Retrieving Information from Subordination

Jean Bertoin and Marc Yor

Abstract We recall some instances of the recovery problem of a signal process hidden in an observation process. Our main focus is then to show that if $(X_s,\ s \geq 0)$ is a right-continuous process, $Y_t = \int_0^t X_s \mathrm{d}s$ its integral process and $\tau = (\tau_u, u \geq 0)$ a subordinator, then the time-changed process $(Y_{\tau_u},\ u \geq 0)$ allows to retrieve the information about $(X_{\tau_v},\ v \geq 0)$ when τ is stable, but not when τ is a gamma subordinator. This question has been motivated by a striking identity in law involving the Bessel clock taken at an independent inverse Gaussian variable.

Keywords Recovery problem • Subordination • Bougerol's identity

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1 Introduction and Motivations Stemming from Hidden Processes

Many studies of random phenomena involve several sources of randomness. To be more specific, a random phenomenon is often modeled as the combination $C = \Phi(X, X')$ of two processes X and X' which can be independent or correlated,

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for some functional Φ acting on pairs of processes. In this framework, it is natural to ask whether one can recover X from C, and if not, what is the information on X that can be recovered from C? We call this the *recovery problem* of X given C. Here are two well-known examples of this problem.

- *Markovian filtering:* There C is the observation process defined for every $t \geq 0$ by $C_t = S_t + B_t$ where $S_t = \int_0^t h(X_s)\mathrm{d}s$ is the signal process arising from a Markov process X and $B = (B_t, t \geq 0)$ is an independent Brownian motion. Then the recovery problem translates in the characterization of the filtering process, that is the conditional law of X_t given the sigma-field $\mathscr{C}_t = \sigma(C_s, s \leq t)$. We refer to Kunita [6] for a celebrated discussion.

  In the simplest case when X remains constant as time passes, which yields $h(X_t) \equiv A$ where A is a random variable, note that A can be recovered in infinite horizon by $A = \lim_{t\to\infty} t^{-1}C_t$, but not in finite horizon. More precisely, it is easily shown that for a Borel function $f \geq 0$, there is the identity

  $$\mathbb{E}(f(A) \mid \mathscr{C}_t) = \frac{\int f(a)\mathscr{E}_t^a \mu(\mathrm{d}a)}{\int \mathscr{E}_t^a \mu(\mathrm{d}a)}$$

  where μ is the law of A and $\mathscr{E}_t^a = \exp(aC_t - ta^2/2)$; see Chap. 1 in [9].
- *Brownian subordination:* An important class of Lévy processes may be represented as

  $$C_t = B_{\tau_t}, \qquad t \geq 0,$$

  where τ a subordinator and B is again a Brownian motion (or more generally a Lévy process) which is independent of τ; see for instance Chap. 6 in [7]. Geman, Madan and Yor [4,5] solved the recovery problem of τ hidden in C; we refer the reader to these papers for the different recovery formulas.

There exist of course other natural examples in the literature; we now say a few words about the specific recovery problem which we will treat here and the organization of the remainder of this paper.

We will consider the recovery problem when the signal is $Y_t = \int_0^t X_s \mathrm{d}s$ and this signal is only perceived at random times induced by a subordinator τ. By this, we mean that the observation process is given by $C = Y \circ \tau$, and we seek to recover the subordinate process $X \circ \tau$. The precise formulation of the framework and our results will be made in Sect. 2. Proofs of the results found in Sect. 2 are presented in Sect. 3. Finally, in Sect. 4, we apply the results of Sect. 2 to an identity in law involving a Bessel process, which is equivalent to Bougerol's identity [2] and has provided the initial motivation of this work.

