (18) with indices 3 instead of 2.

Assuming now that $(c_1, c_2) \in \Gamma_2^+$ one has $v_1 \le u_2 < v_2 < \bar{v}$ due to (10) and Lemma 1. It entails $u_2(x) = v_2(x) = \max(x, v_2)$ and $f_2'(x)$ is given by (12) with $k = 2$. Recall also that v_2 is given by (8) with $k = 1$ in Δ_1 and $v_2 = -\infty$ in A_2.

Clearly,

$$S_3(u) = c_2 - \alpha c_1 + \alpha \int_0^{u-v_2} T_2(u-s)\varphi(s)\, ds,$$

that means $S_3(v_2) = c_2 - \alpha c_1 \le 0$ in $\Delta_0 \cup \Delta_1$, consequently, $v_2 \le u_3$. There are two possibilities: either $u_3 \le \bar{v}$ or $u_3 > \bar{v}$. The first case corresponds to $\Gamma_3^0 \cup \Gamma_3^+$, whereas the second one to Γ_3^-. In the first case $u_3(x) = v_3(x) = \max(x, v_3)$, while in the second one $v_3(x) = \max(x, \bar{v})$ and $u_3(x) = \max(v_3(x), u_3)$. Moreover, in Δ_2^1

$$S_3(u) = c_2 - \alpha p(1+\alpha) + \alpha(p+h)[F^{2*}(u) + \alpha F^{3*}(u)],$$

while

$$T_3(v) = c_1 - p(1+\alpha+\alpha^2) + (p+h)[F(v) + \alpha F^{2*}(v) + \alpha^2 F^{3*}(v)].$$

Thus u_3 and v_3 are given in Δ_2^1 by (15) and (8) respectively with $k = 2$, whereas $u_3 = -\infty$ in A^2 and $v_3 = -\infty$ in A_3. Furthermore, in Δ_2^1

$$g_3'(c_2) = \alpha \frac{\varphi^{2*}(\bar{v}) + \alpha\varphi^{3*}(\bar{v})}{\varphi(\bar{v}) + \alpha\varphi^{2*}(\bar{v}) + \alpha^2\varphi^{3*}(\bar{v})},$$

as well as $g_3(p(1+\alpha+\alpha^2)) = \alpha p(1+\alpha)$ and $\Gamma_3^- \subset \cup_{l=0}^1(\Delta_l^l \cup \Delta_{l+1}^l)$.

Supposing now that the statement of the theorem is true for all $k \le m$ we establish its validity for $k = m+1$. Induction assumption means that

$$\Gamma_k^- = \Gamma_{k-1}^- \cup \Gamma_{k-1}^0 \cup (\Gamma_{k-1}^+ \cap \Gamma_k^-) \subset \cup_{l=0}^{k-2}(\Delta_l^l \cup \Delta_{l+1}^l), \quad k = \overline{2,m},$$

so $\Gamma_2^- \subset \ldots \subset \Gamma_m^-$ and $\Gamma_m^+ \subset \ldots \subset \Gamma_2^+$, moreover, $\Gamma = \Gamma_m^- \cup \Gamma_m^0 \cup \Gamma_m^+$.

Let $(c_1, c_2) \in \Gamma_m^-$, then

$$f_m'(x) = -c_1 + \begin{cases} 0, & x < \bar{v}, \\ K(x), & \bar{v} \le x < u_m, \\ T_m(x), & x \ge u_m, \end{cases} \tag{19}$$

while $f_{m-1}'(x)$ has the form (19) with $m-1$ instead of m, if $(c_1, c_2) \in \Gamma_{m-1}^-$. If $(c_1, c_2) \in \Gamma_{m-1}^0 \cup \Gamma_{m-1}^+$, then $f_{m-1}'(x)$ is given by (12) with $k = m-1$. So, one has either

$$f_m'(x) - f_{m-1}'(x) = \begin{cases} 0, & x < u_{m-1}, \\ -S_{m-1}(x), & u_{m-1} \le x < u_m, \\ S_m(x) - S_{m-1}(x), & x \ge u_m, \end{cases}$$

or

$$f_m'(x) - f_{m-1}'(x) = \begin{cases} 0, & x < v_{m-1}, \\ -T_{m-1}(x), & v_{m-1} \le x < \bar{v}, \\ -S_{m-1}(x), & \bar{v} \le x < u_m, \\ S_m(x) - S_{m-1}(x), & x \ge u_m. \end{cases}$$

It is clear that $H(u_m) < 0$, that means $S_{m+1}(u_m) < 0$ and $u_{m+1} > u_m > \bar{v}$, hence $(c_1, c_2) \in \Gamma_{m+1}^-$.

