Albert N. Shiryaev
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Prokhorov and Contemporary Probability Theory

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Ernst L. Presman
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Prokhorov and Contemporary Probability Theory

In Honor of Yuri V. Prokhorov

Springer

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Preface

This volume is dedicated to the prominent mathematician and leading expert on probability theory and mathematical statistics Yuri Vasilyevich Prokhorov, who celebrated his 80-th birthday on 15 December 2009.

It consists of two parts. The first one contains papers written by his colleagues, friends and pupils who express their deep respect and sincerely love to him and his scientific activity.

The second part contains two interviews with Yu.V. Prokhorov. The first interview was taken by Friedrich Götze and Willem R. van Zwet between November 13 and 28, 2006 at Bielefeld University.

We decided to reproduce also the interview taken by Larry Shepp and published in Statistical Science 7 (1992), 123–130.

Moscow, Russia	Albert Shiryev
New York, USA	S.R.S. Varadhan
Moscow, Russia	Ernst Presman

Contents

Part II Interviews

Yuri Vasilyevich Prokhorov

Yuri Vasil'evich Prokhorov was born in Moscow in the family of a highway construction engineer. In 1944 after finishing high school as an external student at the age of fourteen and a half, he entered the Bauman High School (now Moscow Technical University). In 1945 he voluntary transferred to the Faculty of Mechanics and Mathematics of Moscow State University and graduated in 1949.

Right after graduating he became a research assistant at the Steklov Institute of Mathematics. In 1952 he got a PhD degree, and in 1956 defended a Doctorate dissertation in Physical and Mathematical Sciences.

In 1956 and 1957 he taught at the Moscow Institute of Engineering Physics, and in 1958 he returned to the Steklov Institute, where he continued working since then, replaced A.N. Kolmogorov as the head of the Department of Probability Theory in 1961. In parallel with his work at the Steklov Institute Prokhorov has taught at Moscow State University. Initially he lectured at the Faculty of Mechanics and Mathematics, where he received the title of Professor in 1958. In 1970 he moved to the newly formed Faculty of Computational Mathematics and Cybernetics, where he continues to head the Mathematical Statistics Department. In 1966 Prokhorov was elected a Corresponding Member of Academy of Sciences of the USSR, and a Full Member in 1972.

The scientific activity of Prokhorov began in his student years at the University. During his third year he started to actively participate in the probability theory seminar led by Academician Kolmogorov, and from that time he became one of Kolmogorov's disciples for many years.

In spring of 1948, being a fourth year student at the age of 18 Prokhorov wrote his first research paper which was published in 1949 (see [1]).[1] A detailed version was published next year (see [3]).

[1] All references are given to Sect. II of the List of Publications on pages xix–xxxviii

This paper treated the strong law of large numbers (SLLN) for independent random variables X_k, $k \geq 1$, i. e. the statement that there are centering constants a_n such that $\frac{S_n}{n} - a_n \to 0$ almost surely, where $S_n = \sum_{k=1}^{n} X_k$.

In spite of the author's youth, this paper contained substantial and important advances in the investigation of the SLLN and was the first in a series of papers, the most important results of which are now classical.

Namely, in [1, 3] he gave necessary and sufficient conditions for applicability of the SLLN in terms of the probabilities of large deviations from the medians of the variables

$$Y_r = \frac{1}{2^r} \sum_{k=2^r}^{2^{r+1}} X_k.$$

Using this result Prokhorov obtained also a simpler sufficient condition, which becomes a necessary condition for Gaussian (normally distributed) random variables and for random variables satisfying the condition $X_k = o(k/\log\log k)$. He showed that such variables satisfy the SLLN if and only if

$$\sum_{r=1}^{\infty} \exp\{-\varepsilon/\mathsf{D}Y_r\} < \infty$$

for any $\varepsilon > 0$, where $\mathsf{D}Y_r$ stands for the variance of Y_r. As a corollary the following sufficient condition was obtained from this statement. If $EX_k = 0$ and $\sum_{k=1}^{\infty} E\frac{|X_k|^{2r}}{k^{r+1}} < \infty$ for some $r \geq 1$, then $\frac{S_n}{n} \to 0$, $\mathbf{P}$-a.s.

In subsequent years (mainly in the 1950s) Prokhorov returned more than once to this subject (see [17, 18, 20]) finding new and broader conditions for applicability of the SLLN, some of which are the best possible (for example, $o(k/\log\log k)$ in the above condition was replaced by $O(k/\ln\ln k)$, and the weakened condition is then definitive).

In a later paper related to the law of large numbers (see [53]), Prokhorov discovered a new phenomenon arising when random variables X_n take values in a Hilbert space: there are sequences of independent variables X_n with identical symmetric distributions such that $\frac{\| S_n \|^2}{b_n} \to 1$ in probability as $n \to \infty$ for some sequence of constants $\{b_n\}$ (this is impossible in the finite-dimensional case).

Prokhorov's interest to "exponential" bounds for probabilities of large deviations for sums of independent random variables was methodologically connected with his study of the law of large numbers. For example, in 1968 (see [45]) he was the first to obtain an effective multidimensional generalization of Bernstein's exponential inequality. Further important advances in this direction were later made by his students and followers.

Besides his investigations on the SLLN, Prokhorov actively worked in the 1950s in the classical area of local limit theorems (LLTs) of probability theory. His first paper in this direction, which appeared in 1952 (see [4]), contained the beautiful result that the LLT holds in the mean for sums of independent identically distributed variables X_i (or, which is the same, that the convergence in the integral limit theorem holds in variation) if and only if the integral limit theorem holds and the distribution of the sum $\sum_1^m X_i$ has a non-zero absolutely continuous component for some $m = m_0$. Somewhat later (see [10]) Prokhorov published new effective conditions for the LLT for lattice distributions: he found a simply formulated necessary and sufficient condition for an LLT to hold in a strengthened form for a sequence of independent uniformly bounded integer-valued random variables.

Many Prokhorov's results were connected with approximations of probability distributions and rates of convergence of such approximations. We mention the 1952 papers (see [4–5]) in which the well-known asymptotic expansion refining the central limit theorem under Cramer's condition was generalized to a broad class of discrete distributions. We mention also the remarkable 1953 investigation (see [8]) of the asymptotic behavior of the binomial distribution, where, in particular, the following Prokhorov's transparent and now classical result was obtained on the rate of approximation of the binomial distribution by the Poisson one in the variation distance:

$$\sum_{k=1}^{\infty} |P_n(k) - \pi(k)| \le \frac{2\lambda}{n} \min(2, \lambda),$$

where $\pi(k) = \dfrac{\lambda^k}{k!} e^{-\lambda}$, $P_n(k) = C_n^k p^k (1-p)^{n-k}$, $k = 0, 1, \dots, n$, and $P_n(k) = 0$, $k > n$, with $p = p(n)$ such that $np(n) \to \lambda > 0$.

The following fundamental result may serve as another example.

Let $S_n = \sum\limits_{k=1}^{n} X_k$ be a sum of identically distributed independent random variables with distribution function $F(x)$. Denote $F^n(x) = P(\sum_{i=1}^n X_i \le x)$. The result obtained in [12] says that for any distribution function $F = F(x)$ there exists a sequence of infinitely divisible distribution functions $(G_n)_{n \ge 1}$, such that

$$\rho(F^n, G_n) \to 0,$$

where $\rho(F, G) = \sup_x |F(x) - G(x)|$.

Following Kolmogorov, significant progress was made by Prokhorov in investigating the rate of this convergence. In addition to a sharper upper estimate, Prokhorov obtained for the first time a lower estimate, which required introducing of new ideas (see [21]).

The most profound and important work of Prokhorov is undoubtedly the series of papers on limit theorems for random processes. Short preliminary publications on this topic appeared in 1953–1954 (see [9, 11]).

The 1950s were marked by the creation of functional limit theorems in probability theory, in other words – theorems about weak convergence of probability measures in metric and topological spaces.

Here, the fundamental role belongs to the famous paper by Yu.V. Prokhorov "Convergence of random processes and limit theorems in probability theory", published in 1956 in "Theory of Probability and its Applications". This paper formed his doctoral dissertation defended in the same year in Steklov Mathematical Institute (see [15]).

Paper [14] brought Prokhorov widespread recognition by the international mathematical community and even celebrity among specialists. Most fundamental in this paper was the creation of a method for investigating the convergence of distributions of random processes based on Prokhorov's criterion for compactness of a family of measures on a complete separable metric space E.

Prokhorov Theorem 1. *Let $\mathscr{P} = \{P_\alpha, \alpha \in \mathscr{A}\}$ be a family of measures given on a complete separable metric space $(E, \mathscr{E}, \rho)$. The family $\mathscr{P}$ is relatively compact if and only if it is tight and* $\sup_\alpha P_\alpha(E) < \infty$.

Tightness of the family $\mathscr{P} = \{P_\alpha, \alpha \in \mathscr{A}\}$ means that for any $\varepsilon > 0$ there exists a compact set $K \subseteq \mathscr{C}$ such that $\sup\limits_{\alpha \in \mathscr{A}} P_\alpha(E \setminus K) \leq \varepsilon$. *Relative compactness* of the family $\mathscr{P} = \{P_\alpha, \alpha \in \mathscr{A}\}$ means that for any sequence of measures from $\mathscr{P}$ there exists a subsequence that weakly converges to some measure.

In the process of proving the theorem, in the space $\mathscr{M}$ of all measures on X a metric π was constructed (which came to be called the Prokhorov metric) for which the convergence is equivalent to weak convergence and which turns M into a complete space. In one-dimensional case this metric coincides with the Lévy metric.

A considerable part of [14] is devoted to elaboration (often very delicate technically) of this criterion for diverse function spaces important in applications ($C[0;1]$, $D[0;1]$, and so on).

Of particular interest for Yu.V. Prokhorov was weak convergence in the space of continuous functions $(C, \mathscr{C})$.

Prokhorov Theorem 2. *Let P and P_n, $n \geq 1$, be probability measures on $(C, \mathscr{C})$. If the family $\{P_n\}$ is tight and finite-dimensional restrictions of measures P_n, $n \geq 1$, converge weakly to the corresponding finite-dimensional restrictions of the measure P, then the sequence of measures P_n converge weakly to measure P $(P_n \Rightarrow P)$.*

Thus, for solving the problem of weak convergence $P_n \Rightarrow P$ one needs to find first of all conditions of tightness of the family P_n, $n \geq 1$. The key role plays here the following

Prokhorov Theorem 3. *Let $X = (X_t)_{t \geq 0}$ be a function in C with the module of continuity*

$$w_X(\delta) = \sup_{|t-s|<\delta} |X_t - X_s|.$$

A sequence of probability measures P_n, $n \geq 1$, on $(C, \mathscr{C})$ is tight if and only if the following conditions hold:

1. For any positive b there exists a such that

$$P_n(X\colon |X_0| > a) \leq b, \quad n \geq 1;$$

2. *For any pair of numbers* ε *and* b *there exists* δ, $0 < \delta < 1$, *and an integer* n_0 *such that*
$$P_n(X\colon w_X(\delta) \geq \varepsilon) \leq b, \quad n \geq n_0.$$

Under fairly broad conditions, weak convergence of distributions of random processes implies weak convergence of the distributions of functionals of the processes. This fact, very important in the theory of random processes, is often called the Prokhorov invariance principle. In [14] Prokhorov proved ‘functional central limit theorems’ giving necessary and sufficient conditions for convergence in distribution of random polygonal curves to processes with independent increments. For the case of convergence of random polygons to the Wiener process, the invariance principle is called the Donsker invariance principle. Donsker obtained the corresponding sufficient conditions in 1951 without using the general notion of convergence of measures in functional spaces.

In [14] Prokhorov also obtained an estimate of the convergence rate of random polygons to the Wiener process that was later shown to be the best possible with respect to order.

Together with the results of A.V. Skorokhod on the weak convergence in the space D of discontinuous functions and results of Erdös, Kac, Donsker, Gikhman and others, Prokhorov’s results laid a solid foundations of the theory of functional limit theorems that, for more than 50 years, have been one of the main tools in studying asymptotic properties of random processes.

There is also a criterion in [14] for relative compactness of a family of distributions on a Hilbert space in terms of the characteristic functionals of these distributions. In the harmonic analysis of distributions in linear topological spaces this criterion formed the basis for subsequent major advances, which have found important applications, particularly in the theory of generalized random processes.

On the whole, the methods and results of [14] served as a powerful stimulus to numerous subsequent investigations both in the USSR and abroad. Prokhorov himself extended his work on this topic developing further mainly the method of characteristic functionals (see [23]). In the mid-1960s (see [27, 30, 33]) he also applied the invariance principle to queueing problems in the study of transient phenomena in systems with failures, when the expectation of the time ξ_n between incoming claims and the expectation of the service time η_n approach each other as the number of incoming claims increases. Here the independent variables ξ_n and η_n can have distributions of general form.

In the 1960s Prokhorov became interested in characterization problems of mathematical statistics (see [40, 42]). A statistic Y is said to characterize a class $\mathscr{P}$ of distributions on a sample space if (1) $P_1Y^{-1} = P_2Y^{-1}$ for any $P_1, P_2 \in \mathscr{P}$, and (2) $P \in \mathscr{P}$ when $PY^{-1} = P_1Y^{-1}$. Prokhorov found characterizing statistics for classes $\mathscr{P}$ of general form that are important in applications (while previous results on this problem were only fragmentary). Moreover, in the case when the classes $\mathscr{P}$ are types of distributions he established continuity of the correspondence between $\mathscr{P}$ and the distributions of certain statistics characterizing them (this is the so-called stability of the characterization).

Also in the 1960s Prokhorov investigated the problem of controlling a Wiener process with the purpose of keeping it within specified limits (see [32]). Unlike the partial differential equation approach used usually in this area, which led to the necessity of infinitely frequent switchings for optimal control, he employed direct probabilistic methods and constructed an optimal control with only a bounded number of switchings per unit of time. In the area of control of random processes he also directed a number of investigations by his students.

At the beginning of the 1990s Prokhorov published several papers devoted to estimating the variance of generalized measures occurring in the Edgeworth expansion in finite-dimensional Euclidean spaces (see [57, 60]). Knowledge of such estimates, along with estimates of the remainder terms, is especially important in using the Edgeworth expansion in statistics when the dimension of the observed random vectors is comparable with the number of observations. The investigation of this problem also led to a study of polynomials in random variables having normal or gamma distributions (see [58, 59]).

It was shown there that if Y is a polynomial of degree $n \geq 1$ in a random variable with the standard normal or a gamma distribution (with a certain lower bound on the shape parameter), then $\mathsf{E}|Y|/(\mathsf{E}|Y|^2)^{1/2} \geq c_n$, where c_n depends only on n.

In the 1990s Prokhorov, together with G. Christoph and V.V. Ulyanov, investigated the behavior of the density p of the squared norm of a Gaussian random variable with values in a Hilbert space (see [63, 65, 71–73]). They found a new upper bound for p, and also indicated the range of values of the parameters for which p is at least $1/8$ of its least upper bound. Furthermore, for large values of the argument, p turns out to be asymptotically at least $1/8$ of its least upper bound for all values of the parameters.

In 1994–2002 Prokhorov, together with F. Götze and V.V. Ulyanov, has investigated the behavior of the characteristic functions of polynomials in complex variables (both one- and multi-dimensional) (see [61, 62, 64, 66–70, 76]). In these studies they employed very delicate estimates of trigonometric sums and integrals used in analytic number theory. For instance, they obtained profound results on estimates of characteristic functions of polynomials in normal and asymptotically normal random variables, thereby significantly refining the estimates known earlier. As they themselves have remarked, their estimates 'are analogous in form to the improvements to which the Vinogradov method led for trigonometric sums in comparison with the results of Weyl.'

In 2000s jointly with V.I. Khokhlov and O.V. Viskov a series of papers on analogues of the Chernov inequalitiy for binomial, negative binomial, Poisson and Gamma distributions was published (see [74–75, 82, 85]).

In 2005 Prokhorov together with V.Yu. Korolev and V.E. Bening (who are members of the headed by him Department of Mathematical Statistics of the Faculty of Computational Mathematics of Moscow State University) was awarded M.V. Lomonosov prize for the paper "Analytic methods of the theory of the risk based on a Gaussian mixed models" (see [80]).

It should be mentioned the other papers of this period. The joint papers with his pupil A.A. Kulikova devoted to the estimate of deviation of the distribution of the first digit from the Bedford law (see [79, 81]), and to the investigation of the distribution law of fractional part of random vectors (see [78, 83]).

Along with purely mathematical problems, Prokhorov has always been interested in applied topics. Virtually throughout his career in science he has often consulted specialists in diverse areas of knowledge on applied questions of probability theory and statistics. We have already mentioned his interest in and contributions to queueing theory and the control of random processes. Among his other areas of applied interest we can single out sequential statistical analysis, information theory, and, especially, applications of statistical methods in geology and geochemistry (see [29], [31], [38], [41], [47]). He himself has personally taken part in several geological expeditions.

In the course of nearly half a century of teaching, Prokhorov has had many students, among them more than a few now known as specialists in probability theory and mathematical statistics. In his relations with students he is extremely considerate, he goes through their work in detail, and he demonstrates an uncommon ability to inspire and encourage boldness and independence in creative research, all of which has often proved to be very effective.

Prokhorov has expended and continues to expend much effort on his editorial and publishing activities. Since 1966 (except for 1988–1993) he has been the Editor-in-Chief of Theory of Probability and its Applications, which was founded by Kolmogorov and is one of the world's leading journals in its area. He has also been on the editorial boards of the prestigious journals Zeitschrift für Wahrscheinlichkeitstheorie (now called Probability Theory and Related Fields) and Journal of Applied Probability. He has been especially active in his encyclopedic work: over several decades he has been on the science publishing council of the Great Soviet Encyclopedia (now Great Russian Encyclopedia), he is Deputy Editor-in-Chief of the five-volume Mathematical Encyclopedia, and he is the Editor-in-Chief of the encyclopedic dictionaries Mathematics and Probability and Mathematical Statistics. He is the Deputy Editor of the journal "Mathematical Problems of Cryptography" and the Deputy Editor of the encyclopedia "Discrete Mathematics".

He has put enormous energy and effort into his administrative work related to organization of scientific activities. During the period 1969–1986 he was the deputy director of the Steklov Institute. Since 1966 till 2002 he has been a member of the Office of the Mathematics Branch of the Academy of Sciences, for many years he has headed the Committee on Probability Theory and Mathematical Statistics of the Mathematics Branch (for as long as the committee has existed), from 1975 to 1983 he was a member of the Presidium of the Higher Certification Committee, from 1975 to 1978 he was a member of the Fields Medal Committee of the International Mathematical Union (IMU), and from 1979 to 1982 he was the Vice-President of the IMU.

His role in preventing the canceling of ICM-1982 in Warsaw was described in "Olli Lehto, *Mathematics Without Borders. A History of the International Mathematical Union*, Springer-Verlag New York Inc., 1998, p. 232":

"When the discussion about the Warsaw ICM began on 13 November 1982, at the Collège de France, present were Carleson (Chairman), Lions (Secretary), Bombieri, Cassels, Kneser, Lehto, and Olech. Not much progress had been made, when the door opened and in came Prokhorov, for the first time in attendance at an Executive Committee meeting. He soon ask for the floor and quietly elaborated his view on why holding the Congress would be in the better interests of the IMU than canceling it. He concluded by saying that as regards international contacts, mathematicians in Socialist countries were handicapped. They could not participate in the ICM-1986 at Berkeley in great numbers. Warsaw, in contrast, would provide them an excellent opportunity to meet colleagues from all over the world.

The matter-of-fact performance of Prokhorov was to the taste of the Executive Committee. He certainly contributed to the final decision. After a long discussion, the Executive Committee decided to confirm the organization of the ICM-82 in Warsaw in August 1983".

He also headed the organizing committee of the 1986 First World Congress of the Bernoulli Society bringing together international specialists in probability theory and mathematical statistics; the congress took place in Tashkent and attracted more than a 1,000 participants. He is one of the founders and a member of Academy of Cryptography.

The scientific community and the government has highly valued the research, pedagogical, organizational, and editorial/publishing activities of Prokhorov. He was given the Lenin Prize in 1970, and he has been awarded two Orders of "Trudovogo Krasnogo Znameni" ("Red Banner of Labour"), the Order of "Znak Pochota" ("Sign of Honor"), and also medals.

List of Publications by Yu.V. Prokhorov*

I Monographs and Textbooks

1. *Probability Theory, Basic Concepts, Limit Theorems, Random Processes.* (Russian) Moscow: "Nauka", 1967, 495 pp. (with Yu.A. Rosanov);
   English translation – New York-Heidelberg: Springer-Verlag, 1969. 401 pp.;
   Polish translation – *Rachunek Prawdopodobieństwa*. Warszawa: PWN, 1972, 472 s.
2. *Probability Theory, Basic Concepts, Limit Theorems, Random Processes.* Revised second edition. (Russian) Moscow: "Nauka", 1973, 495 pp. (with Yu.A. Rosanov).
3. *Probability Theory, Basic Concepts, Limit Theorems, Random Processes.* Third edition. Mathematical Reference Library, (Russian), Moscow: "Nauka", 1987, 398 pp. (with Yu.A. Rosanov).
4. *Lectures on Probability Theory and Mathematical Statistics.* (Russian) Moscow: Izdatel'skiy otdel Fakulteta VMiK MGU, 2004, 188 pp.
5. *Lectures on Probability Theory and Mathematical Statistics*. Revised second edition. In Series: Classical University Textbooks. (Russian) Moscow: Izdatel'stvo Moskovskogo Universiteta, 2011, 252 pp.

*This List of Publications is compiled by E.Presman. N.Gamkrelldze participated in the compiling of Sections II and VIII. We use the brochure "Pritulenko L.M.: Yuyi Vasilyevich Prokhorov. Bibliographical index. Edited by Z.G. Vysotskaya, BEN USSR, Moscow, 1979 (in Russian)". We thank for the help K. Borovkov, V. Ulyanov, M. Khatuntseva and T. Tolozova.

II Main Scientific Papers

1949

1. On the strong low of large numbers. (Russian) *Dokl. Akad. Nauk SSSR* (N.S.) 69 (1949), no. 5, 607–610.
2. On sums of a random number of random terms. (Russian) *Uspehi Mat. Nauk* 4 (1949), no. 4 (32), 168–172 (with A.N.Kolmogorov).

1950

3. On the strong low of large numbers. (Russian) *Izv. Akad. Nauk SSSR, Ser. Mat.* 14 (1950), no. 6, 523–536.

1952

4. A local theorem for densities. (Russian) *Dokl. Akad. Nauk SSSR* (N.S.) 83 (1952), no. 6, 797–800.
5. Local limit theorems for sums of independent terms. (Russian) *Uspehi Mat. Nauk* 7 (1952), no. 3 (49), 112.
6. Limit theorems for sums of independent random variables. (Russian) PhD Thesis. Moscow: Steklov Institute of Mathematics, 1952, 42 pp.
7. Some refinements of Lyapunov's theorem. (Russian) *Izv. Akad. Nauk SSSR. Ser. Mat.* 16 (1952), 281–292.

1953

8. Asymptotic behavior of the binomial distribution. (Russian) *Uspehi Mat. Nauk* (N.S.) 8 (1953), no. 3 (55), 135–142; Engl. transl. in *Select. Transl. Math. Statist. and Probability*, 1 (1961), 87–95. Providence, R.I.: Inst. Math. Statist. and Amer. Math. Soc.
9. Probability distributions in functional spaces. (Russian) *Uspehi Mat. Nauk* (N.S.) 8 (1953), no. 3 (55), 165–167.

1954

10. On a local limit theorem for lattice distributions. (Russian) Dokl. Akad. Nauk SSSR (N.S.) 98 (1954), no. 4, 535–538.

1955

11. Methods of functional analysis in limit theorems of theory of probabilities. (Russian) Vestnik Leningradskogo Universiteta 11 (1955), no. 4, 46.
12. On sums of identically distributed random variables. (Russian) *Dokl. Akad. Nauk SSSR* (N.S.) 105 (1955), no. 4, 645–647.

1956

13. Zufällige Funktionen und Grenzverteilungssätze. (German) Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 113–126. Deutscher Verlag der Wissenschaften, Berlin, 1956. (In Gemeinschaft mit A.N. Kolmogorov).
14. Convergence of random processes and limit theorems in probability theory. (Russian) *Teor. Veroyatnost. i Primenen.* 1 (1956), no. 2, 177–238; Engl. transl. in *Theory Probab. Appl.* 1 (1956), no. 2, 157–214.

15. Functional methods in limit theorems of theory of probabilities. Dissertation for degree of Doctor of Phys.-Math. Sci. Moscow: Steklov Institute of Mathematics, 1956, 134 pp.

1957

16. A characterization of normal distributions in Hilbert space. (Russian) *Teor. Veroyatnost. i Primenen.* 2 (1957), no. 4, 475–477; Engl. transl. in *Theory Probab. Appl.* 3 (1957), no. 4, 468–470 (with M. Fish).

1958

17. Strong stability of sums and infinitely divisible distributions. (Russian) *Teor. Veroyatnost. i Primenen.* 3 (1958), no. 2, 153–165; Engl. transl. in *Theory Probab. Appl.* 3 (1958), no. 2, 141–153.
18. Strong stability of sums and infinitely divisible distributions. In: Abstracts of Short Communications, International Congress of Mathematicians, Edinburgh, 1958, p. 125–126.

1959

19. An extremal problem in probability theory. (Russian) *Teor. Veroyatnost. i Primenen.* 4 (1959), no. 2, 211–214; Engl. transl. in *Theory Probab. Appl.* 4 (1959), no. 2, 201–203.
20. Some remarks on the strong law of large numbers. (Russian) *Teor. Veroyatnost. i Primenen.* 4 (1959), no. 2, 215–220; Engl. transl. in *Theory Probab. Appl.* 4 (1959), no. 2, 204–208.

1960

21. On a uniform limit theorem of A.N. Kolmogorov. (Russian) *Teor. Veroyatnost. i Primenen.* 5 (1960), no. 1, 103–113; Engl. transl. in *Theory Probab. Appl.* 5 (1960), no. 1, 98–106.
22. On the necessity of Cramer's conditon. (Russian) *Teor. Veroyatnost. i Primenen.* 5 (1960), no. 2, 262; Engl. transl. in *Theory Probab. Appl.* 5 (1960), no. 2, 236.

1961

23. The method of characteristic functionals, Proc. 4th Berkeley Symp. Math. Statist. and Prob., Vol. II. Probability theory. Berkeley, Calif.: Univ. California Press, 1961, pp. 403–419.
24. Some results associated with Bochner's theorem. (Russian) *Teor. Veroyatnost. i Primenen.* 6 (1961), no. 1, 87–93; Engl. transl. in *Theory Probab. Appl.* 6 (1961), no. 1, 82–87 (with V.V.Sazonov).
25. Random measures on compacta. (Russian) *Dokl. Akad. Nauk SSSR* 138 (1961), no. 1, 53–55.

1962

26. Extremal problems in limit theorems. (Russian) In: *Proc. Sixth All-Union Conf. Theory Prob. and Math. Statist* (Vilnius, 1960). (Russian) 1962, pp. 77–84, Gosudarstv. Izdat. Političesk. i Naučn. Lit. Litovsk. SSR, Vilnius

1963

27. Transition phenomena in queueing processes. I. (Russian). *Litovsk. Mat. Sb.* 3 (1963), no. 1, 199–205.
28. Addition to the Russian translation of the book: *M. Katz. Statistical Independence in Probability Theory, Analysis and Number Theory*. The Mathematical

Assotiation of America, and John Wiley & Sons, 1959, xiv+93 pp.; Moscow: Izd. Inostr. Lit., 1963. pp. 136–153.

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6. Bernoulli random walk, 1 (1977), 418–421; Engl. transl.: 1 (1988), 362–364.
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Part I
Scientific Papers

When Knowing Early Matters: Gossip, Percolation and Nash Equilibria

David J. Aldous

Abstract Continually arriving information is communicated through a network of n agents, with the value of information to the j'th recipient being a decreasing function of j/n, and communication costs paid by recipient. Regardless of details of network and communication costs, the social optimum policy is to communicate arbitrarily slowly. But selfish agent behavior leads to Nash equilibria which (in the $n \to \infty$ limit) may be efficient (Nash payoff = social optimum payoff) or wasteful ($0 <$ Nash payoff $<$ social optimum payoff) or totally wasteful (Nash payoff $= 0$). We study the cases of the complete network (constant communication costs between all agents), the grid with only nearest-neighbor communication, and the grid with communication cost a function of distance. The main technical tool is analysis of the associated first passage percolation process or SI epidemic (representing spread of one item of information) and in particular its "window width", the time interval during which most agents learn the item. In this version (written in July 2007) many arguments are just outlined, not intended as complete rigorous proofs. One of the topics herein (first passage percolation on the $N \times N$ torus with short and long range interactions; Sect. 6.2) has now been studied rigorously by Chatterjee and Durrett [4].

Keywords First passage percolation • Gossip • Information • Nash equilibrium • Rank based • Social network

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1 Introduction

A topic which one might loosely call "random percolation of information through networks" arises in many different contexts, from epidemic models [2] and computer virus models [10] to *gossip algorithms* [8] designed to keep nodes of a decentralized network updated about information needed to maintain the network. This topic differs from *communication networks* in that we envisage information as having a definite source but no definite destination.

In this paper we study an aspect where the vertices of the network are agents, and where there are costs and benefits associated with the different choices that agents may make in communicating information. In such "economic game theory" settings one anticipates a *social optimum* strategy that maximizes the total net payoff to all agents combined, and an (often different) *Nash equilibrium* characterized by the property that no one agent can benefit from deviating from the Nash equilibrium strategy followed by all other agents (so one anticipates that any reasonable process of agents adjusting strategies in a selfish way will lead to some Nash equilibrium). Of course a huge number of different models of costs, benefits and choices could fit the description above, but we focus on the specific setting where the value to you of receiving information depends on how few people know the information before you do. Two familiar real world examples are *gossip* in social networks and *insider trading* in financial markets. In the first, the gossiper gains perceived social status from transmitting information, and so is implicitly willing to pay for communicate to others; in the second the owner of knowledge recognizes its value and implictly expects to be paid for communication onwards. Our basic model makes the simpler assumption that the value to an agent attaches at the time information is received, and subsequently the agent takes no initiative to communicate it to others, but does so freely when requested, with the requester paying the cost of communication. In our model the benefits come from, and communication costs are paid to, the outside world: there are no payments between agents.

Remark. Many arguments are just outlined, not intended as complete rigorous proofs. This version was written in July 2007 to accompany a talk at the ICTP workshop "Common Concepts in Statistical Physics and Computer Science", and intended as a starting point for future thesis projects which could explore these and many variant problems in detail. One of the topics herein (first passage percolation on the $N \times N$ torus with short and long range interactions) has now been studied rigorously by Chatterjee and Durrett [4] (see Sect. 6.2 for their result) and so it seems appropriate to make this version publicly accessible.

1.1 *The General Framework: A Rank-Based Reward Game*

There are n agents (our results are in the $n \to \infty$ limit). The basic two rules are:

Rule 1. New items of information arrive at times of a rate-1 Poisson process; each item comes to one random agent.

Information spreads between agents by virtue of one agent calling another and learning all items that the other knows (details are case-specific, described later), with a (case-specific) communication cost paid by the *receiver* of information.

Rule 2. The j'th person to learn an item of information gets reward $R(\frac{j}{n})$.

Here $R(u),\ 0 < u \leq 1$ is a function such that

$$R(u) \text{ is decreasing; } R(1) = 0; \quad 0 < \bar{R} := \int_0^1 R(u)du < \infty. \tag{1}$$

Assuming information eventually reaches each agent, the total reward from each item will be $\sum_{j=1}^{n} R(\frac{j}{n}) \sim n\bar{R}$. If agents behave in some "exchangeable" way then the average net payoff (per agent per unit time) is

$$\text{payoff} = \bar{R} - (\text{average communication cost per agent per unit time}). \tag{2}$$

Now the average communication cost per unit time can be made arbitrarily small by simply communicating less often (because an agent learns *all* items that another agent knows, for the cost of one call. Note the calling agent does not know in advance whether the other agent has any new items of information). Thus the "social optimum" protocol is to communicate arbitrarily slowly, giving payoff arbitrarily close to $\bar{R}$. But if agents behave selfishly then one agent may gain an advantage by paying to obtain information more quickly, and so we seek to study Nash equilibria for selfish agents. In particular there are three qualitative different possibilities. In the $n \to \infty$ limit, the Nash equilibrium may be

- Efficient (Nash payoff = social optimum payoff)
- Or wasteful (0 < Nash payoff < social optimum payoff)
- Or totally wasteful (Nash payoff = 0).

1.2 *Methodology*

Allowing agents' behaviors to be completely general makes the problems rather complicated (e.g. a subset of agents could seek to coordinate their actions) so in each specific model we restrict agent behavior to be of a specified form, making calls at random times with a rate parameter θ; the agent's "strategy" is just a choice of θ, and for this discussion we assume θ is a single real number. If all agents use the same parameter value θ then the spread of one item of information through the network is as some model-dependent first passage percolation process(see Sect. 2.2). So there is some function $F_{\theta,n}(t)$ giving the proportion of agents who learn the item within time t after the arrival of the information into the network. Now suppose one agent **ego** uses a different parameter value ϕ and gets some payoff-per-unit-time, denoted by payoff(ϕ, θ). The Nash equilibrium value θ^{Nash} is the value of θ

for which **ego** cannot do better by choosing a different value of ϕ, and hence is the solution of

$$\frac{d}{d\phi}\text{payoff}(\phi, \theta)\bigg|_{\phi=\theta} = 0. \qquad (3)$$

Obtaining a formula for payoff(ϕ, θ) requires knowing $F_{\theta,n}(t)$ and knowing something about the geometry of the sets of informed agents at time t – see (19,26) for the two basic examples. The important point is that where we know the exact $n \to \infty$ limit behavior of $F_{\theta,n}(t)$ we get a formula for the exact limit θ^{Nash}, and where we know order of magnitude behavior of $F_{\theta,n}(t)$ we get order of magnitude behavior of θ^{Nash}.

Note that we have assumed that in a Nash equilibrium each agent uses the same strategy. This is only a sensible assumption when the network cost structure has enough symmetry (is *transitive* – see Sect. 7.1) and the non-transitive case is an interesting topic for future study.

It turns out (Sect. 4) that for determining the qualitative behavior of the Nash equilibria, the important aspect is the size of the *window width* $w_{\theta,n}$ of the associated first passage percolation process, that is the time interval over which the proportion of agents knowing the item of information increases from (say) 10 to 90 %. While this is well understood in the simplest examples of first passage percolation on finite sets, it has not been studied for very general models and our game-theoretic questions provide motivation for future such study.

To interpret later formulas it turns out to be convenient to work with the derivative of R. Write $R'(u) = -r(u)$, so that $R(u) = \int_u^1 r(s)ds$ and (1) becomes

$$r(u) \geq 0; \quad 0 < \bar{R} := \int_0^1 ur(u)du < \infty. \qquad (4)$$

1.3 *Summary of Results*

1.3.1 The Complete Graph Case

Network communication model: Each agent i may, at any time, call any other agent j (at cost 1), and learn all items that j knows.

Poisson strategy. The allowed strategy for an agent i is to place calls, at the times of a Poisson (rate θ) process, to a random agent.

Result (Sect. 2). In the $n \to \infty$ limit the Nash equilibrium value of θ is

$$\theta^{\text{Nash}} = \int_0^1 (1 + \log(1-u))R(u)du = \int_0^1 r(u)g(u)du, \qquad (5)$$

where $g(u) = -(1-u)\log(1-u) > 0$.

Our assumptions (1) on $R(u)$ imply $0 < \theta^{\text{Nash}} < \bar{R}$. Because an agent's average cost per unit time equals his value of θ, from (2) the Nash equilibrium payoff $\bar{R} - \theta^{\text{Nash}}$ is strictly less than the social optimum payoff $\bar{R}$ but strictly greater than 0. So this is a "wasteful" case.

1.3.2 The Nearest Neighbor Grid

Network communication model: Agents are at the vertices of the $N \times N$ torus (i.e. the grid with periodic boundary conditions). Each agent i may, at any time, call any of the four neighboring agents j (at cost 1), and learn all items that j knows.

Poisson strategy. The allowed strategy for an agent i is to place calls, at the times of a Poisson (rate θ) process, to a random neighboring agent.

Result (Sect. 3). The Nash equilibrium value of θ is such that as $N \to \infty$

$$\theta_N^{\text{Nash}} \sim N^{-1} \int_0^1 g(u) r(u) du \tag{6}$$

where $g(u) > 0$ is a certain complicated function – see (28).

So here the Nash equilibrium payoff $\bar{R} - \theta_N^{\text{Nash}}$ tends to $\bar{R}$; this is an "efficient" case.

1.3.3 Grid with Communication Costs Increasing with Distance

Network communication model. The agents are at the vertices of the $N \times N$ torus. Each agent i may, at any time, call any other agent j, at cost $c(N, d(i, j))$, and learn all items that j knows.

Here $d(i, j)$ is the distance between i and j. We treat two cases, with different choices of $c(N, d)$. In Sect. 5 we take cost function $c(N, d) = c(d)$ satisfying

$$c(1) = 1; \quad c(d) \uparrow \infty \text{ as } d \to \infty \tag{7}$$

and

Poisson strategy. An agent's strategy is described by a sequence
$(\theta(d);\ d = 1, 2, 3, \ldots)$; where for each d:

at rate $\theta(d)$ the agent calls a random agent at distance d.

In this case a simple abstract argument (Sect. 5) shows that the Nash equilibrium is efficient (without calculating what the equilibrium strategy or payoff actually is) for any $c(d)$ satisfying (7).

In Sect. 6 we take

$$c(N,d) = 1; \quad d = 1$$
$$= c_N; \quad d > 1$$

where $1 \ll c_N \ll N^3$, and

Poisson strategy. An agent's strategy is described by a pair of real numbers $(\theta_{\text{near}}, \theta_{\text{near}}) = \theta$:

at rate θ_{near} the agent calls a random neighbor
at rate θ_{near} the agent calls a random non-neighbor.

In this case we show (42) that the Nash equilibrium strategy satisfies

$$\theta_{\text{near}}^{\text{Nash}} \sim \zeta_1 c_N^{-1/2}; \quad \theta_{\text{far}}^{\text{Nash}} \sim \zeta_2 c_N^{-2}$$

for certain constants ζ_1, ζ_2 depending on the reward function. So the Nash equilibrium cost $\sim \zeta_1 c_N^{-1/2}$, implying that the equilibrium is efficient.

1.3.4 Plan of Paper

The two basic cases (complete graph, nearest-neighbor grid) can be analyzed directly using known results for first passage percolation on these structures; we do this analysis in Sects. 2 and 3. There are of course simple arguments for order-of-magnitude behavior in those cases, which we recall in Sect. 4 (but which the reader may prefer to consult first) as a preliminary to the more complicated model "grid with communication costs increasing with distance", for which one needs to understand orders of magnitude before embarking on calculations.

1.4 *Variant Models and Questions*

These results suggest many alternate questions and models, a few of which are addressed briefly in the sections indicated, the others providing suggestions for future research.

- Are there cases where the Nash equilibrium is totally wasteful? (Sect. 2.1)
- Wouldn't it be better to place calls at regular time intervals? (Sect. 7.2)
- Can one analyze more general strategies?
- In the grid context of Sect. 1.3.3, what is the equilibrium strategy and cost for more general costs $c(N,d)$?
- What about the symmetric model where, when i calls j, they exchange information? (Sect. 7.1)
- In formulas (5,6) we see decoupling between the reward function $r(u)$ and the function $g(u)$ involving the rest of the model – is this a general phenomenon?

- In the nearest-neighbor grid case, wouldn't it be better to cycle calls through the four neighbors?
- What about non-transitive models, e.g. social networks where different agents have different numbers of friends, so that different agents have different strategies in the Nash equilibrium?
- To model gossip, wouldn't it be better to make the reward to agent i depend on the number of other agents who learn the item from agent i? (Sect. 7.3)
- To model insider trading, wouldn't it be better to say that agent j is willing to pay some amount $s(t)$ to agent i for information that i has possessed for time t, the function $s(\cdot)$ not specified in advance but a component of strategy and hence with a Nash equilibrium value?

1.5 *Conclusions*

As the list above suggests, we are only scratching the surface of a potentially large topic. In the usual setting of information communication networks, the goal is to communicate quickly, and our two basic examples (complete graph; nearest-neighbor grid) are the extremes of rapid and slow communication. It is therefore paradoxical that, in our rank-based reward game, the latter is efficient while the former is inefficient. One might jump to the conclusion that in general efficiency in the rank-based reward game was inversely related to network connectivity. But the examples of the grid with long-range interaction show the situation is not so simple, in that agents *could* choose to make long range calls and emulate a highly-connected network, but in equilibrium they do not do so very often.

2 The Complete Graph

The default assumptions in this section are

Network communication model: Each agent i may, at any time, call any other agent j (at cost 1), and learn all items that j knows.

Poisson strategy. The allowed strategy for an agent i is to place calls, at the times of a Poisson (rate θ) process, to a random agent.

2.1 *Finite Number of Rewards*

Before deriving the result (5) in our general framework, let us step outside that framework to derive a very easy variant result. Suppose that only the first two recipients of an item of information receive a reward, of amount w_n say. Agent strategy cannot affect the first recipient, only the second. Suppose **ego** uses rate ϕ and other agents use rate θ. Then (by elementary properties of Exponential distributions)

$$P(\textbf{ego} \text{ is second to receive item}) = \frac{\phi}{\phi + (n-2)\theta}, \tag{8}$$

and so

$$\text{payoff}(\phi, \theta) = \frac{w_n}{n} + \frac{\phi w_n}{\phi + (n-2)\theta} - \phi.$$

We calculate

$$\frac{d}{d\phi}\text{payoff}(\phi, \theta) = \frac{(n-2)\theta w_n}{(\phi + (n-2)\theta)^2} - 1$$

and then the criterion (3) gives

$$\theta_n^{\text{Nash}} = \frac{(n-2)w_n}{(n-1)^2} \sim \frac{w_n}{n}.$$

To compare this variant with the general framework, we want the total reward available from an item to equal n, to make the social optimum payoff $\to$ 1, so we choose $w_n = n/2$. So we have shown that the Nash equilibrium payoff is

$$\text{payoff} = 1 - \theta_n^{\text{Nash}} \to \tfrac{1}{2}. \tag{9}$$

So this is a "wasteful" case.

By the same argument we can study the case where (for fixed $k \geq 2$) the first k recipients get reward n/k. In this case we find

$$\theta_n^{\text{Nash}} \sim \frac{k-1}{k}$$

and the Nash equilibrium payoff is

$$\text{payoff} \to \tfrac{1}{k}\,, \tag{10}$$

while the social optimum payoff $= 1$. Thus by taking $k_n \to \infty$ slowly we have a model in which the Nash equilibrium is "totally wasteful".

2.2 *First Passage Percolation : General Setup*

The classical setting for first passage percolation, surveyed in [11], concerns nearest neighbor percolation on the d-dimensional lattice. Let us briefly state our general setup for first passage percolation (of "information") on a finite graph. There are "rate" parameters $\nu_{ij} \geq 0$ for undirected edges (i, j). There is an initial vertex v_0, which receives the information at time 0. At time t, for each vertex i which has already received the information, and each neighbor j, there is chance $\nu_{ij}\,dt$ that j learns the information from i before time $t + dt$. Equivalently, create independent Exponential(ν_{ij}) random variables V_{ij} on edges (i, j). Then each vertex v receives the information at time

$$T_v = \min\{V_{i_0 i_1} + V_{i_1 i_2} + \ldots + V_{i_{k-1} i_k}\}$$

minimized over paths $v_0 = i_0, i_1, i_2, \ldots, i_k = v$.

2.3 *First Passage Percolation on the Complete Graph*

Let us consider first passage percolation on the complete n-vertex graph with rates $\nu_{ij} = 1/(n-1)$. Pick k random agents and write $\bar{S}^n_{(1)}, \ldots, \bar{S}^n_{(k)}$ for the times at which these k agents receive the information. The key fact for our purposes is that as $n \to \infty$

$$(\bar{S}^n_{(1)} - \log n, \ldots, \bar{S}^n_{(k)} - \log n) \xrightarrow{d} (\xi + S_{(1)}, \ldots, \xi + S_{(k)}) \tag{11}$$

where the limit variables are independent, ξ has double exponential distribution $P(\xi \leq x) = \exp(-e^{-x})$ and each $S_{(i)}$ has the *logistic* distribution with distribution function

$$F_1(x) = \frac{e^x}{1 + e^x}, \quad -\infty < x < \infty. \tag{12}$$

Here $\xrightarrow{d}$ denotes convergence in distribution. To outline a derivation of (11), fix a large integer L and decompose the percolation times as

$$\bar{S}^n_{(i)} - \log n = (\tau_L - \log L) + (\bar{S}^n_{(i)} - \tau_L + \log(L/n)) \tag{13}$$

where τ_L is the time at which some L agents have received the information. By the Yule process approximation (see e.g. [1]) to the fixed-time behavior of the first passage percolation, the number $N(t)$ of agents possessing the information at fixed large time t is approximately distributed as We^t, where W has Exponential(1) distribution, and so

$$P(\tau_L \leq t) = P(N(t) \geq L) \approx P(We^t \geq L) = \exp(-Le^{-t})$$

implying $\tau_L - \log L \approx \xi$ in distribution, explaining the first summand on the right side of (11). Now consider the proportion $H(t)$ of agents possessing the information at time $\tau_L + t$. This proportion follows closely the deterministic logistic equation $H' = H(1 - H)$ whose solution is (12) shifted to satisfy the initial condition $H(0) = L/n$, so this solution approximates the distribution function of $S_{(i)} - \log(L/n)$. Thus the time $\bar{S}^n_{(i)}$ at which a random agent receives the information satisfies

$$(\bar{S}^n_{(i)} - \tau_L + \log(L/n)) \approx S_{(i)} \text{ in distribution}$$

independently as i varies. Now the limit decomposition (11) follow from the finite-n decomposition (13).

We emphasize (11) instead of more elementary derivations (using methods of [9, 13]) of the limit distribution for $\bar{S}^n_{(1)} - \log n$ because (11) gives the correct dependence structure for different agents. Because only *relative* order of gaining information is relevant to us, we may recenter by subtracting ξ and suppose that the times at which different random agents gain information are independent with logistic distribution (12).

2.4 *Analysis of the Rank-Based Reward Game*

We now return to our general reward framework

The j'th person to learn an item of information gets reward $R(\frac{j}{n})$

and give the argument for (5).

Suppose all agents use the Poisson(θ) strategy. In the case $\theta = 1$, the way that a single item of information spreads is exactly as the first passage percolation process above; and the general-θ case is just a time-scaling by θ. So as above, we may suppose that (all calculations in the $n \to \infty$ limit) the recentered time S_θ to reach a random agent has distribution function

$$F_\theta(x) = F_1(\theta x) \tag{14}$$

which is the solution of the time-scaled logistic equation

$$\frac{F'_\theta}{1 - F_\theta} = \theta F_\theta \tag{15}$$

(Recall F_1 is the logistic distribution (12)). Now consider the case where all other agents use a value θ but **ego** uses a different value ϕ. The (limit, recentered) time $T_{\phi,\theta}$ at which **ego** learns the information now has distribution function $G_{\phi,\theta}$ satisfying an analog of (15):

$$\frac{G'_{\phi,\theta}}{1 - G_{\phi,\theta}} = \phi F_\theta. \tag{16}$$

To explain this equation, the left side is the rate at time t at which **ego** learns the information; this equals the rate ϕ of calls by **ego**, times the probability $F_\theta(t)$ that the called agent has received the information. To solve the equation, first we get

$$1 - G_{\phi,\theta} = \exp\left(-\phi \int F_\theta\right).$$

But we know that in the case $\phi = \theta$ the solution is F_θ, that is we know

$$1 - F_\theta = \exp\left(-\theta \int F_\theta\right),$$

and so we have the solution of (16) in the form

$$1 - G_{\phi,\theta} = (1 - F_\theta)^{\phi/\theta}. \tag{17}$$

If **ego** gets the information at time t then his percentile rank is $F_\theta(t)$ and his reward is $R(F_\theta(t))$. So the expected reward to **ego** is

$$ER(F_\theta(T_{\phi,\theta})); \quad \text{where } \mathrm{dist}(T_{\phi,\theta}) = G_{\phi,\theta}.$$

We calculate

$$\begin{aligned} P(F_\theta(T_{\phi,\theta}) \le u) &= G_{\phi,\theta}(F_\theta^{-1}(u)) \\ &= 1 - (1 - F_\theta(F_\theta^{-1}(u)))^{\phi/\theta} \text{ by (17)} \\ &= 1 - (1-u)^{\phi/\theta} \end{aligned} \tag{18}$$

and so

$$ER(F_\theta(T_{\phi,\theta})) = \int_0^1 r(u)\,(1 - (1-u)^{\phi/\theta})du.$$

This is the mean reward to **ego** from one item, and hence also the mean reward per unit time in the ongoing process. So, including the "communication cost" of ϕ per unit time, the net payoff (per unit time) to **ego** is

$$\mathrm{payoff}(\phi, \theta) = -\phi + \int_0^1 r(u)\,(1 - (1-u)^{\phi/\theta})du. \tag{19}$$

Using the fact $\frac{d}{d\phi} x^{\phi/\theta} = \frac{\log x}{\theta} x^{\phi/\theta}$, we have that the criterion (3) for θ to be a Nash equilibrium is,

$$1 = \tfrac{1}{\theta} \int_0^1 r(u)\,(-\log(1-u))\,(1-u)du. \tag{20}$$

This is the second equality in (5), and integrating by parts gives the first equality.

Remark. For the linear reward function

$$R(u) = 2(1-u); \quad \bar{R} = 1$$

result (5) gives Nash payoff $= 1/2$. Consider alternatively

$$R(u) = \tfrac{1}{u_0} 1_{(u \le u_0)}; \quad \bar{R} = 1.$$

Then the $n \to \infty$ Nash equilibrium cost is

$$\theta^{\text{Nash}}(u_0) = \frac{1}{u_0} \int_0^{u_0} (1 + \log(1-u))\, du.$$

In particular, the Nash payoff $1 - \theta^{\text{Nash}}(u_0)$ satisfies

$$1 - \theta^{\text{Nash}}(u_0) \to 0 \text{ as } u_0 \to 0.$$

In words, as the reward becomes concentrated on a smaller and smaller proportion of the population then the Nash equilibrium becomes more and more wasteful. In this sense result (5) in the general framework is consistent with the "finite number of rewards" result (10).

3 The $N \times N$ Torus, Nearest Neighbor Case

Network communication model. There are N^2 agents at the vertices of the $N \times N$ torus. Each agent i may, at any time, call any of the four neighboring agents j (at cost 1), and learn all items that j knows.

Poisson strategy. The allowed strategy for an agent i is to place calls, at the times of a Poisson (rate θ) process, to a random neighboring agent.

We will derive formula (6). As remarked later, the function $g(u)$ is ultimately derived from fine structure of first passage percolation in the plane, and seems impossible to determine as an explicit formula. But of course the main point is that the Nash equilibrium payoff $\bar{R} - \theta_N^{\text{Nash}} = \bar{R} - O(N^{-1})$ tends to the social optimum $\bar{R}$ (in contrast to the complete graph case).

3.1 *Nearest-Neighbor First Passage Percolation on the Torus*

Consider (nearest-neighbor) first passage percolation on the $N \times N$ torus, started at a uniform random vertex, with rates $\nu_{ij} = 1$ for edges (i, j). Write $(T_i^N,\ 1 \le i \le 4)$ for the information receipt times of the four neighbors of the origin (using paths not through the origin), and write $Q^N(t)$ for the number of vertices informed by time t. Write $T_*^N = \min(T_i^N,\ 1 \le i \le 4)$.

The key point is that we expect as $N \to \infty$ limit of the following form

$$(T_i^N - T_*^N, 1 \le i \le 4;\ N^{-2} Q^N(T_*^N);\ (N^{-1}(Q^N(T_*^N + t) - Q^N(T_*^N)), 0 \le t < \infty))$$

$$\xrightarrow{d} (\tau_i, 1 \le i \le 4;\ U;\ (Vt, 0 \le t < \infty)) \tag{21}$$

where $\tau_i, 1 \leq i \leq 4$ are nonnegative with $\min_i \tau_i = 0$; U has uniform$(0,1)$ distribution; $0 < V < \infty$; with a certain complicated joint distribution for these limit quantities.

To explain (21), first note that as $N \to \infty$ the differences $T_i^N - T_*^N$ are stochastically bounded (by the time to percolate through a finite set of edges) but cannot converge to 0 (by linearity of growth rate in the shape theorem below), so we expect some non-degenerate limit distribution $(\tau_i, 1 \leq i \leq 4)$. Next consider the time T_0^N at which the origin is wetted. By uniformity of starting position, $Q^N(T_0^N)$ must have uniform distribution on $\{1, 2, \ldots, N^2\}$, and it follows that $N^{-2}Q^N(T_*^N) \stackrel{d}{\to} U$. The final assertion

$$(N^{-1}(Q^N(T_*^N + t) - Q^N(T_*^N)), 0 \le t < \infty) \stackrel{d}{\to} (Vt, 0 \le t < \infty) \qquad (22)$$

is related to the *shape theorem* [11] for first-pasage percolation on the infinite lattice started at the origin. This says that the random set $\mathscr{B}_s$ of vertices wetted before time s grows linearly with s, and the spatially rescaled set $s^{-1}\mathscr{B}_s$ converges to a limit deterministic convex set $\mathscr{B}$:

$$s^{-1}\mathscr{B}_s \to \mathscr{B}. \qquad (23)$$

It follows that

$$N^{-2}Q^N(sN) \to q(s) \text{ as } N \to \infty$$

where $q(s)$ is the area of $s\mathscr{B}$ regarded as a subset of the continuous torus $[0,1]^2$. Because $N^{-2}Q^N(T_0^N) \stackrel{d}{\to} U$ we have

$$T_*^N \approx T_0^N \approx N^2 q^{-1}(U)$$

where $q^{-1}(\cdot)$ is the inverse function of $q(\cdot)$. Writing $Q'^N(\cdot)$ for a suitably-interpreted local growth rate of $Q^N(\cdot)$ we deduce

$$(N^{-2}Q^N(T_*^N), N^{-1}Q'^N(T_*^N)) \stackrel{d}{\to} (U, q'(q^{-1}(U)))$$

and so (22) holds for $V = q'(q^{-1}(U))$.

3.2 *Analysis of the Rank-Based Reward Game*

We want to study the case where other agents call some neighbor at rate θ but **ego** (at the origin) calls some neighbor at rate ϕ. To analyze rewards, by scaling time we can reduce to the case where other agents call *each* neighbor at rate 1 and **ego** calls each neighbor at rate $\lambda = \phi/\theta$. We want to compare the rank M_λ^N of **ego** (rank = j if **ego** is the j'th person to receive the information) with the rank M_1^N of **ego** in the $\lambda = 1$ case. As noted above, M_1^N is uniform on $\{1, 2, \ldots, N^2\}$. Writing $(\xi_i^\lambda,\ 1 \le i \le 4)$ for independent Exponential(λ) r.v.'s, the time at which the origin receives the information is

$$T_*^N + \min_i (T_i^N - T_*^N + \xi_i^\lambda)$$

and the rank of the origin is

$$M_\lambda^N = Q^N(T_*^N) + N\widetilde{Q}^N(\min_i(T_i^N - T_*^N + \xi_i^\lambda))$$

where

$$\widetilde{Q}^N(t) = N^{-1}(Q^N(T_*^N + t) - Q^N(T_*^N)).$$

Note we can construct $(\xi_i^\lambda,\ 1 \le i \le 4)$ as $(\lambda^{-1}\xi_i^1,\ 1 \le i \le 4)$. Now use (22) to see that as $N \to \infty$

$$(N^{-2}M_1^N, N^{-1}(M_\lambda^N - M_1^N)) \stackrel{d}{\to} (U, VZ(\lambda)) \tag{24}$$

where

$$Z(\lambda) := \min_i(\tau_i + \xi_i^\lambda) - \min_i(\tau_i + \xi_i^1). \tag{25}$$

Now in the setting where **ego** calls at rate ϕ and others at rate θ we have

$$\text{payoff}(\phi, \theta) - \text{payoff}(\theta, \theta) + (\phi - \theta) = E\left[R\left(\frac{M_{\phi/\theta}^N}{N^2}\right) - R\left(\frac{M_1^N}{N^2}\right)\right]$$

and it is straightforward to use (24) to show this

$$\sim N^{-1}\int_0^1 (-r(u))\, z_u(\phi/\theta)du, \text{ for } z_u(\lambda) := E(VZ(\lambda)|U = u). \tag{26}$$

The Nash equilibrium condition

$$\frac{d}{d\phi}\text{payoff}(\phi, \theta)\bigg|_{\phi=\theta} = 0$$

now implies

$$\theta_N^{\text{Nash}} \sim N^{-1}\int_0^1 (-r(u))\, z_u'(1)du. \tag{27}$$

Because $Z(\lambda)$ is decreasing in λ we have $z_u'(1) < 0$ and this expression is of the form (6) with

$$g(u) = -z_u'(1) = -\tfrac{d}{d\lambda}\ E(VZ(\lambda)|U = u)|_{\lambda=1} \tag{28}$$

Remark. The distribution of V depends on the function $q(\cdot)$ which depends on the limit shape in nearest neighbor first passage percolation, which is not explicitly known. Also $Z(\lambda)$ involves the joint distribution of (τ_i), which is not explicitly known, and also is (presumably) correlated with the direction from the percolation source which is in turn not independent of V. This suggests it would be difficult to find an explicit formula for $g(u)$.

4 Order of Magnitude Arguments

Here we mention simple order of magnitude arguments for the two basic cases we have already analyzed. As mentioned in the introduction, what matters is the size of the *window width* $w_{\theta,n}$ of the associated first passage percolation process We will re-use such arguments in Sects. 5 and 6.1, in more complicated settings.

Complete graph. If agents call at rate $\theta = 1$ then by (11) the window width is order 1; so if θ_n is the Nash equilibrium rate then the window width w_n is order $1/\theta_n$. Suppose $w_n \to \infty$. Then **ego** could call at some fixed slow rate ϕ and (because this implies many calls are made near the start of the window) the reward to **ego** will tend to $R(0)$, and **ego**'s payoff $R(0)-\phi$ will be larger than the typical payoff $\bar{R}-\theta_n$. This contradicts the definition of Nash equilibrium. So in fact we must have w_n bounded above, implying θ_n bounded below, implying the Nash equilbrium in wasteful.

Nearest neighbor torus. If agents call at rate $\theta = 1$ then by the shape theorem (23) the window width is order N. The time difference between receipt time for different neighbors of **ego** is order 1, so if **ego** calls at rate 2 instead of rate 1 his rank (and hence his reward) increases by order $1/N$. By scaling, if the Nash equilibrium rate is θ_N and **ego** calls at rate $2\theta_N$ then his increased reward is again of order $1/N$. His increased cost is θ_N. At the Nash equilibrium the increased reward and cost must balance, so θ_N is order $1/N$, so the Nash equilibrium is efficient.

5 The $N \times N$ Torus with General Interactions: A Simple Criterion for Efficiency

Network communication model. The agents are at the vertices of the $N \times N$ torus. Each agent i may, at any time, call any other agent j, at cost $c(d(i,j))$, and learn all items that j knows.

Here $d(i,j)$ is the distance between i and j, and we assume the cost function $c(d)$ satisfies

$$c(1) = 1; \quad c(d) \uparrow \infty \text{ as } d \to \infty. \tag{29}$$

Poisson strategy. An agent's strategy is described by a sequence $(\theta(d);\ d = 1, 2, 3, \ldots)$; and for each d:

at rate $\theta(d)$ the agent calls a random agent at distance d.

A simple argument below shows

$$\text{under condition (29) the Nash equilibrium is efficient.} \tag{30}$$

Consider the Nash strategy, and suppose first that the window width w_N converges to a limit $w_\infty < \infty$. Consider a distance d such that the Nash strategy has $\theta^{\text{Nash}}(d) > 0$. Suppose **ego** uses $\theta(d) = \theta^{\text{Nash}}(d) + \phi$. The increased cost is $\phi c(d)$ while the increased benefit is at most $O(w_\infty \phi)$, because this is the increased chance of getting information earlier. So the Nash strategy must have $\theta^{\text{Nash}}(d) = 0$ for sufficiently large d, not depending on N. But for first passage percolation with bounded range transitions, the shape theorem (23) remains true and implies that w_N scales as N.

This contradiction implies that the window width $w_N \to \infty$. Now suppose the Nash equilibrium were inefficient, with some Nash cost $\bar{\theta} > 0$. Suppose **ego** adopts the strategy of just calling a random neighbor at rate ϕ_N, where $\phi_N \to 0,\ \phi_N w_N \to \infty$. Then **ego** obtains asymptotically the same reward $\bar{R}$ as his neighbor, a typical agent. But **ego**'s cost is $\phi_N \to 0$. This is a contradiction with the assumption of inefficiency. So the conclusion is that the Nash equilibrium is efficient and $w_N \to \infty$.

Remarks. Result (30) is striking. but does not tell us what the Nash equilibrium strategy and cost actually are. It is a natural open problem to study the case of (29) with $c(d) = d^\alpha$. Instead we study a simpler model in the next section.

6 The $N \times N$ Torus with Short and Long Range Interactions

Network communication model. The agents are at the vertices of the $N \times N$ torus. Each agent i may, at any time, call any of the four neighboring agents j (at cost 1), or call any other agent j at cost $c_N \geq 1$, and learn all items that j knows.

Poisson strategy. An agent's strategy is described by a pair of numbers $(\theta_{\text{near}}, \theta_{\text{near}}) = \theta$:

at rate θ_{near} the agent calls a random neighbor
at rate θ_{near} the agent calls a random non-neighbor.

This model obviously interpolates between the complete graph model ($c_N = 1$) and the nearest-neighbor model ($c_N = \infty$).

First let us consider for which values of c_N the nearest-neighbor Nash equilibrium (θ_{near} is order N^{-1}, $\theta_{\text{near}} = 0$) persists in the current setting. When **ego** considers using a non-zero value of θ_{near}, the cost is order $c_N \theta_{\text{near}}$. The time for information to reach a typical vertex is order $N/\theta_{\text{near}} = N^2$, and so the benefit of using a non-zero value of θ_{near} is order $\theta_{\text{near}} N^2$. We deduce that

if $c_N \gg N^2$ then the Nash equilibrium is asymptotically the same as in the nearest-neighbor case; in particular, the Nash equilibrum is efficient.

Let us study the more interesting case

$$1 \ll c_N \ll N^2.$$

The result in this case turns out to be, qualitatively these must balance, so

$$\theta_{\text{near}}^{\text{Nash}} \text{ is order } c_N^{-1/2} \text{ and } \theta_{\text{far}}^{\text{Nash}} \text{ is order } c_N^{-2}. \quad \text{In particular, the Nash equilibrum is efficient.} \tag{31}$$

"Efficient" because the cost $c_N \theta_{\text{near}} + \theta_{\text{near}}$ is order $c_N^{-1/2}$. See (42) for the exact result.

We first do the order-of-magnitude calculation (Sect. 6.1), then analyze the relevant first passage percolation process (Sect. 6.2), and finally do the exact analysis in Sect. 6.3.

6.1 *Order of Magnitude Calculation*

Our order of magnitude argument for (31) uses three ingredients (32, 33, 34). As in Sect. 4 we consider the *window width* w_N of the associated percolation process. Suppose **ego** deviates from the Nash equilibrium $(\theta_{\text{near}}^{\text{Nash}}, \theta_{\text{far}}^{\text{Nash}})$ by setting his $\theta_{\text{far}} = \theta_{\text{far}}^{\text{Nash}} + \delta$. The chance of thereby learning the information earlier, and hence the increased reward to **ego**, is order δw_N and the increased cost is δc_N. At the Nash equilibrium these must balance, so

$$w_N \asymp c_N \tag{32}$$

where $\asymp$ denotes "same order of magnitude". Now consider the difference ℓ_N between the times that different neighbors of **ego** are wetted. Then ℓ_N is order $1/\theta_{\text{near}}^{\text{Nash}}$. Write $\delta = \theta_{\text{near}}^{\text{Nash}}$ and suppose **ego** deviates from the Nash equilibrium by setting his $\theta_{\text{near}} = 2\delta$. The increased benefit to **ego** is order ℓ_N / w_N and the increased cost is δ. At the Nash equilibrium these must balance, so $\delta \asymp \ell_N / w_N$ which becomes

$$\theta_{\text{near}}^{\text{Nash}} \asymp w_N^{-1/2} \asymp c_N^{-1/2}. \tag{33}$$

Finally we need to calculate how the window width w_N for FPP depends on $(\theta_{\text{near}}, \theta_{\text{near}})$, and we show in the next section that

$$w_N \asymp \theta_{\text{near}}{}^{-2/3} \theta_{\text{near}}{}^{-1/3}. \tag{34}$$

Granted this, we substitute (32,33) to get

$$c_N \asymp c_N^{1/3} \theta_{\text{near}}{}^{-1/3}$$

which identifies $\theta_{\text{near}} \asymp c_N^{-2}$ as stated at (31).

6.2 *First Passage Percolation on the N × N Torus with Short and Long Range Interactions*

We study the model (call it *short-long FPP*, to distinguish it from nearest-neighbor FPP) defined by rates

$$\begin{aligned}\nu_{ij} &= \tfrac{1}{4}, \qquad j \text{ a neighbor of } i\\ &= \lambda_N/N^2, \ \ j \text{ not a neighbor of } i\end{aligned}$$

where $1 \gg \lambda_N \gg N^{-3}$.

Recall the shape theorem (23) for nearest neighbor first passage percolation; let A be the area of the limit shape $\mathscr{B}$. Define an artificial distance ρ such that $\mathscr{B}$ is the unit ball in ρ-distance; so nearest neighbor first passage percolation moves at asymptotic speed 1 with respect to ρ-distance. Consider short-long FPP started at a random vertex of the $N \times N$ torus. Write F_{N,λ_N} for the proportion of vertices reached by time t and let $T_{(0,0)}$ be the time at which the origin is reached. The event $\{T_{(0,0)} \le t\}$ corresponds asymptotically to the event that at some time $t-u$ there is percolation across some long edge (i, j) into some vertex j at ρ-distance $\le u$ from $(0, 0)$ (here we use the fact that nearest neighbor first passage percolation moves at asymptotic speed 1 with respect to ρ-distance). The rate of such events at time $t-u$ is approximately

$$N^2 F_{N,\lambda_N}(t-u) \times Au^2 \times \lambda_N/N^2$$

where the three terms represent the number of possible vertices i, the number of possible vertices j, and the percolation rate ν_{ij}. Since these events occur asymptotically as a Poisson process in time, we get

$$1 - F_{N,\lambda_N}(t) \approx P(T_{(0,0)} \le t) \approx \exp\left(-A\lambda_N \int_0^\infty u^2 F_{N,\lambda_N}(t-u)\, du\right). \qquad (35)$$

This motivates study of the equation (for an unknown distribution function F_λ)

$$1 - F_\lambda(t) = \exp\left(-\lambda \int_{-\infty}^t (t-s)^2 F_\lambda(s)\, ds\right), \quad -\infty < t < \infty \qquad (36)$$

whose solution should be unique up to centering. Writing F_1 for the $\lambda = 1$ solution, the general solution scales as

$$F_\lambda(t) := F_1(\lambda^{1/3} t).$$

So by (35), up to centering

$$F_{N,\lambda_N}(t) \approx F_1((A\lambda_N)^{1/3} t). \qquad (37)$$

To translate this result into the context of the rank-based rewards game, suppose each agent uses strategy $\theta_N = (\theta_{N,\text{near}}, \theta_{N,\text{far}})$. Then the spread of one item of information is as first passage percolation with rates

$$\begin{aligned}\nu_{ij} &= \theta_{N,\text{near}}/4, \ \ j \text{ a neighbor of } i\\ &= \theta_{N,\text{far}}/(N^2-5), \ \ j \text{ not a neighbor of } i.\end{aligned}$$

This is essentially the case above with $\lambda_N = \theta_{N,\text{far}}/\theta_{N,\text{near}}$, time-scaled by $\theta_{N,\text{near}}$, and so by (37) the distribution function F_{N,θ_N} for the time at which a typical agent receives the information is

$$F_{N,\theta_N}(t) \approx F_1\left(A^{1/3}\theta_{N,\text{far}}{}^{1/3}\theta_{N,\text{near}}{}^{2/3}t\right). \tag{38}$$

In particular the window width is as stated at (34).

The result of Chatterjee and Durrett [4]. Their paper gives a rigorous proof of analogs of (36, 37) under a slightly different "balloon process" model, consisting of overlapping circular discs in the continuous torus $[0, N]^2$. Disc centers are created at random times and positions with intensity $N^{-\alpha}\times$ (area of covered region); each disc radius then expands linearly and deterministically. Their Theorem 3 proves that the quantity $N^{-2}C_{\psi(s)}$ (in their notation) representing the proportion of the torus covered at time $\psi(s) = R + N^{\alpha/3}s$ satisfies

$$\lim P(\sup_{s\le t} |N^{-2}C_{\psi(s)} - F_1(s)| \ge \delta) = 0 \quad \text{for fixed } (t,\delta).$$

Here

$$R = N^{\alpha/3}[(2-2\alpha/3)\log N - \log M]$$

for M not depending on N. This result pins down the position of the center of the critical window. Additionally, their Theorem 4 analyses the time until the entire torus is covered. These two results provide more detail than our (38).

6.3 *Exact Equations for the Nash Equilibrium*

The equations will involve three quantities:

(i) The solution F_1 of (36).
(ii) The area A of the limit set $\mathscr{B}$ in the shape theorem (23) for nearest-neighbor first passage pecolation.
(iii) The limit distribution (cf. (21))

$$(T_i^r - T_*^r, \ 1 \le i \le 4) \xrightarrow{d} (\tau_i, \ 1 \le i \le 4) \text{ as } r \to \infty \tag{39}$$

for relative receipt times of neighbors of the origin in nearest-neighbor first passage percolation, where now we start the percolation at a random vertex of ρ-distance $\approx r$ from the origin.

To start the analysis, suppose all agents use rates $\theta = (\theta_{N,\text{near}}, \theta_{N,\text{far}})$. Consider the quantities

S is the first time that **ego** receives the information from a non-neighbor

T is the first time that **ego** receives the information from a neighbor

$F = F_{N,\theta_N}$ is the distribution function of T.

We claim that, with probability $\to 1$ as $N \to \infty$ **ego** will actually receive the information first from a neighbor, and so F is asymptotically the distribution function of the time at which **ego** receives the information. To check this we need to show that the chance **ego** receives the information from a non-neighbor during the critical window is $o(1)$. This chance is $O(N^2 \times \theta_{N,\text{far}} / N^2 \times w_N)$, which by the order of magnitude calculations in Sect. 6.1 is $O(c_N^{-1}) = o(1)$.

Now suppose **ego** uses a different rate $\phi_{N,\text{far}} \neq \theta_{N,\text{far}}$ for calling a non-neighbor. This does not affect T but changes the distribution of S to

$$P(S > t) \approx \exp\left(-\phi_{N,\text{far}} \int_{-\infty}^{t} F(s)\, ds\right)$$

by the natural Poisson process approximation. Because $\theta_{N,\text{far}}$ is small we can approximate

$$P(S \leq t) \approx \phi_{N,\text{far}} \int_{-\infty}^{t} F(s)\, ds.$$

The mean reward to **ego** for one item, as a function of $\phi_{N,\text{far}}$, varies as

$$E(R(F(S)) - R(F(T))1_{(S<T)} + \text{ constant}.$$

Because $U = F(T)$ is uniform on $(0, 1)$, in the $N \to \infty$ limit

$$\begin{aligned}
E(R(F(S)) - R(F(T))1_{(S<T)} &= E(R(F(S)) - R(U))1_{(F(S)<U)} \\
&= \int_0^1 du\, E(R(F(S)) - R(u))1_{(F(S)<u)} \\
&= \int_0^1 du\, E \int_{\min(F(S),u}^{u} r(y)dy \\
&= \int_0^1 dy\, (1-y)r(y)P(F(S) \leq y) \\
&= \int_0^1 dy\, (1-y)r(y)P(S \leq F^{-1}(y)) \\
&= \phi_{N,\text{far}} \int_0^1 dy\, (1-y)r(y) \int_{-\infty}^{F^{-1}(y)} F(s)ds.
\end{aligned}$$

The cost associated with using $\phi_{N,\text{far}}$ is $c_N \phi_{N,\text{far}}$, and at the Nash equilibrium the cost and reward must balance, so at the Nash equilibrium $F = F_{N,\theta_N}$ must satisfy

$$c_N \sim \int_0^1 dy\ (1-y)r(y) \int_{-\infty}^{F^{-1}(y)} F(s)ds. \tag{40}$$

Now suppose instead that **ego** uses a different rate $\phi_{N,\text{near}} \neq \theta_{N,\text{near}}$ for calling a neighbor. As in Sect. 3.2, we set $\lambda = \phi_{N,\text{near}}/\theta_{N,\text{near}}$ so that we can use rate-1 nearest-neighbor first passage percolation as comparison. For (τ_i) at (39) and independent Exponential(λ) random variables (ξ_i^λ) write (as at (25))

$$Z(\lambda) := \min_i(\tau_i + \xi_i^\lambda) - \min_i(\tau_i + \xi_i^1).$$

So $Z(\lambda)$ is the time difference for **ego** receiving the information, caused by **ego** using $\phi_{N,\text{near}}$ instead of $\theta_{N,\text{near}}$. This time difference is measured after time-rescaling; in real time units the time difference is $Z(\lambda)/\theta_{N,\text{near}}$.

As above, write T for receipt time for **ego** using $\theta_{N,\text{near}}$, and $F = F_{N,\theta_N}$ for its distribution function. Then receipt time for **ego** using $\phi_{N,\text{near}}$ is $T + Z(\lambda)/\theta_{N,\text{near}}$, so **ego**'s rank becomes $\approx F(T) + F'(T)Z(\lambda)/\theta_{N,\text{near}}$, and setting $U = F(T)$ the rank of **ego** is $\approx U + F'(F^{-1}(U))Z(\lambda)/\theta_{N,\text{near}}$. The associated mean reward change for **ego** is asymptotically

$$\tfrac{z(\lambda)}{\theta_{N,\text{near}}} \times \int_0^1 r(u)F'(F^{-1}(u))\, du; \quad \lambda = \phi_{N,\text{near}}/\theta_{N,\text{near}}$$

where $z(\lambda) = EZ(\lambda)$. Because the cost of using rate $\phi_{N,\text{near}}$ equals $\phi_{N,\text{near}}$, the Nash equilibrium condition (3) implies

$$\theta_{N,\text{near}}{}^2 \sim z'(1) \int_0^1 r(u)F'(F^{-1}(u))\, du. \tag{41}$$

We have now obtained the desired two equations for F_{N,θ_N} at the Nash equilibrium θ_N. Use (38) to rewrite these Eqs. (40,41) in terms of F_1 as

$$c_N \sim A^{-1/3}\theta_{N,\text{far}}{}^{-1/3}\theta_{N,\text{near}}{}^{-2/3} \int_0^1 dy\ (1-y)r(y) \int_{-\infty}^{F_1^{-1}(y)} F_1(s)ds$$

$$\theta_{N,\text{near}}{}^2 \sim A^{1/3}\theta_{N,\text{far}}{}^{1/3}\theta_{N,\text{near}}{}^{2/3} z'(1) \int_0^1 r(u)F_1'(F_1^{-1}(u))\, du.$$

Solving for $\theta_{N,\text{near}}, \theta_{N,\text{far}}$ we find

$$\theta_{N,\text{near}} \sim Q^{1/2}c_N^{1/2}, \quad \theta_{N,\text{far}} \sim A^{-1}Q^{-1}c_N^{-2} \tag{42}$$

for

$$Q = z'(1) \left(\int_0^1 dy\ (1-y) r(y) \int_{-\infty}^{F_1^{-1}(y)} F_1(s) ds \right) \left(\int_0^1 r(u) F_1'(F_1^{-1}(u))\, du \right).$$

7 Variants

7.1 *Transitivity and the Symmetric Variant*

The examples we have studied so far have a certain property called *transitivity* in graph theory [3]. Informally, transitivity means "the network looks the same to each agent"; formally, it means that for any two agents i, j there is an automorphism of the network that preserves the network cost structure and maps i to j. This is what allows us to assume that in a Nash equilibrium each agent uses same strategy.

The general framework of Sect. 1.1 uses the *asymmetric* model in which agent i calls agent j (at a certain cost to i) and learns all items that j knows. In the *symmetric* variant, agent i calls agent j (at a certain cost to i), and each tells the other all items they know.

For the transitive networks we have studied there is a simple relationship between the Nash equilibrium values of the asymmetric and symmetric variants of the Poisson strategies:

$$\theta_{\mathrm{sym}}^{\mathrm{Nash}} = \tfrac{1}{2} \theta_{\mathrm{asy}}^{\mathrm{Nash}}. \tag{43}$$

The point is that the percolation process in the symmetric variant is just the percolation process in the asymmetric variant, run at twice the speed, and this leads to the following relationship between the reward when **ego** uses rate ϕ and other agents use rate θ:

$$\mathrm{reward}_{\mathrm{sym}}(\phi, \theta) = \mathrm{reward}_{\mathrm{asy}}(\phi + \theta, 2\theta).$$

Because $\mathrm{payoff}(\phi, \theta) = \mathrm{reward}(\phi, \theta) - \phi$ in each case, we get

$$\mathrm{payoff}_{\mathrm{sym}}(\phi, \theta) = \mathrm{payoff}_{\mathrm{asy}}(\phi + \theta, 2\theta) + \theta$$

and therefore

$$\frac{d}{d\phi} \mathrm{payoff}_{\mathrm{sym}}(\phi, \theta) = \frac{d}{d\phi} \mathrm{payoff}_{\mathrm{asy}}(\phi + \theta, 2\theta).$$

The criterion (3) leads to (43).

7.2 *Communication at Regular Intervals*

We have studied "Poisson rate θ" calling strategies because these are simplest to analyze explicitly. A natural alternative is the "regular, rate θ" strategy in which agent i calls a random other agent at times

$$U_i, U_i + \tfrac{1}{\theta}, U_i + \tfrac{2}{\theta}, \dots \tag{44}$$

where U_i is uniform on $(0, \frac{1}{\theta})$.

Consider first the complete graph case, and the setting (Sect. 2.1) where (for fixed $k \geq 2$) the first k recipients get reward n/k. In this case, for $k = 2$ formula (8) is replaced by

$$P(\textbf{ego} \text{ is second to receive item}) = \int_0^{\min(\frac{1}{\phi}, \frac{1}{\theta})} (1-\theta u)^{n-2}\, \phi\, du$$

and repeating the analysis in Sect. 2.1 gives exactly the same asymptotics (9,10) as in the Poisson case. Consider instead the general reward framework

The j'th person to learn an item of information gets reward $R(\frac{j}{n})$.

If all agents use rate θ then the distribution function F_θ for receipt time for a typical agent satisfies (as an analog of the logistic equation (15))

$$1 - F_\theta(t) = \int_{t-\frac{1}{\theta}}^{t} \prod_{i\geq 0} \left(1 - F_\theta(s - \tfrac{i}{\theta})\right)\ \theta\, ds. \tag{45}$$

If **ego** switches to rate ϕ then the distribution function $G_{\phi,\theta}$ for **ego**'s receipt time satisfies (as an analog of (16))

$$1 - G_{\phi,\theta}(t) = \int_{t-\frac{1}{\phi}}^{t} \prod_{i\geq 0} \left(1 - F_\theta(s - \tfrac{i}{\phi})\right)\ \phi\, ds. \tag{46}$$

One can now continue the Sect. 2.4 analysis; we do not get useful explicit solutions but the qualitative behavior is similar to the "Poisson calls" case, and in particular the Nash equilibrium is wasteful.

Similarly, on the $N \times N$ grid with nearest neighbor interaction, switching from the "Poisson calls" case to the "regular calls" case preserves the order N^{-1} value of the Nash equilibrium rate θ_N^{Nash} and hence preserves its efficiency.

7.3 *Gossip with Reward Based on Audience Size*

Perhaps a more realistic model for gossip is to replace Rule 2 by

Rule 3. An agent i gets reward c whenever another agent learns an item from i.

For the complete graph and Poisson(θ) strategies we can re-use the Sect. 2.4 analysis to calculate the Nash equilibrium. First suppose all agents use the same rate θ and consider an agent i who receives the information at percentile u. For $j > un$ the j'th agent to receive the information has chance $\frac{1}{j}$ to receive it from agent i, and so the mean reward to agent i is (calculations in the $n \to \infty$ limit) $c \int_u^1 \frac{1}{x}\, dx = -c \log u$. Suppose now **ego** switches to rate ϕ. Then (calls incur unit cost)

$$\text{payoff}(\phi, \theta) = -\phi + cE(-\log F_\theta(T_{\phi,\theta}))$$

where the time $T_{\phi,\theta}$ at which **ego** receives the information has distribution function $G_{\phi,\theta}$ at (17), and where F_θ at (14) is the distribution function of the time at which a typical agent receives the information. Now

$$\begin{aligned} E(-\log F_\theta(T_{\phi,\theta})) &= \int_0^1 \tfrac{1}{u} P(F_\theta(T_{\phi,\theta}) \le u)\, du \\ &= \int_0^1 \tfrac{1}{u} P(G_\theta(T_{\phi,\theta}) \le 1-(1-u)^{\phi/\theta})\, du \\ &= \int_0^1 \tfrac{1}{u} (1-(1-u))^{\phi/\theta}\, du \text{ by (18)} \end{aligned}$$

and then we calculate

$$\frac{d}{d\phi}\text{payoff}(\phi, \theta) = -1 - c\int_0^1 \tfrac{\log(1-u)}{\theta}\, \tfrac{(1-u)^{\phi/\theta}}{u}\, du.$$

Now the Nash equilibrium criterion (3) implies

$$\theta_n^{\text{Nash}} \to -c\int_0^1 \tfrac{1-u}{u}\, \log(1-u)\, du. \tag{47}$$

So switching to this "Rule 3" model preserves the wastefulness of the Nash equilibrium on the compete graph.

However, for the $N \times N$ grid with nearest neighbor interaction, switching to the "Rule 3" models changes the efficient (θ_N^{Nash} is order N^{-1}) Nash equilibrium to a wasteful equilibrium with θ_N^{Nash} becoming order 1.

7.4 *Related Literature*

We do not know any literature closely related to our model. As well as the epidemic and the gossip algorithm topics mentioned in the introduction, and classic applied probability work on *stochastic rumors* [5], other loosely related work includes

- Models where agents form networks under conditions where there are costs for maintaining network edges and benefits from being part of a large network [7].
- Prisoners' Dilemma games between neighboring agents on a graph [6].

One can add many other topics which are harder to model mathematically, e.g. diffusion of technological innovations [12] or of ideologies.

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A Mathematical Model of Investment Incentives

Vadim Arkin and Alexander Slastnikov

Abstract An investment timing problem which takes into account both taxation (including tax exemptions) and financing by credit is considered. This problem is reduced to the optimal stopping of a two-dimensional diffusion process. We give the solution to the investment timing problem as a function of parameters of the model, in particular, of the tax holiday duration and interest rate for borrowing. We study the question whether the higher interest rate for borrowing can be compensated by tax holidays.

Keywords Investment timing problem • Credit • Real options • Optimal stopping • Tax holidays • Compensation of interest rate

Mathematics Subject Classification (2010): 60G40, 91B38, 91B70

1 Introduction

There is an important problem how to attract investments to the real sector of the economy when credit risks are high. Our work is devoted to the analysis of related tax mechanisms for such attraction. In economies with increased risks (political, credit etc.) and other unfavorable factors the following question arises: can tax benefits provide investor with the same conditions for investment as he would have in a "standard" economy without any risks and unfavorable factors. In other words, can tax benefits compensate unfavorable factors?

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In order to compensate risks and other unfavorable factors the following tax benefits are often used to attract investment: tax holidays, i.e. exemption from tax during a certain period, a reduction in tax rate, and accelerated depreciation.

It is worth noting that increased credit risks imply increasing interest rates on credit. In practice, tax holidays are considered as a mechanism which can compensate all arising risks.

Such a compensation problem was formulated and studied in [3, 4], where the risk is modelled by an additive term to the discount rate (a "risk premium"). Tax holidays, depreciation policy and a reduction in profit tax rate were considered as compensating mechanisms.

In the paper, we study a possibility of applying the tax holidays mechanism (on the corporate profit tax) for the compensation of high-level interest rates.

Various problems related to the influence of tax holidays on investment decisions, especially under risk and uncertainty, were studied in a number of papers (see, e.g. [5, 8, 10]). Potential possibilities of tax holidays as a mechanism for maximization of the expected discounted tax payments from the created firm were explored in [4].

