Upper Bounds for Bernstein Basis Functions

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Abstract From Markov's bounds for binomial coefficients (for which a short proof is given) upper bounds are derived for Bernstein basis functions of approximation operators and their maximum. Some related inequalities used in approximation theory and those for concentration functions are discussed.

Keywords Bernstein basis functions for approximation operators • Markov bounds for binomial coefficients • Zeng's upper bounds for binomial probabilities • Extension of upper bounds for binomial probabilities via discretization of the argument. Rogozin's and some other inequalities for concentration functions

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1 Markov's Bounds for Binomial Coefficients. Preliminaries

One can get upper bounds for Bernstein basis functions of approximation operators, i.e., binomial probabilities

$$b(k;n,p) = C_n^k p^k (1-p)^{n-k}, \quad p \in [0,1], \quad k = 0, 1, \ldots, n,$$

using direct analytic or probabilistic methods.

First estimates of $b(k;n,p)$ can be found in "Ars Conjectandi" by J. Bernoulli, see [3] and commentary by Yu.V. Prokhorov "Law of Large Numbers and Estimates for Probabilities of Large Deviations" on pp. 116–155 in the same [3]. Using an additional argument together with one to obtain the Stirling formula Markov proved the double inequality for binomial coefficients C_n^k which we prefer to write in the form of bounds for $b(k;n,p)$ (see [12], pp. 72, 73 or formula (16) on p. 135 in above mentioned commentary in [3]; cf. formula (135) in Chap. IV "The rate of approximation of functions by linear positive operators" of [11]):

Theorem A. *Let* $n \geq 1$, $k \geq 1$, $n-k \geq 1$ *and* $p \in (0,1)$. *Then*

$$e^{\frac{1}{12n}-\frac{1}{12k}-\frac{1}{12(n-k)}} \sqrt{\frac{n}{2\pi k(n-k)}} \left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k}$$
$$< b(k;n,p) < \sqrt{\frac{n}{2\pi k(n-k)}} \left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k} =: \mathrm{Ma}(k;n,p). \quad (1)$$

Let us give a short proof of (1) with $1/(12n+1)$ instead of $1/(12n)$ in the exponent in the left-hand side.

Proof. The proof is based on the double inequality which refines Stirling asymptotics

$$(2\pi)^{1/2} n^{n+1/2} e^{-n+1/(12n+1)} < n! < (2\pi)^{1/2} n^{n+1/2} e^{-n+1/(12n)} \quad (2)$$

(see Feller's book [5], Chap. II, and Robbins' paper [15] referred therein).

Due to (2) we have

$$C_n^k = n!/[k!(n-k)!] < [n/(2\pi\, k(n-k))]^{1/2} n^n\, k^{-k}\, (n-k)^{-(n-k)}$$
$$\times \exp[1/(12n) - 1/(12k+1) - 1/(12(n-k)+1)]. \quad (3)$$

The nominator of the latter exponent equals to

$$(12k+1)(12(n-k)+1) - 12n(12n+2)$$
$$= 144[k(n-k) - (1/4)n^2] - 108n^2 - 12n + 1,$$

which is negative for each $n > 1$ and k. Multiplication of both sides of inequality (3) by $p^k(1-p)^{n-k}$ completes the proof of right-hand inequality of (1). Dealing with the left-hand inequality similarly we find that the exponent is negative, too, both in initial and weakened form.

From (1) immediately follows that for some p and n the binomial probabilities $b(np; n, p)$ is less than its De Moivre–Laplace asymptotic expression.

Corollary 1. (a) *For any rational* $p \in (0, 1)$ *and* n *such that* np *is an integer*

$$b(np; n, p) < \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{np(1-p)}} =: \mathrm{MoLa}(n, p). \tag{4}$$

(b) *Inequality* (4) *is valid for* $b(k_0(n); n, p)$ *with* $p = k_0(n)/n$ *for any integer* $k_0(n)$ *such that* $0 < k_0(n) < n$.
(c) *In both cases* (a) *and* (b) *inequality* (4) *holds for* $b(k; n, p)$ *with any* $k = 0, 1, \ldots, n$.
(d) *The constant* $\frac{1}{\sqrt{2\pi}}$ *in* (4) *is best possible.*

It is worth to mention that in the standard situation when for binomial probabilities Poisson's asymptotic formula is valid, i.e., $b(k; n, p) - \mathrm{Po}(k; np) \to 0$ as $n \to \infty$, $p \to 0$ and np remaining bounded, for any fixed $k \in N = \{0, 1, \ldots\}$ with $\mathrm{Po}(k; \lambda) = \lambda^k e^{-\lambda}/k!$, $\lambda > 0$, one can derive the following representations of $\mathrm{Ma}(k; n, p)$ as upper bounds for $b(k; n, p)$ and $b(n-k; n, p)$ for fixed k and $n-k$ respectively.