2 Framework and Main Statements

We consider on some probability space $(\Omega, \mathscr{A}, \mathbb{P})$ an $\mathbb{R}^d$-valued process $(X_s, s \geq 0)$ with right-continuous sample paths, and its integral process

$$Y_t = \int_0^t X_s \mathrm{d}s, \qquad t \geq 0\,.$$

Let also $(\tau_u,\ u \geq 0)$ denote a stable subordinator with index $\alpha \in (0,1)$. We stress that we do not require X and τ to be independent. We are interested in comparing the information embedded in the processes $\hat{X}$ and $\hat{Y}$ which are obtained from X and Y by the same time-change based on τ, namely

$$\hat{X}_u = X_{\tau_u} \text{ and } \hat{Y}_u = Y_{\tau_u}, \qquad u \geq 0\,.$$

We denote by $\left(\hat{\mathscr{X}}_u\right)_{u\geq 0}$ the usual augmentation of the natural filtration generated by the process $\hat{X}$, i.e. the smallest $\mathbb{P}$-complete and right-continuous filtration to which $\hat{X}$ is adapted. Likewise, we write $\left(\hat{\mathscr{Y}}_u\right)_{u\geq 0}$ for the usual augmentation of the natural filtration of $\hat{Y}$ and state our main result.

Theorem 1. *There is the inclusion* $\hat{\mathscr{X}}_u \subset \hat{\mathscr{Y}}_u$ *for every* $u \geq 0$.

We stress that for $u > 0$, in general $\hat{Y}_u$ cannot be recovered from the sole process $\hat{X}$, and then the stated inclusion is strict. An explicit recovery formula for $\hat{X}_u$ in terms of the jumps of $\hat{Y}$ will be given in the proof of Theorem 1 (see Sect. 3 below).

A perusal of the proof of Theorem 1 shows that it can be extended to the case when it is only assumed that τ is a subordinator such that the tail of its Lévy measure is regularly varying at 0 with index $-\alpha$, which suggests that this result might hold for more general subordinators. On the other hand, if $(N_v,\ v \geq 0)$ is any increasing step-process issued from 0, such as for instance a Poisson process, then the time-changed process $(Y_{N_v},\ v \geq 0)$ stays at 0 until the first jump time of N which is strictly positive a.s. This readily implies that the germ-σ-field

$$\bigcap_{v>0} \sigma(Y_{N_u}, u \leq v)$$

is trivial, in the sense that every event of this field has probability either 0 or 1. Focussing on subordinators with infinite activity, it is interesting to point out that Theorem 1 fails when one replaces the stable subordinator τ by a gamma subordinator, as can be seen from the following observation (choose $X_s \equiv \xi$).

Proposition 1. *Let* $\gamma = (\gamma_t,\ t \geq 0)$ *be a gamma-subordinator and* ξ *a random variable with values in* $(0,\infty)$ *which is independent of* γ. *Then the germ-*σ*-field*

$$\bigcap_{t>0} \sigma(\xi\gamma_s,\ s \leq t)$$

is trivial. On the other hand, we also have

$$\bigcap_{t>0} (\sigma(\xi) \vee \sigma(\gamma_s,\ s \leq t)) = \sigma(\xi)\,.$$

It is natural to investigate a similar question in the framework of stochastic integration. For the sake of simplicity, we shall focus on the one-dimensional case. We thus consider a real valued Brownian motion $(B_t,\ t \geq 0)$ in some filtration$(\mathscr{F}_t)_{t\geq 0}$ and an $(\mathscr{F}_t)$-adapted continuous process $(X_t,\ t \geq 0)$, and consider the stochastic integral

$$I_t = \int_0^t X_s \mathrm{d}B_s, \qquad t \geq 0\,.$$

We claim the following.

Proposition 2. *Fix $\eta > 0$ and assume that the sample paths of $(X_t,\ t \geq 0)$ are Hölder-continuous with exponent η a.s. Suppose also that $(\tau_v,\ v \geq 0)$ is a stable subordinator of index $\alpha \in (0,1)$, which is independent of $\mathscr{F}_\infty$. Then the usual augmentation $(\hat{\mathscr{I}}_v)_{v\geq 0}$ of the natural filtration generated by the subordinate stochastic integral $\left(\hat{I}_v = I_{\tau_v}, v \geq 0\right)$ contains the one generated by $(|X_{\tau_v}|, v \geq 0)$.*

3 Proofs

3.1 *Proof of Theorem 1*

For the sake of simplicity, we henceforth suppose that the tail of the Lévy measure of the stable subordinator τ is $x \mapsto x^{-\alpha}$, which induces no loss of generality. We shall need the following elementary version of the Law of Large Numbers for the jumps $(\Delta\tau_s = \tau_s - \tau_{s-}, s > 0)$ of a stable subordinator.