This paper is organized as follows. Section 2 describes the behavior of an investor under uncertainty and in a fiscal environment, who is interested in investing into the project aimed at creating a new firm and faces the investment timing problem. A solution to this problem (an optimal investment rule) is described in Sect. 3. In Sect. 4 we set the problem whether the higher interest rate for borrowing can be compensated by tax holidays. Some conclusions and simulation results are presented in Sect. 5.

2 The Basic Model

Consider an investment project requiring the creation of a new industrial firm (enterprise). We assume that, at any moment, a decision-maker (investor) can either *accept* the project and proceed with the investment or *delay* the decision until he obtains new information regarding its environment (product and resource prices, product demand, etc.). Thus, the main goal of the decision-maker is to find, using the available information, a "good" time for investing in the project. Thus, this is an investment timing problem.

The real options theory is a convenient and adequate tool for modelling the process of firm creation since it allows us to study the effects connected with a delay in investment (investment waiting). As in the real options literature, we model investment timing problem as an optimal stopping problem for present values of the created firm (see, e.g. [6, 9]).

A creation of an industrial enterprise is usually accompanied by certain tax benefits (in particular, the new firm is exempted from profit taxes during a certain period). We take into account in explicit form some peculiarities of a corporate profit taxation system, including tax exemption. Such an approach was applied by authors for a detailed model of investment project under taxation in [1, 3, 4].

Uncertainty in the economic system is modelled by some probability space $(\Omega, \mathscr{F}, \mathrm{P})$ with filtration $\mathbb{F} = (\mathscr{F}_t,\ t \geq 0)$. $\mathscr{F}_t$ can be interpreted as the observable information about the system up to the time t.

An infinitely-lived investor faces a problem of choosing a stopping time (w.r.t. filtration $\mathbb{F}$) $\tau \geq 0$, when to invest in the creation of a new firm producing some goods. Investment is considered to be instantaneous and irreversible, and an enterprise begins to produce goods just after the investment is made.

The net price for these goods at time t is π_t, and the level of production at time $t \geq \tau$ is ξ_t^τ. So, $p_t^\tau = \pi_t \xi_t^\tau$ is the flow of profits generated by the firm at time $t \geq \tau$.

To launch a firm at time τ and start production, one needs an investment I_τ. We assume that the required investment I_τ is financed by a credit of the duration L and the interest rate λ.

Both the flow of profits p_t^τ and the required investment I_τ are considered as a stochastic processes on the given probability space $(\Omega, \mathscr{F}, \mathrm{P})$.

The principal repayment schedule (without interest repayment) is described by the flow of repayments such that $C_{\tau+t}^\tau \geq 0 : \int_0^L C_{\tau+t}^\tau dt = I_\tau$, and $C_{\tau+t}^\tau = 0$ for $t > L$.

The total repayments (included interest) that the firm pays for borrowing, discounted to the investment time τ are :

$$K_\tau = K_\tau(\lambda) = \int_0^L (C_{\tau+t}^\tau + \lambda R_{\tau+t}^\tau) e^{-\rho t} dt = F_\tau + \frac{\lambda}{\rho}(I_\tau - F_\tau), \qquad (1)$$

where ρ is the discount rate, $R_{\tau+t}^\tau = \displaystyle\int_t^L C_{\tau+s}^\tau\, ds$ is a remaining debt at time $\tau + t$, and $F_\tau = \displaystyle\int_0^L C_{\tau+t}^\tau e^{-\rho t} dt$.

Further, we assume that the total credit repayments $K_\tau(\lambda)$ increase in the interest rate λ. It is a natural economic assumption which allows us to avoid "bad" repayment schemes.

The created firm is granted with tax holidays, during which it does not pay the corporate profit tax. Let γ be a profit tax rate (tax burden), and ν be the duration of the tax holidays.

Interest payments are included in profit tax base, but the maximal value of deductible interest rates is bounded by the limiting value λ_b.

The expected net present value (NPV) of the firm, discounted to the investment time τ is:

$$V_\tau = \mathrm{E}\left(\int\limits_0^\nu p_{\tau+t}^\tau e^{-\rho t} dt + \int\limits_\nu^{\max(\nu,L)} [p_{\tau+t}^\tau - \gamma(p_{\tau+t}^\tau - \bar{\lambda} R_{\tau+t}^\tau)] e^{-\rho t} dt \right.$$

$$\left. + \int\limits_{\max(\nu,L)}^\infty (1-\gamma) p_{\tau+t}^\tau e^{-\rho t} dt \,\middle|\, \mathscr{F}_\tau \right), \qquad (2)$$

where $\bar{\lambda} = \min(\lambda, \lambda_b)$. This formula uses the existing principle of full-loss offset (loss carry forward).

The investor solves the following *investment timing problem* : to find such a stopping time τ (investment rule), that maximizes the NPV from the future firm:

$$\mathbf{E}\,(V_\tau - K_\tau)\,e^{-\rho\tau} \to \max_\tau, \tag{3}$$

where the maximum is taken over all possible stopping times τ (w.r.t. filtration $\mathbb{F}$), and V_τ, K_τ are defined in (1)–(2).

The starting point of this scheme is the known McDonald-Siegel model [9], which was the base for the real option theory (see, e.g., [6, 13]). More complicated variants of this scheme, which take into account a detailed structure of cash-flows as well as a number of different taxes one can find in [3].

3 Solution to the Investment Timing Problem

Main Assumptions

Let $(w_t^i,\ t \geq 0),\ i = 1, 2, 3$ be independent standard Wiener processes on the stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathrm{P})$. These processes are thought as underlying processes modelling economic stochastics. So, we assume that σ-field $\mathscr{F}_t$ is generated by those processes up to t, i.e. $\mathscr{F}_t = \sigma\{(w_s^1, w_s^2, w_s^3),\ s \leq t\}$.

Remind that the flow of profits has the following representation $p_t^\tau = \pi_t \xi_t^\tau$, $t \geq \tau$, and specify its components.

The process of net prices π_t is geometric Brownian motion :

$$d\pi_t = \pi_t(\alpha_1 dt + \sigma_1 dw_t^1), \quad t \geq 0. \tag{4}$$

The level of production ξ_t^u is described by a family of non-negative diffusion processes, homogeneous in $u \geq 0$, defined as the solution (in strong sense) by the stochastic equations

$$\xi_t^u = \xi + \int_u^t a(s-u, \xi_s^u)\,ds + \int_u^t [b_1(s-u, \xi_s^u)\,dw_s^1 + b_2(s-u, \xi_s^u)\,dw_s^2],\ t \geq u, \tag{5}$$

with given functions $a(t, x),\ b_i(t, x),\ i = 1, 2$, which satisfy the standard conditions for the existence of the strongly unique solution – at most linear growth and Lipschitz continuity (see, e.g., [11, Ch.5]).

The fluctuations ξ_t^τ reflects the uncertainty, which can be generated by the firm created at time τ and demand on its production, and are driven by Wiener processes w_t^1 (related to prices) and w_t^2. Obviously, $p_\tau^\tau = \pi_\tau \xi$ for any τ.

The cost of the required investment I_t is also described by the geometric Brownian motion as follows

$$dI_t = I_t(\alpha_2\,dt + \sigma_{21}\,dw_t^1 + \sigma_{22}\,dw_t^3), \quad t \geq 0, \tag{6}$$

where $\sigma_{21} \geq 0$, $\sigma_{22} > 0$. The appearance of the process w_t^3 in (6) means that the cost of investment I_t is correlated with the net price π_t.

The flow of the *principal repayment* at the time t (for the firm created at the time τ) will be represented as:

$$C_t^\tau = I_\tau c_{t-\tau}, \;\; \tau \leq t \leq \tau + L,$$

where $(c_s,\; 0 \leq s \leq L)$ is the "repayment density" (per unit of investment), characterizing a repayment schedule, i.e. non-negative deterministic function such that $\int_0^L c_s\,ds = 1$.

Note that repayment density can depend, in general, on the interest rate λ, i.e. $c_t = c_t(\lambda)$.

Such a scheme covers various schedules of credit repayment, accepted in practice (more exactly, their variants in continuous time). For example, fixed principal repayment can be described by the uniform density $c_t = 1/L$, while the well-known annuity scheme (fixed payments for a principal plus interest during the repayment period) corresponds to exponential density $c_t = \lambda e^{\lambda t}/(e^{\lambda L} - 1) \quad (0 \leq t \leq L)$.

Derivation of the Present Value

The above assumptions allow us to obtain formulas for the present value of the future firm.

At first we need the following assertion about the process $p_t^\tau = \pi_t \xi_t^\tau$.

Lemma 1. *Let τ be a stopping time. Then for all $t \geq 0$*

$$\mathrm{E}(p_{\tau+t}^\tau | \mathscr{F}_\tau) = \pi_\tau B_t, \quad \textit{where} \;\; B_t = \mathrm{E}(\pi_t \xi_t^0)/\pi_0. \tag{7}$$

Proof. From the Dynkin–Hunt theorem follows that for any stopping time τ the processes $\widehat{w}_t^i = w_{\tau+t}^i - w_\tau^i$, $t \geq 0$ $(i = 1, 2)$ are Wiener processes independent on $\mathscr{F}_\tau$.

From representation (5) one can see that

$$\begin{aligned}\xi_{\tau+t}^\tau &= \xi + \int_0^t a(s, \xi_{\tau+s}^\tau)\,ds + \int_0^t [b_1(s, \xi_{\tau+s}^\tau)\,dw_{\tau+s}^1 + b_2(s, \xi_{\tau+s}^\tau)\,dw_{\tau+s}^2] \\ &= \xi + \int_0^t a(s, \xi_{\tau+s}^\tau)\,ds + \int_0^t [b_1(s, \xi_{\tau+s}^\tau)\,d\widehat{w}_s^1 + b_2(s, \xi_{\tau+s}^\tau)\,d\widehat{w}_s^2].\end{aligned}$$

This implies that for any stopping time τ the process $\xi^{\tau}_{\tau+t}$ coincides (a.s.) with the unique (in the strong sense) solution to the stochastic equation

$$\xi_t = \xi + \int_0^t a(s,\xi_s)\,ds + \int_0^t [b_1(s,\xi_s)\,d\widehat{w}^1_s + b_2(s,\xi_s)\,d\widehat{w}^2_s],$$

which is independent on $\mathscr{F}_\tau$.

Then, $p^{\tau}_{\tau+t} = \pi_\tau \Pi^{\tau}_{\tau+t}$, where $\Pi^{\tau}_{\tau+t} = \exp\{(\alpha_1 - \frac{1}{2}\sigma_1^2)t + \sigma_1\widehat{w}^1_t\}\xi^{\tau}_{\tau+t}$ is independent on $\mathscr{F}_\tau$.

Moreover, $\Pi^{\tau}_{t+\tau}$ has the same distribution as $\exp\{(\alpha_1 - \frac{1}{2}\sigma_1^2)t + \sigma_1\widehat{w}_t\}\xi_t$, i.e. as $(\pi_t/\pi_0)\xi^0_t$. Therefore, $\mathrm{E}(p^{\tau}_t|\mathscr{F}_\tau) = \pi_\tau \mathrm{E}\Pi^{\tau}_{t+\tau} = \pi_\tau \mathrm{E}(\pi_t\xi^0_t)/\pi_0$. □

Let us define the following function :

$$B(t) = \int_t^\infty B_s e^{-\rho s}\,ds, \quad t \ge 0, \tag{8}$$

where B_s are defined in (7), and assume that $B(0) < \infty$.

Using Lemma 1 one can derive the following formulae for the present value (2):

$$V_\tau = \mathrm{E}\left(\int_0^\nu p^{\tau}_{\tau+t}e^{-\rho t}dt + (1-\gamma)\int_\nu^\infty p^{\tau}_{\tau+t}e^{-\rho t}dt + \gamma\bar{\lambda}\int_\nu^{\max(\nu,L)} R^{\tau}_{\tau+t}e^{-\rho t}dt \,\middle|\, \mathscr{F}_\tau\right)$$

$$= \pi_\tau[B(0) - \gamma B(\nu)] + \gamma\bar{\lambda}I_\tau D(\nu), \tag{9}$$

where

$$D(\nu) = \int_\nu^{\max(\nu,L)}\left(\int_t^L c_s\,ds\right)e^{-\rho t}dt. \tag{10}$$

Optimal Investment Timing

The above assumptions and formulas show that investment timing problem (3) is reduced to an optimal stopping problem for bivariate geometric Brownian motion and linear reward function. Indeed,

$$K_\tau = I_\tau\,[F + \lambda(1-F)/\rho] = I_\tau K(\lambda), \tag{11}$$

$$V_\tau - K_\tau = \pi_\tau[B(0) - \gamma B(\nu)] - I_\tau\left[K(\lambda) - \gamma\bar{\lambda}D(\nu)\right], \tag{12}$$

where

$$F = \int_0^L c_t e^{-\rho t} dt, \quad K(\lambda) = F + \lambda(1-F)/\rho. \tag{13}$$

Let β be a positive root of the quadratic equation

$$\frac{1}{2}\tilde{\sigma}^2\beta(\beta-1) + (\alpha_1-\alpha_2)\beta - (\rho-\alpha_2) = 0, \tag{14}$$

where $\tilde{\sigma}^2 = (\sigma_1 - \sigma_{21})^2 + \sigma_{22}^2 > 0$ is a "total" volatility of investment project. It is easy to see that $\beta > 1$ whenever $\rho > \max(\alpha_1, \alpha_2)$.

The following theorem characterizes completely an optimal investment time.

Theorem 1. *Let the processes of profits and required investment be described by relations* (4)–(6). *Assume that* $\rho > \max(\alpha_1, \alpha_2)$ *and the following condition is satisfied:*

$$\alpha_1 - \frac{1}{2}\sigma_1^2 \geq \alpha_2 - \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2).$$

Then the optimal investment time for the problem (3) is

$$\tau^* = \min\{t \geq 0 : \ \pi_t \geq \pi^* I_t\}, \tag{15}$$

where

$$\pi^* = \pi^*(\nu, \lambda) = \frac{\beta}{\beta-1} \cdot \frac{K(\lambda) - \gamma\bar{\lambda}D(\nu)}{B(0) - \gamma B(\nu)}, \tag{16}$$

and $B(\cdot),\ D(\cdot),\ K(\cdot)$ *are defined at* (8), (10), (13) *respectively.*

Formulas of the type (15)–(16) for the difference of two geometric Brownian motions was first derived, probably, by McDonald and Siegel [9]. But rigorous proof and precise conditions for its validity appeared a decade later in [7]. It can also be immediately deduced from general results on optimal stopping for two-dimensional geometric Brownian motion and homogeneous reward function (e.g., [2]).

In order to avoid the "trivial" investment time $\tau^* = 0$, we will further suppose that the initial values of the processes satisfy the relation $\pi_0 < \pi^* I_0$.

The optimal investment level π^* characterizes the time when the investor accepts the project and makes the investment. A decrease in π^* implies an earlier investment time, and, on the contrary, an increase in π^* leads to a delayed investment.

Knowing the optimal investment rule, one can derive the expected net present value $N^* = \mathrm{E}\,(V_{\tau^*} - K_{\tau^*})\, e^{-\rho\tau^*}$ under the optimal behavior of the investor. Using the standard technique for boundary value problems (Feynman-Kac formula – see, e.g., [11, 12, Ch.9]), or the results on homogeneous functionals of two-dimensional geometric Brownian motion ([2]), one can obtain the following formula.

Corollary 1. *Under the assumptions of Theorem 1*

$$N^* = N^*(\nu,\lambda) = C\,[B(0) - \gamma B(\nu)]^{\beta}\left[F+\lambda(1-F)/\rho-\gamma\bar{\lambda}D(\nu)\right]^{1-\beta}, \qquad (17)$$

where $C = (\pi_0/\beta)^{\beta}[I_0/(\beta-1)]^{1-\beta}$.

4 Compensation of Interest Rates by Tax Holidays

Now we formulate the problem of compensating a higher interest rate by tax exemptions.

The question is: can one choose such a duration of tax holidays ν that given the index $\mathscr{M}$ (related to the investment project) under a higher interest rate λ will be greater (not less) than those index under "the reference" interest rate λ_0 and without the tax holidays:

$$\mathscr{M}(\nu,\lambda) \geq \mathscr{M}(0,\lambda_0) \quad \text{for some } \nu \geq 0.$$

We consider the following indices:

1. Optimal investment level π^*, that defines the time when an investor accepts the project and makes the investment;
2. Optimal NPV of the investor N^*.

As the reference interest rate we take the limit rate $\lambda_0 = \lambda_b$, which is deducted in profit tax base.

The assumption about an increasing (in interest rate) total payments on credit and explicit formulas (16)–(17) imply that the above indices are monotone in λ. Namely, π^* increases, and N^* decreases. Therefore, it makes sense to consider a compensation problem only for $\lambda > \lambda_0$.

Compensation in Terms of Optimal Investment Level

Let us begin with an *optimal investment level* $\pi^* = \pi^*(\nu,\lambda)$.

We say that an interest rate λ can be compensated in terms of optimal investment level by tax holidays, if $\pi^*(\nu,\lambda) \leq \pi^*(0,\lambda_0)$ for some duration of tax holidays ν, i.e. in other words, if for some duration of tax holidays ν

$$\pi^*(\nu,\lambda) \leq \pi^*(0,\lambda_0). \qquad (18)$$

Since a decrease of π^* implies earlier investment time (for any random event), then a possibility to compensate in terms of an optimal investment level can be interpreted as a possibility to increase investment activity in the real sector. This situation is attractive for the State.

Further, we assume that profits parameters B_t, defined in (8), are such that the function B_t is differentiable and increasing in $t \in (0, L)$. This means that the expected profit of the firm grows in time. We suppose also that the repayment density c_t is continuous in $t \in (0, L)$. These assumptions allow us to avoid some unessential technical difficulties.

The following result is the criterion for the compensation in terms of an optimal investment level.

Theorem 2. *The interest rate λ can be compensated in terms of an optimal investment level by tax holidays if and only if $\lambda \leq \lambda_1$, where λ_1 is a unique root of the equation*

$$(1-\gamma)K(\lambda) = K(\lambda_0) - \gamma\lambda_0(1-F_0)/\rho, \tag{19}$$

and $F_0 = \int_0^L c_t(\lambda_0)e^{-\rho t}dt$ corresponds to the repayment schedule with the interest rate λ_0.

In other words, there is a "critical" value of interest rate λ_1 such that if interest rate is greater than this value, it can not be compensated in terms of optimal investment level by any tax holidays. Note that the "limiting" interest rate $\lambda = \lambda_1$ can be compensated only by tax holidays with infinite duration.

Proof. If $\nu \geq L$ then $D(\nu)=0$, and (16) implies that $\pi^* = \frac{\beta}{\beta-1} \cdot \frac{K(\lambda)}{B(0)-\gamma B(\nu)}$ decreases in ν.

If $\nu < L$ let us denote

$$r_t = \int_t^L c_s ds, \quad \widehat{B}(\nu) = B(0)-\gamma B(\nu), \quad \widehat{D}(\nu) = K-\gamma\bar{\lambda}D(\nu). \tag{20}$$

From (16) we have

$$\widehat{B}^2(\nu)\frac{\partial \pi^*}{\partial \nu} = \frac{\beta}{\beta-1}\left[-\gamma\bar{\lambda}D'(\nu)\widehat{B}(\nu)+\widehat{D}(\nu)\gamma B'(\nu)\right] = \gamma e^{-\rho\nu}\frac{\beta}{\beta-1}Q(\nu), \tag{21}$$

where $Q(\nu) = \bar{\lambda}r_\nu\widehat{B}(\nu) - \widehat{D}(\nu)B_\nu$.

As one can see from (21), the optimal investment level is not, in general, monotone in ν. The sign of its derivative is completely defined by the function $Q(\nu)$. Then

$$\begin{aligned} Q'(\nu) &= \bar{\lambda}\left[r'_\nu\widehat{B}(\nu) + r_\nu\widehat{B}'(\nu)\right] - \left[\widehat{D}(\nu)B'_\nu + B_\nu\widehat{D}'(\nu)\right] \\ &= \bar{\lambda}\left[-c_\nu\widehat{B}(\nu) + r_\nu\gamma B_\nu e^{-\rho\nu}\right] - \left[\widehat{D}(\nu)B'_\nu + \gamma\bar{\lambda}B_\nu r_\nu e^{-\rho\nu}\right] \\ &= -\bar{\lambda}c_\nu\widehat{B}(\nu) - B'_\nu\widehat{D}(\nu) \leq 0, \end{aligned}$$

since $\widehat{B}(\nu) \geq (1-\gamma)B(0) > 0, \quad \widehat{D}(\nu) \geq \int_\nu^L [c_t + \lambda(1-\gamma)r_t]e^{-\rho t}dt \geq 0, \quad B'_\nu \geq 0.$

Hence, if $\dfrac{\partial \pi^*}{\partial \nu} \le 0$ for some $\nu = \nu_0$, then $\dfrac{\partial \pi^*}{\partial \nu} \le 0$ for all $\nu > \nu_0$. So, the function π^* is either decreasing or having a unique maximum in ν.

Therefore, applying formula (16) for an optimal investment level and the inequality $\pi^*(0,\lambda) > \pi^*(0,\lambda_0)$ for $\lambda > \lambda_0$, we have that relation (18) holds if and only if $\pi^*(\infty,\lambda) \le \pi^*(0,\lambda_0)$, i.e.

$$\frac{\beta}{\beta-1}\cdot\frac{K(\lambda)}{B(0)} \le \frac{\beta}{\beta-1}\cdot\frac{K(\lambda_0)-\gamma\lambda_0 D(0)}{(1-\gamma)B(0)}, \tag{22}$$

where

$$D(0)=\int_0^L\left(\int_t^L c_s ds\right)e^{-\rho t}dt = \left(1-\int_0^L c_t e^{-\rho t}dt\right)/\rho = (1-F_0)/\rho$$

and $c_t = c_t(\lambda_0)$ corresponds to repayment schedule with interest rate λ_0.

Now, the statement of Theorem 2 follows from (22). □

In most cases the "critical" value λ_1 can be derived explicitly.

Corollary 2. *Suppose that the schedule of the principal repayments does not depend on the interest rate. Then the interest rate λ can be compensated in terms of optimal investment level by tax holidays if and only if $\lambda \le \lambda_1$, where*

$$\lambda_1 = \lambda_0 + \rho\frac{\gamma}{1-\gamma}\cdot\frac{F}{1-F}, \tag{23}$$

and F is defined in (13).

Proof. The corollary immediately follows from (19) and formula for $K(\lambda)$ (see (13)). □

Compensation in Terms of Optimal Investor's NPV

Now let us consider an optimal investor's NPV $N^* = N^*(\nu,\lambda)$.

We say that interest rate λ can be compensated in terms of optimal investor's NPV by tax holidays, if for some duration of tax holidays ν

$$N^*(\nu,\lambda) \ge N^*(0,\lambda_0). \tag{24}$$

An increase in N^* implies a growth of expected investor's revenue, therefore the possibility to compensate in terms of optimal NPV is attractive for the investor.

The following result is similar to Theorem 2 above.

Theorem 3. *The interest rate λ can be compensated in terms of optimal investor's NPV by tax holidays if and only if $\lambda \leq \lambda_2$, where λ_2 is a unique root of the equation*

$$(1-\gamma)^{\beta/(\beta-1)}K(\lambda) = K(\lambda_0) - \gamma\lambda_0(1-F_0)/\rho, \tag{25}$$

$F_0 = \int_0^L c_t(\lambda_0)e^{-\rho t}dt$ *corresponds to the repayment schedule with the interest rate λ_0, and β is a positive root of the equation (14).*

Proof. The proof of Theorem 3 follows the general scheme of the proof of Theorem 2.

If $\nu \geq L$ then $D(\nu) = 0$, and N^* increases in ν (see (17)).

Using formula (17) and the notations from (20) we have

$$\begin{aligned} C^{-1}\frac{\partial N^*}{\partial \nu} &= \widehat{B}^{\beta-1}(\nu)\widehat{D}^{-\beta}(\nu)\left[\beta\widehat{B}'(\nu)\widehat{D}(\nu) + (1-\beta)\widehat{B}(\nu)\widehat{D}'(\nu)\right] \\ &= \widehat{B}^{\beta-1}(\nu)\widehat{D}^{-\beta}(\nu)\left[\gamma\beta e^{-\rho\nu}B_\nu\widehat{D}(\nu) + (1-\beta)\gamma\bar{\lambda}e^{-\rho\nu}r_\nu\widehat{B}(\nu)\right] \\ &= \gamma e^{-\rho\nu}\widehat{B}^{\beta-1}(\nu)\widehat{D}^{-\beta}(\nu)S(\nu), \end{aligned}$$

where $S(\nu) = \beta B_\nu\widehat{D}(\nu) - (\beta-1)\widehat{B}(\nu)\bar{\lambda}r_\nu$.

Then we have

$$\begin{aligned} S'(\nu) &= \beta B'_\nu\widehat{D}(\nu) + \beta B_\nu\gamma\bar{\lambda}e^{-\rho\nu}r_\nu + (\beta-1)\widehat{B}(\nu)\bar{\lambda}c_\nu - (\beta-1)\bar{\lambda}r_\nu\gamma e^{-\rho\nu}B_\nu \\ &= \beta B'_\nu\widehat{D}(\nu) + (\beta-1)\bar{\lambda}c_\nu\widehat{B}(\nu) + \gamma\bar{\lambda}r_\nu e^{-\rho\nu}B_\nu \geq 0. \end{aligned}$$

Using arguments, similar to those in the proof of Theorem 2, we get that the function N^* is either increasing or having a unique minimum (in ν).

Therefore, like in the above case, one can conclude that relation (24) holds if and only if $N^*(\infty,\lambda) \geq N^*(0,\lambda_0)$, i.e.

$$CB^\beta(0)K^{1-\beta}(\lambda) \geq C(1-\gamma)^\beta B^\beta(0)[K(\lambda_0) - \gamma\lambda_0 D(0)]^{1-\beta}, \tag{26}$$

where $D(0) = (1-F_0)/\rho$ corresponds to the repayment schedule with the interest rate λ_0. This implies the statement of Theorem 3. □

Similarly to the previous case of a compensation in terms of optimal investment level, the "critical" value λ_2 can be derived explicitly when the principal repayments do not depend on the interest rate.

Corollary 3. *Suppose that the schedule of the principal repayments does not depend on the interest rate. Then the interest rate λ can be compensated in terms of optimal investor's NPV by tax holidays if and only if $\lambda \leq \lambda_2$, where*

$$\lambda_2 = \lambda_0(1-\gamma)^{-1/(\beta-1)} + \rho\frac{F}{1-F}\left[(1-\gamma)^{-\beta/(\beta-1)} - 1\right]. \tag{27}$$

5 Concluding Remarks

1. It is interest to compare the obtained "critical" interest rates λ_1 and λ_2 which give limits for the compensation in relevant terms.
   As Theorems 2 and 3 show, the bound λ_1 is a root of the equation
   $$K(\lambda) = \frac{K(\lambda_0) - \gamma\lambda_0 D(0)}{1-\gamma},$$
   and λ_2 is a root of the equation
   $$K(\lambda) = \frac{K(\lambda_0) - \gamma\lambda_0 D(0)}{(1-\gamma)^{\beta/(\beta-1)}}.$$
   Since the function $K(\lambda)$ increases, then $\lambda_2 > \lambda_1$.
   This fact means that interest rates $\lambda < \lambda_1$ can be compensated by tax holidays both in terms of optimal investment level and in terms of investor's NPV. The opposite is not valid, in general, i.e. a compensation in terms of NPV does not always imply a compensation in terms of the investment level, and therefore a growth of investment activity.
2. Note, that the critical bound λ_2 for the compensation in terms of investor's NPV depends (in contrast to the bound λ_1) on the parameters of the project but only through the value β (see (14)). As a consequence, if the volatility of the project σ increases, then the bound λ_2 of the compensation in terms of NPV will increase also.
3. Usually, it is assumed that the reduction in the refinancing (basic) rate λ_{ref} is a positive factor for a revival of investment activity in the real sector. But this differs from the conclusions of our model.
   Indeed, if tax holidays are absent ($\nu = 0$), then an optimal investment level
   $$\pi^* = \pi^*(\lambda_{\text{ref}}) = \frac{\beta}{\beta - 1} \cdot \frac{K(\lambda) - 1.8\gamma\lambda_{\text{ref}} D(0)}{B(0)(1-\gamma)}$$
   decreases in λ_{ref}. So, π^* raises and, hence, investment activity (earlier investor entry) falls when λ_{ref} diminishes.
   Similarly, the optimal investor's NPV increases in λ_{ref}, and therefore decreasing refinancing rate λ_{ref} de-stimulates investor.
   As calculations show when the refinancing rate λ_{ref} falls to two times (from the current value of 8 %) the optimal investment level grows and optimal investor's NPV declines up to 20 %.
4. We performed a number of calculations for a "reasonable" (for Russian economy) data range. Namely, the typical parameters were as follows:

   tax burden $\gamma = 40\,\%$,
   discount rate $\rho = 10\,\%$,

credits with period $L = 10$ (years) and fixed-principal repayment schedule, reference interest rate $\lambda_0 = 1.8 \times$ (refinancing rate of the CB of Russia) $= 14.85\,\%$.

Typical characteristics of profits and investment cost gave us the "aggregated" parameter β in the interval between 3 and 8.

Then, the received estimations for "critical" compensation bounds were the following: $\lambda_1 \approx 25$–$30\,\%$, $\lambda_2 \approx 30$–$40\,\%$. These values seem to be not extremely high (especially, for the current economic situation in Russia).

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The Shape of Asymptotic Dependence

Guus Balkema, Paul Embrechts, and Natalia Nolde

Abstract Multivariate risk analysis is concerned with extreme observations. If the underlying distribution has a unimodal density then both the decay rate of the tails and the asymptotic shape of the level sets of the density are of importance for the dependence structure of extreme observations. For heavy-tailed densities, the sample clouds converge in distribution to a Poisson point process with a homogeneous intensity. The asymptotic shape of the level sets of the density is the common shape of the level sets of the intensity. For light-tailed densities, the asymptotic shape of the level sets of the density is the limit shape of the sample clouds. This paper investigates how the shape changes as the rate of decrease of the tails is varied while the copula of the distribution is preserved. Four cases are treated: a change from light tails to light tails, from heavy to heavy, heavy to light and light to heavy tails.

Keywords Asymptotic dependence • Copula • Level set • Multivariate extremes • Risk • Sample cloud • Shape • Unimodal density

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1 Introduction

Sample clouds evoke densities rather than distribution functions. Here a sample cloud is a finite set of independent observations from a multivariate distribution, treated as a geometric object, such as the set of points on a computer screen for a bivariate sample. Shape is important, the precise scale not.

The classic models such as the multivariate Gaussian distribution and the Student t distributions have continuous unimodal densities, provided the distribution is non-degenerate. These densities are determined by a bounded set, the ellipsoid which describes the shape of the level sets of the density, and by the rate of decrease. In risk analysis one is interested in extreme observations, and it is the asymptotic shape of the level sets and the rate of decrease of the tails of the density that are important. Let us illustrate these two components with a few simple examples.

A *homothetic density* has all level sets of the same shape, scaled copies of some given set. It is completely determined by the set and by the density generator which determines the decay along any ray. Altering the density on compact sets does not affect the asymptotic behaviour. So assume that the density is asymptotic to a homothetic density and impose conditions on the rate of decrease along rays – regular variation, or exponential decay – to ensure a simple asymptotic description of the tails, and also of the shape of large sample clouds (the two limits are related as will be explained later). Within this rather restricted setting of multivariate probability distributions with continuous unimodal densities and level sets with limit shape, we have a simple theory to describe extremes and say something about the asymptotic dependence structure. For a Student t density with spherical level sets the sample clouds, properly scaled, converge to a Poisson point process whose intensity has spherical level sets and decreases like a negative power along rays. For the standard Gaussian density the sample clouds converge onto a ball.

For heavy-tailed dfs F, the scaled sample clouds may converge to a Poisson point process N with mean measure ρ which is *homogeneous*:

$$\rho(rA) = \rho(A)/r^{\lambda}, \qquad r > 0 \tag{1}$$

for all Borel sets A. The measure ρ is infinite, but the complement of any centered ball has finite mass. If ρ has a continuous positive density h then by homogeneity all level sets $\{h > c\}$ have the same shape. They are scaled copies of a bounded open star-shaped set D which contains the origin and which has a continuous boundary. Let $\mathscr{D}$ denote the class of such sets. The function h is the intensity of the point process N. It is completely determined by the set $\{h > 1\} \in \mathscr{D}$ and a parameter $\lambda > 0$ since by homogeneity of ρ it satisfies $h(r\mathbf{w}) = h(\mathbf{w})/r^{\lambda+d}$. The condition $\lambda > 0$ ensures that h is integrable over the complement of the open unit ball B, and hence N almost surely has finitely many points on the complement of centered balls. A continuous density f whose level sets $\{f > c\}$ asymptotically have shape D will lie in the domain of attraction of the measure ρ with density h if on any ray it is asymptotic to $cL(r)/r^{\lambda+d}$ for some slowly varying function L where c depends on the direction. The set of such densities is denoted by $\mathscr{F}_\lambda$. The densities $f \in \mathscr{F}_\lambda$ may be regarded as generalizations of the spherically symmetric Student t density f_λ with λ degrees of freedom. The asymptotic power decrease $c_\lambda/r^{\lambda+d}$ of f_λ is replaced by a regularly varying function $cL(r)/r^{\lambda+d}$; the spherical level sets are replaced by level sets which asymptotically have the shape D for some $D \in \mathscr{D}$. In risk management both the shape D and the parameter λ play a role.

For light-tailed densities there is a similar extended model. The central place is taken by the standard Gaussian density. Here too there is a one-parameter family, the spherically symmetric Weibull-type densities $g_\theta(\mathbf{x}) = c_\theta e^{-r^\theta/\theta}$, $r = \|\mathbf{x}\|$, for $\theta > 0$. One can now introduce the class $\mathscr{G}_\theta$ of continuous densities g asymptotic to a homothetic function whose level sets are scaled copies of a set $D \in \mathscr{D}$, and where g decreases like $ce^{-\psi(r)}$ along rays, with $\psi(r)$ a continuous function which varies regularly with exponent θ. The tails of g decrease rapidly. That implies that sample clouds tend to have a definite shape. For $g \in \mathscr{G}_\theta$ the sample clouds, properly scaled, converge onto the closure of the set D. In general, one may consider light-tailed distributions whose scaled sample clouds converge onto a compact set E. The set E then is star-shaped, but its boundary need not be continuous. The set E may even have empty interior. For light-tailed densities the limit shape D is quite robust. If we multiply the standard Gaussian density $e^{-r^2/2}/2\pi$ by a function like $c(1+x^6)e^{r\sin x^2y^2}$ the new function is integrable and will be a probability density for an appropriate choice of $c > 0$. The auxiliary factor fluctuates wildly, but the new density will have level sets which are asymptotically circular.

The theory so far is geometric. It does not depend on the coordinates. In the light-tailed case the asymptotics are described by a compact star-shaped set E; in the heavy-tailed case by a homogeneous measure ρ. In both cases there is a class of continuous densities whose asymptotic behaviour is determined by a bounded open star-shaped set $D \in \mathscr{D}$, and a positive parameter θ or λ describing the rate of decrease of the tails. The parameter determines the severity of the extremes; the shape tells us where these extremes are more likely to occur. For heavy tails it is the parameter λ which is of greater interest; for light tails the shape becomes increasingly important since new extremes are likely to occur close to the boundary.

Now introduce coordinates. Points in the sample clouds are d-tuples of random variables, $\mathbf{Z} = (Z_1, \dots, Z_d)$. By deleting some of the coordinates the sample is projected onto the lower dimensional space spanned by the remaining coordinates. If we only retain the ith coordinate we have a one-dimensional sample cloud. In the light-tailed situation this univariate cloud converges onto the set E_i, the projection of E onto the ith coordinate. The set E_i is an interval $E_i = [-c_i^-, c_i^+]$ with $c_i^\pm \geq 0$ since E is star-shaped. The d-dimensional coordinate box $[-\mathbf{c}^-, \mathbf{c}^+]$ fits nicely around the limit set E. If E is the closure of a set $D \in \mathscr{D}$, the $2d$ constants $c_i^\pm$ are strictly positive. If desired, one may then scale the sample clouds such that $c_d^+ = 1$. In the heavy-tailed case the univariate sample clouds converge to a one-dimensional Poisson point process on $\mathbb{R}\backslash\{0\}$. The mean measure of this point process is the marginal ρ_i of the homogeneous measure ρ. By the homogeneity property (1),

$$\rho_i(-\infty, -t) = a_i^-/t^\lambda, \qquad \rho_i(t, \infty) = a_i^+/t^\lambda, \qquad t > 0.$$

Here too the balance constants are strictly positive if ρ has a continuous positive density, and one may choose the scaling constants for the sample clouds such that $a_d^+ = 1$. The balance constants $a_i^\pm$ reflect the balance in the upper and lower tails of the margins f_i of the underlying density $f \in \mathscr{F}_\lambda$. There is a slowly varying function $L(t)$ such that $f_i(\pm t) \sim a_i^\pm L(t)/t^{\lambda+1}$, $i = 1, \dots, d$, for $t \to \infty$.

If the balance constants $c_i^\pm$ and $a_i^\pm$ are positive then by the use of simple coordinatewise transformations one may achieve that these constants are one. This may be done with semilinear transformations of the form $t \mapsto pt + q|t|$ with $|q| < p$. The limit set now has symmetric and equal projections $E_i = [-1, 1]$; the homogeneous measure has symmetric and equal margins with tails equal to $1/t^\lambda$. One may go a step further and transform the margins of the light-tailed vector to be standard Gaussian, and those of the heavy-tailed vector to be standard Cauchy. The effect on the limit set and the homogenous limit measure is described in Theorems 1 and 3 below. A vector whose components are independent with the light-tailed Weibull-type density in (9) has as limit shape the closed unit ball in l_p-norm. The transformed density is standard normal. Its level sets are Euclidean balls. For heavy tails there is a simple formula linking the points of the limit Poisson point process of the transformed vectors to the points $\mathbf{W}$ of the original Poisson point process. The new point process has components $W_i^\lambda \mathbf{1}_{[W_i \geq 0]} - |W_i|^\lambda \mathbf{1}_{[W_i < 0]}$.

Since we use coordinatewise transformations, the copula of the underlying distribution is not affected. Coordinatewise transformations are widely used in multivariate extreme value theory (EVT), and for heavy tails the results agree with EVT where it is standard usage to assume that the vectors have positive components with Fréchet margins e^{-1/t^λ} with parameter $\lambda = 1$.

The main focus of the paper however is on continuous densities whose level sets have asymptotic shape $D \in \mathscr{D}$, in particular densities in $\mathscr{F}_\lambda$ and $\mathscr{G}_\theta$. The shape D may be regarded as a geometric expression of the asymptotic dependence. It is natural to ask how the copula changes as one varies the exponent λ or θ or if one goes from heavy tails to light tails or vice versa, but retains the shape D of the level sets. Since we find it difficult to describe "change of the copula" we shall investigate the dual problem: "How does the shape of the level sets change if one keeps the copula constant but varies the tail index of f?"

In answering this question we compare two densities with the same copula but with different rates of decrease in the tails. What happens to the asymptotic shape of the level sets and to the sample clouds if we change the margins? We distinguish four cases: (1) changing from light-tailed margins to light-tailed ones; (2) from heavy to heavy; (3) from heavy to light; and (4) from light to heavy tails. The copula is kept constant. The analysis is presented in Sects. 3–6.

In the next section we give precise definitions, review some results on the limit behaviour of sample clouds, and introduce meta transformations. The paper ends with our conclusions.

1.1 *Notation*

Two positive continuous functions f and g are *asymptotic* and we write $f \sim g$ if $g(\mathbf{z}_n)/f(\mathbf{z}_n) \to 1$ for every sequence $\mathbf{z}_n$ for which $\|\mathbf{z}_n\| \to \infty$. The functions are *weakly asymptotic* and we write $f \asymp g$ if there exists a constant $M > 1$ such that $f/M \le g \le Mf$. We write B for the open unit ball in the Euclidean norm $\|\mathbf{z}\|$, and ∂A for the boundary of the set A. Thus ∂B is the unit sphere. $\mathscr{R}_\theta$ denotes the set of continuous functions f defined on $[0, \infty)$ which vary regularly in infinity with exponent θ, i.e. f is positive eventually and $f(tx)/f(t) \to x^\theta$ for $t \to \infty$ and $x > 0$. The class $\mathscr{D}_d$ of bounded open star-shaped sets in $\mathbb{R}^d$ and the set $\mathscr{F}_\lambda$ of continuous positive densities asymptotic to a homothetic function $f_*(n_D)$ with $f_* \in \mathscr{R}_{-(\lambda+d)}$ and $D \in \mathscr{D}_d$ will be defined in Sect. 2.

2 Preliminaries

This section contains definitions of certain concepts: star-shaped set and its gauge function, sample cloud, homothetic function and its generator, homogeneous measure, von Mises function and its scale function, and meta density. We briefly review the relation between the asymptotic behaviour of multivariate densities and of sample clouds, convergence in distribution and convergence onto a set. Meta densities will play a basic role in our investigation on the relation between shape (of level sets and sample clouds) and copulas. More detailed information may be found in [1] and [2].

2.1 *Definitions and Basics*

A set E in $\mathbb{R}^d$ is *star-shaped* if it contains the origin and if $\mathbf{x} \in E$ implies $r\mathbf{x} \in E$ for $0 < r < 1$. We define $\mathscr{D} = \mathscr{D}_d$ to be the set of all bounded open star-shaped sets D in $\mathbb{R}^d$ whose boundary is continuous. With such a set D we associate the *gauge function* n_D. This is the unique function which satisfies the two conditions

$$D = \{n_D < 1\}, \qquad n_D(r\mathbf{x}) = r n_D(\mathbf{x}), \qquad r \ge 0. \tag{2}$$

If D is convex and $-D = D$ then the gauge function is a norm and D the open unit ball in this norm. A bounded open star-shaped set D has a continuous boundary ∂D if and only if the gauge function is continuous. A continuous positive function f_0 on $\mathbb{R}^d$ is *homothetic* with shape set $D \in \mathscr{D}$ if the level sets $\{f_0 > c\}$ are scaled copies of D for $0 < c < \sup f_0$. One may use the gauge function (like the Euclidean norm $\|\mathbf{z}\|$) to write down explicit expressions for homothetic functions: $f_0(\mathbf{z}) = f_*(n_D(\mathbf{z}))$ for $\mathbf{z} \in \mathbb{R}^d$. The function f_* is called the *generator* of the function f_0. We shall always assume that f_* is a continuous, strictly decreasing, positive function on $[0, \infty)$. This implies that f_0 is continuous and positive on $\mathbb{R}^d$. In order to obtain interesting asymptotics we assume that f_* varies regularly with exponent $-\lambda - d$ with $\lambda > 0$ (to ensure a finite integral) or that f_* varies rapidly.

Write $\mathscr{F}_\lambda$ for the set of all continuous densities f asymptotic to $f_*(n_D)$ with $f_* \in \mathscr{R}_{-\lambda-d}$ and $D \in \mathscr{D}$. Such a density has *heavy tails*. Its asymptotics are described by a function $h : \mathbb{R}^d \setminus \{\mathbf{0}\} \to (0, \infty)$ of the form

$$h(\mathbf{w}) = 1/n_D(\mathbf{w})^{\lambda+d} = \eta(\omega)/r^{\lambda+d}, \qquad r = \|\mathbf{w}\| > 0,\ \omega = \mathbf{w}/r. \tag{3}$$

Here η is a continuous positive function on the unit sphere ∂B. The relation between η and the boundary ∂D is simple:

$$r\omega \in \partial D \qquad \Longleftrightarrow \qquad \eta(\omega) = 1/r^{\lambda+d}.$$

The function h is the intensity of a Poison point process N with mean measure ρ. This measure ρ is a Radon measure on $\mathbb{R}^d \setminus \{\mathbf{0}\}$. It is *homogeneous with exponent* $-\lambda$, see (1). If such a homogeneous measure ρ has a continuous positive density, the density has the form (3). The homogeneity condition (1) implies that the margins ρ_i, $i = 1, \dots, d$, are Radon measures on $\mathbb{R}$ with density $c_i^- \lambda/|t|^{\lambda+1}$ for $t < 0$ and $c_i^+ \lambda/t^{\lambda+1}$ for $t > 0$. (Apply (1) to $A = \{x_i \le -1\}$ or to $A = \{x_i \ge 1\}$.) Let $f \in \mathscr{F}_\lambda$. Then (cf. Proposition 3)

$$h_t(\mathbf{w}) := \frac{f(t\mathbf{w})}{f_*(t)} \to h(\mathbf{w}) = \frac{1}{n_D(\mathbf{w})^{\lambda+d}}, \qquad t \to \infty,\ \mathbf{w} \ne 0. \tag{4}$$

Pointwise convergence follows from regular variation of f_*. An application of Potter's bounds (see [6]) yields $\mathbf{L}^1$ convergence on the complement of centered balls. Choose t_n such that $t_n^d f_*(t_n) = 1/n$. Then h_{t_n} is the density of a measure ρ_{t_n}, of mass n. This measure is the mean measure of the sample cloud

$$N_n = \{\mathbf{Z}_1/t_n, \dots, \mathbf{Z}_n/t_n\}, \tag{5}$$

where $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ are independent observations from the density f. By definition a *sample cloud* is a scaled random sample. The $\mathbf{L}^1$ convergence in (4) on the complement of centered balls implies weak convergence $\rho_{t_n} \to \rho$ on the complement of centered balls and also convergence of the sample clouds: $N_n \Rightarrow N$ weakly on the complement of centered balls. Here N is the Poisson point process with mean measure ρ on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and $\Rightarrow$ denotes convergence in distribution.

If the generator f_* varies rapidly then the asymptotics are different. Again let $\mathbf{Z}_1,\ \mathbf{Z}_2, \dots$ be independent observations from a density f asymptotic to $f_*(n_D)$. Then $h_t(\mathbf{w}) := f(t\mathbf{w})/f_*(t)$ tends to ∞ uniformly on compact sets in D and tends to zero uniformly on the complement of any open set U containing the closure of D. This convergence to zero on U^c also holds in $\mathbf{L}^1$. Hence the measures ρ_t with density h_t satisfy $\rho_t(\mathbf{p} + \epsilon B) \to \infty$, $t \to \infty$, for each point $\mathbf{p}$ in the closure of

D and all $\epsilon > 0$, and $\rho_t(U^c) \to 0$. Choose t_n such that $\rho_{t_n}(\mathbb{R}^d) = n$. Then ρ_{t_n} is the mean measure of the sample cloud N_n in (5), and these sample clouds *converge onto* the compact set $E = \mathrm{cl}(D)$: For $\mathbf{p} \in E$, $\epsilon > 0$, U any open set containing E, and any integer $m \geq 1$

$$\mathbb{P}\{N_n(\mathbf{p} + \epsilon B) \geq m\} \to 1, \qquad \mathbb{P}\{N_n(U^c) = 0\} \to 1.$$

Typical heavy-tailed densities in $\mathscr{F}_\lambda$ are multivariate Student densities with λ degrees of freedom and spherical, elliptical or cubical level sets. In the light-tailed case one may think of generator functions of the form

$$f_*(t) = e^{-\varphi(t)} = at^b e^{-pt^\theta}, \qquad t \geq t_0, \qquad a, p, \theta > 0.$$

These functions with Weibull-type tails vary rapidly. They also have the property that the exponent φ varies regularly with exponent $\theta > 0$, and that f_* is a *von Mises function* with *scale function* $a(t) = 1/\varphi'(t)$:

$$f_* = e^{-\varphi}, \qquad \varphi \in C^2[0, \infty),\ \varphi'(t) > 0,\ a'(t) \to 0, \qquad t \to \infty. \tag{6}$$

Von Mises functions have simple exponential asymptotic behaviour (see e.g. [9]):

$$f_*(t + a(t)v)/f_*(t) \to e^{-v}, \qquad v \in \mathbb{R}, \qquad t \to \infty. \tag{7}$$

Convergence in (7) holds in $\mathbf{L}^1$ on halflines $[c, \infty)$ for all $c \in \mathbb{R}$. The von Mises condition for a df F to lie in the maximum domain attraction of the Gumbel distribution is $(1 - F(t))F''(t)/(F'(t))^2 \to -1$ for $t \to \infty$. This gives (6) for $\varphi = -\log(1 - F)$.

2.2 *Meta Distributions*

It is possible to construct a multivariate df G with Gaussian margins and the copula of a heavy-tailed multivariate elliptical Student t distribution with df F.

For any two continuous strictly increasing dfs F_0 and G_0 on $\mathbb{R}$ there exists a unique increasing transformation K_0 such that $G_0 = F_0 \circ K_0$. Obviously we have $K_0 = F_0^{-1} \circ G_0$. Let the vector $\mathbf{Z} = (Z_1, \ldots, Z_d)$ have df F with continuous strictly increasing margins F_i, and let G_i be continuous strictly increasing univariate dfs. Write $G = F \circ K$, where K is the coordinatewise increasing transformation

$$K : \mathbf{x} \mapsto \mathbf{z} = (K_1(x_1), \ldots, K_d(x_d)), \qquad K_i = F_i^{-1} \circ G_i, \qquad i = 1, \ldots, d.$$

Then G is the *meta distribution with margins G_i based on the df F*. The transformation K is called the *meta transformation*. If $\mathbf{X}$ has df G then $\mathbf{Z} = K(\mathbf{X})$

has df F. The distributions F and G have the same copula. In most of our applications the margins have continuous positive densities.

Proposition 1. *If F has a continuous strictly positive density f with continuous margins f_i, and if the univariate dfs $G_1, \dots, G_d$ have continuous positive densities g_i, then the meta df $G = F \circ K$ with margins G_i based on F has a continuous strictly positive density g. Moreover,*

$$\frac{g(x_1, \dots, x_d)}{g_1(x_1) \cdots g_d(x_d)} = \frac{f(z_1, \dots, z_d)}{f_1(z_1) \cdots f_d(z_d)}, \qquad \mathbf{z} = K(\mathbf{x}) \in \mathbb{R}^d. \tag{8}$$

For the proof of this result and more information on meta distributions we refer to [2].

We shall use the notation F and f for the original df and its density and denote the margins by F_i and f_i. The meta df G based on F is specified by its margins G_i.

We can now become concrete. Define $\mathbf{Z}$ to have df F with density f asymptotic to $f_*(n_D)$ for a set $D \in \mathscr{D}$, where the generator f_* varies regularly or rapidly. Choose a continuous symmetric unimodal positive density g_0 with tails which vary regularly or rapidly. Construct the meta density g with margins g_0 based on f. Then f and g have the same copula. What is the asymptotic shape of the level sets of g? What is the asymptotic behaviour of the sample clouds from the density g?

3 Light Tails to Light Tails

For a multivariate normal vector with independent components, the level sets of the density are balls. If the margins have a Laplace density, the level sets are tetrahedra. For symmetric Weibull-type margins

$$ce^{-|t|^p/p}, \qquad c = p^{1-1/p}/2\Gamma(1/p), \qquad p > 0, \qquad t \in \mathbb{R}, \tag{9}$$

the level sets are open balls in ℓ_p. *Power transformations* J^γ for $\gamma > 0$ are the coordinatewise transformations

$$J^\gamma : \mathbf{x} \mapsto \mathbf{z}, \qquad z_i = x_i^\gamma \mathbf{1}_{[x_i \geq 0]} - |x_i|^\gamma \mathbf{1}_{[x_i < 0]}. \tag{10}$$

They form a group: $J^\alpha J^\beta = J^{\alpha\beta}$ and the inverse of J^γ is $J^{1/\gamma}$. Moreover, they map the unit ball in ℓ_q norm into a unit ball in ℓ_p norm with $p = q/\gamma$. Thus we have $J^\gamma(B_q) = B_{q/\gamma}$ where B_p denotes the open unit ball in ℓ_p since $J^\gamma \mathbf{x} \in B_p$ precisely if $(|x_1|^\gamma)^p + \dots + (|x_d|^\gamma)^p < 1$, i.e. precisely if $\mathbf{x} \in B_{\gamma p}$. Since vectors with independent components have the same copula, the distributions above are linked by meta transformations $K = (K_0, \dots, K_0)$. The K are no power transformations. The square of an exponential variable is not one-sided normal. Power transformations do describe the asymptotic relation between level sets of light-tailed densities though. We need a lemma to link the tail behaviour of the marginal densities and dfs.

Lemma 1. *Let* $e^{-\Psi(s)} = \int_s^\infty e^{-\psi(t)}dt$ *for* $\psi \in \mathscr{R}_\theta$, $\theta > 0$. *There exists* $s_0 > 1$ *such that*

$$\psi(s) - \log(2s) \le \Psi(s) \le \max\{\psi(t) \mid s \le t \le s+1\}, \qquad s \ge s_0.$$

Proof. The second inequality is obvious. For the first one, write

$$\int_s^\infty e^{-\psi(t)}dt = se^{-\psi(s)} \int_1^\infty e^{-(\psi(rs)-\psi(s))}dr,$$

and observe that the Potter bounds (see [6]) yield an $s_1 > 1$ such that we have $(\psi(rs) - \psi(s))/\psi(s) \ge \theta \log r$ for $r \ge 2$ and $s \ge s_1$ (since $\min_{r\ge 2}(r^\theta - 1)/\log r > \theta$ and $\log r << r^{\theta/2}$). Write J_n for the integral on the right over the interval $[n, n+1]$. First assume ψ is increasing. Then

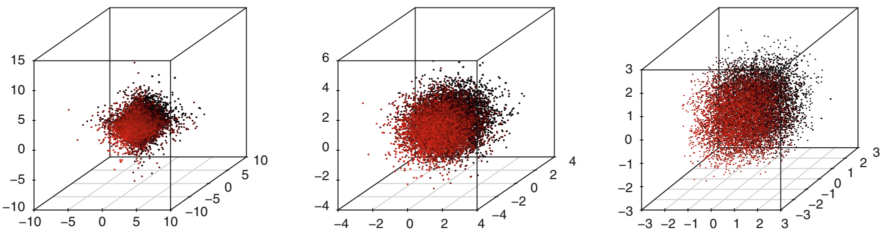
$$J_1 \le 1, \qquad J_m \le e^{-(\psi(ms)-\psi(s))} \le e^{-\theta\psi(s)\log m} < 1/m^2, \qquad m > 1,\ \psi(s) > 2/\theta.$$

Hence the integral on the right is bounded by $\pi^2/6 < 2$. If ψ is not monotone, the bound 2 will do. □

3.1 *Sample Clouds*

We shall first look at sample clouds. They are more intuitive to work with.

Sample clouds from light-tailed unimodal densities tend to have a definite shape. See Fig. 1 for examples of simulated trivariate sample clouds from meta distributions discussed in this section. As the sample size goes to infinity, the scatter plots turn into an octahedron, a Euclidean ball and the ball in ℓ_3, which is halfway between the Euclidean ball and the cube.



(a) $\sigma = 1$, $\tau = 2$ (b) $\sigma = \tau = 2$ (c) $\sigma = 3$, $\tau = 2$

Fig. 1 Sample clouds of 10,000 points from trivariate meta distributions based on the standard normal distribution. The margins are: standard symmetric exponential (Panel (**a**)), standard normal (Panel (**b**)) and symmetric Weibull-type in (9) with shape parameter $p = 3$ (Panel (**c**)). The parameters σ and τ are as defined in Theorem 1

Assume that the sample clouds, suitably scaled, converge onto a compact set E. Such a limit set is star-shaped (see [7]). If E is the closure of a star-shaped open set $D \in \mathscr{D}$ then it is reasonable to model the underlying distribution by a continuous positive density f which is homothetic, or weakly asymptotic to a homothetic density, or to a unimodal density whose level sets have limit shape D. Now consider the meta density with Gaussian margins based on f. What do the sample clouds from the meta density look like? Can they be scaled to converge onto a limit set, and if so, what is the relation between this limit set and the compact star-shaped set E? The answer depends on the tails of the margins.

Theorem 1. *Let $S \in \mathscr{R}_\sigma$ and $T \in \mathscr{R}_\tau$ with $\sigma, \tau > 0$. Let $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ be independent observations from the df F with continuous strictly increasing margins F_i which satisfy*

$$-\log F_i(-t) \sim T(t), \qquad -\log(1 - F_i(t)) \sim T(t), \qquad t \to \infty,\ i = 1, \dots, d. \tag{11}$$

Suppose there exists a compact set E and $a_n > 0$ such that the sample clouds $\{\mathbf{Z}_1/a_n, \dots, \mathbf{Z}_n/a_n\}$ converge onto E. Let E_i denote the projection of E onto the ith coordinate, and assume $\max E_d = 1$. *Then $E_i = [-1, 1]$ for $i = 1, \dots, d$, and $T(a_n) \sim \log n$. Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ be independent observations from the meta df G with continuous strictly increasing margins G_i which satisfy*

$$-\log G_i(-s) \sim S(s), \qquad -\log(1 - G_i(s)) \sim S(s), \qquad s \to \infty,\ i = 1, \dots, d.$$

Let $S(b_n) \sim \log n$. Then the sample clouds $N_n = \{\mathbf{X}_1/b_n, \dots, \mathbf{X}_n/b_n\}$ converge onto the compact star-shaped set $J^\gamma(E)$ with $\gamma = \sigma/\tau$, where J^γ is the power transformation in (10).

Proof. The equality $E_i = [-1, 1]$ and $T(a_n) \sim \log n$ follow from (11) by univariate EVT; see e.g. [9]. Coordinatewise power transformations map rays onto rays, and hence map star-shaped sets into star-shaped sets. Continuity of J^γ ensures that the image $J^\gamma(E)$ is compact. Let K denote the meta transformation with coordinates K_i which satisfy $F_i(K_i) = G_i$. Then we may assume that $\mathbf{X}_n = M(\mathbf{Z}_n)$ for $n = 1, 2, \dots$, where $M = K^{-1}$, see (13). The coordinates $M_i(t)$ and $-M_i(-t)$ are asymptotic to $S^{-1} \circ T$ for $t \to \infty$, and vary regularly with exponent $\gamma = \tau/\sigma > 0$. Let $J_n(\mathbf{w}) = M(a_n \mathbf{w})/b_n$. By regular variation of the coordinates, and the choice of a_n and b_n one finds $J_n(\mathbf{w}) \to J^\gamma(\mathbf{w})$ uniformly on compact sets. Moreover, J_n maps the complement of the cube $[-2, 2]^d$ into the complement of a cube $[-c, c]^d$ for some $c > 1$ eventually. It follows that $J_n(N_n)$ converges onto $J^\gamma(E)$. □

The asymptotic equalities in (11) are not very strong. They hold if the marginal densities $f_i(\pm t)$ are asymptotic to Gamma densities $c_i^\pm t^{b_i^\pm} e^{-t}$ where $c_i^\pm > 0$ and $b_i^\pm$ are arbitrary constants. Yet the implications for the limit set are severe. The projections E_i are symmetric and equal. If we replace the condition on the margins F_i by

$$-\log F_i(-t) \sim a_i^- T(t), \quad -\log(1 - F_i(t)) \sim a_i^+ T(t), \quad s \to \infty,\ a_i^\pm > 0,\ i = 1, \dots, d$$

and similar conditions on the margins G_i with constants $b_i^{\pm} > 0$ we obtain a similar result. For simplicity assume $T = S$. Now $E_i = [-(a_i^-)^{1/\tau}, (a_i^+)^{1/\tau}], i = 1, \dots, d$, and the sample clouds from G converge onto $\Lambda_{\mathbf{c}}(E)$, where $\Lambda_{\mathbf{c}}$ is the coordinatewise semilinear transformation

$$\Lambda_{\mathbf{c}} : \mathbf{u} \mapsto \mathbf{w}, \quad w_i = c_i^- u_i \mathbf{1}_{[u_i < 0]} + c_i^+ u_i \ \mathbf{1}_{[u_i \geq 0]}, \quad c_i^{\pm} = (b_i^{\pm}/a_i^{\pm})^{1/\tau}, \quad i = 1, \dots, d. \tag{12}$$

The proof is similar.

3.2 *Level Sets and Densities*

We now turn our attention to light-tailed densities $f = f_*(n_D)$ and meta densities with light-tailed margins based on f. For instance one could think of Gaussian margins g_0 and a Weibull-type generator $f_*(t) = ce^{-t^\tau/\tau}$. Do the level sets of the meta density have a limit shape? If so, what is the relation between this limit shape and the original set D? The problem here is that we make assumptions about the structure of the density f, but we need information on the margins of f in order to determine the meta transformation K linking the dfs F and G. Recall

$$G = F \circ K, \qquad \mathbf{Z}_n = K(\mathbf{X}_n), \qquad K : \mathbf{x} \mapsto \mathbf{z} = (K_1(x_1), \dots, K_d(x_d)). \tag{13}$$

Under appropriate conditions on the set D and the generator $f_* = e^{-\varphi_*}$, see [4] or Theorem 8.6 in [1], the margins f_i of a continuous positive density $f \sim f_*(n_D)$ satisfy the asymptotic condition:

$$-\log f_i(t) \sim \varphi_*(|t|), \qquad |t| \to \infty, \qquad i = 1, \dots, d. \tag{14}$$

Rather than imposing conditions on D and f_* we shall make assumptions about the margins. We assume that the marginal densities of f are continuous positive functions, $f_i = e^{-\varphi_i}$, and

$$\varphi_i(-t) \sim a_i^- T(t), \qquad \varphi_i(t) \sim a_i^+ T(t), \qquad t \to \infty, \quad a_i^{\pm} > 0, \ i = 1, \dots, d \tag{15}$$

for $T \in \mathscr{R}_\tau$ with $\tau > 0$. We make a similar assumption about the margins $g_i = e^{-\psi_i}$ with respect to $S \in \mathscr{R}_\sigma$ with $\sigma > 0$:

$$\psi_i(-s) \sim b_i^- S(s), \qquad \psi_i(s) \sim b_i^+ S(s), \qquad s \to \infty, \quad b_i^{\pm} > 0, \ i = 1, \dots, d. \tag{16}$$

Theorem 2. *Let $D \in \mathscr{D}_d$. Let $S \in \mathscr{R}_\sigma$ and $T \in \mathscr{R}_\tau$ with $\sigma, \tau > 0$. Let $f = e^{-\varphi}$ be a continuous positive density on $\mathbb{R}^d$ with $\varphi \sim T(n_D)$ and with continuous positive margins $f_i = e^{-\varphi_i}$, $i = 1, \dots, d$. Let $g_i = e^{-\psi_i}$ be continuous positive densities* *on* $\mathbb{R}$*. Assume* (15) *and* (16)*. The level sets of the meta density* $g = e^{-\psi}$ *with margins* g_i *based on* f *then have asymptotic shape* $Q = \Gamma^{-1}(D)$ *where* Γ *is the coordinatewise semi-power transformation in* (19)*. The function* ψ *is asymptotic to* $S(n_Q)$*.*

Proof. The density g by (8) has the form

$$g(\mathbf{x}) = f(\mathbf{z})g_1(x_1)\cdots g_d(x_d)/(f_1(z_1)\cdots f_d(z_d)), \qquad \mathbf{z} = (K_1(x_1), \ldots, K_d(x_d)). \tag{17}$$

The marginal meta transformations K_i are determined by the tails of the dfs, $1 - F_i(K_i(s)) = 1 - G_i(s)$ for all s. Hence $1 - F_i = e^{-\Phi_i}$ and $1 - G_i = e^{-\Psi_i}$ gives $\Phi_i(K_i(s)) = \Psi_i(s)$. Lemma 1 shows that $\Phi_i(t) \sim a_i^+ T(t)$ and $\Psi_i(s) \sim b_i^+ S(s)$, which implies that $K_i \in \mathscr{R}_\gamma$ for $\gamma = \dfrac{\sigma}{\tau}$, and actually $K_i(s) \sim c_i^+ R(s)$ with $c_i^+ = \left(\dfrac{b_i^+}{a_i^+}\right)^{1/\tau}$ and $R = T^{-1} \circ S$ where we assume S and T continuous and strictly increasing. Similarly, $K_i(-s) \sim -c_i^- R(s)$ for $s \to \infty$ with $c_i^- = \left(\dfrac{b_i^-}{a_i^-}\right)^{1/\tau}$. Rewrite (17) as

$$\psi(\mathbf{x}) = \varphi(K(\mathbf{x})) + \delta_1(x_1) + \cdots + \delta_d(x_d), \qquad \delta_i(s) = (\varphi_i(t) - \Phi_i(t)) - (\psi_i(s) - \Psi_i(s))$$

with $t = K_i(s)$, since $\Phi_i(t) = \Psi_i(s)$. We claim that $\psi_0 = \varphi \circ K$ satisfies a simple limit relation and that the δ_i may be neglected. By assumption

$$\varphi(t_n \mathbf{w}_n)/T(t_n) \to n_D^\tau(\mathbf{w}), \qquad \mathbf{w}_n \to \mathbf{w}, \qquad t_n \to \infty. \tag{18}$$

Let $\mathbf{u}_n \equiv \mathbf{u}(n) \to \mathbf{u} \in \mathbb{R}^d$ and $r_n \to \infty$. Then

$$\frac{K_i(r_n u_i(n))}{R(r_n)} \to \Gamma_i(u_i) = c_i^+ u_i^\gamma \mathbf{1}_{[u_i \geq 0]} - c_i^- |u_i|^\gamma \mathbf{1}_{[u_i < 0]}, \qquad c_i^\pm = \left(\frac{b_i^\pm}{a_i^\pm}\right)^{1/\tau}, \ \gamma = \frac{\sigma}{\tau}. \tag{19}$$

Hence $\mathbf{w}_n := K(r_n \mathbf{u}_n)/R(r_n) \to \Gamma(\mathbf{u}) =: \mathbf{w}$, and (18) gives

$$\frac{\psi_0(r_n \mathbf{u}_n)}{S(r_n)} = \frac{\varphi(K(r_n \mathbf{u}_n))}{T(R(r_n))} = \frac{\varphi(R(r_n)\mathbf{w}_n)}{T(R(r_n))} \to n_D^\tau(\mathbf{w}) = n_D^\tau(\Gamma(\mathbf{u})) = n_Q^\sigma(\mathbf{u}). \tag{20}$$

Now let $v_n \to v \in \mathbb{R}$. Then $y_n = K_i(r_n v_n)/R(r_n) \to \Gamma_i(v)$ by (19). Hence for $t_n = R(r_n)$

$$S(r_n) = T(t_n), \quad \psi_i(r_n v_n) - \Psi_i(r_n v_n) = o(S(r_n)), \quad \varphi_i(t_n y_n) - \Phi(t_n y_n) = o(T(t_n))$$

by Lemma 1, (15) and (16). So limit relation (20) also holds for ψ: $\psi \sim S(n_Q)$. The level sets $\{g > e^{-r}\} = \{\psi < r\}$ then have asymptotic shape Q. □

4 Heavy Tails to Heavy Tails

Heavy-tailed distributions have a simple asymptotic theory. There is a nice description of the asymptotic structure. For densities $f \in \mathscr{F}_\lambda$, the asymptotic structure is described by the homogeneous limit function $h = 1/n_D^{\lambda+d}$ in (4), and hence by the shape D and the parameter λ for given dimension d. Figures 2 and 3 below show what happens to the densities and their level sets if we impose Student t margins with μ degrees of freedom on a bivariate spherical Student t density with λ degrees of freedom. The asymptotic shape of the level sets of the new density g is revealed in the limit function $1/n_Q^{\mu+d}$ for g, see (4). Note that the dramatic change in the shape of the level sets for the limit excess densities (which become infinite on the coordinate planes if the tail index λ is decreased, and zero if it is increased) is not visible in the densities themselves even for the large change in λ from 1 to 5 in the figures below.

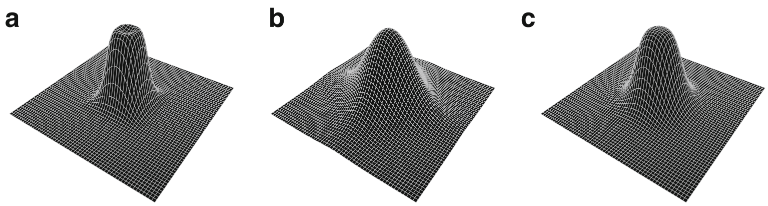


Fig. 2 Bivariate meta densities with standard Student t margins (with μ degrees of freedom) based on the spherical t distribution (with λ degrees of freedom). (**a**) $\lambda = 1, \mu = 5$. (**b**) $\lambda = \mu = 1$. (**c**) $\lambda = 5, \mu = 1$

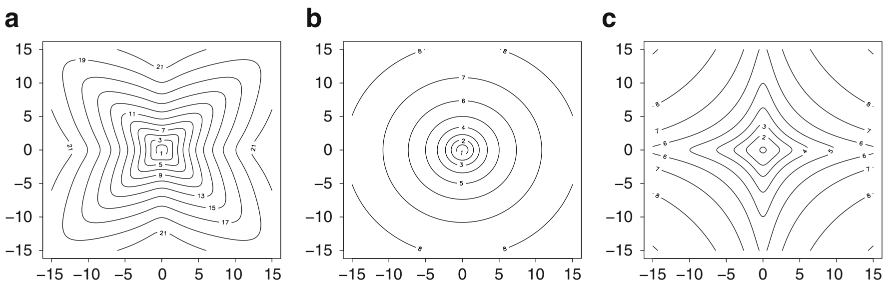


Fig. 3 Level sets of bivariate meta densities with standard Student t margins (with μ degrees of freedom) based on the spherical t distribution (with λ degrees of freedom). Levels are powers of 10^{-1}. (**a**) $\lambda = 1, \mu = 5$. (**b**) $\lambda = \mu = 1$. (**c**) $\lambda = 5, \mu = 1$

Let us first give an overview of the theory. In general, the asymptotic structure is described by a homogeneous measure ρ on $\mathbb{R}^d\backslash\{\mathbf{0}\}$. If $\mathbf{Z}$ with df F lies in the domain of attraction of ρ (see Definition 1 below) then ρ determines the balance conditions of the marginal tails. There is a function $T \in \mathscr{R}_{-\lambda}$ such that

$$F_i(-t)/T(t) \to a_i^-, \qquad (1-F_i(t))/T(t) \to a_i^+, \qquad t \to \infty,\ i = 1,\dots,d, \tag{21}$$

where the non-negative constants $a_i^\pm$ are defined by for $t > 0$ and $i = 1,\dots,d$

$$\rho_i(-\infty,-t] = \rho\{w_i \le -t\} = a_i^-/t^\lambda, \qquad \rho_i[t,\infty) = \rho\{w_i \ge t\} = a_i^+/t^\lambda. \tag{22}$$

These balance conditions not only hold for the components of the vector $\mathbf{Z}$ but for any non-trivial linear combination $Y = \eta\mathbf{Z} = a_1Z_1 + \cdots + a_dZ_d$:

$$\mathbb{P}\{Y \ge t\}/T(t) \to \rho\{\eta \ge 1\}, \qquad t \to \infty, \tag{23}$$

as will be established below. For $f \in \mathscr{F}_\lambda$, the margins f_i satisfy similar balance conditions. Here one may choose $T(t) = t^d f_*(t)$, where f_* is the generator of f. Again, see Proposition 2 below, any non-trivial linear combination $Y = \eta\mathbf{Z}$ has a continuous density f_0 which satisfies

$$f_0(t) \sim \rho\{\eta \ge 1\}\lambda T(t)/t, \qquad t \to \infty. \tag{24}$$

The condition that D contains the origin ensures that $\rho\{\eta \ge 1\}$ is positive, and so are the $2d$ balance constants $a_i^\pm$.

One of the attractive features of this asymptotic theory is that it is geometric. One can first determine the limit ρ and then choose the coordinates. This geometric point of view has an unexpected consequence. If in the bivariate case the measure ρ lives on two lines through the origin then the components of the vector $\mathbf{Z}$ are asymptotically independent if one chooses these lines as coordinate axes; but if one chooses two other lines as the axes, then ρ lives on two lines $v = au$ and $u = bv$ with a, b non-zero and the components of $\mathbf{Z}$ are mixed comonotonic.

There is another reason why the asymptotic theory for heavy-tailed distributions is so rich. There is a close link to multivariate EVT. The non-linear projection

$$\mathbf{z} \mapsto \mathbf{z}^+ = (z_1 \vee 0, \dots, z_d \vee 0)$$

maps $\mathbb{R}^d$ onto $[0,\infty)^d$. The image ρ^+ of the homogeneous limit measure ρ under this projection is the exponent measure of the max-stable limit distribution H for the vector $\mathbf{Z}$:

$$F^n(t_n\mathbf{w}) \to H(\mathbf{w}) \qquad n \to \infty, \qquad T(t_n) = 1/n.$$

The exponent measure ρ^+ determines ρ on $(0,\infty)^d$. By an appropriate sign change, replacing $\mathbf{Z}$ by $\Delta(\mathbf{Z})$ for a diagonal matrix with entries ± 1, one can determine ρ on the other $2^d - 1$ orthants. The 2^d limits for the coordinatewise extremes, maxima and minima, determine ρ. (Mass on coordinate planes will show up in the lower dimensional margins.) See [1], Sect. 17.3. This link allows us to use the *invariance principle* of multivariate EVT. If one applies a coordinatewise strictly increasing continuous transformation of $\mathbb{R}^d$ onto $\mathbb{R}^d$ which transforms the margins of F into univariate dfs G_i whose tails satisfy a balance condition

$$G_i(-s)/S(s) \to b_i^-, \qquad (1-G_i(s))/S(s) \to b_i^+, \qquad s \to \infty,\ i = 1, \dots, d, \tag{25}$$

for $S \in \mathscr{R}_{-\mu}$ with $\mu > 0$, and if all $4d$ balance constants $a_i^{\pm}$ and $b_i^{\pm}$ are positive, then the vector $\mathbf{X}$ with the meta df G with margins G_i based on F lies in the domain of attraction of a max-stable limit law, as do the $2^d - 1$ sign-changed vectors $\Delta(\mathbf{X})$. The exponent measures are related by a power transformation; the df G lies in the domain of attraction of a homogeneous measure σ and we may write $\rho = \Gamma(\sigma)$ where $\Gamma = (\Gamma_1, \dots, \Gamma_d)$ is a coordinatewise transformation whose margins Γ_i are determined by the margins σ_i and ρ_i via $\rho_i = \Gamma_i(\sigma_i)$. We find, cf (19) with τ replaced by $-\lambda$:

$$\frac{b_i^+}{s^\mu} = \sigma_i[s,\infty) = \rho_i[t,\infty) = \frac{a_i^+}{t^\lambda} \Rightarrow t = \Gamma_i(s) = c_i^+ s^\gamma,\ \ c_i^+ = \left(\frac{a_i^+}{b_i^+}\right)^{1/\lambda},\ \ \gamma = \mu/\lambda.$$

The limiting Poisson point processes for the sample clouds from the dfs F and G are linked by Γ.

4.1 *Sample Clouds*

Let us first say what it means that a probability distribution on $\mathbb{R}^d$ lies in the domain of attraction of a homogeneous measure ρ.

Definition 1. A measure ρ on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ is *homogeneous of order* $-\lambda$ and we write $\rho \in \mathscr{H}_\lambda$ if $0 < \rho(B^c) < \infty$ and if ρ satisfies (1) for all Borel sets A in $\mathbb{R}^d \setminus \{\mathbf{0}\}$. A vector $\mathbf{Z}$ with probability distribution π and df F lies in the domain of attraction of this homogeneous measure ρ and we write $\mathbf{Z} \in \mathscr{A}(\rho)$ or $F \in \mathscr{A}(\rho)$ if $p(r) := \mathbb{P}\{\|\mathbf{Z}\| > r\}$ is positive for all $r > 0$ and

$$\rho_r := \gamma_r^{-1}(\pi)/p(r) \to \rho \qquad \text{weakly on } \epsilon B^c, \qquad r \to \infty,\ \epsilon > 0, \tag{26}$$

where γ_r is the scalar expansion $\gamma_r : \mathbf{z} \mapsto r\mathbf{z}$, and hence $\gamma_r^{-1}(\pi)$ is the distribution of $\mathbf{Z}/r$.

Weak convergence in (26) implies for $Y = \eta\mathbf{Z}$ with df F_0 that

$$\mathbb{P}\{Y \geq rs\}/p(r) \to \rho\{\eta \geq s\}/\rho(B^c) = c_0\rho\{\eta \geq 1\}/s^\lambda, \qquad r \to \infty,\ s > 0$$

with $c_0 = \rho(B^c) = 1$. If $\rho\{\eta \geq 1\} = a > 0$ then $(1 - F_0(rs))/(1 - F_0(r)) \to 1/s^\lambda$ and $1 - F_0(r) \sim ap(r)$ for $r \to \infty$ and $s > 0$. Hence, $1 - F_0$ and p vary regularly with exponent $-\lambda$. This establishes (23) and hence (22).

Theorem 3. *Let F be a continuous df in $\mathscr{A}(\rho)$ for a measure $\rho \in \mathscr{H}_\lambda$. Assume (21) with $a_i^\pm > 0$. Let $S \in \mathscr{R}_{-\mu}$ and let $G_1, \ldots, G_d$ be strictly increasing continuous univariate dfs which satisfy (25) with $b_i^\pm > 0$. Let G be the meta df with margins G_i based on F. Then $G \in \mathscr{A}(\sigma)$ where $\sigma \in \mathscr{H}_\mu$.*

Let $\{\mathbf{W}_1, \mathbf{W}_2, \ldots\}$ be the limiting Poisson point process (with mean measure ρ) for the sample clouds from F. Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be independent observations from the meta df G. Let $S(s_n) = 1/n$. Then

$$N_n = \{\mathbf{X}_1/s_n, \ldots, \mathbf{X}_n/s_n\} \Rightarrow N = \{\Gamma^{-1}(\mathbf{W}_1), \Gamma^{-1}(\mathbf{W}_2), \ldots\},$$

where Γ is the coordinatewise semipower transformation in (27).

Proof. This follows from the invariance principle of EVT since the mean measures satisfy $\rho = \Gamma(\sigma)$; see above. □

A more direct proof runs along the lines of the proof of Theorem 1. Regular variation of the tails transforms the relation $\mathbf{z} = K(\mathbf{u})$ into the relation $\mathbf{w} = \Gamma(\mathbf{u})$ since $K(r\mathbf{u})/R(r) \to \Gamma(\mathbf{u})$ gives with the notation in (10) and (12):

$$\Gamma = \Lambda_{\mathbf{a}}^{1/\lambda} J^{\mu/\lambda} \Lambda_{\mathbf{b}}^{-1/\mu} = \Lambda_{\mathbf{c}} J^\gamma, \qquad \gamma = \mu/\lambda, \qquad c_i^\pm = (a_i^\pm / b_i^\pm)^{1/\lambda}. \quad (27)$$

The transformation Γ is homogeneous of degree γ, $\Gamma(r\mathbf{u}) = r^\gamma \Gamma(\mathbf{u})$. Hence

$$n_D(\mathbf{w}) = n_D(\Gamma(\mathbf{u})) = n_Q^\gamma(\mathbf{u}), \qquad Q = \Gamma^{-1}(D). \quad (28)$$

4.2 *Level Sets and Densities*

The theory for the transformation of the level sets of densities for heavy tails is similar to the theory for light tails. Both are based on the limit relation $\dfrac{K(r\mathbf{u})}{R(r)} \to \Gamma(\mathbf{u})$, see (27), resulting from the regular variation of functions associated with the margins. There are two differences. For heavy tails the contribution of the Jacobian is not negligible, far from it, and for heavy tails the tail of the density generator f_* together with the shape D determines the tails of the marginal densities f_i. (The margins inherit the balance conditions from the intensity $1/n_D^{\lambda+d}$, and the slowly varying component from f_*.)

If ρ has density $h(\mathbf{w}) = 1/n_D^{\lambda+d}(\mathbf{w})$ then $\sigma = \Gamma^{-1}(\rho)$ with Γ in (27) has density

$$k(\mathbf{u}) = \frac{\prod_i \Gamma_i'(u_i)}{n_D^{\lambda+d}(\Gamma(\mathbf{u}))} = \frac{\gamma^d \prod_i |u_i|^{\gamma-1}(c_i^- \mathbf{1}_{[u_i<0]} + c_i^+ \mathbf{1}_{[u_i \geq 0]})}{n_{\Gamma^{-1}(D)}^{\gamma\lambda+\gamma d}(\mathbf{u})} = \frac{1}{n_Q^{\mu+d}(\mathbf{u})}, \quad (29)$$

where $c_i^{\pm} = \left(a_i^{\pm}/b_i^{\pm}\right)^{1/\lambda}$, $i = 1, \dots, d$. Here $\gamma = \mu/\lambda$ and Q is an open set in $\mathbb{R}^d$, which may be unbounded (if $\gamma < 1$), which need not contain the origin (if $\gamma > 1$), and which is bounded and contains the origin for $\gamma = 1$, but need not have a continuous boundary. If $\gamma = 1$ and $c_i^- \neq c_i^+$ the intensity k jumps by a factor $c_i^-/c_i^+ \neq 1$ on crossing over from $u_i > 0$ to $u_i < 0$.

The heavy tailed meta density g with margins g_i based on the density $f \in \mathscr{F}_\lambda$ is linked to f by the symmetric relation (8). Write $f_*(t) = T(t)/t^d$ with $T \in \mathscr{R}_{-\lambda}$ for the density generator of f and for $S \in \mathscr{R}_{-\mu}$ define $g_*(r) = S(r)/r^d$. Let r and $t = R(r)$ be linked by $T(t) = S(r)$. Then the limit relation we want to establish reads:

$$\frac{g(r\mathbf{u})}{S(r)/r^d} = \frac{f(t\mathbf{w})}{T(t)/t^d}\frac{r^d}{t^d}\prod_i \frac{g_i(ru_i)}{f_i(tw_i)} \to \frac{\prod_i \Gamma_i'(u_i)}{n_D^{\lambda+d}(\Gamma(\mathbf{u}))} =: k(\mathbf{u}). \tag{30}$$

So assume $\mathbf{Z}$ has density $f \in \mathscr{F}_\lambda$. From Theorem 3, the meta density g lies in the domain of attraction of the homogeneous measure $\sigma \in \mathscr{H}_\mu$. The relation $\rho = \Gamma(\sigma)$ yields the density k of σ in (29). For $\mu < \lambda$, the derivatives Γ_i' are negative powers. The intensity k of the limiting Poisson point process becomes infinite along all coordinate planes. We shall show that even in this case the limit relation (30) holds in $\mathbf{L}^1$ on the complement of centered balls and uniformly on compact sets of $\mathbb{R}^d \setminus \{\mathbf{0}\}$ in the sense that $g(s_n\mathbf{u}_n)/g_*(s_n) \to k(\mathbf{u}) \in [0, \infty]$ holds when $\mathbf{u}_n \to \mathbf{u} \neq \mathbf{0}$ and $s_n \to \infty$. First we show that the margins f_i are well-behaved.

Proposition 2. *Let $Y = \eta\mathbf{Z}$ for a non-trivial linear functional η with df F_0. Then Y has a continuous density f_0 which is asymptotic to $\lambda(1 - F_0(t))/t$ for $t \to \infty$.*

Proof. Think of η as the vertical coordinate and write $\mathbf{z} = (\mathbf{x}, y)$, where $\mathbf{x}$ denotes the horizontal part of the vector. Assume $f = f_*(n_D)$. Then

$$f_0(y) = \int f_*(n_D(\mathbf{x}, y))d\mathbf{x} = y^{d-1}\int f_*(yn_D(\mathbf{u}, 1))d\mathbf{u} = y^{d-1}J(y), \qquad y > 0,$$

by homogeneity of the gauge function n_D. The function $y \mapsto J(y)$ is decreasing since $y \mapsto f_*(ya)$ is for $a \geq 0$. The function $A(t) = (1 - F_0(t))/t^{d-1}$ has derivative $-f_0(t)/t^{d-1} - (d-1)(1 - F_0(t))/t^d$. Since $1 - F_0 \in \mathscr{R}_{-\lambda}$ by the arguments in the previous section, the second term varies regularly, and hence so does its integral $I(t) \sim ((d-1)/(d-1+\lambda))A(t)$. The function $A(t) + I(t) \in \mathscr{R}_{-(\lambda+d-1)}$ has a monotone derivative $-f_0(t)/t^{d-1}$, and $f_0(t)/t^{d-1}$ then varies regularly by the Monotone Density Theorem in [6] (where the case of slow variation has to be excluded!). Regular variation of f_0 gives the desired asymptotic equality. □

Proposition 3. *Let the df F have density f in $\mathscr{F}_\lambda$. The function f satisfies* (4) *pointwise on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and in $\mathbf{L}^1$ on the complement of centered balls ϵB, $\epsilon > 0$. The margins F_i satisfy* (21)*, where $T(t) \sim t^d f_*(t)$ lies in $\mathscr{R}_{-\lambda}$, and $a_i^{\pm}$ are positive constants depending on D and λ.*

Proof. Regular variation of f_* follows by writing this relation out for $(st, \mathbf{w})$ and $(t, s\mathbf{w})$ and using the homogeneity of n_D. This implies regular variation of T. The $2d$ constants $a_i^{\pm}$ in (22) are positive since D contains the origin. Let $\mathbf{Z}$ have df F. Then $\mathbf{W} = \mathbf{Z}/t$ has density $g_t(\mathbf{w}) = t^d f(t\mathbf{w})$ and $g_t \sim T(t)h$ by (4) implies that $F_i(-st) = \mathbb{P}\{W_i < -s\} \sim T(t)\rho\{w_i \leq -s\} \sim a_i^- T(ts)$ for $t \to \infty$. This gives (21). □

Theorem 4. *Let $f \in \mathscr{F}_\lambda$. Let $S \in \mathscr{R}_{-\mu}$ for some $\mu > 0$ and let $g_1, \ldots, g_d$ be continuous positive densities such that*

$$g_i(-s) \sim b_i^- \mu S(s)/s, \qquad g_i(s) \sim b_i^+ \mu S(s)/s, \qquad s \to \infty,\ i = 1, \ldots, d$$

with $b_i^{\pm} > 0$. The meta density g with margins g_i based on f is continuous. Set $g_*(s) = S(s)/s^d$. Then $g(s\mathbf{u})/g_*(s) \to k(\mathbf{u}) := 1/n_Q^{\mu+d}(\mathbf{u})$. Here Q is an open set. Convergence holds uniformly on compact sets which are disjoint from the coordinate planes and in $\mathbf{L}^1$ on the complement of centered balls.