Corollary 2. *If* k *is fixed, then for* $n > k$

$$\begin{aligned} b(k; n, p) &< \ell\mathrm{Ma}(k; n, p) \\ &:= \mathrm{Po}(k; np) \sqrt{\frac{n}{n-k}} \frac{k!}{\sqrt{2\pi k}(k/e)^k} e^{np-k} \left(1 + \frac{k-np}{n-k}\right)^{n-k}. \end{aligned} \tag{5}$$

If $l = n - k$ *is fixed, then for* $n > l$

$$b(l; n, p) = b(n-l; n, 1-p) < \ell\mathrm{Ma}(n-l; n, 1-p) =: \mathrm{rMa}(l; n, p). \tag{6}$$

The chain of results which has inspired our small contribution has began by the inequality established and used by Guo [7], to estimate the rate of convergence of the Durrmeyer operators for functions of bounded variation. His proof was based on the Berry–Esseen theorem; Guo obtained the inequality

$$b(k; n, p) \le \frac{C}{\sqrt{np(1-p)}}, \quad p \in (0, 1), \quad 0 \le k \le n,$$

with $C = 5/2$. In the year 1998, Zeng [17] has improved this bound having proved the following assertion.

Theorem B. *For a fixed $j \in N$ and*

$$C_j = ((j + 1/2)^{j+1/2}/j!)e^{-(j+1/2)} \tag{7}$$

for all k, p such that $j \le k \le n - j$, $p \in (0, 1)$, there holds

$$b(k; n, p) < \frac{C_j}{\sqrt{np(1-p)}} =: Z_j(n, p). \tag{8}$$

Moreover, the coefficient C_j is best possible (that is to say, for arbitrary $\varepsilon > 0$, it can not be replaced by $C_j - \varepsilon$), and the estimate order $n^{-1/2}$ is the optimal also.

The sequence of constants C_j decreases strictly and

$$\lim_{j\to\infty} C_j = \frac{1}{\sqrt{2\pi}}.$$

Hence for all $j \in N$, there holds

$$\frac{1}{\sqrt{2\pi}} < C_j \le C_0 = \frac{1}{\sqrt{2e}}. \tag{9}$$

In particular, for $j = 0$ (8) reduces to

$$b(k; n, p) < \frac{1}{\sqrt{2enp(1-p)}} = Z_0(n, p), \quad p \in (0, 1), \quad 0 \le k \le n. \tag{10}$$

Bastien and Rogalski solved in [2] a problem posed by V. Gupta in a private communication, having given there another proof that the upper bound (10) obtained by Zeng [17] is the optimum.

In the year 2001 Zeng and Zhao [18] have obtained the bound (4) for Bernstein basis functions (in fact assertions (b), (c) and (d) of our Corollary 1 of Theorem A from [11] and [3]).

In [1, 9] and [8] upper bound (10) is used to obtain the rate of convergence for Bernstein–Durrmeyer operators. Here we present the result of our collaboration to investigate the above mentioned problem concerning the optimal constant in the inequality (10).

Our first observation is that the inequalities given by Corollary 1 and Theorem B, namely relations (4) and (10) in fact are estimates for maximal probability of binomial distribution

$$b(n, p) = \max_{0\le k\le n} b(k; n, p).$$

It is well-known that due to De Moivre–Laplace local limit theorem, for $p \in (0, 1)$ $b(n, p)$ is equivalent to

$$(2\pi np(1-p))^{-1/2}$$

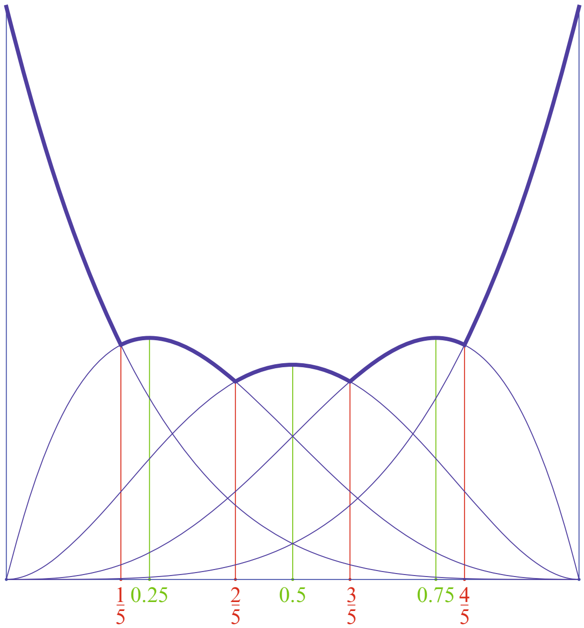

Fig. 1 Graphs of $b(k; n, p)$ as functions of $p \in [0, 1]$, $k = 0, \ldots, n$ for $n = 4$, their maxima and intersection points. In this figure, $b(n, p)$ is drawn by a *thick line*