Fix any $\beta > 2/\alpha$ and introduce for any given $b \in \mathbb{R}$ and $\varepsilon > 0$

$$N_{\varepsilon,b} = \mathrm{Card}\{s \leq \varepsilon : b\Delta\tau_s > \varepsilon^\beta\}\,.$$

Note that $N_{\varepsilon,b} \equiv 0$ for $b \leq 0$.

Lemma 1. *We have*

$$\mathbb{P}\left(\lim_{n\to\infty} n^{1-\alpha\beta} N_{1/n,b} = b^\alpha \qquad \textit{for all } b > 0\right) = 1.$$

Remark. The rectangles $[0,\varepsilon] \times (\varepsilon^\beta, \infty)$ neither increase nor decrease with ε for $\varepsilon > 0$, so Lemma 1 does not reduce to the classical Law of Large Numbers for Poisson point processes. This explains the requirement that $\beta > 2/\alpha$.

Proof. Recall that for $b > 0$, $N_{\varepsilon,b}$ is a Poisson variable with parameter

$$\varepsilon(\varepsilon^\beta/b)^{-\alpha} = b^\alpha \varepsilon^{1-\beta\alpha}\,.$$

Chebychev's inequality thus yields the bound

$$\mathbb{P}\left(\left|n^{1-\alpha\beta}N_{1/n,b}-b^{\alpha}\right|>\frac{1}{\ln n}\right)\leq b^{2\alpha}n^{1-\alpha\beta}\ln^2 n$$

and since $1-\alpha\beta<-1$, we deduce from the Borel-Cantelli lemma that for each fixed $b>0$,

$$\lim_{n\to\infty} n^{1-\alpha\beta}N_{1/n,b}=b^{\alpha} \qquad \text{almost surely.}$$

We can then complete the proof with a standard argument of monotonicity. □

We now tackle the proof of Theorem 1 by verifying first that X_0 is $\hat{\mathscr{Y}}_0$-measurable. Let us assume that the process X is real-valued as the case of higher dimensions will then follow by considering coordinates. Set

$$J_\varepsilon=\mathrm{Card}\{s\leq\varepsilon:\Delta\hat{Y}_s>\varepsilon^{\beta}\},$$

where as usual $\Delta\hat{Y}_s=\hat{Y}_s-\hat{Y}_{s-}$. We note that

$$\Delta\hat{Y}_s-X_0\Delta\tau_s=\int_{\tau_{s-}}^{\tau_s}(X_u-X_0)\mathrm{d}u\,.$$

Hence if we set $a_\varepsilon=\sup_{0\leq u\leq\tau_\varepsilon}|X_u-X_0|$, then

$$(X_0-a_\varepsilon)\,\Delta\tau_s\leq\Delta\hat{Y}_s\leq(X_0+a_\varepsilon)\,\Delta\tau_s,$$

from which we deduce $N_{\varepsilon,X_0-a_\varepsilon}\leq J_\varepsilon\leq N_{\varepsilon,X_0+a_\varepsilon}$.

Since X has right-continuous sample paths a.s., we have $\lim_{\varepsilon\to0}a_\varepsilon=0$ a.s., and taking $\varepsilon=1/n$, we now deduce from Lemma 1 that

$$\lim_{n\to\infty} n^{1-\alpha\beta}J_{1/n}=(X_0^+)^{\alpha} \qquad \text{almost surely.}$$

Hence X_0^+ is $\hat{\mathscr{Y}}_0$-measurable, and the same argument also shows that X_0^- is $\hat{\mathscr{Y}}_0$-measurable.