Now if $(c_1, c_2) \in \Gamma_m^+ \cup \Gamma_m^0$, then $f_k'(x)$ has the form (12) for $k \le m$, with $v_k = -\infty$ for $k \le l$ and v_{l+1} given by (8) with $k = l$ in Δ_l. This entails

$$S_{m+1}(u) = c_2 - \alpha c_1 + \alpha \int_0^{u - v_m} T_m(u-s)\varphi(s)\, ds$$

and $S_{m+1}(v_m) = c_2 - \alpha c_1 \le 0$, whence it is obvious that $v_m \le u_{m+1}$. As a result one has two possibilities: either $u_{m+1} \le \bar{v}$, that is, $(c_1, c_2) \in \Gamma_{m+1}^+ \cup \Gamma_{m+1}^0$, or $\bar{v} < u_{m+1}$, namely, $(c_1, c_2) \in \Gamma_{m+1}^-$. In the first case there exists $v_{m+1} \in (u_{m+1}, \bar{v})$ and $f_{m+1}'(x)$ is given by (12). Furthermore, v_{m+1} satisfies (8) with $k = m$ in Δ_m. In the second case $f_{m+1}'(x)$ has the form (19) with indices $m+1$ instead of m. Thus,

$$S_n(u) = R^\infty(x) + \alpha \int_0^{u-u_{n-1}} S_{n-1}(u-s)\varphi(s)\,ds \geq R^\infty(u)$$

and $u_n \leq u^\infty$ for $n > 2$. It is simple to prove, as in Theorem 2, that $u^\infty = \lim_{n\to\infty} u_n$. □

Corollary 1. *If* $(c_1 - p)^+ \leq c_2 \leq \beta_k c_1$ *with* $\beta_k = \sum_{i=1}^{k-1}\alpha^i / \sum_{i=0}^{k-1}\alpha^i$*, then* $(c_1, c_2) \in \Gamma_k^-$*,* $k \geq 2$*.*

Remark 3. As follows from Theorem 3, for the parameters set Γ^α one uses two suppliers or only the first one. The order sizes are regulated by critical levels u_n and $\bar{v}$ or v_n respectively, according to values of cost parameters. More precisely, if $\alpha c_1 > c_2 \geq (c_1 - p)^+$, then there exists $n_0(c_1, c_2)$ such that for $n > n_0$ it is optimal to use both suppliers, whereas for $n \leq n_0$ only the first supplier may be used.

2.4 *Order Constraints*

Turning to the results with order constraints we begin by the study of the first restriction impact.

Theorem 4. *Let* $a_1 < \infty$*,* $a_2 = \infty$*, then the optimal decision at the first step of* n*-step process has the form*

$$z_n^{(1)}(x) = \min[a_1, (\min(v_n, \bar{v}) - x)^+], \quad z_n^{(2)}(x) = (u_n - x - z_n^{(1)})^+. \tag{20}$$

The sequences $\{u_n\}$ *and* $\{v_n\}$ *defined by (6) are non-decreasing. There exists* $\lim_{n\to\infty} u_n$ *equal to* u^{a_1} *in* Γ *and* u^0 *in* Γ^-*.*

Proof. As previously, we proceed by induction. At first let us take $n = 1$. Since $S_1(u) = c_2 > 0$, that is, $u_1 = -\infty$, it is optimal to put $u = v$. On the other hand, $T_1(v) = c_1 - p + (p+h)F(v)$, therefore $v_1 = -\infty$ in A_1 and in Δ_0 there exists $v_1 \in [0, \bar{v}]$ satisfying (8) with $k = 0$. In the former case $u_1(x) = v_1(x) = x$ for all x and in the latter case $u_1(x) = v_1(x) = x + a_1$ for $x < v_1 - a_1$, $u_1(x) = v_1(x) = v_1$ for $x \in [v_1 - a_1, v_1)$ and $u_1(x) = v_1(x) = x$ for $x \geq v_1$.