If $\mu > \lambda$, convergence holds uniformly on compact sets in $\mathbb{R}^d \setminus \{\mathbf{0}\}$. If $\mu \neq \lambda$ then $k_{s_n}(\mathbf{u}_n) \to k(\mathbf{u})$ for $s_n \to \infty$ and $\mathbf{u}_n \to \mathbf{u} \neq \mathbf{0}$, where the limit is infinite if $\mu < \lambda$ and $\mathbf{u}$ lies on a coordinate plane. If $\mu = \lambda$ and the balance in each coordinate is preserved, $b_i^-/b_i^+ = a_i^-/a_i^+$ for $i = 1, \ldots, d$, then Γ in (27) is a linear map and $Q = c\Gamma^{-1}(D)$ with $c > 0$. If $\mu = \lambda$ and the balance condition for the index i is violated there is a jump discontinuity over the corresponding coordinate plane by a factor $\neq 1$.

Proof. We may and shall assume that T and S are strictly decreasing, continuous and map $(0, \infty)$ onto itself. Write $S = T \circ R$. Then $R \in \mathscr{R}_\gamma$ with $\gamma = \mu/\lambda$ is continuous and strictly increasing. Let $t_n = R(s_n) \to \infty$. Let $\mathbf{u}_n \to \mathbf{u}$ and set $\mathbf{w}_n = K(s_n\mathbf{u}_n)/t_n$. Then in (30) the factor $f(t_n\mathbf{w}_n)/(T(t_n)/t_n^d)$ tends to $h(\Gamma(\mathbf{u}))$ uniformly on compact sets in $\mathbb{R}^d \setminus \{\mathbf{0}\}$ by Proposition (3) since $f_*(t) \sim T(t)/t^d$ and $\mathbf{w}_n = K(r_n\mathbf{u}_n)/R(r_n) \to \Gamma(\mathbf{u})$, see (27). Convergence of the product on the right is less obvious. Let $i \in \{1, \ldots, d\}$. Write $Q_n = g_i(s_nu_n)/f_i(t_nw_n)$. Claim:

$$u_n \to u \neq 0 \quad \Rightarrow \quad Q_n s_n/t_n \sim \mu w_n/\lambda u_n \to \gamma \Gamma_i(u)/u,$$

and the left side converges to zero for $u_n \to 0$ provided $\gamma = \mu/\lambda > 1$. The asymptotic relations between the tails of the density and the df give the asymptotic equality provided $s_nu_n \to \infty$ since $G_i(s_nu_n) = F_i(t_nw_n)$. Then $w_n = K_n(s_nu_n)/t_n \to \Gamma_i(u)$ gives the limit for $u \neq 0$. Now assume $\gamma > 1$, We have to prove $Q_ns_n/t_n \to 0$ for $u_n \to 0$. First assume $s_nu_n \to \infty$. Then $K_i(s) \sim c_i^+ R(s)$ and Potter's bounds, see [6], give $w_n = K_i(s_nu_n)/R(s_n) \leq 2c_i^+ u_n^{(1+\gamma)/2}$ for $s_nu_n \geq M_0$. A similar bound holds for $s_nu_n \leq -M_0$. It is possible that $u_n = 0$ and $w_n \neq 0$ if $K_i(0) \neq 0$. But K_i is a homeomorphism. Hence if $|s_nu_n| \leq M_0$ then A_n is bounded and $R \in \mathscr{R}_\gamma$ with $\gamma > 1$ implies $s_n/t_n = s_n/R(s_n) \to 0$ for $s_n \to \infty$ and hence in this case also $Q_ns_n/t_n \to 0$. By symmetry, a similar result holds for $\mu < \lambda$, with the limit value ∞. $\mathbf{L}^1$ convergence on the complement of centered balls follows from the almost sure convergence of the densities and the weak convergence of the measures by Fatou's Lemma as in the proof of Scheffé's Theorem. □

The change in the intensity on decreasing the parameter λ is dramatic. The spikes of the new level sets may perhaps be interpreted as an indication of extra asymptotic independence for lighter tails. Student densities with spherical level sets tend to complete independence of the coordinates as the exponent λ goes to ∞, and the Student distribution converges to the Gaussian distribution.

5 Heavy Tails to Light Tails

The transformation from heavy to light tails gives new and unexpected results. If we import the copula of the Student t density $f = f_*(n_D)$ with the function $f_*(r) = \frac{c}{(\lambda + r^2)^{(\lambda+d)/2}} \sim \frac{c}{r^{\lambda+d}}$ and D a centered ellipsoid into a density with standard Gaussian margins, the resulting density is continuous and its level sets have an asymptotic shape D_λ, whose boundary is given by a quadratic expression. In Fig. 4 this shape is clearly visible in a sample cloud of a ten thousand points. The shape depends only on the parameter λ. All other information is lost in the transformation from heavy to light tails. All $f \in \mathscr{F}_\lambda$ yield the same shape D_λ. We restrict ourselves here to citing the corresponding theorem from [2], where the proof may be found and a discussion.

Theorem 5. *Suppose f is a density on $\mathbb{R}^d$ in $\mathscr{F}_\lambda$ for some $\lambda > 0$, and g_0 is a continuous, positive, symmetric density on $\mathbb{R}$ asymptotic to a von Mises function $e^{-\psi}$ with $\psi \in RV_\theta$ for some $\theta > 0$. Let $\psi(r_n) = \log n$, and let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be independent observations from the meta density g based on f with equal margins g_0.*

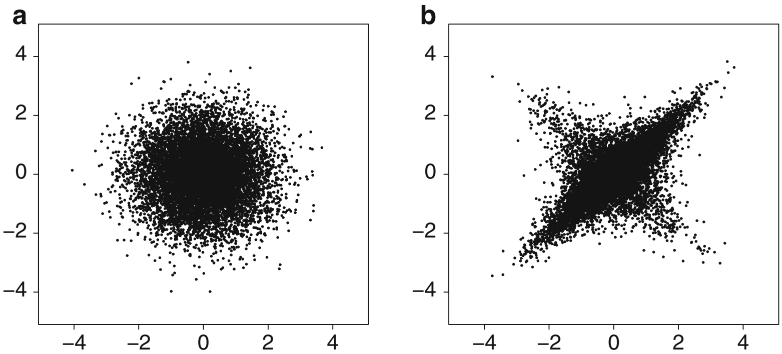


Fig. 4 Bivariate sample clouds of 10,000 points from (**a**) the standard normal distribution, and (**b**) the meta-Cauchy distribution with standard normal margins based on the Cauchy density with level sets shaped like the ellipse $5x^2 + 6xy + 5y^2 = 1$

Set $c(r) = g(r, \dots, r)$. *Then the level sets* $\{g > c(r)\}$, *scaled by* r, *converge to the limit set* $D_{\lambda,\theta} = \{\chi < \lambda\}$ *where*

$$\chi(\mathbf{u}) = (\lambda + d)\|\mathbf{u}\|_\infty^\theta - (|u_1|^\theta + \cdots + |u_d|^\theta). \tag{31}$$

The sample clouds $N_n = \{\mathbf{X}_1/r_n, \dots, \mathbf{X}_n/r_n\}$ *from* g *converge onto the closure of* $D_{\lambda,\theta}$.

The shape of the limit set $D_{\lambda,\theta} \in \mathscr{D}$ varies continuously in λ. For fixed θ the shape $D_{\lambda,\theta}$ reflects the change in the copula as the tail parameter λ varies over $(0, \infty)$ for $f \in \mathscr{F}_\lambda$ with the shape D of the level sets fixed. The good behaviour of the function $\lambda \mapsto D_{\lambda,\theta}$ unfortunately is unstable. One can alter the bivariate circle symmetric Cauchy density without affecting the limit measure $\rho \in \mathscr{H}_1$ so that the sample clouds from the meta density with Gaussian margins based on the perturbed density converge onto the diagonal cross $E_\times$, the union of the two diagonals of the square $[-1, 1]^2$. See [3] for details.

6 Light Tails to Heavy Tails

Assume the level sets of a light-tailed density g may be scaled to converge to a set $D \in \mathscr{D}$, or more generally, assume the sample clouds from the light-tailed distribution dG converge onto the closure of D. Turn to the meta distribution dF with heavy-tailed margins. Can one describe the tails of dF asymptotically by a homogeneous measure ρ? Is there a limiting point process N for the sample clouds?

The limit shape of the level sets of the light-tailed meta densities in the previous section gives no information on the asymptotic shape of the original heavy-tailed density. In the transition from heavy to light tails, information about the limit shape is blurred to such an extent that one cannot go back from the asymptotic shape of the level sets for light-tailed density to the asymptotic shape for heavy-tailed one. Yet we do have some results. As in EVT all that matters is the shape of D in the vertices of the circumscribed coordinate box.

6.1 *Asymptotic Independence*

Theorem 6. *Let* $\mathbf{X}$ *have a positive continuous density* g. *Suppose there exist* $c_n > 0$ *and* $0 < r_n \to \infty$ *such that* $c_{n+1}/c_n \to 0$, $r_{n+1}/r_n \to 1$ *and* $\{g > c_n\}/r_n \to D \in \mathscr{D}$. *Let* $F_1, \dots, F_d$ *be strictly increasing continuous dfs such that*

$$F_i(-t) \sim a_i^- T(t), \qquad 1 - F_i(t) \sim a_i^+ T(t), \qquad t \to \infty$$

for positive constants $a_i^\pm$ *and* $T \in \mathscr{R}_{-\lambda}$, $\lambda > 0$. *Let* $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ *be independent observations from the meta df* F *with margins* F_i *based on the density* g. *Suppose*

D is convex with a C^1 boundary (in each boundary point there is a unique tangent plane). Choose $a_n > 0$ such that $nT(a_n) \to 1$. Then the sample clouds converge:

$$N_n = \{\mathbf{Z}_1/a_n, \ldots, \mathbf{Z}_n/a_n\} \Rightarrow N \quad \text{weakly on } \epsilon B^c, \qquad \epsilon > 0.$$

The limit N is a Poisson point process with mean measure ρ which lives on the $2d$ halfaxes. It is determined by

$$\rho\{x_i < -t\} = a_i^-/t^\lambda, \qquad \rho\{x_i > t\} = a_i^+/t^\lambda, \qquad t > 0.$$

Proof. The condition on the shape of D implies asymptotic independence of all coordinates, both positive and negative. See [10] and [5]. □

The charm of the condition in the theorem above is that it is geometric. Any two linear combinations of the coordinates $X = \xi\mathbf{X}$ and $Y = \eta\mathbf{X}$ are asymptotically independent provided the linear functionals ξ and η are linearly independent.

6.2 *Asymptotic Dependence and Homothetic Densities*

We now turn to bivariate distributions. For asymptotic independence it suffices that the limit set $\text{cl}(D)$ of the sample clouds does not contain the coordinatewise supremum of the points in D: $\sup D \notin \text{cl}(D)$. Such sets D are called *blunt*. This condition ensures that for large sample clouds the maximal horizontal and the maximal vertical coordinate come from sample points in disjoint subsets. See [5] for details. Now suppose D is the triangle with vertices $(1, 1), (-1, 0), (0, -1)$. This set certainly is not blunt. Yet there exist continuous positive densities g with light tails and convex level sets which, properly scaled, converge to D such that the vector $\mathbf{X}$ with density g has asymptotically independent components. (The level sets are triangles tD with a tip of size $\sqrt{t}$ cut off to blunt them. See [5] for details.) For asymptotic dependence we need strong conditions. So assume $g \sim g_*(n_D)$ where g_* is a von Mises function. We focus on the positive quadrant. So one could restrict D to the positive quadrant or assume that D and g are invariant under sign changes, of assume that the behaviour of D outside the positive quadrant is harmless. We shall do the latter.

For $\tilde{a}, a \in [0, 1]$ with $\tilde{a}a < 1$ define $\mathscr{D}_{\tilde{a},a}$ as the set of all $D \in \mathscr{D}_2$ whose closure intersects the lines $x_1 = 1$ and $x_2 = 1$ only in the one point $\mathbf{e} = (1, 1)$, and whose closure does not contain the point $\inf D$. Moreover in the point $\mathbf{e}$ the set D has tangents with slope $\tilde{a}$ and $1/a$. In geometric terms this condition means that the sets $n(D - \mathbf{e})$ converge to the open sector

$$C = \{\mathbf{x} \in (-\infty, 0)^2 \mid x_1/a < x_2 < \tilde{a}x_1\}. \tag{32}$$

In analytic terms the functions $t \mapsto n_D(t,1)$ and $t \mapsto n_D(1,t)$ have a left derivative in $t = 1$. We shall see that the heavy-tailed meta density then lies in the domain of a homogeneous measure ρ with a scaled *power density* on the first quadrant:

$$r(x,y) = c_0 r_0(a_0 x, b_0 y), \quad r_0(x,y) = \begin{cases} |x|^{\tilde{\alpha}-1}/|y|^{\tilde{\beta}+1}, & |x| \le |y|, \quad 0 < \tilde{\alpha} = \tilde{\beta} - \lambda; \\ |y|^{\alpha-1}/|x|^{\beta+1}, & |y| \le |x|, \quad 0 < \alpha = \beta - \lambda. \end{cases} \tag{33}$$

Power densities are associated with exponential densities $h(u,v)$ on the plane which satisfy

$$h(u+t, v+t) = e^{-t} h(u,v), \qquad u, v \in \mathbb{R}, \qquad \{h > 1\} = C, \tag{34}$$

where C is an open sector in the negative quadrant.

Theorem 7. *Let $\lambda > 0$ and $0 \le a \le \tilde{a} \le 1$ with $\tilde{a}a < 1$. Let f_λ denote the standard Student t density with λ degrees of freedom. Let $g \sim g_*(n_D)$ be a continuous positive density on $\mathbb{R}^2$ where g_* is a von Mies function and $D \in \mathscr{D}_{\tilde{a}a}$. The meta density f with margins f_λ based on g is continuous and positive. It lies in the domain of the measure $\rho \in \mathscr{H}_\lambda$ whose margins ρ_1 and ρ_2 have density $\lambda/|t|^{\lambda+1}$ on $\mathbb{R}\backslash\{0\}$. If $\tilde{a}a = 0$ then ρ lives on the two axes. If $\tilde{a}a$ is positive then ρ lives on the two negative halfaxes and on the positive quadrant, where it has the power density r in (33).*

Let L denote the ray through the point $(1,b)$ with $b = (\tilde{a}/a)^{1/\lambda} \ge 1$. The level set D_λ of r containing $(1,b)$ as a boundary point is bounded by the power curves $y = bx^{\tilde{\mu}}$ above L and $y = bx^{1/\mu}$ below L with $\mu = (1-\alpha)/(1+\beta)$ and $\tilde{\mu}$ defined similarly. The constants α, β and μ are determined by a and λ:

$$\alpha = \frac{a\lambda}{1-a}, \qquad \beta = \frac{\lambda}{1-a}, \qquad \mu = \frac{\alpha - 1}{\beta + 1} = \frac{a\lambda - (1-a)}{\lambda + (1-a)}, \qquad b = \left(\frac{\tilde{a}}{a}\right)^{1/\lambda}.$$

The same expressions define $\tilde{\alpha}, \tilde{\beta}, \tilde{\mu}$ in terms of $\tilde{a}$ and λ. If $\tilde{a} = 1$ then $\tilde{\mu} = 1$ and the upper boundary of D_λ is the line segment from $(0,0)$ to $(1,b)$ along L, and r vanishes above L. The sets D_λ satisfy $D_{\lambda_1} \supset D_{\lambda_2}$ for $0 < \lambda_1 < \lambda_2$.

Proof. The proof proceeds in three steps. (i) The limit relation (4) holds for g if we replace the scaling by affine normalizations. The limit function h is the density of the exponent measure σ on $\mathbb{R}^2$ associated with a max-stable limit law with Gumbel margins. (ii) By multivariate EVT, the exponent measure ρ of the meta df is the image $K(\sigma)$ under the coordinatewise exponential map

$$K : (u,v) \mapsto (x,y) = (e^{u/\lambda}, e^{v/\lambda}) \in (0,\infty)^2, \qquad (u,v) = \lambda(\log x, \log y), \tag{35}$$

with the coordinates scaled by a diagonal linear transformation to ensure that $\rho\{y \ge 1\} = \rho\{x \ge 1\} = 1$. (iii) A computation gives the results.

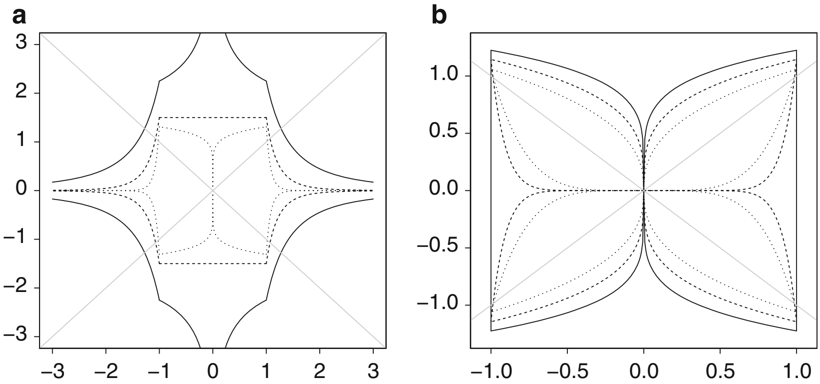


Fig. 5 Level sets of the density r in (33) with parameters $\tilde{a} = 1/2$, $a = 1/3$ and λ. The values of λ are $\{1/2, 1, 3/2\}$ in Panel (**a**) and $\{2, 3, 7\}$ in Panel (**b**) corresponding to *solid, dashed* and *dotted curves,* respectively. The original density g in Theorem 7 now is assumed symmetric for sign changes. Transformations between successive level sets are described in Sect. 4

Let $a(t)$ be the scale function of the von Mises function g_*. It is known that $a(t)/t \to 0$; see e.g. [9]. The set $D_0 = D - (1,1)$, scaled by $t/a(t)$, converges to the open sector C in (32). Hence

$$h_t(\mathbf{w}) = \frac{g((t,t) + a(t)\mathbf{w})}{g(t,t)} \to h(\mathbf{w}), \qquad t \to \infty. \tag{36}$$

Here h is the exponential function in (34). It is continuous if $\tilde{a} < 1$, and then convergence holds uniformly on compact sets in the plane. If $\tilde{a} = 1$, it vanishes above the diagonal and convergence is uniform on compact sets disjoint from the diagonal. In both cases, convergence holds in $\mathbf{L}^1$ on halfplanes $\{(u,v) \mid c_1 u + c_2 v \geq c\}$ with $c_1, c_2 > 0$ and $c \in \mathbb{R}$. If $a = 0$ then the measure σ on $\mathbb{R}^2$ with density h is infinite on horizontal halfplanes, $\{v \geq 0\}$, and the positive coordinates of the vector (Z_1, Z_2) with density f are asymptotically independent, see [5]. So assume $0 < a \leq \tilde{a}$. Then $\sigma\{v \geq c\}$ and $\sigma\{u \geq c\}$ are finite for all $c \in \mathbb{R}$. In this case, the partial maxima converge in distribution to a max-stable limit vector with Gumbel margins if we apply the coordinatewise affine transformations $(t_n + a(t_n)u, t_n + a(t_n)v)$ for a suitable sequence $t_n \to \infty$, and σ is the exponent measure associated with this max-stable limit law. The coordinatewise maxima from f then converge in distribution to a max-stable limit vector with Fréchet margins e^{-1/t^λ} on $(0,\infty)$. The associated exponent measure ρ satisfies $\rho\{x > t\} = \rho\{y > t\} = 1/t^\lambda$. It is a positive diagonal linear transformation of the measure $\rho_1 = K(\sigma)$ with K in (35). The measure ρ_1 has density $\lambda^2 r_0$, where r_0 is a power function in (33).

Here are some details. Write $h(u,0) = e^{\tilde{p}u}\mathbf{1}_{[u\leq 0]} + e^{-(1+p)u}\mathbf{1}_{[u>0]}$. Then from (34)

$$h(u,v) = e^{-v}h(u-v,0) = e^{\tilde{p}u-(1+\tilde{p})v}\mathbf{1}_{[u\leq v]} + e^{pv-(1+p)u}\mathbf{1}_{[u>v]}.$$

The sector C is bounded by the lines $\tilde{p}u = (1+\tilde{p})v$ and $pv = (1+p)u$, which gives $1/a = 1 + 1/p$ and a similar expression for $\tilde{a}$. The measure $\rho_1 = K(\sigma)$ has density $\lambda^2 r_0$, where r_0 is the power function in (33) with $\alpha = p\lambda$, $\beta = (1+p)\lambda$, $\tilde{\alpha} = \tilde{p}\lambda$ and $\tilde{\beta} = (1+\tilde{p})\lambda$. The level set $\{r_0 > 1\}$ is bounded by two curves, $y = x^{\tilde{\mu}}$ and $x = y^{\mu}$ which meet at $(1,1)$, with $\mu = \dfrac{\alpha - 1}{\beta + 1} = \dfrac{a\lambda - (1-a)}{\lambda + (1-a)}$ and a similar expression for $\tilde{\mu}$. Then $\rho_1\{y \geq 1\} = \tilde{A} := (1 - a\tilde{a})/\tilde{a}$ and $\rho_1\{x \geq 1\} = A := (1 - a\tilde{a})/a$. Now rescale by $Q = \mathrm{diag}(q, \tilde{q})$ with $q = 1/A^{1/\lambda}$ and $\tilde{q} = 1/\tilde{A}^{1/\lambda}$. Then $\rho\{x \geq 1\} = \rho\{y \geq 1\} = 1$ for $\rho = Q(\rho_1)$ and the positive diagonal maps into the ray L through the point $(1, b)$ with $b = (\tilde{a}/a)^{1/\lambda}$. The level sets of the density r of ρ are scaled copies of the set $D_\lambda \subset (0,\infty)^2$ bounded by the two curves $y = bx^{\tilde{\mu}}$ and $y = bx^{1/\mu}$ which meet at $(1, b)$. □

There is a discontinuity in the description of the asymptotic behaviour when a vanishes. For $a = 0$, the right tangent to D is vertical. Assume $\tilde{a} = 1/2$ and $\lambda = 2$. The exponential density h is well-defined and continuous for $a = 0$ and the limit (36) holds uniformly on compact sets in the plane, but the mass of vertical halfspaces is infinite since $h \equiv e^{-u}$ below the diagonal. For $a = 0$, the level set $\{r_0 > 1\}$ of the power function r_0 is bounded above by a continuous unimodal curve:

$$0 < y < x^{\tilde{\mu}} \mathbf{1}_{[0<x<1]} + x^{1/\mu} \mathbf{1}_{[1 \leq x]}, \qquad \tilde{\mu} = 1/5,\ 1/\mu = -3. \tag{37}$$

The ray L becomes the positive vertical axis as $a \to 0$. The normalized density r yields standard marginal densities $2/t^3$, but r vanishes uniformly on compact subsets of the positive quadrant for $a \to 1$. The information contained in r_0 and h is lost by the normalization, which pulls down the measure onto the one-dimensional axes.

7 Conclusion

The relation between the asymptotic geometric structure of multivariate densities and the copula is not as intuitive as one might hope. As observed in [8] the rate of decrease of the tails plays a crucial role.

This paper presents a systematic investigation of the relation between copula and shape of the level sets as the decay rate of the tails of the density is varied, both for light and heavy tails. The most striking results are the loss of information on shape as one passes from a heavy-tailed density to a light-tailed density, while preserving the copula, and vice versa passing from light to heavy tails – in both cases there is a reduction to a finite dimensional parametric family; and the explosive change in the asymptotic shape of the level sets of heavy-tailed densities as the tail exponent is varied.

The paper compares the asymptotic shape of level sets of two multivariate densities with the same copula but different tails. In the light-tailed case, the shape is stable. We start with a density with level sets whose asymptotic shape is a bounded open star-shaped set with a continuous boundary and which contains the origin. The asymptotic shape for the new density will have the same properties. The new shape is the image of the old shape under a coordinatewise semi-linear power transformation. In the heavy-tailed case, the change is more dramatic. Assume regular variation of the tails. A change in the slowly varying component has no effect on the shape. A change in the balance between the $2d$ marginal tails will have an effect. The new shape is still bounded and contains the origin as interior point, but the boundary is no longer continuous. As a result, the intensity also has discontinuities. If the exponent of regular variation is decreased, the new limit shape is no longer bounded and the new intensity is infinite along the coordinate planes.

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Limit Theorems for Functionals of Higher Order Differences of Brownian Semi-Stationary Processes

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Abstract We present some new asymptotic results for functionals of higher order differences of Brownian semi-stationary processes. In an earlier work [8] we have derived a similar asymptotic theory for first order differences. However, the central limit theorems were valid only for certain values of the smoothness parameter of a Brownian semi-stationary process, and the parameter values which appear in typical applications, e.g. in modeling turbulent flows in physics, were excluded. The main goal of the current paper is the derivation of the asymptotic theory for the whole range of the smoothness parameter by means of using second order differences. We present the law of large numbers for the multipower variation of the second order differences of Brownian semi-stationary processes and show the associated central limit theorem. Finally, we demonstrate some estimation methods for the smoothness parameter of a Brownian semi-stationary process as an application of our probabilistic results.

Keywords Brownian semi-stationary processes • Central limit theorem • Gaussian processes • High frequency observations • Higher order differences • Multipower variation • Stable convergence

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1 Introduction

Brownian semi-stationary processes ($\mathscr{BSS}$) has been originally introduced in [2] for modeling turbulent flows in physics. This class consists of processes $(X_t)_{t\in\mathbb{R}}$ of the form

$$X_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s W(ds) + \int_{-\infty}^{t} q(t-s)a_s ds, \tag{1}$$

where μ is a constant, $g, q : \mathbb{R}_{>0} \to \mathbb{R}$ are memory functions, $(\sigma_s)_{s\in\mathbb{R}}$ is a càdlàg *intermittency* process, $(a_s)_{s\in\mathbb{R}}$ a càdlàg *drift* process and W is the Wiener measure. When $(\sigma_s)_{s\in\mathbb{R}}$ and $(a_s)_{s\in\mathbb{R}}$ are stationary then the process $(X_t)_{t\in\mathbb{R}}$ is also stationary, which explains the name Brownian semi-stationary processes. In the following we concentrate on $\mathscr{BSS}$ models without the drift part (i.e. $a \equiv 0$), but we come back to the original process (1) in Example 1.

The path properties of the process $(X_t)_{t\in\mathbb{R}}$ crucially depend on the behaviour of the weight function g near 0. When $g(x) \simeq x^{\beta}$ (here $g(x) \simeq h(x)$ means that $g(x)/h(x)$ is slowly varying at 0) with $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, X has r-Hölder continuous paths for any $r < \beta + \frac{1}{2}$ and, more importantly, X is not a semimartingale, because g' is not square integrable in the neighborhood of 0 (see e.g. [10] for a detailed study of conditions under which Brownian moving average processes are semimartingales). In the following, whenever $g(x) \simeq x^{\beta}$, the index β is referred to as the *smoothness parameter* of X.

In practice the stochastic process X is observed at high frequency, i.e. the data points $X_{i\Delta_n}$, $i = 0, \ldots, [t/\Delta_n]$ are given, and we are in the framework of *infill asymptotics*, that is $\Delta_n \to 0$. For modeling and for practical applications in physics it is extremely important to infer the integrated powers of intermittency, i.e.

$$\int_0^t |\sigma_s|^p ds, \qquad p > 0,$$

and to estimate the smoothness parameter β. A very powerful instrument for analyzing those estimation problems is the normalized *multipower variation* that is defined as

$$MPV(X, p_1, \ldots, p_k)_t^n = \Delta_n \tau_n^{-p^+} \sum_{i=1}^{[t/\Delta_n]-k+1} |\Delta_i^n X|^{p_1} \cdots |\Delta_{i+k-1}^n X|^{p_k}, \tag{2}$$

where $\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$, $p_1, \ldots, p_k \geq 0$ and $p^+ = \sum_{l=1}^k p_l$, and τ_n is a certain normalizing sequence which depends on the weight function g and n (to be defined later). The concept of multipower variation has been originally introduced in [3] for the semimartingale setting. Power and multipower variation of semimartingales has been intensively studied in numerous papers; see e.g. [3–6, 13, 15, 17, 22] for theory and applications.

However, as mentioned above, $\mathscr{BSS}$ processes of the form (1) typically do not belong to the class of semimartingales. Thus, different probabilistic tools are required to determine the asymptotic behaviour of the multipower variation $MPV(X, p_1, \ldots, p_k)_t^n$ of $\mathscr{BSS}$ processes. In [8] we applied techniques from Malliavin calculus, which has been originally introduced in [18, 19] and [20], to show the consistency, i.e.

$$MPV(X, p_1, \ldots, p_k)_t^n - \rho^n_{p_1,\ldots,p_k} \int_0^t |\sigma_s|^{p^+} ds \xrightarrow{\text{u.c.p.}} 0,$$

where $\rho^n_{p_1,\ldots,p_k}$ is a certain constant and $Y^n \xrightarrow{\text{u.c.p.}} Y$ stands for $\sup_{t\in[0,T]} |Y_t^n - Y_t| \xrightarrow{\mathbb{P}} 0$ (for all $T > 0$). This holds for all smoothness parameters $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, and we proved the associated (stable) central limit theorem for $\beta \in (-\frac{1}{2}, 0)$.

Unfortunately, the restriction to $\beta \in (-\frac{1}{2}, 0)$ in the central limit theorem is not satisfactory for applications as in turbulence we usually have $\beta \in (0, \frac{1}{2})$ at ultra high frequencies. The theoretical reason for this restriction is two-fold: (i) long memory effects which lead to non-normal limits for $\beta \in (\frac{1}{4}, \frac{1}{2})$ and more importantly (ii) a hidden drift in X which leads to an even stronger restriction $\beta \in (-\frac{1}{2}, 0)$.

The main aim of this paper is to overcome both problems by considering multipower variations of higher order differences of $\mathscr{BSS}$ processes. We will show the law of large numbers and prove the associated central limit theorem for all values of the smoothness parameter $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$. Furthermore, we discuss possible extensions to other type of processes. We apply the asymptotic results to estimate the smoothness parameter β of a $\mathscr{BSS}$ process X. Let us mention that the idea of using higher order differences to diminish the long memory effects is not new; we refer to [12, 16] for theoretical results in the Gaussian framework. However, the derivation of the corresponding theory for $\mathscr{BSS}$ processes is more complicated due to their more involved structure.

This paper is organized as follows: in Sect. 2 we introduce our setting and present the main assumptions on the weight function g and the intermittency σ. Section 3 is devoted to limit theorems for the multipower variation of the second order differences of $\mathscr{BSS}$ processes. In Sect. 4 we apply our asymptotic results to derive three estimators (the realised variation ratio, the modified realised variation ratio and the change-of-frequency estimator) for the smoothness parameter. Finally, all proofs are collected in Sect. 5.

2 The Setting and the Main Assumptions

We consider a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ on which we define a $\mathscr{BSS}$ process $X = (X_t)_{t\in\mathbb{R}}$ without a drift as

$$X_t = \mu + \int_{-\infty}^t g(t-s)\sigma_s W(ds), \tag{3}$$

where W is an $\mathbb{F}$-adapted Wiener measure, σ is an $\mathbb{F}$-adapted càdlàg processes and $g \in \mathbb{L}^2(\mathbb{R}_{>0})$. We assume that

$$\int_{-\infty}^{t} g^2(t-s)\sigma_s^2 ds < \infty \quad \text{a.s.}$$

to ensure that $X_t < \infty$ almost surely. We introduce a Gaussian process $G = (G_t)_{t\in\mathbb{R}}$, that is associated to X, as

$$G_t = \int_{-\infty}^{t} g(t-s)W(ds). \tag{4}$$

Notice that G is a stationary process with the autocorrelation function

$$r(t) = \mathrm{corr}(G_s, G_{s+t}) = \frac{\int_0^\infty g(u)g(u+t)du}{||g||_{\mathbb{L}^2}^2}. \tag{5}$$

We also define the variance function $\overline{R}$ of the increments of the process G as

$$\overline{R}(t) = \mathbb{E}(|G_{s+t} - G_s|^2) = 2||g||_{\mathbb{L}^2}^2(1 - r(t)). \tag{6}$$

Now, we assume that the process X is observed at time points $t_i = i\Delta_n$ with $\Delta_n \to 0$, $i = 0, \ldots, [t/\Delta_n]$, and define the second order differences of X by

$$\diamondsuit_i^n X = X_{i\Delta_n} - 2X_{(i-1)\Delta_n} + X_{(i-2)\Delta_n}. \tag{7}$$

Our main object of interest is the multipower variation of the second order differences of the $\mathscr{BSS}$ process X, i.e.

$$MPV^{\diamondsuit}(X, p_1, \ldots, p_k)_t^n = \Delta_n(\tau_n^{\diamondsuit})^{-p^+} \sum_{i=2}^{[t/\Delta_n]-2k+2} \prod_{l=0}^{k-1} |\diamondsuit_{i+2l}^n X|^{p_l}, \tag{8}$$

where $(\tau_n^{\diamondsuit})^2 = \mathbb{E}(|\diamondsuit_i^n G|^2)$ and $p^+ = \sum_{l=1}^k p_l$. To determine the asymptotic behaviour of the functional $MPV^{\diamondsuit}(X, p_1, \ldots, p_k)^n$ we require a set of assumptions on the memory function g and the intermittency process σ. Below, the functions $L_{\overline{R}}, L_{\overline{R}^{(4)}}, L_g, L_{g^{(2)}} : \mathbb{R}_{>0} \to \mathbb{R}$ are assumed to be continuous and slowly varying at 0, $f^{(k)}$ denotes the k-th derivative of a function f and β denotes a number in $(-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$.

Assumption 1. It holds that

(i) $g(x) = x^\beta L_g(x)$.
(ii) $g^{(2)} = x^{\beta-2}L_{g^{(2)}}(x)$ and, for any $\varepsilon > 0$, we have $g^{(2)} \in \mathbb{L}^2((\varepsilon, \infty))$. Furthermore, $|g^{(2)}|$ is non-increasing on the interval (a, ∞) for some $a > 0$.

(iii) For any $t > 0$

$$F_t = \int_1^\infty |g^{(2)}(s)|^2 \sigma_{t-s}^2 ds < \infty. \tag{9}$$

Assumption 2. For the smoothness parameter β from Assumption 1 it holds that

(i) $\overline{R}(x) = x^{2\beta+1} L_{\overline{R}}(x)$.
(ii) $\overline{R}^{(4)}(x) = x^{2\beta-3} L_{\overline{R}^{(4)}}(x)$.
(iii) There exists a $b \in (0,1)$ such that

$$\limsup_{x\to 0} \sup_{y\in[x,x^b]} \left| \frac{L_{\overline{R}^{(4)}}(y)}{L_{\overline{R}}(x)} \right| < \infty.$$

Assumption 3-γ. For any $p > 0$, it holds that

$$\mathbb{E}(|\sigma_t - \sigma_s|^p) \le C_p |t-s|^{\gamma p} \tag{10}$$

for some $\gamma > 0$ and $C_p > 0$.

Some remarks are in order to explain the rather long list of conditions.

- *The memory function* g: We remark that $g(x) \simeq x^\beta$ implies $g^{(2)}(x) \simeq x^{\beta-2}$ under rather weak assumptions on g (due to the *Monotone Density Theorem*; see e.g. [11, p. 38]). Furthermore, Assumption 1(ii) and *Karamata's Theorem* (see again [11]) imply that

$$\int_\varepsilon^1 |g(x+2\Delta_n) - 2g(x+\Delta_n) + g(x)|^2 dx \simeq \varepsilon^{2\beta-3} \Delta_n^4 \tag{11}$$

for any $\varepsilon \in [\Delta_n, 1)$. This fact will play an important role in the following discussion. Finally, let us note that Assumptions 1(i)–(ii) and 2 are satisfied for the parametric class

$$g(x) = x^\beta \exp(-\lambda x),$$

where $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$ and $\lambda > 0$, which is used to model turbulent flows in physics (see [2]). This class constitutes the most important example in this paper. □

- *The central decomposition and the concentration measure*: Observe the decomposition

$$\begin{aligned} \Diamond_i^n X &= \int_{(i-1)\Delta_n}^{i\Delta_n} g(i\Delta_n - s)\sigma_s W(ds) \\ &+ \int_{(i-2)\Delta_n}^{(i-1)\Delta_n} \Big(g(i\Delta_n - s) - 2g((i-1)\Delta_n - s)\Big)\sigma_s W(ds) \\ &+ \int_{-\infty}^{(i-2)\Delta_n} \Big(g(i\Delta_n - s) - 2g((i-1)\Delta_n - s) + g((i-2)\Delta_n - s)\Big)\sigma_s W(ds), \end{aligned} \tag{12}$$

and the same type of decomposition holds for $\diamondsuit_i^n G$. We deduce that

$$(\tau_n^\diamondsuit)^2 = \int_0^{\Delta_n} g^2(x)dx + \int_0^{\Delta_n} \Big(g(x+\Delta_n) - 2g(x)\Big)^2 dx$$
$$+ \int_0^\infty \Big(g(x+2\Delta_n) - 2g(x+\Delta_n) + g(x)\Big)^2 dx.$$

One of the most essential steps in proving the asymptotic results for the functionals $MPV^\diamondsuit(X, p_1, \ldots, p_k)^n$ is the approximation $\diamondsuit_i^n X \approx \sigma_{(i-2)\Delta_n} \diamondsuit_i^n G$. The justification of this approximation is not trivial: while the first two summands in the decomposition (12) depend only on the intermittency σ around $(i-2)\Delta_n$, the third summand involves the whole path $(\sigma_s)_{s \le (i-2)\Delta_n}$. We need to guarantee that the influence of the intermittency path outside of $(i-2)\Delta_n$ on the third summand of (12) is asymptotically negligible. For this reason we introduce the measure

$$\pi_n^\diamondsuit(A) = \frac{\int_A \Big(g(x+2\Delta_n) - 2g(x+\Delta_n) + g(x)\Big)^2 dx}{(\tau_n^\diamondsuit)^2} < 1, \qquad A \in \mathscr{B}(\mathbb{R}_{>0}), \tag{13}$$

and define $\overline{\pi}_n^\diamondsuit(x) = \pi_n^\diamondsuit((x, \infty))$. To justify the negligibility of the influence of the intermittency path outside of $(i-2)\Delta_n$ we need to ensure that

$$\overline{\pi}_n^\diamondsuit(\varepsilon) \to 0$$

for all $\varepsilon > 0$. Indeed, this convergence follows from Assumptions 1(i)–(ii) (due to (11)). □

- *The correlation structure*: By the stationarity of the process G we deduce that

$$r_n^\diamondsuit(j) = \mathrm{corr}(\diamondsuit_i^n G, \diamondsuit_{i+j}^n G)$$
$$= \frac{-\overline{R}((j+2)\Delta_n) + 4\overline{R}((j+1)\Delta_n) - 6\overline{R}(j\Delta_n) + 4\overline{R}(|j-1|\Delta_n) - \overline{R}(|j-2|\Delta_n)}{(\tau_n^\diamondsuit)^2}. \tag{14}$$

Since $(\tau_n^\diamondsuit)^2 = 4\overline{R}(\Delta_n) - \overline{R}(2\Delta_n)$ we obtain by Assumption 2(i) the convergence

$$r_n^\diamondsuit(j) \to \rho^\diamondsuit(j)$$
$$= \frac{-(j+2)^{1+2\beta} + 4(j+1)^{1+2\beta} - 6j^{1+2\beta} + 4|j-1|^{1+2\beta} - |j-2|^{1+2\beta}}{2\Big(4 - 2^{1+2\beta}\Big)}. \tag{15}$$

We remark that $\rho^\diamondsuit$ is the correlation function of the normalized second order fractional noise $\Big(\diamondsuit_i^n B^H / \sqrt{\mathrm{var}(\diamondsuit_i^n B^H)}\Big)_{i \ge 2}$, where B^H is a fractional Brownian motion with Hurst parameter $H = \beta + \frac{1}{2}$. Notice that

$$|\rho^\diamondsuit(j)| \sim j^{2\beta - 3},$$

where we write $a_j \sim b_j$ when a_j/b_j is bounded. In particular, it implies that $\sum_{j=1}^{\infty} |\rho^{\diamond}(j)| < \infty$. This absolute summability has an important consequence: it leads to standard central limit theorems for the appropriately normalized version of the functional $MPV^{\diamond}(G, p_1, \ldots, p_k)^n$ for all $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$. □

- *Sufficient conditions*: Instead of considering Assumptions 1 and 2, we can alternatively state sufficient conditions on the correlation function $r_n^{\diamond}$ and the measure $\pi_n^{\diamond}$ directly, as it has been done for the case of first order differences in [8]. To ensure the consistency of $MPV^{\diamond}(X, p_1, \ldots, p_k)_t^n$ we require the following assumptions: there exists a sequence $h(j)$ with

$$|r_n^{\diamond}| \leq h(j), \qquad \Delta_n \sum_{j=1}^{[1/\Delta_n]} h^2(j) \to 0, \tag{16}$$

and $\overline{\pi}_n^{\diamond}(\varepsilon) \to 0$ for all $\varepsilon > 0$ (cf. condition (LLN) in [8]). For the proof of the associated central limit theorem we need some stronger conditions: $r_n^{\diamond}(j) \to \rho^{\diamond}(j)$ for all $j \geq 1$, there exists a sequence $h(j)$ with

$$|r_n^{\diamond}| \leq h(j), \qquad \sum_{j=1}^{\infty} h^2(j) < \infty, \tag{17}$$

Assumption 3-γ holds for some $\gamma \in (0, 1]$ with $\gamma(p \wedge 1) > \frac{1}{2}$, $p = \max_{1 \leq i \leq k}(p_i)$, and there exists a constant $\lambda > 1/(p \wedge 1)$ such that for all $\kappa \in (0, 1)$ and $\varepsilon_n = \Delta_n^{\kappa}$ we have

$$\overline{\pi}_n^{\diamond}(\varepsilon_n) = O\left(\Delta_n^{\lambda(1-\kappa)}\right). \tag{18}$$

(cf. condition (CLT) in [8]). In Sect. 5 we will show that Assumptions 1 and 2 imply the conditions (16)–(18). □

3 Limit Theorems

In this section we present the main results of the paper. Recall that the multipower variation process is defined in (8) as

$$MPV^{\diamond}(X, p_1, \ldots, p_k)_t^n = \Delta_n (\tau_n^{\diamond})^{-p^+} \sum_{i=2}^{[t/\Delta_n]-2k+2} \prod_{l=0}^{k-1} | \diamond_{i+2l}^n X|^{p_l}$$

with $\tau_n^2 = \mathbb{E}(|\diamondsuit_i^n G|^2)$ and $p^+ = \sum_{l=1}^k p_l$. We introduce the quantity

$$\rho_{p_1,\ldots,p_k}^n = \mathbb{E}\Big(\prod_{l=0}^{k-1}\Big|\frac{\diamondsuit_{i+2l}^n G}{\tau_n^\diamondsuit}\Big|^{p_l}\Big). \tag{19}$$

Notice that in the case $k = 1$, $p_1 = p$ we have that $\rho_p^n = \mathbb{E}(|U|^p)$ with $U \sim N(0,1)$. We start with the consistency of the functional $MPV^\diamondsuit(X, p_1, \ldots, p_k)_t^n$.

Theorem 1. *Let the Assumptions 1 and 2 hold. Then we obtain*

$$MPV^\diamondsuit(X, p_1, \ldots, p_k)_t^n - \rho_{p_1,\ldots,p_k}^n \int_0^t |\sigma_s|^{p^+} ds \xrightarrow{u.c.p.} 0. \tag{20}$$

Proof. See Sect. 5. □

As we have mentioned in the previous section, under Assumption 2(i) we deduce the convergence $r_n^\diamondsuit(j) \to \rho^\diamondsuit(j)$ for all $j \geq 1$ (see (15)). Consequently, it holds that

$$\rho_{p_1,\ldots,p_k}^n \to \rho_{p_1,\ldots,p_k} = \mathbb{E}\Big(\prod_{l=0}^{k-1}\Big|\frac{\diamondsuit_{i+2l}^n B^H}{\sqrt{\mathrm{var}(\diamondsuit_{i+2l}^n B^H)}}\Big|^{p_l}\Big), \tag{21}$$

where B^H is a fractional Brownian motion with Hurst parameter $H = \beta + \frac{1}{2}$ (notice that the right-hand side of (21) does not depend on n, because B^H is a self-similar process). Thus, we obtain the following result.

Lemma 1. *Let the Assumptions 1 and 2 hold. Then we obtain*

$$MPV^\diamondsuit(X, p_1, \ldots, p_k)_t^n \xrightarrow{u.c.p.} \rho_{p_1,\ldots,p_k} \int_0^t |\sigma_s|^{p^+} ds. \tag{22}$$

Next, we present a multivariate stable central limit theorem for the family $(MPV^\diamondsuit(X, p_1^j, \ldots, p_k^j)^n)_{1\leq j\leq d}$ of multipower variations. We say that a sequence of d-dimensional processes Z^n converges stably in law to a d-dimensional process Z, where Z is defined on an extension $(\Omega', \mathscr{F}', \mathbb{P}')$ of the original probability $(\Omega, \mathscr{F}, \mathbb{P})$, in the space $\mathscr{D}([0,T])^d$ equipped with the uniform topology ($Z^n \xrightarrow{st} Z$) if and only if

$$\lim_{n\to\infty} \mathbb{E}(f(Z^n)V) = \mathbb{E}'(f(Z)V)$$

for any bounded and continuous function $f : \mathscr{D}([0,T])^d \to \mathbb{R}$ and any bounded $\mathscr{F}$-measurable random variable V. We refer to [1, 14] or [21] for a detailed study of stable convergence.

Theorem 2. *Let the Assumptions 1, 2 and 3-γ be satisfied for some $\gamma \in (0,1]$ with $\gamma(p \wedge 1) > \frac{1}{2}$, $p = \max_{1\leq i\leq k, 1\leq j\leq d}(p_i^j)$. Then we obtain the stable convergence*

$$\Delta_n^{-1/2}\Big(MPV^{\diamond}(X,p_1^j,\ldots,p_k^j)_t^n-\rho_{p_1^j,\ldots,p_k^j}^n\int_0^t|\sigma_s|^{p_j^+}\,ds\Big)_{1\le j\le d}\xrightarrow{st}\int_0^t A_s^{1/2}dW_s', \tag{23}$$

where W' is a d-dimensional Brownian motion that is defined on an extension of the original probability space $(\Omega,\mathscr{F},\mathbb{P})$ and is independent of $\mathscr{F}$, A is a $d\times d$-dimensional process given by

$$A_s^{ij}=\mu_{ij}|\sigma_s|^{p_i^+ + p_j^+},\qquad 1\le i,j\le d, \tag{24}$$

and the $d\times d$ matrix $\mu=(\mu_{ij})_{1\le i,j\le d}$ is defined as

$$\mu_{ij}=\lim_{n\to\infty}\Delta_n^{-1}cov\Big(MPV^{\diamond}(B^H,p_1^i,\ldots,p_k^i)_1^n,MPV^{\diamond}(B^H,p_1^j,\ldots,p_k^j)_1^n\Big) \tag{25}$$

with B^H being a fractional Brownian motion with Hurst parameter $H=\beta+\frac{1}{2}$.

Proof. See Sect. 5. □

We remark that the conditions of Theorem 2 imply that $\max_{1\le i\le k,1\le j\le d}(p_i^j)>\dfrac{1}{2}$ since $\gamma\in(0,1]$.

Remark 1. Notice that the limit process in (23) is *mixed normal*, because the Brownian motion W' is independent of the process A. In fact, we can transform the convergence result of Theorem 2 into a standard central limit theorem due to the properties of stable convergence; we demonstrate this transformation in Sect. 4. We remark that the limit in (25) is indeed finite; see Theorem 2 in [8] and its proof for more details. □

Remark 2. In general, the convergence in (23) does not remain valid when $\rho^n_{p_1^j,\ldots,p_k^j}$ is replaced by its limit $\rho_{p_1^j,\ldots,p_k^j}$ defined by (21). However, when the rate of convergence associated with (21) is faster than $\Delta_n^{-1/2}$, we can also use the quantity $\rho_{p_1^j,\ldots,p_k^j}$ without changing the stable central limit theorem in (23). This is the case when the convergence

$$\Delta_n^{-1/2}(r_n^{\diamond}(j)-\rho^{\diamond}(j))\to 0$$

holds for any $j\ge 1$. Obviously, the latter depends on the behaviour of the slowly varying function $L_{\overline{R}}$ from Assumption 2(i) near 0. It can be shown that for our main example

$$g(x)=x^{\beta}\exp(-\lambda x),$$

where $\beta\in(-\frac{1}{2},0)\cup(0,\frac{1}{4})$ and $\lambda>0$, $\rho^n_{p_1^j,\ldots,p_k^j}$ can indeed be replaced by the quantity $\rho_{p_1^j,\ldots,p_k^j}$ without changing the limit in Theorem 2. □

Remark 3 (*Second order differences vs. increments*). Let us demonstrate some advantages of using second order differences $\diamond_i^n X$ instead of using first order increments $\Delta_i^n X$.

(i) First of all, taking second order differences weakens the value of autocorrelations which leads to normal limits for the normalized version of the functional $MPV^{\diamond}(G, p_1, \dots, p_k)^n$ (and hence to mixed normal limits for the value of $MPV^{\diamond}(X, p_1, \dots, p_k)^n$) for all $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$. This can be explained as follows: to obtain normal limits it has to hold that

$$\sum_{j=1}^{\infty} |\rho^{\diamond}(j)|^2 < \infty$$

where $\rho^{\diamond}(j)$ is defined in formula (15) (it relies on the fact that the function $|x|^p - \mathbb{E}(|N(0,1)|^p)$ has *Hermite rank* 2; see also condition (17)). This is clearly satisfied for all $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, because we have $|\rho^{\diamond}(j)| \sim j^{2\beta-3}$.

In the case of using first order increments $\Delta_i^n X$ we obtain the correlation function ρ of the fractional noise $(B_i^H - B_{i-1}^H)_{i \geq 1}$ with $H = \beta + \frac{1}{2}$ as the limit autocorrelation function (see e.g. (4.15) in [8]). As $|\rho(j)| \sim j^{2\beta-1}$ it holds that

$$\sum_{j=1}^{\infty} |\rho(j)|^2 < \infty$$

only for $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{4})$. □

(ii) As we have mentioned in the previous section, we need to ensure that $\overline{\pi}_n^{\diamond}(\varepsilon) \to 0$, where the measure $\pi_n^{\diamond}$ is defined by (13), for all $\varepsilon > 0$ to show the law of large numbers. But for proving the central limit theorem we require a more precise treatment of the quantity

$$\overline{\pi}_n^{\diamond}(\varepsilon) = \frac{\int_\varepsilon^\infty \Big(g(x + 2\Delta_n) - 2g(x + \Delta_n) + g(x)\Big)^2 dx}{(\tau_n^{\diamond})^2}.$$

In particular, we need to show that the above quantity is small enough (see condition (18)) to prove the negligibility of the error that is due to the first order approximation $\diamond_i^n X \approx \sigma_{(i-2)\Delta_n} \diamond_i^n G$. The corresponding term in the case of increments is essentially given as

$$\overline{\pi}_n(\varepsilon) = \frac{\int_\varepsilon^\infty \Big(g(x + \Delta_n) - g(x)\Big)^2 dx}{\tau_n^2},$$

where $\tau_n^2 = \mathbb{E}(|\Delta_i^n G|^2)$ (see [8]). Under the Assumptions 1 and 2 the denominators $(\tau_n^{\diamond})^2$ and τ_n^2 have the same order, but the nominator of $\overline{\pi}_n^{\diamond}(\varepsilon)$ is much smaller than the nominator of $\overline{\pi}_n(\varepsilon)$. This has an important consequence: the central limit theorems for the multipower variation of the increments of X hold only for $\beta \in (-\frac{1}{2}, 0)$ while the corresponding results for the second order differences hold for all $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$. □

Another advantage of using second order differences $\diamond_i^n X$ is the higher robustness to the presence of smooth drift processes. Let us consider the process

$$Y_t = X_t + D_t, \qquad t \geq 0, \tag{26}$$

where X is a $\mathscr{BSS}$ model of the form (3) and D is a stochastic drift. We obtain the following result.

Proposition 1. *Assume that the conditions of Theorem 2 hold and $D \in C^v(\mathbb{R}_{\geq 0})$ for some $v \in (1,2)$, i.e. $D \in C^1(\mathbb{R}_{\geq 0})$ (a.s.) and D' has $(v-1)$-Hölder continuous paths (a.s.). When $v - \beta > 1$ then*

$$\Delta_n^{-1/2}\Big(MPV^{\diamond}(Y, p_1^j, \ldots, p_k^j)_t^n - \rho_{p_1^j,\ldots,p_k^j}^n \int_0^t |\sigma_s|^{p_j^+} ds\Big)_{1\leq j\leq d} \xrightarrow{st} \int_0^t A_s^{1/2} dW_s',$$

where the limit process is given in Theorem 2. That is, the central limit theorem is robust to the presence of the drift D.

Proof. Proposition 1 follows by a direct application of the Cauchy-Schwarz and Minkovski inequalities (see Proposition 6 in [8] for more details). □

The idea behind Proposition 1 is rather simple. Notice that $\diamond_i^n X = O_{\mathbb{P}}(\Delta_n^{\beta+\frac{1}{2}})$ (this follows from Assumption 2) whereas $\diamond_i^n D = O_{\mathbb{P}}(\Delta_n^v)$. It can be easily seen that the drift process D does not influence the central limit theorem if $v - \beta - \frac{1}{2} > \frac{1}{2}$, because $\Delta_n^{-1/2}$ is the rate of convergence; this explains the condition of Proposition 1.

Notice that we obtain better robustness properties than in the case of first order increments: we still have $\Delta_i^n X = O_{\mathbb{P}}(\Delta_n^{\beta+\frac{1}{2}})$, but now $\Delta_i^n D = O_{\mathbb{P}}(\Delta_n)$. Thus, the drift process D is negligible only when $\beta < 0$, which is obviously a more restrictive condition.

Example 1. Let us come back to the original $\mathscr{BSS}$ process from (1), which is of the form (26) with

$$D_t = \int_{-\infty}^t q(t-s) a_s ds.$$

For the ease of exposition we assume that

$$q(x) = x^{\overline{\beta}} 1_{\{x \in (0,1)\}}, \qquad \overline{\beta} > -1,$$

and the drift process a is càdlàg and bounded. Observe the decomposition

$$D_{t+\varepsilon} - D_t = \int_t^{t+\varepsilon} q(t+\varepsilon-s) a_s ds + \int_{-\infty}^t (q(t+\varepsilon-s) - q(t-s)) a_s ds.$$

We conclude that the process D has Hölder continuous paths of order $(\overline{\beta}+1)\wedge 1$. Consequently, Theorem 1 is robust to the presence of the drift process D when $\overline{\beta} > \beta - \frac{1}{2}$. Furthermore, for $\overline{\beta} \geq 0$ we deduce that

$$D'_t = q(0)a_t + \int_0^\infty q'(s)a_{t-s}ds.$$

By Proposition 1 we conclude that Theorem 2 is robust to the presence of D when the process a has Hölder continuous paths of order bigger than β. □

Remark 4 (*Higher order differences*). Clearly, we can also formulate asymptotic results for multipower variation of q-order differences of $\mathscr{BSS}$ processes X. Define

$$MPV^{(q)}(X, p_1, \ldots, p_k)^n_t = \Delta_n(\tau_n^{(q)})^{-p^+} \sum_{i=q}^{[t/\Delta_n]-qk+q} \prod_{l=0}^{k-1} |\Delta_{i+ql}^{(q)n} X|^{p_l},$$

where $\Delta_i^{(q)n}X$ is the q-order difference starting at $i\Delta_n$ and $(\tau_n^{(q)})^2 = \mathbb{E}(|\Delta_i^{(q)n}G|^2)$. Then the results of Theorems 1 and 2 remain valid for the class $MPV^{(q)}(X, p_1, \ldots, p_k)^n$ with $\rho^n_{p_1,\ldots,p_k}$ defined as

$$\rho^n_{p_1,\ldots,p_k} = \mathbb{E}\Big(\prod_{l=0}^{k-1}\Big|\frac{\Delta_{i+ql}^{(q)n}G}{\tau_n^{(q)}}\Big|^{p_l}\Big).$$

The Assumptions 1 and 2 have to be modified as follows: (a) $g^{(2)}$ has to be replaced by $g^{(q)}$ in Assumption 1(ii) and 1(iii), and (b) $\overline{R}^{(4)}$ has to be replaced by $\overline{R}^{(2q)}$ in Assumption 2(ii).

However, let us remark that going from second order differences to q-order differences with $q > 2$ does not give any new theoretical advantages (with respect to robustness etc.). It might though have some influence in finite samples. □

Remark 5 (*An extension to other integral processes*). In [8] and [9] we considered processes of the form

$$Z_t = \mu + \int_0^t \sigma_s dG_s, \tag{27}$$

where $(G_s)_{s\geq 0}$ is a Gaussian process with centered and stationary increments. Define

$$\overline{R}(t) = \mathbb{E}(|G_{s+t} - G_s|^2)$$

and assume that Assumption 2 holds for $\overline{R}$ (we use the same notations as for the process (3) to underline the parallels between the models (27) and (3)). We remark that the integral in (27) is well-defined in the Riemann-Stieltjes sense when the process σ has finite r-variation with $r < 1/(1/2 - \beta)$ (see [8] and [23]), which we assume in the following discussion. We associate $\tau_n^\diamond$ and $MPV^\diamond(Z, p_1, \ldots, p_k)_t^n$ with the process Z by (8). Then Theorem 1 remains valid for the model (27) and Theorem 2 also holds if we further assume that Assumption 3-γ is satisfied for some $\gamma \in (0, 1]$ with $\gamma(p \wedge 1) > \frac{1}{2}$, $p = \max_{1 \le i \le k, 1 \le j \le d}(p_i^j)$.

We remark that the justification of the approximation $\diamond_i^n Z = \sigma_{(i-2)\Delta_n} \diamond_i^n G$ is easier to provide for the model (27) (see e.g. [8]). All other proof steps are performed in exactly the same way as for the model (3). □

Remark 6 (*Some further extensions*). We remark that the use of the power functions in the definition of $MPV^\diamond(X, p_1, \ldots, p_k)_t^n$ is not essential for the proof of Theorems 1 and 2. In principle, both theorems can be proved for a more general class of functionals

$$MPV^\diamond(X, H)_t^n = \Delta_n \sum_{i=2}^{[t/\Delta_n]-2k+2} H\Big(\frac{\diamond_i^n X}{\tau_n^\diamond}, \ldots, \frac{\diamond_{i+2(k-1)}^n X}{\tau_n^\diamond}\Big),$$

where $H : \mathbb{R}^k \to \mathbb{R}$ is a measurable *even* function with polynomial growth (cf. Remark 2 in [8]). However, we dispense with the exact exposition.

Another useful extension of Theorem 2 is a joint central limit theorem for functionals $MPV^\diamond(X, p_1, \ldots, p_k)_t^n$ computed at different frequencies (this result will be applied in Sect. 4.3). For $r \ge 1$, define the multipower variation computed at frequency $r\Delta_n$ as

$$MPV_r^\diamond(X, p_1, \ldots, p_k)_t^n = \Delta_n (\tau_{n,r}^\diamond)^{-p^+} \sum_{i=2r}^{[t/\Delta_n]-2k+2} \prod_{l=0}^{k-1} |\diamond_{i+2lr}^{n,r} X|^{p_l}, \tag{28}$$

where $\diamond_i^{n,r} X = X_{i\Delta_n} - 2X_{(i-r)\Delta_n} + X_{(i-2r)\Delta_n}$ and $(\tau_{n,r}^\diamond)^2 = \mathbb{E}(|\diamond_i^{n,r} G|^2)$. Then, under the conditions of Theorem 2, we obtain the stable central limit theorem

$$\Delta_n^{-1/2} \begin{pmatrix} MPV_{r_1}^\diamond(X, p_1, \ldots, p_k)_t^n - \rho_{p_1,\ldots,p_k}^{n,r_1} \int_0^t |\sigma_s|^{p^+} ds \\ MPV_{r_2}^\diamond(X, p_1, \ldots, p_k)_t^n - \rho_{p_1,\ldots,p_k}^{n,r_2} \int_0^t |\sigma_s|^{p^+} ds \end{pmatrix} \xrightarrow{st} \int_0^t |\sigma_s|^{p^+} \mu^{1/2} dW_s', \tag{29}$$

where W' is a 2-dimensional Brownian motion independent of $\mathscr{F}$,

$$\rho_{p_1,\ldots,p_k}^{n,r} = \mathbb{E}\Big(\prod_{l=0}^{k-1} \Big|\frac{\diamond_{i+2lr}^{n,r} G}{\tau_{n,r}^\diamond}\Big|^{p_l}\Big)$$

and the 2×2 matrix $\mu = (\mu_{ij})_{1 \le i,j \le 2}$ is defined as

$$\mu_{ij} = \lim_{n\to\infty} \Delta_n^{-1} \mathrm{cov}\Big(MPV_{r_i}^\diamond(B^H, p_1, \ldots, p_k)_1^n, MPV_{r_j}^\diamond(B^H, p_1, \ldots, p_k)_1^n\Big)$$

with B^H being a fractional Brownian motion with Hurst parameter $H = \beta + \frac{1}{2}$.

Clearly, an analogous result can be formulated for any d-dimensional family $(r_j; p_1^j, \ldots, p_k^j)_{1 \le j \le d}$. □

4 Estimation of the Smoothness Parameter

In this section we apply our probabilistic results to obtain consistent estimates of the smoothness parameter $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$. We propose three different estimators for β: the *realised variation ratio* ($RVR^{\diamond}$), the *modified realised variation ratio* ($\overline{RVR}^{\diamond}$) and the *change-of-frequency* estimator ($COF^{\diamond}$). Throughout this section we assume that

$$\Delta_n^{-1/2}(r_n^{\diamond}(j) - \rho^{\diamond}(j)) \to 0 \tag{30}$$

for any $j \geq 1$, where $r_n^{\diamond}(j)$ and $\rho^{\diamond}(j)$ are defined in (14) and (15), respectively. This condition guarantees that $\rho^n_{p_1^j,\ldots,p_k^j}$ can be replaced by the quantity $\rho_{p_1^j,\ldots,p_k^j}$ in Theorem 2 without changing the limit (see Remark 2). Recall that the condition (30) holds for our canonical example

$$g(x) = x^{\beta} \exp(-\lambda x)$$

when $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{4})$ and $\lambda > 0$.

4.1 *The Realised Variation Ratio*

We define the realised variation ratio based on the second order differences as

$$RVR_t^{\diamond n} = \frac{MPV^{\diamond}(X, 1, 1)_t^n}{MPV^{\diamond}(X, 2, 0)_t^n}. \tag{31}$$

This type of statistics has been successfully applied in semimartingale models to test for the presence of the jump part (see e.g. [4]). In the $\mathscr{BSS}$ framework the statistic $RVR_t^{\diamond n}$ is used to estimate the smoothness parameter β.

Let us introduce the function $\psi : (-1, 1) \to (\frac{2}{\pi}, 1)$ given by

$$\psi(x) = \frac{2}{\pi}(\sqrt{1 - x^2} + x \arcsin x). \tag{32}$$

We remark that $\psi(x) = \mathbb{E}(U_1 U_2)$, where U_1, U_2 are two standard normal variables with correlation x. Let us further notice that while the computation of the value of $MPV^{\diamond}(X, p_1, \ldots, p_k)_t^n$ requires the knowledge of the quantity $\tau_n^{\diamond}$ (and hence the knowledge of the memory function g), the statistic $RVR_t^{\diamond n}$ is purely observation based since

$$RVR_t^{\diamond n} = \frac{\sum_{i=2}^{[t/\Delta_n]-2} |\diamond_i^n X||\diamond_{i+2}^n X|}{\sum_{i=2}^{[t/\Delta_n]} |\diamond_i^n X|^2}.$$

Our first result is the consistency of $RVR_t^{\diamond n}$, which follows directly from Theorem 1 and Lemma 1.

Proposition 2. *Assume that the conditions of Theorem 1 hold. Then we obtain*

$$RVR_t^{\diamond n} \xrightarrow{u.c.p.} \psi(\rho^{\diamond}(2)), \tag{33}$$

where $\rho^{\diamond}(j)$ is defined by (15).

Note that

$$\rho^{\diamond}(2) = \frac{-4^{1+2\beta} + 4 \cdot 3^{1+2\beta} - 6 \cdot 2^{1+2\beta} + 4}{2\left(4 - 2^{1+2\beta}\right)},$$

$\rho^{\diamond}(2) = \rho^{\diamond}_{\beta}(2)$ is invertible as a function of $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, it is positive for $\beta \in (-\frac{1}{2}, 0)$ and negative for $\beta \in (0, \frac{1}{2})$.

Obviously, the function ψ is only invertible on the interval $(-1, 0)$ or $(0, 1)$. Thus, we can recover the absolute value of $\rho^{\diamond}(2)$, but not its sign (which is not a big surprise, because we use absolute values of the second order differences in the definition of $RVR_t^{\diamond n}$). In the following proposition we restrict ourselves to $\beta \in (0, \frac{1}{2})$ as those values typically appear in physics.

Proposition 3. *Assume that the conditions of Theorems 2 and (30) hold. Let $\beta \in (0, \frac{1}{2})$, $\rho^{\diamond}_{\beta}(2) : (0, \frac{1}{2}) \to (-1, 0)$, $\psi : (-1, 0) \to (\frac{2}{\pi}, 1)$ and set $f = \psi \circ \rho^{\diamond}_{\beta}(2)$. Then we obtain for $h = f^{-1}$*

$$h(RVR_t^{\diamond n}) \xrightarrow{u.c.p.} \beta, \tag{34}$$

and

$$\frac{\Delta_n^{-1/2}(h(RVR_t^{\diamond n}) - \beta) MPV^{\diamond}(X, 2, 0)_t^n}{\sqrt{\frac{1}{3}|h'(RVR_t^{\diamond n})|(1, -RVR_t^{\diamond n})\mu(1, -RVR_t^{\diamond n})^T MPV^{\diamond}(X, 4, 0)_t^n}} \xrightarrow{d} N(0, 1), \tag{35}$$

for any $t > 0$, where $\mu = (\mu_{ij})_{1 \le i, j \le 2}$ is given by

$$\mu_{11} = \lim_{n \to \infty} \Delta_n^{-1} var\left(MPV^{\diamond}(B^H, 1, 1)_1^n\right),$$

$$\mu_{12} = \lim_{n \to \infty} \Delta_n^{-1} cov\left(MPV^{\diamond}(B^H, 1, 1)_1^n, MPV^{\diamond}(B^H, 2, 0)_1^n\right),$$

$$\mu_{22} = \lim_{n \to \infty} \Delta_n^{-1} var\left(MPV^{\diamond}(B^H, 2, 0)_1^n\right),$$

with $H = \beta + \frac{1}{2}$.

Proposition 3 is a direct consequence of Theorem 2, of the delta-method for stable convergence and of the fact that the true centering $\psi(r_n^{\diamond}(2))$ in (23) can be replaced by its limit $\psi(\rho^{\diamond}(2))$, because of the condition (30) (see Remark 2). We note that the normalized statistic in (35) is again self-scaling, i.e. we do not require the knowledge of $\tau_n^{\diamond}$, and consequently we can immediately build confidence regions for the smoothness parameter $\beta \in (0, \frac{1}{2})$.

Remark 7. The constants β_{ij}, $1 \le i, j \le 2$, can be expressed as

$$\mu_{11} = \mathrm{var}(|Q_1||Q_3|) + 2\sum_{k=1}^{\infty} \mathrm{cov}(|Q_1||Q_3|, |Q_{1+k}||Q_{3+k}|),$$

$$\mu_{12} = \mathrm{cov}(Q_2^2, |Q_1||Q_3|) + 2\sum_{k=0}^{\infty} \mathrm{cov}(Q_1^2, |Q_{1+k}||Q_{3+k}|),$$

$$\mu_{22} = \mathrm{var}(Q_1^2) + 2\sum_{k=1}^{\infty} \mathrm{cov}(Q_1^2, Q_{1+k}^2) = 2 + 4\sum_{k=1}^{\infty} |\rho^{\diamond}(k)|^2,$$

with $Q_i = \diamond_i^n B^H / \sqrt{\mathrm{var}(\diamond_i^n B^H)}$. The above quantities can be computed using formulas for absolute moments of the multivariate normal distributions. □

4.2 *The Modified Realised Variation Ratio*

Recall that the restriction $\beta \in (0, \frac{1}{2})$ is required to formulate Proposition 3. To obtain estimates for all values $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$ let us consider a modified (and, in fact, more natural) version of $RVR_t^{\diamond n}$:

$$\overline{RVR}_t^{\diamond n} = \frac{\sum_{i=2}^{[t/\Delta_n]-2} \diamond_i^n X \diamond_{i+2}^n X}{\sum_{i=2}^{[t/\Delta_n]} |\diamond_i^n X|^2}. \tag{36}$$

Notice that $\overline{RVR}_t^{\diamond n}$ is an analogue of the classical autocorrelation estimator. The following result describes the asymptotic behaviour of $\overline{RVR}_t^{\diamond n}$.

Proposition 4. *Assume that the conditions of Theorems 2 and (30) hold, and let $h = (\rho_\beta^{\diamond}(2))^{-1}$. Then we obtain*

$$h(\overline{RVR}_t^{\diamond n}) \xrightarrow{u.c.p.} \beta, \tag{37}$$

and, with $\overline{MPV}^{\diamond}(X,1,1)_t^n = \Delta_n(\tau_n^{\diamond})^{-2} \sum_{i=2}^{[t/\Delta_n]-2} \diamond_i^n X \diamond_{i+2}^n X$,

$$\frac{\Delta_n^{-1/2}(h(\overline{RVR}_t^{\diamond n}) - \beta)MPV^{\diamond}(X,2,0)_t^n}{\sqrt{\frac{1}{3}|h'(\overline{RVR}_t^{\diamond n})|(1, -\overline{RVR}_t^{\diamond n})\mu(1, -\overline{RVR}_t^{\diamond n})^T MPV^{\diamond}(X,4,0)_t^n}} \xrightarrow{d} N(0,1), \tag{38}$$

for any $t > 0$*, where* $\mu = (\mu_{ij})_{1\le i,j\le 2}$ *is given by*

$$\mu_{11} = \lim_{n\to\infty} \Delta_n^{-1} var\Big(\overline{MPV}^{\diamond}(B^H,1,1)_1^n\Big),$$

$$\mu_{12} = \lim_{n\to\infty} \Delta_n^{-1} cov\Big(\overline{MPV}^{\diamond}(B^H,1,1)_1^n, MPV^{\diamond}(B^H,2,0)_1^n\Big),$$

$$\mu_{22} = \lim_{n\to\infty} \Delta_n^{-1} var\Big(MPV^{\diamond}(B^H,2,0)_1^n\Big),$$

with $H = \beta + \frac{1}{2}$.

Remark 8. Note that Proposition 4 follows from Remark 6, because the function $H(x,y) = xy$ is even one. In fact, its proof is much easier than the corresponding result of Theorem 2. The most essential step is the joint central limit theorem for the nominator and the denominator of $\overline{RVR}_t^{\diamond n}$ when $X = G$ (i.e. $\sigma \equiv 1$). The latter can be shown by using Wiener chaos expansion and Malliavin calculus. Let $\mathbb{H}$ be a separable Hilbert space generated by the triangular array $(\diamond_i^n G/\tau_n^{\diamond})_{n\ge1, 1\le i\le[t/\Delta_n]}$ with scalar product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ induced by the covariance function of the process $(\diamond_i^n G/\tau_n^{\diamond})_{n\ge1, 1\le i\le[t/\Delta_n]}$. Setting $\chi_i^n = \diamond_i^n G/\tau_n^{\diamond}$ we deduce the identities

$$\Delta_n^{1/2} \sum_{i=2}^{[t/\Delta_n]-2} \Big(\chi_i^n \chi_{i+2}^n - \rho^{\diamond}(2)\Big) = I_2(f_n^{(1)}), \quad f_n^{(1)} = \Delta_n^{1/2} \sum_{i=2}^{[t/\Delta_n]-2} \chi_i^n \otimes \chi_{i+2}^n,$$

$$\Delta_n^{1/2} \sum_{i=2}^{[t/\Delta_n]} \Big(|\chi_i^n|^2 - 1\Big) = I_2(f_n^{(2)}), \quad f_n^{(2)} = \Delta_n^{1/2} \sum_{i=2}^{[t/\Delta_n]} (\chi_i^n)^{\otimes 2},$$

where I_2 is the second multiple integral. The joint central limit theorem for the above statistics follows from [19] once we show the contraction conditions

$$||f_n^{(1)} \otimes_1 f_n^{(1)}||_{\mathbb{H}^{\otimes 2}} \to 0, \qquad ||f_n^{(2)} \otimes_1 f_n^{(2)}||_{\mathbb{H}^{\otimes 2}} \to 0,$$

and identify the asymptotic covariance structure by computing $2\lim_{n\to\infty}\langle f_n^{(i)}, f_n^{(j)}\rangle_{\mathbb{H}^{\otimes 2}}$ for $1 \le i,j \le 2$. We refer to the appendix of [7] for a more detailed proof of such central limit theorems. □

Remark 9. The constants β_{ij}, $1 \le i,j \le 2$, are now much easier to compute. They are given as

$$\mu_{11} = \text{var}(Q_1 Q_3) + 2\sum_{k=1}^{\infty} \text{cov}(Q_1 Q_3, Q_{1+k} Q_{3+k})$$

$$= 1 + |\rho^{\diamond}(2)|^2 + 2\sum_{k=1}^{\infty} (|\rho^{\diamond}(k)|^2 + \rho^{\diamond}(k+2)\rho^{\diamond}(|k-2|),$$

$$\mu_{12} = \mathrm{cov}(Q_2^2, Q_1 Q_3) + 2\sum_{k=0}^{\infty} \mathrm{cov}(Q_1^2, Q_{1+k} Q_{3+k})$$

$$= 2|\rho^{\diamond}(1)|^2 + 4\sum_{k=1}^{\infty} \rho^{\diamond}(k)\rho^{\diamond}(k+2),$$

$$\mu_{22} = \mathrm{var}(Q_1^2) + 2\sum_{k=1}^{\infty} \mathrm{cov}(Q_1^2, Q_{1+k}^2) = 2 + 4\sum_{k=1}^{\infty} |\rho^{\diamond}(k)|^2,$$

with $Q_i = \diamond_i^n B^H / \sqrt{\mathrm{var}(\diamond_i^n B^H)}$. This follows from a well-known formula

$$\mathrm{cov}(Z_1 Z_2, Z_3 Z_4) = \mathrm{cov}(Z_1, Z_3)\mathrm{cov}(Z_2, Z_4) + \mathrm{cov}(Z_2, Z_3)\mathrm{cov}(Z_1, Z_4)$$

whenever (Z_1, Z_2, Z_3, Z_4) is normal. □

4.3 *Change-of-Frequency Estimator*

Another idea of estimating β is to change the frequency Δ_n at which the second order differences are built. We recall that $(\tau_n^{\diamond})^2 = 4\overline{R}(\Delta_n) - \overline{R}(2\Delta_n)$ and consequently we obtain the relationship

$$(\tau_n^{\diamond})^2 \simeq \Delta_n^{2\beta+1}$$

by Assumption 2(i). Observing the latter we define the statistic

$$COF_t^n = \frac{\sum_{i=4}^{[t/\Delta_n]} |\diamond_i^{n,2} X|^2}{\sum_{i=2}^{[t/\Delta_n]} |\diamond_i^n X|^2}, \tag{39}$$

that is essentially the ratio of $MPV^{\diamond}(X, 2, 0)_t^n$ computed at frequencies Δ_n and $2\Delta_n$. Recall that $(\tau_{n,2}^{\diamond})^2 = \mathbb{E}(|\diamond_i^{n,2} G|^2) = 4\overline{R}(2\Delta_n) - \overline{R}(4\Delta_n)$ and observe

$$\frac{(\tau_{n,2}^{\diamond})^2}{(\tau_n^{\diamond})^2} \to 2^{2\beta+1}.$$

As a consequence we deduce the convergence

$$COF_t^n \xrightarrow{\text{u.c.p.}} 2^{2\beta+1}.$$

The following proposition is a direct consequence of (29) and the properties of stable convergence.

Proposition 5. *Assume that the conditions of Theorems 2 and (30) hold, and let* $h(x) = (\log_2(x) - 1)/2$. *Then we obtain*

$$h(COF_t^n) \xrightarrow{u.c.p.} \beta, \tag{40}$$

and

$$\frac{\Delta_n^{-1/2}(h(COF_t^n) - \beta)MPV^{\diamond}(X,2,0)_t^n}{\sqrt{\frac{1}{3}|h'(COF_t^n)|(1,-COF_t^n)\mu(1,-COF_t^n)^T MPV^{\diamond}(X,4,0)_t^n}} \xrightarrow{d} N(0,1), \tag{41}$$

for any $t > 0$*, where* $\mu = (\mu_{ij})_{1\le i,j\le 2}$ *is given by*

$$\mu_{11} = \lim_{n\to\infty} \Delta_n^{-1} var\Big(MPV_2^{\diamond}(B^H,2,0)_1^n\Big),$$

$$\mu_{12} = \lim_{n\to\infty} \Delta_n^{-1} cov\Big(MPV_2^{\diamond}(B^H,2,0)_1^n, MPV^{\diamond}(B^H,2,0)_1^n\Big),$$

$$\mu_{22} = \lim_{n\to\infty} \Delta_n^{-1} var\Big(MPV^{\diamond}(B^H,2,0)_1^n\Big),$$

with $H = \beta + \frac{1}{2}$.

Let us emphasize that the normalized statistic in (41) is again self-scaling. We recall that the approximation

$$\frac{(\tau_{n,2}^{\diamond})^2}{(\tau_n^{\diamond})^2} - 2^{2\beta+1} = o(\Delta_n^{1/2}),$$

which follows from (30), holds for our main example $g(x) = x^{\beta}\exp(-\lambda x)$ when $\beta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{4})$ and $\lambda > 0$.

Remark 10. Observe the identity

$$X_{i\Delta_n} - 2X_{(i-2)\Delta_n} + X_{(i-4)\Delta_n} = \diamond_i^n X - 2\diamond_{i-1}^n X + \diamond_{i-2}^n X.$$

The latter implies that

$$\mu_{11} = 2 + 2^{-4\beta}\sum_{k=1}^{\infty} |\rho^{\diamond}(k+2) - 4\rho^{\diamond}(k+1) + 6\rho^{\diamond}(k) - 4\rho^{\diamond}(|k-1|)$$
$$+\rho^{\diamond}(|k-2|)|^2,$$

$$\mu_{12} = 2^{-2\beta}(\rho^{\diamond}(1)-1) + 2^{1-2\beta}\sum_{k=0}^{\infty}|\rho^{\diamond}(k+2) - 2\rho^{\diamond}(k+1) + \rho^{\diamond}(k)|^2,$$

$$\mu_{22} = 2 + 4\sum_{k=1}^{\infty}|\rho^{\diamond}(k)|^2.$$

□

5 Proofs

Let us start by noting that the intermittency process σ is assumed to be càdlàg, and thus σ_- is locally bounded. Consequently, w.l.o.g. σ can be assumed to be bounded on compact intervals by a standard localization procedure (see e.g. Sect. 3 in [5] for more details). We also remark that the process F defined by (9) is continuous. Hence, F is locally bounded and can be assumed to be bounded on compact intervals w.l.o.g. by the same localization procedure.

Below, all positive constants are denoted by C or C_p if they depend on some parameter p. In the following we present three technical lemmas.

Lemma 2. *Under Assumption 1 we have that*

$$\mathbb{E}(|\diamond_i^n X|^p) \le C_p(\tau_n^{\diamond})^p, \qquad i = 2, \ldots, [t/\Delta_n] \tag{42}$$

for all $p > 0$.

Proof of Lemma 2: Recall that due to Assumption 1(ii) the function $|g^{(2)}|$ is non-increasing on (a, ∞) for some $a > 0$ and assume w.l.o.g. that $a > 1$. By the decomposition (12) and Burkholder's inequality we deduce that

$$\mathbb{E}(|\diamond_i^n X|^p) \le C_p\Bigg((\tau_n^{\diamond})^p + \mathbb{E}\Big(\int_0^{\infty}\Big(g(s+2\Delta_n) - 2g(s+\Delta_n) + g(s)\Big)^2\sigma^2_{(i-2)\Delta_n - s}ds\Big)^{p/2}\Bigg),$$

since σ is bounded on compact intervals. We immediately obtain the estimates

$$\int_0^1\Big(g(s+2\Delta_n) - 2g(s+\Delta_n) + g(s)\Big)^2\sigma^2_{(i-2)\Delta_n - s}ds \le C(\tau_n^{\diamond})^2,$$

$$\int_1^a\Big(g(s+2\Delta_n) - 2g(s+\Delta_n) + g(s)\Big)^2\sigma^2_{(i-2)\Delta_n - s}ds \le C\Delta_n^2,$$

because $g^{(2)}$ is continuous on $(0,\infty)$ and σ is bounded on compact intervals. On the other hand, since $|g^{(2)}|$ is non-increasing on (a,∞), we deduce that

$$\int_a^\infty \Big(g(s+2\Delta_n) - 2g(s+\Delta_n) + g(s)\Big)^2 \sigma^2_{(i-2)\Delta_n - s} ds \le \Delta_n^2 F_{(i-2)\Delta_n}.$$

Finally, the boundedness of the process F implies (42). □

Next, for any stochastic process f and any $s>0$, we define the (possibly infinite) measure

$$\pi^{\diamond n}_{f,s}(A) = \frac{\int_A \Big(g(x+2\Delta_n) - 2g(x+\Delta_n) + g(x)\Big)^2 f^2_{s-x} dx}{(\tau_n^\diamond)^2}, \qquad A \in \mathscr{B}(\mathbb{R}_{>0}), \tag{43}$$

and set $\overline{\pi}^{\diamond n}_{f,s}(x) = \pi^n_{f,s}(\{y:\ y>x\})$.

Lemma 3. *Under Assumption 1 it holds that*

$$\sup_{s\in[0,t]} \overline{\pi}^{\diamond n}_{\sigma,s}(\varepsilon) \le C\overline{\pi}^{\diamond}_n(\varepsilon) \tag{44}$$

for any $\varepsilon>0$*, where the measure* $\pi^\diamond_n$ *is given by* (13).

Proof of Lemma 3: Recall again that $|g^{(2)}|$ is non-increasing on (a,∞) for some $a>0$, and assume w.l.o.g. that $a>\varepsilon$. Since the processes σ and F are bounded we deduce exactly as in the previous proof that

$$\int_\varepsilon^\infty \Big(g(x+2\Delta_n) - 2g(x+\Delta_n) + g(x)\Big)^2 \sigma^2_{s-x} dx$$

$$= \int_\varepsilon^a \Big(g(x+2\Delta_n) - 2g(x+\Delta_n) + g(x)\Big)^2 \sigma^2_{s-x} dx$$

$$+ \int_a^\infty \Big(g(x+2\Delta_n) - 2g(x+\Delta_n) + g(x)\Big)^2 \sigma^2_{s-x} dx \le C(\overline{\pi}^\diamond_n(\varepsilon) + \Delta_n^2).$$

This completes the proof of Lemma 3. □

Finally, the last lemma gives a bound for the correlation function $r_n^\diamond(j)$.