Now that we have shown that X_0 is $\hat{\mathscr{Y}}_0$-measurable, it follows immediately that for every $v\geq0$, the variable $\hat{X}_v$ is $\hat{\mathscr{Y}}_v$-measurable. Indeed, define $\tau'_u=\tau_{v+u}-\tau_v$ and $X'_v=X_{v+\tau_v}$. Then τ' is again a stable(α) subordinator and X' a right-continuous process, and

$$\hat{Y}_{v+u}-\hat{Y}_v=\int_0^{\tau'_u}X'_s\mathrm{d}s\,.$$

Hence $X'_0=\hat{X}_v$ is measurable with respect to the $\mathbb{P}$-complete germ-σ-field generated by the process $(\hat{Y}_{v+u}-\hat{Y}_v,\ u\geq0)$, and *a fortiori* to $\hat{\mathscr{Y}}_v$.

Thus we have shown that the process $\hat{X}$ is adapted to the right-continuous filtration $\left(\hat{\mathscr{Y}}_v\right)_{v\geq0}$. Since by definition the latter is $\mathbb{P}$-complete and right-continuous, Theorem 1 is established. □

3.2 *Proof of Proposition 1*

Here it is convenient to agree that Ω denotes the space of càdlàg paths $\omega : [0,\infty) \to \mathbb{R}_+$ endowed with the right-continuous filtration $(\mathscr{A}_t)_{t\geq 0}$ generated by the canonical process $\omega_t = \omega(t)$. We write $\mathbb{Q}$ for the law on Ω of the process $(\xi\gamma_t,\, t \geq 0)$.

It is well known that for every $x > 0$ and $t > 0$, the distribution of the process $(x\gamma_s, 0 \leq s \leq t)$ is absolutely continuous with respect to that of the gamma process $(\gamma_s, 0 \leq s \leq t)$ with density $x^{-t}\exp((1-1/x)\gamma_t)$. Because ξ and γ are independent, this implies that for any event $\Lambda \in \mathscr{A}_r$ with $r < t$

$$\mathbb{Q}(\Lambda) = \mathbb{E}\left(\xi^{-t}\exp((1-1/\xi)\gamma_t)\,\mathbf{1}_{\{\gamma\in\Lambda\}}\right) .$$

Observe that

$$\lim_{t\to 0+} \xi^{-t}\exp((1-1/\xi)\gamma_t) = 1 \qquad \text{a.s.}$$

and the convergence also holds in $L^1(\mathbb{P})$ by an application of Scheffé's lemma (alternatively, one may also invoke the convergence of backwards martingales). We deduce that for every $\Lambda \in \mathscr{A}_0$, we have $\mathbb{Q}(\Lambda) = \mathbb{P}(\gamma \in \Lambda)$ and the right-hand-side must be 0 or 1 because the gamma process satisfies the Blumenthal's 0-1 law. On the other hand, the independence of ξ and γ yields that the second germ sigma field is $\sigma(\xi)$. □

Remarks. We point out that Proposition 1 holds more generally when γ is replaced by a subordinator with logarithmic singularity, also called of class $(\mathscr{L})$, in the sense that the drift coefficient is zero and the Lévy measure is absolutely continuous with density g such that $g(x) = g_0x^{-1} + G(x)$ where g_0 is some strictly positive constant and $G : (0,\infty) \to \mathbb{R}$ a measurable function such that

$$\int_0^1 |G(x)|\mathrm{d}x < \infty\,,\ g(x) \geq 0\,,\ \text{and} \int_1^\infty g(x)\mathrm{d}x < \infty\,.$$

Indeed, it has been shown by von Renesse et al. [8] that such subordinators enjoy a quasi-invariance property analogous to that of the gamma subordinator, and this is the key to Proposition 1.

Thanks to Theorem 1, if we replace in Proposition 1 the gamma process by τ, a stable subordinator, then both germ sigma fields are equal to $\sigma(\xi)$.

3.3 *Proof of Proposition 2*

The guiding line is similar to that of the proof of Theorem 1. In particular it suffices to verify that $|X_0|$ is measurable with respect to the germ-σ-field $\hat{\mathscr{F}}_0$.