Thus, $f_1'(x) = L'(x) = -p + (p+h)F(x)$ in A_1, whereas in Δ_0

$$f_1'(x) = -c_1 + \begin{cases} T_1(x + a_1), & x < v_1 - a_1, \\ 0, & v_1 - a_1 \leq x < v_1, \\ T_1(x), & x \geq v_1. \end{cases} \tag{21}$$

It is obvious that $f_1'(x)$ is non-decreasing, hence the same is true of $S_2(u)$ and $T_2(v)$ taking values in $[c_2 - \alpha p, c_2 + \alpha h]$ and $[c_1 - p(1+\alpha), c_1 + h(1+\alpha)]$ respectively. Hence, $u_2 = -\infty$ in A^1, $v_2 = -\infty$ in A_2, so for $n = 2$ the optimal decision is $u_2(x) = v_2(x) = x$ if $(c_1, c_2) \in D_2 = A_2 \cap A^1$.

Proceeding in the same way we establish that in $D_k = A_k \cap A^{k-1}$, $k > 2$, one has $u_n = v_n = -\infty$, $n \le k$, so $u_n(x) = v_n(x) = x$ is optimal for all $n \le k$ and

$$f_n'(x) = -p \sum_{i=0}^{n-1} \alpha^i + (p+h) \sum_{i=1}^{n} \alpha^{i-1} F^{i*}(x).$$

Moreover, in Δ_k^{k-1} there exist $u_{k+1} \ge \underline{\kappa}$ and $v_{k+1} \ge \underline{\kappa}$ given by (15) and (8) respectively.

Next consider the set Γ. For each $k > 1$ it is divided into subsets Γ_k^- and Γ_k^+ by a curve $c_2 = g_k(c_1)$ defined implicitly by equality $S_k(\bar{v}) = 0$. The point $(p\sum_{i=0}^{k-1} \alpha^i, p\sum_{i=1}^{k-1} \alpha^i)$ on the boundary of Γ, corresponding to $\bar{v} = \underline{\kappa}$, belongs to $g_k(c_1)$, since $S_k(\underline{\kappa}) = T_k(\underline{\kappa}) = 0$ for such (c_1, c_2) from Δ_{k-1}^{k-2}. According to the rule of implicit function differentiation and the form of $S_k(\cdot)$ in Δ_{k-1}^{k-2}, we get

$$g_k'(c_1) = \frac{\sum_{i=2}^{k-1} \alpha^{i-1} \varphi^{i*}(\bar{v})}{\sum_{i=1}^{k-1} \alpha^{i-1} \varphi^{i*}(\bar{v})},$$

whence it is obvious that $g_k'(c_1) \in [0, 1]$. The last result is valid for other values of c_1 although expression of $g_k'(c_1)$ is more complicated.

Suppose $(c_1, c_2) \in \Gamma_k^+ \subset \Gamma_{k-1}^+$ and

$$f_k'(x) = -c_1 + \begin{cases} K(x+a_1), & x < u_k - a_1, \\ T_k(x+a_1), & u_k - a_1 \le x < v_k - a_1, \\ 0, & v_k - a_1 \le x < v_k, \\ T_k(x), & x \ge v_k. \end{cases}$$

It is not difficult to verify that $u_{k+1} > u_k$ and $v_{k+1} > v_k$ and $\Gamma_{k+1}^+ \subset \Gamma_k^+$.

Now let $(c_1, c_2) \in \Gamma_k^-$, then

$$f_k'(x) = -c_1 + \begin{cases} K(x+a_1), & x < \bar{v} - a_1, \\ 0, & \bar{v} - a_1 \le x < \bar{v}, \\ K(x), & \bar{v} \le x < u_k, \\ T_n(x), & x \ge u_k. \end{cases}$$

It is easy to check that $\Gamma_n^- \subset \Gamma_{n+1}^-$ for any $n \ge k$ and

$$S_{n+1}(u) = R^{a_1}(u) + \alpha \int_0^{u-u_n} S_n(u-s)\varphi(s)\,ds \ge R^{a_1}(u),$$

entailing $u_n \le u^{a_1}$ for all n.

Since $R^{\infty}(u) \geq R^{a_1}(u) \geq R^0(u)$, for any u and $a_1 > 0$, one has $u^{\infty} < u^{a_1} < u^0$. It is not difficult to establish that $\lim_{n\to\infty} u_n = u^{a_1}$ where u^{a_1} is defined by (6).