Lemma 4. *Under Assumption 2 there exists a sequence* $(h(j))_{j\ge 1}$ *such that*

$$|r_n^\diamond(j)| \le h(j), \qquad \sum_{j=1}^\infty h(j) < \infty, \tag{45}$$

for all $j\ge 1$.

Proof of Lemma 4: This result follows directly from Lemma 1 in [7]. Recall that $r_n^{\diamond}(j) \to \rho^{\diamond}(j)$ and $\sum_{j=1}^{\infty} |\rho^{\diamond}(j)| < \infty$, so the assertion is not really surprising. □

Observe that Lemma 4 implies the conditions (16) and (17).

5.1 *Proof of Theorem 1*

In the following we will prove Theorems 1 and 2 only for $k = 1$, $p_1 = p$. The general case can be obtained in a similar manner by an application of the Hölder inequality.

Note that $MPV^{\diamond}(X, p)_t^n$ is increasing in t and the limit process of (22) is continuous in t. Thus, it is sufficient to show the pointwise convergence

$$MPV^{\diamond}(X, p)_t^n \xrightarrow{\mathbb{P}} m_p \int_0^t |\sigma_s|^p ds,$$

where $m_p = \mathbb{E}(|N(0,1)|^p)$. We perform the proof of Theorem 1 in two steps.

- *The crucial approximation*: First of all, we prove that we can use the approximation $\diamond_i^n X \approx \sigma_{(i-2)\Delta_n} \diamond_i^n G$ without changing the limit of Theorem 1, i.e. we show that

$$\Delta_n (\tau_n^{\diamond})^{-p} \sum_{i=2}^{[t/\Delta_n]} \Big(| \diamond_i^n X|^p - |\sigma_{(i-2)\Delta_n} \diamond_i^n G|^p \Big) \xrightarrow{\mathbb{P}} 0. \tag{46}$$

  An application of the inequality $||x|^p - |y|^p| \le p|x-y|(|x|^{p-1} + |y|^{p-1})$ for $p > 1$ and $||x|^p - |y|^p| \le |x-y|^p$ for $p \le 1$, (42) and the Cauchy-Schwarz inequality implies that the above convergence follows from

$$\Delta_n (\tau_n^{\diamond})^{-2} \sum_{i=2}^{[t/\Delta_n]} \mathbb{E}(| \diamond_i^n X - \sigma_{(i-2)\Delta_n} \diamond_i^n G|^2) \longrightarrow 0. \tag{47}$$

  Observe the decomposition

$$\diamond_i^n X - \sigma_{(i-2)\Delta_n} \diamond_i^n G = A_i^n + B_i^{n,\varepsilon} + C_i^{n,\varepsilon}$$

  with

$$A_i^n = \int_{(i-1)\Delta_n}^{i\Delta_n} g(i\Delta_n - s)(\sigma_s - \sigma_{(i-2)\Delta_n}) W(ds)$$
$$+ \int_{(i-2)\Delta_n}^{(i-1)\Delta_n} \Big(g(i\Delta_n - s) - 2g((i-1)\Delta_n - s) \Big) (\sigma_s - \sigma_{(i-2)\Delta_n}) W(ds)$$

$$B_i^{n,\varepsilon} = \int_{(i-2)\Delta_n-\varepsilon}^{(i-2)\Delta_n} \Big(g(i\Delta_n - s) - 2g((i-1)\Delta_n - s) + g((i-2)\Delta_n - s)\Big)\sigma_s W(ds)$$

$$- \sigma_{(i-2)\Delta_n} \int_{(i-2)\Delta_n-\varepsilon}^{(i-2)\Delta_n} g(i\Delta_n - s) - 2g((i-1)\Delta_n - s) + g((i-2)\Delta_n - s)W(ds)$$

$$C_i^{n,\varepsilon} = \int_{-\infty}^{(i-2)\Delta_n-\varepsilon} \Big(g(i\Delta_n - s) - 2g((i-1)\Delta_n - s) + g((i-2)\Delta_n - s)\Big)\sigma_s W(ds)$$

$$- \sigma_{(i-2)\Delta_n} \int_{-\infty}^{(i-2)\Delta_n-\varepsilon} g(i\Delta_n - s) - 2g((i-1)\Delta_n - s) + g((i-2)\Delta_n - s)W(ds)$$

Lemma 3 and the boundedness of σ imply that

$$\Delta_n(\tau_n^{\diamond})^{-2} \sum_{i=2}^{[t/\Delta_n]} \mathbb{E}(|C_i^{n,\varepsilon}|^2) \le C\overline{\pi}_n^{\diamond}(\varepsilon), \tag{48}$$

and by (11) and Assumption 2(i) we deduce that

$$\Delta_n(\tau_n^{\diamond})^{-2} \sum_{i=2}^{[t/\Delta_n]} \mathbb{E}(|C_i^{n,\varepsilon}|^2) \longrightarrow 0,$$

as $n \to \infty$, for all $\varepsilon > 0$. Next, set $v(s,\eta) = \sup\{|\sigma_s - \sigma_r|^2 |\ r \in [-t,t],\ |r-s| \le \eta\}$ for $s \in [-t,t]$ and denote by $\Delta\sigma$ the jump process associated with σ. We obtain the inequality

$$\Delta_n(\tau_n^{\diamond})^{-2} \sum_{i=2}^{[t/\Delta_n]} \mathbb{E}(|A_i^n|^2) \le \Delta_n \sum_{i=2}^{[t/\Delta_n]} \mathbb{E}(v((i-2)\Delta_n, 2\Delta_n)) \tag{49}$$

$$\le \lambda + \Delta_n \mathbb{E}\Big(\sum_{s\in[-t,t]} |\Delta\sigma_s|^2 1_{\{|\Delta\sigma_s|\ge\lambda\}}\Big) = \theta(\lambda, n)$$

for any $\lambda > 0$. We readily deduce that

$$\lim_{\lambda\to 0} \limsup_{n\to\infty} \theta(\lambda, n) = 0.$$

Next, observe the decomposition $B_i^{n,\varepsilon} = B_i^{n,\varepsilon}(1) + B_i^{n,\varepsilon}(2)$ with

$$B_i^{n,\varepsilon}(1) = \int_{(i-2)\Delta_n-\varepsilon}^{(i-2)\Delta_n} \Big(g(i\Delta_n - s) - 2g((i-1)\Delta_n - s) + g((i-2)\Delta_n - s)\Big)$$

$$\times (\sigma_s - \sigma_{(i-2)\Delta_n-\varepsilon})W(ds)$$

$$B_i^{n,\varepsilon}(2) = (\sigma_{(i-2)\Delta_n-\varepsilon} - \sigma_{(i-2)\Delta_n}) \times \int_{(i-2)\Delta_n-\varepsilon}^{(i-2)\Delta_n} g(i\Delta_n - s) - 2g((i-1)\Delta_n - s) + g((i-2)\Delta_n - s)W(ds).$$

We deduce that

$$\Delta_n(\tau_n^{\diamond})^{-2} \sum_{i=2}^{[t/\Delta_n]} \mathbb{E}(|B_i^{n,\varepsilon}(1)|^2) \leq \Delta_n \sum_{i=2}^{[t/\Delta_n]} \mathbb{E}(v((i-2)\Delta_n, \varepsilon)),$$

$$\Delta_n(\tau_n^{\diamond})^{-2} \sum_{i=2}^{[t/\Delta_n]} \mathbb{E}(|B_i^{n,\varepsilon}(2)|^2) \leq \Delta_n \sum_{i=2}^{[t/\Delta_n]} \mathbb{E}(v((i-2)\Delta_n, \varepsilon)^2)^{\frac{1}{2}}. \tag{50}$$

By using the same arguments as in (49) we conclude that both terms converge to zero and we obtain (47), which completes the proof of Theorem 1. □

- *The blocking technique*: Having justified the approximation $\diamond_i^n X \approx \sigma_{(i-2)\Delta_n} \diamond_i^n G$ in the previous step, we now apply a blocking technique for $\sigma_{(i-2)\Delta_n} \diamond_i^n G$: we divide the interval $[0,t]$ into big sub-blocks of the length l^{-1} and freeze the intermittency process σ at the beginning of each big sub-block. Later we let l tend to infinity.

  For any fixed $l \in \mathbb{N}$, observe the decomposition

$$MPV^{\diamond}(X,p)_t^n - m_p \int_0^t |\sigma_s|^p ds = \Delta_n(\tau_n^{\diamond})^{-p} \sum_{i=2}^{[t/\Delta_n]} \left(|\diamond_i^n X|^p - |\sigma_{(i-2)\Delta_n} \diamond_i^n G|^p \right) + R_t^{n,l},$$

where

$$\begin{aligned} R_t^{n,l} &= \Delta_n(\tau_n^{\diamond})^{-p} \Big(\sum_{i=2}^{[t/\Delta_n]} |\sigma_{(i-2)\Delta_n} \diamond_i^n G|^p - \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \sum_{i \in I_l(j)} |\diamond_i^n G|^p \Big) \\ &+ \Big(\Delta_n(\tau_n^{\diamond})^{-p} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \sum_{i \in I_l(j)} |\diamond_i^n G|^p - m_p l^{-1} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \Big) \\ &+ m_p \Big(l^{-1} \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p - \int_0^t |\sigma_s|^p ds \Big), \end{aligned}$$

and

$$I_l(j) = \Big\{ i \,|\, i\Delta_n \in \Big(\frac{j-1}{l}, \frac{j}{l} \Big] \Big\}, \qquad j \geq 1.$$

Notice that the third summand in the above decomposition converges to 0 in probability due to Riemann integrability of σ. By Theorem 1 in [8] we know that

$MPV^{\diamond}(G,p)_t^n \xrightarrow{\text{u.c.p.}} m_p t$, because the condition (16) is satisfied (see Lemma 4). This implies the negligibility of the second summand in the decomposition when we first let $n \to \infty$ and then $l \to \infty$. As σ is càdlàg and bounded on compact intervals, we finally deduce that

$$\lim_{l\to\infty} \limsup_{n\to\infty} \mathbb{P}(|R_t^{n,l}| > \varepsilon) = 0,$$

for any $\varepsilon > 0$. This completes the proof of the second step and of Theorem 1. □

5.2 *Proof of Theorem 2*

Here we apply the same scheme of the proof as for Theorem 1. We start with the justification of the approximation $\diamond_i^n X \approx \sigma_{(i-2)\Delta_n} \diamond_i^n G$ and proceed with the blocking technique.

- *The crucial approximation:* Here we prove that

$$\Delta_n^{1/2}(\tau_n^{\diamond})^{-p} \sum_{i=2}^{[t/\Delta_n]} \Big(|\diamond_i^n X|^p - |\sigma_{(i-2)\Delta_n} \diamond_i^n G|^p\Big) \xrightarrow{\mathbb{P}} 0. \tag{51}$$

Again we apply the inequality $||x|^p - |y|^p| \le p|x-y|(|x|^{p-1} + |y|^{p-1})$ for $p > 1$, $||x|^p - |y|^p| \le |x-y|^p$ for $p \le 1$ and (42) to deduce that

$$\Delta_n^{1/2}(\tau_n^{\diamond})^{-p} \sum_{i=2}^{[t/\Delta_n]} \mathbb{E}\Big(\Big|| \diamond_i^n X|^p - |\sigma_{(i-2)\Delta_n} \diamond_i^n G|^p\Big|\Big)| \le \Delta_n^{1/2}(\tau_n^{\diamond})^{-(p\wedge 1)}$$

$$\times \sum_{i=2}^{[t/\Delta_n]} \big(\mathbb{E}(| \diamond_i^n X - \sigma_{(i-2)\Delta_n} \diamond_i^n G|^2)\big)^{\frac{p\wedge 1}{2}} .$$

Now we use a similar decomposition as in the proof of Theorem 1:

$$\diamond_i^n X - \sigma_{(i-2)\Delta_n} \diamond_i^n G = A_i^n + B_i^{n,\varepsilon_n^{(1)}} + \sum_{j=1}^{l} C_i^{n,\varepsilon_n^{(j)},\varepsilon_n^{(j+1)}},$$

where A_i^n, $B_i^{n,\varepsilon_n^{(1)}}$ are defined as above, $0 < \varepsilon_n^{(1)} < \cdots < \varepsilon_n^{(l)} < \varepsilon_n^{(l+1)} = \infty$ and

$$C_i^{n,\varepsilon_n^{(j)},\varepsilon_n^{(j+1)}} = \int_{(i-2)\Delta_n-\varepsilon_n^{(j+1)}}^{(i-2)\Delta_n-\varepsilon_n^{(j)}} \Big(g(i\Delta_n - s) - 2g((i-1)\Delta_n - s) + g((i-2)\Delta_n - s)\Big)\sigma_s W(ds)$$
$$-\sigma_{(i-2)\Delta_n} \int_{(i-2)\Delta_n-\varepsilon_n^{(j+1)}}^{(i-2)\Delta_n-\varepsilon_n^{(j)}} g(i\Delta_n - s) - 2g((i-1)\Delta_n - s) + g((i-2)\Delta_n - s)W(ds).$$

An application of Assumptions 1, 2 and 3-γ, for $\gamma \in (0,1]$ with $\gamma(p \wedge 1) > \frac{1}{2}$, and Lemma 3 implies that (recall that σ is bounded on compact intervals)

$$\Delta_n^{1/2}(\tau_n^{\diamond})^{-p} \sum_{i=2}^{[t/\Delta_n]} \big(\mathbb{E}(|A_i^n|^2)\big)^{\frac{p\wedge 1}{2}} \leq C\Delta_n^{\gamma(p\wedge 1)-\frac{1}{2}},$$
$$\Delta_n^{1/2}(\tau_n^{\diamond})^{-p} \sum_{i=2}^{[t/\Delta_n]} \Big(\mathbb{E}(|B_i^{n,\varepsilon_n^{(1)}}|^2)\Big)^{\frac{p\wedge 1}{2}} \leq C\Delta_n^{-1/2}|\varepsilon_n^{(1)}|^{\gamma(p\wedge 1)},$$
$$\Delta_n^{1/2}(\tau_n^{\diamond})^{-p} \sum_{i=2}^{[t/\Delta_n]} \Big(\mathbb{E}(|C_i^{n,\varepsilon_n^{(j)},\varepsilon_n^{(j+1)}}|^2)\Big)^{\frac{p\wedge 1}{2}} \leq$$
$$\leq C\Delta_n^{-1/2}|\varepsilon_n^{(j+1)}|^{\gamma(p\wedge 1)}|\overline{\pi}_n^{\diamond}(\varepsilon_n^{(j+1)}) - \overline{\pi}_n^{\diamond}(\varepsilon_n^{(j)})|^{\frac{p\wedge 1}{2}},$$
$$\Delta_n^{1/2}(\tau_n^{\diamond})^{-p} \sum_{i=2}^{[t/\Delta_n]} \Big(\mathbb{E}(|C_i^{n,\varepsilon_n^{(l)},\varepsilon_n^{(l+1)}}|^2)\Big)^{\frac{p\wedge 1}{2}} \leq C\Delta_n^{-1/2}\overline{\pi}_n^{\diamond}(\varepsilon_n^{(l)})^{\frac{p\wedge 1}{2}}, \tag{52}$$

for $1 \leq j \leq l-1$. In [8] (see Lemma 3 therein) we have proved the following result: if the condition (18) is satisfied then there exist sequences

$$0 < \varepsilon_n^{(1)} < \cdots < \varepsilon_n^{(l)} < \varepsilon_n^{(l+1)} = \infty$$

such that all terms on the right-hand side of (52) converge to 0.

Set $\lambda = (3-2\beta)(1-\delta)$ for some $\delta > 0$ such that $\lambda > 1/(p \wedge 1)$. This is possible, because $3 - 2\beta \in (2,4)$ and the assumptions of Theorem 2 imply that $p > 1/2$. We obtain that

$$\overline{\pi}_n^{\diamond}(\varepsilon_n) \leq C\Delta_n^{\lambda(1-\kappa)},$$

for any $\varepsilon_n = \Delta_n^{\kappa}$, $\kappa \in (0,1)$, by (11) and Assumption 2(i). Thus, we deduce (18) which implies the convergence of (51). □

- *The blocking technique*: Again we only consider the case $d = 1$, $k = 1$ and $p_1 = p$. We recall the decomposition from the proof of Theorem 1:

$$\Delta_n^{-1/2}\Big(MPV^{\diamond}(X,p)_t^n - m_p\int_0^t |\sigma_s|^p ds\Big)$$
$$= \Delta_n^{-1/2}\Big(\Delta_n(\tau_n^{\diamond})^{-p}\sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \sum_{i\in I_l(j)} |\diamond_i^n G|^p - m_p l^{-1}\sum_{j=1}^{[t/l\Delta_n]} |\sigma_{\frac{j-1}{l}}|^p\Big)$$
$$+\Delta_n^{1/2}(\tau_n^{\diamond})^{-p}\sum_{i=2}^{[t/\Delta_n]}\Big(|\diamond_i^n X|^p - |\sigma_{(i-2)\Delta_n}\diamond_i^n G|^p\Big) + \overline{R}_t^{n,l}, \tag{53}$$

where

$$\overline{R}_t^{n,l} = \Delta_n^{1/2}(\tau_n^{\diamond})^{-p}\Big(\sum_{i=2}^{[t/\Delta_n]} |\sigma_{(i-2)\Delta_n} \diamond_i^n G|^p - \sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \sum_{i\in I_l(j)} |\diamond_i^n G|^p\Big)$$

$$+m_p \Delta_n^{-1/2}\Big(l^{-1}\sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p - \int_0^t |\sigma_s|^p ds\Big).$$

Note that the negligibility of the second summand in the decomposition (53) has been shown in the previous step. The convergence

$$\lim_{l\to\infty} \limsup_{n\to\infty} \mathbb{P}(|\overline{R}_t^{n,l}| > \varepsilon) = 0,$$

for any $\varepsilon > 0$, has been shown in [7] (see the proof of Theorem 7 therein). Finally, we concentrate on the first summand of the decomposition (53). By Remark 11 in [8] we know that $(G_t, \Delta_n^{-1/2}(MPV^{\diamond}(G,p)_t^n - m_p t)) \Rightarrow (G_t, \sqrt{\mu}W_t')$, where μ is defined by (25), because $r_n^{\diamond}(j) \to \rho^{\diamond}(j)$ and condition (17) holds (see again Lemma 4). An application of the condition D'' from Proposition 2 in [1] shows that

$$\Delta_n^{-1/2}(MPV^{\diamond}(G,p)_t^n - m_p t) \xrightarrow{st} \sqrt{\mu}W_t'.$$

Now we deduce by the properties of stable convergence:

$$\Delta_n^{-1/2}\Big(\Delta_n(\tau_n^{\diamond})^{-p}\sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \sum_{i\in I_l(j)} |\diamond_i^n G|^p - m_p l^{-1}\sum_{j=1}^{[t/l\Delta_n]} |\sigma_{\frac{j-1}{l}}|^p\Big)$$

$$\xrightarrow{st} \sqrt{\mu}\sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \Delta_j^l W',$$

for any fixed l. On the other hand, we have that

$$\sqrt{\mu}\sum_{j=1}^{[lt]} |\sigma_{\frac{j-1}{l}}|^p \Delta_j^l W' \xrightarrow{\mathbb{P}} \sqrt{\mu}\int_0^t |\sigma_s|^p dW_s'$$

as $l \to \infty$. This completes the proof of Theorem 2. □

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Retrieving Information from Subordination

Jean Bertoin and Marc Yor

Abstract We recall some instances of the recovery problem of a signal process hidden in an observation process. Our main focus is then to show that if $(X_s,\ s \geq 0)$ is a right-continuous process, $Y_t = \int_0^t X_s \mathrm{d}s$ its integral process and $\tau = (\tau_u, u \geq 0)$ a subordinator, then the time-changed process $(Y_{\tau_u},\ u \geq 0)$ allows to retrieve the information about $(X_{\tau_v},\ v \geq 0)$ when τ is stable, but not when τ is a gamma subordinator. This question has been motivated by a striking identity in law involving the Bessel clock taken at an independent inverse Gaussian variable.

Keywords Recovery problem • Subordination • Bougerol's identity

Mathematics Subject Classification (2010): 60G35, 60G51

1 Introduction and Motivations Stemming from Hidden Processes

Many studies of random phenomena involve several sources of randomness. To be more specific, a random phenomenon is often modeled as the combination $C = \Phi(X, X')$ of two processes X and X' which can be independent or correlated,

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for some functional Φ acting on pairs of processes. In this framework, it is natural to ask whether one can recover X from C, and if not, what is the information on X that can be recovered from C? We call this the *recovery problem* of X given C. Here are two well-known examples of this problem.

- *Markovian filtering:* There C is the observation process defined for every $t \geq 0$ by $C_t = S_t + B_t$ where $S_t = \int_0^t h(X_s)\mathrm{d}s$ is the signal process arising from a Markov process X and $B = (B_t, t \geq 0)$ is an independent Brownian motion. Then the recovery problem translates in the characterization of the filtering process, that is the conditional law of X_t given the sigma-field $\mathscr{C}_t = \sigma(C_s, s \leq t)$. We refer to Kunita [6] for a celebrated discussion.

  In the simplest case when X remains constant as time passes, which yields $h(X_t) \equiv A$ where A is a random variable, note that A can be recovered in infinite horizon by $A = \lim_{t\to\infty} t^{-1}C_t$, but not in finite horizon. More precisely, it is easily shown that for a Borel function $f \geq 0$, there is the identity

$$\mathbb{E}(f(A) \mid \mathscr{C}_t) = \frac{\int f(a)\mathscr{E}_t^a \mu(\mathrm{d}a)}{\int \mathscr{E}_t^a \mu(\mathrm{d}a)}$$

  where μ is the law of A and $\mathscr{E}_t^a = \exp(aC_t - ta^2/2)$; see Chap. 1 in [9].
- *Brownian subordination:* An important class of Lévy processes may be represented as

$$C_t = B_{\tau_t}, \qquad t \geq 0,$$

  where τ a subordinator and B is again a Brownian motion (or more generally a Lévy process) which is independent of τ; see for instance Chap. 6 in [7]. Geman, Madan and Yor [4, 5] solved the recovery problem of τ hidden in C; we refer the reader to these papers for the different recovery formulas.

There exist of course other natural examples in the literature; we now say a few words about the specific recovery problem which we will treat here and the organization of the remainder of this paper.

We will consider the recovery problem when the signal is $Y_t = \int_0^t X_s \mathrm{d}s$ and this signal is only perceived at random times induced by a subordinator τ. By this, we mean that the observation process is given by $C = Y \circ \tau$, and we seek to recover the subordinate process $X \circ \tau$. The precise formulation of the framework and our results will be made in Sect. 2. Proofs of the results found in Sect. 2 are presented in Sect. 3. Finally, in Sect. 4, we apply the results of Sect. 2 to an identity in law involving a Bessel process, which is equivalent to Bougerol's identity [2] and has provided the initial motivation of this work.

2 Framework and Main Statements

We consider on some probability space $(\Omega, \mathscr{A}, \mathbb{P})$ an $\mathbb{R}^d$-valued process $(X_s, s \geq 0)$ with right-continuous sample paths, and its integral process

$$Y_t = \int_0^t X_s \mathrm{d}s, \qquad t \geq 0\,.$$

Let also $(\tau_u,\ u \geq 0)$ denote a stable subordinator with index $\alpha \in (0,1)$. We stress that we do not require X and τ to be independent. We are interested in comparing the information embedded in the processes $\hat{X}$ and $\hat{Y}$ which are obtained from X and Y by the same time-change based on τ, namely

$$\hat{X}_u = X_{\tau_u} \text{ and } \hat{Y}_u = Y_{\tau_u}, \qquad u \geq 0\,.$$

We denote by $\left(\hat{\mathscr{X}}_u\right)_{u\geq 0}$ the usual augmentation of the natural filtration generated by the process $\hat{X}$, i.e. the smallest $\mathbb{P}$-complete and right-continuous filtration to which $\hat{X}$ is adapted. Likewise, we write $\left(\hat{\mathscr{Y}}_u\right)_{u\geq 0}$ for the usual augmentation of the natural filtration of $\hat{Y}$ and state our main result.

Theorem 1. *There is the inclusion $\hat{\mathscr{X}}_u \subset \hat{\mathscr{Y}}_u$ for every $u \geq 0$.*

We stress that for $u > 0$, in general $\hat{Y}_u$ cannot be recovered from the sole process $\hat{X}$, and then the stated inclusion is strict. An explicit recovery formula for $\hat{X}_u$ in terms of the jumps of $\hat{Y}$ will be given in the proof of Theorem 1 (see Sect. 3 below).

A perusal of the proof of Theorem 1 shows that it can be extended to the case when it is only assumed that τ is a subordinator such that the tail of its Lévy measure is regularly varying at 0 with index $-\alpha$, which suggests that this result might hold for more general subordinators. On the other hand, if $(N_v,\ v \geq 0)$ is any increasing step-process issued from 0, such as for instance a Poisson process, then the time-changed process $(Y_{N_v},\ v \geq 0)$ stays at 0 until the first jump time of N which is strictly positive a.s. This readily implies that the germ-σ-field

$$\bigcap_{v>0} \sigma(Y_{N_u}, u \leq v)$$

is trivial, in the sense that every event of this field has probability either 0 or 1. Focussing on subordinators with infinite activity, it is interesting to point out that Theorem 1 fails when one replaces the stable subordinator τ by a gamma subordinator, as can be seen from the following observation (choose $X_s \equiv \xi$).

Proposition 1. *Let $\gamma = (\gamma_t,\ t \geq 0)$ be a gamma-subordinator and ξ a random variable with values in $(0,\infty)$ which is independent of γ. Then the germ-σ-field*

$$\bigcap_{t>0} \sigma(\xi\gamma_s,\ s \leq t)$$

is trivial. On the other hand, we also have

$$\bigcap_{t>0} (\sigma(\xi) \vee \sigma(\gamma_s,\ s \leq t)) = \sigma(\xi)\,.$$

It is natural to investigate a similar question in the framework of stochastic integration. For the sake of simplicity, we shall focus on the one-dimensional case. We thus consider a real valued Brownian motion $(B_t,\ t \geq 0)$ in some filtration$(\mathscr{F}_t)_{t\geq 0}$ and an $(\mathscr{F}_t)$-adapted continuous process $(X_t,\ t \geq 0)$, and consider the stochastic integral

$$I_t = \int_0^t X_s \mathrm{d}B_s, \qquad t \geq 0\,.$$

We claim the following.

Proposition 2. *Fix $\eta > 0$ and assume that the sample paths of $(X_t,\ t \geq 0)$ are Hölder-continuous with exponent η a.s. Suppose also that $(\tau_v,\ v \geq 0)$ is a stable subordinator of index $\alpha \in (0,1)$, which is independent of $\mathscr{F}_\infty$. Then the usual augmentation $(\hat{\mathscr{I}}_v)_{v\geq 0}$ of the natural filtration generated by the subordinate stochastic integral $\left(\hat{I}_v = I_{\tau_v}, v \geq 0\right)$ contains the one generated by $(|X_{\tau_v}|, v \geq 0)$.*

3 Proofs

3.1 *Proof of Theorem 1*

For the sake of simplicity, we henceforth suppose that the tail of the Lévy measure of the stable subordinator τ is $x \mapsto x^{-\alpha}$, which induces no loss of generality. We shall need the following elementary version of the Law of Large Numbers for the jumps $(\Delta\tau_s = \tau_s - \tau_{s-}, s > 0)$ of a stable subordinator.

Fix any $\beta > 2/\alpha$ and introduce for any given $b \in \mathbb{R}$ and $\varepsilon > 0$

$$N_{\varepsilon,b} = \mathrm{Card}\{s \leq \varepsilon : b\Delta\tau_s > \varepsilon^\beta\}\,.$$

Note that $N_{\varepsilon,b} \equiv 0$ for $b \leq 0$.

Lemma 1. *We have*

$$\mathbb{P}\left(\lim_{n\to\infty} n^{1-\alpha\beta} N_{1/n,b} = b^\alpha \qquad \textit{for all } b > 0\right) = 1.$$

Remark. The rectangles $[0,\varepsilon] \times (\varepsilon^\beta, \infty)$ neither increase nor decrease with ε for $\varepsilon > 0$, so Lemma 1 does not reduce to the classical Law of Large Numbers for Poisson point processes. This explains the requirement that $\beta > 2/\alpha$.

Proof. Recall that for $b > 0$, $N_{\varepsilon,b}$ is a Poisson variable with parameter

$$\varepsilon(\varepsilon^\beta/b)^{-\alpha} = b^\alpha \varepsilon^{1-\beta\alpha}\,.$$

Chebychev's inequality thus yields the bound

$$\mathbb{P}\left(\left|n^{1-\alpha\beta}N_{1/n,b}-b^{\alpha}\right|>\frac{1}{\ln n}\right)\leq b^{2\alpha}n^{1-\alpha\beta}\ln^2 n$$

and since $1-\alpha\beta<-1$, we deduce from the Borel-Cantelli lemma that for each fixed $b>0$,

$$\lim_{n\to\infty} n^{1-\alpha\beta}N_{1/n,b}=b^{\alpha}\qquad \text{almost surely.}$$

We can then complete the proof with a standard argument of monotonicity. □

We now tackle the proof of Theorem 1 by verifying first that X_0 is $\hat{\mathscr{Y}}_0$-measurable. Let us assume that the process X is real-valued as the case of higher dimensions will then follow by considering coordinates. Set

$$J_\varepsilon=\mathrm{Card}\{s\leq\varepsilon:\Delta\hat{Y}_s>\varepsilon^{\beta}\},$$

where as usual $\Delta\hat{Y}_s=\hat{Y}_s-\hat{Y}_{s-}$. We note that

$$\Delta\hat{Y}_s-X_0\Delta\tau_s=\int_{\tau_{s-}}^{\tau_s}(X_u-X_0)\mathrm{d}u\,.$$

Hence if we set $a_\varepsilon=\sup_{0\leq u\leq\tau_\varepsilon}|X_u-X_0|$, then

$$(X_0-a_\varepsilon)\,\Delta\tau_s\leq\Delta\hat{Y}_s\leq(X_0+a_\varepsilon)\,\Delta\tau_s,$$

from which we deduce $N_{\varepsilon,X_0-a_\varepsilon}\leq J_\varepsilon\leq N_{\varepsilon,X_0+a_\varepsilon}$.

Since X has right-continuous sample paths a.s., we have $\lim_{\varepsilon\to0}a_\varepsilon=0$ a.s., and taking $\varepsilon=1/n$, we now deduce from Lemma 1 that

$$\lim_{n\to\infty} n^{1-\alpha\beta}J_{1/n}=(X_0^+)^{\alpha}\qquad\text{almost surely.}$$

Hence X_0^+ is $\hat{\mathscr{Y}}_0$-measurable, and the same argument also shows that X_0^- is $\hat{\mathscr{Y}}_0$-measurable.

Now that we have shown that X_0 is $\hat{\mathscr{Y}}_0$-measurable, it follows immediately that for every $v\geq0$, the variable $\hat{X}_v$ is $\hat{\mathscr{Y}}_v$-measurable. Indeed, define $\tau'_u=\tau_{v+u}-\tau_v$ and $X'_v=X_{v+\tau_v}$. Then τ' is again a stable(α) subordinator and X' a right-continuous process, and

$$\hat{Y}_{v+u}-\hat{Y}_v=\int_0^{\tau'_u}X'_s\mathrm{d}s\,.$$

Hence $X'_0=\hat{X}_v$ is measurable with respect to the $\mathbb{P}$-complete germ-σ-field generated by the process $(\hat{Y}_{v+u}-\hat{Y}_v,\ u\geq0)$, and *a fortiori* to $\hat{\mathscr{Y}}_v$.

Thus we have shown that the process $\hat{X}$ is adapted to the right-continuous filtration $\left(\hat{\mathscr{Y}}_v\right)_{v\geq0}$. Since by definition the latter is $\mathbb{P}$-complete and right-continuous, Theorem 1 is established. □

3.2 *Proof of Proposition 1*

Here it is convenient to agree that Ω denotes the space of càdlàg paths $\omega : [0,\infty) \to \mathbb{R}_+$ endowed with the right-continuous filtration $(\mathscr{A}_t)_{t\geq 0}$ generated by the canonical process $\omega_t = \omega(t)$. We write $\mathbb{Q}$ for the law on Ω of the process $(\xi\gamma_t,\, t \geq 0)$.

It is well known that for every $x > 0$ and $t > 0$, the distribution of the process $(x\gamma_s, 0 \leq s \leq t)$ is absolutely continuous with respect to that of the gamma process $(\gamma_s, 0 \leq s \leq t)$ with density $x^{-t} \exp((1-1/x)\gamma_t)$. Because ξ and γ are independent, this implies that for any event $\Lambda \in \mathscr{A}_r$ with $r < t$

$$\mathbb{Q}(\Lambda) = \mathbb{E}\left(\xi^{-t} \exp((1-1/\xi)\gamma_t)\, \mathbf{1}_{\{\gamma\in\Lambda\}}\right) .$$

Observe that

$$\lim_{t\to 0+} \xi^{-t} \exp((1-1/\xi)\gamma_t) = 1 \qquad \text{a.s.}$$

and the convergence also holds in $L^1(\mathbb{P})$ by an application of Scheffé's lemma (alternatively, one may also invoke the convergence of backwards martingales). We deduce that for every $\Lambda \in \mathscr{A}_0$, we have $\mathbb{Q}(\Lambda) = \mathbb{P}(\gamma \in \Lambda)$ and the right-hand-side must be 0 or 1 because the gamma process satisfies the Blumenthal's 0-1 law. On the other hand, the independence of ξ and γ yields that the second germ sigma field is $\sigma(\xi)$. □

Remarks. We point out that Proposition 1 holds more generally when γ is replaced by a subordinator with logarithmic singularity, also called of class $(\mathscr{L})$, in the sense that the drift coefficient is zero and the Lévy measure is absolutely continuous with density g such that $g(x) = g_0 x^{-1} + G(x)$ where g_0 is some strictly positive constant and $G : (0,\infty) \to \mathbb{R}$ a measurable function such that

$$\int_0^1 |G(x)|\mathrm{d}x < \infty\,,\ g(x) \geq 0\,,\ \text{and} \int_1^\infty g(x)\mathrm{d}x < \infty\,.$$

Indeed, it has been shown by von Renesse et al. [8] that such subordinators enjoy a quasi-invariance property analogous to that of the gamma subordinator, and this is the key to Proposition 1.

Thanks to Theorem 1, if we replace in Proposition 1 the gamma process by τ, a stable subordinator, then both germ sigma fields are equal to $\sigma(\xi)$.

3.3 *Proof of Proposition 2*

The guiding line is similar to that of the proof of Theorem 1. In particular it suffices to verify that $|X_0|$ is measurable with respect to the germ-σ-field $\hat{\mathscr{I}}_0$.

Because Brownian motion B and subordinator τ are independent, the subordinate Brownian motion $(\hat{B}_v = B_{\tau_v},\, v \geq 0)$ is a symmetric stable Lévy process with index

2α. With no loss of generality, we may suppose that the tail of its Lévy measure Π is given by $\Pi(\mathbb{R}\backslash[-x,x]) = x^{-2\alpha}$. As a consequence, for every $\beta > 2/\alpha$ and $\varepsilon > 0$ and $b \in \mathbb{R}$, if one defines

$$N_{\varepsilon,b} = \mathrm{Card}\{s \le \varepsilon : |b\Delta \hat{B}_s|^2 > \varepsilon^\beta\},$$

then $N_{\varepsilon,b}$ is a Poisson variable with parameter $|b|^{2\alpha}\varepsilon^{1-\alpha\beta}$, and this readily yields (see Lemma 1)

$$\lim_{n\to\infty} n^{1-\alpha\beta} N_{1/n,b} = |b|^{2\alpha} \qquad \text{for all } b \in \mathbb{R}, \text{ almost-surely.} \tag{1}$$

Next set

$$J_\varepsilon = \mathrm{Card}\{s \le \varepsilon : |\Delta \hat{I}_s|^2 > \varepsilon^\beta\},$$

where as usual $\hat{I}_s = I_{\tau_s}$, and observe that

$$\Delta \hat{I}_s = X_0 \Delta \hat{B}_s + (X_{\tau_{s-}} - X_0)\Delta \hat{B}_s + \int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}})\mathrm{d}B_u\,. \tag{2}$$

Recall the assumption that the paths of X are Hölder-continuous with exponent $\eta > 0$, so the $(\mathscr{F}_t)$-stopping time

$$T = \inf\left\{u > 0 : \sup_{0\le v<u} (u-v)^{-\eta}|X_u - X_v|^2 > 1\right\}$$

is strictly positive a.s. In particular, if we write $\Lambda_\varepsilon = \{\tau_\varepsilon < T\}$, then $\mathbb{P}(\Lambda_\varepsilon)$ tends to 1 as $\varepsilon \to 0+$.

We fix $a > 0$, we consider

$$K_{\varepsilon,a} = \mathrm{Card}\left\{s \le \varepsilon : \left|\int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}})\mathrm{d}B_u\right|^2 > a\varepsilon^\beta\right\},$$

and we claim that

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha\beta-1}\mathbb{E}(K_{\varepsilon,a}, \Lambda_\varepsilon) = 0\,. \tag{3}$$

If we take (3) for granted, then we can complete the proof by an easy adaptation of the argument in Theorem 1. Indeed, we can then find a strictly increasing sequence of integers $(n(k), k \in \mathbb{N})$ such that with probability one, for all rational numbers $a > 0$

$$\lim_{k\to\infty} n(k)^{1-\alpha\beta} K_{1/n(k),a} = 0\,. \tag{4}$$

We observe from (2) that for any $a \in (0, 1/2)$, if $|\Delta \hat{I}_s|^2 > \varepsilon^\beta$, then necessarily either

$$|X_0 \Delta \hat{B}_s|^2 > (1-2a)^2\varepsilon^\beta,$$

or

$$|(X_{\tau_{s-}} - X_0)\varDelta \hat{B}_s|^2 > a^2\varepsilon^\beta,$$

or

$$\left|\int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}})\mathrm{d}B_u\right|^2 > a^2\varepsilon^\beta\,.$$

As

$$\lim_{\varepsilon\to 0+} \sup_{0\le s\le\varepsilon} |X_{\tau_{s-}} - X_0| = 0,$$

this easily entails, using (1) and (4), that

$$\begin{aligned}\limsup_{k\to\infty} n(k)^{1-\alpha\beta} J_{1/n(k)} &\le \lim_{k\to\infty} n(k)^{1-\alpha\beta} N_{1/n(k),(1-2a)^{-1}|X_0|}\\ &= (1-2a)^{-2\alpha}|X_0|^{2\alpha}, \qquad \text{a.s.}\end{aligned}$$

where the identity in the second line stems from (1). A similar argument also gives

$$\liminf_{k\to\infty} n(k)^{1-\alpha\beta} J_{1/n(k)} \ge (1+2a)^{-2\alpha}|X_0|^{2\alpha}, \qquad \text{a.s.,}$$

and as a can be chosen arbitrarily close to 0, we conclude that

$$\lim_{k\to\infty} n(k)^{1-\alpha\beta} J_{1/n(k)} = |X_0|^{2\alpha}, \qquad \text{a.s.}$$

Hence $|X_0|$ is $\hat{\mathscr{I}}_0$-measurable.

Thus we need to establish (3). As τ is independent of $\mathscr{F}_\infty$, we have by an application of Markov's inequality that for every $s \le \varepsilon$

$$\begin{aligned}&\mathbb{P}\left(\left|\int_{\tau_{s-}}^{\tau_s} (X_u - X_{\tau_{s-}})\mathrm{d}B_u\right|^2 > a\varepsilon^\beta, \varLambda_\varepsilon \mid \tau\right)\\ &\le \frac{1}{a\varepsilon^\beta}\int_0^{\varDelta\tau_s} v^\eta \mathrm{d}v \;\le\; \frac{(\varDelta\tau_s)^{1+\eta}}{a\varepsilon^\beta}\,.\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}(K_{\varepsilon,a}, \varLambda_\varepsilon) &\le \mathbb{E}\left(\sum_{s\le\varepsilon}\left(\frac{(\varDelta\tau_s)^{1+\eta}}{a\varepsilon^\beta}\wedge 1\right)\right)\\ &= \varepsilon c\int_{(0,\infty)} x^{-1-\alpha}\left(\frac{x^{1+\eta}}{a\varepsilon^\beta}\wedge 1\right)\mathrm{d}x \;=\; O(\varepsilon^{1-\alpha\beta/(1+\eta)}),\end{aligned}$$

where for the second line we used the fact that the Lévy measure of τ is $cx^{-1-\alpha}\mathrm{d}x$ for some unimportant constant $c > 0$. This establishes (3) and hence completes the proof of our claim. □

4 Application to an Identity of Bougerol

In this section, we answer a question raised by Dufresne and Yor [3], which has motivated this work.

A result due to Bougerol [2] (see also Alili et al. [1]) states that for each fixed $t \geq 0$ there is the identity in distribution

$$\sinh(B_t) \stackrel{\text{(law)}}{=} \int_0^t \exp(B_s)\mathrm{d}W_s \tag{5}$$

where B and W are two independent one-dimensional Brownian motions. Consider now a two-dimensional Bessel process $(R_u,\ u \geq 0)$ issued from 1 and the associated clock

$$H_t = \int_0^t R_u^{-2}\mathrm{d}u, \qquad t \geq 0\,.$$

Let also $(\tau_s,\ s \geq 0)$ denote a stable $(1/2)$ subordinator independent from the Bessel process R.

In Dufresne and Yor [3], it was remarked that by combining Bougerol's identity (5) and the symmetry principle of Désiré André, there is the identity in distribution for every fixed $s \geq 0$

$$H_{\tau_s} \stackrel{\text{(law)}}{=} \tau_{a(s)}, \tag{6}$$

where $a(s) = \mathrm{Argsinh}(s) = \log\left(s + \sqrt{1+s^2}\right)$.

In [3], the authors wondered whether (6) extends at the level of processes indexed by $s \geq 0$, or equivalently whether $(\hat{H}_s = H_{\tau_s},\ s \geq 0)$ has independent increments. Theorem 1 entails that this is not the case. Indeed, it implies that the usual augmentation $\left(\hat{\mathscr{H}}_s\right)_{s\geq 0}$ of the filtration generated by $\hat{H}$ contains the one generated by $\left(\hat{R}_s = R_{\tau_s},\ s \geq 0\right)$. On the other hand, (R, H) is a Markov (additive) process, and since subordination by an independent stable subordinator preserves the Markov property, $(\hat{R}, \hat{H})$ is Markovian in its own filtration, which coincides with $\left(\hat{\mathscr{H}}_s\right)_{s\geq 0}$ by Theorem 1. It is readily seen that for any $v > 0$, the conditional distribution of $H_{\tau_{s+v}}$ given (R_{τ_s}, H_{τ_s}) does not only depend on H_{τ_s}, but on R_{τ_s} as well. Consequently the process $\hat{H}$ is not Markovian and *a fortiori* does not have independent increments.

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Asymptotic Expansions for Distributions of Sums of Independent Random Vectors

Algimantas Bikelis

Abstract We consider the asymptotic behavior of the convolution $P^{*n}(\sqrt{n}A)$ of a k-dimensional probability distribution $P(A)$ as $n \to \infty$ for A from the σ-algebra $\mathfrak{M}^k$ of Borel subsets of Euclidian space R^k or from its subclasses (often appearing in mathematical statistics). We will deal with two questions: construction of asymptotic expansions and estimating the remainder terms by using necessary and sufficient conditions. The most widely and deeply investigated cases are those where $P^{*n}(\sqrt{n}A)$ are approximated by the k-dimensional normal laws $\Phi^{*n}(A\sqrt{n})$ or by the accompanying ones $e^{n(P-E_0)}$. In this and other papers, estimating the remainder terms, we extensively use the method developed in the candidate thesis of Yu. V. Prokhorov (Limit theorems for sums of independent random variables. Candidate Thesis, Moscow, 1952) (adviser A. N. Kolmogorov) and there obtained necessary and sufficient conditions (see also Prokhorov (Dokl Akad Nauk SSSR 83(6):797–800 (1952) (in Russian); 105:645–647, 1955 (in Russian)).

Keywords Probability distributions in R^k • Convolutions • Bergström identity • Appell polynomials • Chebyshev–Cramer asymptotic expansion

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1 Introduction

We first present three theorems from the thesis of Yu. V. Prokhorov [10]. Let

$$\xi_1, \xi_2, \ldots, \xi_n, \ldots$$

be a sequence of independent identically distributed random variables with distribution function $F(x) = P\{\xi_1 < x\}$.

Theorem P4. *Let $F(x)$ satisfy one of the following two conditions:*

1. *$F(x)$ is a discrete distribution function;*
2. *There exists an integer n_0 such that $F^{*n_0}(x)$ has an absolutely continuous component.*

Then there exists a sequence $\{G_n(x)\}$ of infinitely divisible distribution functions such that

$$\|F^{*n}(x) - G_n(x)\| \to 0 \quad as \quad n \to \infty,$$

where $\| \cdot \|$ stands for the total variation.

Theorem P5. *In order that*

$$\|F^{*n}(xB_n + A_n) - G(x)\| \to 0, \quad n \to \infty,$$

for some appropriately chosen constants $B_n > 0$ and A_n and a stable distribution function $G(x)$, the following conditions are necessary and sufficient:

1. *$F^{*n}(xB_n + A_n) \to G(x), \quad n \to \infty, \quad x \in R^1$;*
2. *There exists n_0 such that*

$$\int_{-\infty}^{\infty} p_{n_0}(x)\,dx > 0,$$

where $p_{n_0}(x) = \frac{d}{dx} F^{*n_o}_{(x)}$.

Theorem P6. *Suppose that ξ_1 takes only the values $m = 0, \pm 1, \ldots$ and that the stable distribution function $G(x)$ has a density $g(x)$. Then*

$$\sum_m \left| P\{\xi_1 + \cdots + \xi_n = m\} - \frac{1}{B_n} g\Big(\frac{m - A_n}{B_n}\Big) \right| \to \infty$$

if and only if the following two conditions are satisfied:

1. *$F^{*n}(xB_n + A_n) \to G(x), \quad n \to \infty, \quad x \in R^1$;*
2. *The maximal step of the distribution of ξ_1 equals 1.*

In the case where $G(x) = \Phi(x)$ is the standard Gaussian distribution function, the following statement is proved.

Theorem 1. *Let* ξ_1 *have* 0 *mean and unit variance. In order that*

$$\|F^{*n}(x\sqrt{n}) - \Phi(x)\| = O(n^{-\delta/2}), \quad n \to \infty,$$

for some $\delta \in (0, 1]$*, the following two conditions are necessary and sufficient:*

1. $\sup_x \left|F^{*n}(x\sqrt{n}) - \Phi(x)\right| = O(n^{-\delta/2}), \quad n \to \infty;$
2. *There exists* n_0 *such that the distribution function* $F^{*n_0}(x)$ *has an absolutely continuous component.*

The theorem is proved in [2]. In the same paper, a sequence of random variables $\xi_1, \dots, \xi_n, \dots$ with values $m = 0, \pm 1, \pm 2, \dots$ is also considered. In this case, the following statement is proved.

Theorem 2. *In order that*

$$\sum_m \left| P\{\xi_1 + \cdots + \xi_n = m\} - \frac{1}{\sqrt{2\pi n}}\, e^{-m^2/2n} \right| = O(n^{-\delta/2})$$

for some $\delta \in (0, 1]$*, the following two conditions are necessary and sufficient*:

1. $\sup_x \left|F^{*n}(x\sqrt{n}) - \Phi(x)\right| = O(n^{-\delta/2}), \quad n \to \infty;$
2. *The maximal step of the distribution of* ξ_1 *is 1.*

In the case where $P(A)$ is a probability distribution defined in the k-dimensional space R^k, and $\Phi(A)$ is the standard k-dimensional normal distribution, the following theorem is proved in [3].

Theorem 3. *In order that*

$$\sup_{A \in \mathfrak{M}^k} \left|P^{*n}(A\sqrt{n}) - \Phi(A)\right| = O(n^{-\delta/2}),$$

the following two conditions are necessary and sufficient:

1. $\sup_{\|\mathbf{t}\|=1} \sup_{x \in R^1} \left|P^{*n}(\sqrt{n} A_x(\mathbf{t})) - \Phi(A_x(\mathbf{t}))\right| = O(n^{-\delta/2})$ *as* $n \to \infty$, *where* $A_x(\mathbf{t}) = \{\mathbf{u} : (\mathbf{t}, \mathbf{u}) < x\}$*,* $\|\mathbf{t}\|$ *is the length of a vector* $\mathbf{t} \in R^k$*, and* $(\mathbf{u}, \mathbf{t})$ *denotes the inner product in* R^k;
2. *There exists* n_0 *such that the distribution function* F^{*n_0} *has a absolutely continuous component.*

The statements of Theorems 1–3 remain valid if one replaces $\Phi(A)$ by "long" Chebyshev–Cramer asymptotic expansions with appropriate changes in condition (1) and with no changes in the Prokhorov conditions (in the theorems, conditions (2)); see [3].

2 Appell Polynomials

Recall that a sequence of polynomials $g_n(x)$, $n = 1, 2, \ldots$, is called an Appell polynomial set if

$$\frac{d}{dx} g_n(x) = n g_{n-1}(x),\ n = 1, 2, \ldots,\ x \in R^1;$$

see [6], p. 242.

Often, by Appell polynomials are meant the polynomials

$$A_j(z) = (-1)^j z^{j+1} \sum_{l=0}^{j-1} q_{jl} z^l \tag{1}$$

defined by

$$\left(1 + \frac{z}{\tau}\right)^{\tau} = e^z \left(1 + \sum_{j=1}^{\infty} \left(\frac{1}{\tau}\right)^j A_j(z)\right) \tag{2}$$

for $|z| < \tau$ (see [5, 8]).

The coefficients $q_{j\,l}$ satisfy the recursion formula

$$q_{jl} = \frac{(j+l) q_{j-1,l} + q_{j-1,l-1}}{j+l+1} \tag{3}$$

for $j = 1, 2, \ldots$, $l = 1, 2, \ldots, j-2$ (see [8]). For $l < 0$, $q_{jl} = 0$, and

$$q_{j0} = \frac{1}{j+1}, \qquad q_{j,j-1} = \frac{1}{2^j j!}.$$

It is known [8] that

$$q_{jl} = \sum_{\substack{\nu_1 + 2\nu_2 + \cdots + j\nu_j = j \\ \nu_1 + \nu_2 + \cdots + \nu_j = l+1}} \prod_{i=1}^{j} \left[\frac{1}{\nu_1!} \left(\frac{1}{i+1}\right)^{\nu_i}\right]$$

for $j = 1, 2, \ldots$, $l = 0, 1, \ldots, j-1$.

Estimating the remainder terms of asymptotic expansions, we will use the following lemma.

Lemma 1. *We have*

$$\sum_{l=0}^{j-1} q_{jl} \leqslant \frac{1}{2}, \quad j = 1, 2, \ldots. \tag{4}$$

The lemma can be proved by induction using (3).

Let us now estimate the remainder term

$$R_s(z,\tau) = \Big(1+\frac{z}{\tau}\Big)^{\tau} - e^{z}\Big(1+\sum_{j=1}^{s}\Big(\frac{1}{\tau}\Big)^{j} A_j(z)\Big).$$

Here z may be a complex number, e.g., the difference of characteristic functions of random vectors, $\tau > 0$, and $|z| < \tau$; $s = 1, 2, \ldots$.

Lemma 2. *We have*

$$|R_s(z,\tau)| \leqslant \begin{cases} \frac{1}{2}\Big(\frac{1}{\tau}\Big)^{s}\frac{1}{\tau-|z|}|z^{s+2}e^{z}| & \text{if } |z| < \tau, \\ \frac{1}{2}\Big(\frac{1}{\tau}\Big)^{s}\frac{1}{\tau-1}|z^{s+2}e^{z}| & \text{if } |z| = 1 \text{ and } \tau > 1, \\ \frac{1}{2}\Big(\frac{|z|}{\tau}\Big)^{s+1}\frac{\tau|z^{s+2}e^{z}|}{2(|z|-1)(\tau-|z|^2)} & \text{if } 1 < |z| < \sqrt{\tau}. \end{cases}$$

Proof. From (2) and (3) it follows that

$$R_s(z,\tau) = \sum_{j=s+1}^{\infty}\Big(\frac{1}{\tau}\Big)^{j} A_j(z)e^{z} = \Big(\frac{1}{\tau}\Big)^{s+1} z^{s+2}e^{z}\sum_{r=1}^{\infty}\Big(-\frac{z}{\tau}\Big)^{r}\sum_{l=0}^{r+s} q_{r+s+1,l}\, z^{l}.$$

Now it remains to apply inequality (4), and the lemma follows after a simple calculation.

3 Expansion of Convolutions of Measures by Appell Polynomials

Consider the convolutions of generalized finite-variation measures $\mu(B)$, $B \in \mathfrak{M}^k$:

$$\Big(\mu_0+\frac{\mu}{n}\Big)^{*n}(B) = \int_{R^k}\Big(\mu_0+\frac{\mu}{n}\Big)(B-\mathbf{x})\Big(\mu_0+\frac{\mu}{n}\Big)^{*(n-1)}(d\mathbf{x}),$$

where μ_0 is the Dirac measure, $\mathbf{0} = (0, 0, \ldots, 0) \in R^k$, $n = 1, 2, \ldots$;

$$\Big(\mu_0+\frac{\mu}{n}\Big)^{*0} = \mu_0; \qquad \mu_0 * \mu = \mu.$$

It is obvious that

$$\Big\|\Big(\mu_0+\frac{\mu}{n}\Big)^{*n}\Big\| \leqslant \Big(\Big\|\mu_0+\frac{\mu}{n}\Big\|\Big)^{n}.$$

Theorem 4. *If $\|\mu\| < n$, we have the asymptotic expansion*

$$\left(\mu_0 + \frac{\mu}{n}\right)^{*n} = e^{\mu} * \left\{\mu_0 + \sum_{j=1}^{\infty} \left(\frac{1}{n}\right)^j A_j(\mu)\right\}$$

where $n = 1, 2, \ldots$, and

$$A_j(\mu) = (-1)^j \mu^{*(j+1)} * \sum_{l=0}^{j-1} q_{jl}\, \mu^{*l}$$

is the Appell polynomial with the powers of μ are understood in the convolution sense.

Proof. Obviously,

$$\left(\mu_0 + \frac{\mu}{n}\right)^{*n} = \sum_{\nu=0}^{n} \left(\frac{1}{n}\right)^{\nu} \binom{n}{\nu} \mu^{*\nu} = \sum_{\nu=0}^{\infty} \frac{\mu^{*\nu}}{n^{\nu}} \frac{n(n-1)\ldots(n-\nu+1)}{\nu!},$$

where

$$n(n-1)\ldots(n-\nu+1) = \sum_{j=0}^{\nu-1} (-1)^j n^{\nu-j} C_{\nu}^{(j)},$$

and $C_{\nu}^{(j)}$ is the Stirling number of the first kind.

From the last two equalities it follows that

$$\begin{aligned}\left(\mu_0 + \frac{\mu}{n}\right)^{*n} &= \mu_0 + \sum_{\nu+1}^{\infty} \frac{\mu^{*\nu}}{\nu! n^{\nu}} \sum_{j=0}^{\nu-1} (-1)^j C_{\nu}^{(j)} n^{\nu-j} \\ &= \mu_0 + \sum_{j=0}^{\infty} \left(-\frac{1}{n}\right)^j \sum_{\nu=j+1}^{\infty} \frac{1}{\nu!} \mu^{*\nu} C_{\nu}^{(j)}.\end{aligned}$$

Since $C_{\nu}^{(0)} = 1$ and

$$C_{\nu}^{(j)} = \sum_{l=0}^{j-1} q_{jl}\, \nu(\nu-1)\ldots(\nu-j-l),$$

we obtain

$$\left(\mu_0+\frac{\mu}{n}\right)^{*n}=\mu_0+\sum_{\nu+1}^{\infty}\frac{1}{n!}\mu^{*\nu}C_\nu^{(0)}+$$

$$+\sum_{j=1}^{\infty}\left(-\frac{1}{n}\right)^j\sum_{\nu=j+1}^{\infty}\frac{1}{\nu!}\mu^{*\nu}\sum_{k=0}^{j-1}q_{jk}\nu(\nu-1)\dots(\nu-j-k)=$$

$$=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\mu^{*\nu}+\sum_{j=1}^{\infty}\left(-\frac{1}{n}\right)^j\sum_{k=0}^{j-1}q_{jk}\sum_{\nu=j+k+1}^{\infty}\frac{1}{(\nu-j-k-1)!}\mu^{*\nu}=$$

$$=e^{\mu}+\sum_{j=1}^{\infty}\left(-\frac{1}{n}\right)^j\sum_{k=0}^{j-1}q_{jk}\mu^{*(j+k+1)}*\left(\sum_{l=0}^{\infty}\frac{1}{l!}\mu^l\right)=$$

$$=e^{\mu}*\left\{\mu_0+\sum_{j=1}^{\infty}\left(-\frac{1}{n}\right)^j\mu^{*(j+1)}*\left(\sum_{k+1}^{j-1}q_{jk}\mu^{*k}\right)\right\}=$$

$$=e^{\mu}*\left\{\mu_0+\sum_{j=1}^{\infty}\left(\frac{1}{n}\right)^jA_j(\mu)\right\}.$$

The theorem is proved.

Theorem 5. *Let μ and μ_1 be generalized finite-variation measures in R^k. Then, for every Borel set $B\in\mathfrak{M}^k$, we have*

$$\left|\left(\mu_1*\left(\mu_0+\frac{\mu}{n}\right)\right)^{*n}(B)-\mu_1^{*n}*e^{\mu}*\left\{\mu_0+\sum_{j=1}^{s}\left(\frac{1}{n}\right)^jA_j(\mu)\right\}(B)\right|$$

$$\leqslant\begin{cases}\frac{1}{2}\left(\frac{1}{n}\right)^s\Delta(B) & \textit{if } \|\mu\|<1,\\ \frac{1}{2(n-1)}\left(\frac{1}{n}\right)^s\Delta(B) & \textit{if } \|\mu\|=1,\\ \frac{1}{2}\left(\frac{\|\mu\|}{n}\right)^s\frac{\Delta(B)}{(\|\mu\|-1)(n-\|\mu\|^2)} & \textit{if } 1<\|\mu\|<\sqrt{n},\end{cases}$$

where $n=1,2,\dots$, and

$$\Delta(B)=\sup_{\mathbf{x}}\left|\mu_1^{*n}*\mu^{*(s+2)}*e^{\mu}(B-\mathbf{x})\right|.$$

Proof. When $\|\mu\| < n$, the remainder term is

$$r_{s+1}(B) = \sum_{j=s+1}^{\infty} \Big(\frac{1}{n}\Big)^j (-1)^j e^{\mu} * \mu_1^{*n} * \mu^{*(j+1)} \sum_{l=0}^{j-1} q_{jl}\mu^{*l}(B) =$$

$$= e^{\mu} * \mu_1^{*n} * \mu^{*(s+2)} \Big(-\frac{1}{n}\Big)^{s+1} * \sum_{r=0}^{\infty} \Big(-\frac{\mu}{n}\Big)^{*r} \sum_{l=0}^{r+s} q_{r+s+1,l}\mu^{*l} =$$

$$= \Big(-\frac{1}{n}\Big)^{s+1} \int_{R^k} e^{\mu} * \mu_1^{*n} * \mu^{*(s+2)}(B-\mathbf{x})$$

$$\left(\sum_{r=0}^{\infty} \Big(-\frac{\mu}{n}\Big)^{*r} \sum_{l=0}^{r+s} q_{r+s+1,l}\mu^{*l}\right)(d\mathbf{x}).$$

From this it follows that

$$|r_{s+1}(B)| \leqslant \Big(\frac{1}{n}\Big)^{s+1} \Delta(B) \sum_{r=0}^{\infty} \left(\frac{\|\mu\|}{n}\right)^r \sum_{l=0}^{r+s} q_{r+s+1,l}(\|\mu\|)^l. \tag{5}$$

Here,

$$\sum_{r=0}^{\infty} \left(\frac{\|\mu\|}{n}\right)^r \sum_{l=0}^{r+s} q_{r+s+1,l}(\|\mu\|)^l \leqslant \begin{cases} \dfrac{1}{2}\dfrac{n}{n-\|\mu\|} & \text{if } \|\mu\| < 1, \\ \dfrac{1}{2}\dfrac{n}{n-1} & \text{if } \|\mu\| = 1, \\ \dfrac{1}{2}\dfrac{\|\mu\|^{s+1}}{\|\mu\|-1}\dfrac{n}{n-\|\mu\|^2} & \text{if } 1 < \|\mu\| < \sqrt{n}. \end{cases}$$

From this and from (5) the theorem follows.

Suppose that the probability distribution has an inverse generalized measure G^{-*}, i.e.,

$$G * G^{-*} = G^{-*} * G = E_0,$$

where E_0 is the degenerate k-dimensional measure concentrated at $\mathbf{0} \in R^k$. Such a property is possessed by accompanying probability distributions e^{F-E_0}, i.e., $G^{-*} = e^{-(F-E_0)}$.

Theorem 6. *Let F be a k-dimensional probability distribution, let a probability distribution G have an inverse G^{-*}, and let $\varrho = \|(F-G) * G^{-*}\| < 1$. Then*

$$F^{*n} = G^{*n} * e^{n(F-G)*G^{-*}} * \left\{E_0 + \sum_{j=1}^{\infty} \Big(\frac{1}{n}\Big)^j A_j(n(F-G) * G^{-*})\right\}, \tag{6}$$

where

$$A_j(n(F-G)*G^{-*}) = (-1)^j(n(F-G)*G^{-*})^{*(j+1)} * \sum_{l=0}^{j-1} q_{jl}(n(F-G)*G^{-*})^{*l}.$$

To estimate the remainder term

$$r_{s+1}(B) = \sum_{j=s+1}^{\infty} \left(\frac{1}{n}\right)^j G^{*n} * e^{n(F-G)*G^{-*}} * A_j\big(n(F-G)*G^{-*}\big)(B),$$

we use

$$\Delta(B) = \sup_{\mathbf{x}} \left| G^{*n} * e^{n(F-G)*G^{-*}} * \big(n(F-G)*G^{-*}\big)^{*(s+2)}(B-\mathbf{x})\right|$$

and

$$L = \left(\frac{1}{n}\right)^{s+1} \sum_{r=0}^{\infty} \varrho^r \sum_{l=0}^{r+s} q_{r+s+1,l}\big(\|n\varrho\|\big)^l.$$

It is obvious that

$$|r_{s+1}(B)| \leqslant L\Delta(B), \tag{7}$$

where

$$L \leqslant \begin{cases} \dfrac{1}{2(1-\varrho)}\left(\dfrac{1}{n}\right)^{s+1} & \text{if } n\varrho < 1, \\ \dfrac{1}{2}\left(\dfrac{1}{n}\right)^{s}\dfrac{1}{n-1} & \text{if } n\varrho = 1, \\ \dfrac{1}{2}\dfrac{\varrho^{s+1}}{(1-n\varrho^2)} & \text{if } \frac{1}{n} < \varrho < \frac{1}{\sqrt{n}}. \end{cases} \tag{8}$$

From (7) and (8) there follows an estimate of the remainder term in the asymptotic expansion (6).

4 Expansion of a Convolution by Accompanying Probability Measures

Every k-dimensional probability measure P satisfies the identity

$$P = e^{P-E_0} * \big(E_0 - (P-E_0)^{*2} * E(E_0-P)^{*\xi_1}\big), \tag{9}$$

where

$$E(E_0-P)^{*\xi_1} = \sum_{m=0}^{\infty} P\{\xi_1 = m\}(E_0-P)^{*m} \tag{10}$$

with $P\{\xi_1 = m\} = \dfrac{m+1}{(m+2)!}$, $m = 0, 1, 2, \ldots$.

From (9) and (10) it follows that

$$\big((P-e^{P-E_0})*e^{-(P-E_0)}\big)^{*l}=(-1)^l(P-E_0)^{*2l}*E(E_0-P)^{*z_l} \tag{11}$$

for $l=1,2,\ldots$, where $z_l=\xi_1+\xi_2+\cdots+\xi_l$ -is the sum of i.i.d. random variables $\xi_1,\xi_2,\ldots,\xi_l$.

It is obvious that, for all P,

$$\big\|\big((P-e^{P-E_0})*e^{-(P-E_0)}\big)^{*l}\big\|\leqslant\left(\frac{1+e^2}{4}\right)^l,\qquad l=1,2,\ldots.$$

If $\|(P-E_0)^{*2}\|<\frac{4}{1+e^2}$, then

$$\varrho=\big\|(P-e^{P-E_0})*e^{-(P-E_0)}\big\|<1,$$

and for the convolution P^{*n}, we can apply Theorem 6:

$$P^{*n}=e^{n(P-E_0)}*e^{n\mu}*\left\{E_0+\sum_{j=1}^{\infty}\left(\frac{1}{n}\right)^jA_j(n\mu)\right\}, \tag{12}$$

where

$$A_j(n\mu)=(-1)^j(n\mu)^{*(j+1)}*\sum_{l=0}^{j-1}q_{jl}\,(n\mu)^{*l}$$

and

$$\mu=(P-e^{P-E_0})*e^{-(P-E_0)}.$$

Let us estimate the remainder term

$$r_{s+1}(B)=e^{n(P-E_0)}*e^{n\mu}*\left(E_0+\sum_{j=s+1}^{\infty}\left(\frac{1}{n}\right)^jA_j(n\mu)\right)(B).$$

From (11) it follows that

$$\mu^{*l}=(-1)^l(P-E_0)^{*2l}*E(E_0-P)^{*z_l}$$

and

$$\begin{aligned}r_{s+1}(B)=&(-n)^{s+2}(P-E_0)^{*2(s+2)}*E(E_0-P)^{*z_{s+2}}*e^{n(P-E_0)}*\\&*\exp\big\{-n(P-E_0)^{*2}E(E_0-P)^{*\xi_1}\big\}*\sum_{r=0}^{\infty}(-\mu)^{*r}*\sum_{l=0}^{r+s}q_{r+s+1,l}(n\mu)^{*l}.\end{aligned}$$

Theorem 7. *Suppose that* $\|(P - E_0)^{*2}\| < \frac{4}{1+e^2}$. *Then, for all Borel sets* $B \in \mathfrak{M}^k$,

$$|r_{s+1}(B)| \leqslant \Delta(B) \cdot L,$$

where

$$\Delta(B) = \sup_{\mathbf{x}} \Big| \big(n(P - E_0)^{*2}\big)^{*(s+2)} * E(E_0 - P)^{*(z_s+2)} * \\ * \exp\big\{n(P - E_0) * (E_0 - (P - E_0) * E(E_0 - P)^{*\xi_1}\big\}(B - \mathbf{x})\Big|,$$

$\varrho = \|\mu\|$, *and*

$$L = \begin{cases} \dfrac{1}{2(1-\varrho)}\left(\dfrac{1}{n}\right)^{s+1} & \text{if } n\varrho < 1, \\ \dfrac{1}{2}\left(\dfrac{1}{n}\right)^{s}\dfrac{1}{n-1} & \text{if } n\varrho = 1, \\ \dfrac{1}{2}\dfrac{\varrho^{s+1}}{1-n\varrho^2} & \text{if } 1 < n\varrho < \sqrt{n}. \end{cases}$$

The theorem follows from inequalities (7) and (8).

5 Asymptotic Bergström Expansion

For any k-dimensional probability distributions P and Q,

$$P^{*n} = \sum_{\nu=0}^{s} C_n^{\nu} Q^{*(n-\nu)} * (P - Q)^{*\nu} + r_n^{(s+1)}$$

(the Bergström identity). Here, for $s + 1 < n$,

$$r_n^{(s+1)} = \sum_{m=s+1}^{n} C_{m-1}^{s} P^{*(n-m)} * (P - Q)^{*(s+1)} * Q^{*(m-s-1)}.$$

Let Θ be a negative hypergeometric random variable taking the natural values $m = s + 1, s + 2, \ldots, n$ with probabilities

$$P\{\Theta = m\} = \frac{C_{m-1}^{s}}{C_n^{s+1}}.$$

Then we can rewrite the remainder term as

$$r_n^{(s+1)} = C_n^{s+1}(P - Q)^{*(s+1)} * E(P^{*(n-\Theta)} * Q^{*(\Theta-s-1)}),$$

where

$$E(P^{*(n-\Theta)} * Q^{*(\Theta-s-1)}) = \sum_{m=s+1}^{n} P\{\Theta = m\} P^{*(n-m)} * Q^{*(m-s-1)}.$$

Lemma 3. *Suppose that P and Q have finite jth-order absolute moments and that*

$$\int_{R^k} (\mathbf{t}, \mathbf{x})^r d(P-Q)(\mathbf{x}) = 0$$

for $r = 1, 2, \dots, j$ and $\mathbf{t} \in R^k$. Then

$$\int_{R^k} (\mathbf{t}, \mathbf{x})^l d(P-Q)^{*m}(\mathbf{x}) = 0$$

for $l = 0, 1, \dots, (j+1)m - 1$ and $\mathbf{t} \in R^k$.

Remark. If the first moments of P and Q coincide, then

$$\int_{R^k} (\mathbf{t}, \mathbf{x})^l d(P-Q)^{*m}(\mathbf{x}) = 0$$

for $l = 0, 1, \dots, 3m - 1$.

The lemma is proved by using characteristic functions and the Faa de Bruno formula that can be found, e.g., in [8].

Since

$$C_n^{\nu} = \frac{n^{\nu}}{\nu!}\left(1 + \sum_{j=1}^{\nu-1}\left(-\frac{1}{n}\right)^j C_{\nu}^{(j)}\right),$$

where $C_{\nu}^{(j)}$ is the Stirling number of the first kind, $C_{\nu}^{(0)} = 1$, and

$$C_{\nu}^{(j)} = \nu(\nu-1)\cdots(\nu-j)\sum_{l=0}^{j-1} q_{jl}(\nu-j-1)\cdots(\nu-j-l),$$

we have, for $1 \leqslant s < n$,

$$A_n^{(s)}(B) = Q^{*n} + \sum_{\nu=1}^{s} C_n^{\nu} Q^{*(n-\nu)} * (P-Q)^{*\nu}(B) =$$

$$= Q^{*n}(B) + \sum_{j=0}^{s-1}\left(\frac{1}{n}\right)^j \sum_{\nu=j+1}^{s} (-1)^j \frac{1}{\nu!} C_{\nu}^{(j)} Q^{*(n-\nu)} * (n(P-Q))^{*\nu}(B).$$

Now, the Bergström identity becomes

$$P^{*n}(B) = Q^{*n}(B) + \sum_{j=0}^{s-1} \left(\frac{1}{n}\right)^j \sum_{\nu=j+1}^{s} \frac{(-1)^j}{\nu!} C_\nu^{(j)} Q^{*(n-\nu)} * (n(P-Q))^{*\nu}(B) +$$

$$+ C_n^\nu (P-Q)^{*(s+1)} * E(P^{*(n-\Theta)} * Q^{*(\Theta-s-1)})(B). \tag{13}$$

Let us now consider the cases where $Q(B)$ is the normal k-dimensional distribution $\Phi(B) = P\{\boldsymbol{\xi} \in B\}$, $\boldsymbol{\xi} \sim N_k(\mathbf{0}, \Sigma)$, where Σ is a nondegenerate matrix of second moments.

Suppose that the expectation vectors and second-moment matrices of $P(B)$ and $\Phi(B)$ coincide.

Theorem 8. *Suppose that the probability distribution* $P(B) = P\{\boldsymbol{\eta} \in B\}$ *has finite absolute moments of order* $2+\delta$ *with* $0 < \delta \leqslant 1$. *Then there exists a constant* C*, depending only on* k*,* s*, and* δ*, such that*

$$\sup_{B \in \mathfrak{M}^k} \left|\Phi^{*(n-\nu)} * \big(n(P-\Phi)\big)^{*\nu}(B\sqrt{n})\right| \leqslant \left(\frac{CE\big[(\boldsymbol{\eta}^T \Sigma^{-1} \boldsymbol{\eta})^{\frac{2+\delta}{2}}\big]}{n^{\delta/2}}\right)^\nu$$

for $1 \leqslant \nu \leqslant s$*, where*

$$E(\boldsymbol{\eta}^T \Sigma^{-1} \boldsymbol{\eta})^{\frac{2+\delta}{2}} = \int_{R^k} (\mathbf{x}^T \Sigma^{-1} \mathbf{x})^{\frac{2+\delta}{2}} dP(\mathbf{x}),$$

and $\boldsymbol{\eta}^{\boldsymbol{T}}$ *is the transpose of the vector* $\boldsymbol{\eta}$.

Theorem 8 is proved in [4]. H. Bergström proved that (see [1])

$$\sup_{B \in \mathfrak{M}^k} \left|\Phi^{*(n-\nu)} * (n(P-\Phi))^{*\nu}(B\sqrt{n})\right| = O\left(\frac{(\ln n)^{k/2}}{n^{\delta/2}}\right)^\nu.$$

We will estimate the remainder term $r_n^{(s+1)}(B)$ for all convex Borel sets $B \in \mathfrak{N}^k$.

Theorem 9. *Suppose that the assumptions of Theorem 8 are satisfied and that the characteristic function of the random vector* $\boldsymbol{\eta}_1$ *satisfies Cramer condition* (C):

$$\overline{\lim}_{\|\mathbf{t}\| \to \infty} |Ee^{i(\mathbf{t}, \boldsymbol{\eta}_1)}| < 1.$$

Then

$$\sup_{B \in \mathfrak{N}^k} |r_n^{(s+1)}(B)| = o(n^{-(\delta/2)s}).$$

The theorem is proved in [4].

In the one-dimensional case ($k = 1$), Bergström [1] proved that from his asymptotic expansion there follows the Chebyshev–Cramer asymptotic expansion.

For $k > 1$ and $Q(B) = \Phi(B)$, from (13) it follows that

$$P^{*n}(B\sqrt{n}) = \Phi(B) + \sum_{j=0}^{s-1}\left(\frac{1}{n}\right)^j \sum_{\nu=j+1}^{s} \frac{(-1)^j}{\nu!} C_\nu^{(j)} \Phi^{*(n-\nu)} * (n(P-\Phi))^{*\nu}(B(\sqrt{n})$$
$$+ C_n^{s+1}(P-\Phi)^{*(s+1)} * E\big(P^{*(n-\Theta)} * \Phi^{*(\Theta-s-1)}(B\sqrt{n})\big). \qquad (14)$$

The formal asymptotic expansion of the density $p_\nu(\mathbf{y})$ of the convolution

$$\Phi^{*(n-\nu)} * (n(P-\Phi))^{*\nu}(B\sqrt{n}) = \int_B p_\nu(\mathbf{y})d\mathbf{y}$$

is

$$p_\nu(\mathbf{y}) \approx \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{1}{\sqrt{n}}\right)^{l+\nu+2m} \frac{(-\nu)^m}{m!(3\nu+l)!} \frac{\partial^{m+3\nu+l}}{\partial\varepsilon^m \partial\varrho^{3\nu+l}} \times$$
$$\times \left[\int_{x\in R^k} \left(\frac{1}{\sqrt{(1+\varepsilon)2\pi}}\right)^k \frac{1}{\sqrt{|\Sigma|}} \times\right.$$
$$\left.\times \exp\left\{-\frac{1}{2(1+\varepsilon)}(\mathbf{y}-\mathbf{x}\varrho)^T \Sigma^{-1}(\mathbf{y}-\mathbf{x}\varrho)\right\} d(P-\Phi)^{*\nu}(\mathbf{x})\right]\Bigg|_{\substack{\varepsilon=0\\ \varrho=0}} \qquad (15)$$

for $1 \leqslant \nu \leqslant s$, where $|\Sigma|$ denotes the determinant of the matrix Σ.

Let $\boldsymbol{\xi}_\varepsilon \sim N_k(\mathbf{0}, (1+\varepsilon)\Sigma))$ be a k-dimensional normal random vector. If $\varepsilon = 0$, then

$$\boldsymbol{\xi}_0 = \boldsymbol{\xi} \sim N_k(\mathbf{0}, \Sigma).$$

From (14) and (15) we get the following formal expansion of the convolution $P^{*n}(B\sqrt{n})$:

$$P^{*n}(B\sqrt{n}) \approx \Phi(B) + \sum_{j=0}^{s-1}\left(\frac{1}{n}\right)^j \sum_{\nu=j+1}^{s} \frac{(-1)^j}{\nu!} C_\nu^{(j)} \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{1}{\sqrt{n}}\right)^{l+\nu+2m} \frac{(-\nu)^m}{m!(3\nu+l)!}\cdot$$
$$\cdot \frac{\partial^{m+3\nu+l}}{\partial\varepsilon^m \partial\varrho^{3\nu+l}}\left[\int_{\mathbf{x}\in R^k} P\{\boldsymbol{\xi}_\varepsilon + \mathbf{x}\varrho \in B\} d(P-\Phi)^{*\nu}(\mathbf{x})\right]\Bigg|_{\substack{\varepsilon=0\\ \varrho=0}} + \cdots =$$
$$= P\{\boldsymbol{\xi} \in B\} + \sum_{r=1}^{\infty}\left(\frac{1}{\sqrt{n}}\right)^r \sum_{\substack{j=0\\ 2j+l+\nu+2m=r}}^{s-1} \sum_{\nu=j+1}^{s} \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+m}\nu^m}{\nu! m!(3\nu+l)!} C_\nu^{(j)}\cdot$$
$$\cdot \frac{\partial^{m+3\nu+l}}{\partial\varepsilon^m \partial\varrho^{3\nu+l}}\left[\int_{\mathbf{x}\in R^k} P\{\boldsymbol{\xi}_\varepsilon + \mathbf{x}\varrho \in B\} d(P-\Phi)^{*\nu}(\mathbf{x})\right]\Bigg|_{\substack{\varepsilon=0\\ \varrho=0}} + \cdots,$$

where

$$P\{\boldsymbol{\xi}_\varepsilon + \mathbf{x}_\varrho < \mathbf{z}\} = \left(\frac{1}{\sqrt{2\pi(1+\varepsilon)}}\right)^k \frac{1}{\sqrt{|\Sigma|}} \cdot$$

$$\cdot \int_{\mathbf{y}<\mathbf{z}} \exp\left\{-\frac{1}{2(1+\varepsilon)}(\mathbf{y}-\mathbf{x}_\varrho)^T \Sigma^{-1}(\mathbf{y}-\mathbf{x}_\varrho)\right\} d\mathbf{y}.$$

The formal expansions are obtained by means of the characteristic functions.

6 Expansion of a Convolution by χ^2-Distributions

Let $\boldsymbol{\xi}_\mu \sim N_k(\boldsymbol{\mu}, \Sigma)$ be a normal k-dimensional random vector with nondegenerate covariation? matrix Σ. The random variable

$$\chi_k^2 = (\boldsymbol{\xi}_\mu - \boldsymbol{\mu})^T \Sigma^{-1}(\boldsymbol{\xi}_\mu - \boldsymbol{\mu})$$

has the χ^2-distribution with k degrees of freedom, and the random variable

$$\chi_k^2(\delta) = (\boldsymbol{\xi}_\mu - \boldsymbol{\nu})^T \Sigma^{-1}(\boldsymbol{\xi}_\mu - \boldsymbol{\nu})$$

has the noncentral χ^2-distribution with k degrees of freedom and noncentrality parameter

$$\delta = (\boldsymbol{\mu} - \boldsymbol{\nu})^T \Sigma^{-1}(\boldsymbol{\mu} - \boldsymbol{\nu}).$$

The distribution function of $\chi_k^2(\delta)$ is

$$P\{\chi_k^2(\delta) < x\} = \sum_{j=0}^{\infty} \left[\frac{(\delta/2)^j}{j!} e^{-\delta/2}\right] P\{\chi_{k+2j}^2 < x\} \tag{16}$$

(see [7, 9]).

Let

$$\mathbf{S}_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \boldsymbol{\eta}_j$$

be the sum of i.i.d. k-dimensional vectors $\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_n, \ldots$ with zero mean vector $\mathbf{0} \in R^k$ and nondegenerate covariation matrix Σ. Let $\boldsymbol{\xi} \sim N_k(\mathbf{0}, \Sigma)$ and

$$A_x = \{\mathbf{y} \in R^k : \mathbf{y}^T \Sigma^{-1} \mathbf{y} < x\}, \quad x > 0.$$

We are interested in an asymptotic expansion of

$$P\{\mathbf{S}_n \in A_x\} = P\{\mathbf{S}_n^T \Sigma^{-1} \mathbf{S}_n < x\} = P^{*n}(\sqrt{n} A_x),$$

where $P(\sqrt{n}A_x) = P\left\{\frac{\eta_1}{\sqrt{n}} \in A_x\right\}$, i.e., the difference

$$P\{\mathbf{S}_n^T \Sigma^{-1}\mathbf{S}_n < x\} - P\{\boldsymbol{\xi}^T \Sigma^{-1}\boldsymbol{\xi} < x\} = P^{*n}(\sqrt{n}A_x) - P\{\chi_k^2 < x\} \tag{17}$$

for $x > 0$.