as $n \to \infty$ (a nice proof is given in Feller's book [5], Chap. VII). It turns out that the latter expression is at the same time an upper bound for $b(n, p)$ for rational p and n such that np is an integer. The above equivalence shows that dependence on n and the constant in this upper bound are optimal. The fine structure of the system of modal binomial values $m = [(n+1)p]$, where $[\cdot]$ denotes the integer part, leads to an immediate upper bound for any n and p by substitution of p with the step function $p^* = m/n$; see Fig. 1 and a few useful facts concerning m, namely:

(a) The most probable value (or modal value or mode) m of the binomial distribution is defined by the inequality

$$(n+1)p - 1 < m \leq (n+1)p, \tag{11}$$

if $m = (n+1)p$, there are two modal values $b(m-1; n, p) = b(m; n, p)$.

(b) The suitable binomial probability is not greater than maximum of $b(m; n, p)$ in p attained at $p = p^* = m/n$, that is

$$b(m; n, p) \leq b(m; n, p^*). \tag{12}$$

2 Bounds for $b(n, p)$

The following proposition is in fact a reformulation of Corollary 1 for $b(n, p)$.

Proposition 1. *For any* $k_0 = k_0(n)$ *such that* $0 < k_0(n) < n$ *and* $p = k_0(n)/n$ *there holds*

$$b(n, k_0(n)/n) < \mathrm{MoLa}(n, p). \tag{13}$$

The estimate coefficient $\frac{1}{\sqrt{2\pi}}$ is the best possible.

In particular, for a constant rational probability $p, 0 < p < 1$, and n such that np is an integer, for $b(n, p) = b(np; n, p)$ inequality (13) holds true.

The right-hand side of inequality (12) is covered by Proposition 1. Thus we obtain

Proposition 2. *Define for* $0 < p < 1$ *the function* $p^* = p^*(p) = m/n$, *where* $m = m(p) = [(n + 1)p]$ *is the (maximal) mode of binomial distribution (*m/n *is equal to* 0 *on* $(0, 1/(n + 1))$, *to* $1/n$ *on* $[1/(n + 1), 2/(n + 1)), \ldots$ *and to* 1 *on* $[n/(n + 1), 1)$*). Then for any* n *and* $1/(n + 1) \le p < n/(n + 1)$ *the inequality*

$$b(n, p) < (2\pi np^*(1 - p^*))^{-1/2} = \mathrm{MoLa}(n, p^*) \tag{14}$$

holds.

Proposition 3. *For any* $p \in [1/(n + 1), n/(n + 1))$ *we have*

$$b(n, p) < \mathrm{Ma}(m; n, p) = \frac{\left(\frac{p}{p^*}\right)^{np^*}\left(\frac{1-p}{1-p^*}\right)^{n(1-p^*)}}{\sqrt{2\pi np^*(1 - p^*)}}. \tag{15}$$

Let us now try to discuss whether Propositions 2 and 3 have some advantage in approximation theory compared with the curves

$$z_0(n, p) = 1 \vee Z_0(n, p) = 1 \vee (2enp(1 - p))^{-1/2}, p \in (0, 1),$$
$$z_1(n, p) = 1 \vee Z_1(n, p) = 1 \vee C_1(np(1 - p))^{-1/2}, p \in (0, 1)$$

(cf.(5) and (10); see (4) and (9) for C_1), which seem natural to be introduced as $b(n, p)$ does not exceed 1.

Denote

$$v(n, p) = 1 \vee \mathrm{MoLa}(n, p), \quad p \in (0, 1).$$

Our results make it meaningful to consider the function $v^*(n, p)$ as $v(n, p^*)$ which reduces the interval $(0, 1)$ for p to

$$1/(n + 1) \le p < n/(n + 1);$$

out of this range lie the values of p for which $m = 0$ or $m = n$ which correspond to the values 0 and 1 for p^* excluded in the proposition. So we are motivated to