Turning to $\Gamma^- \subset A_1$ we get, for $n > k$,

$$f_n'(x) = -c_2 + L'(x) + \begin{cases} 0, & x < u_n, \\ S_n(x), & x \geq u_n, \end{cases}$$

if $(c_1, c_2) \in \Gamma^- \cap \Delta^k$. Verifying that $f_n'(x) - f_{n-1}'(x) < 0$ for $x < u_n$, one obtains $u_{n+1} > u_n$. Furthermore, for all u and $n > k$,

$$S_{n+1}(u) = R^0(u) + \alpha \int_0^{u-u_n} S_n(u-s)\varphi(s)\,ds \geq R^0(u).$$

So, $u_n \leq u^0$ for all n. Obviously, there exists $\lim_{n\to\infty} u_n$ and it is easy to show that it is equal to u^0.

Finally, if u_n and v_n are finite then for Γ^+ it is optimal to take $v_n(x) = x + a_1$, $u_n(x) = u_n$ for $x < u_n - a_1$; $u_n(x) = v_n(x) = x + a_1$ for $x \in [u_n - a_1, v_n - a_1)$; $u_n(x) = v_n(x) = v_n$ for $x \in [v_n - a_1, v_n)$; and $u_n(x) = v_n(x) = x$ for $x \geq v_n$. □

To study the impact of the other constraint we formulate at first the almost obvious

Corollary 2. *If $c_2 > \alpha c_1$ the optimal behaviour for $a_1 = \infty$, $a_2 < \infty$ has the same form as that for $a_1 = a_2 = \infty$ in Theorem 1.*

Proof. Proceeding in the same way as in Theorem 1 we easily get the result. □

Theorem 5. *Let $a_1 \leq \infty$, $a_2 < \infty$ and $(c_1, c_2) \in \Gamma^-$. Then the optimal decision at the first step of n-step process is given by*

$$z_n^{(1)}(x) = \min(a_1, (w_n - x)^+), \quad z_n^{(2)}(x) = \min(a_2, (u_n - x - z_n^{(1)}(x))^+),$$

where w_n and u_n are defined by (6). There exist $\lim_{n\to\infty} u_n \geq \widehat{v}$ and $\lim_{n\to\infty} w_n \leq \widehat{v}$ with $\widehat{v}$ defined by (7).

Proof. Begin by treating the case $a_1 = \infty$, $a_2 < \infty$. It follows easily from assumptions that $u_1 = v_1 = w_1 = -\infty$ and $f_1'(x) = L'(x)$. Moreover,

$$S_2(u) = c_2 - \alpha p + (p+h)F^{2*}(u), \quad T_2(c) = c_1 - p - \alpha p + (p+h)[F(v) + \alpha F^{2*}(v)]$$

and

$$B_2(v) = c_1 - p - \alpha p + (p+h)[F(v) + \alpha F^{2*}(v + a_2)].$$

Since $T_2(v) < B_2(v)$ and $S_2(v + a_2) < B_2(v) < T_2(v + a_2)$ it follows from here that $w_2 < v_2 < w_2 + a_2 < u_2$. It is clear that $w_2 > -\infty$ in Δ_1^0

$$f_2(x) = -c_1 x + \begin{cases} G_2(w_2, w_2 + a_2), & x < w_2, \\ G_2(x, x + a_2), & w_2 \le x < u_2 - a_2, \\ G_2(x, u_2), & u_2 - a_2 \le x < u_2, \\ G_2(x, x), & x \ge u_2. \end{cases}$$

It follows immediately that

$$f_2'(x) - f_1'(x) = \begin{cases} -T_1(x), & x < w_2, \\ -c_2 + S_2(x + a_2), & w_2 \le x < u_2 - a_2, \\ -c_2, & u_2 - a_2 \le x < u_2, \\ -c_2 + S_2(x), & x \ge u_2. \end{cases}$$

So, $f_2'(x) - f_1'(x) < 0$ for $x \le u_2$. This entails the following inequalities $w_2 < w_3$, $v_2 < v_3$, $u_2 < u_3$.