Denote by $\widehat{P}(\mathbf{t})$ and $\widehat{\Phi}(\mathbf{t})$ the characteristic functions of the vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\xi}$. From the Bergström identity (13) it follows that

$$\left(\widehat{P}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right)^n = \widehat{\Phi}(\mathbf{t}) + \sum_{\nu=0}^{s-1}\left(\frac{1}{n}\right)^j \sum_{\nu=j+1}^{s} \frac{(-1)^j}{\nu!} C_\nu^{(j)} \cdot$$

$$\cdot \left(\widehat{\Phi}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right)^{n-\nu} \left(\left(\widehat{P}\left(\frac{\mathbf{t}}{\sqrt{n}}\right) - \widehat{\Phi}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right)n\right)^\nu +$$

$$+ C_n^{s+1}\left(\widehat{P}\left(\frac{\mathbf{t}}{\sqrt{n}}\right) - \widehat{\Phi}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right)^{s+1} \cdot$$

$$\cdot E\left[\left(\widehat{P}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right)^{n-\Theta}\left(\widehat{\Phi}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right)^{\Theta-s-1}\right], \tag{18}$$

where

$$\left(\widehat{\Phi}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right)^{n-\nu}\left(n\left(\widehat{P}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right) - \widehat{\Phi}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right)^\nu =$$

$$= \int_{\mathbf{x}\in R^k} e^{i(\mathbf{t}/\sqrt{n},\mathbf{x})}\left(\widehat{\Phi}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right)^{n-\nu} d(n(P-\Phi))^{*\nu}(\mathbf{x}).$$

The Fourier transform is

$$p_\nu(\mathbf{y}) = \left(\frac{1}{2\pi}\right)^k \int_{\mathbf{t}\in R^k} e^{-i(\mathbf{t},\mathbf{y})}\left(\widehat{\Phi}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right)^{n-\nu}\left(n\left(\widehat{P}\left(\frac{\mathbf{t}}{\sqrt{n}}\right) - \widehat{\Phi}\left(\frac{\mathbf{t}}{\sqrt{n}}\right)\right)\right)^\nu d\mathbf{t} =$$

$$= \int_{\mathbf{x}\in R^k}\left(\frac{1}{2\pi}\right)^k \int_{\mathbf{t}\in R^k} e^{-i(\mathbf{t},\mathbf{y}-\mathbf{x}/\sqrt{n})} \exp\left\{-\frac{1}{2}\frac{n-\nu}{n}\mathbf{t}^T\Sigma\mathbf{t}\right\} d\mathbf{t}\, d(n(P-\Phi))^{*\nu}(\mathbf{x}).$$

By the change of variables $\mathbf{v} = \sqrt{\dfrac{n-\nu}{n}}\mathbf{t}$ we obtain

$$p_\nu(\mathbf{y}) = \int_{\mathbf{x}\in R^k} \left(\sqrt{\frac{n}{n-\nu}}\frac{1}{2\pi}\right)^k \int_{\mathbf{v}\in R^k} \exp\left\{-i\left(\mathbf{v}, \sqrt{\frac{n}{n-\nu}}\mathbf{y} - \frac{\mathbf{x}}{\sqrt{n-\nu}}\right)\right\}\cdot$$

$$\cdot \exp\left\{-\frac{1}{2}\mathbf{v}^T\Sigma\mathbf{v}\right\} d\mathbf{v}\, d(n(P-\Phi))^{*\nu}(\mathbf{x}), \tag{19}$$

where

$$\left(\sqrt{\frac{n}{n-\nu}}\frac{1}{2\pi}\right)^k \int_{\mathbf{v}\in R^k} \exp\left\{-i\left(\mathbf{v}, \sqrt{\frac{n}{n-\nu}}\mathbf{y} - \frac{\mathbf{x}}{\sqrt{n-\nu}}\right)\right\} \exp\left\{-\frac{1}{2}\mathbf{v}^T\Sigma\mathbf{v}\right\} d\mathbf{v} =$$

$$= \left(\sqrt{\frac{n}{n-\nu}}\right)^k \left(\frac{1}{\sqrt{2\pi}}\right)^k \frac{1}{\sqrt{|\Sigma|}}$$

$$\exp\left\{-\frac{1}{2}\left(\mathbf{y}\sqrt{\frac{n}{n-\nu}} - \frac{\mathbf{x}}{\sqrt{n-\nu}}\right)^T \Sigma^{-1}\left(\mathbf{y}\sqrt{\frac{n}{n-\nu}} - \frac{\mathbf{x}}{\sqrt{n-\nu}}\right)\right\}. \tag{20}$$

From (17) to (20) after the change of variables $\mathbf{u} = \mathbf{y}\sqrt{\frac{n}{n-\mu}} - \frac{\mathbf{x}}{\sqrt{n-\nu}}$ it follows that

$$\int_{\mathbf{y}^T\Sigma^{-1}\mathbf{y}<x} p_\nu(\mathbf{y})\, d\mathbf{y} = \int_{\mathbf{x}\in R^k} \left(\frac{1}{\sqrt{2\pi}}\right)^k \frac{1}{\sqrt{|\Sigma|}}$$

$$\int_{(\mathbf{u}+\frac{\mathbf{x}}{\sqrt{n-\nu}})^T\Sigma^{-1}(\mathbf{u}+\frac{\mathbf{x}}{\sqrt{n-\nu}})<x\frac{n}{n-\nu}} e^{-1/2\mathbf{u}^T\Sigma^{-1}\mathbf{u}} d\mathbf{u}\, d(n(P-\Phi))^{*\nu}(\mathbf{x}) =$$

$$= \int_{\mathbf{x}\in R^k} P\left\{\left(\boldsymbol{\xi} + \frac{\mathbf{x}}{\sqrt{n-\nu}}\right)^T \Sigma^{-1}\left(\xi + \frac{\mathbf{x}}{\sqrt{n-\nu}}\right) < x\frac{n}{n-\nu}\right\} d(n(P-\Phi))^{*\nu}(\mathbf{x}) =$$

$$= \int_{\mathbf{x}\in R^k} P\left\{\chi_k^2(\delta(\mathbf{x})) < x\frac{n}{n-\nu}\right\} d(n(P-\Phi))^{*\nu}(\mathbf{x}),$$

where

$$\delta(\mathbf{x}) = \frac{1}{n-\nu}\mathbf{x}^T\Sigma^{-1}\mathbf{x}$$

is the noncentrality parameter of the $\chi_k^2(\delta(\mathbf{x}))$-distribution. From (16) it follows that

$$P\{\chi_k^2(\delta(\mathbf{x})) < x\} = \sum_{j=0}^{\infty} \left(\frac{1}{n-\nu}\right)^j \frac{(\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x})^j}{j!} e^{-\frac{1}{2}\frac{1}{n-\nu}\mathbf{x}^T\Sigma^{-1}\mathbf{x}} P\left\{\chi_{k+2j}^2 < x\frac{n}{n-\nu}\right\}.$$

Now the asymptotic Bergström expansion writes as

$$P\{\boldsymbol{\xi}^T\Sigma^{-1}\boldsymbol{\xi} < x\} + \sum_{j=0}^{s-1}\Big(-\frac{1}{n}\Big)^j \sum_{\nu=j+1}^{s} \frac{1}{\nu!} C_\nu^{(j)}$$

$$\int_{\mathbf{x}\in R^k} P\Big\{\chi_k^2(\delta(\mathbf{x})) < x\frac{n}{n-\nu}\Big\} d(n(P-\Phi))^{*\nu}(\mathbf{x}).$$

To estimate the remainder term, we applied Theorem 9. Thus, we have proved the following:

Theorem 10. *Suppose that a random vector $\boldsymbol{\eta}_1$ has a finite absolute moment of order $2+\delta$ for some $0<\delta\leqslant 1$ and that the characteristic function $\widehat{P}(\mathbf{t})$ satisfies the Cramer condition*

$$\overline{\lim_{\|\mathbf{t}\|\to\infty}} |\widehat{P}(\mathbf{t})| < 1. \tag{C}$$

Then

$$P\{\mathbf{S}_n^T\Sigma^{-1}\mathbf{S}_n < x\} = P\{\chi_k^2 < x\} +$$

$$+\sum_{\nu=0}^{s-1}\Big(-\frac{1}{n}\Big)^j \sum_{\nu=j+1}^{s} \frac{1}{\nu!} C_\nu^{(j)} \sum_{r=0}^{\infty}\Big(\frac{1}{n-\nu}\Big)^r P\Big\{\chi_{k+2r}^2 < x\frac{n}{n-\nu}\Big\}\frac{1}{r!}\times$$

$$\times\int_{\mathbf{x}\in R^k}\Big(\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\Big)^r e^{-\frac{1}{2}\frac{1}{n-\nu}\mathbf{x}^T\Sigma^{-1}\mathbf{x}} d(n(P-\Phi))^{*\nu}(\mathbf{x}) + o(n^{-(\delta/2)s})$$

for all $x>0$ and $s=1,2,\ldots$.

If instead of considering the χ_k^2 random variable, we change t, F, μ, etc., then we have also to change the definition of the set A_x and to replace the Cramer condition (C) by the Prokhorov [11,12] condition that there exists n_0 such that the convolution $P^{*n_0}(\mathbf{x})$ has an absolutely continuous component.

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An Extension of the Concept of Slowly Varying Function with Applications to Large Deviation Limit Theorems

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Abstract Karamata's integral representation for slowly varying functions is extended to a broader class of the so-called ψ-locally constant functions, i.e. functions $f(x) > 0$ having the property that, for a given non-decreasing function $\psi(x)$ and any fixed v, $f(x + v\psi(x))/f(x) \to 1$ as $x \to \infty$. We consider applications of such functions to extending known theorems on large deviations of sums of random variables with regularly varying distribution tails.

Keywords Slowly varying function • Locally constant function • Large deviation probabilities • Random walk

Mathematics Subject Classification (2010): Primary 60F10; Secondary 26A12

1 Introduction

Let $L(x)$ be a slowly varying function (s.v.f.), i.e. a positive measurable function such that, for any fixed $v \in (0, \infty)$ holds $L(vx) \sim L(x)$ as $x \to \infty$:

$$\lim_{x\to\infty} \frac{L(vx)}{L(x)} = 1. \tag{1}$$

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Among the most important and often used results on s.v.f.'s are the Uniform Convergence Theorem (see property **(U)** below) and the Integral Representation Theorem (property **(I)**), the latter result essentially relying on the former. These theorems, together with their proofs, can be found e.g. in monographs [1] (Theorems 1.2.1 and 1.3.1) and [2] (see §1.1).

(U) *For any fixed* $0 < v_1 < v_2 < \infty$, *convergence (1) is uniform in* $v \in [v_1, v_2]$.
(I) *A function* $L(x)$ *is an s.v.f. iff the following representation holds true:*

$$L(x) = c(x) \exp\left\{ \int_1^x \frac{\varepsilon(t)}{t}\, dt \right\}, \qquad x \geq 1, \tag{2}$$

where $c(t) > 0$ *and* $\varepsilon(t)$ *are measurable functions*, $c(t) \to c \in (0, \infty)$ *and* $\varepsilon(t) \to 0$ *as* $t \to \infty$.

The concept of a s.v.f. is closely related to that of a regularly varying function (r.v.f.) $R(x)$, which is specified by the relation

$$R(x) = x^\alpha L(x), \qquad \alpha \in \mathbb{R},$$

where L is an s.v.f. and α is called the index of the r.v.f.'R. The class of all r.v.f.'s we will denote by $\mathscr{R}$.

R.v.f.'s are characterised by the relation

$$\lim_{x\to\infty} \frac{R(vx)}{R(x)} = v^\alpha, \quad v \in (0, \infty). \tag{3}$$

For them, convergence (3) is also uniform in v on compact intervals, while representation (2) holds for r.v.f.'s with $\varepsilon(t) \to \alpha$ as $t \to \infty$.

In Probability Theory there exists a large class of limit theorems on large deviations of sums of random variable whose distributions $\mathbf{F}$ have the property that their right tails $F_+(x) := \mathbf{F}\big([x, \infty)\big)$ are r.v.f.'s. The following assertion (see e.g. Theorem 4.4.1 in [2]) is a typical representative of such results. Let $\xi, \xi_1, \xi_2, \ldots$ be independent identically distributed random variables, $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 < \infty$, $S_n := \sum_{k=1}^n \xi_k$ and $\overline{S}_n := \max_{k\leq n} S_k$.

Theorem A. *If* $F_+(t) = \mathbf{P}(\xi \geq t)$ *is an r.v.f. of index* $\alpha < -2$ *then, as* $x \to \infty$, $x(n \ln n)^{-1/2} \to \infty$, *one has*

$$\mathbf{P}(S_n \geq x) \sim nF_+(x), \qquad \mathbf{P}(\overline{S}_n \geq x) \sim nF_+(x). \tag{4}$$

Similar assertions hold true under the assumption that the distributions of the scaled sums S_n tend to a stable law (see Chaps. 2 and 3 in [2]).

There arises the natural question of how essential the condition $F_+ \in \mathscr{R}$ is for (4) to hold. It turns out that this condition can be significantly relaxed.

The aim of the present paper is to describe and study classes of functions that are wider than $\mathscr{R}$ and have the property that the condition that F_+ belongs to such a class, together with some other natural conditions, would ensure the validity of limit laws of the form (4).

In Sect. 2 of the present note we give the definitions of the above-mentioned broad classes of functions which we call asymptotically ψ-locally constant functions. The section also contains assertions in which conditions sufficient for relations (4) are given in terms of these functions. Section 3 presents the main results on characterisation of asymptotically ψ-locally constant functions. Section 4 contains the proofs of these results.

2 The Definitions of Asymptotically Locally Constant Functions. Applications to Limit Theorems on Large Deviations

Following §1.2 in [2], we will call a positive function $g(x)$ an *asymptotically locally constant function* (l.c.f.) if, for any fixed $v \in (-\infty, \infty)$,

$$\lim_{x \to \infty} \frac{g(x+v)}{g(x)} = 1 \tag{5}$$

(the function $g(x)$, as all the other functions appearing in the present note, will be assumed measurable; assumptions of this kind will be omitted for brevity's sake).

If one puts $x = \ln y$, $v = \ln u$, then $g(x+v) = g(\ln yu)$, so that the composition $L = g \circ \ln$ will be an s.v.f. by virtue of (5) and (1). From here and the equality $g(x) = L(e^x)$ it follows that an l.c.f. g will have the following properties:

($\mathbf{U_1}$) *For any fixed $-\infty < v_1 < v_2 < \infty$, convergence (5) is uniform in $v \in [v_1, v_2]$.*
($\mathbf{I_1}$) *A function $g(x)$ is an l.c.f. iff it admits a representation of the form*

$$g(x) = c(x) \exp\left\{ \int_1^{e^x} \frac{\varepsilon(t)}{t}\, dt \right\}, \qquad x \geq 1, \tag{6}$$

where $c(t)$ and $\varepsilon(t)$ have the same properties as in **(I)**.

Probability distributions $\mathbf{F}$ on $\mathbb{R}$ such that $F_+(t) := \mathbf{F}\big([t,\infty)\big)$ is an l.c.f. are sometimes referred to as long-tailed distributions, or class $\mathscr{L}$ distributions. Such distributions often appear in papers on limit theorems for sums of random variables with "heavy tails". Examples of l.c.f.'s are provided by r.v.f.'s and functions of the form $\exp\{x^\alpha L(x)\}$, where L is an s.v.f., $\alpha \in (0, 1)$.

It is not hard to see that, by virtue of property (U_1), definition (5) of an l.c.f. is equivalent to the following one: for any fixed $v \in (-\infty, \infty)$ and function $v(x) \to v$ as $x \to \infty$, one has

$$\lim_{x\to\infty} \frac{g(x+v(x))}{g(x)} = 1. \tag{7}$$

Now we will consider a broader concept, which includes both s.v.f.'s and l.c.f.'s as special cases.

Let $\psi(t) > 1$ be a fixed non-decreasing function.

Definition 1. (See also Definition 1.2.7 in [2].) A function $g(x) > 0$ is said to be an *asymptotically* ψ*-locally constant function* (ψ-l.c.f.) if, for any fixed $v \in (-\infty, \infty)$ such that $x + v\psi(x) \ge cx$ for some $c > 0$ and all large enough x, one has

$$\lim_{x\to\infty} \frac{g(x+v\psi(x))}{g(x)} = 1. \tag{8}$$

If $\psi(x) \equiv 1$ then the class of ψ-l.c.f.'s coincides with the class of l.c.f.'s, while if $\psi(x) \equiv x$ then the class of ψ-l.c.f.'s coincides with the class of s.v.f.'s. In the case when $\psi(x) \to \infty$ and $\psi(x) = o(x)$ as $x \to \infty$, the class of ψ-l.c.f.'s occupies, in a sense, an intermediate (in terms of the zone where its functions are locally constant) place between the classes of s.v.f.'s and l.c.f.'s.

Clearly, all functions from $\mathscr{R}$ are ψ-l.c.f.'s for any function $\psi(x) = o(x)$.

Note that the concept of ψ-l.c.f.'s is closely related to that of h-insensitive functions extensively used in [5] (see Definition 2.18 therein). Our Theorem 1 below shows that, under broad conditions, a ψ-l.c.f. will he h-insensitive with $\psi = h$.

We will also need the following

Definition 2. (See also Definition 1.2.20 in [2].) We will call a function g an *upper-power function* if it is an l.c.f. and, for any $p \in (0, 1)$, there exists a constant $c(p)$, $\inf_{p\in(p_1,1)} c(p) > 0$ for any $p_1 \in (0, 1)$, such that

$$g(t) \ge c(p)g(pt), \qquad t > 0.$$

It is clear that all r.v.f.'s are upper-power functions.

The concept of ψ-l.c.f.'s and that of an upper-power function enable one to substantially extend the assertion of Theorem A. It is not hard to derive from Theorem 4.8.1 in [2] the following result.

Let $h(v) > 0$ be a non-decreasing function such that $h(v) \gg \sqrt{v \ln v}$ as $v \to \infty$. Such a function always has a generalised inverse $h^{(-1)}(t) := \inf\{v : h(v) \ge t\}$.

Theorem B. *Assume that* $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 < \infty$ *and that the following conditions are satisfied:*

1. $F_+(t) \le V(t) = t^\alpha L(t)$*, where* $\alpha < -2$ *and* L *is an s.v.f.*
2. *The function* $F_+(t)$ *is upper-power and a* ψ*-l.c.f. for* $\psi(t) = \sqrt{h^{(-1)}(t)}$.

Then relations (4) *hold true provided that* $x \to \infty$, $x \ge h(n)$ *and*

$$nV^2(x) = o\big(F_+(x)\big). \tag{9}$$

In particular, if $x = h(n) \sim cn^\beta$ as $n \to \infty$, $\beta > 1/2$, then one can put $\psi(t) := t^{1/2\beta}$ ($\psi(t) := \sqrt{t}$ if $x \sim cn$).

Condition (9) is always met provided that $F_+(t) \ge cV(t)t^{-\varepsilon}$ for some $\varepsilon > 0$, $\varepsilon < -\alpha - 2$, and $c =$ const. Indeed, in this case, for $x \ge \sqrt{n}$, $x \to \infty$,

$$nV^2(x) \le c^{-1}x^{2+\varepsilon}V(x)F_+(x) = o\big(F_+(x)\big).$$

Now consider the case where $\mathbf{E}\xi^2 = \infty$. Let, as before, $V(t) = t^\alpha L(t)$ is an r.v.f. and set $\sigma(v) := V^{(-1)}(1/v))$. Observe that $\sigma(v)$ is also an r.v.f. (see e.g. Theorem 1.1.4 in [2]). Further, let $h(v) > 0$ be a non-decreasing function such that $h(v) \gg \sigma(v)$ as $v \to \infty$. Employing Theorem 4.8.6 in [2] (using this opportunity, note that there are a couple of typos in the formulation of that theorem: the text "with $\psi(t) = \sigma(t) = V^{(-1)}(1/t)$" should be omitted, while the condition "$x \gg \sigma(n)$" must be replaced with "$x \gg \sigma(n) = V^{(-1)}(1/n)$") it is not difficult to establish the following result.

Theorem C. *Let* $\mathbf{E}\xi = 0$ *and the following conditions be met:*

1. $F_+(t) \le V(t) = t^\alpha L(t)$, *where* $-\alpha \in (1, 2)$ *and* L *is an s.v.f.*
2. $\mathbf{P}(\xi < -t) \le cV(t)$ *for all* $t > 0$.
3. *The function* F_+ *is upper-power and a* ψ*-l.c.f. for* $\psi(t) = \sigma\big(h^{(-1)}(t)\big)$.

Then relations (4) *hold true provided that* $x \to \infty$, $x \ge h(n)$ *and relation* (9) *is satisfied.*

If, for instance, $V(t) \sim c_1 t^\alpha$ as $t \to \infty$, $x \sim c_2 n^\beta$ as $n \to \infty$, $c_i =$ const, $i = 1, 2$, and $\beta > -1/\alpha$, then one can put $\psi(t) := t^{-1/(\alpha\beta)}$.

Condition (9) of Theorem C is always satisfied provided that $x \ge n^{\delta-(1/\alpha)}$, $F_+(t) \ge cV(t)t^{-\varepsilon}$ for some $\delta > 0$ and $\varepsilon < \alpha^2\delta/(1-\alpha\delta)$. Indeed, in this case $n \le x^{-\alpha/(1-\alpha\delta)}$ and

$$nV^2(x) \le c^{-1}F_+(x)x^{\varepsilon-\alpha/(1-\alpha\delta)}V(x) = o\big(F_+(x)\big).$$

Note also that the conditions of Theorems B and C do not stipulate that $n \to \infty$.

The proofs of Theorems B and C basically consist in verifying, for the indicated choice of functions ψ, the conditions of Theorems 4.8.1 and 4.8.6 in [2], respectively. We will omit them.

It is not hard to see (e.g. from the representation theorem on p. 74 in [1]) that Theorems B and C include, as special cases, situations when the right tail of $\mathbf{F}$ satisfies the condition of *extended regular variation*, i.e. when, for any $b > 1$ and some $0 < \alpha_1 \le \alpha_2 < \infty$,

$$b^{-\alpha_2} \le \liminf_{x\to\infty} \frac{F_+(bx)}{F_+(x)} \le \limsup_{x\to\infty} \frac{F_+(bx)}{F_+(x)} \le b^{-\alpha_1}. \tag{10}$$

Under the assumption that the random variable $\xi = \xi' - \mathbf{E}\xi'$ was obtained by centering a non-negative random variable $\xi' \geq 0$, the former of the asymptotic relations (4) was established in the above-mentioned case in [3]. One could mention here some further efforts aimed at extending the conditions of Theorem A that ensure the validity of (4), see e.g. [4, 6].

In conclusion of this section, we will make a remark showing that the presence of the condition that $F_+(t)$ is a ψ-l.c.f. in Theorems B and C is quite natural. Moreover, it also indicates that any further extension of this condition in the class of "sufficiently regular" functions is hardly possible. If we turn, say, to the proof of Theorem 4.8.1 in [2], we will see that when $x \sim cn$, the main term in the asymptotic representation for $\mathbf{P}(S_n \geq x)$ is given by

$$n \int_{-N\sqrt{n}}^{N\sqrt{n}} \mathbf{P}(S_{n-1} \in dt) F_+(x-t), \tag{11}$$

where $N \to \infty$ slowly enough as $n \to \infty$. It is clear that, by virtue of the Central Limit Theorem, the integral in this expression is asymptotically equivalent to $F_+(t)$ (implying that the former relation in (4) will hold true), provided that $F_+(t)$ is a ψ-l.c.f. for $\psi(t) = \sqrt{t}$.

Since $\mathbf{E}S_{n-1} = 0$, one might try to obtain such a result in the case when $F_+(t)$ belongs to a broader class of "asymptotically ψ-locally linear functions", i.e. such functions that, for any fixed v and $t \to \infty$,

$$F_+(t + v\psi(t)) = F_+(t)(1 - cv + o(1)), \quad c = \text{const} > 0.$$

However, such a representation is impossible as $1 - cv < 0$ when $v > 1/c$.

3 The Chracterization of ψ-l.c.f.'s

The aim of the present section is to prove that, for any ψ-l.c.f. g, convergence (8) is uniform in v on any compact set and, moreover, that g admits an integral representation similar to (2) and (6). To do that, we will need some restrictions on the function ψ.

We assume that ψ is a non-decreasing function such that $\psi(x) = o(x)$ as $x \to \infty$. For such functions, we introduce the following condition:

(A) *For any fixed* $v > 0$, *there exists a value* $a(v) \in (0, \infty)$ *such that*

$$\frac{\psi(x - v\psi(x))}{\psi(x)} \geq a(v) \quad \text{for all sufficiently large } x. \tag{12}$$

Letting $y := x + v\psi(x) > x$ and using the monotonicity of ψ, one has

$$\psi(y - v\psi(y)) \le \psi(x).$$

Therefore, relation (12) implies that, for all large enough x,

$$\frac{\psi(x + v\psi(x))}{\psi(x)} \equiv \frac{\psi(y)}{\psi(x)} \le \frac{\psi(y)}{\psi(y - v\psi(y))} \le \frac{1}{a(v)} \in (0, \infty).$$

Thus, any function ψ satisfying condition **(A)** will also satisfy the following relation: for any fixed $v > 0$,

$$\frac{\psi(x + v\psi(x))}{\psi(x)} \le \frac{1}{a(v)} \quad \text{for all sufficiently large } x. \tag{13}$$

Observe that the converse is not true: it is not hard to construct an example of a (piece-wise linear, globally Lipschitz) non-decreasing function ψ which satisfies condition of the form (13), but for which condition **(A)** will hold for no finite function $a(v)$.

It is clear that if ψ is a ψ-l.c.f., $\psi(x) = o(x)$, then ψ satisfies condition **(A)**.

Introduce class $\mathscr{K}$ consisting of non-decreasing functions $\psi(x) \ge 1$, $x \ge 0$, that satisfy condition **(A)** for a function $a(v)$ such that

$$\int_0^\infty a(u)\,du = \infty. \tag{14}$$

Class $\mathscr{K}_1$ we define as the class of continuous r.v.f.'s $\psi(x) = x^\alpha L(x)$ with index $\alpha < 1$ and such that $x/\psi(x) \uparrow \infty$ as $x \to \infty$ and the following "asymptotic smoothness" condition is met: for any fixed v,

$$\psi(x + \Delta) = \psi(x) + \frac{\alpha\Delta\psi(x)}{x}(1 + o(1)) \quad \text{as } x \to \infty, \quad \Delta = v\psi(x). \tag{15}$$

Clearly, $\mathscr{K}_1 \subset \mathscr{K}$. Condition (15) is always met for any $\Delta \ge c_1 = \text{const}$, $\Delta = o(x)$, provided that the function $L(x)$ is differentiable and $L'(x) = o\left(L(x)/x\right)$ as $x \to \infty$.

In the assertions to follow, it will be assumed that ψ belongs to the class $\mathscr{K}$ or $\mathscr{K}_1$. We will not dwell on how far the conditions $\psi \in \mathscr{K}$ or $\psi \in \mathscr{K}_1$ can be extended. The function ψ specifies the "asymptotic local constancy zone width" of the function g under consideration, and what matters for us is just the growth rate of $\psi(x)$ as $x \to \infty$. All its other properties (smoothness, presence of oscillations etc.) are for us to choose, and so we can assume the function ψ to be as smooth as we need. In this sense, the assumption that ψ belongs to the class $\mathscr{K}$ or $\mathscr{K}_1$ is not restrictive. For example, it is quite natural to assume in Theorems B and C from Sect. 2 that $\psi \in \mathscr{K}_1$.

The following assertion establishes the uniformity of convergence in (8).

Theorem 1. *If g is a ψ-l.c.f. with $\psi \in \mathscr{K}$, then convergence in (8) is uniform: for any fixed real numbers $v_1 < v_2$,*

$$(\mathbf{U}_\psi) \qquad \lim_{x\to\infty} \sup_{v_1\le v\le v_2} \left| \frac{g\big(x+v\psi(x)\big)}{g(x)} - 1 \right| = 0. \tag{16}$$

Observe that, for monotone g, the condition $\psi \in \mathscr{K}$ in Theorem 1 is superfluous. Indeed, assume for definiteness that g is a non-decreasing ψ-l.c.f. Then, for any v and $v(x) \to v$, there is a $v_0 > v$ such that, for all sufficiently large x, one has $v(x) < v_0$, and therefore

$$\limsup_{x\to\infty} \frac{g(x+v(x)\psi(x))}{g(x)} \le \limsup_{x\to\infty} \frac{g(x+v_0\psi(x))}{g(x)} = 1. \tag{17}$$

A converse inequality for lim inf is established in a similar way. As a consequence,

$$\lim_{x\to\infty} \frac{g(x+v(x)\psi(x))}{g(x)} = 1, \tag{18}$$

which is easily seen to be equivalent to (16) (cf. (7)).

Note also that it is not hard to see that monotonicity property required to derive (17) and (18), could be somewhat relaxed.

Now set

$$\gamma(x) := \int_1^x \frac{dt}{\psi(t)}. \tag{19}$$

Theorem 2. *Let $\psi \in \mathscr{K}$. Then g is a ψ-l.c.f. iff it admits a representation of the form*

$$(\mathbf{I}_\psi) \qquad g(x) = c(x) \exp\left\{ \int_1^{e^{\gamma(x)}} \frac{\varepsilon(t)}{t}\, dt \right\}, \qquad x \ge 1, \tag{20}$$

where $c(t)$ and $\varepsilon(t)$ have the same properties as in (**I**).

Since, for any $\varepsilon > 0$ and all large enough x,

$$\int_1^{e^{\gamma(x)}} \frac{\varepsilon(t)}{t}\, dt < \varepsilon \ln e^{\gamma(x)} = \varepsilon\gamma(x)$$

and a similar lower bound holds true, Theorem 2 implies the following result.

Corollary 1. *If $\psi \in \mathscr{K}$ and g is a ψ-l.c.f., then*

$$g(x) = e^{o(\gamma(x))}, \qquad x \to \infty.$$

For $\psi \in \mathscr{K}_1$ we put

$$\theta(x) := \frac{x}{\psi(x)}.$$

Clearly, $\theta(x) \sim (1-\alpha)\gamma(x)$ as $x \to \infty$.

Theorem 3. *Let $\psi \in \mathscr{K}_1$. Then the assertion of Theorem 2 holds true with $\gamma(x)$ replaced by $\theta(x)$.*

Corollary 2. *If $\psi \in \mathscr{K}_1$ and g is a ψ-l.c.f., then*

$$g(x) = e^{o(\theta(x))}, \qquad x \to \infty.$$

Since the function $\theta(x)$ has a "more explicit" representation in terms of ψ than the function $\gamma(x)$, the assertions of Theorem 3 and Corollary 2 display the asymptotic properties ψ-l.c.f.'s in a more graphical way than those of Theorem 2 and Corollary 1. A deficiency of Theorem 3 is the fact that the condition $\psi \in \mathscr{K}_1$ is more restrictive than the condition that $\psi \in \mathscr{K}$. It is particularly essential that, in the former condition, the equality $\alpha = 1$ is excluded for the index α of the r.v.f. ψ.

4 Proofs

Proof of Theorem 1. Our proof will use an argument modifying H. Delange's proof of property **(U)** (see e.g. p. 6 in [1] or §1.1 in [2]).

Let $l(x) := \ln g(x)$. It is clear that (8) is equivalent to the convergence

$$l(x + v\psi(x)) - l(x) \to 0, \qquad x \to \infty, \tag{21}$$

for any fixed $v \in \mathbb{R}$. To prove the theorem, it suffices to show that

$$H_{v_1,v_2}(x) := \sup_{v_1 \le v \le v_2} \left| l(x + v\psi(x)) - l(x) \right| \to 0, \qquad x \to \infty.$$

It is not hard to see that the above relation will follow from the convergence

$$H_{0,1}(x) \to 0, \qquad x \to \infty. \tag{22}$$

Indeed, let $v_1 < 0$ (for $v_1 \ge 0$ the argument will be even simpler) and

$$x_0 := x + v_1\psi(x), \qquad x_k := x_0 + k\psi(x_0), \qquad k = 1, 2, \ldots$$

By virtue of condition **(A)**, one has $\psi(x_0) \ge a(-v_1)\psi(x)$ with $a(-v_1) > 0$. Therefore, letting $n := \lfloor (v_2 - v_1)/a(-v_1) \rfloor + 1$, where $\lfloor x \rfloor$ denotes the integer part of x, we obtain

$$H_{v_1,v_2}(x) \le \sum_{k=0}^{n} H_{0,1}(x_k),$$

which establishes the required implication.

Assume without loss of generality that $\psi(0) = 1$. To prove (22), fix an arbitrary small $\varepsilon \in \big(0, a(1)/(1 + a(1))\big)$ and set

$$I_x := [x, x + 2\psi(x)], \qquad I_x^* := \{y \in I_x : |l(y) - l(x)| \ge \varepsilon/2\},$$

$$I_{0,x}^* := \{u \in I_0 : |l(x + u\psi(x)) - l(x)| \ge \varepsilon/2\}.$$

One can easily see that all these sets are measurable and

$$I_x^* = x + \psi(x) I_{0,x}^*,$$

so that for the Lebesgue measure $\mu(\cdot)$ on $\mathbb{R}$ we have

$$\mu(I_x^*) = \psi(x)\mu(I_{0,x}^*). \tag{23}$$

It follows from (21) that, for any $u \in I_0$, the value of the indicator $\mathbf{1}_{I_{0,x}^*}(u)$ tends to zero as $x \to \infty$. Therefore, by the dominated convergence theorem,

$$\int_{I_0} \mathbf{1}_{I_{0,x}^*}(u)\, du \to 0, \qquad x \to 0.$$

From here and (23) we see that there exists an $x_{(\varepsilon)}$ such that

$$\mu(I_x^*) \le \frac{\varepsilon}{2}\,\psi(x), \qquad x \ge x_{(\varepsilon)}.$$

Now observe that, for any $s \in [0, 1]$, the set $I_x \cap I_{x+s\psi(x)} = [x + s\psi(x), x + 2\psi(x)]$ has the length $(2 - s)\psi(x) \ge \psi(x)$. Hence for $x \ge x_{(\varepsilon)}$ the set

$$J_{x,s} := \big(I_x \cap I_{x+s\psi(x)}\big) \setminus \big(I_x^* \cup I_{x+s\psi(x)}^*\big)$$

will have the length

$$\begin{aligned}\mu(J_{x,s}) &\ge \psi(x) - \tfrac{\varepsilon}{2}\big[\psi(x) + \psi(x + s\psi(x))\big] \\ &\ge \psi(x) - \frac{\varepsilon}{2}\left(1 + \frac{1}{a(1)}\right)\psi(x) \ge \frac{1}{2}\psi(x) \ge \frac{1}{2},\end{aligned}$$

where we used relation (13) to establish the second inequality. Therefore $J_{x,s} \ne \varnothing$ and one can choose a point $y \in J_{x,s}$. Then $y \notin I_x^*$ and $y \notin I_{x+s\psi(x)}^*$, so that

$$|l(x + s\psi(x)) - l(x)| \le |l(x + s\psi(x)) - l(y)| + |l(y) - l(x)| < \varepsilon.$$

Since this relation holds for any $s \in [0, 1]$, the required convergence (22) and hence the assertion of Theorem 1 are proved. □

Proof of Theorem 2. First let g be a ψ-l.c.f. with $\psi \in \mathscr{K}$. Since $\psi(t) = o(t)$, one has $\gamma(x) \uparrow \infty$ as $x \uparrow \infty$ (see (19)). Moreover, the function $\gamma(x)$ is continuous and so always has an inverse $\gamma^{(-1)}(t) \uparrow \infty$ as $t \to \infty$, so that we can consider the composition function

$$g_\gamma(t) := (g \circ \gamma^{(-1)})(t).$$

If we show that g_γ is an l.c.f. then representation (20) will immediately follow from the relation $g(x) = g_\gamma(\gamma(x))$ and property $(\mathbf{I_1})$.

By virtue of the uniformity property $(\mathbf{U}_\psi)$ which holds for g by Theorem 1, for any bounded function $r(x)$ one has

$$g_\gamma\big(\gamma(x)\big) \equiv g(x) \sim g\big(x + r(x)\psi(x)\big) = g_\gamma\big(\gamma(x + r(x)\psi(x))\big). \tag{24}$$

Next we will show that, for a given v (let $v > 0$ for definiteness), there is a bounded (as $x \to \infty$) value $r(x, v)$ such that

$$\gamma(x + r(x, v)\psi(x)) = \gamma(x) + v. \tag{25}$$

Indeed, we have

$$\gamma(x + r\psi(x)) - \gamma(x) = \int_x^{x+r\psi(x)} \frac{dt}{\psi(t)} = \int_0^r \frac{\psi(x)\,dz}{\psi(x + z\psi(x))} =: I(r, x),$$

where, by Fatou's lemma and relation (13),

$$\liminf_{x\to\infty} I(r, x) \ge \int_0^r \liminf_{x\to\infty} \frac{\psi(x)}{\psi(x + z\psi(x))}\,dz \ge I(r) := \int_0^r a(z)dz \uparrow \infty$$

as $r \uparrow \infty$ (see (14)). Since, moreover, for any x the function $I(r, x)$ is continuous in r, there exists $r(v, x) \le r_v < \infty$ such that $I\big(r(v, x), x\big) = v$, where r_v is the solution of the equation $I(r) = v$.

Now choosing $r(x)$ in (24) to be the function $r(x, v)$ from (25) we obtain that

$$g_\gamma\big(\gamma(x)\big) \sim g_\gamma\big(\gamma(x) + v\big)$$

as $x \to \infty$, which means that g_γ is an l.c.f. and hence (20) holds true.

Conversely, let representation (20) be true. Then, for a fixed $v \ge 0$, any $\varepsilon > 0$ and $x \to \infty$, one has

$$\begin{aligned}\left|\ln \frac{g\big(x + v\psi(x)\big)}{g(x)}\right| &\le \int_{e^{\gamma(x)}}^{e^{\gamma(x+v\psi(x))}} \frac{|\varepsilon(t)|}{t}\,dt + o(1) \le \big(\gamma(x + v\psi(x)) - \gamma(x)\big)\varepsilon + o(1)\\ &\le \varepsilon \int_0^v \frac{\psi(x)ds}{\psi(x + s\psi(x))} + o(1) \le \varepsilon v + o(1).\end{aligned} \tag{26}$$

This clearly means that the left-hand side of this relation is $o(1)$ as $x \to \infty$.

If $v = -u < 0$ then, bounding in a similar fashion the integral

$$\int_{e^{\gamma(x-u\psi(x))}}^{e^{\gamma(x)}} \frac{|\varepsilon(t)|dt}{t} \le \varepsilon \int_0^u \frac{\psi(x)ds}{\psi(x - s\psi(x))},$$

we will obtain from condition **(A)** that

$$\limsup_{x\to\infty} \left| \ln \frac{g(x + v(\psi(x)))}{g(x)} \right| \le \varepsilon \int_0^u \limsup_{x\to\infty} \frac{\psi(x)ds}{\psi(x - s\psi(x))} \le \varepsilon \int_0^u \frac{ds}{a(s)},$$

so that the left-hand side of (26) is still $o(1)$ as $x \to \infty$. Therefore $g(x + v\psi(x)) \sim g(x)$ and hence g is a ψ-l.c.f. Theorem 2 is proved. □

It is evident that the assertion of Theorem 2 can also be stated as follows: for $\psi \in \mathscr{K}$, a function g is a ψ-l.c.f. iff $g_\gamma(x)$ is an l.c.f. (which, in turn, holds iff $g_\gamma(\ln x)$ is an s.v.f.).

Proof of Theorem 3. One can employ an argument similar to the one used to prove Theorem 2.

Since the function $\theta(x)$ is continuous and increasing, it has an inverse $\theta^{(-1)}(t)$. It is not hard to see that if ψ has property (15), then the function $\theta(x) = x/\psi(x)$ also possesses a similar property: for a fixed v and $\Delta = v\psi(x)$, $x \to \infty$, one has

$$\theta(x + \Delta) = \theta(x) + \frac{(1-\alpha)\Delta\theta(x)}{x}(1 + o(1)). \tag{27}$$

Therefore, as $x \to \infty$,

$$\theta(x + v\psi(x)) = \theta(x) + (1-\alpha)v(1 + o(1)).$$

As the function θ is monotone and continuous, this relation means that, for any v, there is a function $v(x) \to v$ as $x \to \infty$ such that

$$\theta(x + v(x)\psi(x)) = \theta(x) + (1-\alpha)v. \tag{28}$$

Let g be a ψ-l.c.f. Then, for the function $g_\theta := g \circ \theta^{(-1)}$ we obtain by virtue of (28) that

$$g_\theta(\theta(x)) \equiv g(x) \sim g(x + v(x)\psi(x)) = g_\theta\left(\theta(x + v(x)\psi(x))\right) = g_\theta(\theta(x) + (1-\alpha)v).$$

Since $\theta(x) \to \infty$ as $x \to \infty$, the relation above means that g_θ is an l.c.f. The direct assertion of the integral representation theorem follows from here and (6).

The converse assertion is proved in the same way as in Theorem 2. Theorem 3 is proved. □

Similarly to our earlier argument, it follows from Theorem 3 that if $\psi \in \mathscr{K}_1$ then g is a ψ-l.c.f. iff g_θ is an l.c.f. (and $g_\theta(\ln x)$ is an s.v.f.).

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Optimal and Asymptotically Optimal Control for Some Inventory Models

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Abstract A multi-supplier discrete-time inventory model is considered as illustration of problems arising in applied probability. Optimal and asymptotically optimal control is established for all values of parameters involved. The model stability is also investigated.

Keywords Optimal and asymptotically optimal policies • Discrete-time inventory models • Stability

Mathematics Subject Classification (2010): Primary 90B05, Secondary 90C31

1 Introduction

It was my scientific adviser Professor Yu.V. Prokhorov who proposed optimal control of some inventory systems as a topic of my Phd thesis. At the time it was a new research direction. The subject of my habilitation thesis was stochastic inventory models. So I decided to return to these problems in the paper devoted to jubilee of academician of Russian Academy of Sciences Yu.V. Prokhorov.

Optimal control of inventory systems is a particular case of decision making under uncertainty (see, e.g., [5]). It is well known that construction of a mathematical model is useful to investigate a real life process or system and make a correct decision.

There always exist a lot of models describing the process under consideration more or less precisely. Therefore it is necessary to choose an appropriate model.

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Usually the model depends on some parameters not known exactly. So they are estimated on the base of previous observations. The same is true of underlying processes distributions. Hence, the model must be stable with respect to small parameters fluctuations and processes perturbations (see, e.g., [6]).

To illustrate the problems arising and the methods useful for their solution, a multi-supplier inventory model is considered.

2 Main Results

The aim of investigation is to establish optimal and asymptotically optimal control. It is reasonable to begin by some definitions.

2.1 *Definitions*

To describe any applied probability model one needs to know the following elements: the planning horizon T, input process $Z = \{Z(t), t \in [0, T]\}$, output process $Y = \{Y(t), t \in [0, T]\}$ and control $U = \{U(t), t \in [0, T]\}$. The system state is given by $X = \Psi(Z, Y, U)$ where functional Ψ represents the system configuration and operation mode. Obviously, one has also $X = \{X(t), t \in [0, T]\}$. Moreover, processes Z, Y, U and X can be multi-dimensional and their dimensions may differ. For evaluation of the system performance quality it is necessary to introduce an objective function $\mathscr{L}(Z, Y, U, X, T)$. For brevity it will be denoted by $\mathscr{L}_T(U)$. So, a typical applied probability model is described by a six-tuple $(Z, Y, U, \Psi, \mathscr{L}, T)$.

Such description is useful for models classification. It also demonstrates the similarity of models arising in different applied probability domains such as inventory and dams theory, insurance and finance, queueing and reliability theory, as well as population growth and many others (see, e.g., [7]). One only gives another interpretation to processes Z, Y, X in order to switch from one research domain to another. Thus, input to inventory system is replenishment delivery (or production) and output is demand, whereas for a queueing system it is arrival and departure of customers respectively (for details see, e.g., [6]).

Definition 1. A control $U_T^* = \{U^*(t), t \in [0, T]\}$ is called *optimal* if

$$\mathscr{L}_T(U_T^*) = \inf_{U_T \in \mathscr{U}_T} \mathscr{L}_T(U_T) \quad (\text{or} \quad \mathscr{L}_T(U_T^*) = \sup_{U_T \in \mathscr{U}_T} \mathscr{L}_T(U_T)), \tag{1}$$

where $\mathscr{U}_T$ is a class of all feasible controls. Furthermore, $U^* = \{U_T^*, T \geq 0\}$ is called an *optimal policy*.

The choice of inf or sup in (1) is determined by the problem we want to solve. Namely, if we are interested in minimization of losses (or ruin probability) we use the first expression, whereas for profit (or system life-time) maximization we use the second one in (1).

Since extremum in (1) may be not attained we introduce the following

Definition 2. A control U_T^ε is *ε-optimal* if

$$\mathscr{L}_T(U_T^\varepsilon) < \inf_{U_T \in \mathscr{U}_T} \mathscr{L}_T(U_T) + \varepsilon \qquad (\text{or} \quad \mathscr{L}_T(U_T^\varepsilon) > \sup_{U_T \in \mathscr{U}_T} \mathscr{L}_T(U_T) - \varepsilon).$$

Definition 3. A policy $\tilde{U} = \{\tilde{U}_T, T \geq 0\}$ is *stationary* if for any $T, S \geq 0$

$$\tilde{U}_T(t) = \tilde{U}_S(t), \quad t \leq \min(T, S).$$

Definition 4. A policy $\widehat{U} = (\widehat{U}_T, T \geq 0)$ is *asymptotically optimal* if

$$\lim_{T\to\infty} T^{-1}\mathscr{L}_T(\widehat{U}_T) = \lim_{T\to\infty} T^{-1}\mathscr{L}_T(U_T^*).$$

The changes necessary for discrete-time models are obvious.

2.2 *Model Description*

Below we consider a discrete-time multi-supplier one-product inventory system. It is supposed that a store created to satisfy the customers demand can be replenished periodically. Namely, at the end of each period (e.g., year, month, week, day etc.) an order for replenishment of inventory stored can be sent to one of m suppliers or to any subset of them. The i-th supplier delivers an order with $(i-1)$-period delay, $i = \overline{1, m}$. Let a_i be the maximal order possible at the i-th supplier, and the ordering price is c_i per unit, $i = \overline{1, m}$. For simplicity, the constant delivery cost associated with each order is ignored. However we take into account holding cost h per unit stored per period and penalty p for deficit of unit per period.

Let the planning horizon be equal to n periods. The demand is described by a sequence of independent identically distributed nonnegative random variables $\{\xi_k\}_{k=1}^n$. Here ξ_k is amount demanded during the k-th period. Assume $F(x)$ to be the distribution function of ξ_k having a density $\varphi(s) > 0$ for $s \in [\underline{\kappa}, \overline{\kappa}] \subset [0, \infty)$. It is also supposed that there exists $\mathsf{E}\xi_k = \mu$, $k = \overline{1, n}$.

Unsatisfied demand is backlogged. That means, the inventory level x_k at the end of the k-th period can be negative. In this case $|x_k|$ is the deficit amount.

Expected discounted n-period costs are chosen as objective function. We denote by $f_n(x, y_1, \ldots, y_{m-2})$ the minimal value of objective function if inventory on hand (or initial inventory level) is x and y_i is already ordered (during previous periods) quantity to be delivered i periods later, $i = \overline{1, m-2}$.

2.2.1 Notation and Preliminary Results

It is supposed that the order amounts at the end of each period depend on the inventory level x and yet undelivered quantities $y_1, \ldots, y_{m-2}$. Using the Bellman optimality principle (see, e.g., [2]) it is possible to obtain, for $n \geq 1$, the following functional equation

$$f_n(x, y_1, \ldots, y_{m-2}) = \min_{0 \leq z_i \leq a_i, i=\overline{1,m}} [\sum_{i=1}^{m} c_i z_i + L(x + z_1) + \tag{2}$$

$$+\alpha \mathsf{E} f_{n-1}(x + y_1 + z_1 + z_2 - \xi_1, y_2 + z_3, \ldots, y_{m-2} + z_{m-1}, z_m)].$$

Here α is the discount factor, E stands for mathematical expectation and z_i is the order size at the first step of n-step process from the i-th supplier, $i = \overline{1,m}$. Furthermore, the one-period mean holding and penalty costs are represented by

$$L(v) = \mathsf{E}[h(v - \xi_1)^+ + p(\xi_1 - v)^+], \quad \text{with} \quad a^+ = \max(a, 0),$$

if inventory level available to satisfy demand is equal to v.

The calculations for arbitrary m being too cumbersome, we treat below in detail the case $m = 2$. Then we need to know only the initial level x and Eq. (2) takes the form

$$f_n(x) = \min_{0 \leq z_i \leq a_i, i=1,2} [c_1 z_1 + c_2 z_2 + L(x + z_1) + \alpha \mathsf{E} f_{n-1}(x + z_1 + z_2 - \xi_1)] \tag{3}$$

with $f_0(x) \equiv 0$. Let us introduce the following notation $v = x + z_1$, $u = v + z_2$ and

$$G_n(v, u) = (c_1 - c_2)v + c_2 u + L(v) + \alpha \mathsf{E} f_{n-1}(u - \xi_1).$$

Then Eq. (3) can be rewritten as follows

$$f_n(x) = -c_1 x + \min_{(v,u) \in D_x} G_n(v, u) \tag{4}$$

where $D_x = \{x \leq v \leq x + a_1, v \leq u \leq v + a_2\}$.

The minimum in (4) can be attained either inside of D_x or at its boundary.

To formulate the main results we need the following functions

$$\frac{\partial G_n}{\partial v}(v, u) = c_1 - c_2 + L'(v) := K(v),$$

$$\frac{\partial G_n}{\partial u}(v, u) = c_2 + \alpha \int_0^{\infty} f'_{n-1}(u - s)\varphi(s)\, ds := S_n(u).$$

Moreover, $T_n(v) = S_n(v) + K(v)$ and $B_n(v) = S_n(v + a_2) + K(v)$ represent $\dfrac{dG_n(v,v)}{dv}$ and $\dfrac{dG_n(v,v+a_2)}{dv}$ respectively, whereas

$$R^a(u) = c_2 - \alpha c_1 + \alpha \int_0^{u-\bar{v}} K(u-s)\varphi(s)\,ds + \alpha \int_{u+a-\bar{v}}^{\infty} K(u+a-s)\varphi(s)\,ds. \quad (5)$$

Let $\bar{v}$, u_n, v_n, w_n and u^a be the roots of the following equations

$$K(\bar{v}) = 0, \quad S_n(u_n) = 0, \quad T_n(v_n) = 0, \quad B_n(w_n) = 0, \quad R^a(u^a) = 0, \quad (6)$$

provided the solutions exist for a given set of cost parameters. In particular, $\bar{v} \in [\underline{\kappa}, \overline{\kappa}]$ is given by

$$F(\bar{v}) = \frac{p - c_1 + c_2}{p + h}$$

if $(c_1, c_2) \in \Gamma = \{(c_1, c_2) : (c_1 - p)^+ \le c_2 \le c_1 + h\}$. Otherwise, we set $\bar{v} = -\infty$, if $K(v) > 0$ for all v, that is, $(c_1, c_2) \in \Gamma^- = \{(c_1, c_2) : c_2 < (c_1 - p)^+\}$, and $\bar{v} = +\infty$, if $K(v) < 0$ for all v, that is, $(c_1, c_2) \in \Gamma^+ = \{(c_1, c_2) : c_2 > c_1 + h\}$. A similar assumption holds for $S_n(u)$, $T_n(v)$, $B_n(w)$ and $u_n, v_n, w_n, n \ge 1$, as well as $R^a(u)$ and u^a. Below we are going to use also the following notation. For $k \ge 0$ set

$$\Delta_k = \{(c_1, c_2) : p \sum_{i=0}^{k-1} \alpha^i < c_1 \le p \sum_{i=0}^{k} \alpha^i\}, \quad \Delta^k = \{(c_1, c_2) : p \sum_{i=1}^{k} \alpha^i < c_2 \le p \sum_{i=1}^{k+1} \alpha^i\},$$

where as always the sum over empty set is equal to 0,

$$\Delta_k^l = \Delta_k \cap \Delta^l,\ A_k = \cup_{l \ge k} \Delta_l,\ A^k = \cup_{l \ge k} \Delta^l,\ \Gamma^\alpha = \{(c_1, c_2) : (c_1 - p)^+ \le c_2 \le \alpha c_1\},$$

$$\Gamma_n^- = \{(c_1, c_2) \in \Gamma : S_n(\bar{v}) < 0\},\ \Gamma_n^+ = \{(c_1, c_2) \in \Gamma : S_n(\bar{v}) > 0\},$$

whereas $\Gamma_n^0 = \{(c_1, c_2) \in \Gamma : S_n(\bar{v}) = 0\}$. As usual dealing with dynamic programming all the proofs are carried out by induction.

Thus, it will be proved that functions $f_n'(x)$ are non-decreasing as well as $K(v)$, $S_n(v)$, $T_n(v)$ and $B_n(v)$. Moreover, to establish that sequences $\{u_n\}$, $\{v_n\}$, $n \ge 1$, are non-decreasing it is enough to check that $f_n'(x) - f_{n-1}'(x) \le 0$ for $x \le \max(u_n, v_n)$, since $S_{n+1}(u) = S_n(u) + \alpha H_n(u)$ and $T_{n+1}(v) = T_n(v) + \alpha H_n(v)$ where $H_n(u) = (f_n' - f_{n-1}') * F(u)$, here and further on $*$ denotes the convolution.

The crucial role for classification of possible variants of optimal behaviour plays the following

Lemma 1. *If* $(c_1, c_2) \in \Gamma_n^-$, *then* $\bar{v} < v_n < u_n$; *if* $(c_1, c_2) \in \Gamma_n^+$, *then* $\bar{v} > v_n > u_n$, *whereas* $\bar{v} = v_n = u_n$ *if* $(c_1, c_2) \in \Gamma_n^0$, *and* $u_n, v_n, \bar{v}$ *are defined by (6). Moreover, if* $(c_1, c_2) \in \Gamma^-$, *then* $v_n < u_n$ *and* $v_n > u_n$, *if* $(c_1, c_2) \in \Gamma^+$, *for all* n.

Proof. The statement is obvious, since functions $K(v)$, $S_n(v)$ and $T_n(v)$ are non-decreasing in v, $T_n(v) = S_n(v) + K(v)$ and $K(v) < 0$ for $v < \bar{v}$, while $K(v) > 0$ for $v > \bar{v}$. □

2.3 *Optimal Control*

We begin by treating the case without constraints on order sizes. Although Corollary 1 was already formulated in [4] (under assumption $\alpha = 1$) a more thorough investigation undertaken here lets clarify the situation and provides useful tools for the case with order constraints.

2.3.1 Unrestricted Order Sizes

At first we suppose that the order size at both suppliers may assume any value, that is, $a_i = \infty$, $i = 1, 2$.

Theorem 1. *If* $c_2 > \alpha c_1$, *the optimal behaviour at the first step of* n*-step process has the form* $u_n(x) = v_n(x) = \max(x, v_n)$. *The sequence* $\{v_n\}$ *of critical levels given by (6) is non-decreasing and there exists* $\lim_{n\to\infty} v_n = \widehat{v}$ *satisfying the following relation*

$$F(\widehat{v}) = \frac{p - c_1(1-\alpha)}{p+h}. \tag{7}$$

Moreover, for $(c_1, c_2) \in \Delta_k$, $k = 0, 1, \ldots$, *one has* $v_n = -\infty$, $n \le k$ *and* v_{k+1} *is a solution of the equation*

$$\sum_{i=1}^{k+1} \alpha^{i-1} F^{i*}(v_{k+1}) = \frac{p\sum_{i=0}^{k} \alpha^i - c_1}{p+h}. \tag{8}$$

Proof. For $n = 1$ it is optimal to take $u = v$, since $S_1(u) = c_2 > 0$ for all u, that means $u_1 = -\infty$. On the other hand, $T_1(v) = c_1 - p + (p+h)F(v)$, therefore $v_1 = -\infty$ in A_1, whereas in Δ_0 there exists $v_1 \in [0, \bar{v}]$ such that $F(v_1) = (p - c_1)/(p+h)$. Thus, the optimal decision has the form $u_1(x) = v_1(x) = \max(x, v_1)$.

For further investigation we need only to know

$$f_1'(x) = -c_1 + \begin{cases} 0, & x < v_1, \\ T_1(x), & x \ge v_1, \end{cases} = \begin{cases} -c_1, & x < v_1, \\ L'(x), & x \ge v_1. \end{cases} \tag{9}$$

It is obvious that $f_1'(x)$ is non-decreasing, the same being true of

$$S_2(u) = c_2 - \alpha c_1 + \alpha \int_0^{u-v_1} T_1(u-s)\varphi(s)\,ds \tag{10}$$

and

$$T_2(v) = Q(v) + \alpha \int_0^{v-v_1} T_1(v-s)\varphi(s)\,ds \tag{11}$$

with $Q(v) = c_1(1-\alpha) + L'(v)$. Note that in the case $v_1 = -\infty$ the meaning of $\int_0^{u-v_1}$ in (10) and $\int_0^{v-v_1}$ in (11) is $\int_0^{\infty}$. The same agreement will be used further on.

Thus, $S_2(u) > 0$ for all u under assumption $c_2 > \alpha c_1$, that is, $u_2 = -\infty$. Since $T_2(v) \geq Q(v)$, it follows immediately that $v_2 \leq \widehat{v}$ and $\widehat{v}$ is given by (7), hence $\widehat{v} < \bar{v}$. Moreover, $\widehat{v} = -\infty$ for $c_1 > p(1-\alpha)^{-1}$. It is also clear that $v_2 > v_1$ in Δ_0 because $T_2(v_1) = -\alpha c_1$. Recalling that in A_1

$$T_2(v) = c_1 + L'(v) + \alpha \int_0^{\infty} L'(v-s)\varphi(s)\,ds$$

we get

$$F(v_2) + \alpha F^{2*}(v_2) = \frac{p(1+\alpha) - c_1}{p+h} \quad \text{in} \quad \Delta_1,$$

whereas $v_2 = -\infty$ in A_2. Hence, $u_2(x) = v_2(x) = \max(x, v_2)$.

Assuming now the statement of the theorem to be valid for $k \leq m$, one has

$$f_k'(x) = -c_1 + \begin{cases} 0, & x < v_k, \\ T_k(x), & x \geq v_k, \end{cases} \tag{12}$$

and

$$f_m'(x) - f_{m-1}'(x) = \begin{cases} 0, & x < v_{m-1}, \\ -T_{m-1}(x), & v_{m-1} \leq x < v_m, \\ T_m(x) - T_{m-1}(x), & x \geq v_m. \end{cases} \tag{13}$$

Thus, $S_{m+1}(u) > 0$ for all u, that entails $u_{m+1} = -\infty$. Moreover, $T_{m+1}(v) \geq Q(v)$ and $H_m(v_m) \leq 0$. Hence, $v_m < v_{m+1} \leq \widehat{v}$ in $\cup_{k=0}^{m-1} \Delta_k$ and v_{m+1} satisfies (8) with $k = m$ in Δ_m, whereas $v_{m+1} = -\infty$ in A_{m+1}. That means, the theorem statement is valid for $m+1$.

The sequence $\{v_n\}$ is non-decreasing and bounded. Consequently there exists $\lim_{n\to\infty} v_n = \breve{v} \leq \widehat{v}$. It remains to prove that $\breve{v} = \widehat{v}$. In fact, for $n > k+1$

$$T_n(v) = Q(v) + \alpha \int_0^{v-v_{n-1}} T_{n-1}(v-s)\varphi(s)\,ds \quad \text{in} \quad \Delta_k, \quad k \geq 0,$$

so

$$|Q(v_n)| = \alpha \int_0^{v_n - v_{n-1}} T_{n-1}(v_n - s)\varphi(s)\,ds$$

$$\le T_{n-1}(\widehat{v})\alpha \int_0^{v_n - v_{n-1}} \varphi(s)\,ds \le \ldots \le T_k(\widehat{v})\alpha^{n-k} \int_0^{v_n - v_{n-1}} \varphi(s)\,ds$$

where $T_k(\widehat{v}) \le c_1 + h\sum_{i=0}^{k} \alpha^i \le c_1 + h(1-\alpha)^{-1}$.

Hence, $Q(v_n) \to 0 = Q(\widehat{v})$, as $n \to \infty$. On the other hand, $Q(v_n) \to Q(\breve{v})$, therefore $\breve{v} = \widehat{v}$. It is clear that this result is true for any $0 < \alpha \le 1$. □

Remark 1. The main result of Theorem 1 can be reformulated in the following way:

$$z_n^{(1)}(x) = z_n^{(2)}(x) = 0 \quad \text{for} \quad n \le k$$

and

$$z_n^{(1)}(x) = (v_n - x)^+, \quad z_n^{(2)}(x) = 0 \quad \text{for} \quad n > k,$$

if $(c_1, c_2) \in \Delta_k$, $k = 0, 1, \ldots$.

That means, for $c_2 > \alpha c_1$ it is optimal to use only the first supplier. The inventory level is raised up to a prescribed critical value v_n if the initial level x at the first step of n-step process is less than v_n. Nothing is ordered for $x \ge v_n$. Furthermore, if $c_1 > p\sum_{i=0}^{k-1} \alpha^i$ then for $n \le k$ nothing is ordered for all initial inventory levels x at the first step of n-step process. If $c_1 > p(1-\alpha)^{-1}$, it is optimal never to order for any initial level.

Theorem 2. *If $c_2 < (c_1 - p)^+$, the optimal behaviour at the first step of n-step process has the form $v_n(x) = x$, $u_n(x) = \max(x, u_n)$. The sequence $\{u_n\}$ defined by (6) is non-decreasing and there exists $\lim_{n\to\infty} u_n = u^0$, where u^0 is given by*

$$F^{2*}(u^0) = \frac{\alpha p - c_2(1-\alpha)}{\alpha(p+h)}. \tag{14}$$

Moreover, for $(c_1, c_2) \in \Delta^{k-1}$, $k = 1, 2, \ldots$, one has $u_n = -\infty$, $n \le k$, and

$$\sum_{i=2}^{k+1} \alpha^{i-2} F^{i*}(u_{k+1}) = \frac{p\sum_{i=1}^{k} \alpha^i - c_2}{\alpha(p+h)}. \tag{15}$$

Proof. Recall that $\bar{v} = -\infty$ in Γ^- and $\Gamma^- \subset A_1$. It follows immediately from here that $u_1(x) = v_1(x) = x$ and $f_1'(x) = L'(x)$. Now turn to $n = 2$. Since

$$S_2(u) = c_2 - \alpha p + \alpha(p+h)F^{2*}(u),$$

it is obvious that $S_2(u) > 0$ for all u (that is, $u_2 = -\infty$) in A^1 and there exists $u_2 \ge 0$ satisfying (15) with $k = 1$ in Δ^0. According to Lemma 1 one has $v_2 < u_2$, therefore it is optimal to have $v_2(x) = x$ and $u_2(x) = \max(x, u_2)$. Thus,

$$f_2'(x) = -c_1 + \begin{cases} K(x), & x < u_2, \\ T_2(x), & x \geq u_2, \end{cases} = -c_2 + L'(x) + \begin{cases} 0, & x < u_2, \\ S_2(x), & x \geq u_2, \end{cases}$$

and

$$S_3(u) = R^0(u) + \alpha \int_0^{u-u_2} S_2(u-s)\varphi(s)\,ds$$

where $R^0(u)$ given by (5) with $a = 0$ has the form

$$c_2(1-\alpha) + \alpha \int_0^{\infty} L'(u-s)\varphi(s)\,ds = c_2(1-\alpha) - \alpha p + \alpha(p+h)F^{2*}(u). \quad (16)$$

It is clear that there exists u^0 satisfying $R^0(u^0) = 0$. For $c_2 \leq \alpha p(1-\alpha)^{-1}$ it is given by (14), otherwise $u^0 = -\infty$. Since $S_3(u) \geq R^0(u)$ and $S_3(u_2) = -\alpha c_2$ in Δ^0, one has $u_2 < u_3 \leq u^0$. Moreover, in Δ^1 there exists u_3 satisfying (15) with $k = 2$, whereas $u_3 = -\infty$ in A^2.

Assuming the statement of the theorem to be valid for $k \leq m$ one has

$$f_k'(x) = -c_1 + K(x) + \begin{cases} 0, & x < u_k, \\ S_k(x), & x \geq u_k, \end{cases} \quad (17)$$

and

$$f_m'(x) - f_{m-1}'(x) = \begin{cases} 0, & x < u_{m-1}, \\ -S_{m-1}(x), & u_{m-1} \leq x < u_m, \\ S_m(x) - S_{m-1}(x), & x \geq u_m. \end{cases}$$

That means $S_{m+1}(u) \geq R^0(u)$ for all u and $H_m(u_m) < 0$. Thus, $u_m < u_{m+1} \leq u^0$ in $\cup_{k=0}^{m-2}\Delta^k$ and $u_{m+1} \leq u^0$ satisfies (15) with $k = m$ in Δ^{m-1}, whereas $u_{m+1} = -\infty$ in A^m. It follows immediately that $v_{m+1}(x) = x$ and $u_{m+1}(x) = \max(x, u_{m+1})$. Clearly, the theorem statement is valid for $m+1$.

The sequence $\{u_n\}$ is non-decreasing and bounded, consequently there exists $\breve{u} = \lim_{n\to\infty} u_n$. It remains to prove that $\breve{u} = u^0$. In fact, for $n > k+2$,

$$S_n(u) = R^0(u) + \alpha \int_0^{u-u_{n-1}} S_{n-1}(u-s)\varphi(s)\,ds \quad \text{in} \quad \Delta^k, \quad k = 0, 1, \ldots,$$

and

$$|R^0(u_n)| = \alpha \int_0^{u_n-u_{n-1}} S_{n-1}(u_n-s)\varphi(s)\,ds$$

$$\leq S_{n-1}(u^0)\alpha \int_0^{u_n-u_{n-1}} \varphi(s)\,ds \leq \ldots \leq S_k(u^0)\alpha^{n-k} \int_0^{u_n-u_{n-1}} \varphi(s)\,ds,$$

where $S_k(u^0) \leq c_2 + h\sum_{i=1}^{k+1}\alpha^i \leq c_2 + h(1-\alpha)^{-1}$.

It is clear that $R^0(u_n) \to 0 = R^0(u^0)$, as $n \to \infty$, hence, $\breve{u} = u^0$ for $0 < \alpha \leq 1$.

□

Remark 2. In other words, Theorem 2 states that for $c_2 < c_1 - p$ one has to use only the second supplier, the order sizes being

$$z_n^{(1)}(x) = 0, \quad z_n^{(2)}(x) = 0, \quad n \le k + 1,$$

and

$$z_n^{(1)}(x) = 0, \quad z_n^{(2)}(x) = (u_n - x)^+, \quad n > k + 1,$$

if $(c_1, c_2) \in \Delta^k$, $k = 0, 1, \dots$.

Now let us turn to the last and most complicated case.

Theorem 3. *If* $(c_1, c_2) \in \Gamma^\alpha$*, the optimal behaviour at the first step of n-step process has the form* $v_n(x) = \max(x, \min(v_n, \bar{v}))$*,* $u_n(x) = max(v_n(x), u_n)$*. The sequence* $\{u_n\}$ *is non-decreasing and there exists* $\lim_{n\to\infty} u_n = u^\infty$ *where* u^∞ *is given by* (5) and (6) *with* $a = \infty$.

Proof. It is obvious that $\Gamma^\alpha \subset \cup_{i=0}^{\infty}(\Delta_i^i \cup \Delta_{i+1}^i)$ and $\bar{v} \ge \underline{\kappa}$ in Γ^α. As in Theorem 1, for $n = 1$ one has $u_1(x) = v_1(x) = \max(x, v_1)$ where v_1 is given by (8) with $k = 0$ in Δ_0 and $v_1 = -\infty$ in A_1. Thus $f_1'(x)$ has the form (9). Note also that

$$\Gamma_1^0 = \{(c_1, c_2) : 0 \le c_1 \le p, c_2 = 0\}.$$

Moreover, $S_2(u)$ is given by (10) and $u_2 \ge v_1$. It is also clear that $\{(c_1, c_2) : c_2 = \alpha c_1\} \subset \Gamma_2^+$. On the other hand, $\Gamma_1^0 \subset \Gamma_2^-$, since $S_2(\bar{v}) = -\alpha c_1$ in Γ_1^0. Furthermore, in Δ_0^0 the function $c_2 = g_2(c_1)$ is defined implicitly by

$$S_2(\bar{v}) = c_2 - \alpha c_1 + \alpha \int_0^{\bar{v}-v_1} T_1(\bar{v} - s)\varphi(s)\, ds = 0,$$

whence it follows $g_2(0) = 0$ and

$$g_2'(c_1) = \alpha \frac{\varphi(\bar{v}) \int_{\bar{v}-v_1}^{\infty} \varphi(s)\, ds + \int_0^{\bar{v}-v_1} \varphi(\bar{v} - s)\varphi(s)\, ds}{\varphi(\bar{v}) + \alpha \int_0^{\bar{v}-v_1} \varphi(\bar{v} - s)\varphi(s)\, ds}.$$

Thus, it is clear that $1 \ge g_2'(c_1) \ge 0$ and $g_2'(0) = \alpha$, since $\bar{v} = v_1$ for $c_1 = c_2 = 0$.

For $c_1 = p$ two expressions for $S_2(\bar{v})$ coincide because

$$S_2(\bar{v}) \to c_2 - \alpha p + \alpha(p + h)F^{2*}(\bar{v}), \quad \text{as} \quad c_1 \uparrow p,$$

and in Δ_1^0 one has $S_2(u) = c_2 - \alpha p + \alpha(p + h)F^{2*}(u)$. It is easy to get that u_2 is determined by (15) with $k = 1$ in Δ_1^0 and $u_2 = -\infty$ in Δ^1. We have also

$$g_2'(c_1) = \frac{\alpha\varphi^{2*}(\bar{v})}{\varphi(\bar{v}) + \alpha\varphi^{2*}(\bar{v})} \quad \text{in} \quad \Delta_1^0$$

and $\{(c_1, c_2) \in \Delta_1^0 : c_2 = c_1 - p\} \subset \Gamma_2^- \cup \Gamma_2^0$, more precisely, $g_2(p(1+\alpha)) = \alpha p$.

Hence, $\Gamma_2^- \subset \Delta_0^0 \cup \Delta_1^0$, moreover, we are going to establish that $\Gamma_2^- \subset \Gamma_3^-$, whereas $\Gamma_3^+ \subset \Gamma_2^+$. In fact, due to Lemma 1 one has $\bar{v} < v_2 < u_2$ in Γ_2^-. It follows immediately that $v_2(x) = \max(x, \bar{v})$ and $u_2(x) = \max(v_2(x), u_2)$. That means,

$$f_2'(x) = -c_1 + \begin{cases} 0, & x < \bar{v}, \\ K(x), & \bar{v} \le x < u_2, \\ T_2(x), & x \ge u_2, \end{cases} \tag{18}$$

and

$$S_3(u) = R^{\infty}(u) + \alpha \int_0^{u-u_2} S_2(u-s)\varphi(s)\,ds$$

with $R^{\infty}(u)$ given by (5) with $a = \infty$. In other words, we have

$$R^{\infty}(u) = c_2 - \alpha c_1 + \alpha \int_0^{u-\bar{v}} K(u-s)\varphi(s)\,ds.$$

Since $S_3(\bar{v}) = c_2 - \alpha c_1 < 0$ in Γ_2^-, it is clear that $\Gamma_2^- \subset \Gamma_3^-$. From (9) and (18) one gets

$$f_2'(x) - f_1'(x) = \begin{cases} 0, & x < v_1, \\ -T_1(x), & v_1 \le x < \bar{v}, \\ -c_2, & \bar{v} \le x < u_2, \\ T_2(x) - T_1(x), & x \ge u_2. \end{cases}$$

Thus, $f_2'(x) - f_1'(x) \le 0$ for $x \le u_2$, that is, $H_2(u_2) < 0$ and $u_2 < u_3$. As soon as $S_3(u) \ge R^{\infty}(u)$, it is obvious that $u_3 \le u^{\infty}$. Hence, $f_3'(x)$ has the form (18) with indices 3 instead of 2.

Assuming now that $(c_1, c_2) \in \Gamma_2^+$ one has $v_1 \le u_2 < v_2 < \bar{v}$ due to (10) and Lemma 1. It entails $u_2(x) = v_2(x) = \max(x, v_2)$ and $f_2'(x)$ is given by (12) with $k = 2$. Recall also that v_2 is given by (8) with $k = 1$ in Δ_1 and $v_2 = -\infty$ in A_2.

Clearly,

$$S_3(u) = c_2 - \alpha c_1 + \alpha \int_0^{u-v_2} T_2(u-s)\varphi(s)\,ds,$$

that means $S_3(v_2) = c_2 - \alpha c_1 \le 0$ in $\Delta_0 \cup \Delta_1$, consequently, $v_2 \le u_3$. There are two possibilities: either $u_3 \le \bar{v}$ or $u_3 > \bar{v}$. The first case corresponds to $\Gamma_3^0 \cup \Gamma_3^+$, whereas the second one to Γ_3^-. In the first case $u_3(x) = v_3(x) = \max(x, v_3)$, while in the second one $v_3(x) = \max(x, \bar{v})$ and $u_3(x) = \max(v_3(x), u_3)$. Moreover, in Δ_2^1

$$S_3(u) = c_2 - \alpha p(1+\alpha) + \alpha(p+h)[F^{2*}(u) + \alpha F^{3*}(u)],$$

while

$$T_3(v) = c_1 - p(1+\alpha+\alpha^2) + (p+h)[F(v) + \alpha F^{2*}(v) + \alpha^2 F^{3*}(v)].$$

Thus u_3 and v_3 are given in Δ_2^1 by (15) and (8) respectively with $k = 2$, whereas $u_3 = -\infty$ in A^2 and $v_3 = -\infty$ in A_3. Furthermore, in Δ_2^1

$$g_3'(c_2) = \alpha \frac{\varphi^{2*}(\bar{v}) + \alpha\varphi^{3*}(\bar{v})}{\varphi(\bar{v}) + \alpha\varphi^{2*}(\bar{v}) + \alpha^2\varphi^{3*}(\bar{v})},$$

as well as $g_3(p(1+\alpha+\alpha^2)) = \alpha p(1+\alpha)$ and $\Gamma_3^- \subset \cup_{l=0}^{1}(\Delta_l^l \cup \Delta_{l+1}^l)$.

Supposing now that the statement of the theorem is true for all $k \le m$ we establish its validity for $k = m+1$. Induction assumption means that

$$\Gamma_k^- = \Gamma_{k-1}^- \cup \Gamma_{k-1}^0 \cup (\Gamma_{k-1}^+ \cap \Gamma_k^-) \subset \cup_{l=0}^{k-2}(\Delta_l^l \cup \Delta_{l+1}^l), \quad k = \overline{2,m},$$

so $\Gamma_2^- \subset \ldots \subset \Gamma_m^-$ and $\Gamma_m^+ \subset \ldots \subset \Gamma_2^+$, moreover, $\Gamma = \Gamma_m^- \cup \Gamma_m^0 \cup \Gamma_m^+$.

Let $(c_1, c_2) \in \Gamma_m^-$, then

$$f_m'(x) = -c_1 + \begin{cases} 0, & x < \bar{v}, \\ K(x), & \bar{v} \le x < u_m, \\ T_m(x), & x \ge u_m, \end{cases} \tag{19}$$

while $f_{m-1}'(x)$ has the form (19) with $m-1$ instead of m, if $(c_1, c_2) \in \Gamma_{m-1}^-$. If $(c_1, c_2) \in \Gamma_{m-1}^0 \cup \Gamma_{m-1}^+$, then $f_{m-1}'(x)$ is given by (12) with $k = m-1$. So, one has either

$$f_m'(x) - f_{m-1}'(x) = \begin{cases} 0, & x < u_{m-1}, \\ -S_{m-1}(x), & u_{m-1} \le x < u_m, \\ S_m(x) - S_{m-1}(x), & x \ge u_m, \end{cases}$$

or

$$f_m'(x) - f_{m-1}'(x) = \begin{cases} 0, & x < v_{m-1}, \\ -T_{m-1}(x), & v_{m-1} \le x < \bar{v}, \\ -S_{m-1}(x), & \bar{v} \le x < u_m, \\ S_m(x) - S_{m-1}(x), & x \ge u_m. \end{cases}$$

It is clear that $H(u_m) < 0$, that means $S_{m+1}(u_m) < 0$ and $u_{m+1} > u_m > \bar{v}$, hence $(c_1, c_2) \in \Gamma_{m+1}^-$.

Now if $(c_1, c_2) \in \Gamma_m^+ \cup \Gamma_m^0$, then $f_k'(x)$ has the form (12) for $k \le m$, with $v_k = -\infty$ for $k \le l$ and v_{l+1} given by (8) with $k = l$ in Δ_l. This entails

$$S_{m+1}(u) = c_2 - \alpha c_1 + \alpha \int_0^{u-v_m} T_m(u-s)\varphi(s)\,ds$$

and $S_{m+1}(v_m) = c_2 - \alpha c_1 \le 0$, whence it is obvious that $v_m \le u_{m+1}$. As a result one has two possibilities: either $u_{m+1} \le \bar{v}$, that is, $(c_1, c_2) \in \Gamma_{m+1}^+ \cup \Gamma_{m+1}^0$, or $\bar{v} < u_{m+1}$, namely, $(c_1, c_2) \in \Gamma_{m+1}^-$. In the first case there exists $v_{m+1} \in (u_{m+1}, \bar{v})$ and $f_{m+1}'(x)$ is given by (12). Furthermore, v_{m+1} satisfies (8) with $k = m$ in Δ_m. In the second case $f_{m+1}'(x)$ has the form (19) with indices $m+1$ instead of m. Thus,

$$S_n(u) = R^{\infty}(x) + \alpha \int_0^{u-u_{n-1}} S_{n-1}(u-s)\varphi(s)\,ds \geq R^{\infty}(u)$$

and $u_n \leq u^{\infty}$ for $n > 2$. It is simple to prove, as in Theorem 2, that $u^{\infty} = \lim_{n\to\infty} u_n$. □

Corollary 1. *If* $(c_1 - p)^+ \leq c_2 \leq \beta_k c_1$ *with* $\beta_k = \sum_{i=1}^{k-1} \alpha^i / \sum_{i=0}^{k-1} \alpha^i$, *then* $(c_1, c_2) \in \Gamma_k^-$, $k \geq 2$.

Remark 3. As follows from Theorem 3, for the parameters set Γ^{α} one uses two suppliers or only the first one. The order sizes are regulated by critical levels u_n and $\bar{v}$ or v_n respectively, according to values of cost parameters. More precisely, if $\alpha c_1 > c_2 \geq (c_1 - p)^+$, then there exists $n_0(c_1, c_2)$ such that for $n > n_0$ it is optimal to use both suppliers, whereas for $n \leq n_0$ only the first supplier may be used.

2.4 *Order Constraints*

Turning to the results with order constraints we begin by the study of the first restriction impact.

Theorem 4. *Let* $a_1 < \infty$, $a_2 = \infty$, *then the optimal decision at the first step of* n*-step process has the form*

$$z_n^{(1)}(x) = \min[a_1, (\min(v_n, \bar{v}) - x)^+], \quad z_n^{(2)}(x) = (u_n - x - z_n^{(1)})^+. \tag{20}$$

The sequences $\{u_n\}$ *and* $\{v_n\}$ *defined by (6) are non-decreasing. There exists* $\lim_{n\to\infty} u_n$ *equal to* u^{a_1} *in* Γ *and* u^0 *in* Γ^-.

Proof. As previously, we proceed by induction. At first let us take $n = 1$. Since $S_1(u) = c_2 > 0$, that is, $u_1 = -\infty$, it is optimal to put $u = v$. On the other hand, $T_1(v) = c_1 - p + (p+h)F(v)$, therefore $v_1 = -\infty$ in A_1 and in Δ_0 there exists $v_1 \in [0, \bar{v}]$ satisfying (8) with $k = 0$. In the former case $u_1(x) = v_1(x) = x$ for all x and in the latter case $u_1(x) = v_1(x) = x + a_1$ for $x < v_1 - a_1$, $u_1(x) = v_1(x) = v_1$ for $x \in [v_1 - a_1, v_1)$ and $u_1(x) = v_1(x) = x$ for $x \geq v_1$.

Thus, $f_1'(x) = L'(x) = -p + (p+h)F(x)$ in A_1, whereas in Δ_0

$$f_1'(x) = -c_1 + \begin{cases} T_1(x + a_1), & x < v_1 - a_1, \\ 0, & v_1 - a_1 \leq x < v_1, \\ T_1(x), & x \geq v_1. \end{cases} \tag{21}$$

It is obvious that $f_1'(x)$ is non-decreasing, hence the same is true of $S_2(u)$ and $T_2(v)$ taking values in $[c_2 - \alpha p, c_2 + \alpha h]$ and $[c_1 - p(1+\alpha), c_1 + h(1+\alpha)]$ respectively. Hence, $u_2 = -\infty$ in A^1, $v_2 = -\infty$ in A_2, so for $n = 2$ the optimal decision is $u_2(x) = v_2(x) = x$ if $(c_1, c_2) \in D_2 = A_2 \cap A^1$.

Proceeding in the same way we establish that in $D_k = A_k \cap A^{k-1}$, $k > 2$, one has $u_n = v_n = -\infty$, $n \le k$, so $u_n(x) = v_n(x) = x$ is optimal for all $n \le k$ and

$$f_n'(x) = -p\sum_{i=0}^{n-1}\alpha^i + (p+h)\sum_{i=1}^{n}\alpha^{i-1}F^{i*}(x).$$

Moreover, in Δ_k^{k-1} there exist $u_{k+1} \ge \underline{\kappa}$ and $v_{k+1} \ge \underline{\kappa}$ given by (15) and (8) respectively.

Next consider the set Γ. For each $k > 1$ it is divided into subsets Γ_k^- and Γ_k^+ by a curve $c_2 = g_k(c_1)$ defined implicitly by equality $S_k(\bar{v}) = 0$. The point $(p\sum_{i=0}^{k-1}\alpha^i, p\sum_{i=1}^{k-1}\alpha^i)$ on the boundary of Γ, corresponding to $\bar{v} = \underline{\kappa}$, belongs to $g_k(c_1)$, since $S_k(\underline{\kappa}) = T_k(\underline{\kappa}) = 0$ for such (c_1, c_2) from Δ_{k-1}^{k-2}. According to the rule of implicit function differentiation and the form of $S_k(\cdot)$ in Δ_{k-1}^{k-2}, we get

$$g_k'(c_1) = \frac{\sum_{i=2}^{k-1}\alpha^{i-1}\varphi^{i*}(\bar{v})}{\sum_{i=1}^{k-1}\alpha^{i-1}\varphi^{i*}(\bar{v})},$$

whence it is obvious that $g_k'(c_1) \in [0, 1]$. The last result is valid for other values of c_1 although expression of $g_k'(c_1)$ is more complicated.

Suppose $(c_1, c_2) \in \Gamma_k^+ \subset \Gamma_{k-1}^+$ and

$$f_k'(x) = -c_1 + \begin{cases} K(x+a_1), & x < u_k - a_1, \\ T_k(x+a_1), & u_k - a_1 \le x < v_k - a_1, \\ 0, & v_k - a_1 \le x < v_k, \\ T_k(x), & x \ge v_k. \end{cases}$$

It is not difficult to verify that $u_{k+1} > u_k$ and $v_{k+1} > v_k$ and $\Gamma_{k+1}^+ \subset \Gamma_k^+$.

Now let $(c_1, c_2) \in \Gamma_k^-$, then

$$f_k'(x) = -c_1 + \begin{cases} K(x+a_1), & x < \bar{v} - a_1, \\ 0, & \bar{v} - a_1 \le x < \bar{v}, \\ K(x), & \bar{v} \le x < u_k, \\ T_n(x), & x \ge u_k. \end{cases}$$

It is easy to check that $\Gamma_n^- \subset \Gamma_{n+1}^-$ for any $n \ge k$ and

$$S_{n+1}(u) = R^{a_1}(u) + \alpha\int_0^{u-u_n} S_n(u-s)\varphi(s)\,ds \ge R^{a_1}(u),$$

entailing $u_n \le u^{a_1}$ for all n.

Since $R^{\infty}(u) \geq R^{a_1}(u) \geq R^0(u)$, for any u and $a_1 > 0$, one has $u^{\infty} < u^{a_1} < u^0$. It is not difficult to establish that $\lim_{n\to\infty} u_n = u^{a_1}$ where u^{a_1} is defined by (6).

Turning to $\Gamma^- \subset A_1$ we get, for $n > k$,

$$f_n'(x) = -c_2 + L'(x) + \begin{cases} 0, & x < u_n, \\ S_n(x), & x \geq u_n, \end{cases}$$

if $(c_1, c_2) \in \Gamma^- \cap \Delta^k$. Verifying that $f_n'(x) - f_{n-1}'(x) < 0$ for $x < u_n$, one obtains $u_{n+1} > u_n$. Furthermore, for all u and $n > k$,

$$S_{n+1}(u) = R^0(u) + \alpha \int_0^{u-u_n} S_n(u-s)\varphi(s)\, ds \geq R^0(u).$$

So, $u_n \leq u^0$ for all n. Obviously, there exists $\lim_{n\to\infty} u_n$ and it is easy to show that it is equal to u^0.

Finally, if u_n and v_n are finite then for Γ^+ it is optimal to take $v_n(x) = x + a_1$, $u_n(x) = u_n$ for $x < u_n - a_1$; $u_n(x) = v_n(x) = x + a_1$ for $x \in [u_n - a_1, v_n - a_1)$; $u_n(x) = v_n(x) = v_n$ for $x \in [v_n - a_1, v_n)$; and $u_n(x) = v_n(x) = x$ for $x \geq v_n$. □

To study the impact of the other constraint we formulate at first the almost obvious

Corollary 2. *If $c_2 > \alpha c_1$ the optimal behaviour for $a_1 = \infty$, $a_2 < \infty$ has the same form as that for $a_1 = a_2 = \infty$ in Theorem 1.*

Proof. Proceeding in the same way as in Theorem 1 we easily get the result. □

Theorem 5. *Let $a_1 \leq \infty$, $a_2 < \infty$ and $(c_1, c_2) \in \Gamma^-$. Then the optimal decision at the first step of n-step process is given by*

$$z_n^{(1)}(x) = \min(a_1, (w_n - x)^+), \quad z_n^{(2)}(x) = \min(a_2, (u_n - x - z_n^{(1)}(x))^+),$$

where w_n and u_n are defined by (6). There exist $\lim_{n\to\infty} u_n \geq \widehat{v}$ and $\lim_{n\to\infty} w_n \leq \widehat{v}$ with $\widehat{v}$ defined by (7).

Proof. Begin by treating the case $a_1 = \infty$, $a_2 < \infty$. It follows easily from assumptions that $u_1 = v_1 = w_1 = -\infty$ and $f_1'(x) = L'(x)$. Moreover,

$$S_2(u) = c_2 - \alpha p + (p+h)F^{2*}(u), \quad T_2(c) = c_1 - p - \alpha p + (p+h)[F(v) + \alpha F^{2*}(v)]$$

and

$$B_2(v) = c_1 - p - \alpha p + (p+h)[F(v) + \alpha F^{2*}(v + a_2)].$$

Since $T_2(v) < B_2(v)$ and $S_2(v + a_2) < B_2(v) < T_2(v + a_2)$ it follows from here that $w_2 < v_2 < w_2 + a_2 < u_2$. It is clear that $w_2 > -\infty$ in Δ_1^0

$$f_2(x) = -c_1 x + \begin{cases} G_2(w_2, w_2 + a_2), & x < w_2, \\ G_2(x, x + a_2), & w_2 \le x < u_2 - a_2, \\ G_2(x, u_2), & u_2 - a_2 \le x < u_2, \\ G_2(x, x), & x \ge u_2. \end{cases}$$

It follows immediately that

$$f_2'(x) - f_1'(x) = \begin{cases} -T_1(x), & x < w_2, \\ -c_2 + S_2(x + a_2), & w_2 \le x < u_2 - a_2, \\ -c_2, & u_2 - a_2 \le x < u_2, \\ -c_2 + S_2(x), & x \ge u_2. \end{cases}$$

So, $f_2'(x) - f_1'(x) < 0$ for $x \le u_2$. This entails the following inequalities $w_2 < w_3$, $v_2 < v_3$, $u_2 < u_3$.