Because Brownian motion B and subordinator τ are independent, the subordinate Brownian motion $(\hat{B}_v = B_{\tau_v},\, v \geq 0)$ is a symmetric stable Lévy process with index

2α. With no loss of generality, we may suppose that the tail of its Lévy measure Π is given by $\Pi(\mathbb{R}\backslash[-x,x]) = x^{-2\alpha}$. As a consequence, for every $\beta > 2/\alpha$ and $\varepsilon > 0$ and $b \in \mathbb{R}$, if one defines

$$N_{\varepsilon,b} = \mathrm{Card}\{s \le \varepsilon : |b\Delta\hat{B}_s|^2 > \varepsilon^\beta\},$$

then $N_{\varepsilon,b}$ is a Poisson variable with parameter $|b|^{2\alpha}\varepsilon^{1-\alpha\beta}$, and this readily yields (see Lemma 1)

$$\lim_{n\to\infty} n^{1-\alpha\beta} N_{1/n,b} = |b|^{2\alpha} \qquad \text{for all } b \in \mathbb{R}, \text{ almost-surely.} \tag{1}$$

Next set

$$J_\varepsilon = \mathrm{Card}\{s \le \varepsilon : |\Delta\hat{I}_s|^2 > \varepsilon^\beta\},$$

where as usual $\hat{I}_s = I_{\tau_s}$, and observe that

$$\Delta\hat{I}_s = X_0\Delta\hat{B}_s + (X_{\tau_{s-}} - X_0)\Delta\hat{B}_s + \int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}})\mathrm{d}B_u\,. \tag{2}$$

Recall the assumption that the paths of X are Hölder-continuous with exponent $\eta > 0$, so the $(\mathscr{F}_t)$-stopping time

$$T = \inf\left\{u > 0 : \sup_{0\le v<u} (u-v)^{-\eta}|X_u - X_v|^2 > 1\right\}$$

is strictly positive a.s. In particular, if we write $\Lambda_\varepsilon = \{\tau_\varepsilon < T\}$, then $\mathbb{P}(\Lambda_\varepsilon)$ tends to 1 as $\varepsilon \to 0+$.

We fix $a > 0$, we consider

$$K_{\varepsilon,a} = \mathrm{Card}\left\{s \le \varepsilon : \left|\int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}})\mathrm{d}B_u\right|^2 > a\varepsilon^\beta\right\},$$

and we claim that

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha\beta-1}\mathbb{E}(K_{\varepsilon,a}, \Lambda_\varepsilon) = 0\,. \tag{3}$$

If we take (3) for granted, then we can complete the proof by an easy adaptation of the argument in Theorem 1. Indeed, we can then find a strictly increasing sequence of integers $(n(k), k \in \mathbb{N})$ such that with probability one, for all rational numbers $a > 0$

$$\lim_{k\to\infty} n(k)^{1-\alpha\beta} K_{1/n(k),a} = 0\,. \tag{4}$$

We observe from (2) that for any $a \in (0, 1/2)$, if $|\Delta\hat{I}_s|^2 > \varepsilon^\beta$, then necessarily either

$$|X_0\Delta\hat{B}_s|^2 > (1-2a)^2\varepsilon^\beta,$$

or

$$|(X_{\tau_{s-}} - X_0)\Delta \hat{B}_s|^2 > a^2\varepsilon^\beta,$$

or

$$\left|\int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}})\mathrm{d}B_u\right|^2 > a^2\varepsilon^\beta .$$

As

$$\lim_{\varepsilon\to 0+} \sup_{0\le s\le\varepsilon} |X_{\tau_{s-}} - X_0| = 0,$$

this easily entails, using (1) and (4), that

$$\begin{aligned}\limsup_{k\to\infty} n(k)^{1-\alpha\beta} J_{1/n(k)} &\le \lim_{k\to\infty} n(k)^{1-\alpha\beta} N_{1/n(k),(1-2a)^{-1}|X_0|}\\ &= (1-2a)^{-2\alpha}|X_0|^{2\alpha}, \qquad \text{a.s.}\end{aligned}$$

where the identity in the second line stems from (1). A similar argument also gives

$$\liminf_{k\to\infty} n(k)^{1-\alpha\beta} J_{1/n(k)} \ge (1+2a)^{-2\alpha}|X_0|^{2\alpha}, \qquad \text{a.s.},$$

and as a can be chosen arbitrarily close to 0, we conclude that

$$\lim_{k\to\infty} n(k)^{1-\alpha\beta} J_{1/n(k)} = |X_0|^{2\alpha}, \qquad \text{a.s.}$$

Hence $|X_0|$ is $\hat{\mathscr{I}}_0$-measurable.