Then if $(c_1, c_2) \in \Delta_1^0$, it is not difficult to verify by induction that there exist finite u_n and w_n, $n \ge 2$. Furthermore, one has $w_n < v_n < w_n + a_2 < u_n$. Hence, it is optimal to take $v_n(x) = w_n$, $u_n(x) = w_n + a_2$ for $x < w_n$; $v_n(x) = x$, $u_n(x) = x + a_2$ for $x \in [w_n, u_n - a_2)$; $v_n(x) = x$, $u_n(x) = u_n$ for $x \in [u_n - a_2, u_n)$ and $u_n(x) = v_n(x) = x$ for $x \ge u_n$. Consequently, one gets

$$f_n'(x) = -c_1 + \begin{cases} 0, & x < w_n, \\ B_n(x), & w_n \le x < u_n - a_2, \\ K(x), & u_n - a_2 \le x < u_n, \\ T_n(x), & x \ge u_n, \end{cases} = -c_2 + L'(x) + \begin{cases} -K(x), \\ S_n(x + a_2), \\ 0, \\ S_n(x), \end{cases} \tag{22}$$

and $B_n(v) \ge c_1(1 - \alpha) + L'(x)$. That means, $w_n \le \widehat{v}$ for all n and a_2. Using (22) one also obtains $\lim_{n\to\infty} u_n \ge \widehat{v}$.

If $(c_1, c_2) \in \Delta_l^0$, there exists $w_{l+1} > -\infty$, whereas $w_m = -\infty$ for $m \le l$. Thus,

$$f_n'(x) = -c_1 + \begin{cases} B_n(x), & x < u_n - a_2, \\ K(x), & u_n - a_2 \le x < u_n, \\ T_n(x), & x \ge u_n, \end{cases}$$

for $1 < n \le l$ and $f_n'(x)$ has the form (22) for $n > l$.

The subsets Δ_l^k corresponding to $k \ge 1$ are treated in the same way giving also $z_n^{(1)}(x) = (w_n - x)^+$, $z_n^{(2)} = \min(a_2, (u_n - x - z_n^{(1)}(x))^+)$.

Changes necessary under assumption $a_1 < \infty$ are almost obvious, so the details are omitted. □

2.5 *Sensitivity Analysis*

We begin studying the impact of model parameters on the optimal decision by the motivating

Example. Assume $\underline{\kappa} = 0, \overline{\kappa} = d$ and $\varphi(s) = d^{-1}$, $s \in [\underline{\kappa}, \overline{\kappa}]$, that is, distribution of ξ_i is uniform. Obviously, $F(u) = u/d$, $u \in [0, d]$, and $\overline{v} = d(p + c_2 - c_1)/(p + h)$, while $F^{2*}(u) = u^2/2d^2$, $u \in [0, d]$, $F^{2*}(u) = 1 - (u - 2d)^2/2d^2$, $u \in [d, 2d]$. Suppose also $a_1 < \infty$ and $\alpha = 1$.

According to (21) the form of $g_2(c_1)$, given by the relation $S_2(\overline{v}) = 0$, depends on a_1 for $(c_1, c_2) \in \Delta_0^0$. Moreover, $c_2 - p + (p + h)F^{2*}(u) = S_2^{(0)}(u) \leq S_2^{(a_1)}(u)$ and $S_2^{(a_1)}(u) \leq S_2^{(\infty)}(u) = c_2 + \int_0^{u-v_1} L'(u - s)\varphi(s)\, ds$, whence it follows that the domain Γ_2^- decreases as a_1 increases.

On the other hand, the curve $g_2(c_1)$ is the same for all a_1 if $(c_1, c_2) \in \Delta_1^0$. It is determined by equation $S_2^{(0)}(\overline{v}) = 0$, which can be rewritten in the form

$$2(p + h)(p - c_2) = (p + c_2 - c_1)^2, \quad \text{for} \quad h \geq p.$$

Thus, $g_2^{(0)}(c_1)$ does not depend on d. It starts from the point $c_1 = 2p$, $c_2 = p$ and crosses the line $c_1 = p$ at $c_2 = -(2p + h) + \sqrt{5p^2 + 4ph + h^2}$ and then the line $c_2 = c_1$ at $c_2 = p[1 - p/2(p + h)]$. For $h = p$ these values of c_2 are equal to $p(\sqrt{10} - 3)$ and $3p/4$ respectively.

Next, if $c_1 = 0$ one has $c_2 = (p+h)[\sqrt{1 + 2p(p + h)^{-1}} - 1]$ equal to $p(2\sqrt{3} - 3)$ for $h = p$. However, the set $\Gamma_2^- \cap \{c_2 > c_1\}$ is empty when $a_1 = \infty$.