Then if $(c_1, c_2) \in \Delta_1^0$, it is not difficult to verify by induction that there exist finite u_n and w_n, $n \ge 2$. Furthermore, one has $w_n < v_n < w_n + a_2 < u_n$. Hence, it is optimal to take $v_n(x) = w_n$, $u_n(x) = w_n + a_2$ for $x < w_n$; $v_n(x) = x$, $u_n(x) = x + a_2$ for $x \in [w_n, u_n - a_2)$; $v_n(x) = x$, $u_n(x) = u_n$ for $x \in [u_n - a_2, u_n)$ and $u_n(x) = v_n(x) = x$ for $x \ge u_n$. Consequently, one gets

$$f_n'(x) = -c_1 + \begin{cases} 0, & x < w_n, \\ B_n(x), & w_n \le x < u_n - a_2, \\ K(x), & u_n - a_2 \le x < u_n, \\ T_n(x), & x \ge u_n, \end{cases} = -c_2 + L'(x) + \begin{cases} -K(x), \\ S_n(x + a_2), \\ 0, \\ S_n(x), \end{cases} \tag{22}$$

and $B_n(v) \ge c_1(1 - \alpha) + L'(x)$. That means, $w_n \le \widehat{v}$ for all n and a_2. Using (22) one also obtains $\lim_{n\to\infty} u_n \ge \widehat{v}$.

If $(c_1, c_2) \in \Delta_l^0$, there exists $w_{l+1} > -\infty$, whereas $w_m = -\infty$ for $m \le l$. Thus,

$$f_n'(x) = -c_1 + \begin{cases} B_n(x), & x < u_n - a_2, \\ K(x), & u_n - a_2 \le x < u_n, \\ T_n(x), & x \ge u_n, \end{cases}$$

for $1 < n \le l$ and $f_n'(x)$ has the form (22) for $n > l$.

The subsets Δ_l^k corresponding to $k \ge 1$ are treated in the same way giving also $z_n^{(1)}(x) = (w_n - x)^+$, $z_n^{(2)} = \min(a_2, (u_n - x - z_n^{(1)}(x))^+)$.

Changes necessary under assumption $a_1 < \infty$ are almost obvious, so the details are omitted. □

2.5 Sensitivity Analysis

We begin studying the impact of model parameters on the optimal decision by the motivating

Example. Assume $\underline{\kappa} = 0, \overline{\kappa} = d$ and $\varphi(s) = d^{-1}$, $s \in [\underline{\kappa}, \overline{\kappa}]$, that is, distribution of ξ_i is uniform. Obviously, $F(u) = u/d$, $u \in [0, d]$, and $\overline{v} = d(p + c_2 - c_1)/(p + h)$, while $F^{2*}(u) = u^2/2d^2$, $u \in [0, d]$, $F^{2*}(u) = 1 - (u - 2d)^2/2d^2$, $u \in [d, 2d]$. Suppose also $a_1 < \infty$ and $\alpha = 1$.

According to (21) the form of $g_2(c_1)$, given by the relation $S_2(\overline{v}) = 0$, depends on a_1 for $(c_1, c_2) \in \Delta_0^0$. Moreover, $c_2 - p + (p + h)F^{2*}(u) = S_2^{(0)}(u) \le S_2^{(a_1)}(u)$ and $S_2^{(a_1)}(u) \le S_2^{(\infty)}(u) = c_2 + \int_0^{u-v_1} L'(u - s)\varphi(s)\,ds$, whence it follows that the domain Γ_2^- decreases as a_1 increases.

On the other hand, the curve $g_2(c_1)$ is the same for all a_1 if $(c_1, c_2) \in \Delta_1^0$. It is determined by equation $S_2^{(0)}(\overline{v}) = 0$, which can be rewritten in the form

$$2(p + h)(p - c_2) = (p + c_2 - c_1)^2, \quad \text{for} \quad h \ge p.$$

Thus, $g_2^{(0)}(c_1)$ does not depend on d. It starts from the point $c_1 = 2p$, $c_2 = p$ and crosses the line $c_1 = p$ at $c_2 = -(2p + h) + \sqrt{5p^2 + 4ph + h^2}$ and then the line $c_2 = c_1$ at $c_2 = p[1 - p/2(p + h)]$. For $h = p$ these values of c_2 are equal to $p(\sqrt{10} - 3)$ and $3p/4$ respectively.

Next, if $c_1 = 0$ one has $c_2 = (p+h)[\sqrt{1 + 2p(p + h)^{-1}} - 1]$ equal to $p(2\sqrt{3} - 3)$ for $h = p$. However, the set $\Gamma_2^- \cap \{c_2 > c_1\}$ is empty when $a_1 = \infty$.

As usual for dynamic programming, the optimal control depends on the planning horizon. Moreover, for n fixed there exist stability domains of cost parameters $(\Gamma_n^-, \Gamma_n^+, \Gamma^- \cap \Delta^k, \Gamma^+ \cap \Delta_l, k, l \ge 0)$ where the optimal behaviour has the same type, that is determined by the same set of critical levels u_n, v_n, w_n, $n \ge 2$, and $\bar{v}$.

Fortunately, using the ε-optimal and asymptotically optimal stationary controls one can reduce the number of stability domains and exclude dependence on n.

We prove below only the simplest results demonstrating the reasoning necessary for the general case.

Theorem 6. *Let* $0 < \alpha < 1$, $a_1 = \infty$, $a_2 \le \infty$ *and* $c_2 > \alpha c_1$. *Then for any* $\varepsilon > 0$ *there exists* $n_0 = n_0(\varepsilon, k)$ *such that it is* ε*-optimal to use* $u_n(x) = v_n(x) = \max(x, \widehat{v})$ *at the first step of* n*-step process with* $n > n_0$ *if* $(c_1, c_2) \in \Delta_k$, $k = 0, 1, \ldots$. *The critical level* $\widehat{v}$ *is given by* (7).

Proof. Put for brevity $g_n(x) = G_n(x, x)$. According to Theorem 1 and Corollary 2 we can write for $n > k + 1$

$$f_n(x) = -c_1 x + \begin{cases} g_n(v_n), & x < v_n, \\ g_n(x), & x \ge v_n, \end{cases}$$

and

$$f_n(x) - f_{n-1}(x) = \begin{cases} g_n(v_n) - g_{n-1}(v_{n-1}), & x < v_{n-1}, \\ g_n(v_n) - g_{n-1}(x), & v_{n-1} \le x < v_n, \\ g_n(x) - g_{n-1}(x), & x \ge v_n, \end{cases}$$

if $(c_1, c_2) \in \Delta_k$, $k = 0, 1, \ldots$.

Taking into account that $g_n(v_n) = \min_x g_n(x)$ one easily gets

$$\max_{x \le z} |f_n(x) - f_{n-1}(x)| \le \max_{v_{n-1} \le x \le \max(z, \widehat{v})} |g_n(x) - g_{n-1}(x)|.$$

Recalling that $g_n(x) = c_1 x + L(x) + \alpha \int_0^\infty f_{n-1}(x-s)\varphi(s)\,ds$ it is possible to write for $z > \widehat{v}$ the following chain of inequalities

$$\max_{x \le z} |f_n(x) - f_{n-1}(x)| \le \alpha \max_{x \le z} |f_{n-1}(x) - f_{n-2}(x)| \le \ldots \le \alpha^{n-k} \delta_k(z).$$

Here $\delta_k(z) = \max_{v_{k+1} \le x \le z} |\int_0^\infty (f_{k+1}(x-s) - f_k(x-s)\varphi(s)\,ds| < \infty$ in Δ_k, $k = 0, 1, \ldots$, in particular, $\delta_0(z) = c_1\mu + \max(L(z), L(v_1))$.

Clearly, we have established that $f_n(x)$ tends uniformly to a limit $f(x)$ on any half-line $\{x \le z\}$. This enables us to state that continuous function $f(x)$ satisfies the following functional equation

$$f(x) = -c_1 x + \min_{v \ge x}[c_1 v + L(v) + \alpha \int_0^\infty f(v-s)\varphi(s)\,ds].$$

Furthermore, if the planning horizon is infinite the optimal behaviour at each step is determined by a critical level $\widehat{v}$.

Since $u_n(x) = v_n(x) = x$ for all n, if $x \ge \widehat{v}$, it follows immediately that for any $\varepsilon > 0$ one can find $n_0(\varepsilon, c_1)$ such that ordering $(\widehat{v} - x)^+$ at the first step of n-step process with $n > n_0$ we obtain an ε-optimal control. It is obvious that $n_0(\varepsilon, c_1)$ can be chosen the same for the parameter set Δ_k, that is, $n_0 = n_0(\varepsilon, k)$. □

As follows from Definitions 3 and 4, a control is stationary if it prescribes the same behaviour at each step and it is asymptotically optimal if

$$\lim_{n\to\infty} n^{-1} \widehat{f}_n(x) = \lim_{n\to\infty} n^{-1} f_n(x)$$

where $\widehat{f}_n(x)$ represents the expected n-step costs under this control.

Theorem 7. *If $\alpha = 1$, $a_1 = \infty$, $a_2 \le \infty$ and $c_2 > c_1$, it is asymptotically optimal to take $z_n^{(1)}(x) = (\bar{t} - x)^+$, $z_n^{(2)}(x) = 0$ for all n with $\bar{t}$ given by $L'(\bar{t}) = 0$.*

Proof. Denote by $f_n^l(x)$ the expected n-step costs if $\bar{t}$ is applied during the first l steps, whereas the critical levels v_k, $k \le n-l$, optimal under the assumptions made, are used during the other steps.

It is clear that $f_n^n(x) = \widehat{f}_n(x)$ and $f_n^0(x) = f_n(x)$, hence

$$\widehat{f}_n(x) - f_n(x) = \sum_{l=1}^{n} (f_n^l(x) - f_n^{l-1}(x)). \tag{23}$$

Suppose for simplicity that $c_1 < p$, that is, v_1 is finite.

Since $v_n \le v_{n+1}$, $n \ge 1$, and $v_n \to \bar{t}$, as $n \to \infty$, one can find, for any $\varepsilon > 0$, such $\widehat{n} = n(\varepsilon)$ that $\bar{t} - \varepsilon < v_n \le \bar{t}$, if $n \ge \widehat{n}$. Furthermore, we have

$$\max_x |f_n^l(x) - f_n^{l-1}(x)| \le \max_x |f_{n-l+1}^1 - f_{n-l+1}^0(x)|$$

and

$$f_k^1(x) - f_k^0(x) = \begin{cases} c_1(\bar{t} - v_k) + L(\bar{t}) - L(v_k) + V(v_k), & x < v_k, \\ c_1(\bar{t} - x) + L(\bar{t}) - L(x) + V(x), & v_k \le x < \bar{t}, \\ 0, & x \ge \bar{t}, \end{cases}$$

where $V(x) = \int_0^\infty (f_{k-1}(\bar{t} - s) - f_{k-1}(x - s))\varphi(s)\, ds$.

Obviously, $k - 1 = n - l \ge \widehat{n}$ for $l \le n - \widehat{n}$, therefore

$$\max_x |f_k^1(x) - f_k^0(x)| \le d\varepsilon \quad \text{with} \quad d = 2(c_1 + \max(p, h))$$

and

$$\sum_{l=1}^{n-\widehat{n}} |f_n^l(x) - f_n^{l-1}(x)| \le (n - \widehat{n}) d\varepsilon. \tag{24}$$

On the other hand,

$$\sum_{l=n-\widehat{n}+1}^{n} |f_n^l(x) - f_n^{l-1}(x)| \le \widehat{n} b(x) \tag{25}$$

where $b(x) = \max_{k \le \widehat{n}} |f_k^1(x) - f_k^0(x)| \le L(v_1) + d\bar{t} < \infty$, for all x.

It follows immediately from (23) to (25) that

$$n^{-1}(\widehat{f}_n(x) - f_n(x)) \to 0, \quad \text{as} \quad n \to \infty.$$

To complete the proof we have to verify that there exists, for all x,

$$\lim_{n\to\infty} n^{-1} \widehat{f}_n(x) = c_1\mu + L(\bar{t}), \quad \mu = \mathsf{E}\xi_k,\ k \ge 1. \tag{26}$$

This is obvious for $x \le \bar{t}$, since in this case

$$\widehat{f}_n(x) = c_1(\bar{t} - x) + c_1 \sum_{k=1}^{n-1} \mathsf{E}\xi_k + nL(\bar{t}).$$

Now let $x > \bar{t}$. Then we do not order during the first m_x steps where

$$m_x = \inf\{k : \sum_{i=1}^{k} \xi_i > x - \bar{t}\}.$$

In other words, we wait until the inventory falls below the level $\bar{t}$ proceeding after that as in the previous case. Hence,

$$\widehat{f}_n(x) = L(x) + \mathsf{E} \sum_{i=1}^{m_x - 1} L(x - \sum_{k=1}^{i} \xi_k) + c_1 \mathsf{E} \left[\zeta_x + \sum_{i=m_x+1}^{n-1} \xi_i \right] + \mathsf{E}(n - m_x)L(\bar{t})$$

here $\zeta_x = \sum_{i=1}^{m_x} \xi_i - (x - \bar{t})$ is the overshot of the level $x - \bar{t}$ by the random walk with jumps ξ_i, $i \geq 1$.

Thus, it is possible to rewrite $\widehat{f}_n(x)$ as follows

$$\widehat{f}_n(x) = n(c_1\mu + L(\bar{t})) + W(x).$$

Using Wald's identity and renewal processes properties (see, e.g. [1]), as well as, the fact that $L(\bar{t})$ is the minimum of $L(x)$ it is possible to establish that $|W(x)| < \infty$ for a fixed x. So (26) follows immediately.

The same result is valid for $c_1 \geq p$. The calculations being long and tedious are omitted. □

Remark 4. For the parameter sets treated in Theorems 2 and 3 the asymptotically optimal policy is also of threshold type being based either on u^0 or $\bar{v}$ and u^∞.

Since $\bar{t} = g(p, h)$, with $g(a_1, a_2) = F^{inv}(a_1/(a_1 + a_2))$, it is useful to check its sensitivity with respect to small fluctuations of parameters p and h and perturbations of distribution F.

We apply the local technique, more precisely, differential importance measure (DIM) introduced in [3] is used. Let $a^0 = (a_1^0, a_2^0)$ be the base-case values of parameters, reflecting the decision maker (researcher) knowledge of assumptions made. The (DIM) for parameter a_s, $s = 1, 2$, is defined as follows

$$D_s(a^0, da) = g'_{a_s}(a^0)\, da_s \left(\sum_{j=1}^{2} g'_{a_j}(a^0)\, da_j \right)^{-1} \quad (= dg_s(a^0)/dg(a^0)$$

if $dg(a^0) \neq 0$. Whence, for uniform parameters changes: $da_s = u$, $s = 1, 2$, we get

$$D1_s(a^0) = g'_{a_s}(a^0) \Big/ \sum_{j=1}^{2} g'_{a_j}(a^0). \tag{27}$$

Theorem 8. *Under assumptions of Theorem 7, (DIM)s for parameters p and h do not depend on distribution F.*

Proof. The result follows immediately from (27) and definition of function g. Since

$$g'_{a_1}(a^0) = \varphi^{-1}(\bar{t}^0)a_2^0/(a_1^0 + a_2^0)^2, \quad g'_{a_2}(a^0) = -\varphi^{-1}(\bar{t}^0)a_1^0/(a_1^0 + a_2^0)^2,$$

it is clear that

$$D1_1(a^0) = \frac{a_2^0}{a_2^0 - a_1^0}, \quad D1_2(a^0) = -\frac{a_1^0}{a_2^0 - a_1^0} = 1 - D1_1(a^0).$$

Thus, they are well defined for $a_1^0 \neq a_2^0$ and do not depend on F. Moreover, $D1_1(a^0) > 1$, $D1_2(a^0) < 0$ for $a_2^0 > a_1^0$ and $D1_1(a^0) < 0$, $D1_2(a^0) > 1$ for $a_2^0 < a_1^0$. □

Note that a similar result is valid for $\widehat{v}$ if $0 < \alpha < 1$.

Now we can establish that the asymptotically optimal policy is stable with respect to small perturbations of distribution F.

Denote by $\bar{t}_k$ value of $\bar{t}$ corresponding to distribution $F_k(t)$. Moreover, set

$$\gamma(F_k, F) = \sup_t |F_k(t) - F(t)|,$$

that is, γ is the Kolmogorov (or uniform) metric.

Lemma 2. *Let distribution function $F(t)$ be continuous and strictly increasing. Then $\bar{t}_k \to \bar{t}$, provided $\gamma(F_k, F) \to 0$, as $k \to \infty$.*

Proof. According to assumptions $F_k(\bar{t}_k) = F(\bar{t})$ and $|F_k(\bar{t}_k) - F(\bar{t}_k)| \leq \gamma(F, F_k)$. Hence $|F(\bar{t}) - F(\bar{t}_k)| \leq \gamma(F, F_k)$. That means, $\bar{t}_k \to \bar{t}$, as $k \to \infty$. □

This result is also important for construction of asymptotically optimal policies under assumption of none a priori information about distribution F.

3 Conclusion

We have treated in detail the case of two suppliers and obtained the explicit form of optimal, ε-optimal and asymptotically optimal policies for various sets of cost parameters. Stability of model to small fluctuations of parameters and perturbations of underlying process is also established. The case of m suppliers, $m > 2$, can be investigated using induction procedure and numerical methods. Due to lack of space the results will be published in a forthcoming paper.

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Levy Preservation and Associated Properties for f-Divergence Minimal Equivalent Martingale Measures

Suzanne Cawston and Lioudmila Vostrikova

Abstract We study such important properties of f-divergence minimal martingale measure as Levy preservation property, scaling property, invariance in time property for exponential Levy models. We give some useful decomposition for f-divergence minimal martingale measures and we answer on the question which form should have f to ensure mentioned properties. We show that f is not necessarily common f-divergence. For common f-divergences, i.e. functions verifying $f''(x) = ax^\gamma$, $a > 0$, $\gamma \in \mathbb{R}$, we give necessary and sufficient conditions for existence of f-minimal martingale measure.

Keywords f-divergence • Exponential Levy models • Minimal martingale measures • Levy preservation property

Mathematics Subject Classification (2010): 60G07, 60G51, 91B24

1 Introduction

This article is devoted to some important and exceptional properties of f-divergences. As known, the notion of f-divergence was introduced by Ciszar [3] to measure the difference between two absolutely continuous probability measures by mean of the expectation of some convex function f of their Radon-Nikodym density. More precisely, let f be a convex function and $Z = \frac{dQ}{dP}$ be a Radon-Nikodym density of

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two measures Q and P, $Q \ll P$. Supposing that $f(Z)$ is integrable with respect to P, f-divergence of Q with respect to P is defined as

$$f(Q\|P) = E_P[f(Z)].$$

One can remark immediately that this definition cover such important cases as variation distance when $f(x) = |x-1|$, as Hellinger distance when $f(x) = (\sqrt{x}-1)^2$ and Kulback-Leibler information when $f(x) = x\ln(x)$. Important as notion, f-divergence was studied in a number of books and articles (see for instance [14, 19])

In financial mathematics it is of particular interest to consider measures Q^* which minimise on the set of all equivalent martingale measures the f-divergence. This fact is related to the introducing and studying so called incomplete models, like exponential Levy models (see [2,6,7,20,22]). In such models contingent claims cannot, in general, be replicated by admissible strategies. Therefore, it is important to determine strategies which are, in a certain sense optimal. Various criteria are used, some of which are linked to risk minimisation (see [9, 25, 26]) and others consisting in maximizing certain utility functions (see [1, 11, 16]). It has been shown (see [11, 18]) that such questions are strongly linked via Fenchel-Legendre transform to dual optimisation problems, namely to f-divergence minimisation on the set of equivalent martingale measures, i.e. the measures Q which are equivalent to the initial physical measure P and under which the stock price is a martingale.

Mentioned problems has been well studied in the case of relative entropy, when $f(x) = x\ln(x)$ (cf. [10, 21]), also for power functions $f(x) = x^q$, $q > 1$ or $q < 0$ (cf. [15]), $f(x) = -x^q$, $0 < q < 1$ (cf. [4, 5]) and for logarithmic divergence $f(x) = -\ln(x)$ (cf. [17]), called common f-divergences. Note that the three mentioned functions all satisfy $f''(x) = ax^{\gamma}$ for an $a > 0$ and a $\gamma \in \mathbb{R}$. The converse is also true, any function which satisfies $f''(x) = ax^{\gamma}$ is, up to linear term, a common f-divergence. It has in particular been noted that for these functions, the f-divergence minimal equivalent martingale measure, when it exists, preserves the Levy property, that is to say that the law of Levy process under initial measure P remains a law of Levy process under the f-divergence minimal equivalent martingale measure Q^*.

The aim of this paper is to study the questions of preservation of Levy property and associated properties such as scaling property and invariance in time property for f-divergence minimal martingale measures when P is a law of d-dimensional Levy process X and Q^* belongs to the set of so called equivalent martingale measures for exponential Levy model, i.e. measures under which the exponential of X is a martingale. More precisely, let fix a convex function f defined on $\mathbb{R}^{+,*}$ and denote by $\mathscr{M}$ the set of equivalent martingale measures associated with exponential Levy model related to X. We recall that an equivalent martingale measure Q^* is f-divergence minimal if $f(Z^*)$ is integrable with respect to P where Z^* is the Radon-Nikodym density of Q^* with respect to P, and

$$f(Q^*\|P) = \min_{Q\in\mathcal{M}} f(Q\|P).$$

We say that Q^* preserves Levy property if X remains Levy process under Q^*. The measure Q^* is said to be scale invariant if for all $x \in \mathbb{R}^+$, $E_P|f(xZ^*)| < \infty$ and

$$f(xQ^*\|P) = \min_{Q\in\mathcal{M}} f(xQ\|P).$$

We also recall that an equivalent martingale measure Q^* is said to be time invariant if for all $T > 0$, and the restrictions Q_T,P_T of the measures P, Q on time interval $[0, T]$, $E_P|f(Z_T^*)| < \infty$ and

$$f(Q_T^*\|P_T) = \min_{Q\in\mathcal{M}} f(Q_T\|P_T)$$

In this paper we study the shape of f belonging to the class of strictly convex tree times continuously differentiable functions and ones used as f-divergence, gives an equivalent martingale measure which preserves Levy property. More precisely, we consider equivalent martingale measures Q belonging to the class $\mathcal{K}^*$ such that for all compact sets K of $\mathbb{R}^{+,*}$

$$E_P|f(\frac{dQ_T}{dP_T})| < +\infty, \quad E_Q|f'(\frac{dQ_T}{dP_T})| < +\infty, \quad \sup_{t\le T}\sup_{\lambda\in K} E_Q[f''(\lambda\frac{dQ_t}{dP_t})\frac{dQ_t}{dP_t}] < +\infty.$$

We denote by Z_T^* Radon-Nikodym density of Q_T^* with respect to P_T and by β^* and Y^* the corresponding Girsanov parameters of an f-divergence minimal measure Q^* on $[0, T]$, which preserves the Levy property and belongs to $\mathcal{K}^*$.

To precise the shape of f we obtain fundamental equations which necessarily verify f. Namely, in the case $\overset{\circ}{supp}\ (\nu) \neq \emptyset$, for a.e. $x \in supp(Z_T^*)$ and a.e. $y \in supp(\nu)$, we prove that

$$f'(xY^*(y)) - f'(x) = \Phi(x)\sum_{i=1}^{d}\alpha_i(e^{y_i}-1) \tag{1}$$

where Φ is a continuously differentiable function defined on the set on $\overset{\circ}{\text{supp}}\ (Z_T^*)$ and $y = {}^\top(y_1, y_2, \cdots y_d), \alpha = {}^\top(\alpha_1, \alpha_2, \cdots \alpha_d)$ are vectors of $\mathbb{R}^d$. Furthermore, if $c \neq 0$, for a.e. $x \in \text{supp}(Z_T^*)$ and a.e. $y \in supp(\nu)$, we get that

$$f'(xY^*(y)) - f'(x) = xf''(x)\sum_{i=1}^{d}\beta_i^*(e^{y_i}-1) + \sum_{j=1}^{d}V_j(e^{y_j}-1) \tag{2}$$

where $\beta^* =^\top (\beta_1^*, \cdots, \beta_d^*)$ is a first Girsanov parameter and $V = {}^\top(V_1, \cdots, V_d)$ is a vector which belongs to the kernel of the matrix c, i.e. $cV = 0$.

Mentioned above equations permit us to precise the form of f. Namely, we prove that if the set $\{\ln Y^*(y),\ y \in supp(\nu)\}$ is of non-empty interior and it contains zero, then there exists $a > 0$ and $\gamma \in \mathbb{R}$ such that for all $x \in supp(Z_t^*)$,

$$f''(x) = ax^\gamma. \tag{3}$$

Taking in account the known results we conclude that in considered case the relation (3) is necessary and sufficient condition for f-divergence minimal martingale measure to preserve Levy property. In addition, as we will see, such f-divergence minimal measure will be also scale and time invariant.

In the case when ${}^\top\beta^* c\beta^* \neq 0$ and support of ν is nowhere dense but when there exists at least one $y \in supp(\nu)$ such that $\ln(Y^*(y)) \neq 0$, we prove that there exist $n \in \mathbb{N}$, the real constants $b_i, \tilde{b}_i$, $1 \leq i \leq n$, and $\gamma \in \mathbb{R}$, $a > 0$ such that

$$f''(x) = ax^\gamma + x^\gamma \sum_{i=1}^{n} b_i (\ln(x))^i + \frac{1}{x} \sum_{i=1}^{n} \tilde{b}_i (\ln(x))^{i-1}$$

The case when ${}^\top\beta^* c\beta^* = 0$ and $supp(\nu)$ is nowhere dense, is not considered in this paper, and from what we know, form an open question.

We underline once more the exceptional properties of the class of functions such that:

$$f''(x) = ax^\gamma$$

and called common f-divergences. This class of functions is exceptional in a sense that they verify also scale and time invariance properties for all Levy processes. As well known, Q^* does not always exist. For some functions, in particular $f(x) = x\ln(x)$, or for some power functions, some necessary and sufficient conditions of existence of a minimal measure have been given (cf. [13, 15]). We will give a unified version of these results for all functions which satisfy $f''(x) = ax^\gamma$, $a > 0$, $\gamma \in \mathbb{R}$. We give also an example to show that the preservation of Levy property can have place not only for the functions verifying $f''(x) = ax^\gamma$.

The paper is organized in the following way: in Sect. 2 we recall some known facts about exponential Levy models and f-divergence minimal equivalent martingale measures. In Sect. 3 we give some known useful for us facts about f-divergence minimal martingale measures. In Sect. 4 we obtain fundamental equations for Levy preservation property (Theorem 3). In Sect. 5 we give the result about the shape of f having Levy preservation property for f-divergence minimal martingale measure (Theorem 5). In Sect. 6 we study the common f-divergences, i.e. with f verifying $f''(x) = ax^\gamma$, $a > 0$. Their properties are given in Theorem 6.

2 Some Facts About Exponential Levy Models

Let us describe our model in more details. We assume the financial market consists of a bank account B whose value at time t is

$$B_t = B_0 e^{rt},$$

where $r \geq 0$ is the interest rate which we assume to be constant. We also assume that there are $d \geq 1$ risky assets whose prices are described by a d-dimensional stochastic process $S = (S_t)_{t\geq 0}$,

$$S_t = {}^\top(S_0^{(1)} e^{X_t^{(1)}}, \cdots, S_0^{(d)} e^{X_t^{(d)}})$$

where $X = (X_t)_{t\geq 0}$ is a d-dimensional Levy process, $X_t = {}^\top(X_t^{(1)}, \cdots, X_t^{(d)})$ and $S_0 = {}^\top(S_0^{(1)}, \cdots, S_0^{(d)})$. We recall that Levy processes form the class of processes with stationary and independent increment and that the characteristic function of the law of X_t is given by the Levy-Khintchine formula: for all $t \geq 0$, for all $u \in \mathbb{R}$,

$$E[e^{i<u,X_t>}] = e^{t\psi(u)}$$

where

$$\psi(u) = i < u, b > -\frac{1}{2}\,{}^\top u c u + \int_{\mathbb{R}^d} [e^{i<u,y>} - 1 - i < u, h(y) >]\nu(dy)$$

where $b \in \mathbb{R}^d$, c is a positive $d \times d$ symmetric matrix, h is a truncation function and ν is a Levy measure, i.e. positive measure on $\mathbb{R}^d \setminus \{0\}$ which satisfies

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(dy) < +\infty.$$

The triplet (b, c, ν) entirely determines the law of the Levy process X, and is called the characteristic triplet of X. From now on, we will assume that the interest rate $r = 0$ as this will simplify calculations and the more general case can be obtained by replacing the drift b by $b - r$. We also assume for simplicity that $S_0 = 1$.

We will denote by $\mathscr{M}$ the set of all locally equivalent martingale measures:

$$\mathscr{M} = \{Q \overset{loc}{\sim} P,\ S \text{ is a martingale under } Q\}.$$

We will assume that this set is non-empty, which is equivalent to assuming the existence of $Q \overset{loc}{\sim} P$ such that the drift of S under Q is equal to zero. We consider our model on finite time interval $[0, T]$, $T > 0$, and for this reason the distinction between locally equivalent martingale measures and equivalent martingale measures does not need to be made. We recall that the density Z of any equivalent to P measure can be written in the form $Z = \mathscr{E}(M)$ where $\mathscr{E}$ denotes the Doleans-Dade exponential and $M = (M_t)_{t\geq 0}$ is a local martingale. It follows from Girsanov theorem theorem that there exist predictable functions $\beta = {}^{\top}(\beta^{(1)}, \cdots \beta^{(d)})$ and Y verifying the integrability conditions: for $t \geq 0$ (P-a.s.)

$$\int_0^t {}^{\top}\beta_s c \beta_s ds < \infty,$$

$$\int_0^t \int_{\mathbb{R}^d} |\, h(y)\,(Y_s(y) - 1)\,| \nu^{X,P}(ds, dy) < \infty,$$

and such that

$$M_t = \sum_{i=1}^{d} \int_0^t \beta_s^{(i)} dX_s^{c,(i)} + \int_0^t \int_{\mathbb{R}^d} (Y_s(y) - 1)(\mu^X - \nu^{X,P})(ds, dy) \qquad (4)$$

where μ^X is a jumps measure of the process X and $\nu^{X,P}$ is its compensator with respect to P and the natural filtration $\mathbb{F}$, $\nu^{X,P}(ds, dy) = ds\, \nu(dy)$ (for more details see [14]). We will refer to (β, Y) as the Girsanov parameters of the change of measure from P into Q. It is known from Grigelionis result [12] that a semi-martingale is a process with independent increments under Q if and only if their semi-martingale characteristics are deterministic, i.e. the Girsanov parameters do not depend on ω, i.e. β depends only on time t and Y depends on time and jump size (t, x). Since Levy process is homogeneous process, it implies that X will remain a Levy process under Q if and only if there exists $\beta \in \mathbb{R}$ and a positive measurable function Y such that for all $t \leq T$ and all ω, $\beta_t(\omega) = \beta$ and $Y_t(\omega, y) = Y(y)$.

We recall that if Levy property is preserved, S will be a martingale under Q if and only if

$$b + \frac{1}{2} diag(c) + c\beta + \int_{\mathbb{R}^d} [(e^y - 1)Y(y) - h(y)]\nu(dy) = 0 \qquad (5)$$

where e^y is a vector with components e^{y_i}, $1 \leq i \leq d$, and $y = {}^{\top}(y_1, \cdots, y_d)$. This follows again from Girsanov theorem and reflects the fact that under Q the drift of S is equal to zero.

3 Properties of f-Divergence Minimal Martingale Measures

Here we consider a fixed strictly convex continuously differentiable on $\mathbb{R}^{+,*}$ function f and a time interval $[0, T]$. We recall in this section a few known and useful results about f-divergence minimisation on the set of equivalent martingale measures. Let $(\Omega, \mathscr{F}, \mathbb{F}, P)$ be a probability filtered space with the natural filtration $\mathbb{F} = (\mathscr{F}_t)_{t \geq t}$ satisfying usual conditions and let $\mathscr{M}$ be the set of equivalent martingale measures. We denote by Q_t, P_t the restrictions of the measures Q, P on $\mathscr{F}_t$. We introduce Radon-Nikodym density process $Z = (Z_t)_{t \geq 0}$ related to Q, an equivalent martingale measure, where for $t \geq 0$

$$Z_t = \frac{dQ_t}{dP_t}.$$

We denote by Z^* Radon-Nikodym density process related with f-divergence minimal equivalent martingale measure Q^*.

Definition 1. An equivalent martingale measure Q^* is said to be f-divergence minimal on the time interval $[0, T]$ if $E_P|f(Z_T^*)| < \infty$ and

$$E_P[f(Z_T^*)] = \min_{Q \in \mathscr{M}} E_P[f(Z_T)]$$

where $\mathscr{M}$ is a class of locally equivalent martingale measures.

Then we introduce the subset of equivalent martingale measures

$$\mathscr{K} = \{Q \in \mathscr{M} \mid E_P|f(Z_T)| < +\infty \text{ and } E_Q[|f'(Z_T)|] < +\infty.\} \tag{6}$$

We will concentrate ourselves on the case when the minimal measure, if it exists, belongs to $\mathscr{K}$. Note that for a certain number of functions this is necessarily the case.

Lemma 1 (cf. [19], Lemma 8.7). *Let f be a convex continuously differentiable on $\mathbb{R}^{+,*}$ function. Assume that for $c > 1$ there exist positive constants c_0, c_1, c_2, c_3 such that for $u > c_0$,*

$$f(cu) \leq c_1 f(u) + c_2 u + c_3 \tag{7}$$

Then a measure $Q \in \mathscr{M}$ which is f-divergence minimal necessarily belongs to $\mathscr{K}$.

We now recall the following necessary and sufficient condition for a martingale measure to be minimal.

Theorem 1 (cf. [11], Theorem 2.2). *Consider $Q^* \in \mathscr{K}$. Then, Q^* is minimal if and only if for all $Q \in \mathscr{K}$,*

$$E_{Q^*}[f'(Z_T^*)] \leq E_Q[f'(Z_T^*)].$$

This result is in fact true in the much wider context of semi-martingale modelling. We will mainly use it here to check that a candidate is indeed a minimal measure. We will also use extensively another result from [11] in order to obtain conditions that must be satisfied by minimal measures.

Theorem 2 (cf. [11], Theorem 3.1). *Assume $Q^* \in \mathscr{K}$ is an f-divergence minimal martingale measure. Then there exists $x_0 \in \mathbb{R}$ and a predictable d-dimensional process ϕ such that*

$$f'(\frac{dQ_T^*}{dP_T}) = x_0 + \sum_{i=1}^{d} \int_0^T \phi_t^{(i)} \, dS_t^{(i)}$$

and such that $\sum_{i=1}^d \int_0^\cdot \phi_t^{(i)} \, dS_t^{(i)}$ defines a martingale under the measure Q^*.

4 A Fundamental Equation for f-Divergence Minimal Levy Preserving Martingale Measures

Our main aim in this section is to obtain an equation satisfied by the Radon-Nikodym density of f-divergence minimal equivalent martingale measures. This result will both enable us to obtain information about the Girsanov parameters of f-divergence minimal equivalent martingale measures and also to determine conditions which must be satisfied by the function f in order to a f-minimal equivalent martingale measure exists. Let us introduce the class $\mathscr{K}^*$ of locally equivalent martingale measures verifying: for all compact sets K of $\mathbb{R}^{+,*}$

$$E_P|f(Z_T)| < +\infty, \quad E_Q|f'(Z_T)| < +\infty, \quad \sup_{t \le T} \sup_{\lambda \in K} E_Q[f''(\lambda Z_t^*) Z_t^*] < +\infty. \tag{8}$$

Theorem 3. *Let f be strictly convexe $\mathscr{C}^3(\mathbb{R}^{+,*})$ function. Let Z^* be the density of an f-divergence minimal measure Q^* on $[0,T]$, which preserves the Levy property and belongs to $\mathscr{K}^*$. We denote by (β^*, Y^*) its Girsanov parameters. Then, if $\overset{\circ}{supp}$ $(\nu) \neq \emptyset$, for a.e. $x \in supp(Z_T^*)$ and a.e. $y \in supp(\nu)$, we have*

$$f'(xY^*(y)) - f'(x) = \Phi(x) \sum_{i=1}^{d} \alpha_i (e^{y_i} - 1) \tag{9}$$

where Φ is a continuously differentiable function defined on the set $\overset{\circ}{supp}$ (Z_T^*) and $\alpha = {}^\top(\alpha_1, \alpha_2, \cdots \alpha_d)$ is a vector of $\mathbb{R}^d$. Furthermore, if $c \neq 0$, for a.e. $x \in supp(Z_T^*)$ and a.e. $y \in supp(\nu)$, we have

$$f'(xY^*(y)) - f'(x) = xf''(x) \sum_{i=1}^{d} \beta_i^* (e^{y_i} - 1) - \sum_{j=1}^{d} V_j (e^{y_j} - 1) \tag{10}$$

where $\beta^* = {}^\top (\beta_1^*, \cdots, \beta_d^*)$ and $V = {}^\top(V_1, \cdots, V_d)$ belongs to the kernel of the matrix c, i.e. $cV = 0$.

We recall that for all $t \leq T$, since Q^* preserves Levy property, Z^*_t and $\frac{Z^*_T}{Z^*_t}$ are independent under P and that $\mathscr{L}(\frac{Z^*_T}{Z^*_t}) = \mathscr{L}(Z^*_{T-t})$. Therefore denoting

$$\rho(t,x) = E_{Q^*}[f'(xZ^*_{T-t})],$$

and taking cadlag versions of processes, we deduce that Q^*-a.s. for all $t \leq T$

$$E_{Q^*}[f'(Z^*_T)|\mathscr{F}_t] = \rho(t, Z^*_t)$$

We note that the proof of Theorem 3 is based on the identification using Theorem 2 and an application of decomposition formula to function ρ. However, the function ρ is not necessarily twice continuously differentiable in x and once continuously differentiable in t. So, we will proceed by approximations, by application of Ito formula to specially constructed function ρ_n. In order to do this, we need a number of auxiliary lemmas given in the next section.

Since the result of Theorem 3 is strongly related to the support of Z^*_T, we are also interested with the question: when this support is an interval? This question has been well studied in [24, 27] for infinitely divisible distributions. In our case, the specific form of the Girsanov parameters following from preservation of Levy property allow us to obtain the following result proved in Sect. 4.3.

Proposition 1. *Let Z^* be the density of an f-divergence minimal equivalent martingale measure on $[0,T]$, which preserves the Levy property and belongs to $\mathscr{K}^*$. Then*

(i) *If $^\top\beta^* c \beta^* \neq 0$, then $supp(Z^*_T) = \mathbb{R}^{+,*}$.*
(ii) *If $^\top\beta^* c \beta^* = 0$, $\overset{\circ}{supp}\ (\nu) \neq \emptyset$, $0 \in supp(\nu)$ and Y^* is not identically 1 on $\overset{\circ}{supp}\ (\nu)$, then*
   (j) *In the case $\ln(Y(y)) > 0$ for all $y \in supp(\nu)$, there exists $A > 0$ such that $supp(Z^*_T) = [A, +\infty[$;*
   (jj) *In the case $\ln(Y(y)) < 0$ for all $y \in supp(\nu)$ there exists $A > 0$ such that $supp(Z^*_T) =]0, A]$;*
   (jjj) *In the case when there exist $y, \bar{y} \in supp(\nu)$ such that $\ln(Y^*(y)).\ln(Y^*(\bar{y})) < 0$, we have $supp(Z^*_T) = \mathbb{R}^{+,*}$.*

4.1 *Some Auxiliary Lemmas*

We begin with approximation lemma. Let a strictly convex tree times continuously differentiable on $\mathbb{R}^{+,*}$ function f be fixed.

Lemma 2. *There exists a sequence of bounded functions $(\phi_n)_{n\geq 1}$, which are of class $\mathscr{C}^2$ on $\mathbb{R}^{+*}$, increasing, such that for all $n \geq 1$, ϕ_n coincides with f' on* *the compact set* $[\frac{1}{n}, n]$ *and such that for sufficiently big* n *the following inequalities hold for all* $x, y > 0$ *:*

$$|\phi_n(x)| \leq 4|f'(x)| + \alpha \ , \ |\phi_n'(x)| \leq 3f''(x) \ , \ |\phi_n(x) - \phi_n(y)| \leq 5|f'(x) - f'(y)| \tag{11}$$

where α *is a real positive constant.*

Proof. We set, for $n \geq 1$,

$$A_n(x) = f'(\frac{1}{n}) - \int_{x \vee \frac{1}{2n}}^{\frac{1}{n}} f''(y)(2ny-1)^2(5-4ny)dy$$

$$B_n(x) = f'(n) + \int_n^{x \wedge (n+1)} f''(y)(n+1-y)^2(1+2y-2n)dy$$

and finally

$$\phi_n(x) = \begin{cases} A_n(x) \text{ if } 0 \leq x < \frac{1}{n}, \\ f'(x) \text{ if } \frac{1}{n} \leq x \leq n, \\ B_n(x) \text{ if } x > n. \end{cases}$$

Here A_n and B_n are defined so that ϕ_n is of class $\mathscr{C}^2$ on $\mathbb{R}^{+,*}$. For the inequalities we use the fact that f' is increasing function as well as the estimations: $0 \leq (2nx-1)^2(5-4nx) \leq 1$ for $\frac{1}{2n} \leq x \leq \frac{1}{n}$ and $0 \leq (n+1-x)^2(1+2x-2n) \leq 3$ for $n \leq x \leq n+1$. □

Let Q be Levy property preserving locally equivalent martingale measure and (β, Y) its Girsanov parameters when change from P into Q. We use the function

$$\rho_n(t,x) = E_Q[\phi_n(xZ_{T-t})]$$

to obtain the following analog to Theorem 4, replacing f' with ϕ_n.

For this let us denote for $0 \leq t \leq T$

$$\xi_t^{(n)}(x) = E_Q[\phi_n'(xZ_{T-t})\, Z_{T-t}] \tag{12}$$

and

$$H_t^{(n)}(x,y) = E_Q[\phi_n(xZ_{T-t}Y(y)) - \phi_n(xZ_{T-t})] \tag{13}$$

Lemma 3. *We have* Q^**-a.s., for all* $t \leq T$,

$$\rho_n(t, Z_t) = E_Q[\phi_n(Z_T)] + \tag{14}$$

$$\sum_{i=1}^d \beta_i \int_0^t \xi_s^{(n)}(Z_{s-})\, Z_{s-} dX_s^{(c),Q,i} + \int_0^t \int_{\mathbb{R}^d} H_s^{(n)}(Z_{s-}, y)\, (\mu^X - \nu^{X,Q})(ds, dy)$$

where $\beta = {}^{\top}(\beta_1, \cdots, \beta_d)$ *and* $\nu^{X,Q}$ *is a compensator of the jump measure* μ^X *with respect to* $(\mathbb{F}, Q)$.

Proof. In order to apply the Ito formula to ρ_n, we need to show that ρ_n is twice continuously differentiable with respect to x and once with respect to t and that the corresponding derivatives are bounded for all $t \in [0, T]$ and $x \geq \epsilon$, $\epsilon > 0$. First of all, we note that from the definition of ϕ_n for all $x \geq \epsilon > 0$

$$|\frac{\partial}{\partial x}\phi_n(xZ_{T-t})| = |Z_{T-t}\phi_n'(xZ_{T-t})| \leq \frac{(n+1)}{\epsilon} \sup_{z>0} |\phi_n'(z)| < +\infty.$$

Therefore, ρ_n is differentiable with respect to x and we have

$$\frac{\partial}{\partial x}\rho_n(t, x) = E_Q[\phi_n'(xZ_{T-t})\, Z_{T-t}].$$

Moreover, the function $(x, t) \mapsto \phi_n'(xZ_{T-t})Z_{T-t}$ is continuous P-a.s. and bounded. This implies that $\frac{\partial}{\partial x}\rho_n$ is continuous and bounded for $t \in [0, T]$ and $x \geq \epsilon$.
In the same way, for all $x \geq \epsilon > 0$

$$|\frac{\partial^2}{\partial x^2}\phi_n(xZ_{T-t})| = Z_{T-t}^2\phi_n''(xZ_{T-t}) \leq \frac{(n+1)^2}{\epsilon^2} \sup_{z>0} \phi_n''(z) < +\infty.$$

Therefore, ρ_n is twice continuously differentiable in x and

$$\frac{\partial^2}{\partial x^2}\rho_n(t, x) = E_Q[\phi_n''(xZ_{T-t})Z_{T-t}^2]$$

We can verify easily that it is again continuous and bounded function. In order to obtain differentiability with respect to t, we need to apply the Ito formula to ϕ_n:

$$\begin{aligned}\phi_n(xZ_t) = \phi_n(x) &+ \sum_{i=1}^{d}\int_0^t x\phi_n'(xZ_{s-})\beta_i Z_{s-}dX_s^{(c),Q,i} \\ &+ \int_0^t \int_{\mathbb{R}^d} \phi_n(xZ_{s-}Y(y)) - \phi_n(xZ_{s-})\,(\mu^X - \nu^{X,Q})(ds, dy) \\ &+ \int_0^t \psi_n(x, Z_{s-})ds\end{aligned}$$

where

$$\begin{aligned}\psi_n(x, Z_{s-}) = {}&{}^{\top}\beta c\beta[xZ_{s-}\phi_n'(xZ_{s-}) + \frac{1}{2}x^2 Z_{s-}^2\phi_n''(xZ_{s-})] \\ &+ \int_{\mathbb{R}^d} [(\phi_n(xZ_{s-}Y(y)) - \phi_n(xZ_{s-}))\, Y(y) - x\phi_n'(xZ_{s-})Z_{s-}(Y(y) - 1)]\nu(dy).\end{aligned}$$

Therefore,

$$E_Q[\phi_n(xZ_{T-t})] = \int_0^{T-t} E_Q[\psi_n(x, Z_{s-})]ds$$

so that ρ_n is differentiable with respect to t and

$$\frac{\partial}{\partial t}\rho_n(t,x) = -E_Q[\psi_n(x, Z_{s-})]_{|_{s=(T-t)}}$$

We can also easily verify that this function is continuous and bounded. For this we take in account the fact that ϕ_n, ϕ_n' and ϕ_n'' are bounded functions and also that the Hellinger process of Q_T and P_T of the order $1/2$ is finite.

We can finally apply the Ito formula to ρ_n. For that we use the stopping times

$$s_m = \inf\{t \geq 0 \,|\, Z_t \leq \frac{1}{m}\},$$

with $m \geq 1$ and $\inf\{\emptyset\} = +\infty$. Then, from Markov property of Lévy process we have :

$$\rho_n(t \wedge s_m, Z_{t\wedge s_m}) = E_Q(\phi_n(\lambda Z_T) \,|\, \mathscr{F}_{t\wedge s_m})$$

We remark that $(E_Q(\phi_n(\lambda Z_T) \,|\, \mathscr{F}_{t\wedge s_m})_{t\geq 0}$ is Q-martingale, uniformly integrable with respect to m. From Ito formula we have:

$$\rho_n(t \wedge s_m, Z_{t\wedge s_m}) = E_Q(\phi_n(\lambda Z_T)) + \int_0^{t\wedge s_m} \frac{\partial \rho_n}{\partial s}(s, Z_{s-})ds$$

$$+ \int_0^{t\wedge s_m} \frac{\partial \rho_n}{\partial x}(s, Z_{s-})dZ_s + \frac{1}{2}\int_0^{t\wedge s_m} \frac{\partial^2 \rho_n}{\partial x^2}(s, Z_{s-})d < Z^c >_s$$

$$+ \sum_{0\leq s\leq t\wedge s_m} \rho_n(s, Z_s) - \rho_n(s, Z_{s-}) - \frac{\partial \rho_n}{\partial x}(s, Z_{s-})\Delta Z_s$$

where $\Delta Z_s = Z_s - Z_{s-}$. After some standard simplifications, we see that

$$\rho_n(t \wedge s_m, Z_{t\wedge s_m}) = A_{t\wedge s_m} + M_{t\wedge s_m}$$

where $(A_{t\wedge s_m})_{0\leq t\leq T}$ is predictable process, which is equal to zero,

$$A_{t\wedge s_m} = \int_0^{t\wedge s_m} \frac{\partial \rho_n}{\partial s}(s, Z_{s-})ds + \frac{1}{2}\int_0^{t\wedge s_m} \frac{\partial^2 \rho_n}{\partial x^2}(s, Z_{s-})d < Z^c >_s +$$

$$\int_0^{t\wedge s_m}\int_{\mathbb{R}} [\rho_n(s, Z_{s-} + x) - \rho_n(s, Z_{s-}) - \frac{\partial \rho_n}{\partial x}(s, Z_{s-})x]\nu^{Z,Q}(ds, dx)$$

and $(M_{t\wedge s_m})_{0\le t\le T}$ is a Q-martingale,

$$M_{t\wedge s_m} = E_Q(\phi_n(\lambda Z_T)) + \int_0^{t\wedge s_m} \frac{\partial \rho_n}{\partial x}(s, Z_{s-})dZ_s^c +$$

$$\int_0^{t\wedge s_m} \int_{\mathbb{R}} [\rho_n(s, Z_{s-} + x) - \rho_n(s, Z_{s-})](\mu^Z(ds, dx) - \nu^{Z,Q}(ds, dx))$$

Then, we pass to the limit as $m \to +\infty$. We remark that the sequence $(s_m)_{m\ge 1}$ is going to $+\infty$ as $m \to \infty$. From [23], Corollary 2.4, p. 59, we obtain that

$$\lim_{m\to\infty} E_Q(\phi_n(Z_T) \,|\, \mathscr{F}_{t\wedge s_m}) = E_Q(\phi_n(Z_T) \,|\, \mathscr{F}_t)$$

and by the definition of local martingales we get:

$$\lim_{m\to\infty} \int_0^{t\wedge s_m} \frac{\partial \rho_n}{\partial x}(s, Z_{s-})dZ_s^c = \int_0^t \frac{\partial \rho_n}{\partial x}(s, Z_{s-})dZ_s^c = \int_0^t \lambda \xi_s^{(n)}(Z_{s-})dZ_s^c$$

and

$$\lim_{m\to\infty} \int_0^{t\wedge s_m} \int_{\mathbb{R}} [\rho_n(s, Z_{s-} + x) - \rho_n(s, Z_{s-})](\mu^Z(ds, dx) - \nu^{Z,Q}(ds, dx)) =$$

$$\int_0^t \int_{\mathbb{R}} [\rho_n(s, Z_{s-} + x) - \rho_n(s, Z_{s-})](\mu^Z(ds, dx) - \nu^{Z,Q}(ds, dx))$$

Now, in each stochastic integral we pass from the integration with respect to the process Z to the one with respect to the process X. For that we remark that

$$dZ_s^c = \sum_{i=1}^d \beta^{(i)} Z_{s-} dX_s^{c,Q,i}, \;\; \Delta Z_s = Z_{s-} Y(\Delta X_s).$$

Lemma 3 is proved. □

4.2 *A Decomposition for the Density of Levy Preserving Martingale Measures*

This decomposition will follow from a previous one by a limit passage. Let again Q be Levy property preserving locally equivalent martingale measure and (β, Y) the corresponding Girsanov parameters when passing from P to Q. We introduce cadlag versions of the following processes: for $t > 0$

$$\xi_t(x) = E_Q[f''(x Z_{T-t}) Z_{T-t}]$$

and

$$H_t(x,y) = E_Q[f'(xZ_{T-t}Y(y)) - f'(xZ_{T-t})] \tag{15}$$

Theorem 4. *Let Z be the density of a Levy preserving equivalent martingale measure Q. Assume that Q belongs to $\mathscr{K}^*$. Then we have Q- a.s, for all $t \le T$,*

$$E_Q[f'(Z_T)|\mathscr{F}_t] = E_Q[f'(Z_T)] + \tag{16}$$

$$\sum_{i=1}^{d} \beta_i \int_0^t \xi_s(Z_{s-})Z_{s-}dX_s^{(c),Q,i} + \int_0^t \int_{\mathbb{R}^d} H_s(Z_{s-},y)\,(\mu^X - \nu^{X,Q})(ds,dy)$$

We now turn to the proof of Theorem 4. In order to obtain the decomposition for f', we obtain convergence in probability of the different stochastic integrals appearing in Lemma 3.

Proof of Theorem 4. For a $n \ge 1$, we introduce the stopping times

$$\tau_n = \inf\{t \ge 0 \,|\, Z_t \ge n \text{ or } Z_t \le \frac{1}{n}\} \tag{17}$$

where $\inf\{\emptyset\} = +\infty$ and we note that $\tau_n \to +\infty$ (P-a.s.) as $n \to \infty$. First of all, we note that

$$|E_Q[f'(Z_T)|\mathscr{F}_t] - \rho_n(t,Z_t)| \le E_Q[|f'(Z_T) - \phi_n(Z_T)||\mathscr{F}_t]$$

As f' and ϕ_n coincide on the interval $[\frac{1}{n}, n]$, it follows from Lemma 3 that

$$\begin{aligned}|E_Q[f'(Z_T)|\mathscr{F}_t] - \rho_n(t,Z_t)| &\le E_Q[|f'(Z_T) - \phi_n(Z_T)|\mathbf{1}_{\{\tau_n \le T\}}|\mathscr{F}_t] \\ &\le E_Q[(5|f'(Z_T)| + \alpha)\mathbf{1}_{\{\tau_n \le T\}}|\mathscr{F}_t].\end{aligned}$$

Now, for every $\epsilon > 0$, by Doob inequality and Lebesgue dominated convergence theorem we get:
$\lim_{n\to+\infty} Q(\sup_{t\le T} E_Q[(5|f'(Z_T)| + \alpha)\mathbf{1}_{\{\tau_n \le T\}}|\mathscr{F}_t] > \epsilon)$

$$\le \lim_{n\to+\infty} \frac{1}{\epsilon} E_Q[(5|f'(Z_T)| + \alpha)\mathbf{1}_{\{\tau_n \le T\}}] = 0$$

Therefore, we have

$$\lim_{n\to+\infty} Q(\sup_{t\le T} |E_Q[f'(Z_T) - \rho_n(t,Z_t)|\mathscr{F}_t]| > \epsilon) = 0.$$

We now turn to the convergence of the three elements of the right-hand side of (14). We have almost surely $\lim_{n\to+\infty} \phi_n(Z_T) = f'(Z_T)$, and for all $n \ge 1$, $|\phi_n(Z_T)| \le 4|f'(Z_T)| + \alpha$. Therefore, it follows from the dominated convergence theorem that,

$$\lim_{n\to+\infty} E_Q[\phi_n(Z_T)] = E_Q[f'(Z_T)].$$

We prove now the convergence of continuous martingale parts of (14). It follows from Lemma 2 that

$$Z_t\,|\xi_t^{(n)}(Z_t) - \xi_t(Z_t)| \le E_Q[Z_T|\phi_n'(Z_T) - f''(Z_T)|\,|\mathscr{F}_t] \le \\ 4E_Q[Z_T|f''(Z_T)|\mathbf{1}_{\{\tau_n\le T\}}|\mathscr{F}_t].$$

Hence, we have as before for $\epsilon > 0$

$$\lim_{n\to+\infty} Q(\sup_{t\le T} Z_t\,|\xi_t^{(n)}(Z_t) - \xi_t(Z_t)| > \epsilon) \le \lim_{n\to+\infty} \frac{4}{\epsilon} E_Q[Z_T f''(Z_T)\mathbf{1}_{\{\tau_n\le T\}}] = 0$$

Therefore, it follows from the Lebesgue dominated convergence theorem for stochastic integrals (see [14], Theorem I.4.31, p. 46) that for all $\epsilon > 0$ and $1 \le i \le d$

$$\lim_{n\to+\infty} Q(\sup_{t\le T} \Big| \int_0^t Z_{s-}\,(\xi_s^{(n)}(Z_{s-}) - \xi_s(Z_{s-}))dX_s^{(c),Q,i}\Big| > \epsilon) = 0.$$

It remains to show the convergence of the discontinuous martingales to zero as $n \to \infty$. We start by writing

$$\int_0^t \int_{\mathbb{R}^d} [H_s^{(n)}(Z_{s-},y) - H_s(Z_{s-},y)](\mu^X - \nu^{X,Q})(ds,dy) = M_t^{(n)} + N_t^{(n)}$$

with

$$M_t^{(n)} = \int_0^t \int_{\mathscr{A}} [H_s^{(n)}(Z_{s-},y) - H_s(Z_{s-},y)](\mu^X - \nu^{X,Q})(ds,dy),$$
$$N_t^{(n)} = \int_0^t \int_{\mathscr{A}^c} [H_s^{(n)}(Z_{s-},y) - H_s(Z_{s-},y)](\mu^X - \nu^{X,Q})(ds,dy),$$

where $\mathscr{A} = \{y : |Y(y) - 1| < \frac{1}{4}\}$.

For $p \ge 1$, we consider the sequence of stopping times τ_p defined by (17) with replacing n by real positive p. We introduce also the processes

$$M^{(n,p)} = (M_t^{(n,p)})_{t\ge 0},\;\; N^{(n,p)} = (N_t^{(n,p)})_{t\ge 0}$$

with $M_t^{(n,p)} = M_{t\wedge\tau_p}^{(n)}$, $N_t^{(n,p)} = N_{t\wedge\tau_p}^{(n)}$. We remark that for $p \ge 1$ and $\epsilon > 0$

$$Q(\sup_{t\le T} |M_t^{(n)} + N_t^{(n)}| > \epsilon) \le Q(\tau_p \le T) + Q(\sup_{t\le T} |M_t^{(n,p)}| > \frac{\epsilon}{2}) + Q(\sup_{t\le T} |N_t^{(n,p)}| > \frac{\epsilon}{2}).$$

Furthermore, we obtain from Doob martingale inequalities that

$$Q(\sup_{t\leq T}|M_t^{(n,p)}| > \frac{\epsilon}{2}) \leq \frac{4}{\epsilon^2}\mathbb{E}_Q[(M_T^{(n,p)})^2] \tag{18}$$

and

$$Q(\sup_{t\leq T}|N_t^{(n,p)}| > \frac{\epsilon}{2}) \leq \frac{2}{\epsilon}\mathbb{E}_Q|N_T^{(n,p)}| \tag{19}$$

Since $\tau_p \to +\infty$ as $p \to +\infty$ it is sufficient to show that $E_Q[M_T^{(n,p)}]^2$ and $E_Q|N_T^{(n,p)}|$ converge to 0 as $n \to \infty$.

For that we estimate $E_Q[(M_T^{(n,p)})^2]$ and prove that

$$E_Q[(M_T^{(n,p)})^2] \leq \\ C\Big(\int_0^T \sup_{v\in K}\mathbb{E}_Q^2[Z_s\, f''(vZ_s)\mathbf{1}_{\{\tau_{q_n}<s\}}]ds\Big)\Big(\int_{\mathscr{A}}(\sqrt{Y(y)}-1)^2\nu(dy)\Big)$$

where C is a constant, K is some compact set of $\mathbb{R}^{+,*}$ and $q_n = \frac{n}{4p}$.

First we note that on stochastic interval $[\![0, T\wedge\tau_p)]\!]$ we have $1/p \leq Z_{s-} \leq p$, and, hence,

$$E_Q[(M_T^{(n,p)})^2] = E_Q[\int_0^{T\wedge\tau_p}\int_{\mathscr{A}}|H_s^{(n)}(Z_{s-},y) - H_s(Z_{s-},y)|^2\, Y(y)\nu(dy)ds] \leq \\ \int_0^T\int_{\mathscr{A}}\sup_{1/p\leq x\leq p}|H_{T-s}^{(n)}(x,y) - H_{T-s}(x,y)|^2\, Y(y)\nu(dy)ds$$

To estimate the difference $|H_{T-s}^{(n)}(x,y) - H_{T-s}(x,y)|$ we note that

$$H_{T-s}^{(n)}(x,y) - H_{T-s}(x,y) = E_Q[\phi_n(xZ_sY(y)) - \phi_n(xZ_s) - f'(xZ_sY(y)) + f'(xZ_s)]$$

From Lemma 2 we deduce that if $xZ_sY(y) \in [1/n, n]$ and $xZ_s \in [1/n, n]$ then the expression on the right-hand side of the previous expression is zero. But if $y \in \mathscr{A}$ we also have: $3/4 \leq Y(y) \leq 5/4$ and, hence,

$$|H_{T-s}^{(n)}(x,y) - H_{T-s}(x,y)| \leq \\ |E_Q[\mathbf{1}_{\{\tau_{q_n}\leq s\}}|\phi_n(xZ_sY(y)) - \phi_n(xZ_s) - f'(xZ_sY(y)) + f'(xZ_s)|].$$

Again from the inequalities of Lemma 2 we get:

$$|H_{T-s}^{(n)}(x,y) - H_{T-s}(x,y)| \leq 6E_Q[\mathbf{1}_{\{\tau_{q_n}\leq s\}}|f'(xZ_sY(y)) - f'(xZ_s)|].$$

Writing

$$f'(xZ_sY(y)) - f'(xZ_s) = \int_1^{Y(y)} xZ_s f''(xZ_s\theta)d\theta$$

we finally get

$$|H^{(n)}_{T-s}(x,y) - H_{T-s}(x,y)| \le 6 \sup_{3/4\le u\le 5/4} E_Q[\mathbf{1}_{\{\tau_{q_n}\le s\}}\, xZ_s\, f''(xuZ_s)]|Y(y)-1|$$

and this gives us the estimation of $E_Q[(M_T^{(n,p)})^2]$ cited above.

We know that $P_T \sim Q_T$ and this means that the corresponding Hellinger process of order 1/2 is finite:

$$h_T(P,Q,\frac{1}{2}) = \frac{T}{2}{}^\top\beta c\beta + \frac{T}{8}\int_{\mathbb{R}}(\sqrt{Y(y)}-1)^2\nu(dy) < +\infty.$$

Then

$$\int_{\mathscr{A}}(\sqrt{Y(y)}-1)^2\nu(dy) < +\infty.$$

From Lebesgue dominated convergence theorem and (8) we get:

$$\int_0^T \sup_{v\in K}\mathbb{E}^2_Q[Z_s f''(vZ_s)\mathbf{1}_{\{\tau_{q_n}\le s\}}]ds \to 0$$

as $n \to +\infty$ and this information together with the estimation of $E_Q[(M_T^{(n,p)})^2]$ proves the convergence of $E_Q[(M_T^{(n,p)})^2]$ to zero as $n \to +\infty$.

We now turn to the convergence of $E_Q|N_T^{(n,p)}|$ to zero as $n \to +\infty$. For this we prove that

$$E_Q|N_T^{(n,p)}| \le 2TE_Q[\mathbf{1}_{\{\tau_n\le T\}}(5|f'(Z_T)|+\alpha)]\int_{\mathscr{A}^c} Y(y)d\nu$$

We start by noticing that

$$E_Q|N_T^{(n,p)}| \le 2E_Q[\int_0^{T\wedge\tau_p}\int_{\mathscr{A}^c}|H_s^{(n)}(Z_{s-},y) - H_s(Z_{s-},y)|\,Y(y)\nu(dy)ds] \le$$

$$2\int_0^T\int_{\mathscr{A}^c} E_Q[|H_s^{(n)}(Z_{s-},y) - H_s(Z_{s-},y)|\,Y(y)\nu(dy)ds]$$

To evaluate the right-hand side of previous inequality we write

$$|H_s^{(n)}(x,y) - H_s(x,y)|$$
$$\leq E_Q|\phi_n(xZ_{T-s}Y(y)) - f'(xZ_{T-s}Y(y))| + E_Q|\phi_n(xZ_{T-s}) - f'(xZ_{T-s})|.$$

We remark that in law with respect to Q

$$|\phi_n(xZ_{T-s}Y(y)) - f'(xZ_{T-s}Y(y))| = E_Q[|\phi_n(Z_T) - f'(Z_T)|\,|\,Z_s = x\,Y(y)]$$

and

$$|\phi_n(xZ_{T-s}) - f'(xZ_{T-s})| = E_Q[|\phi_n(Z_T) - f'(Z_T)|\,|\,Z_s = x]$$

Then

$$H_s^{(n)}(x,y) - H_s(x,y)| \leq 2E_Q|\phi_n(Z_T) - f'(Z_T)|$$

From Lemma 2 we get:

$$E_Q|\phi_n(xZ_T) - f'(xZ_T)| \leq E_Q[\mathbf{1}_{\{\tau_n \leq T\}}|\phi_n(Z_T) - f'(Z_T)|]$$
$$\leq E_Q[\mathbf{1}_{\{\tau_n \leq T\}}(5|f'(Z_T)| + \alpha)]$$

and is proves the estimation for $E_Q|N_T^{(n,p)}|$.

Then, Lebesgue dominated convergence theorem applied for the right-hand side of the previous inequality shows that it tends to zero as $n \to \infty$. On the other hand, from the fact that the Hellinger process is finite and also from the inequality $(\sqrt{Y(y)} - 1)^2 \geq Y(y)/25$ verifying on $\mathscr{A}^c$ we get

$$\int_{A^c} Y(y)d\nu < +\infty$$

This result with previous convergence prove the convergence of $E_Q|N_T^{(n,p)}|$ to zero as $n \to \infty$. Theorem 4 is proved. □

4.3 *Proof of Theorem 3 and Proposition 1*

Proof of Theorem 3. We define a process $\hat{X} = {}^\top(\hat{X}^{(1)}, \cdots \hat{X}^{(d)})$ such that for $1 \leq i \leq d$ and $t \in [0,T]$

$$S_t^{(i)} = \mathscr{E}(\hat{X}^{(i)})_t$$

where $\mathscr{E}(\cdot)$ is Dolean-Dade exponential. We remark that if X is a Levy process then $\hat{X}$ is again a Levy process and that

$$d\,S_t^{(i)} = S_{t-}^{(i)}\,d\hat{X}_t^{(i)}.$$

In addition, for $1 \leq i \leq d$ and $t \in [0,T]$

$$\hat{X}_t^{(c),i} = X_t^{(c),i}$$
$$\nu^{\hat{X}^{(i)},Q^*} = (e^{y_i} - 1) \cdot \nu^{X^{(i)},Q^*}.$$

Replacing in Theorem 2 the process S by the process $\hat{X}$ we obtain Q-a.s. for all $t \leq T$:

$$E_{Q^*}[f'(Z_T^*)|\mathscr{F}_t] = x_0 + \sum_{i=1}^{d} [\int_0^t \phi_s^{(i)} S_{s-}^{(i)} d\hat{X}_s^{(c),Q^*,i} + \int_0^t \int_{\mathbb{R}^d} \phi_s^{(i)} S_{s-}^{(i)} d(\mu^{\hat{X}^{(i)}} - \nu^{\hat{X}^{(i)},Q^*})] \tag{20}$$

Then it follows from (20), Theorem 4 and the unicity of decomposition of martingales on continuous and discontinuous parts, that $Q^* - a.s.$, for all $s \leq T$ and all $y \in supp(\nu)$,

$$H_s(Z_{s-}^*, y) = \sum_{i=1}^{d} \phi_s^{(i)} S_{s-}^{(i)} (e^{y_i} - 1) \tag{21}$$

and for all $t \leq T$

$$\sum_{i=1}^{d} \int_0^t \xi_s(Z_{s-}^*)\, Z_{s-}^*\, \beta_i^*\, dX_s^{(c),Q^*,i} = \sum_{i=1}^{d} \int_0^t \phi_s^{(i)} S_{s-}^{(i)} dX_s^{(c),Q^*,i}. \tag{22}$$

We remark that $Q^* - a.s.$ for all $s \leq T$

$$H_s(Z_s^*, y) = E_{Q^*}(f'(Y^*(y)Z_T^*) - f'(Z_T^*)\,|\,\mathscr{F}_s).$$

Moreover, $H_s(Z_{s-}^*, y)$ coincide with $H_s(Z_s^*, y)$ in points of continuity of Z^*. Taking the sequence of continuity points of Z^* tending to T and using that $Z_T = Z_{T-}$ (Q^*-a.s.) we get that $Q^* - a.s.$ for $y \in supp(\nu)$

$$f'(Z_T^* Y^*(y)) - f'(Z_T^*) = \sum_{i=1}^{d} \phi_{T-}^{(i)} S_{T-}^{(i)} (e^{y_i} - 1) \tag{23}$$

We fix an arbitrary $y_0 \in \overset{\circ}{supp}\ (\nu)$. Differentiating with respect to y_i, $i \leq d$, we obtain that

$$Z_T^* \frac{\partial}{\partial y_i} Y^*(y_0) f''(Z_T^* Y^*(y_0)) = \phi_{T-}^{(i)} S_{T-}^{(i)} e^{y_{0,i}}$$

We also define:

$$\Phi(x) = x f''(x Y^*(y_0))$$

and

$$\alpha_i = e^{-y_{0,i}} \frac{\partial}{\partial y_i} Y^*(y_0).$$

We then have $\phi_{T-}^{(i)} S_{T-}^{(i)} = \Phi(Z_T^*)\alpha_i$, and inserting this in (23), we obtain (9).

Taking quadratic variation of the difference of the right-hand side and left-hand side in (22), we obtain that $Q^* - a.s.$ for all $s \leq T$

$$^\top[\xi_s(Z_{s-}^*)\, Z_{s-}^*\, \beta^* - S_{s-}\phi_s]\, c\, [\xi_s(Z_{s-}^*)\, Z_{s-}^*\, \beta^* - S_{s-}\phi_s] = 0$$

where by convention $S_{s-}\phi_s = (S_{s-}^{(i)}\phi_s^{(i)})_{1\leq i\leq d}$. Now, we remark that $Q^* - a.s.$ for all $s \leq T$

$$Z_s^*\, \xi_s(Z_s^*) = E_{Q^*}(f''(Z_T^*)Z_T^* \mid \mathscr{F}_s)$$

and that it coincides with $\xi(Z_{s-}^*)$ in continuity points of Z^*. We take a set of continuity points of Z^* which goes to T and we obtain since Levy process has no predictable jumps that $Q^* - a.s.$

$$^\top[Z_T^* f''(Z_T^*)\beta^* - S_{T-}\phi_{T-}]\, c\, [Z_T^* f''(Z_T^*)\beta^* - S_{T-}\phi_{T-}] = 0$$

Hence, if $c \neq 0$,

$$Z_T^* f''(Z_T^*)\beta^* - S_{T-}\phi_{T-} = V$$

where $V \in \mathbb{R}^d$ is a vector which satisfies $cV = 0$. Inserting this in (23) we obtain (10). Theorem 3 is proved. □

Proof of Proposition 1. Writing Ito formula we obtain P-a.s. for $t \leq T$:

$$\begin{aligned}\ln(Z_t^*) = \sum_{i=1}^d \beta_i^* X_t^{(c),i} + \int_0^t \int_{\mathbb{R}^d} \ln(Y^*(y)) d(\mu^X - \nu^{X,P}) \\ \left[-\frac{t}{2}\, {}^\top\beta^* c \beta^* + t \int_{\mathbb{R}^d} [\ln(Y^*(y)) - (Y^*(y) - 1)\right] \nu(dy)\end{aligned} \tag{24}$$

As we have assumed Q^* to preserve the Levy property, the Girsanov parameters (β^*, Y^*) are independent from (ω, t), and the process $\ln(Z^*) = (\ln(Z_t^*))_{0\leq t\leq T}$ is a Levy process with the characteristics:

$$\begin{aligned} b^{\ln Z^*} &= [-\frac{1}{2}\, {}^\top\beta^* c \beta^* + \int_{\mathbb{R}^d} [\ln(Y^*(y)) - (Y^*(y) - 1)]\, \nu(dy), \\ c^{\ln Z^*} &= {}^\top\beta^* c \beta^*, \\ d\nu^{\ln Z^*} &= \ln(Y^*(y)\, \nu(dy). \end{aligned}$$

Now, as soon as ${}^\top\beta^* c\beta^* \neq 0$, the continuous component of $\ln(Z^*)$ is non zero, and from Theorem 24.10 in [24] we deduce that $\mathrm{supp}(Z_T^*) = \mathbb{R}^{+,*}$ and, hence, i).

If $Y^*(y)$ is not identically 1 on $\overset{\circ}{supp}\ (\nu)$, then in (9) the α_i, $1 \le i \le d$, are not all zeros, and hence, the set $supp(\nu^{\ln(Z^*)}) = \{\ln Y^*(y), y \in supp(\nu)\}$ contains an interval. It implies that $\overset{\circ}{supp}\ (\nu^{\ln Z^*}) \neq \emptyset$. Since $0 \in supp(\nu)$, again from (9) it follows that $0 \in supp(\nu^{\ln Z^*})$. Then ii) is a consequence of Theorem 24.10 in [24].

□

5 So Which f Can Give MEMM Preserving Levy Property?

If one considers some simple models, it is not difficult to obtain f-divergence minimal equivalent martingale measures for a variety of functions. In particular, one can see that the f-divergence minimal measure does not always preserve the Levy property. What can we claim for the functions f such that f-divergence minimal martingale measure exists and preserve Levy property?

Theorem 5. *Let $f : \mathbb{R}^{+*} \to \mathbb{R}$ be a strictly convex function of class $\mathscr{C}^3$ and let X be a Levy process given by its characteristics (b, c, ν). Assume there exists an f-divergence minimal martingale measure Q^* on a time interval $[0, T]$, which preserves the Levy property and belongs to $\mathscr{K}^*$.*

Then, if $supp(\nu)$ is of the non-empty interior, it contains zero and Y is not identically 1, there exists $a > 0$ and $\gamma \in \mathbb{R}$ such that for all $x \in supp(Z_T^*)$,

$$f''(x) = ax^\gamma.$$

If ${}^\top\beta^* c\beta^* \neq 0$ and there exists $y \in supp(\nu)$ such that $Y^*(y) \neq 1$, then there exist $n \in \mathbb{N}$, $\gamma \in \mathbb{R}$, $a > 0$ and the real constants $b_i, \tilde{b}_i$, $1 \le i \le n$, such that

$$f''(x) = ax^\gamma + x^\gamma \sum_{i=1}^{n} b_i (\ln(x))^i + \frac{1}{x} \sum_{i=1}^{n} \tilde{b}_i (\ln(x))^{i-1}$$

We deduce this result from the equations obtained in Theorem 3. We will successively consider the cases when $\overset{\circ}{supp}\ (\nu) \neq \emptyset$, then when c is invertible, and finally when c is not invertible.

5.1 *First Case: The Interior of $supp(\nu)$ Is Not Empty*

Proof of Theorem 5. We assume that $\overset{\circ}{supp}\ (\nu) \neq \emptyset$, $0 \in supp(\nu)$, Y^* is not identically 1 on $supp(\nu)$. According to the Proposition 1 it implies in both cases ${}^\top\beta^* c\beta^* \neq 0$ and ${}^\top\beta^* c\beta^* = 0$, that $supp(Z_T^*)$ is an interval, say J. Since the interior of $supp(\nu)$ is not empty, there exist open non-empty intervals $I_1, \dots I_d$ such that $I = I_1 \times \dots \times I_d \subseteq \overset{\circ}{supp}\,(\nu)$. Then it follows from Theorem 3 that for all $(x, y) \in J \times I$,

$$f'(xY^*(y)) - f'(x) = \Phi(x) \sum_{i=1}^{d} \alpha_i (e^{y_i} - 1) \tag{25}$$

where Φ is a differentiable on $\overset{\circ}{J}$ function and $\alpha \in \mathbb{R}^d$. If we now fix $x_0 \in \overset{\circ}{J}$, we obtain

$$Y^*(y) = \frac{1}{x_0}(f')^{-1}(f'(x_0) + \Phi(x_0) \sum_{i=1}^{d} \alpha_i (e^{y_i} - 1))$$

and so Y^* is differentiable and monotonous in each variable. Since Y^* is not identically 1 on $\overset{\circ}{supp}\,(\nu)$ we get that $\alpha \neq 0$. We may now differentiate (25) with respect to y_i corresponding to $\alpha_i \neq 0$, to obtain for all $(x, y) \in J \times I$,

$$\Psi(x_0) f''(xY^*(y)) = \Psi(x) f''(x_0 Y^*(y)), \tag{26}$$

where $\Psi(x) = \frac{\Phi(x)}{x}$. Differentiating this new expression with respect to x on the one hand, and with respect to y_i on the other hand, we obtain the system

$$\begin{cases} \Psi(x_0) Y^*(y) f'''(xY^*(y)) = f''(x_0 Y^*(y)) \Psi'(x) \\ \Psi(x_0) x f'''(xY^*(y)) = x_0 f'''(x_0 Y^*(y)) \Psi(x) \end{cases} \tag{27}$$

In particular, separating the variables, we deduce from this system that there exists $\gamma \in \mathbb{R}$ such that for all $x \in \overset{\circ}{J}$,

$$\frac{\Psi'(x)}{\Psi(x)} = \frac{\gamma}{x}.$$

Hence, there exists $a > 0$ and $\gamma \in \mathbb{R}$ such that for all $x \in \overset{\circ}{J}$, $\Psi(x) = ax^\gamma$. It then follows from (26) and (27) that for all $(x, y) \in J \times I$,

$$\frac{f'''(xY^*(y))}{f''(xY^*(y))} = \frac{\gamma}{xY^*(y)}$$

and hence that $f''(xY^*(y)) = a(xY^*(y))^\gamma$.

We take now the sequence of $(y_m)_{m \geq 1}$, $y_m \in supp(\nu)$, going to zero. Then, the sequence $(Y^*(y_m))_{m \geq 1}$ according to the formula for Y^*, is going to 1. Inserting y_m in previous expression and passing to the limit we obtain that for all $x \in \overset{\circ}{J}$,

$$\frac{f'''(x)}{f''(x)} = \frac{\gamma}{x}$$

and it proves the result on $\overset{\circ}{supp}\ (Z_T^*)$. The final result on $supp(Z_T^*)$ can be proved again by limit passage. □

5.2 *Second Case: c Is Invertible and ν Is Nowhere Dense*

In the first case, the proof relied on differentiating the function Y^*. This is of course no longer possible when the support of ν is nowhere dense. Howerever, since ${}^\top\beta^* c\beta^* \neq 0$, we get from Proposition 1 that $supp(Z^*) = \mathbb{R}^{+,*}$. Again from Theorem 3 we have for all $x > 0$ and $y \in supp(\nu)$,

$$f'(xY^*(y)) - f'(x) = xf''(x)\sum_{i=1}^{d}\beta_i^*(e^{y_i} - 1). \tag{28}$$

We will distinguish two similar cases: $b > 1$ and $0 < b < 1$. For $b > 1$ we fix ϵ, $0 < \epsilon < 1$, and we introduce for $a \in \mathbb{R}$ the following vector space:

$$V_{a,b} = \{\phi \in \mathscr{C}^1([\epsilon(1\wedge b), \frac{1\vee b}{\epsilon}]),\ \text{such that for } x \in [\epsilon, \frac{1}{\epsilon}], \phi(bx)-\phi(x) = ax\phi'(x)\}$$

with the norm

$$\|\phi\|_\infty = \sup_{x\in[\epsilon,\frac{1}{\epsilon}]} |\phi(x)| + \sup_{x\in[\epsilon,\frac{1}{\epsilon}]} |\phi(bx)|$$

It follows from (28) that $f' \in V_{a,b}$ with $b = Y^*(y)$ and $a = \sum_{i=1}^d \beta_i^*(e^{y_i} - 1)$. The condition that there exist $y \in supp(\nu)$ such that $Y^*(y) \neq 1$ insure that $\sum_{i=1}^d \beta_i^*(e^{y_i} - 1) \neq 0$.

Lemma 4. *If $a \neq 0$ then $V_{a,b}$ is a finite dimensional closed in $\|\cdot\|_\infty$ vector space.*

Proof. It is easy to verify that $V_{a,b}$ is a vector space. We show that $V_{a,b}$ is a closed vector space: if we consider a sequence $(\phi_n)_{n\geq 1}$ of elements of $V_{a,b}$ which converges to a function ϕ, we denote by ψ the function such that $\psi(x) = \frac{\phi(bx)-\phi(x)}{ax}$. We then have

$$\lim_{n\to+\infty} ||\phi_n' - \psi||_\infty \leq \frac{1}{\epsilon|a|(1\wedge b)} \lim_{n\to+\infty} ||\phi_n - \phi||_\infty = 0$$

Therefore, ϕ is differentiable and we have $\phi' = \psi$. Therefore, ϕ is of class $\mathscr{C}^1$ and belongs to $V_{a,b}$. Hence, $V_{a,b}$ is a closed in $\|\cdot\|_\infty$ vector space. Now, for $\phi \in V_{a,b}$ and $x, y \in [\epsilon, \frac{1}{\epsilon}]$, we have

$$|\phi(x)-\phi(y)| \leq \sup_{u\in[\epsilon,\frac{1}{\epsilon}]} |\phi'(u)||x-y| \leq \sup_{u\in[\epsilon,\frac{1}{\epsilon}]} \frac{|\phi(bu)-\phi(u)|}{|au|}|x-y| \leq \frac{||\phi||_\infty}{|a|\epsilon}|x-y|$$

Therefore, the unit ball of $V_{a,b}$ is equi-continuous, hence, by Ascoli theorem, it is relatively compact, and now it follows from the Riesz Theorem that $V_{a,b}$ is a finite dimensional vector space. □

We now show that elements of $V_{a,b}$ belong to a specific class of functions.

Lemma 5. *All elements of $V_{a,b}$ are solutions to a Euler type differential equation, that is to say there exists $m \in \mathbb{N}$ and real numbers $(\rho_i)_{0\le i\le m}$ such that*

$$\sum_{i=0}^{m} \rho_i x^i \phi^{(i)}(x) = 0. \tag{29}$$

Proof. It is easy to see from the definition of $V_{a,b}$ that if $\phi \in V_{a,b}$, then the function $x \mapsto x\phi'(x)$ also belongs to $V_{a,b}$. If we now denote by $\phi^{(i)}$ the derivative of order i of ϕ, we see that the span of $(x^i\phi^{(i)}(x))_{i\ge 0}$ must be a subvector space of $V_{a,b}$ and in particular a finite dimensional vector space. In particular, there exists $m \in \mathbb{N}$ and real constants $(\rho_i)_{0\le i\le m}$ such that (29) holds. □

Proof of Theorem 5. The previous result applies in particular to the function f' since f' verify (28). As a consequence, f' satisfy Euler type differential equation. It is known that the change of variable $x = \exp(u)$ reduces this equation to a homogeneous differential equation of order m with constant coefficients. It is also known that the solution of such equation can be written as a linear combination of the solutions corresponding to different roots of characteristic polynomial. These solutions being linearly independent, we need only to considerer a generic one, say f'_λ, λ being the root of characteristic polynomial. If the root of characteristic polynomial λ is real and of the multiplicity n, $n \le m$, then

$$f'_\lambda(x) = a_0 x^\lambda + x^\lambda \sum_{i=1}^{n} b_i (\ln(x))^i$$

and if this root is complex then

$$f'_\lambda(x) = x^{Re(\lambda)} \sum_{i=0}^{n} [c_i \cos(\ln(Im(\lambda)x)) + d_i \sin(\ln(Im(\lambda)x))] \ln(x)^i$$

where a_0, b_i, c_i, d_i are real constants. Since f' is increasing, we must have for all $i \le n$, $c_i = d_i = 0$. But f is strictly convex and the last case is excluded. Putting

$$f'_\lambda(x) = a_0 x^\lambda + x^\lambda \sum_{i=1}^{n} b_i (\ln(x))^i$$

into the equation

$$f'(bx) - f'(x) = axf''(x) \tag{30}$$

we get using linear independence of mentioned functions that

$$a_0(b^\lambda - a\lambda - 1) + b^\lambda \sum_{i=1}^{n} b_i (\ln b)^i - ab_1 = 0 \tag{31}$$

and that for all $1 \leq i \leq n$,

$$\sum_{k=i}^{n} b^\lambda b_k C_k^i (\ln(b))^{k-i} - b_i(1 + a\lambda) - ab_{i+1}(i+1) = 0 \tag{32}$$

with $b_{n+1} = 0$. We remark that the matrix corresponding to (32) is triangular matrix M with $b^\lambda - 1 - a\lambda$ on the diagonal. If $b^\lambda - 1 - a\lambda \neq 0$, then the system of equations has unique solution. This solution should also verify:for all $x > 0$

$$f_\lambda''(x) > 0 \tag{33}$$

If $b^\lambda - 1 - a\lambda = 0$, then $rang(M) = 0$, and b_i are free constants. Finally, we conclude that there exist a solution

$$f_\lambda'(x) = ax^\lambda + x^\lambda \sum_{i=1}^{n} b_i (\ln(x))^i$$

verifying (33) with any λ verifying $b^\lambda - 1 - a\lambda = 0$. □

5.3 *Third Case: c Is Non Invertible and ν Is Nowhere Dense*

We finally consider the case of Levy models which have a continuous component but for which the matrix c is not invertible. It follows from Theorem 3 that in this case we have for all $x \in supp(Z^*)$ and $y \in supp(\nu)$

$$f'(xY^*(y)) - f'(x) = xf''(x) \sum_{i=1}^{d} \beta_i^* (e^{y_i} - 1) - \sum_{j=1}^{d} V_j (e^{y_j} - 1) \tag{34}$$

where $cV = 0$.