Thus we need to establish (3). As τ is independent of $\mathscr{F}_\infty$, we have by an application of Markov's inequality that for every $s \le \varepsilon$

$$\begin{aligned}&\mathbb{P}\left(\left|\int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}})\mathrm{d}B_u\right|^2 > a\varepsilon^\beta, \Lambda_\varepsilon \mid \tau\right)\\ &\le \frac{1}{a\varepsilon^\beta}\int_0^{\Delta\tau_s} v^\eta \mathrm{d}v \le \frac{(\Delta\tau_s)^{1+\eta}}{a\varepsilon^\beta} .\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}(K_{\varepsilon,a}, \Lambda_\varepsilon) &\le \mathbb{E}\left(\sum_{s\le\varepsilon}\left(\frac{(\Delta\tau_s)^{1+\eta}}{a\varepsilon^\beta}\wedge 1\right)\right)\\ &= \varepsilon c\int_{(0,\infty)} x^{-1-\alpha}\left(\frac{x^{1+\eta}}{a\varepsilon^\beta}\wedge 1\right)\mathrm{d}x = O(\varepsilon^{1-\alpha\beta/(1+\eta)}),\end{aligned}$$

where for the second line we used the fact that the Lévy measure of τ is $cx^{-1-\alpha}\mathrm{d}x$ for some unimportant constant $c > 0$. This establishes (3) and hence completes the proof of our claim. □

4 Application to an Identity of Bougerol

In this section, we answer a question raised by Dufresne and Yor [3], which has motivated this work.

A result due to Bougerol [2] (see also Alili et al. [1]) states that for each fixed $t \geq 0$ there is the identity in distribution

$$\sinh(B_t) \overset{\text{(law)}}{=} \int_0^t \exp(B_s)\mathrm{d}W_s \tag{5}$$

where B and W are two independent one-dimensional Brownian motions. Consider now a two-dimensional Bessel process $(R_u,\ u \geq 0)$ issued from 1 and the associated clock

$$H_t = \int_0^t R_u^{-2}\mathrm{d}u, \qquad t \geq 0\,.$$

Let also $(\tau_s,\ s \geq 0)$ denote a stable $(1/2)$ subordinator independent from the Bessel process R.

In Dufresne and Yor [3], it was remarked that by combining Bougerol's identity (5) and the symmetry principle of Désiré André, there is the identity in distribution for every fixed $s \geq 0$

$$H_{\tau_s} \overset{\text{(law)}}{=} \tau_{a(s)}, \tag{6}$$

where $a(s) = \mathrm{Argsinh}(s) = \log\left(s + \sqrt{1+s^2}\right)$.

In [3], the authors wondered whether (6) extends at the level of processes indexed by $s \geq 0$, or equivalently whether $(\hat{H}_s = H_{\tau_s},\ s \geq 0)$ has independent increments. Theorem 1 entails that this is not the case. Indeed, it implies that the usual augmentation $\left(\hat{\mathscr{H}}_s\right)_{s\geq 0}$ of the filtration generated by $\hat{H}$ contains the one generated by $\left(\hat{R}_s = R_{\tau_s},\ s \geq 0\right)$. On the other hand, (R, H) is a Markov (additive) process, and since subordination by an independent stable subordinator preserves the Markov property, $(\hat{R}, \hat{H})$ is Markovian in its own filtration, which coincides with $\left(\hat{\mathscr{H}}_s\right)_{s\geq 0}$ by Theorem 1. It is readily seen that for any $v > 0$, the conditional distribution of $H_{\tau_{s+v}}$ given (R_{τ_s}, H_{τ_s}) does not only depend on H_{τ_s}, but on R_{τ_s} as well. Consequently the process $\hat{H}$ is not Markovian and *a fortiori* does not have independent increments.

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