As usual for dynamic programming, the optimal control depends on the planning horizon. Moreover, for n fixed there exist stability domains of cost parameters $(\Gamma_n^-, \Gamma_n^+, \Gamma^- \cap \Delta^k, \Gamma^+ \cap \Delta_l, k, l \geq 0)$ where the optimal behaviour has the same type, that is determined by the same set of critical levels u_n, v_n, w_n, $n \geq 2$, and $\bar{v}$.

Fortunately, using the ε-optimal and asymptotically optimal stationary controls one can reduce the number of stability domains and exclude dependence on n.

We prove below only the simplest results demonstrating the reasoning necessary for the general case.

Theorem 6. *Let* $0 < \alpha < 1$*,* $a_1 = \infty$*,* $a_2 \leq \infty$ *and* $c_2 > \alpha c_1$. *Then for any* $\varepsilon > 0$ *there exists* $n_0 = n_0(\varepsilon, k)$ *such that it is* ε*-optimal to use* $u_n(x) = v_n(x) = \max(x, \widehat{v})$ *at the first step of* n*-step process with* $n > n_0$ *if* $(c_1, c_2) \in \Delta_k$*,* $k = 0, 1, \ldots$*. The critical level* $\widehat{v}$ *is given by* (7).

Proof. Put for brevity $g_n(x) = G_n(x, x)$. According to Theorem 1 and Corollary 2 we can write for $n > k + 1$

$$f_n(x) = -c_1 x + \begin{cases} g_n(v_n), & x < v_n, \\ g_n(x), & x \geq v_n, \end{cases}$$

and

$$f_n(x) - f_{n-1}(x) = \begin{cases} g_n(v_n) - g_{n-1}(v_{n-1}), & x < v_{n-1}, \\ g_n(v_n) - g_{n-1}(x), & v_{n-1} \le x < v_n, \\ g_n(x) - g_{n-1}(x), & x \ge v_n, \end{cases}$$

if $(c_1, c_2) \in \Delta_k$, $k = 0, 1, \ldots$.

Taking into account that $g_n(v_n) = \min_x g_n(x)$ one easily gets

$$\max_{x \le z} |f_n(x) - f_{n-1}(x)| \le \max_{v_{n-1} \le x \le \max(z, \widehat{v})} |g_n(x) - g_{n-1}(x)|.$$

Recalling that $g_n(x) = c_1 x + L(x) + \alpha \int_0^\infty f_{n-1}(x - s)\varphi(s)\, ds$ it is possible to write for $z > \widehat{v}$ the following chain of inequalities

$$\max_{x \le z} |f_n(x) - f_{n-1}(x)| \le \alpha \max_{x \le z} |f_{n-1}(x) - f_{n-2}(x)| \le \ldots \le \alpha^{n-k} \delta_k(z).$$

Here $\delta_k(z) = \max_{v_{k+1} \le x \le z} |\int_0^\infty (f_{k+1}(x - s) - f_k(x - s)\varphi(s)\, ds| < \infty$ in Δ_k, $k = 0, 1, \ldots$, in particular, $\delta_0(z) = c_1 \mu + \max(L(z), L(v_1))$.

Clearly, we have established that $f_n(x)$ tends uniformly to a limit $f(x)$ on any half-line $\{x \le z\}$. This enables us to state that continuous function $f(x)$ satisfies the following functional equation

$$f(x) = -c_1 x + \min_{v \ge x} [c_1 v + L(v) + \alpha \int_0^\infty f(v - s)\varphi(s)\, ds].$$

Furthermore, if the planning horizon is infinite the optimal behaviour at each step is determined by a critical level $\widehat{v}$.

Since $u_n(x) = v_n(x) = x$ for all n, if $x \ge \widehat{v}$, it follows immediately that for any $\varepsilon > 0$ one can find $n_0(\varepsilon, c_1)$ such that ordering $(\widehat{v} - x)^+$ at the first step of n-step process with $n > n_0$ we obtain an ε-optimal control. It is obvious that $n_0(\varepsilon, c_1)$ can be chosen the same for the parameter set Δ_k, that is, $n_0 = n_0(\varepsilon, k)$. □

As follows from Definitions 3 and 4, a control is stationary if it prescribes the same behaviour at each step and it is asymptotically optimal if

$$\lim_{n \to \infty} n^{-1} \widehat{f}_n(x) = \lim_{n \to \infty} n^{-1} f_n(x)$$

where $\widehat{f}_n(x)$ represents the expected n-step costs under this control.