Proof of Theorem 5. First of all, we note that if f' satisfies (34) then $\phi : x \mapsto xf''(x)$ satisfies (30). The conclusions of the previous section then hold for ϕ. □

6 Minimal Equivalent Measures When $f''(x) = ax^\gamma$

Our aim in this section is to consider in more detail the class of minimal martingale measures for the functions which satisfy $f''(x) = ax^\gamma$. First of all, we note that these functions are those for which there exists $A > 0$ and real B,C such that

$$f(x) = Af_\gamma(x) + Bx + C$$

where

$$f_\gamma(x) = \begin{cases} c_\gamma x^{\gamma+2} \text{ if } \gamma \neq -1, -2, \\ x \ln(x) \text{ if } \gamma = -1, \\ -\ln(x) \text{ if } \gamma = -2. \end{cases} \tag{35}$$

and $c_\gamma = \text{sign}[(\gamma+1)/(\gamma+2)]$. In particular, the minimal measure for f will be the same as that for f_γ. Minimal measures for the different functions f_γ have been well studied. It has been shown in [8, 15, 16] that in all these cases, the minimal measure, when it exists, preserves the Levy property.

Sufficient conditions for the existence of a minimal measure and an explicit expression of the associated Girsanov parameters have been given in the case of relative entropy in [10, 13] and for power functions in [15]. It was also shown in [13] that these conditions are in fact necessary in the case of relative entropy or for power functions when $d = 1$. Our aim in this section is to give a unified expression of such conditions for all functions which satisfy $f''(x) = ax^\gamma$ and to show that, under some conditions, they are necessary and sufficient, for all d-dimensional Levy models.

We have already mentioned that f-divergence minimal martingale measures play an important role in the determination of utility maximising strategies. In this context, it is useful to have further invariance properties for the minimal measures such as scaling and time invariance properties. This is the case when $f''(x) = ax^\gamma$.

Theorem 6. *Consider a Levy process X with characteristics (b, c, ν) and let f be a function such that $f''(x) = ax^\gamma$, where $a > 0$ and $\gamma \in \mathbb{R}$. Suppose that $c \neq 0$ or $\overset{\circ}{\text{supp}}\ (\nu) \neq \emptyset$. Then there exists an f-divergence minimal equivalent to P martingale measure Q preserving Levy properties if and only if there exist $\gamma, \beta \in \mathbb{R}^d$ and measurable function $Y : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^+$ such that*

$$Y(y) = (f')^{-1}(f'(1) + \sum_{i=1}^{d} \gamma_i (e^{y_i} - 1)) \tag{36}$$

and such that the following properties hold:

$$Y(y) > 0 \ \ \nu - a.e., \tag{37}$$

$$\sum_{i=1}^{d} \int_{|y|\geq 1} (e^{y_i} - 1) Y(y) \nu(dy) < +\infty. \tag{38}$$

$$b + \frac{1}{2} diag(c) + c\beta + \int_{\mathbb{R}^d} ((e^y - 1) Y(y) - h(y)) \nu(dy) = 0. \tag{39}$$

If such a measure exists the Girsanov parameters associated with Q are β and Y, and this measure is scale and time invariant.

We begin with some technical lemmas.

Lemma 6. *Let Q be the measure preserving Levy property. Then, $Q_T \sim P_T$ for all $T > 0$ iff*

$$Y(y) > 0 \;\; \nu - a.e., \tag{40}$$

$$\int_{\mathbb{R}^d} (\sqrt{Y(y)} - 1)^2 \nu(dy) < +\infty. \tag{41}$$

Proof. See Theorem 2.1, p. 209 of [14]. □

Lemma 7. *Let $Z_T = \frac{dQ_T}{dP_T}$. Under $Q_T \sim P_T$, the condition $E_P|f(Z_T)| < \infty$ is equivalent to*

$$\int_{\mathbb{R}^d} [f(Y(y)) - f(1) - f'(1)(Y(y) - 1)] \nu(dy) < +\infty \tag{42}$$

Proof. In our particular case, $E_P|f(Z_T)| < \infty$ is equivalent to the existence of $E_P f(Z_T)$. We use Ito formula to express this integrability condition in predictable terms. Taking for $n \geq 1$ stopping times

$$s_n = \inf\{t \geq 0 : Z_t > n \text{ or } Z_t < 1/n\}$$

where $\inf\{\emptyset\} = +\infty$, we get for $\gamma \neq -1, -2$ and $\alpha = \gamma + 2$ that P-a.s.

$$Z^{\alpha}_{T\wedge s_n} = 1 + \int_0^{T\wedge s_n} \alpha\, Z^{\alpha}_{s-} \beta dX^c_s + \int_0^{T\wedge s_n} \int_{\mathbb{R}^d} Z^{\alpha}_{s-} (Y^{\alpha}(y) - 1)(\mu^X - \nu^{X,P})(ds, dy)$$
$$+ \frac{1}{2} \alpha(\alpha - 1) \beta^2 c \int_0^{T\wedge s_n} Z^{\alpha}_{s-}\, ds + \int_0^{T\wedge s_n} \int_{\mathbb{R}^d} Z^{\alpha}_{s-} [Y^{\alpha}(y) - 1 - \alpha(Y(y) - 1)] ds\;\nu(dy)$$

Hence,

$$Z^{\alpha}_{T\wedge s_n} = \mathscr{E}(N^{(\alpha)} + A^{(\alpha)})_{T\wedge s_n} \tag{43}$$

where

$$N_t^{(\alpha)} = \int_0^t \alpha\, \beta d X_s^c + \int_0^t (Y^\alpha(y) - 1)(\mu^X - \nu^{X,P})(ds, dy)$$

and

$$A_t^{(\alpha)} = \int_0^t \int_{\mathbb{R}^d} [Y^\alpha(y) - 1 - \alpha(Y(y) - 1)] ds\ \nu(dy)$$

Since $[N^{(\alpha)}, A^{(\alpha)}]_t = 0$ for each $t \geq 0$ we have

$$Z^\alpha_{T\wedge s_n} = \mathscr{E}(N^{(\alpha)})_{T\wedge s_n} \mathscr{E}(A^{(\alpha)})_{T\wedge s_n}$$

If $E_P Z_T^\alpha < \infty$, then by Jensen inequality

$$0 \leq Z^\alpha_{T\wedge s_n} \leq E_P(Z_T^\alpha \mid \mathscr{F}_{T\wedge s_n})$$

and since the right-hand side of this inequality form uniformly integrable sequence, $(Z^\alpha_{T\wedge s_n})_{n\geq 1}$ is also uniformly integrable. We remark that in the case $\alpha > 1$ and $\alpha < 0$, $A_t^{(\alpha)} \geq 0$ for all $t \geq 0$ and

$$\mathscr{E}(A^{(\alpha)})_{T\wedge s_n} = \exp(A^{(\alpha)}_{T\wedge s_n}) \geq 1.$$

It means that $(\mathscr{E}(N^{(\alpha)})_{T\wedge s_n})_{n\in\mathbb{N}^*}$ is uniformly integrable and

$$E_P(Z_T^\alpha) = \exp(A_T^{(\alpha)}). \tag{44}$$

If (42) holds, then by Fatou lemma and since $\mathscr{E}(N^{(\alpha)})$ is a local martingale we get

$$E_P(Z_T^\alpha) \leq \underline{\lim}_{n\to\infty} E_P(Z_{T\wedge s_n}) \leq \exp(A_T^{(\alpha)}).$$

For $0 < \alpha < 1$, we have again

$$Z^\alpha_{T\wedge s_n} = \mathscr{E}(N^{(\alpha)})_{T\wedge s_n} \mathscr{E}(A^{(\alpha)})_{T\wedge s_n}$$

with uniformly integrable sequence $(Z^\alpha_{T\wedge s_n})_{n\geq 1}$. Since

$$\mathscr{E}(A^{(\alpha)})_{T\wedge s_n} = \exp(A^{(\alpha)}_{T\wedge s_n}) \geq \exp(A_T^{(\alpha)}),$$

the sequence $(\mathscr{E}(N^{(\alpha)})_{T\wedge s_n})_{n\in\mathbb{N}^*}$ is uniformly integrable and

$$E_P(Z_T^\alpha) = \exp(A_T^{(\alpha)}). \tag{45}$$

For $\gamma = -2$ we have that $f(x) = x\ln(x)$ up to linear term and

$$Z_{T\wedge s_n}\ln(Z_{T\wedge s_n}) = \int_0^{T\wedge s_n}(\ln(Z_{s-})+1)Z_{s-}\beta dX_s^c$$
$$+\int_0^{T\wedge s_n}\int_{\mathbb{R}^d}[\ln(Z_{s-})(Y(y)-1)-Y(y)\ln(Y(y))](\mu^X-\nu^{X,P})(ds,dy)$$
$$+\frac{1}{2}\beta^2 c\int_0^{T\wedge s_n}Z_{s-}\,ds+\int_0^{T\wedge s_n}\int_{\mathbb{R}^d}Z_{s-}[Y(y)\ln(Y(y))-Y(y)+1]ds\,\nu(dy)$$

Taking mathematical expectation we obtain:

$$E_P[Z_{T\wedge s_n}\ln(Z_{T\wedge s_n})] = E_P\int_0^{T\wedge s_n}\int_{\mathbb{R}^d}Z_{s-}[Y(y)\ln(Y(y))-Y(y)+1]ds\ \nu(dy) \tag{46}$$

If $E_P[Z_T\ln(Z_T)] < \infty$, then the sequence $(Z_{T\wedge s_n}\ln(Z_{T\wedge s_n}))_{n\in\mathbb{N}^*}$ is uniformly integrable and $E_P(Z_{s-})=1$ and we obtain applying Lebesgue convergence theorem that

$$E_P[Z_T\ln(Z_T)] = \frac{T}{2}\beta^2 c + T\int_{\mathbb{R}^d}[Y(y)\ln(Y(y))-Y(y)+1]\nu(dy) \tag{47}$$

and this implies (42). If (42), then by Fatou lemma from (46) we deduce that $E_P[Z_T\ln(Z_T)] < \infty$.

For $\gamma = -1$, we have $f(x) = -\ln(x)$ and exchanging P and Q we get:

$$E_P[-\ln(Z_T)] = E_Q[\tilde{Z}_T\ln(\tilde{Z}_T)] = \frac{T}{2}\beta^2 c + T\int_{\mathbb{R}^d}[\tilde{Y}(y)\ln(\tilde{Y}(y))-\tilde{Y}(y)+1]\nu^Q(dy)$$

where $\tilde{Z}_T = 1/Z_T$ and $\tilde{Y}(y) = 1/Y(y)$. But $\nu^Q(dy) = Y(y)\nu(dy)$ and, finally,

$$E_P[-\ln(Z_T)] = \frac{T}{2}\beta^2 c + T\int_{\mathbb{R}^d}[-\ln(Y(y))+Y(y)-1]\nu(dy) \tag{48}$$

Again by Fatou lemma we get that $E_P[-\ln(Z_T)] < \infty$ which implies (42). □

Lemma 8. *If the second Girsanov parameter Y has a particular form* (36) *then the condition*

$$\sum_{i=1}^d\int_{|y|\geq 1}(e^{y_i}-1)Y(y)\nu(dy) < +\infty \tag{49}$$

implies the conditions (40) *and* (42).

Proof. We can cut each integral in (40) and (42) on two parts and integrate on the sets $\{|y| \leq 1\}$ and $\{|y| > 1\}$. Then we can use a particular form of Y and conclude easily writing Taylor expansion of order 2. □

Proof of Theorem 6. *Necessity*. We suppose that there exist f-divergence minimal equivalent martingale measure Q preserving Levy property of X. Then, since

$Q_T \sim P_T$, the conditions (37) and (40) follow from Theorem 2.1, p. 209 of [14]. From Theorem 3 we deduce that (36) holds. Then, the condition (38) follows from the fact that S is a martingale under Q. Finally, the condition (39) follows from Girsanov theorem since Q is a martingale measure and, hence, the drift of S under Q is zero.

Sufficiency. We take β and Y verifying the conditions (37)–(39) and we construct

$$M_t = \sum_{i=1}^{d} \int_0^t \beta^{(i)} dX_s^{c,(i)} + \int_0^t \int_{\mathbb{R}^d} (Y(y) - 1)(\mu^X - \nu^{X,P})(ds, dy) \tag{50}$$

As known from Theorem 1.33, p. 72–73, of [14], the last stochastic integral is well defined if

$$C(W) = T \int_{\mathbb{R}^d} (Y(y) - 1)^2 I_{\{|Y(y)-1| \le 1\}} \nu(dy) < \infty,$$

$$C(W') = T \int_{\mathbb{R}^d} |Y(y) - 1| I_{\{|Y(y)-1| > 1\}} \nu(dy) < \infty.$$

But the condition (38), the relation (36) and Lemma 8 implies (40). Consequently, $(Y - 1) \in G_{loc}(\mu^X)$ and M is local martingale. Then we take

$$Z_T = \mathscr{E}(M)_T$$

and this defines the measure Q_T by its Radon-Nikodym density. Now, the conditions (37) and (38) together with the relation (36) and Lemma 8 imply (40), and, hence, from Lemma 6 we deduce $P_T \sim Q_T$.

We show that $E_P|f(Z_T)| < \infty$. Since $P_T \sim Q_T$, the Lemma 7 gives needed integrability condition.

Now, since (39) holds, Q is martingale measure, and it remains to show that Q is indeed f-divergence minimal. For that we take any equivalent martingale measure $\bar{Q}$ and we show that

$$E_Q f'(Z_T) \le E_{\bar{Q}} f'(Z_T). \tag{51}$$

If the mentioned inequality holds, the Theorem 1 implies that Q is a minimal.

In the case $\gamma \neq -1, -2$ we obtain from (43) replacing α by $\gamma + 1$:

$$Z_T^{\gamma+1} = \mathscr{E}(N^{(\gamma+1)})_T \ \exp(A_T^{(\gamma+1)})$$

and using a particular form of f' and Y we get that for $0 \le t \le T$

$$N_t^{(\gamma+1)} = \sum_{i=1}^{d} \theta^{(i)} \hat{X}_t^{(i)}$$

where $\theta = \beta$ if $c \neq 0$ and $\theta = \gamma$ if $c = 0$, and $\hat{X}^{(i)}$ is a stochastic logarithm of $S^{(i)}$. So, $\mathscr{E}(N^{(\gamma+1)})$ is a local martingale and we get

$$E_{\bar{Q}} Z_T^{\gamma+1} \leq \exp(A_T^{(\gamma+1)}) = E_Q Z_T^{\gamma+1}$$

and, hence, (51).

In the case $\gamma = -1$ we prove using again a particular form of f' and Y that

$$f'(Z_T) = E_Q(f'(Z_T)) + \sum_{i=1}^{d} \theta^{(i)} \hat{X}_T^{(i)}$$

with $\theta = \beta$ if $c \neq 0$ and $\theta = \gamma$ if $c = 0$. Since $E_{\bar{Q}} \hat{X}_T = 0$ we get that

$$E_{\bar{Q}}(f'(Z_T)) \leq E_Q(f'(Z_T))$$

and it proves that Q is f-divergence minimal.

The case $\gamma = -2$ can be considered in similar way.

Finally, note that the conditions which appear in Theorem 6 do not depend in any way on the time interval which is considered and, hence, the minimal measure is time invariant. Furthermore, if Q^* is f-divergence minimal, the equality

$$f(cx) = Af(x) + Bx + C$$

with A, B, C constants, $A > 0$, gives

$$E_P[f(c\frac{d\bar{Q}}{dP})] = AE_P[f(\frac{d\bar{Q}}{dP})] + B + C \geq AE_P[f(\frac{dQ}{dP})] + B + C = E_P[f(c\frac{dQ}{dP})]$$

and Q is scale invariant. □

6.1 *Example*

We now give an example of a Levy model and a convex function which does not satisfy $f''(x) = ax^\gamma$ yet preserves the Levy property. We consider the function $f(x) = \frac{x^2}{2} + x\ln(x) - x$ and the $\mathbb{R}^2$-valued Levy process given by equality $X_t = (W_t + \ln(2)P_t, W_t + \ln(3)P_t - t)$, where W is a standard one-dimensional Brownian motion and P is a standard one-dimensional Poisson process. Note that the covariance matrix

$$c = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is not invertible. The support of the Levy measure is the singleton $a = (\ln(2), \ln(3))$, and is in particular nowhere dense. Let Q be a martingale measure for this model, and (β, Y) its Girsanov parameters, where ${}^{\top}\beta = (\beta_1, \beta_2)$. In order for Q to be a martingale measure preserving Levy property, we must have

$$\begin{aligned} \ln(2) + \frac{1}{2} + \beta_1 + \beta_2 + Y(a) = 0, \\ \ln(3) - \frac{1}{2} + \beta_1 + \beta_2 + 2Y(a) = 0, \end{aligned} \tag{52}$$

and, hence, $Y(a) = 1 - \ln(\frac{3}{2})$. Now, it is not difficult to verify using Ito formula that the measure Q satisfy: $E_P Z_T^2 < \infty$ and, hence, $E_P|f(Z_T)| < +\infty$. Moreover, the conditions (37) and (38) are satisfied meaning that $P_T \sim Q_T$.

Furthermore, in order for Q to be minimal we must have according to Theorem 3:

$$f'(xY(y)) - f'(x) = xf''(x)\sum_{i=1}^{2}\beta_i(e^{a_i} - 1) + \sum_{i=1}^{2} v_i(e^{a_i} - 1)$$

with $a_1 = \ln 2, a_2 = \ln 3$ and $V = {}^{\top}(v_1, v_2)$ such that $cV = 0$. We remark that $v_2 = -v_1$. Then for $x \in supp(Z_T)$

$$\ln(Y(a)) + x(Y(a) - 1) = (x + 1)(\beta_1 + 2\beta_2) - v_1$$

and since $supp(Z_T) = \mathbb{R}^{+,*}$ we must have

$$\beta_1 + 2\beta_2 = Y(a) - 1 \text{ and } \beta_1 + 2\beta_2 - v_1 = \ln(Y(a))$$

Using (52), this leads to

$$\begin{cases} v_1 = -\ln(1 - \ln(\frac{3}{2})) - \ln(\frac{3}{2}) \\ \beta_1 = 3\ln(3) - 5\ln(2) - 3 \\ \beta_2 = \frac{3}{2} + 3\ln(2) - 2\ln(3) \end{cases}$$

We now need to check that the martingale measure given by these Girsanov parameters is indeed minimal. Note that the decomposition of Theorem 4 can now be written

$$f'(Z_T) = E_Q[f'(Z_T)] + \sum_{i=1}^{2}\int_0^T \left[\beta_i(\frac{1}{Z_{s-}} + E_Q[Z_{T-s}]) + v_i\right]\frac{dS_s^i}{S_{s-}^i}$$

But for $s \geq 0$

$$\frac{dS_s^i}{S_{s-}^i} = \hat{X}_s^i$$

and right-hand side of previous equality is a local martingale with respect to any martingale measure $\bar{Q}$. Taking a localising sequence and then the expectation with respect to $\bar{Q}$ we get after limit passage that

$$E_{\bar{Q}}[f'(Z_T)] \leq E_Q[f'(Z_T)],$$

and so, it follows from Theorem 1 that the measure Q is indeed minimal.

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Non-standard Limit Theorems in Number Theory

Francesco Cellarosi and Yakov G. Sinai

Dedicated to Yu.V. Prokhorov on the occasion of his 80-th birthday

Abstract We prove a non-standard limit theorem for a sequence of random variables connected with the classical Möbius function. The so-called *Dickman-De Bruijn distribution* appears in the limit. We discuss some of its properties, and we provide a number of estimates for the error term in the limit theorem.

Keywords Möbius function • Limit theorems • Infinite divisibility • Dickman-De Bruijn distribution

Mathematics Subject Classification (2010): 60F05, 11K65

1 Introduction

There are many unusual limit theorems in Number Theory which are well-known to experts in the field but not so well-known to probabilists. The purpose of this paper is to discuss some examples of such theorems. They were chosen in order to be close to the field of interest of Yu.V. Prokhorov.

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One of the main objects in Number Theory is the so-called Möbius function. It is defined as follows

$$\mu(n) = \begin{cases} 1 & \text{if } n\text{=1;} \\ 0 & \text{if } n \text{ is not square-free;} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes.} \end{cases}$$

Throughout the paper, when we write $n = p_1 p_2 \cdots p_k$ we assume that $p_1 < \ldots < p_k$ are the first k prime numbers. Many properties of the Möbius function are connected with the Riemann zeta function. For example, while the Prime Number Theorem is equivalent to the fact that

$$\sum_{n \le N} \mu(n) = o(N),$$

the Riemann Hypothesis is equivalent to

$$\sum_{n \le N} \mu(n) = O_\varepsilon\big(N^{1/2+\varepsilon}\big)$$

for every $\varepsilon > 0$.

Recently, a conjecture by Sarnak [15] has fostered a great interest towards the connections between the Möbius function and Ergodic Theory, and in particular the works of Furstenberg [7] and Green and Tao [10].

2 A Probabilistic Model for Square-Free Numbers

Fix $m > 1$ and introduce the set Ω_m, whose elements have the form $n = \prod_{j=1}^m p_j^{\nu_j}$, where $\nu_j \in \{0, 1\}$. Then $\mu(n) = \pm 1$ iff $n \in \Omega_m$ for some m. Define on Ω_m the probability distribution Π_m for which

$$\pi_m(n) = \frac{1}{Z_m}\frac{1}{n} = \frac{1}{Z_m \prod_{j=1}^m p_j^{\nu_j}}, \tag{1}$$

In (1) Z_m is the normalizing factor and

$$Z_m = \sum_{\nu_1,\ldots,\nu_m} \frac{1}{\prod_{j=1}^m p_j^{\nu_j}} = \prod_{j=1}^m \left(1 + \frac{1}{p_j}\right) = \exp\left\{\sum_{j=1}^m \ln\left(1 + \frac{1}{p_j}\right)\right\} =$$

$$= \exp\left\{O(1) + \sum_{j=1}^m \frac{1}{p_j}\right\}$$

as $m \to \infty$. Denote by $N(t)$ the number of primes which are less or equal than t. The Prime Number Theorem says that $N(t) \sim \frac{t}{\ln t}$ as $t \to \infty$ and a slightly stronger version asserts that

$$N(t) - \frac{t}{\ln t} = O\left(\frac{t}{\ln^2 t}\right). \tag{2}$$

We can write, by summation by parts,

$$\sum_{j=1}^{m} \frac{1}{p_j} = \sum_{t=1}^{p_m} \frac{1}{t}\left(N(t) - N(t-1)\right) = \frac{N(p_m)}{p_m+1} + \sum_{t=1}^{p_m} \frac{N(t)}{t(t+1)} =$$

$$= \frac{m}{p_m+1} + \sum_{t=1}^{p_m} N(t)\left(\frac{1}{t^2} + O\left(\frac{1}{t^3}\right)\right) = O(1) + \sum_{t=2}^{p_m} \frac{1}{t\ln t} = O(1) + \ln\ln p_m,$$

i.e. $Z_m \sim O(1)\ln p_m$. A more precise asymptotic follows from Mertens' product formula [13]

$$\lim_{n\to\infty} \ln n \prod_{p\le n}\left(1 - \frac{1}{p}\right) = e^{-\gamma} \approx 0.561459,$$

where γ is Euler-Mascheroni constant. In fact

$$\frac{1}{\ln n}\prod_{p\le n}\left(1+\frac{1}{p}\right) = \frac{\left(\prod_{p\le n}\frac{1}{1-p^{-2}}\right)^{-1}}{\ln n \prod_{p\le n}\left(1-\frac{1}{p}\right)} \longrightarrow \frac{\zeta(2)^{-1}}{e^{-\gamma}} \qquad \text{as } n \to \infty.$$

Thus

$$Z_m \sim \frac{e^{\gamma}}{\zeta(2)} \ln p_m. \tag{3}$$

By analogy with Statistical Physics, Z_m is called *partition function*.

It is easy to check that w.r.t. Π_m, the random variables ν_j are independent and

$$\Pi_m\{\nu_j = 0\} = \frac{p_j}{1+p_j}, \qquad \Pi_m\{\nu_j = 1\} = \frac{1}{1+p_j}, \qquad 1 \le j \le m.$$

Indeed,

$$\Pi_m\{\nu_j = 0\} = \frac{1}{Z_m}\sum_{\nu_1,\dots,\nu_{j-1}}\sum_{\nu_{j+1},\dots,\nu_m} \frac{1}{\prod_{l=1}^{j-1} p_l^{\nu_l}\prod_{l=j+1}^{m} p_l^{\nu_l}} =$$

$$= \frac{\prod_{l=1}^{j-1}\left(1+\frac{1}{p_l}\right)\prod_{l=j+1}^{m}\left(1+\frac{1}{p_l}\right)}{\prod_{l=1}^{m}\left(1+\frac{1}{p_l}\right)} = \frac{p_j}{1+p_j}.$$

Since

$$n = \prod_{j=1}^{m} p_j^{\nu_j} = \exp\left\{\sum_{j=1}^{m} \nu_j \ln p_j\right\},$$

the statistical properties of n with respect to Π_m are determined by the properties of $\sum_{j=1}^{m} \nu_j \ln p_j$, which are sums of independent random variables. However the Central Limit Theorem cannot be applied here because ν_j are not identically distributed. Instead, the following limit theorem is valid.

Theorem 1. *Let* $\zeta_m = \frac{1}{\ln p_m}\sum_{j=1}^{m} \nu_j \ln p_j$. *As* $m \to \infty$ *the distributions of* ζ_m *converge weakly to the infinitely divisible distribution whose characteristic function* $\varphi(\lambda)$ *has the form*

$$\varphi(\lambda) = \exp\left\{\int_0^1 \frac{e^{i\lambda v} - 1}{v} \mathrm{d}v\right\}. \tag{4}$$

Proof. The characteristic function φ_m of ζ_m is

$$\varphi_m(\lambda) = \mathrm{e}e^{i\lambda\zeta_m} = \mathrm{e}\exp\left\{\frac{i\lambda}{\ln p_m}\sum_{j=1}^{m} \nu_j \ln p_j\right\} = \prod_{j=1}^{m}\left(\frac{p_j}{1+p_j} + \frac{1}{1+p_j} e^{\frac{i\lambda \ln p_j}{\ln p_m}}\right) =$$

$$= \prod_{j=1}^{m}\left(1 + \frac{1}{1+p_j}\left(e^{\frac{i\lambda \ln p_j}{\ln p_m}} - 1\right)\right) =$$

$$= \exp\left\{\sum_{t=1}^{p_m} (N(t) - N(t-1)) \ln\left(1 + \frac{1}{1+t}\left(e^{\frac{i\lambda \ln t}{\ln p_m}} - 1\right)\right)\right\} =$$

$$= \exp\left\{f_m(p_m+1)N(p_m) - \sum_{t=1}^{p_m} N(t-1)(f_m(t+1) - f_m(t))\right\},$$

by summation by parts, where $f_m(s) = \ln\left(1 + \frac{1}{1+s}\left(e^{\frac{i\lambda \ln s}{\ln p_m}} - 1\right)\right)$. Since f_m is complex-valued, the identity $f_m(t+1) - f_m(t) = f_m'(t+\tau)$ for some $0 < \tau < 1$ does not follow from the mean value theorem and we have to work with the real and imaginary parts separately. Writing $f_m = \Re f_m + i\Im f_m$ we have

$$\Re f_m(s) = \ln\left|1 + \frac{1}{1+s}\left(e^{\frac{i\lambda \ln s}{\ln p_m}} - 1\right)\right| = \frac{1}{2}\ln\left(\frac{s^2 + 2s\cos\left(\frac{\lambda \ln s}{\ln p_m}\right) + 1}{(1+s)^2}\right)$$

and (by choosing the principal branch of the natural logarithm)

$$\Im f_m(s) = \arg\left(1 + \frac{1}{1+s}\left(e^{\frac{i\lambda \ln s}{\ln p_m}} - 1\right)\right) = \arctan\left(\frac{\sin\left(\frac{\lambda \ln s}{\ln p_m}\right)}{s + \cos\frac{\lambda \ln s}{\ln p_m}}\right).$$

Now, by applying the mean value theorem twice to $\Re f_m$ and $\Im f_m$ separately, we get

$$\Re f_m(t+1) - \Re f_m(t) = (\Re f_m)'(t+\tau_1) = (\Re f_m)'(t) + \tau_1 (\Re f_m)''(t+\tau_1')$$

for some $0 < \tau_1' < \tau_1 < 1$, and

$$\Im f_m(t+1) - \Im(f_m)(t) = (\Im f_m)'(t+\tau_2) = (\Im f_m)'(t) + \tau_2 (\Im f_m)''(t+\tau_2')$$

for some $0 < \tau_2' < \tau_2 < 1$. Thus

$$\ln \varphi_m(\lambda) = f_m(p_m+1)N(p_m) - \\ - \sum_{t=1}^{p_m} N(t-1)\left(f_m'(t) + \tau_1(\Re f_m)''(t+\tau_1') + \tau_2(\Im f_m)''(t+\tau_2')\right). \quad (5)$$

We claim that the sum involving $f_m'(t)$ gives the main term. In fact, the first term and the other sums in (5) tend to zero as $m \to \infty$ (see Appendix). Thus, the main term comes from the following sum:

$$-\sum_{t=1}^{p_m} N(t-1) f_m'(t) = -\sum_{t=2}^{p_m} \left(\frac{t}{\ln t} + O\left(\frac{t}{\ln^2 t}\right)\right) \frac{1}{1 + \frac{1}{1+t}\left(e^{i\lambda \frac{i\lambda \ln t}{\ln p_m}} - 1\right)} \cdot \\ \cdot \left[-\frac{1}{(t+1)^2}\left(e^{i\lambda \frac{i\lambda \ln t}{\ln p_m}} - 1\right) + \frac{1}{t(t+1)} e^{i\lambda \frac{i\lambda \ln t}{\ln p_m}} \frac{i\lambda}{\ln p_m}\right] = \\ = \sum_{t=1}^{p_m} \left(\frac{1}{t \ln t} + O\left(\frac{1}{t \ln^2 t}\right)\right)\left(1 + \frac{1 - e^{\frac{i\lambda \ln t}{\ln p_m}}}{t + e^{\frac{i\lambda \ln t}{\ln p_m}}}\right) \cdot \\ \cdot \left[\left(e^{\frac{i\lambda \ln t}{\ln p_m}} - 1\right) - \frac{(2t+1)\left(e^{\frac{i\lambda \ln t}{\ln p_m}} - 1\right)}{(t+1)^2} - \frac{i\lambda}{\ln p_m} \frac{t}{t+1} e^{\frac{i\lambda \ln t}{\ln p_m}}\right]. \quad (6)$$

By opening the brackets in (6) we obtain 12 sums. Let us look at the first sum and consider the change of variables (which will be used in the Appendix too) $v = v(t) = \frac{\ln t}{\ln p_m}$ for which $\mathrm{d}v = v(t+1) - v(t) = v'(t+\tau_3) = v'(t) + \tau_3 v''(t+\tau_3')$ for some $0 < \tau_3' < \tau_3 < 1$. We get

$$\sum_{t=2}^{p_m} \frac{1}{t \ln t}\left(e^{\frac{i\lambda \ln t}{\ln p_m}} - 1\right) = \sum_v \left(\mathrm{d}v + \frac{\tau}{(t+\tau')^2 \ln p_m}\right) \frac{e^{i\lambda v} - 1}{v} \longrightarrow \int_0^1 \frac{e^{i\lambda v} - 1}{v} \mathrm{d}v$$

as $m \to \infty$ since for some $C > 0$

$$\left|\sum_{t=2}^{p_m} \frac{\tau}{(t+\tau')^2 \ln p_m} \frac{e^{i\lambda v}-1}{v}\right| \le \frac{C|\lambda|}{\ln p_m} \sum_{t=2}^{p_m} \frac{1}{t^2} \longrightarrow 0.$$

All the remaining 11 sums coming from (6) tend to zero (see Appendix).

To show that distribution corresponding to $\varphi(\lambda)$ is infinitely divisible, we use the characterization due to Kolmogorov (see, e.g. [8] for a detailed account on infinite divisibility). He proved [12] that a probability distribution P_ξ over $\mathbb{R}$ with finite variance is infinitely divisible if and only if its characteristic function $\varphi(\lambda)$ has the form

$$\ln \varphi(\lambda) = i\kappa\lambda + \int_{\mathbb{R}} (e^{i\lambda v} - 1 - i\lambda v) \frac{\mathrm{d}K(v)}{v^2}, \tag{7}$$

where κ is a constant and $v \mapsto K(v)$ is a non-decreasing function of bounded variation satisfying $\lim_{v\to-\infty} K(v) = 0$. It easy to check that $\kappa = \int_{\mathbb{R}} x \mathrm{d}P(x) = \mathbb{E}\xi$ and $\lim_{v\to\infty} K(v) = \mathbb{E}(\xi - \mathbb{E}\xi)^2$. In our case

$$\kappa = e^{-\gamma} \int_0^\infty \rho(t) = 1,$$

$$\lim_{v\to\infty} K(v) = -\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\varphi(\lambda)|_{\lambda=0} - \kappa^2 = \frac{3}{2} - 1 = \frac{1}{2},$$

and by choosing

$$K(v) = \begin{cases} 0 & \text{if } v < 0; \\ \frac{v^2}{2} & \text{if } 0 \le v \le 1; \\ \frac{1}{2} & \text{if } v > 1 \end{cases}$$

in (7) we obtain (4). This concludes the proof of Theorem 1. □

Notice that

$$\int \frac{\cos(\lambda v) - 1}{v} \mathrm{d}v = -\int_{|\lambda| v}^\infty \frac{\cos u}{u} \mathrm{d}u - \ln v \quad \text{and}$$

$$\lim_{x\to 0+} \left(-\int_x^\infty \frac{\cos u}{u} \mathrm{d}u - \ln x\right) = \gamma,$$

where γ is the Euler-Mascheroni constant as before. Therefore the improper integral $\int_0^1 \frac{\cos(\lambda v)-1}{v} \mathrm{d}v$ converges to $-\gamma - \int_{|\lambda|}^\infty \frac{\cos u}{u}\mathrm{d}u - \ln|\lambda|$. On the other hand

$$\int \frac{\sin(\lambda v)}{v} \mathrm{d}v = \mathrm{sgn}(\lambda) \int_0^{|\lambda| v} \frac{\sin u}{u} \mathrm{d}u \quad \text{gives}$$

$$\int_0^1 \frac{\sin(\lambda v)}{v} \mathrm{d}v = \mathrm{sgn}(\lambda) \int_0^{|\lambda|} \frac{\sin u}{u} \mathrm{d}u.$$

This shows that

$$\varphi(\lambda) = \begin{cases} \exp\left\{-\left(\gamma + \int_{\lambda}^{\infty} \frac{\cos u}{u} \mathrm{d}u + \ln \lambda\right) + i \int_0^{\lambda} \frac{\sin u}{u} \mathrm{d}u\right\} & \lambda > 0\,, \\ 1 & \lambda=0\,, \\ \exp\left\{-\left(\gamma + \int_{-\lambda}^{\infty} \frac{\cos u}{u} \mathrm{d}u + \ln(-\lambda)\right) - i \int_0^{-\lambda} \frac{\sin u}{u} \mathrm{d}u\right\} & \lambda < 0. \end{cases}$$

It is known (see [1]) that $\varphi(\lambda)$ is the characteristic function of the *Dickman-De Bruijn distribution*, with density $e^{-\gamma}\rho(t)$, where $\rho(t)$ is determined by the initial condition

$$\rho(t) = \begin{cases} 0, & t \le 0; \\ 1, & 0 < t \le 1, \end{cases} \tag{8}$$

and the integral equation

$$t\rho(t) = \int_{t-1}^{t} \rho(s)\mathrm{d}s, \qquad t \in \mathbb{R}.$$

It also satisfies the delay differential equation

$$t\rho'(t) + \rho(t-1) = 0$$

for $t \ge 1$ (at $t = 1$ we consider the right derivative) and for every $k = 1, 2, 3, \dots$ there is an analytic function $\rho_k(t)$ that gives $\rho(t)$ on $k - 1 \le t \le k$. For example, $\rho_1 \equiv 1$, $\rho_2(t) = 1 - \ln t$ and $\rho_3(t) = 1 - \ln t + \int_2^t \ln(u-1)\frac{\mathrm{d}u}{u}$. It is also easy to see that $\rho \in C^k([k,\infty))$ for each k.

Among other properties of $\rho(t)$ one can mention that it is log-concave on $[1,\infty)$ and

$$\rho(t) = \exp\left\{-t\left(\ln t + \ln\ln t - 1 + \frac{\ln\ln t}{\ln t} + O\left(\frac{(\ln\ln t)^2}{(\ln t)^2}\right)\right)\right\}$$

as $t \to \infty$. In other words, the limiting density $e^{-\gamma}\rho(t)$ is constant on the interval $(0, 1]$, where it takes the value $e^{-\gamma}$, and decays faster then exponentially on $(1,\infty)$, like Poisson distribution. In particular, all its moments exist.

The Dickman-De Bruijn function ρ first appeared in the theory of *smooth numbers* (i.e. numbers with small prime factors). Let $\Psi(x, y)$ denote the number of integers $\le x$ whose prime factors are $\le y$ (such numbers are called y-*smooth*). Dickman [4] showed that $\Psi(x, x^{1/u}) \sim x\rho(u)$ as $x \to \infty$. The range of y such that the asymptotic formula $\Psi(x, y) \sim x\rho(u)$, where $x = y^u$, has been significantly enlarged by De Bruijn [1–3] ($y \ge \exp((\ln x)^{5/8+\varepsilon})$) and Hildebrand [11] ($y \ge \exp((\ln\ln x)^{5/3+\varepsilon})$). Our ensemble Ω_m coincides (as a set) with the intersection of the set of all square-free numbers with the set of p_m-smooth numbers less or equal than $p_1 p_2 \cdots p_m$. Therefore we are in the case when $y \sim \ln x$. In this regime Erdös [6] showed that $\ln \Psi(x, \ln x) \sim \dfrac{\ln 4 \ln x}{\ln \ln x}$ as $x \to \infty$ and therefore the asymptotic is no longer given by the function ρ. In other words a *phase transition* occurs in the asymptotic behavior of $\Psi(x, y)$. For a survey on the theoretical and computational aspects of smooth numbers see [9]. In our problem, we still get the Dickman-De Bruijn distribution in the limit because of the probability distribution we put on Ω_m.

It is worth to mention that in many limit theorems in Number Theory there appear limiting densities which are constants on some interval starting at 0. An example can be found in the work of Elkies and McMullen [5] on the distribution of the gaps in the sequence $\{\sqrt{n} \mod 1\}$.

Here is another example from Probability Theory where the Dickman-De Bruijn distribution appears. Let $\{\eta_j\}_{j\geq 1}$ be a sequence of independent random variables such that

$$P\{\eta_k = k\} = \frac{1}{k} \qquad \text{and} \qquad P\{\eta_k = 0\} = 1 - \frac{1}{k},$$

and let $\theta_n = \sum_{j=1}^{n} \eta_j$ then

$$\lim_{n\to\infty} P\{n^{-1}\theta_n < x\} = e^{-\gamma} \int_0^x \rho(t)\mathrm{d}t.$$

Theorem 1 has several important corollaries and applications. An immediate consequence of (4) is that

$$\Pi_m\{n \leq p_m^s\} = \sum_{n\leq p_m^s,\, n\in\Omega_m} \pi_m(x) \longrightarrow e^{-\gamma} \int_0^s \rho(t)\mathrm{d}t$$

as $m \to \infty$. For instance, for $s = 2$ we get $e^{-\gamma}(3 - \ln 4) \approx 0.90603$. In other words, despite the fact that the largest element of our ensemble Ω_m is of order m^m, approximately 90 % of the "mass" of our probability distribution Π_m is concentrated on numbers less than p_m^2 for large m.

Let us fix $0 < \sigma \leq 1$ and decompose the interval $(0, \sigma)$ onto K equal intervals (δ_k, δ_{k+1}), $\delta_k = \frac{\sigma k}{K}$, $k = 0, \ldots, K-1$. For fixed K, Theorem 1 states that

$$\Pi_m\left\{\delta_k < \frac{\ln n}{\ln p_m} < \delta_{k+1}\right\} \longrightarrow \frac{e^{-\gamma}\sigma}{K} \tag{9}$$

as $m \to \infty$. Let us consider the error term in (9)

$$E_m^{(\sigma)}(k, K) := \Pi_m\left\{\delta_k < \frac{\ln n}{\ln p_m} < \delta_{k+1}\right\} - \frac{e^{-\gamma}\sigma}{K}.$$

In the rest of this paper we provide some estimates about the error terms $E_m^{(\sigma)}(k, K)$ when K grows with n. We prove the following

Theorem 2. *For every $\varepsilon > 0$ and every function $K(m)$ such that $\lim_{m\to\infty} \frac{\ln^3 p_m}{K(m)^2} = c \geq 0$ there exists $m^* = m^*(\varepsilon, K)$ such that the inequalities*

$$-\frac{c\sigma^3}{12\zeta(2)} - \varepsilon \leq \frac{Z_m}{p_m^\sigma} \sum_{k=0}^{K(m)-1} p_m^{\delta_k} E_m^{(\sigma)}(k, K(m)) \leq \frac{c\sigma^3}{12\zeta(2)} + \varepsilon \qquad (10)$$

hold for every $m \geq m^*$ and every $0 < \sigma \leq 1$.

An important tool in the proof of Theorem 2 is given by the counting function

$$M_m(t) = \#\{n \leq t : n \in \Omega_m\}.$$

This is analogous to the classical quantity

$$M(t) = \#\{n \leq t : \mu(n) \neq 0\},$$

for which the asymptotic

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 0.607927.$$

holds (see, e.g., [14]). Even though the ensemble Ω_m is very sparse, its initial segment of length p_m contains all square-free numbers less or equal than p_m. In particular $\lim_{m\to\infty} \frac{M_m(p_m^\sigma)}{p_m^\sigma} = \frac{1}{\zeta(2)}$ for every $0 < \sigma \leq 1$. For $\sigma = 1$ this fact can be rephrased as

$$\lim_{m\to\infty} \frac{1}{p_m} \sum_{n \leq p_m} \mu^2(n) = \frac{1}{\zeta(2)}$$

and can be compared with

$$\lim_{m\to\infty} \frac{1}{\ln p_m} \sum_{n \leq p_m} \frac{\mu^2(n)}{n} = \frac{1}{\zeta(2)},$$

which is a corollary of our Theorem 1 and (3).

The following Lemma provides some simple estimates that will be used in the proof of Theorem 2.

Lemma 1. *The following inequalities hold:*

$$0 < \sum_{k=0}^{K-1} p_m^{\delta_{k+1}} \frac{\sigma}{K} - \frac{p_m^\sigma - 1}{\ln p_m} \leq \frac{\sigma^3 p_m^\sigma \ln^2 p_m}{12K^2} + \frac{\sigma(p_m^\sigma - 1)}{2K}, \qquad (11)$$

$$-\frac{\sigma^3 p_m^\sigma \ln^2 p_m}{12K^2} - \frac{\sigma(p_m^\sigma - 1)}{2K} \leq \sum_{k=0}^{K-1} p_m^{\delta_k} \frac{\sigma}{K} - \frac{p_m^\sigma - 1}{\ln p_m} < 0. \qquad (12)$$

Proof. The right (resp. left) Riemann sum $\sum_{k=0}^{K-1} p_m^{\delta_{k+1}} \frac{\sigma}{K}$ (resp. $\sum_{k=0}^{K-1} p_m^{\delta_k} \frac{\sigma}{K}$) converges as $K \to \infty$ to the integral $\int_0^\sigma e^{\delta \ln p_m} \mathrm{d}\delta = \frac{p_m^\sigma - 1}{\ln p_m}$. Moreover, since the function $t \mapsto p_m^t$ is increasing, the right (resp. left) sum is strictly bigger (resp. smaller) than the integral. This proves the first inequality in (11) and the second inequality in (12). A classical result from Calculus states that in the absolute value of the error performed by approximating the integral $\int_a^b f(x)\mathrm{d}x$ by the trapezoidal Riemann sum

$$\left(\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + \ldots + f(x_{K-1}) + \frac{1}{2} f(x_K)\right) \Delta x,$$

$x_k = a + k\frac{b-a}{K}$ is bounded by $\frac{M(b-a)^3}{12K^2}$ where $\sup_{a \le x \le b} |f''(x)| \le M$. This implies that the error for the right Riemann sum

$$(f(x_1) + \ldots + f(x_K))\, \Delta x$$

is bounded from above by $\frac{M(b-a)^3}{12K^2} + (f(b) - f(a))\frac{b-a}{2K}$ and gives the second inequality of (11) when applied to the function $t \mapsto p_m^t$ over the interval $[0, \sigma]$. On the other hand, the error given by the left Riemann sum

$$(f(x_0) + \ldots + f(x_{K-1}))\, \Delta x$$

is bounded from below by $-\frac{M(b-a)^3}{12K^2} - (f(b) - f(a))\frac{b-a}{2K}$ and this gives the first inequality in (12). □

Proof (of Theorem 2). A direct estimate yields

$$\frac{M_m(p_m^\sigma)}{p_m^\sigma} = \frac{Z_m}{p_m^\sigma} \sum_{\substack{n \in \Omega_m \\ n \le p_m^\sigma}} n\pi_m(n) = \frac{Z_m}{p_m^\sigma} \sum_{k=0}^{K-1} \sum_{\substack{n \in \Omega_m \\ p_m^{\delta_k} < n \le p_m^{\delta_{k+1}}}} n\pi_m(n) \le$$

$$\le \frac{Z_m}{p_m^\sigma} \sum_{k=0}^{K-1} p_m^{\delta_{k+1}} \sum_{\substack{n \in \Omega_m \\ p_m^{\delta_k} < n \le p_m^{\delta_{k+1}}}} \pi_m(n) =$$

$$= \frac{Z_m}{p_m^\sigma} e^{-\gamma} \sum_{k=0}^{K-1} p_m^{\delta_{k+1}} \frac{\sigma}{K} + \frac{Z_m}{p_m^\sigma} \sum_{k=0}^{K-1} p_m^{\delta_{k+1}} E_m^{(\sigma)}(k, K)$$

Applying Lemma 1 to the right Riemann sum $\sum_{k=0}^{K-1} p_m^{\delta_{k+1}} \frac{\sigma}{K}$ we obtain the estimate

$$\frac{M_m(p_m^\sigma)}{p_m^\sigma} \le \frac{e^{-\gamma} Z_m}{\ln p_m} \frac{p_m^\sigma - 1}{p_m^\sigma} + \frac{e^{-\gamma} Z_m}{\ln p_m} \left(\frac{\sigma^3 \ln^3 p_m}{12K^2} + \frac{p_m^\sigma - 1}{p_m^\sigma} \frac{\sigma \ln p_m}{2K} \right) +$$

$$+ \frac{Z_m}{p_m^\sigma} \sum_{k=0}^{K-1} p_m^{\delta_{k+1}} E_m^{(\sigma)}(k, K),$$

which is true for every m and K.

Since, as $m \to \infty$, $\frac{M_m(p_m^\sigma)}{p_m^\sigma} \to \frac{1}{\zeta(2)}$, $\frac{Z_m}{\ln p_m} \to \frac{e^\gamma}{\zeta(2)}$, and by hypothesis $\frac{\ln^3 p_m}{K(m)^2} \to c$ (and thus $\frac{\ln p_m}{K(m)} \to 0$), then for every $\varepsilon > 0$ the inequality

$$\frac{Z_m}{p_m^\sigma} \sum_{k=1}^{K(m)-1} p_m^{\delta_k+1} E_m^{(\sigma)}(k, K(m)) \geq -\frac{c\sigma^3}{12\zeta(2)} - \varepsilon$$

holds true for sufficiently large m. By noticing that $p_m^{\delta_k+1} = p_m^{\delta_k}\left(1 + (e^{\frac{\sigma \ln p_m}{K(m)}} - 1)\right)$ and $0 \leq (e^{\frac{\sigma \ln p_m}{K(m)}} - 1) \to 0$ as $m \to \infty$, we obtain the first inequality of (10). On the other hand

$$\frac{M_m(p_m^\sigma)}{p_m^\sigma} \geq \frac{Z_m}{p_m^\sigma} \sum_{k=0}^{K-1} p_m^{\delta_k} \sum_{\substack{n \in \Omega_m \\ p_m^{\delta_k} < n \leq p_m^{\delta_k+1}}} \pi_m(n) = \frac{Z_m}{p_m^\sigma} e^{-\gamma} \sum_{k=0}^{K-1} p_m^{\delta_k} \frac{\sigma}{K} +$$

$$+ \frac{Z_m}{p_m^\sigma} \sum_{k=0}^{K-1} p_m^{\delta_k} E_m^{(\sigma)}(k, K)$$

and applying Lemma 1 to the left Riemann sum $\sum_{k=0}^{K-1} p_m^{\delta_k} \frac{\sigma}{K}$ we obtain the estimate

$$\frac{M_m(p_m^\sigma)}{p_m^\sigma} \geq \frac{e^{-\gamma} Z_m}{\ln p_m} \frac{p_m^\sigma - 1}{p_m^\sigma} - \frac{e^{-\gamma} Z_m}{\ln p_m} \left(\frac{\sigma^3 \ln^3 p_m}{12K^2} + \frac{p_m^\sigma - 1}{p_m^\sigma} \frac{\sigma \ln p_m}{2K} \right) +$$

$$\frac{Z_m}{p_m^\sigma} \sum_{k=0}^{K-1} p_m^{\delta_k} E_m^{(\sigma)}(k, K),$$

which is true for every m and K. Proceeding as above we have that for every $\varepsilon > 0$ the inequality

$$\frac{Z_m}{p_m^\sigma} \sum_{k=0}^{K(m)-1} p_m^{\delta_k} E_m^{(\sigma)}(k, K(m)) \leq \frac{c\sigma^3}{12\zeta(2)} + \varepsilon$$

holds for sufficiently large m and we have the second inequality of (10). □

An immediate consequence of Theorem 2 is the following

Corollary 1. *Consider a function $K(m)$ such that*

$$\lim_{m \to \infty} \frac{\ln^3 p_m}{K(m)^2} = c \geq 0.$$

Then the sum of the error terms coming from (9), with weights $p_m^{-\sigma+\delta_k}$, satisfies the asymptotic estimate

$$\sum_{k=0}^{K(m)-1} \frac{E_m^{(\sigma)}(k, K(m))}{p_m^{\sigma-\delta_k}} = \begin{cases} O\left(\frac{1}{\ln p_m}\right) & \text{if } c > 0; \\ o\left(\frac{1}{\ln p_m}\right) & \text{if } c = 0; \end{cases} \tag{13}$$

as $m \to \infty$ *for every* $0 < \sigma \leq 1$.

Notice that implied constant in the O-notation depends explicitly on c and σ by (3) and (10). Moreover, as k ranges from 0 to $K(m) - 1$, the weights vary from $p_m^{-\sigma}$ ($\to 0$ as $m \to \infty$) to $e^{-\sigma \frac{\ln p_m}{K(m)}}$ ($\to 1$ as $m \to \infty$). This means that the error terms $E_m^{(\sigma)}(k, K(m))$ corresponding to small values of k are allowed to be larger in absolute value.

In order to get estimates on the mean value of the error term (for which al weights are equal to $\frac{1}{K(m)}$) we just replace the weights $p_m^{-\sigma+\delta_k}$ by either $p^{-\sigma}$ or 1 in (10). This yields, for every ε and sufficiently large m,

$$\frac{1}{Z_m}\left(-\frac{c\sigma^3}{12\zeta(2)} - \varepsilon\right) \leq \sum_{k=0}^{K(m)-1} E_m^{(\sigma)}(k, K(m)) \leq \frac{p_m^\sigma}{Z_m}\left(\frac{c\sigma^3}{12\zeta(2)} + \varepsilon\right).$$

In particular we get, as $m \to \infty$,

$$\langle E_m^{(\sigma)} \rangle := \frac{1}{K(m)} \sum_{k=0}^{K(m)-1} E_m^{(\sigma)}(k, K(m)) = \begin{cases} O\left(\frac{p_m^\sigma}{\ln^{5/2} p_m}\right) & \text{if } c > 0; \\ o\left(\frac{p_m^\sigma}{K(m) \ln p_m}\right) & \text{if } c = 0. \end{cases}$$

Let us point out that, even though by (9) the error term $E_m^{(\sigma)}(k, K(m))$ tends to zero as $m \to \infty$ for each k, it is not a priori true that $\langle E_m^{(\sigma)} \rangle$ tends to zero as well. It follows from our Theorem 2 that this is indeed the case when $\frac{p_m^\sigma}{K(m)\ln p_m}$ remains bounded (i.e. a particular case of $c = 0$). Let us summarize this fact in the following

Corollary 2. *Let* $0 < \sigma \leq 1$ *and consider a function* $K(m)$ *such that*

$$\lim_{m\to\infty} \frac{p_m^\sigma}{K(m) \ln p_m} < \infty.$$

Then, as $m \to \infty$,

$$\langle E_m^{(\sigma)} \rangle = o\left(\frac{p_m^\sigma}{K(m) \ln p_m}\right). \tag{14}$$

In other words, if K grows sufficiently fast (namely as $const \cdot \frac{p_m^\sigma}{\ln p_m}$ or faster), then the mean value of the error $\langle E_m^{(\sigma)} \rangle$ tends to zero as $m \to \infty$ and the rate of convergence to zero is controlled explicitly in terms of σ and K.

Notice that one would expect the error term $E_m^{(\sigma)}(k, K(m))$ in (9) to be $o\left(\frac{1}{K(m)}\right)$, however we could only derive the weaker asymptotic estimates (13) and (14) from Theorem 2. A possible approach to further investigate the size of the error term in (9) would be to first prove an analogue of Theorem 1 for shrinking intervals. This is, however, beyond the aim of this paper.

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Appendix

This Appendix contains the estimates for the error terms in the proof Theorem 1. By C_j, $j = 1, \ldots, 21$, we will denote some positive constants.

The first term of (5) tends to zero as $m \to \infty$ uniformly in λ. In fact using (2) we obtain

$$\begin{aligned}
&\Re f_m(p_m+1)N(p_m) = \\
&= \frac{N(p_m)}{2}\ln\left(\frac{(p_m+1)^2 + 2(p_m+1)\cos\left(\frac{\lambda \ln(p_m+1)}{\ln p_m}\right)+1}{(p_m+2)^2}\right) = \\
&= \frac{N(p_m)}{2}\left(\ln\left(1+O\left(\frac{1}{p_m}\right)\right) - \ln\left(1+O\left(\frac{1}{p_m}\right)\right)\right) = O\left(\frac{p_m}{\ln p_m}\right)O\left(\frac{1}{p_m}\right) = \\
&= O\left(\frac{1}{\ln p_m}\right),
\end{aligned}$$

and

$$\begin{aligned}
\Im f_m(p_m+1)N(p_m) &= N(p_m)\arctan\left(\frac{\sin\left(\lambda\frac{\ln(p_m+1)}{\ln p_m}\right)}{p_m+1+\cos\left(\lambda\frac{\ln(p_m+1)}{\ln p_m}\right)}\right) = \\
&= O\left(\frac{p_m}{\ln p_m}\right)O\left(\frac{1}{p_m}\right) = O\left(\frac{1}{\ln p_m}\right)
\end{aligned}$$

as $m \to \infty$, and the implied constants do not depend on λ. An explicit computation shows that

$$(\Re f_m)''(s) = f_m^{(1)}(s) + f_m^{(2)}(s) + f_m^{(3)}(s),$$

where

$$f_m^{(1)}(s) = -\lambda^2 \frac{2s + (1+s^2)\cos\left(\frac{\lambda \ln s}{\ln p_m}\right)}{s\left(s^2 + 2s\cos\left(\frac{\lambda \log s}{\log p_m}\right) + 1\right)^2 \ln^2 p_m},$$

$$f_m^{(2)}(s) = \lambda \frac{\left(3s^2 + 2s\cos\left(\frac{\lambda \ln s}{\ln p_m}\right) - 1\right)\sin\left(\frac{\lambda \ln s}{\ln p_m}\right)}{s\left(s^2 + 2s\cos\left(\frac{\lambda \log s}{\log p_m}\right) + 1\right)^2 \ln p_m},$$

$$f_m^{(3)}(s) = \frac{2\left(\cos\left(\frac{\lambda \ln s}{\ln p_m}\right) - 1\right)\left(s^3 - s^2 - s - 1 + (s^2 - 2s - 1)\cos\left(\frac{\lambda \ln s}{\ln p_m}\right)\right)}{(1+s)^2\left(s^2 + 2s\cos\left(\frac{\lambda \log s}{\log p_m}\right) + 1\right)^2}.$$

We have

$$\left|f_m^{(1)}(s)\right| \le \frac{C_1\lambda^2}{s^3 \ln^2 p_m}, \qquad \left|f_m^{(2)}(s)\right| \le \frac{C_2|\lambda|}{s^3 \ln p_m}$$

and thus

$$\left|\sum_{t=1}^{p_m} N(t-1)\tau_1 f_m^{(1)}(t+\tau_1')\right| \le \frac{C_3\lambda^2}{\ln^2 p_m}\sum_{t=2}^{p_m}\frac{1}{t^2 \ln t} \longrightarrow 0 \quad \text{and}$$

$$\left|\sum_{t=1}^{p_m} N(t-1)\tau_1 f_m^{(2)}(t+\tau_1')\right| \le \frac{C_4|\lambda|}{\ln p_m}\sum_{t=2}^{p_m}\frac{1}{t^2 \ln t} \longrightarrow 0 \quad \text{as } m \to \infty.$$

The third function satisfies the estimate

$$\left|f_m^{(3)}(s)\right| \le \frac{s^3\left|2\cos\left(\frac{\lambda \ln s}{\ln p_m}\right) - 2\right| + s^2 C_5}{(1+s)^2(1-s)^4} \le \frac{C_6\left(1 - \cos\left(\frac{\lambda \ln s}{\ln p_m}\right)\right)}{s^3}.$$

We now perform the same change of variables $v = v(t) = \frac{\ln t}{\ln p_m}$ as before (using τ_3 and τ_3' as in the proof of Theorem 1). We get

$$\left|\sum_{t=1}^{p_m} N(t-1)\tau_1 f_m^{(3)}(t+\tau_1')\right| \le C_7 \sum_{t=2}^{p_m} \frac{1 - \cos\left(\frac{\lambda \ln t}{\ln p_m}\right)}{t^2 \ln t} \le$$

$$\le C_8 \sum_v \left(\mathrm{d}v + \frac{\tau_3}{(t+\tau_3')^2 \ln p_m}\right)\frac{1 - \cos(\lambda u)}{t\, v} \longrightarrow 0$$

as $m \to \infty$. Another explicit computation shows that

$$(\Im f_m)''(s) = f_m^{(4)}(s) + f_m^{(5)}(s) + f_m^{(6)}(s),$$

where

$$f_m^{(4)}(s) = \lambda^2 \frac{(s^2-1)\sin\left(\frac{\lambda \log s}{\log p_m}\right)}{s\left(s^2+2s\cos\left(\frac{\lambda \log s}{\log p_m}\right)+1\right)^2 \ln^2 p_m},$$

$$f_m^{(5)}(s) = -\lambda \frac{1+5s^2+2s^2\cos^2\left(\frac{\lambda \log s}{\log p_m}\right)+(3s^3+5s)\cos\left(\frac{\lambda \log s}{\log p_m}\right)}{s^2\left(s^2+2s\cos\left(\frac{\lambda \log s}{\log p_m}\right)+1\right)^2 \ln p_m}$$

$$f_m^{(6)}(s) = \frac{2\left(s+\cos\left(\frac{\lambda \log s}{\log p_m}\right)\right)\sin\left(\frac{\lambda \log s}{\log p_m}\right)}{\left(s^2+2s\cos\left(\frac{\lambda \log s}{\log p_m}\right)+1\right)^2}.$$

We have the estimates

$$\left|f_m^{(4)}(s)\right| \le \frac{C_{10}\lambda^2}{s^3 \ln^2 p_m}, \qquad \left|f_m^{(5)}(s)\right| \le \frac{C_{11}|\lambda|}{s^3 \ln p_m}$$

and thus

$$\left|\sum_{t=1}^{p_m} N(t-1)\tau_1 f_m^{(4)}(t+\tau_1')\right| \le \frac{C_{12}\lambda^2}{\ln^2 p_m}\sum_{t=2}^{p_m}\frac{1}{t^2\ln t} \longrightarrow 0 \quad \text{and}$$

$$\left|\sum_{t=1}^{p_m} N(t-1)\tau_1 f_m^{(5)}(t+\tau_1')\right| \le \frac{C_{13}|\lambda|}{\ln p_m}\sum_{t=2}^{p_m}\frac{1}{t^2\ln t} \longrightarrow 0 \quad \text{as } m\to\infty.$$

The estimate

$$\left|f_m^{(6)}(s)\right| \le \frac{C_{14}s\sin\left(\frac{\lambda \log s}{\log p_m}\right)}{(s-1)^4} \le \frac{C_{15}\sin\left(\frac{\lambda \log s}{\log p_m}\right)}{s^3}$$

yields, as $m \to \infty$,

$$\left|\sum_{t=1}^{p_m} N(t-1)\tau_1 f_m^{(6)}(t+\tau_1')\right| \le C_{15}\sum_{t=2}^{p_m}\frac{\sin\left(\frac{\lambda \ln t}{\ln p_m}\right)}{t^2 \ln t} \le$$

$$\le C_{16}\sum_{v}\left(\mathrm{d}v + \frac{\tau_3}{(t+\tau_3')^2\ln p_m}\right)\frac{\sin(\lambda v)}{t\,v} \to 0.$$

This concludes the analysis of the error terms coming from (5).

Let us now deal with the error terms coming from (6). One sum (giving the main term) is already discussed in the proof of Theorem 1. Amongst the remaining 11 sums coming from (6), it is enough to check that the following three tend to zero as $m \to \infty$ (the other 8 being dominated by these):

$$\left| \sum_{t=2}^{p_m} \frac{1}{t \ln t} \frac{(2t-1)\left(e^{\frac{i\lambda \ln t}{\ln p_m}} - 1\right)}{(t+1)^2} \right| \leq C_{17} \sum_{t=1}^{p_m} \left| \frac{e^{\frac{i\lambda \ln t}{\ln p_m}} - 1}{\frac{\ln t}{\ln p_m}} \right| \frac{1}{t^2 \ln p_m} \leq \frac{C_{18}|\lambda|}{\ln p_m} \sum_{t=2}^{p_m} \frac{1}{t^2} \to 0,$$

$$\left| \frac{i\lambda}{\ln m} \sum_{t=2}^{p_m} \frac{1}{t \ln t} \frac{t}{t+1} e^{\frac{i\lambda \ln t}{\ln p_m}} \right| \leq \frac{C_{19}|\lambda|}{\ln m} \sum_{v} \left(\mathrm{d}v + \frac{\tau}{(t+\tau')^2 \ln p_m} \right) \frac{e^{i\lambda v}}{v} \to 0,$$

$$\left| \sum_{t+2}^{p_m} \frac{1}{t \ln t} \frac{\left(e^{\frac{i\lambda \ln t}{\ln p_m}} - 1\right)^2}{t + e^{\frac{i\lambda \ln t}{\ln p_m}}} \right| \leq C_{20} \sum_{t=2}^{p_m} \left| \frac{\left(e^{\frac{i\lambda \ln t}{\ln p_m}} - 1\right)^2}{\left(\frac{\ln t}{\ln p_m}\right)^2} \right| \frac{\ln t}{t^2 \ln^2 p_m} \leq \frac{C_{21}\lambda^2}{\ln^2 p_m} \sum_{t=2}^{p_m} \frac{\ln t}{t^2} \to 0.$$

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Additive Functions and Gaussian Measures

Linan Chen and Daniel W. Stroock

Abstract In this paper we examine infinite dimensional analogs of the measure theoretic variations of Cauchy's classical functional equation for additive functions. In particular, we show that the a naïve generalization of the finite dimensional statement fails in infinite dimensions and show how it has to be altered to make it true. In the process, we develop various techniques which lead naturally to results about the structure of abstract Wiener spaces.

Keywords Abstract Wiener spaces

Mathematics Subject Classification (2010): 60G15, 60G60

1 Introduction

The classical Cauchy functional equation

$$f(x + y) = f(x) + f(y) \tag{1}$$

has a rich history. When $f : \mathbb{R} \longrightarrow \mathbb{R}$, the problem of determining which functions satisfy (1) is well understood. It is easy to see that the only continuous solutions are linear. In the absence of any further conditions, all that one can say is that $f(qx) = qf(x)$ for all $q \in \mathbb{Q}$ (the field of rational numbers) and $x \in \mathbb{R}$. In fact, as an application of Zorn's Lemma, one can construct solutions which are $\mathbb{Q}$-valued and

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therefore certainly not continuous. On the other hand, if f is Lebesgue measurable solution, then it must be linear. The argument is simple but worth repeating. Namely, given a Lebesgue measurable solution, choose $R > 0$ so that $\Gamma \equiv \{x : |f(x)| \le R\}$ has positive Lebesgue measure. Then, by the lemma of Vitali on which the standard example of a non-measurable set relies, $\Delta = \Gamma - \Gamma$ contains an interval $[-\delta, \delta]$ for some $\delta > 0$, and clearly $|f(x)| \le 2R$ for all $x \in \Delta$. Further, for any $x \in \mathbb{R} \setminus \{0\}$, one can find a positive $q \in \mathbb{Q}$ such that $q \ge \frac{\delta}{2|x|}$ and $qx \in \Delta$. Hence, $|f(x)| = \frac{|f(qx)|}{q} \le \frac{4R|x|}{\delta}$, which means that f is continuous at 0 and therefore everywhere.

When f is a map from one real Banach space E into a second F, the $\mathbb{R}$-valued result shows that the only Borel measurable solutions to (1) must be linear. If one combines this with L. Schwartz's result (cf. [3] or, for a proof which is more in keeping with the present paper, [6]) which says that all Borel measurable, linear functions are continuous, then one arrives at the conclusion that the only Borel measurable solutions $f : E \longrightarrow F$ to (1) are continuous, linear maps. In particular, when $F = \mathbb{R}$, $f(x) = \langle x, x^* \rangle$ for some $x^* \in E^*$.

P. Erdös asked what could be said when $f : \mathbb{R} \longrightarrow \mathbb{R}$ and one replaces (1) by

$$f(x + y) = f(x) + f(y) \quad \text{for Lebesgue-almost every } (x, y) \in \mathbb{R}^2. \tag{2}$$

A definite answer was given by N.G. de Bruijn [1] and W.B. Jurkat [2] who showed that, even if f is not measurable, every solution to (2) is almost everywhere equal to an additive function, and therefore every Lebesgue measurable solution to (2) is almost everywhere equal to a linear function.

In this article, we will study the analogous problem for maps between Banach spaces. Of course, since there is no Lebesgue measure on an infinite dimensional space, (2) as it stands makes no sense there. Thus, instead of Lebesgue measure, we take a Gaussian measure as the reference measure. That is, when E and F are separable, real Banach spaces and $\mathscr{W}$ is a non-degenerate, centered Gaussian measure on E, we will investigate the $\mathscr{W}$-measurable functions $f : E \longrightarrow F$ which satisfy

$$f(x + y) = f(x) + f(y) \quad \mathscr{W}^2\text{-almost surely.} \tag{3}$$

Among other things, we will show that there are solutions to (3) which are not $\mathscr{W}$-almost surely equal to a linear function. On the other hand, if (3) is replaced by

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \mathscr{W}^2\text{-almost surely,} \tag{4}$$

for some pair $(\alpha, \beta) \in (0, 1)^2$ satisfying $\alpha^2 + \beta^2 = 1$, we will show that there is a dense, Borel measurable, linear subspace L of E and a Borel measurable linear map $\ell : L \longrightarrow F$ such that $\mathscr{W}(L) = 1$ and $f \upharpoonright L = \ell$ $\mathscr{W}$-almost surely. In general, the linear map ℓ will not be continuous or even admit an extension to E, and so we investigate how E can be modified so that ℓ becomes continuous.

2 Wiener Maps

Let E be a separable, real Banach space. A non-degenerate, centered Gaussian measure $\mathscr{W}$ on E is a Borel probability measure with the property that, for each $x^* \in E^*\setminus\{0\}$, $x \rightsquigarrow \langle x, x^*\rangle$ is a non-degenerate, centered Gaussian random variable under $\mathscr{W}$. When E is finite dimensional, any such $\mathscr{W}$ is equivalent to Lebesgue measure. However, when E is infinite dimensional, there are uncountably many, mutually singular choices of $\mathscr{W}$. Indeed, given a $\mathscr{W}$ and an $\alpha \in \mathbb{R}$, let $\mathscr{W}_\alpha$ denote the distribution of $x \rightsquigarrow \alpha x$ under $\mathscr{W}$. Then for any $\alpha \notin \{-1, 1\}$, $\mathscr{W}_\alpha$ is singular to $\mathscr{W}$. As a consequence, the distribution of $(x, y) \in E^2 \longmapsto x + y \in E$ under $\mathscr{W}^2$ is also singular to $\mathscr{W}$. In particular, the $\mathscr{W}$-analog

$$f(x+y) = f(x) + f(y) \quad \text{for } \mathscr{W}^2\text{-almost every } (x, y) \in E^2 \tag{5}$$

of (2) is somewhat suspect, and so it is not too surprising that there are Borel measurable, $\mathbb{R}$-valued solutions to (5) which are very far from being linear.

To produce a highly non-linear, Borel measurable solution to (5), assume that E is infinite dimensional, and choose $\{x_m^* : m \geq 0\} \subseteq E^*$ so that

$$\mathbb{E}^{\mathscr{W}}\big[\langle \cdot, x_m^*\rangle\langle \cdot, x_n^*\rangle\big] = \delta_{m,n}.$$

Then $\{\langle \cdot, x_m^*\rangle : m \geq 0\}$ is a sequence of mutually independent, standard Gaussian random variables under $\mathscr{W}$, and so, by the strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1}\langle x, x_m^*\rangle^2 = 1 \quad \text{for } \mathscr{W}\text{-almost every } x \in E.$$

Now let A be the set of $x \in E$ for which $\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1}\langle x, x_m^*\rangle^2$ exists in $\mathbb{R}$. Obviously, A is a Borel measurable subset of E, and so the function $f : E \longrightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1}\langle x, x_m^*\rangle^2 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is Borel measurable. Next, take A_0 to be the subset of $x \in A$ for which $f(x) = 1$, and let B denote the subset of $(x, y) \in A_0 \times A_0$ for which

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1}\langle x, x_m^*\rangle\langle y, x_m^*\rangle = 0.$$

Clearly B is Borel measurable, and another application of the strong law shows that $\mathscr{W}^2(B) = 1$. In addition, $f(x+y) = f(x) + f(y)$ for $(x, y) \in B$. On the other hand, if $x \in A_0$, then $f(2x) = 4 \neq 2 = 2f(x)$, and so f is $\mathscr{W}$-almost everywhere non-linear.

The preceding example shows that, in infinite dimensions, (5) is not a good replacement for (2). A more satisfactory replacement is provided by the notion of a *Wiener map*. To describe this, say that (α, β) is a Pythagorian pair if $(\alpha, \beta) \in (0, 1)^2$ and $\alpha^2 + \beta^2 = 1$. Then a Wiener map $f : E \longrightarrow F$ is a $\mathscr{W}$-measurable map which satisfies

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \text{for } \mathscr{W}^2\text{-almost all } (x, y) \in E^2 \tag{6}$$

for some Pythagorian pair (α, β). Notice that, since the distribution of $(x, y) \rightsquigarrow \alpha x + \beta y$ under $\mathscr{W}^2$ is $\mathscr{W}$, (6) makes perfectly good sense even though, in general, f is well defined only up to a set of $\mathscr{W}$-measure 0. For this reason, one should suspect that (6) has virtues which (5) does not possess.

To fully describe these virtues, it is necessary to introduce a little terminology. For a given $\mathscr{W}$ on E, there is a unique Hilbert space, known as the *Cameron–Martin space*, H continuously embedded as a dense subspace of E which has the property that, for each $x^* \in E^*$, $\|h_{x^*}\|_H^2$ is the variance of $\langle \cdot, x^* \rangle$ under $\mathscr{W}$, where h_{x^*} is the element of H determined by $(h, h_{x^*})_H = \langle h, x^* \rangle$ for $h \in H$. In particular, because $\{h_{x^*} : x^* \in E^*\}$ is a dense subspace of H, there is a unique isometry, known as the Paley–Wiener map, $\mathscr{I} : H \longrightarrow L^2(\mathscr{W}; \mathbb{R})$ such that $\mathscr{I}(h_{x^*}) = \langle \cdot, x^* \rangle$ for all $x^* \in E^*$. Moreover, the image of H under $\mathscr{I}$ is a centered Gaussian family in $L^2(\mathscr{W}; \mathbb{R})$.

The following statement is essentially the same as the one of Theorem 2.5 in [5].

Theorem 1. *If $f : E \longrightarrow F$ is $\mathscr{W}$-measurable, then f is a Wiener map if and only if there is a bounded, linear map $A : H \longrightarrow F$ such that $\langle f, y^* \rangle = \mathscr{I}(A^\top y^*)$ $\mathscr{W}$-almost surely for each $y^* \in F^*$, where $A^\top : F^* \longrightarrow H$ is the adjoint of A. Moreover, if A exists, then it is unique, it is continuous from the weak* topology on H into the strong topology on F, and, for any orthonormal basis $\{h_k : k \geq 1\}$,*

$$f = \sum_{k=1}^{\infty} \mathscr{I}(h_k) A h_k \quad \mathscr{W}\text{-almost surely,} \tag{7}$$

where the convergence is $\mathscr{W}$-almost sure as well as in $L^p(\mathscr{W}; \mathbb{R})$ for each $p \in [1, \infty)$. In particular, if F_A is the closure in F of the range AH of A, then $f(x) \in F_A$ $\mathscr{W}$-almost surely.

In [5], this theorem was proved under slightly different hypotheses. For one thing, f was assumed there to be Borel measurable, but, because, as was pointed out above, the $\mathscr{W}^2$-distribution of $(x, y) \rightsquigarrow \alpha x + \beta y$ is $\mathscr{W}$, assuming that f is Borel measurable causes no loss in generality. Second, and more significant, is the difference between the definition of a Wiener map here and the one there. Namely, in [5] it was assumed that $\alpha = 2^{-\frac{1}{2}} = \beta$. However, the modification of the proof given in [5] which is required to cover the generalization to arbitrary Pythagorian pairs is trivial. In addition, the conclusion drawn in Theorem 1 shows that if f is a Wiener map relative to one Pythagorian pair, then it is a Wiener map relative to any other Pythagorian pair.

As an immediate corollary to Theorem 1, we have the following.

Corollary 1. *If $f : E \longrightarrow F$ is a Wiener map, then there is a dense, linear subspace L of $\mathscr{W}$-measure 1 and a Borel measurable linear map $\ell : L \longrightarrow F$ such that $f \restriction L = \ell$ $\mathscr{W}$-almost surely.*

Proof. Simply take L to be the set of $x \in E$ for which the series on the right hand side of (7) converges in F, and define ℓ on L to be the sum of that series.

Corollary 1 represents the best approximation we have in infinite dimensions to the result, alluded to earlier, proved by de Bruijn and Jurkat in the real-valued setting. In fact, at least when f is assumed to be Lebesgue measurable, Corollary 1 contains their result. To see this, what one has to show is that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lebesgue measurable function satisfying (2), then f is a Wiener map. To this end, first argue that, for each $n \geq 1$, $f(nx + y) = nf(x) + f(y)$ for almost every $(x, y) \in \mathbb{R}^2$. Indeed, there is nothing to do when $n = 1$, and, assuming that it holds for n, one has that, for almost every (x, y),

$$f\big((n+1)x+y\big) = f\big(nx+(x+y)\big) = f(nx)+f(x+y) = (n+1)f(x)+f(y),$$

where we have used the fact that the Lebesgue distribution of $(x, y) \rightsquigarrow (nx, x+y)$ is equivalent to that of (x, y). At the same time, because $(x, y) \rightsquigarrow \big((n+1)x, y\big)$ has the same Lebesgue distribution as (x, y), $f\big((n+1)x+y\big) = f\big((n+1)x\big) + f(y)$ for almost every (x, y). Hence, by Fubini's Theorem, $f\big((n+1)x\big) = (n+1)f(x)$ almost everywhere. Knowing that $f(nx) = nf(x)$ almost everywhere, one can repeat the same sort of argument to show first that $f(qx) = qf(x)$ almost everywhere for all $q \in \mathbb{Q}^+ \equiv \mathbb{Q} \cap (0, \infty)$ and then that $f(q_1x + q_2y) = q_1f(x) + q_2f(y)$ almost everywhere for all $(q_1, q_2) \in (\mathbb{Q}^+)^2$. In particular, by taking $q_1 = \frac{3}{5}$ and $q_2 = \frac{4}{5}$, one concludes that f is a Wiener map. Finally, any non-degenerate Gaussian measure on $\mathbb{R}$ is equivalent to Lebesgue measure and because $\mathbb{R}$ itself is the only subspace of $\mathbb{R}$ to which a non-degenerate Gaussian measure assigns measure 1, we arrive at the conclusion that there is a linear function to which f is almost everywhere equal.

Remark. It should be clear where the preceding line of reasoning breaks down in the infinite dimensional setting. Specifically, in infinite dimensions, there is no counterpart of the equivalence of measures assertions which were crucial in the proof that f is a Wiener map if it satisfies (2).

3 A Refinement

By the result, alluded to earlier, in [6], the linear map ℓ in Corollary 1 can be extended as a $\mathscr{W}$-measurable, linear map on the whole of E if and only if it is continuous. Thus, because $\mathscr{W}$ gives positive mass to every non-empty open subset of E and $\mathscr{W}(E \setminus L) = 0$, there is at most one such extension. In this section, we will show how to modify E so that, for a given $\mathscr{W}$-measurable $f : E \longrightarrow \mathbb{R}$ satisfying (6), a continuous extension will exist.

By Theorem 1, for each Wiener map $f : E \longrightarrow \mathbb{R}$ there is a $g \in H$ such that $f = \mathscr{I}(g)$. Thus, what we need to show is that for each $g \in H$ there is a Banach space E_g which is a Borel measurable subset of E such that $\mathscr{W}(E_g) = 1$, $\big(H, E_g, \mathscr{W} \restriction E_g\big)$ is an abstract Wiener space, and

$$\sup\big\{\big|(g,h)_H\big| : \|h\|_{E_g} \le 1\big\} < \infty. \tag{8}$$

Indeed, if (8) holds, then, because H is dense in E_g, there is a unique $x^* \in E_g^*$ such that $(g,h)_H = {}_{E_g}\langle h, x^*\rangle_{E_g^*}$ for all h and therefore $\mathscr{I}(g) = {}_{E_g}\langle \cdot\,, x^*\rangle_{E_g^*}$ $\mathscr{W}$-almost surely.

The construction of E_g mimicks a line of reasoning introduced by L. Gross when he proved (cf. Corollary 8.3.10 in [4]) that one can always find a Banach space $E_0 \subseteq E$ such that bounded subsets of E_0 is relatively compact in E and $\big(H, E_0, \mathscr{W} \restriction E_0\big)$ is an abstract Wiener space. To be precise, given a finite dimensional subspace L of H, there is a $\mathscr{W}$-almost surely unique $P_L : E \longrightarrow H$ such that, for each $h \in H$, $\mathscr{I}(h) \circ P_L = \mathscr{I}(\Pi_L h)$ $\mathscr{W}$-almost surely, where Π_L denotes orthogonal projection from H onto L. In fact, given any orthonormal basis $\{b_1, \dots, b_m\}$ for L, one can take $P_L x = \sum_{\ell=1}^m [\mathscr{I}(b_\ell)](x) b_\ell$. Now choose $\{x_n^* : n \ge 0\} \subseteq E^*$ so that $\{h_n : n \ge 0\}$ is an orthonormal basis in H when $h_n = h_{x_n^*}$. Using Theorem 8.3.9 in [4], one can find a strictly increasing sequence $\{n_m : m \ge 0\} \subseteq \mathbb{N}$ so that $n_0 = 0$ and, for $m \ge 1$, $\mathbb{E}^{\mathscr{W}}\big[\|P_L x\|_E^2\big] \le 4^{-m}$ for all finite dimensional $L \perp \{h_0, \dots, h_{n_m}\}$. Now define

$$Q_0 x = \langle x, x_0^*\rangle \quad \text{and, for } m \ge 1, \quad Q_m x = \sum_{n_{m-1}+1}^{n_m} \langle x, x_n^*\rangle h_n,$$

and set $S_m = \sum_{\ell=0}^m Q_\ell$ and

$$s_m(x) = \big(g, S_m(x)\big)_H = \sum_{\ell=0}^{n_m} \langle x, x_\ell^*\rangle \big(g, h_\ell\big)_H.$$

By Theorem 8.3.3 in [4], $S_m(x) \longrightarrow x$ and $s_m(x) \longrightarrow [\mathscr{I}(g)](x)$ for $\mathscr{W}$-almost every $x \in E$. In addition, in both cases, the convergence takes place in L^2. In particular, by passing to another subsequence if necessary, we may and will assume that $\{n_m : m \ge 0\}$ has been chosen so that $\mathbb{E}^{\mathscr{W}}\big[|s_m - s_{m-1}|^2\big] \le 4^{-m}$ for $m \ge 1$. Finally, take E_g to be the set of $x \in E$ such that $S_m(x) \longrightarrow x$ and

$$\|x\|_{E_g} \equiv \sum_{m=0}^\infty \Big(\|Q_m x\|_E + \big|\big(g, Q_m x\big)_H\big|\Big) < \infty.$$

Repeating the argument given to prove Theorem 8.3.10 cited above, one can show that E_g is a dense, measurable, subspace with $\mathscr{W}(E_g)=1$, E_g with norm $\|\cdot\|_{E_g}$ is a Banach space which is continuously embedded in E, and $\big(H, E_g, \mathscr{W} \restriction E_g\big)$ is an abstract Wiener space. In addition, it is an easy matter to check that, for each $x \in E_g$, $S_m(x) \longrightarrow x$ in E_g and $\{s_m(x) : m \geq 0\}$ converges in $\mathbb{R}$, and that, for each $h \in H$, $|(g,h)_H| \leq \|h\|_{E_g}$. Thus, we have justified the following statement.

Theorem 2. *Let $(H, E, \mathscr{W})$ be an abstract Wiener space, and let $\{x_n^* : n \geq 0\} \subseteq E^*$ be chosen so that $\{h_{x_n^*} : n \geq 0\}$ is an orthonormal basis in H. Then for each $g \in H$ there is a Banach space E_g, a unique element $x^* \in E_g^*$, and a strictly increasing subsequence $\{n_m : m \geq 0\} \subseteq \mathbb{N}$ such that*

1. *E_g is continuously embedded in E as a measurable subspace with $\mathscr{W}(E_g) = 1$,*
2. *$\big(H, E_g, \mathscr{W} \restriction E_g\big)$ is an abstract Wiener space,*
3. *$\lim_{m\to\infty} \sum_{\ell=0}^{n_m} \langle g, x_\ell^* \rangle \langle x, x_\ell^* \rangle = {}_{E_g}\langle x, x^* \rangle_{E_g^*}$ for each $x \in E_g$.*

In particular, $[\mathscr{I}(g)](x) = {}_{E_g}\langle x, x^* \rangle_{E_g^*}$ for $\mathscr{W}$-almost every $x \in E_g$.

Corollary 2. *Let $(H, E, \mathscr{W})$ be an abstract Wiener space and $f : E \longrightarrow \mathbb{R}$ a Wiener map. Then there is a Banach space E_f which is continuously embedded as a dense, measurable subspace of E with $\mathscr{W}(E_f) = 1$ and a unique $x^* \in E_f^*$ such that $\big(H, E_f, \mathscr{W} \restriction E_f\big)$ is an abstract Wiener space and $f(x) = {}_{E_f}\langle x, x^* \rangle_{E_f^*}$ for $\mathscr{W}$-almost every $x \in E_f$.*

4 Concluding Considerations

The result in Theorem 2 has an interesting application to the structure of abstract Wiener spaces. Namely, it gives a simple proof of the fact that

$$H = \bigcap \big\{E : \big(H, E, \mathscr{W} \restriction E\big) \text{ is an abstract Wiener space}\big\}. \tag{9}$$

In fact, given an abstract Wiener space $(H, E, \mathscr{W})$, choose $\{x_n^* : n \geq 0\} \subseteq E^*$ so that $\{h_{x_n^*} : n \geq 0\}$ is an orthonormal basis in H, and, for each $g \in H$, let E_g be taken accordingly, as in that theorem. If $x \in \bigcap_{g\in H} E_g$, then, for each $g \in H$, $\Lambda(g,x) = \lim_{n\to\infty} \Lambda_n(g,x)$ exists, where

$$\Lambda_n(g,x) \equiv \sum_{\ell=0}^{n} \langle g, x_\ell^* \rangle \langle x, x_\ell^* \rangle.$$

To see this, suppose that $\lim_{n\to\infty} \Lambda_n(g,x)$ fails to exist for some $g \in H$. Then

$$\sum_{\ell=0}^{\infty} \big| \langle g, x_\ell^* \rangle \langle x, x_\ell^* \rangle \big| = \infty.$$

But, if $h \in H$ is determined so that $\langle h, x_\ell^* \rangle = \pm\langle g, x_\ell^* \rangle$, where the $+$ sign is chosen if $\langle g, x_\ell^* \rangle\langle x, x_\ell^* \rangle \geq 0$ and the $-$ sign is chosen if $\langle g, x_\ell^* \rangle\langle x, x_\ell^* \rangle < 0$, then we have the contradiction that

$$\left\{ \sum_{\ell=0}^{n} \langle h, x_\ell^* \rangle \langle x, x_\ell^* \rangle \, : \, n \geq 0 \right\}$$

has no convergent subsequence.