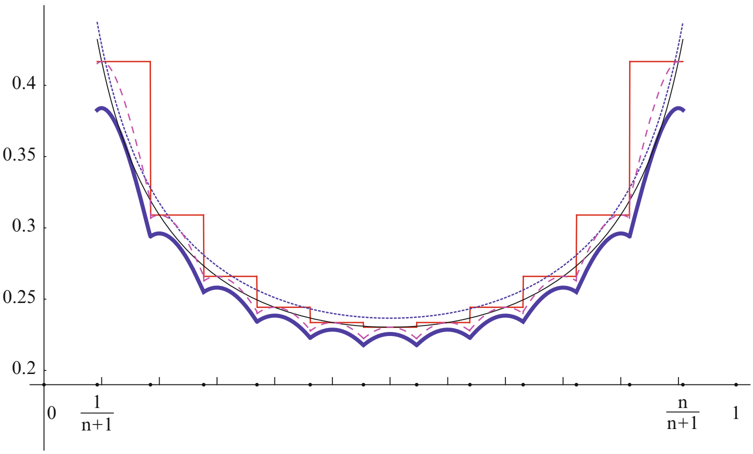

Fig. 2 Approximations of $b(n, p)$ ($\frac{1}{n+1} \leq p < \frac{n}{n+1}$, $n = 12$) *Thick line:* $b(n, p)$, *Dashed line:* $\mathrm{Ma}(m; n, p)$, *Step line:* $\mathrm{MoLa}(n, p^*)$, *Pointed line:* $Z_1(n, p)$, *Thin line:* $\mathrm{MoLa}(n, p)$

introduce probabilities $b(0; n, p)$ and $b(n; n, p)$ on corresponding intervals for p as extra summands into modified $v^*(n, p)$:

$$v^{**}(n, p) = v^*(n, p) + (1 - p)^n I_{(0,1/(n+1))}(p) + p^n I_{[n/(n+1),1)}(p),$$

where $I_E(p)$ stands for the indicator of a set E.

Figure 2 illustrates the fact that at least for p from some neighborhood of 1/2 the curves $z_0(n, p)$ and $z_1(n, p)$ lie over $v^{**}(n, p)$. In the same sense $\mathrm{Ma}(m; n, p)$ behaves much better.

In all the papers where Zeng's inequality (10) is used to obtain approximation estimations, see, e.g., [1, 8, 9], those will be evidently improved using inequalities (14) and (15).

As for each fixed k and l $b(k; n, k/n)$ and $b(n - l; n, 1 - l/n)$, according to Prokhorov's famous result (1953) [14], is better to treat via Poisson approximation than by normal one, this way may lead to better estimates useful for approximation theory.

Being motivated by this advantage for p close to 0 or 1, we tried to explore the following expression, using for $\mathrm{Ma}(k; n, p)$ the representations $\ell\,\mathrm{Ma}(k; n, p)$ for $k < n/2, 0 < p < 1/2$ and $r\,\mathrm{Ma}(k; n, p)$ for $k \geq n/2, 1/2 \leq p < 1$ (see relations (5) and (6)), each without two factors tending to one from three such ones:

$$\mathrm{Ma}^*(n, p) = I_{(0,1/2)}(p) \max_{0 \leq k < n/2} \sqrt{n/(n - k)}\,\mathrm{Po}(k; np) + I_{[1/2,1)}(p) \max_{n/2 \leq k < n} \sqrt{n/k}\,\mathrm{Po}(n - k; n(1 - p)).$$

Computer experiment shows that $\mathrm{Ma}^*(n, p)$ fits with $b(n, p)$ much better than $\mathrm{Ma}(m, n, p)$. This phenomenon is to be explained with theoretical argument.

An alternative way to construct estimates $b(n, p) = O(n^{-1/2})$ for Bernstein basis functions and similar ones for some other basis functions goes via inequalities for concentration functions of the sum S_n of the integer-valued i.i.d. random variables $\xi_1, \ldots, \xi_n$, namely for maximal probabilities of such a sum. For example, Rogozin gave in [16] the estimate which implies that

$$\max_k P(S_n = k) \le c((1 - p_0)n)^{-1/2}, \tag{16}$$

where p_0 stands for the maximal probability of each summand and c is an absolute constant.

In the case of binomial distribution $p_0 = p \vee (1-p)$ and as $1-p_0 = p \wedge (1-p)$, we have $p(1-p) < 1-p_0$ in (0,1) and thus dependence on p in Rogozin's inequality turns out to be better. As for the constant c its comparison with De Moivre–Laplace asymptotic expression shows that $c \ge 1/\pi^{1/2}$. The upper bound 2π for this constant is available from [13] (the suitable inequality is wrongly reproduced in the Russian translation of [10]). A general explanation of optimality of the order $n^{-1/2}$ in bounds of type of (16) can be found in [4] (see also [10] and [6]).

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