Theorem 7. *If $\alpha = 1$, $a_1 = \infty$, $a_2 \le \infty$ and $c_2 > c_1$, it is asymptotically optimal to take $z_n^{(1)}(x) = (\bar{t} - x)^+$, $z_n^{(2)}(x) = 0$ for all n with $\bar{t}$ given by $L'(\bar{t}) = 0$.*

Proof. Denote by $f_n^l(x)$ the expected n-step costs if $\bar{t}$ is applied during the first l steps, whereas the critical levels v_k, $k \le n - l$, optimal under the assumptions made, are used during the other steps.

It is clear that $f_n^n(x) = \widehat{f}_n(x)$ and $f_n^0(x) = f_n(x)$, hence

$$\widehat{f}_n(x) - f_n(x) = \sum_{l=1}^{n} (f_n^l(x) - f_n^{l-1}(x)). \tag{23}$$

Suppose for simplicity that $c_1 < p$, that is, v_1 is finite.

Since $v_n \le v_{n+1}$, $n \ge 1$, and $v_n \to \bar{t}$, as $n \to \infty$, one can find, for any $\varepsilon > 0$, such $\widehat{n} = n(\varepsilon)$ that $\bar{t} - \varepsilon < v_n \le \bar{t}$, if $n \ge \widehat{n}$. Furthermore, we have

$$\max_x |f_n^l(x) - f_n^{l-1}(x)| \le \max_x |f_{n-l+1}^1 - f_{n-l+1}^0(x)|$$

and

$$f_k^1(x) - f_k^0(x) = \begin{cases} c_1(\bar{t} - v_k) + L(\bar{t}) - L(v_k) + V(v_k), & x < v_k, \\ c_1(\bar{t} - x) + L(\bar{t}) - L(x) + V(x), & v_k \le x < \bar{t}, \\ 0, & x \ge \bar{t}, \end{cases}$$

where $V(x) = \int_0^\infty (f_{k-1}(\bar{t} - s) - f_{k-1}(x - s))\varphi(s)\, ds$.

Obviously, $k - 1 = n - l \ge \widehat{n}$ for $l \le n - \widehat{n}$, therefore

$$\max_x |f_k^1(x) - f_k^0(x)| \le d\varepsilon \quad \text{with} \quad d = 2(c_1 + \max(p, h))$$

and

$$\sum_{l=1}^{n-\widehat{n}} |f_n^l(x) - f_n^{l-1}(x)| \le (n - \widehat{n})d\varepsilon. \tag{24}$$

On the other hand,

$$\sum_{l=n-\widehat{n}+1}^{n} |f_n^l(x) - f_n^{l-1}(x)| \le \widehat{n} b(x) \tag{25}$$

where $b(x) = \max_{k \le \widehat{n}} |f_k^1(x) - f_k^0(x)| \le L(v_1) + d\bar{t} < \infty$, for all x.

It follows immediately from (23) to (25) that

$$n^{-1}(\widehat{f}_n(x) - f_n(x)) \to 0, \quad \text{as} \quad n \to \infty.$$

To complete the proof we have to verify that there exists, for all x,

$$\lim_{n\to\infty} n^{-1}\widehat{f}_n(x) = c_1\mu + L(\bar{t}), \quad \mu = \mathsf{E}\xi_k,\ k \ge 1. \tag{26}$$

This is obvious for $x \le \bar{t}$, since in this case

$$\widehat{f}_n(x) = c_1(\bar{t} - x) + c_1 \sum_{k=1}^{n-1} \mathsf{E}\xi_k + nL(\bar{t}).$$

Now let $x > \bar{t}$. Then we do not order during the first m_x steps where

$$m_x = \inf\{k : \sum_{i=1}^{k} \xi_i > x - \bar{t}\}.$$

In other words, we wait until the inventory falls below the level $\bar{t}$ proceeding after that as in the previous case. Hence,

$$\widehat{f}_n(x) = L(x) + \mathsf{E} \sum_{i=1}^{m_x - 1} L(x - \sum_{k=1}^{i} \xi_k) + c_1 \mathsf{E} \left[\zeta_x + \sum_{i=m_x+1}^{n-1} \xi_i \right] + \mathsf{E}(n - m_x) L(\bar{t})$$

here $\zeta_x = \sum_{i=1}^{m_x} \xi_i - (x - \bar{t})$ is the overshot of the level $x - \bar{t}$ by the random walk with jumps ξ_i, $i \geq 1$.

Thus, it is possible to rewrite $\widehat{f}_n(x)$ as follows

$$\widehat{f}_n(x) = n(c_1\mu + L(\bar{t})) + W(x).$$

Using Wald's identity and renewal processes properties (see, e.g. [1]), as well as, the fact that $L(\bar{t})$ is the minimum of $L(x)$ it is possible to establish that $|W(x)| < \infty$ for a fixed x. So (26) follows immediately.

The same result is valid for $c_1 \geq p$. The calculations being long and tedious are omitted. □

Remark 4. For the parameter sets treated in Theorems 2 and 3 the asymptotically optimal policy is also of threshold type being based either on u^0 or $\bar{v}$ and u^∞.

Since $\bar{t} = g(p, h)$, with $g(a_1, a_2) = F^{inv}(a_1/(a_1 + a_2))$, it is useful to check its sensitivity with respect to small fluctuations of parameters p and h and perturbations of distribution F.

We apply the local technique, more precisely, differential importance measure (DIM) introduced in [3] is used. Let $a^0 = (a_1^0, a_2^0)$ be the base-case values of parameters, reflecting the decision maker (researcher) knowledge of assumptions made. The (DIM) for parameter a_s, $s = 1, 2$, is defined as follows

$$D_s(a^0, da) = g'_{a_s}(a^0)\, da_s \left(\sum_{j=1}^{2} g'_{a_j}(a^0)\, da_j \right)^{-1} \quad (= dg_s(a^0)/dg(a^0)$$

if $dg(a^0) \neq 0$. Whence, for uniform parameters changes: $da_s = u$, $s = 1, 2$, we get

$$D1_s(a^0) = g'_{a_s}(a^0) \Big/ \sum_{j=1}^{2} g'_{a_j}(a^0). \tag{27}$$

Theorem 8. *Under assumptions of Theorem 7, (DIM)s for parameters p and h do not depend on distribution F.*

Proof. The result follows immediately from (27) and definition of function g. Since

$$g'_{a_1}(a^0) = \varphi^{-1}(\bar{t}^0)a_2^0/(a_1^0 + a_2^0)^2, \quad g'_{a_2}(a^0) = -\varphi^{-1}(\bar{t}^0)a_1^0/(a_1^0 + a_2^0)^2,$$

it is clear that

$$D1_1(a^0) = \frac{a_2^0}{a_2^0 - a_1^0}, \quad D1_2(a^0) = -\frac{a_1^0}{a_2^0 - a_1^0} = 1 - D1_1(a^0).$$

Thus, they are well defined for $a_1^0 \neq a_2^0$ and do not depend on F. Moreover, $D1_1(a^0) > 1$, $D1_2(a^0) < 0$ for $a_2^0 > a_1^0$ and $D1_1(a^0) < 0$, $D1_2(a^0) > 1$ for $a_2^0 < a_1^0$. □

Note that a similar result is valid for $\widehat{v}$ if $0 < \alpha < 1$.

Now we can establish that the asymptotically optimal policy is stable with respect to small perturbations of distribution F.

Denote by $\bar{t}_k$ value of $\bar{t}$ corresponding to distribution $F_k(t)$. Moreover, set

$$\gamma(F_k, F) = \sup_t |F_k(t) - F(t)|,$$

that is, γ is the Kolmogorov (or uniform) metric.

Lemma 2. *Let distribution function $F(t)$ be continuous and strictly increasing. Then $\bar{t}_k \to \bar{t}$, provided $\gamma(F_k, F) \to 0$, as $k \to \infty$.*

Proof. According to assumptions $F_k(\bar{t}_k) = F(\bar{t})$ and $|F_k(\bar{t}_k) - F(\bar{t}_k)| \leq \gamma(F, F_k)$. Hence $|F(\bar{t}) - F(\bar{t}_k)| \leq \gamma(F, F_k)$. That means, $\bar{t}_k \to \bar{t}$, as $k \to \infty$. □

This result is also important for construction of asymptotically optimal policies under assumption of none a priori information about distribution F.

3 Conclusion

We have treated in detail the case of two suppliers and obtained the explicit form of optimal, ε-optimal and asymptotically optimal policies for various sets of cost parameters. Stability of model to small fluctuations of parameters and perturbations of underlying process is also established. The case of m suppliers, $m > 2$, can be investigated using induction procedure and numerical methods. Due to lack of space the results will be published in a forthcoming paper.

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