Since $\Lambda(\cdot, x)$ is the weak limit of the continuous, linear functionals $\Lambda_n(\cdot, x)$ on H, $\Lambda(\cdot, x)$ is itself a continuous linear functional on H. Equivalently, there is a $C_x < \infty$ such that $|\Lambda(g, x)| \leq C_x \|g\|_H$, and from this is clear first that $\sum_{\ell=0}^{\infty} \langle x, x_\ell^* \rangle^2 \leq C_x^2 < \infty$ and then (cf. Lemma 8.2.3 in [4]) that $x \in H$. Hence, we have shown that $H = \bigcap_{g \in H} E_g$, which certainly implies (9).

We close with an observation which, in some sense, complements (9). Namely, given a separable, real Banach space E,

$$E = \bigcup \big\{ H \, : \, H \text{ is the Cameron-Martin space for some } \mathscr{W} \text{ on } E \big\}. \tag{10}$$

Theorem 3. *Suppose that $\{L_n : n \geq 1\}$ is a non-decreasing sequence of finite dimensional subspaces of E and that $L \equiv \bigcup_{n=1}^{\infty} L_n$ is dense in E. Then there exists an abstract Wiener space $(H, E, \mathscr{W})$ and a sequence $\{x_n^* : n \geq 1\} \subseteq E^*$ such that, for each $n \geq 1$ and $x \in L_n$, $\langle x, x_{n+1}^* \rangle = 0$ and $x = h_{x^*}$, where $x^* = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle x_n^*$. In particular, $L \subseteq \{h_{x^*} : x^* \in E^*\} \subseteq H$.*

Proof. Without loss in generality, we will assume that $\dim(L_n) = n$. We now apply a Gram–Schmit orthogonalization procedure to produce $\{x_n : n \geq 1\} \subseteq E$ and $\{x_n^* : n \geq 1\} \subseteq E^*$ so that $\{x_1, \ldots, x_n\}$ is a basis for L_n, $\|x_n\|_E = \frac{1}{n^2}$, and $\langle x_m, x_n^* \rangle = \delta_{m,n}$. That is, choose $x_1 \in L_1$ with $\|x_1\|_E = 1$ and $x_1^* \in E^*$ so that $\langle x_1, x_1^* \rangle = 1$. Given $\{x_1, \ldots, x_n\}$ and $\{x_1^*, \ldots, x_n^*\}$, choose $y_{n+1} \in L_{n+1} \setminus L_n$, and set

$$x_{n+1} = \frac{y_{n+1} - \sum_{m=1}^{n} \langle y_{n+1}, x_m^* \rangle x_m}{(n+1)^2 \left\| y_{n+1} - \sum_{m=1}^{n} \langle y_{n+1}, x_m^* \rangle x_m \right\|_E}.$$

Finally, choose $x_{n+1}^* \in E^*$ so that $\langle x_{n+1}, x_{n+1}^* \rangle = 1$ and $\langle x_m, x_{n+1}^* \rangle = 0$ for $1 \leq m \leq n$.

Now let γ denote the standard Gauss measure on $\mathbb{R}$. Then, because $\sum_{n=1}^{\infty} \|x_n\|_E < \infty$, $\sum_{n=1}^{\infty} |\omega_n| \|x_n\|_E < \infty$ for $\mathbb{P} \equiv \gamma^{\mathbb{Z}^+}$-almost every $\omega \in \mathbb{R}^{\mathbb{Z}^+}$. Thus, there is a random variable $X : \mathbb{R}^{\mathbb{Z}^+} \longrightarrow E$ such that $X(\omega) = \sum_{n=1}^{\infty} \omega_n x_n$ for $\mathbb{P}$-almost every ω. Let $\mathscr{W}$ denote the distribution of X under $\mathbb{P}$. Then

$$\mathbb{E}^{\mathscr{W}}\left[\langle x, x^* \rangle^2 \right] = \sum_{n=1}^{\infty} \langle x_n, x^* \rangle^2,$$

and so, since L is dense in E, $\mathscr{W}$ is a non-degenerate, centered Gaussian measure on E. In addition,

$$h_{x^*} = \int_E \langle x, x^* \rangle x \, \mathscr{W}(dx) = \sum_{n=1}^{\infty} \langle x_n, x^* \rangle x_n,$$

from which is clear that $x_n = h_{x_n^*}$. Finally, if $x \in L_n$, then $\langle x, x_m^* \rangle = 0$ when $m > n$ and $x = \sum_{m=1}^{n} \langle x, x_m^* \rangle x_m$.

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Free Infinitely Divisible Approximations of n-Fold Free Convolutions

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Abstract Based on the method of subordinating functions we prove a free analog of error bounds in classical Probability Theory for the approximation of n-fold convolutions of probability measures by infinitely divisible distributions.

Keywords Additive free convolution • Cauchy's transform • Free infinitely divisible probability measures • n-fold additive free convolutions of probability measures

Mathematics Subject Classification (2010): 46L53, 46L54, 60E07

1 Introduction

In recent years a number of papers are investigating limit theorems for the free convolution of probability measures defined by D. Voiculescu. The key concept of this definition is the notion of freeness, which can be interpreted as a kind of independence for noncommutative random variables. As in the classical probability where the concept of independence gives rise to the classical convolution, the concept of freeness leads to a binary operation on the probability measures on the real line, the free convolution. Classical results for the convolution of probability

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measures have their counterpart in this new theory, such as the law of large numbers, the central limit theorem, the Lévy-Khintchin formula and others. We refer to Voiculescu, Dykema and Nica [26] and Hiai and Petz [17] for introduction to these topics. Bercovici and Pata [10] established the distributional behavior of sums of free identically distributed random variables and described explicitly the correspondence between limits laws for free and classical additive convolution. Chistyakov and Götze [14] generalized the results of Bercovici and Pata to the case of free non-identically distributed random variables. They showed that the parallelism found by Bercovici and Pata holds in the general case of free non-identically distributed random variables. Using the method of subordination functions they proved the semi-circle approximation theorem (an analog of the Berry-Esseen inequality). See Kargin's paper [18] as well.

In the classical probability Doeblin [15] showed that it is possible to construct independent identically distributed random variables $X_1, X_2, \dots$ such that the distribution of the centered and normalized sum $b_{n_k}^{-1}(X_1 + \dots + X_{n_k} - a_{n_k})$ does not converge to any nondegenerate distribution, whatever the choice of the constants a_n and b_n and of the sequence $n_1 < n_2 < \dots$. Kolmogorov [19] initiated the study of approximations of sequences $\{\mu^{n*}\}_{n=1}^{\infty}$ of convolutions of some distribution μ by elements of the class of infinitely divisible distributions in some metric as $n \to \infty$. Prokhorov [23] and Kolmogorov [20] studied this problem which subsequently led to seminal results by Arak and Zaitsev in their monograph [5] on this problem.

Due to the Bercovici–Pata parallelism between limits laws for free and classical additive convolution results like those of Doeblin should hold for free random variables as well, which we discuss in Sect. 2. Thus Kolmogorov's approach would be natural in free Probability Theory as well but has not been done yet and we would like to start research in this direction. In particular in this paper we study the problem of approximating n-fold additive free convolutions of probability measures by additive free infinitely divisible probability measures.

The paper is organized as follows. In Sect. 2 we formulate and discuss the main results of the paper. In Sect. 3 we formulate auxiliary results. Section 4 contains a upper bound in the approximation problem.

2 Results

Denote by $\mathscr{M}$ the family of all Borel probability measures defined on the real line $\mathbb{R}$. On $\mathscr{M}$ define the associative composition laws denoted $*$ and $\boxplus$ as follows. For $\mu_1, \mu_2 \in \mathscr{M}$ let a probability measure $\mu_1 * \mu_2$ denote the classical convolution of μ_1 and μ_2. In probabilistic terms, $\mu_1 * \mu_2$ is the probability distribution of $X+Y$, where X and Y are (commuting) independent random variables with distributions μ_1 and μ_2, respectively. A measure $\mu_1 \boxplus \mu_2$ on the other hand denotes the free (additive) convolution of μ_1 and μ_2 introduced by Voiculescu [25] for compactly supported measures. Free convolution was extended by Maassen [21] to measures with finite variance and by Bercovici and Voiculescu [9] to the class $\mathcal{M}$. Thus, $\mu_1 \boxplus \mu_2$ is the probability distribution of $X + Y$, where X and Y are free random variables with distributions μ_1 and μ_2, respectively.

Let $\rho(\mu, \nu)$ be the Kolmogorov distance between probability measures μ and ν, i.e.,

$$\rho(\mu, \nu) = \sup_{x \in \mathbb{R}} |\mu((-\infty, x)) - \nu((-\infty, x))|.$$

In 1955 Prokhorov [23] proved that

$$\rho(\mu^{n*}, \mathbf{D}^*) := \inf_{\nu \in \mathbf{D}^*} \rho(\mu^{n*}, \nu) \to 0, \quad n \to \infty, \tag{1}$$

for any $\mu \in \mathcal{M}$, where μ^{n*} denotes the n-fold convolution of the probability measure μ and $\mathbf{D}^*$ denotes the set of infinitely divisible probability measures (with respect to classical convolution). Kolmogorov [20] noted that the convergence in (1) is uniform with respect to μ throughout the class $\mathcal{M}$. Work by a number of researchers (a detailed history of the problem may be found in [5]) eventually proved upper and lower bounds for the function $\psi(n) := \sup_{\mu \in \mathcal{M}} \rho(\mu^{n*}, \mathbf{D}^*)$. A final answer was given by Arak [3, 4], who proved the following bound:

$$c_1 n^{-2/3} \le \psi(n) \le c_2 n^{-2/3},$$

where c_1 and x_2 are absolute positive constants. Chistyakov [12] returned to Prohorov's result [23] and studied the problem of determining the possible rate of convergence of $\rho(\mu^{n*}, \mathbf{D}^*)$ to zero as $n \to \infty$ for probability measures $\mu \notin \mathbf{D}^*$.

Define the distance in variation between two signed measures μ and ν by

$$\rho_{var}(\mu, \nu) := \sup_{S \in \mathcal{B}} |\mu(S) - \nu(S)|,$$

where $\mathcal{B}$ denotes the σ-algebra of Borel subsets of $\mathbb{R}$. If μ, ν are probability measures, then $\rho_{var}(\mu, \nu) := var(\mu - \nu)/2$. It is natural to consider the analogous problems for the distance in variation. Prokhorov's paper [23] states that the quantity

$$\rho_{var}(\mu^{n*}, \mathbf{D}^*) := \inf_{\nu \in \mathbf{D}^*} \rho_{var}(\mu^{n*}, \nu) \tag{2}$$

tends to zero as $n \to \infty$ if $\mu \in \mathcal{M}$ is discrete or it has a nondegenerate absolutely continuous component.

Zaitsev [28] proved that there exist probability measures μ whose set of n-fold convolutions is uniformly separated from the set of infinitely divisible measures in the sense of the variation distance.

Let $\mu \in \mathcal{M}$, denote $\mu^{n\boxplus} := \mu \boxplus \cdots \boxplus \mu$ (n times). Recall that $\nu \in \mathcal{M}$ is $\boxplus$-infinitely divisible if, for every $n \in \mathbb{N}$, there exists $\nu_n \in \mathcal{M}$ such that $\nu = \nu_n^{n\boxplus}$. In the sequel we will write in this case that $\nu \in \mathbf{D}^{\boxplus}$.

Fix now $\mu, \nu \in \mathscr{M}$. We will say that μ belongs to the partial $*$-domain of attraction (resp., partial $\boxplus$-domain of attraction) of ν if there exist measures $\mu_1, \mu_2, \dots$ equivalent to μ, and natural numbers $k_1 < k_2 < \dots$ such that

$$\mu_n * \mu_n * \dots * \mu_n, \quad (k_n \text{ times}) \quad \Big(\text{resp.,} \quad \mu_n \boxplus \mu_n \boxplus \dots \boxplus \mu_n, \quad (k_n \text{ times})\Big)$$

converges weakly to ν as $n \to \infty$. Recall that μ_j and μ are equivalent if there exist real numbers a, b with $a > 0$, such that $\mu_j(S) = \mu(aS + b)$ for every $S \in \mathscr{B}$. Denote by $\mathscr{P}_*(\nu)$ (resp., $\mathscr{P}_\boxplus(\nu)$) the partial $*$-domain of attraction (resp., partial $\boxplus$-domain of attraction) of ν. Khinchin proved the following result for the classical convolution (for free convolution it was proved by Pata [22]).

A measure $\nu \in \mathscr{M}$ is $*$-infinitely divisible (resp., $\boxplus$-infinitely divisible) if and only if $\mathscr{P}_*(\nu)$ (resp., $\mathscr{P}_\boxplus(\nu)$) is not empty.

The next result is due to Bercovici and Pata [10] and is known as the Bercovici-Pata bijection.

There exists a bijection $\nu \leftrightarrow \nu'$ between $*$-infinitely divisible measures ν and $\boxplus$-infinitely divisible measures ν' such that $\mathscr{P}_*(\nu) = \mathscr{P}_\boxplus(\nu')$. More precisely, let $\mu_n \in \mathscr{M}$, let $k_1 < k_2 < \dots$ be positive integers, and set

$$\nu_n = \mu_n * \mu_n * \dots * \mu_n \quad (k_n \text{ times}), \quad \nu_n' = \mu_n \boxplus \mu_n \boxplus \dots \boxplus \mu_n \quad (k_n \text{ times}).$$

Then ν_n converges weakly to ν if and only if ν_n' converges weakly to ν'.

We return to Doeblin's result [15]. Using this result and the two last results about $\mathscr{P}_*(\nu)$ and $\mathscr{P}_\boxplus(\nu)$ we see that there exist free identically distributed random variables $X_1, X_2, \dots$ such that the distribution of the centered and normalized sum $b_{n_k}^{-1}(X_1 + \dots + X_{n_k} - a_{n_k})$ does not converge weakly to any nondegenerate distribution, whatever the choice of the constants a_n and b_n and of the sequence $n_1 < n_2 < \dots$.

Introduce the quantity

$$\rho_{var}(\mu^{n\boxplus}, \mathbf{D}^\boxplus) := \inf_{\nu \in \mathbf{D}^\boxplus} \rho_{var}(\mu^{n\boxplus}, \nu)$$

and raise the question of the behavior of this quantity when $n \to \infty$.

In the sequel we denote by $c(\mu), c_1(\mu)$ positive constants depending on μ only, while $c(\mu)$ is used to denote either generic constants for cases where we are not interested in particular values.

In order to formulate our main result we introduce the following notation

$$c_1(\mu) := \Im\Big(1 \Big/ \int_{\mathbb{R}} \frac{\mu(dt)}{i - t}\Big) - 1.$$

It is easy to see that $c_1(\mu) > 0$ if and only if $\mu \neq \delta_b$ with $b \in \mathbb{R}$, where δ_b denotes the Dirac measure concentrated at the point b.

Theorem 1. *Let $\mu \in \mathscr{M}$ and $c_1(\mu) > 0$. Then*

$$\rho_{var}(\mu^{n\boxplus}, \mathbf{D}^{\boxplus}) \le c(\mu)\Big(\frac{1}{\sqrt{n}} \int\limits_{[-N_n/8, N_n/8]} |u| \, \mu(du) + \mu(\mathbb{R} \setminus [-N_n/8, N_n/8])\Big), \quad n \in \mathbb{N}, \tag{3}$$

where $N_n := \sqrt{c_1(\mu)(n-1)}$.

It was proved in [6] that $\mu^{n\boxplus}$ is Lebesgue absolutely continuous when n is sufficiently large, provided that $\mu \ne \delta_b$ for any b. Therefore we immediately obtain from Theorem 1 a free analog of Prokhorov's result (2).

Corollary 1. *For $\mu \in \mathscr{M}$,*

$$\rho_{var}(\mu^{n\boxplus}, \mathbf{D}^{\boxplus}) \to 0, \quad as \quad n \to \infty.$$

In addition this corollary shows that in contrast to the classical case the approximation of n-fold free additive convolutions by free infinitely divisible probability measures can be shown in variation distance for all $\mu \in \mathscr{M}$.

Denote by $\mathscr{M}_d$, $d \ge 0$, the set of probability measures such that

$$\beta_d(\mu) := \int\limits_{\mathbb{R}} |x|^d \, \mu(dx) < \infty.$$

We easily obtain from Theorem 1 the following upper bound.

Corollary 2. *Let $\mu \in \mathscr{M}_d$ with some $d > 0$. Then*

$$\rho_{var}(\mu^{n\boxplus}, \mathbf{D}^{\boxplus}) \le c(\mu, d) n^{-\min\{d/2, 1/2\}}, \quad n \in \mathbb{N}, \tag{4}$$

where $c(\mu, d)$ denotes a constant depending on μ and d only.

From this corollary it follows that for all $\mu \in \mathscr{M}_1$ the order of approximation of $\mu^{n\boxplus}$ by free infinitely divisible measures is of order $n^{-1/2}$ in variation distance.

In the classical case there exist results with a rate of approximation in the Kolmogorov metric which depend on the number of existing moments. See, for example, the paper of Zaitsev [29].

Proof. Let $d_0 := \min\{1, d\}$ and $c_1(\mu) > 0$. We have

$$\int\limits_{[-N_n/8, N_n/8]} |u| \, \mu(du) \le \Big(\frac{N_n}{8}\Big)^{1-d_0} \int\limits_{[-N_n/8, N_n/8]} |u|^{d_0} \, \mu(du) \le \beta_{d_0}(\mu)\Big(\frac{c_1(\mu)n}{64}\Big)^{\frac{1-d_0}{2}}. \tag{5}$$

In addition

$$\mu(\mathbb{R} \setminus [-N_n/8, N_n/8]) \le \Big(\frac{8}{N_n}\Big)^{d_0} \int\limits_{|u| > N_n/8} |u|^{d_0} \, \mu(du) \le \beta_{d_0}(\mu)\Big(\frac{64}{c_1(\mu)(n-1)}\Big)^{\frac{d_0}{2}}. \tag{6}$$

Now we see that (4) follows immediately from (3), (5) and (6). □

3 Auxiliary Results

We shall need some results about some classes of analytic functions (see [1], Sect. 3, and [2], Sect. 6, §59).

Let $\mathbb{C}^+$ denote the open upper half of the complex plane. The class $\mathcal{N}$ (Nevanlinna, R.) denotes the class of analytic functions $f(z) : \mathbb{C}^+ \to \{z : \Im z \geq 0\}$. For such functions there is an integral representation

$$f(z) = a+bz+\int_{\mathbb{R}} \frac{1+uz}{u-z}\,\tau(du) = a+bz+\int_{\mathbb{R}} \Big(\frac{1}{u-z}-\frac{u}{1+u^2}\Big)(1+u^2)\,\tau(du) \quad (7)$$

for $z \in \mathbb{C}^+$, where $b \geq 0$, $a \in \mathbb{R}$, and τ is a nonnegative finite measure. Moreover, $a = \Re f(i)$ and $\tau(\mathbb{R}) = \Im f(i) - b$. From this formula it follows that

$$f(z) = (b + o(1))z \quad (8)$$

for $z \in \mathbb{C}^+$ such that $|\Re z|/\Im z$ stays bounded as $|z|$ tends to infinity (in other words $z \to \infty$ non-tangentially to $\mathbb{R}$). Hence if $b \neq 0$, then f has a right inverse $f^{(-1)}$ defined on the region

$$\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}^+ : |\Re z| < \alpha \Im z,\ \Im z > \beta\}$$

for any $\alpha > 0$ and some positive $\beta = \beta(f, \alpha)$.

A function $f \in \mathcal{N}$ admits the representation

$$f(z) = \int_{\mathbb{R}} \frac{\sigma(du)}{u-z}, \quad z \in \mathbb{C}^+, \quad (9)$$

where σ is a finite nonnegative measure, if and only if $\sup_{y\geq 1} |yf(iy)| < \infty$ and $\sigma(\mathbb{R}) = \lim_{y\to\infty} y \Im f(iy)$.

For $\mu \in \mathcal{M}$, define its Cauchy transform by

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{\mu(dt)}{z-t}, \qquad z \in \mathbf{C}^+. \quad (10)$$

The measure μ can be recovered from $G_\mu(z)$ as the weak limit of the measures

$$\mu_y(dx) = -\frac{1}{\pi}\Im G_\mu(x+iy)\,dx, \quad x \in \mathbb{R},\ y > 0,$$

as $y \downarrow 0$. If the function $\Im G_\mu(z)$ is continuous at $x \in \mathbb{R}$, then the probability distribution function $D_\mu(t) = \mu((-\infty, t))$ is differentiable at x and its derivative is given by

$$D'_\mu(x) = -\Im G_\mu(x)/\pi. \quad (11)$$

This inversion formula allows to extract the density function of the measure μ from its Cauchy transform.

Following Maassen [21] and Bercovici and Voiculescu [9], we shall consider in the following the *reciprocal Cauchy transform*

$$F_\mu(z) = \frac{1}{G_\mu(z)}. \tag{12}$$

The corresponding class of reciprocal Cauchy transforms of all $\mu \in \mathscr{M}$ will be denoted by $\mathscr{F}$. This class coincides with the subclass of Nevanlinna functions f for which $f(z)/z \to 1$ as $z \to \infty$ non-tangentially to $\mathbb{R}$. Indeed, reciprocal Cauchy transforms of probability measures have obviously such property. Let $f \in \mathscr{N}$ and $f(z)/z \to 1$ as $z \to \infty$ non-tangentially to $\mathbb{R}$. Then, by (8), f admits the representation (7) with $b = 1$. By (8) and (9), $-1/f(z)$ admits the representation (9) with $\sigma \in \mathscr{M}$.

The function $\phi_\mu(z) = F_\mu^{(-1)}(z) - z$ is called the Voiculescu transform of μ. It is not difficult to show that $\phi_\mu(z)$ is an analytic function on $\Gamma_{\alpha,\beta}$ and $\Im \phi_\mu(z) \le 0$ for $z \in \Gamma_{\alpha,\beta}$, where ϕ_μ is defined. Furthermore, note that $\phi_\mu(z) = o(z)$ as $|z| \to \infty$, $z \in \Gamma_{\alpha,\beta}$.

Voiculescu [27] showed that for compactly supported probability measures there exist unique functions $Z_1, Z_2 \in \mathscr{F}$ such that $G_{\mu_1 \boxplus \mu_2}(z) = G_{\mu_1}(Z_1(z)) = G_{\mu_2}(Z_2(z))$ for all $z \in \mathbb{C}^+$. Maassen [21] proved the similar result for probability measures with finite variance. Using Speicher's combinatorial approach [24] to freeness, Biane [11] proved this result in the general case.

Chistyakov and Götze [13], Bercovici and Belinschi [7] and Belinschi [8], proved, using methods from complex analysis, that there exist unique functions $Z_1(z)$ and $Z_2(z)$ in the class $\mathscr{F}$ such that, for $z \in \mathbb{C}^+$,

$$z = Z_1(z) + Z_2(z) - F_{\mu_1}(Z_1(z)) \quad \text{and} \quad F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z)). \tag{13}$$

The function $F_{\mu_1}(Z_1(z))$ belongs again to the class $\mathscr{F}$ and there exists a probability measure μ such that $F_{\mu_1}(Z_1(z)) = F_\mu(z)$, where $F_\mu(z) = 1/G_\mu(z)$ and $G_\mu(z)$ is the Cauchy transform as in (10).

Specializing to $\mu_1 = \mu_2 = \ldots = \mu_n = \mu$ write $\mu_1 \boxplus \cdots \boxplus \mu_n = \mu^{n\boxplus}$. The relation (13) admits the following consequence (see for example [13]).

Proposition 1. *Let $\mu \in \mathscr{M}$. There exists a unique function $Z_n(z) \in \mathscr{F}$ such that*

$$z = nZ_n(z) - (n-1)F_\mu(Z_n(z)), \quad z \in \mathbb{C}^+, \tag{14}$$

and $F_{\mu^{n\boxplus}}(z) = F_\mu(Z_n(z))$.

Using the last proposition we now state and prove some auxiliary results about the behavior of the function $Z_n(z)$.

From (14) we obtain the formula

$$Z_n^{(-1)}(z) = nz - (n-1)F_\mu(z) \tag{15}$$

for $z \in \Gamma_{\alpha,\beta}$ with some $\alpha, \beta > 0$. This equation provides an analytic continuation of the function $Z_n^{(-1)}(z)$ defined on $\mathbb{C}^+$. By (7), we have the following representation for the function $F_\mu(z)$

$$F_\mu(z) = c + z + \int_{\mathbb{R}} \frac{1 + uz}{u - z}\, \tau(du), \quad z \in \mathbb{C}^+, \tag{16}$$

where $c \in \mathbb{R}$, and τ is a nonnegative finite measure. Moreover, $c = \Re F_\mu(i)$ and $\tau(\mathbb{R}) = \Im F_\mu(i) - 1 = \Im(1/G_\mu(i)) - 1 = c_1(\mu)$.

Bercovici and Voiculescu [9] proved the following result.

Proposition 2. *A probability measure μ is $\boxplus$-infinitely divisible if and only if the function $\phi_\mu(z)$ has an analytic continuation defined on $\mathbb{C}^+$, with values in $\mathbb{C}^- \cup \mathbb{R}$, such that*

$$\lim_{y \to +\infty} \frac{\phi_\mu(iy)}{y} = 0. \tag{17}$$

It follows from Proposition 2 and (15), (16) that a probability measure ν_n such that $F_{\nu_n}(z) = Z_n(z), z \in \mathbb{C}^+$, is $\boxplus$-infinitely divisible.

The next lemma was proved in [13].

Lemma 1. *Let $g : \mathbb{C}^+ \to \mathbb{C}^-$ be analytic with*

$$\liminf_{y \to +\infty} \frac{|g(iy)|}{y} = 0. \tag{18}$$

Then the function $f : \mathbb{C}^+ \to \mathbb{C}$ defined via $z \mapsto z + g(z)$ takes every value in $\mathbb{C}^+$ precisely once. The inverse $f^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$ thus defined is in the class $\mathscr{F}$.

This lemma generalizes a result of Maassen [21] (see Lemma 2.3). Maassen proved Lemma 1 under the additional restriction $|g(z)| \le c(g)/\Im z$ for $z \in \mathbb{C}^+$, where $c(g)$ is a constant depending on g.

Denote $z = x + iy$, where $x, y \in \mathbb{R}$. Using the representation (16) for $F_\mu(z)$ we see that, for $\Im z > 0$,

$$\Im\Big(nz - (n-1)F_\mu(z)\Big) = y\Big(1 - (n-1)I_\mu(x, y)\Big),$$

where

$$I_\mu(x, y) := \int_{\mathbb{R}} \frac{(1 + u^2)\, \tau(du)}{(u - x)^2 + y^2}.$$

For every real fixed x, consider the equation

$$y\Big(1 - (n-1)I_\mu(x, y)\Big) = 0, \quad y > 0. \tag{19}$$

Since in the case $\tau(\mathbb{R}) \neq 0$ $y \mapsto I_\mu(x,y)$, $y > 0$, is positive and monotone, and decreases to 0 as $y \to \infty$, it is clear that the Eq. (19) has at most one positive solution. If such a solution exists, denote it by $y_n(x)$. Note that (19) does not have a solution $y > 0$ for any given $x \in \mathbb{R}$ if and only if $I_\mu(x,0) \leq 1/(n-1)$. Consider the set $S := \{x \in \mathbb{R} : I_\mu(x,0) \leq 1/(n-1)\}$. We put $y_n(x) = 0$ for $x \in S$. By Fatou's lemma, $I_\mu(x_0,0) \leq \liminf_{x\to x_0} I_\mu(x,0)$ for any given $x_0 \in \mathbb{R}$, hence the set S is closed. Therefore $\mathbb{R} \setminus S$ is the union of finitely or countably many intervals (x_k, x_{k+1}), $x_k < x_{k+1}$. The function $y_n(x)$ is continuous on the interval (x_k, x_{k+1}). Since the set $\{z \in \mathbb{C}^+ : n\Im z - (n-1)\Im F_\mu(z) > 0\}$ is open, we see that $y_n(x) \to 0$ if $x \downarrow x_k$ and $x \uparrow x_{k+1}$. Hence the curve γ_n given by the equation $z = x + iy_n(x)$, $x \in \mathbb{R}$, is a Jordan curve. In the case $\tau(\mathbb{R}) = 0$ we put $y_n(x) := 0$ for all $x \in \mathbb{R}$.

Consider the open domain $D_n := \{z = x + iy,\ x, y \in \mathbb{R} : y > y_n(x)\}$.

Lemma 2. *Let $Z_n(z)$ be the solution of the Eq. (14). The map $Z_n(z) : \mathbb{C}^+ \mapsto D_n$ is univalent. Moreover the function $Z_n(z)$, $z \in \mathbb{C}^+$, is continuous up to the real axis and it maps the real axis bicontinuously onto the curve γ_n.*

Proof. Using the formula (15) for $z \in \Gamma_{\alpha,\beta}$ with some $\alpha, \beta > 0$, we see that the function $Z_n^{(-1)}(z)$ has an analytic continuation defined on $\mathbb{C}^+$. In view of the representation (16) for the function $F_\mu(z)$, we note that $Z_n^{(-1)}(z) = z + g(z)$, $z \in \mathbb{C}^+$, where $g(z)$ is analytic on $\mathbb{C}^+$ and satisfies the assumptions of Lemma 1. By Lemma 1, we conclude that the function $Z_n^{(-1)}(z)$ takes every value in $\mathbb{C}^+$ precisely once. Moreover, as it is easy to see, $Z_n^{(-1)}(D_n) = \mathbb{C}^+$ and $\Re Z_n^{(-1)}(x + iy_n(x)) \to \pm\infty$ as $x \to \pm\infty$. The inverse $Z_n(z)$ gives us a conformal mapping of $\mathbb{C}^+$ onto D_n. By well-known results of the theory of analytic functions (see [16]), $Z_n(z)$ is continuous up to the real axis and it maps the real axis bicontinuously onto the curve γ_n. ☐

Lemma 3. *Let $c_1(\mu) > 0$ and let $Z_n(z)$ be the solution of the Eq. (14). Then the following lower bound holds*

$$|Z_n(z)| \geq \frac{1}{4}\sqrt{c_1(\mu)(n-1)}, \quad z \in \mathbb{C}^+, \quad n \geq c(\mu). \tag{20}$$

Proof. We shall prove that, for real x such that $|x| \leq \frac{1}{4}N_n = \frac{1}{4}\sqrt{c_1(\mu)(n-1)}$, the lower bound $y_n(x) > \frac{1}{2}N_n$ holds. Indeed, for $|x| \leq \frac{1}{4}N_n$ and $|u| \leq \frac{1}{4}N_n$, the inequality $(u-x)^2 + y_n^2(x) \leq \frac{1}{4}c_1(\mu)(n-1) + y_n^2(x) = \frac{1}{4}\tau(\mathbb{R})(n-1) + y_n^2(x)$ is valid. Therefore, using (19), we deduce the following chain of inequalities

$$\frac{1}{n-1}\int\limits_{[-N_n/4,N_n/4]} \frac{\tau(du)}{\frac{1}{4}\tau(\mathbb{R}) + \frac{1}{n-1}y_n^2(x)} \leq \int\limits_{[-N_n/4,N_n/4]} \frac{\tau(du)}{(u-x)^2 + y_n^2(x)}$$
$$\leq \int\limits_{\mathbb{R}} \frac{(1+u^2)\,\tau(du)}{(u-x)^2 + y_n^2(x)} \leq \frac{1}{n-1}. \tag{21}$$

Assume that there exists an $x_0 \in [\mathbb{N}_n/4, N_n/4]$ such that $0 \le y_n(x_0) \le N_n/2$. Then it follows from (21) that

$$\frac{\tau([-N_n/4, N_n/4])}{\frac{1}{4}\tau(\mathbb{R}) + \frac{1}{4}\tau(\mathbb{R})} \le 1. \qquad (22)$$

Since, for all sufficiently large $n \ge c(\mu)$, the lower bound $\tau([-N_n/4, N_n/4]) \ge \frac{3}{4}\tau(\mathbb{R})$ holds, we arrive at contradiction.

Finally note that the assertion of the lemma follows from Lemma 2. □

4 A Upper Bound in the Approximation Theorem

Proof of Theorem 1. By Proposition 1 there exists a unique function $Z_n(z) \in \mathscr{F}$ such that (14) holds and $G_{\mu^{n\boxplus}}(z) = G_\mu(Z_n(z))$, $z \in \mathbb{C}^+$. We have shown in Sect. 3 that the function Z_n satisfies $1/Z_n(z) = G_{\nu_n}(z)$, $z \in \mathbb{C}^+$, where ν_n is an $\boxplus$-infinitely divisible probability measure. Our aim is to estimate $\rho_{var}(\mu^{n\boxplus}, \nu_n)$ for all $n \in \mathbb{N}$. For any $z \in \mathbb{C}^+$, we may represent $G_{\mu^{n\boxplus}}(z)$ as

$$G_{\mu^{n\boxplus}}(z) = I_{n1}(z) + I_{n2}(z) := \Big(\int\limits_{[-N_n/8, N_n/8]} + \int\limits_{\mathbb{R}\setminus[-N_n/8, N_n/8]} \Big) \frac{\mu(du)}{Z_n(z) - u}. \qquad (23)$$

Since $Z_n(z) \in \mathscr{F}$, by (8), we have $Z_n(iy) = (1 + o(1))iy$ as $y \to \infty$. Therefore

$$-y\Im\frac{1}{Z_n(iy) - u} = y\frac{\Im Z_n(iy)}{|Z_n(iy) - u|^2} = 1 + o(1)$$

as $y \to \infty$ for all fixed $u \in \mathbb{R}$, and, by the inequality $\Im Z_n(iy) \ge y$, $y > 0$,

$$-y\Im\frac{1}{Z_n(iy) - u} \le \frac{y}{\Im Z_n(iy)} \le 1, \quad u \in \mathbb{R}, y > 0.$$

By Lebesgue's theorem, we easily deduce the relations

$$\lim_{y\to\infty}(-y\Im I_{n1}(iy)) = \mu([-N_n/8, N_n/8])$$

and

$$\lim_{y\to\infty}(-y\Im I_{n2}(iy)) = \mu(\mathbb{R} \setminus [-N_n/8, N_n/8]).$$

Therefore, by (9),

$$I_{nj}(z) = \int\limits_{\mathbb{R}} \frac{\sigma_{nj}(dt)}{z - t}, \quad z \in \mathbb{C}^+, \quad j = 1, 2,$$

where σ_{nj}, $j = 1, 2$, denote nonnegative measures such that $\sigma_{n1}(\mathbb{R}) = \mu([-N_n/8, N_n/8])$ and $\sigma_{n2}(\mathbb{R}) = \mu(\mathbb{R} \setminus [-N_n/8, N_n/8])$.

By Lemma 2, the map $Z_n(z) : \mathbb{C}^+ \mapsto D_n$ is univalent. Moreover the function $Z_n(z)$ is continuous on $\mathbb{C}^+ \cup \mathbb{R}$ and it maps $\mathbb{R}$ bicontinuously onto the curve γ_n. The function $F_\mu(z)$ admits the representation (16), where, by the assumption of the theorem, $\tau(\mathbb{R}) = c_1(\mu) > 0$.

By (20), we have

$$|Z_n(x + i\varepsilon) - u| \geq |Z_n(x + i\varepsilon)| - |u| \geq \frac{1}{8} N_n \tag{24}$$

for $x \in \mathbb{R}$, $\varepsilon \in (0, 1]$ and $u \in [-N_n/8, N_n/8]$. Therefore, by Lemmas 2 and (24), $\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \Im(1/(u - Z_n(x + i\varepsilon)))$ exists for every $x \in \mathbb{R}$, $u \in [-N_n/8, N_n/8]$, and this limit is a continuous probability density for every fixed $u \in [-N_n/8, N_n/8]$. By Lebesgue's theorem the measure σ_{n1} is absolutely continuous and its density $p_1(x)$ has the form

$$p_1(x) = \frac{1}{\pi} \int\limits_{[-N_n/8, N_n/8]} \lim_{\varepsilon \downarrow 0} \Im \frac{1}{u - Z_n(x + i\varepsilon)} \mu(du).$$

The probability measure ν_n is absolutely continuous as well with a density

$$p_2(x) = -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \Im \frac{1}{Z_n(x + i\varepsilon)}.$$

Since

$$\Im\Big(\frac{1}{Z_n(x + i\varepsilon) - u} - \frac{1}{Z_n(x + i\varepsilon)}\Big) = u \frac{2\Re Z_n(x + i\varepsilon) - u}{|Z_n(x + i\varepsilon) - u|^2} \Im \frac{1}{Z_n(x + i\varepsilon)},$$

we obtain, using (24),

$$\Big|\Im\Big(\frac{1}{Z_n(x + i\varepsilon) - u} - \frac{1}{Z_n(x + i\varepsilon)}\Big)\Big| \leq -c(\mu) \frac{|u|}{\sqrt{n}} \Im \frac{1}{Z_n(x + i\varepsilon)}$$

for all $x \in \mathbb{R}$, $u \in [-N_n/8, N_n/8]$ and $\varepsilon \in (0, 1]$. From this bound we conclude that

$$\int\limits_{\mathbb{R}} |p_1(x) - \mu([-N_n/8, N_n/8]) p_2(x)| \, dx$$
$$= \frac{1}{\pi} \int\limits_{\mathbb{R}} \Big| \int\limits_{[-N_n/8, N_n/8]} \lim_{\varepsilon \downarrow 0} \Im\Big(\frac{1}{u - Z_n(x + i\varepsilon)} + \frac{1}{Z_n(x + i\varepsilon)}\Big) \mu(du)\Big| \, dx$$

$$\le \frac{1}{\pi}\int_{\mathbb{R}}\int_{[-N_n/8,N_n/8]} \limsup_{\varepsilon\downarrow 0}\Big|\Im\Big(\frac{1}{u-Z_n(x+i\varepsilon)}+\frac{1}{Z_n(x+i\varepsilon)}\Big)\Big|\,\mu(du)\,dx$$

$$\le \frac{c(\mu)}{\sqrt{n}}\int_{-N_n/8}^{N_n/8}|u|\,\mu(du)\int_{\mathbb{R}}\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\Big|\Im\frac{1}{Z_n(x+i\varepsilon)}\Big|\,dx \le \frac{c(\mu)}{\sqrt{n}}\int_{-N_n/8}^{N_n/8}|u|\,\mu(du). \tag{25}$$

In view of the relation

$$\begin{aligned} var(\mu^{n\boxplus}-\nu_n) &= var(\sigma_{n1}+\sigma_{n2}-\nu_n)\\ &\le var(\sigma_{n1}-\mu([-N_n/8,N_n/8])\nu_n)+var\,\sigma_{n2}\\ &\quad+\mu(\mathbb{R}\setminus[-N_n/8,N_n/8])\,var\,\nu_n, \end{aligned}$$

we have

$$\begin{aligned} \rho_{var}(\mu^{n\boxplus},\nu_n) &\le \frac{1}{2}\int_{\mathbb{R}}|p_1(x)-\mu([-N_n/8,N_n/8])\,p_2(x)|\,dx\\ &\quad+\frac{1}{2}\sigma_{n2}(\mathbb{R})+\frac{1}{2}\mu(\mathbb{R}\setminus[-N_n/8,N_n/8])\\ &\le \frac{1}{2}\int_{\mathbb{R}}|p_1(x)-\mu([-N_n/8,N_n/8])\,p_2(x)|\,dx+\mu(\mathbb{R}\setminus[-N_n/8,N_n/8]). \end{aligned} \tag{26}$$

Note that the statement of the theorem now follows immediately from (25) and (26). □

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Accurate Approximation of Correlation Coefficients by Short Edgeworth-Chebyshev Expansion and Its Statistical Applications

Gerd Christoph, Vladimir V. Ulyanov, and Yasunori Fujikoshi

Abstract In Christoph, Prokhorov and Ulyanov (Theory Probab Appl 40(2):250–260, 1996) we studied properties of high-dimensional Gaussian random vectors. Yuri Vasil'evich Prokhorov initiated these investigations. In the present paper we continue these investigations. Computable error bounds of order $O(n^{-3})$ or $O(n^{-2})$ for the approximations of sample correlation coefficients and the angle between high-dimensional Gaussian vectors by the standard normal law are obtained. We give some numerical results as well. Moreover, different types of Bartlett corrections are suggested.

Keywords High-dimensional Gaussian random vectors • Sample correlation coefficient • Short Edgeworth-Chebyshev expansions • Computable error bound • Bartlett correction • Fisher transform

Mathematics Subject Classification (2010): Primary 62H10; Secondary 62E20

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1 Introduction

In the present paper we continue to study properties of high-dimensional Gaussian random vectors. We get new results for basic statistics connected with high-dimensional vectors. In Christoph, Prokhorov and Ulyanov [2] two-sided bounds were constructed for a probability density function $p(u, a)$ of a random variable $|Y - a|^2$, where Y is a Gaussian random element with zero mean in a Hilbert space H. The constructed bounds are sharp in the sense that starting from large enough u a ratio of upper bound to lower one equals 8 and does not depend on any parameters of a distribution of $|Y - a|^2$. The results hold for finite-dimensional space $H = \mathbf{R}^d$ as well provided that its dimension $d \geq 3$. In Kawaguchi, Ulyanov and Fujikoshi [8] geometric representation of N observations on n variables were studied. It is useful to describe asymptotic behavior of the following statistics:

- Length of n-dimensional observation vector,
- Distance between two independent observation vectors and
- Angle between these vectors.

In Hall, Marron and Neeman [6] the asymptotic distributions of these statistics were pointed out in a high-dimensional framework when the dimension n tends to infinity while the sample size N is fixed. In Kawaguchi, Ulyanov and Fujikoshi [8] we obtained the computable error bounds for approximations of the length and the distance. The aim of the present paper is to get a computable error bounds for the angle. Moreover, in order to construct the bounds we study approximations for the sample correlation coefficients. Assuming that $\mathbf{X}_1, \ldots, \mathbf{X}_N$ is a sample from a normal distribution $N(0, I_n)$ with zero mean and identity covariance matrix I_n. Hall, Marron and Neeman [6] showed that

$$\theta = \operatorname{ang}(\mathbf{X}_i, \mathbf{X}_j) = \frac{1}{2}\pi + O_p(n^{-1/2}), \quad i, j = 1, \ldots, N, \ i \neq j, \tag{1}$$

where O_p denotes the stochastic order. Since

$$\cos\theta = \frac{\|\mathbf{X}_i\|^2 + \|\mathbf{X}_j\|^2 - \|\mathbf{X}_i - \mathbf{X}_j\|^2}{2\,\|\mathbf{X}_i\|\ \|\mathbf{X}_j\|} = R_{ij},$$

where R_{ij} is the sample correlation coefficient for the vectors $\mathbf{X}_i$ and $\mathbf{X}_j$, the computable error bounds for θ will follow from computable error bounds for R_{ij}. Below we omit the indices i and j and write simply $R = R_{ij}$. There are many results about asymptotic properties of R, see e.g. Johnson, Kotz and Balakrishnan [7], Chap. 32. Some of the most precise approximations of the distributions of R and Fisher's normalizing and variance stabilizing z-transform

$$Z(R) = (1/2)\ln\{(1 + R)/(1 - R)\} \tag{2}$$

by short Edgeworth-Chebyshev expansions were suggested by Konishi [9]. The remainder terms have the order $O(n^{-3/2})$. The accuracy of the proposed approximations is examined comparing the normal short Edgeworth-Chebyshev expansions with the exact values due to David [4]. However, our paper is first one containing the computable error bounds of approximations.

The structure of the paper is the following. In Sect. 2 we consider the sample correlation coefficient and the angle between the involved vectors. In Sect. 3 some asymptotes for the constant factor with the Gamma-functions in the density function of the correlation coefficient are given. Computable error bounds of order $O(n^{-3})$ or $O(n^{-2})$ are constructed in Sect. 4 when the distributions of R or the angle between the vectors are approximated by short asymptotic expansions using one of the representations for the probability density of R. In Sect. 5 some Bartlett-type corrections are considered. A new transform of R similar to Fisher transform is constructed. This transform can be approximated by normal distribution up to order $O(n^{-2})$. In Sect. 6 we give an error bound also of order $O(n^{-2})$ as corollary of general results for scale-mixed distributions, see Fujikoshi, Ulyanov and Shimizu [5], Chap. 13, and the fact that $\sqrt{n-2}\, R \,/\, \sqrt{1-R^2}$ has Student's t-distribution with $n-2$ degrees of freedom. The last Sect. 7 contains the proofs.

2 Sample Correlation Coefficient and Angle Between Vectors

Let $\mathbf{X} = (X_1, \ldots, X_n)^T$, and $\mathbf{Y} = (Y_1, \ldots, Y_n)^T$ be two vectors from an n-dimensional normal distribution $N(0, I_n)$ with zero mean, identity covariance matrix I_n and the sample correlation coefficient

$$R = R(\mathbf{X}, \mathbf{Y}) = \frac{\sum_{k=1}^n X_k\, Y_k}{\sqrt{\sum_{k=1}^n X_k^2\ \sum_{k=1}^n Y_k^2}}.$$

The so-called null density function $p_R(r;n)$ of R is given in Johnson, Kotz and Balakrishnan [7], Chap. 32, formula (32.7):

$$p_R\,(r;n) = \frac{\Gamma((n-1)/2)}{\sqrt{\pi}\ \Gamma((n-2)/2)}\,\left(1\,-\,r^2\right)^{(n-4)/2}\ I_{(-1,\,1)}(r), \quad n \geq 5, \qquad (3)$$

where $I_A(x)$ denotes indicator function of set A.

R is two point distributed with $P(R = -1) = P(R = 1) = 1/2$ if $n = 2$ and it is U-shaped for $n = 3$ with density $p_R(r;3) = \Gamma^{-2}(1/2)\ (1-r^2)^{-1/2}\ I_{(-1,1)}(r)$. The sample correlation coefficient R is uniform for $n = 4$: $p_R(r;4) = 1/2\ I_{(-1,1)}(r)$. Moreover, for $n \geq 5$ the density function $p_R(r;n)$ is unimodal.

Consider now the standardized correlation coefficient $\overline{R} = \sqrt{n-c}\ \ R$ with some correcting real constant $c < n$ having density

$$p_{\overline{R}}(r;n,c) = \frac{\Gamma((n-1)/2)}{\sqrt{n-c}\,\sqrt{\pi}\,\Gamma((n-2)/2)}\left(1-\frac{r^2}{n-c}\right)^{(n-4)/2} I_{\{|r|<\sqrt{n-c}\}}(r), \quad (4)$$

which converges to the standard normal density $\varphi(r) = e^{-r^2/2} / \sqrt{2\pi}$, $r \in \mathbf{R}^1$ for $c = O(1)$ as $n \to \infty$ and by Konishi [9]

$$F_n^*(x) := P\left(\sqrt{n-2}\ R \le x\right) = \Phi(x) + \frac{1}{n-2}\left(-\frac{x}{4}+\frac{x^3}{4}\right)\varphi(x) + O(n^{-3/2}) \quad (5)$$

and

$$F_n(x) := P\left(\sqrt{n-2.5}\ R \le x\right) = \Phi(x) + \frac{1}{n-2.5}\,\frac{x^3}{4}\,\varphi(x) + O(n^{-3/2}). \quad (6)$$

as $n \to \infty$, where $\Phi(x) = \int_{-\infty}^x \varphi(r)dr$ is standard normal distribution function. Note that in Konishi [9] the sample size (in our case the dimension of vectors) is $n+1$ and our $c = 1+2\Delta$ with Konishi's correcting constant Δ. Moreover (5) and (6) are corollaries for independent components in the pairs $(X_k,\ Y_k)$, $k = 1,\ldots,n$ from the more general Theorem 2.2 in the mentioned paper.

Usually the asymptotic for $\overline{R}$ is (5), where $c = 2$ since it is related to the t-distributed statistic $\sqrt{n-2}\,R\,/\,\sqrt{1-R^2}$. With the correcting constant $c = 2.5$, one term in the asymptotic in (6) vanishes.

Let us consider now the connection between the correlation coefficient R and the angle θ of the involved vectors, defined in (1). For any fixed constant $c < n$, and arbitrary x with $|x|/\sqrt{n-c} < \pi/2$ we write for the angle $\theta : 0 < \theta < \pi$:

$$\begin{aligned} P\left(\sqrt{n-c}(\theta-\pi/2) \le x\right) &= P\left(\theta \le \pi/2 + x/\sqrt{n-c}\right) \\ &= P\left(\cos\theta \ge \cos(\pi/2 + x/\sqrt{n-c})\right) \\ &= P\left(R \ge -\sin(x/\sqrt{n-c})\right) \\ &= P\left(\sqrt{n-c}\ R \le \sqrt{n-c}\ \sin(x/\sqrt{n-c})\right) \quad (7) \end{aligned}$$

because R is symmetric and $P(R \le x) = P(-R \le x)$.

3 Some Preliminaries and Remarks

Before we calculate the error bounds in (5) and (6), we prove some estimates for the constant factor with the Gamma-functions in the density (4) which are of independent interest. Define for arbitrary correcting real constant $c < n$

$$A_n(c) := \frac{\sqrt{2}\ \Gamma((n-1)/2)}{\sqrt{n-c}\ \ \Gamma((n-2)/2)}, \quad A_n := A_n(2.5) \quad \text{and} \quad A_n^* := A_n(2). \quad (8)$$

Lemma 1. *For $n \geq 7$ we have with $c = 2$*

$$\left| A_n^* - 1 + \frac{1}{4\,(n-2)} - \frac{1}{32\,(n-2)^2} \right| \leq \frac{23}{360\,(n-2)^3}. \tag{9}$$

and with $c = 2.5$

$$\left| A_n - 1 - \frac{1}{16\,(n-2.5)^2} \right| \leq \frac{319}{2{,}880\,(n-2.5)^3}. \tag{10}$$

Remark 1. Stirling's formula, see Abramowitz and Stegun [1], formula (3.6.37), allows us to find the asymptotic of function A_n given in (8) with $c = 2.5$ as $n \to \infty$:

$$A_n = 1 + \frac{1}{16\,(n-2.5)^2} + \frac{1}{8\,(n-2.5)^3} + \frac{77}{512\,(n-2.5)^4} + O\left(n^{-5}\right).$$

Remark 2. Consider now an arbitrary c in (8), then we may obtain the following asymptotic behavior as $n \to \infty$:

$$A_n(c) = 1 + \frac{2c-5}{4\,(n-2)} + \frac{1 - 4(c-2) + 12(c-2)^2}{32\,(n-2)^2} + O\Big(n^{-3}\Big). \tag{11}$$

Only for $c = 5/2$ the term with $1/(n-2)$ in the asymptotic expansion (11) vanishes.

From (4) and (8) it follows that

$$p_{\overline{R}}(r; n, c) = \frac{A_n(c)}{\sqrt{2\pi}} \left(1 - \frac{r^2}{n-c}\right)^{(n-4)/2} I_{\{|r|<\sqrt{n-c}\,\}}(r). \tag{12}$$

Define

$$q_{\overline{R}}(r; n, c) := \frac{1}{\sqrt{2\pi}} \left(1 - \frac{r^2}{n-c}\right)^{(n-4)/2} I_{\{|r|<\sqrt{n-c}\,\}}(r). \tag{13}$$

Then we obtain

$$p_{\overline{R}}(r; n, c) - q_{\overline{R}}(r; n, c) = \frac{A_n(c) - 1}{\sqrt{2\pi}} \left(1 - \frac{r^2}{n-c}\right)^{(n-4)/2} I_{\{|r|<\sqrt{n-c}\,\}}(r) \tag{14}$$

and $A_n(c) - 1$ can be estimated with Lemma 1.

Remark 3. Equation (14) permits a non-uniform bound. Using $1 - z \leq e^{-z}$we find

$$\left(1 - \frac{r^2}{n-c}\right)^{(n-4)/2} \leq \exp\left\{-\frac{r^2\,(n-4)}{2\,(n-c)}\right\} \quad \text{for} \quad |r| < \sqrt{n-c}.$$

For $|r| \leq \sqrt{n-c}$ and $n > 4$ Lemma 1 leads to

$$\left| p_{\overline{R}}(r;n,2.5) - \left(1 + \frac{1}{16\,(n-2.5)^2}\right) q_{\overline{R}}(r;n,2.5) \right| \le \frac{319\; q_{\overline{R}}(r;n,2.5)}{2{,}880\,(n-2.5)^3} \qquad (15)$$

for $c = 2.5$, whereas in case of $c = 2$ we obtain

$$\left| p_{\overline{R}}(r;n,2) - \left(1 - \frac{1}{4\,(n-2)} + \frac{1}{32\,(n-2)^2}\right) q_{\overline{R}}(r;n,2) \right| \le \frac{23\, q_{\overline{R}}(r;n,2)}{360\,(n-2)^3}.$$

Finally let us consider upper and lower bounds for Mills' ratio with the standard normal law, which follow from formula (7.1.13) of Abramowitz and Stegun [1]:

$$\frac{2\,e^{-r^2/2}}{r + \sqrt{r^2+4}} \le \int_r^\infty e^{-t^2/2}\,dt \le \frac{2\,e^{-r^2/2}}{r + \sqrt{r^2 + 8/\pi}}. \qquad (16)$$

Using $E(Y^{2k}) = (2k-1)!!$ if Y is standard normal distributed and integrating by parts for $k = 6, 5, 4, 3, 2$ together with the lower bound of (16) for $k = 0$, we find

$$\int_0^A r^{2k}\varphi(r)\,dr = \int_0^\infty r^{2k}\,\varphi(r)\,dr - \int_A^\infty r^{2k}\,\varphi(r)\,dr \le \frac{(2k-1)!\,!}{2} - U_{2k}(A), \qquad (17)$$

where

$$U_{12}(A) = \Big(A^{11} + 11\,A^9 + 99\,A^7 + 693\,A^5 + 3{,}465\,A^3 + 10{,}395\,A$$
$$+\,20{,}790/(A + \sqrt{A^2+4})\Big)\,\varphi(A),$$
$$U_{10}(A) = \Big(A^9 + 9A^7 + 63\,A^5 + 315\,A^3 + 945\,A$$
$$+\,1{,}890/(A + \sqrt{A^2+4})\Big)\,\varphi(A),$$
$$U_8(A) = \Big(A^7 + 7\,A^5 + 35\,A^3 + 105\,A + 210/(A + \sqrt{A^2+4})\Big)\,\varphi(A) \quad \text{and}$$
$$U_6(A) = \Big(A^5 + 5\,A^3 + 15\,A + 30/(A + \sqrt{A^2+4})\Big)\,\varphi(A),$$

Usually, the integral on the left hand side of (17) will be estimated by the $2\,k$-th moment of the normal random variable. The term $U_{2k}(A)$ decreases exponentially fast, nevertheless it has a remarkable influence on the numerical values of the bounds for $A < 200$, for larger A its influence is not remarkable. Using (17) in Christoph and Ulyanov [3] we could significantly decrease the numerical constants in similar results for the standardized chi-squared distribution, obtained in Ulyanov, Christoph and Fujikoshi [11].

4 Main Results

First we prove an estimate for the standardized correlation coefficient with a second order Edgeworth-Chebyshev expansion, which leads to smaller numerical constants in the error bounds in (5) and (6). Define $Q_2(x) = (-3\,x^7 + 13\,x^5 + 2\,x^3 + 6\,x)\,\varphi(x)$.

Theorem 1. *Let R be the sample correlation coefficient with density (3). Then for any $n \geq 7$ and any $\lambda : 0 < \lambda < 1$ with $\lambda\sqrt{n-2.5} \geq 1.7$ we have*

$$\sup_x \left| P\left(\sqrt{n-2.5}\ R \leq x\right) - \Phi(x) - \frac{x^3\,\varphi(x)}{4\,(n-2.5)} - \frac{Q_2(x)}{96\,(n-2.5)^2} \right| \leq \frac{C_n(\lambda)}{(n-2.5)^3}, \tag{18}$$

where with $N = n - 2.5$ and $n_\lambda = \lambda\sqrt{N}$

$$C_n(\lambda) = \left(1 + \frac{1}{16\,N^2}\right)\Bigg[e^{9/(16N)+9/(32N^2)+\lambda^6/4}\Big(\frac{105 - 2U_8(n_\lambda)}{16} + \frac{945 - 2U_{10}(n_\lambda)}{20\,N\,(1-\lambda^2)}\Big)$$
$$+ \frac{1.148999}{2{,}304\ N}\Big(10{,}6785 - 32\,U_{12}(n_\lambda) + 144\,U_{10}(n_\lambda) - 162\,U_8(n_\lambda)\Big)$$
$$+ \frac{1}{384}\Big(2{,}161.560294 - U_{12}(n_\lambda) + 9\,U_{10}(n_\lambda) - 27\,U_8(n_\lambda) + 27\,U_6(n_\lambda)\Big)$$
$$+ \frac{1}{96}\Big(990.574299 - 4\,U_{10}(n_\lambda) + 21\,U_8(n_\lambda) - 27\,U_6(n_\lambda)\Big)$$
$$+ \frac{N^{3.5}}{\sqrt{2\pi}\,\lambda\,(N+1/2)}\left(1-\lambda^2\right)^{N/2+1/4} + \frac{2\,N^{2.5}\,\varphi(n_\lambda)\big(1 + 1/(16\,N^2)\big)}{\lambda\left(1 + \sqrt{1 + 8/(\pi\,\lambda^2 t)}\right)}$$
$$+ \frac{\lambda^3\,N^{2.5}\,\varphi(n_\lambda)}{4} + N\,\frac{|Q_2(n_\lambda)|}{96}\Bigg] + 0.062610 + 0.009614/N.$$

Theorem 1 leads to computable error bounds in (5) and (6) of order $O(n^{-2})$.

Theorem 2. *Let R be the sample correlation coefficient with density (3). Then for any $n \geq 7$ and any $\lambda : 0 < \lambda < 1$ with $\lambda\sqrt{n-2.5} \geq 1.7$ we have*

$$\sup_x \left| P\left(\sqrt{n-2.5}\ R \leq x\right) - \Phi(x) - \frac{x^3\,\varphi(x)}{4\,(n-2.5)} \right| \leq \frac{B_n(\lambda)}{(n-2.5)^2} \tag{19}$$

and

$$\sup_x \left| P\left(\sqrt{n-2}\ R \leq x\right) - \Phi(x) - \frac{(-x + x^3)\,\varphi(x)}{4\,(n-2)} \right| \leq \frac{B_n^*(\lambda)}{(n-2)^2}, \tag{20}$$

where $\quad B_n(\lambda) := 0.15372984 + C_n(\lambda)\,/\,(n-2.5)$ $\quad$ *and*

Table 1 Numerical values of $C_n(\lambda)$, $B_n(\lambda)$, $B_n^*(\lambda)$, $D_n(\lambda)$ and $D_n^*(\lambda)$ for some n

n	7	10	25	50	75	100	500	1000
λ	0.8854	0.8481	0.7609	0.6717	0.6071	0.5580	0.3103	0.2341
$C_n(\lambda)$	7.74551	14.64497	28.22116	26.87359	25.61820	25.00013	23.78460	23.66564
$B_n(\lambda)$	1.875004	2.106477	1.408088	0.719574	0.507169	0.410226	0.201623	0.177539
$B_n^*(\lambda)$	2.083101	2.298199	1.584401	0.891985	0.678381	0.580864	0.370904	0.346655
$D_n(\lambda)$	5.240261	5.471734	4.226421	2.705223	2.234399	2.011559	1.509904	1.449459
$D_n^*(\lambda)$	1.929972	2.161445	1.463056	0.774542	0.562137	0.465194	0.256591	0.232507

$$B_n^*(\lambda) := B_n(\lambda) + \frac{0.036471}{1-1/(2n-4)} + \frac{0.014454}{(1-1/(2n-4))^2} + \frac{0.114414}{(1-1/(2n-4))^{5/2}}.$$

Equation (7) shows the connection between the correlation coefficient R and the angle θ among the vectors involved.

Theorem 3. *Let θ be the angle between two vectors, defined in* (1). *Then for any $n \geq 7$ and any $\lambda : 0 < \lambda < 1$ with $\lambda\sqrt{n-2.5} \geq 1.7$ we have for $|x| \leq \pi\sqrt{n-2.5}/2$*

$$\sup_x \left| P\left(\sqrt{n-2.5}(\theta - \pi/2) \leq x\right) - \Phi(x) - \frac{x^3\,\varphi(x)}{12\,(n-2.5)}\right| \leq \frac{D_n(\lambda)}{(n-2.5)^2}, \tag{21}$$

where with $B_n(\lambda)$ given in Theorem 2

$$D_n(\lambda) = B_n(\lambda) + 1.084341 + \min\{2.280916,\ 0.151842 + 35.597236/(n-2.5)\}.$$

If only the domain $|x| \leq \pi\sqrt{n-2.5}/6$ is considered and the supremum in (21) *is taken in that interval, then $D_n(\lambda)$ may be replaced by $D_n^*(\lambda) = B_n(\lambda) +$* 0.054968.

Note that for $n \geq 20$ the second term in the $\min\{.,.\}$ in $D_n(x)$ is the smaller one.

In the Table 1 some numerical calculation performed by MAPLE are given. For fixed n we found optimal λ to calculate the constants $C_n(\lambda)$ by n^{-3} and then $B_n(\lambda)$, $B_n^*(\lambda)$, $D_n(\lambda)$ and $D_n^*(\lambda)$ by n^{-2} in the bounds in Theorems 1–3.

Remark 4. The relationship (7) leads to

$$\begin{aligned} P\left(\sqrt{n-c}\,(\theta - \pi/2) \leq x\right) &= P\left(R \leq \sin(x/\sqrt{n-c}\,)\right) \\ &= P\left(\sqrt{n-c}\ \arcsin(R) \leq x\right). \end{aligned}$$

Hence, Theorem 3 gives also an countable error bound for the arcsin-transform of the sample correlation coefficient $\arcsin(R)$, investigated in Konishi [9], formula (4.3) in the mentioned paper with an error term of $O(n^{-2})$.

Remark 5. Many authors used Fisher transform $Z(R)$, given in (2) to calculate quantiles of sample correlation coefficient R. The inverse to Z function is

$$Z^{-1}(u) = (e^{2u} - 1)/(e^{2u} + 1) = u - u^3/3 + O(|u|^5) \quad \text{as} \quad u \to 0.$$

It follows from Theorem 2 with $x = \sqrt{n-c}\, Z^{-1}(y/\sqrt{n-c})$ and Taylor expansion that with the correcting constants $c = 2.5$ or $c = 2$ as $n \to \infty$

$$P(\sqrt{n-c}\; Z(R) \le y) = \Phi(y) - \frac{(6(2.5-c)\, y + y^3)\, \varphi(y)}{12\,(n-2.5)} + O(n^{-2}), \quad (22)$$

which improves the error rate for the approximation of Fisher transform $Z(R)$ in Konishi [9], formula (4.2). Using (22) with $y = y(x) = \sqrt{n-c}\, Z(x/\sqrt{n-c})$ we find $P(\sqrt{n-c}\; R \le x) = \Phi_c\big(y(x), n\big) + O(n^{-2})$ as $n \to \infty$ and for $c = 2.5$

$$\Phi_{2.5}(y,n) := \Phi(y) - \frac{y^3\, \varphi(y)}{12\,(n-2.5)} \quad \text{with} \quad y = \sqrt{n-2.5}\, Z(x/\sqrt{n-2.5}) \quad (23)$$

or for $c = 2$

$$\Phi_2(y,n) := \Phi(y) - \frac{3y + y^3\, \varphi(y)}{12\,(n-2)} \quad \text{with} \quad y = \sqrt{n-2}\, Z(x/\sqrt{n-2}). \quad (24)$$

5 Bartlett Type Corrections

Assume the distribution function of some statistic S admits an asymptotic expansion

$$P(S \le x) = \Phi(x) + p_n(x)\, \varphi(x) + O(n^{-\alpha-1/2}) \quad \text{as} \quad n \to \infty, \quad (25)$$

where $p_n(x) = O(n^{-\alpha})$ is a polynomial usually with $\alpha = 1/2$ or 1. Then we understand under Bartlett type correction of S a monotone transformation T such that

$$P(T(S) \le x) = \Phi(x) + O(n^{-\alpha-1/2}) \quad \text{as} \quad n \to \infty. \quad (26)$$

The following elementary proposition gives this kind of transformations, see more advanced discussion in Sect. 5.7 in Fujikoshi, Ulyanov and Shimizu [5].

Theorem 4. *Let the distribution function of statistic S admit asymptotic expansion* (25) *and the function $x + p_n(x)$ be increasing. Then the transformation*

$$T(z) = z + p_n(z) \quad (27)$$

is a Bartlett type correction, i.e. (26) *holds for the distribution function of $T(S)$.*

The expansions (6) and (5) allow transformations like (27):

$$T(x) = x + \frac{1}{n-2.5}\,\frac{x^3}{4} \quad \text{or} \quad T_1(x) = x + \frac{1}{n-2}\,\frac{x^3 - x}{4}. \tag{28}$$

With the correcting constant $c = 2.5$ in (4) one term in the expansion (6) is removed by comparison with (5). It follows from Theorem 4 that

$$P(\sqrt{n-2.5}\,T(R) \le x) = \Phi(x) + O(n^{-2}) \text{ and } P(\sqrt{n-2}\,T_1(R) \le x) = \Phi(x) + O(n^{-2}).$$

A Fisher-like transform leads to another kind of Bartlett type correction. Consider the transformation

$$F(y) = \frac{1}{\sqrt{3}} \ln\left(\frac{1+\sqrt{3}\;y/2}{1-\sqrt{3}\;y/2}\right) \quad \text{for} \quad -1 \le y \le 1. \tag{29}$$

The function $F(y)$ is increasing and $F(y) = y + y^3/4 + O(|y|^5)$ as $y \to 0$. Having in mind (25) and (26), we may use our Fisher-like transform $F(y)$ also as Bartlett correction function.

Theorem 5. *Let R be sample correlation coefficient with density* (3). *Then as* $n \to \infty$

$$P(\sqrt{n-2.5}\;F(R)\;\le x) = \Phi(x) + O(n^{-2}), \tag{30}$$

and

$$P(\sqrt{n-2.5}\;R\;\le x) = \Phi(\sqrt{n-2.5}\;F(x/\sqrt{n-2.5})) + O(n^{-2}) \tag{31}$$

Remark 6. In difference to the distribution of $Z(R)$ in (22) the distribution of $F(R)$ in (30) may be approximated only by the normal law $\Phi(.)$ up to the order $O(n^{-2})$.

Remark 7. In this Section we showed how to apply Bartlett type correction in order to improve approximation. Another approach can be find in Niki and Konishi [10]. It is connected with the fact that approximate formulae using a large number of terms of Edgeworth asymptotic expansions for the distributions of statistics often produce spurious oscillations and give poor fits to the exact distribution functions in parts of the tails. A general method for suppressing these oscillations and leading to more accurate approximations see in Niki and Konishi [10], in particular for Fisher statistics based on sample correlation coefficient.

In order to compare the results some numerical calculation performed by MAPLE are given in Tables 2 and 4 for $c = 2.5$ and in Table 3 for $c = 2$.

Remark 8. The Tables 2–4 show the accuracy of asymptotic approximations to the exact values for the quantiles of the sample correlation coefficient, given in the columns A. The normal approximation (columns N) can remarkable be improved by adding the term by $1/n$ (columns $N1$) and the term by $1/n^2$ (columns $N2$) of the Edgeworth-Chebyshev expansions. The columns E and $E1$ of Table 3 show that the

Table 2 Numerical values of $A = P(\sqrt{n-2.5}R \leq x), N = \Phi(x), N1 = \Phi(x) + \frac{x^3\varphi(x)}{4(n-2.5)}$, $N2 = N1 + \frac{Q_2(x)}{96(n-2.5)^2}$, $E1 = \Phi_{2.5}(y(x), n)$, $U = \Phi(T(x))$ and $G = \Phi\left(\sqrt{n-2.5}\, F\left(\frac{x}{\sqrt{n-2.5}}\right)\right)$ with $n = 50$ for some values x, where $\Phi_{2.5}$, T and F are defined in (23), (28) and (29)

x	A	N	$N1$	$N2$	$E1$	U	G
0.1	0.539831	0.539828	0.539830	0.539831	0.539829	0.539830	0.539830
0.3	0.617969	0.617911	0.617966	0.617969	0.617966	0.617966	0.617966
1.0	0.842638	0.841345	0.842618	0.842638	0.842628	0.842615	0.842627
2.26147	0.990000	0.988135	0.990017	0.990003	0.989996	0.989893	0.989977
2.48824	0.995000	0.993581	0.995045	0.995004	0.995000	0.994905	0.994980
2.94093	0.999000	0.998363	0.999071	0.999003	0.999005	0.998947	0.998989

Table 3 Numerical values of $A = P(\sqrt{n-2}R \leq x), N = \Phi(x), N1 = \Phi(x) + \frac{(x^3-x)\varphi(x)}{4(n-2)}$, $E = \Phi(\sqrt{n-2}\, Z(x/\sqrt{n-2}))$ $E1 = \Phi_2(y(x), n)$ and $U = \Phi(T_1(x))$ with $n = 50$ for some values x, where Φ_2 and T_1 are defined in (24) and (28)

x	A	N	$N1$	E	$E1$	U
0.1	0.539624	0.539828	0.539623	0.539831	0.539623	0.539623
0.3	0.617370	0.617911	0.617369	0.617983	0.617369	0.617369
1.0	0.841358	0.841345	0.841345	0.843040	0.841354	0.841345
2.27334	0.990000	0.988497	0.989983	0.990878	0.990014	0.989902
2.50130	0.995000	0.993813	0.995010	0.995595	0.995016	0.994918
2.95636	0.999000	0.998444	0.999045	0.999207	0.999013	0.998949

Table 4 Numerical values of $A = P(\sqrt{n-2.5}R \leq x), N = \Phi(x), N1 = \Phi(x) + \frac{x^3\varphi(x)}{4(n-2.5)}$, $N2 = N1 + \frac{Q_2(x)}{96(n-2.5)^2}$, $E1 = \Phi_{2.5}(y(x), n)$, $U = \Phi(T(x))$ and $G = \Phi\left(\sqrt{n-2.5}\, F\left(\frac{x}{\sqrt{n-2.5}}\right)\right)$ with $n = 25$ for some values x, where $\Phi_{2.5}$, T and F are defined in (23), (28) and (29)

x	A	N	$N1$	$N2$	$E1$	U	G
0.1	0.539837	0.539828	0.539832	0.539837	0.539832	0.539832	0.539832
0.3	0.618041	0.617911	0.618026	0.618040	0.618026	0.618026	0.618026
1.0	0.844124	0.841345	0.844033	0.844123	0.844077	0.844018	0.844073
2.19256	0.99000	0.988135	0.990053	0.990025	0.989980	0.989547	0.989894
2.39629	0.995000	0.993581	0.995173	0.995038	0.995001	0.994601	0.994909
2.78844	0.999000	0.998363	0.999321	0.999039	0.999020	0.998775	0.998949

additional term by $1/n$ improves significant the accuracy of the Fisher transform. Moreover, the first order Fisher approximations and the given Bartlett corrections lead to approximations like the first order normal approximation in $N1$.

6 Estimates Followed from Approximations for Scale-Mixed Distributions

As it was noted in Introduction the random variable $\sqrt{n-2}R/\sqrt{1-R^2}$ has Student's t-distribution with $n-2$ degrees of freedom. At the same time the ratio

$$T_n = Z/\sqrt{\chi_n^2/n}$$

has t-distribution with n degrees of freedom, where Z is standard normal random variable, χ_n^2 is chi-squared random variable with n degrees of freedom and Z and χ_n^2 are independent. The ratio T_n can be considered as scale mixture of two distributions. Therefore, we can apply general theory of approximations for the distributions of scale mixtures, see Fujikoshi, Ulyanov and Shimizu [5], Sect. 13.2.

Let $G_n(x)$ be a distribution function of random variable T_n. It follows from Theorem 13.2.3 (cf. Example 13.2.1) in Fujikoshi, Ulyanov and Shimizu [5] that

$$\sup_x |G_n(x) - \Phi_{-1,4}(n,x)| \le \frac{6\,(n+4)}{n^3}, \tag{32}$$

where

$$\Phi_{-1,4}(n,x) = \Phi(x) - \Big(\frac{x^3+x}{4\,n} - \frac{x^5+2x^3+3x}{6\,n^2}\Big)\varphi(x).$$

Let us fix any natural $n > 2$ and put for $x : -1 < x < 1$,

$$g(x) = \frac{x\,\sqrt{n-2}}{\sqrt{1-x^2}}.$$

Since the function $g(x)$ is increasing, we have for any constant $c : c < n$

$$P\Big(\sqrt{n-c}\;R \le x\Big) = P\Big(g(R) \le g(x/\sqrt{n-c})\Big) = P\Big(T_{n-2} \le g(x/\sqrt{n-c})\Big).$$

Therefore, by (32) we get

$$\sup\Big|P\Big(\sqrt{n-c}\;R \le x\Big) - \Phi_{-1,4}(n-2, g(x/\sqrt{n-c}))\Big| \le \frac{6\,(n+2)}{(n-2)^3}. \tag{33}$$

Using (33), we can obtain results similar to Theorems 1 and 2. However, the upper bounds for errors of approximation, say M_n, will be worse comparing with right hand sides in the inequalities (19) and (20). In fact, according to (33) we shall have for all $n > 2$ that

$$(n-2)^2\, M_n \ge 6.$$

Compare it with values for $B_n(\lambda)$ in Table 1.

It is not surprising that (33) implies the worse result because in Theorems 1 and 2 we have used essentially the representation (4) and, in particular, the properties of Gamma-function while Theorem 13.2.3 in Fujikoshi, Ulyanov and Shimizu [5] is obtained for the general distributions of scale mixtures.

7 Proofs

Proof of Lemma 1. The error term estimations for asymptotic expansions of logarithm of Gamma function in Abramowitz and Stegun [1], formula (6.1.42), imply

$$\frac{1}{12\,x} - \frac{1}{360\,x^3} \le \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln(2\,\pi) \le \frac{1}{12\,x}, \quad x > 0. \tag{34}$$

Consider now for $x \ge 1$ the function

$$h(x) := \ln \frac{\Gamma(x)}{\Gamma(x-1/2)} - \left(x - \frac{1}{2}\right) \ln x + (x-1) \ln\left(x - \frac{1}{2}\right) + \frac{1}{2}. \tag{35}$$

Taking into account (34) and similar inequalities for the argument $x - 1/2$ we find

$$\begin{aligned} a(x) := \frac{1}{12\,x} - \frac{1}{360\,x^3} - \frac{1}{12\,(x-1/2)} \\ \le h(x) \le \frac{1}{12\,x} - \frac{1}{12\,(x-1/2)} + \frac{1}{360\,(x-1/2)^3} =: b(x). \end{aligned} \tag{36}$$

Using $\frac{1}{x} - \frac{1}{x-1/2} = -\frac{1}{2\,x\,(x-1/2)} = -\frac{1}{2\,(x-1/2)^2} + \frac{1}{4\,x\,(x-1/2)^2}$ we obtain for $x \ge 1$

$$a(x) = -\frac{1}{24\,(x-1/2)^2} + \frac{1}{48\,x\,(x-1/2)^2} - \frac{1}{360\,x^3} \tag{37}$$

and

$$b(x) = -\frac{1}{24(x-1/2)^2} + \frac{17}{720(x-1/2)^3} - \frac{1}{96\,x\,(x-1/2)^3}. \tag{38}$$

Remember some well-known inequalities where k is an integer:

$$0 \le -\ln(1-z) - z - \ldots - \frac{z^k}{k} \le \frac{z^{k+1}}{k+1} + \frac{z^{k+2}}{(k+2)\,(1-z)}, \quad 0 \le z < 1, \;\; k \ge 1, \tag{39}$$

$$0 \le \ln(1+z) - z + z^2/2 - z^3/3 + z^4/4 \le z^5/5, \quad 0 \le z < 1, \tag{40}$$

and for integer $k \ge 0$

$$0 \le \operatorname{sgn}^{k+1}(z)\left(e^z - 1 - z - \ldots - \frac{z^k}{k!}\right) \le \begin{cases} z^{k+1}\, e^z \,/\, (k+1)!, & z \ge 0 \\ (-z)^{k+1} \,/\, (k+1)!, & z < 0 \end{cases}. \tag{41}$$

Using (35) for $x = y + 1/2$ and $y > 1$ we define the function

$$g(y) := h(y+1/2) - \ln\frac{\Gamma(y+1/2)}{\Gamma(y)} + \frac{1}{2}\ln(y) \;=\; \frac{1}{2} - y\ln\left(1 + \frac{1}{2\,y}\right).$$

The inequalities (40) for $z = 1/(2y)$ lead to upper and lower bounds for $g(y)$:

$$-\frac{1}{160\,y^4} \le g(y) - \frac{1}{8\,y} + \frac{1}{24\,y^2} - \frac{1}{64\,y^3} \le 0, \quad y > 1. \tag{42}$$

Next we are going to estimate the function

$$R(y) := h(y+1/2) - g(y) \;=\; \ln\frac{\Gamma(y+1/2)}{\Gamma(y)} - \frac{1}{2}\ln(y).$$

Suppose $m := n-2 \ge 5$. Using (36)–(38) and (42) with $y = x - 1/2 = m/2$ and

$$\frac{1}{6\,(m+1)\,m^2} - \frac{1}{45\,(m+1)^3} - \frac{1}{8\,m^3} = \frac{7u^3 + 90u^2 + 300u + 80}{360(m+1)^3 m^3} > 0, \;\; u = m-5,$$

to obtain the lower bound we find

$$-1 < -\frac{1}{4\,m} \le R(m/2) \le -\frac{1}{4\,m} + \frac{23}{360\,m^3} < 0. \tag{43}$$

Since $A_n^* = e^{R((n-2)/2)} = e^{R(m/2)}$ with $-1 < R(m/2) < 0$ we find $A_n^* < 1$ and define

$$r_1(m) := e^{R(m/2)} - 1 - R(m/2) - R^2(m/2)/2 \;\text{ and }\; r_2(m) := R(m/2) + 1/(4\,m).$$

Making use of (41) with $k = 2$ for $-1 < z < 0$ and (43) we find

$$\frac{-1}{384\,m^3} \le r_1(m) \le 0, \;\; 0 \le r_2(m) \le \frac{23}{360\,m^3}, \;\; \frac{-23}{1{,}440\,m^4} \le \frac{R^2(m/2)}{2} - \frac{1}{32\,m^2} \le 0,$$

which lead to (9) for $m = n-2 \ge 5$.

Let now $c = 5/2$ and put $N = n - 2.5$. Note $A_n = \sqrt{1 + 1/(2N)}\, A_n^*$, then by (9)

$$\left| A_n - \left(1 + \frac{1}{2N}\right)^{1/2} + \frac{1}{4N}\left(1 + \frac{1}{2N}\right)^{-1/2} - \frac{1}{32\,N^2}\left(1 + \frac{1}{2N}\right)^{-3/2} \right| \leq \frac{23}{360\,N^3}.$$

The binomial series $(1 + x)^\alpha$ for $0 \leq x \leq 1$ and $\alpha \in \{-3/2, -1/2, 1/2\}$, see Abramowitz and Stegun [1], formula (3.6.9), imply

$$-\frac{3}{4\,N} \leq \left(1 + \frac{1}{2\,N}\right)^{-3/2} - 1 \leq 0, \quad 0 \leq \left(1 + \frac{1}{2\,N}\right)^{-1/2} - 1 - \frac{1}{4\,N} \leq \frac{1}{32\,N^2}$$

and

$$0 \leq \left(1 + \frac{1}{2\,N}\right)^{1/2} - 1 - \frac{1}{4\,N} + \frac{1}{32\,N^2} \leq \frac{1}{128\,N^3}.$$

Hence (10) holds for $n \geq 7$. □

Proof of Theorem 1. Let $F_n(x)$ be the distribution function of the standardized correlation coefficient $\overline{R}$ having density (4) with $c = 2.5$, see (6). Put

$$\Phi_n(x) := \Phi(x) + \varphi(x)\left(\frac{x^3}{4\,(n - 2.5)} + \frac{-3x^7 + 13x^5 + 2x^3 + 6x}{96\,(n - 2.5)^2}\right).$$

Our aim is to estimate $F_n(x) - \Phi_n(x)$ with an error having the order $C/(n - 2.5)^3$. Note $F_n(0) - \Phi_n(0) = 0$, therefore we suppose $x \neq 0$. Moreover, we may consider only case $x > 0$ since $p_{\overline{R}}(r; n, 2.5)$, $q_{\overline{R}}(r; n, 2.5)$ and $\varphi(r)$ are symmetric functions, hence $|F_n(x) - \Phi_n(x)| = |F_n(-x) - \Phi_n(-x)|$.

Using (13) define for $x > 0$ with $N = n - 2.5$

$$H_n(-x) = 1 - H_n(x) := \left(1 + \frac{1}{16\,N^2}\right) \int_x^{\sqrt{N}} q_{\overline{R}}(r; n, 2.5)\, dr.$$

Then we have

$$|F_n(x) - \Phi_n(x)| \leq |F_n(x) - H_n(x)| + |H_n(x) - \Phi_n(x)|. \tag{44}$$

For $0 \leq x \leq \sqrt{N}$ with (15), (12) and (10) we find $A_n \geq 1$ and

$$\left| F_n(x) - H_n(x) \right| \leq \left| \int_x^{\sqrt{N}} \left(p_{\overline{R}}(r; n, 2.5) - \left(1 + \frac{1}{16\,N^2}\right) q_{\overline{R}}(r; n, 2.5) \right) dr \right|$$
$$\leq \frac{319}{2{,}880\,N^3} \frac{2\,A_n}{2\,A_n\,\sqrt{2\pi}} \int_0^{\sqrt{N}} \left(1 - \frac{r^2}{N}\right)^{N/2 - 3/4} dr \leq \frac{319}{5{,}760\,A_n\,N^3} \leq \frac{319}{5{,}760\;N^3}. \tag{45}$$

Now we have to estimate $|H_n(x) - \Phi_n(x)|$. Define $\overline{Q_2}(x) := Q_2(x) + 6(1 - \Phi(x))$,

$$\varphi_n(x) := \frac{d}{dx}\Phi_n(x) = \varphi(x)\Big(1 - \frac{x^4 - 3x^2}{4\,N} + \frac{3x^8 - 34x^6 + 63x^4 + 6}{96\,N^2}\Big)$$

and $\overline{\varphi_n}(x) := \varphi_n(x) - \varphi(x)\,/\,(16\,N^2)$. Then we obtain for $x > 0$

$$|H_n(x) - \Phi_n(x)| \le K_1 + \frac{1}{16\,N^2}\,K_2 \le K_1\Big(1 + \frac{1}{16\,N^2}\Big) + \frac{0.462541}{64\,N^3} + \frac{14.766155}{1{,}536\,N^4}, \tag{46}$$

where

$$K_1 := \left|\int_x^\infty (q_{\overline{R}}(r;n) - \overline{\varphi_n}(r))dr\right|, \quad K_2 = \left|\int_x^\infty (q_{\overline{R}}(r;n) - \varphi(r))dr\right| \le K_1 + K_3$$

$$K_3 = \left|\int_x^\infty \varphi(x)\Big(-\frac{x^4 - 3x^2}{4\,N} + \frac{3x^8 - 34x^6 + 63x^4}{96\,N^2}\Big)dr\right| \le \frac{x^3\varphi(x)}{4\,N} + \frac{|\overline{Q_2}(x)|}{96\,N^2}$$

$$\sup_{x>0} x^3\varphi(x) = \frac{(3/e)^{3/2}}{\sqrt{2\,\pi}} \le 0.462541 \quad \text{and} \quad \sup_{x>0}|\overline{Q_2}(x)| = \overline{Q_2}(3) \le 14.766155.$$

Now we have to estimate $K_1 \le J_1 + J_2 + J_3$, where with $\lambda \in (0,\ 1)$

$$J_1 := I_{[0,\,\lambda\,\sqrt{N}\,)}(x)\ \int_x^{\lambda\sqrt{N}}\left|\frac{1}{\sqrt{2\pi}}\Big(1 - \frac{r^2}{N}\Big)^{N/2-3/4} - \overline{\varphi_n}(r)\right|\,dr,$$

$$J_2 := \int_{\lambda\sqrt{N}}^{\sqrt{N}} \frac{1}{\sqrt{2\pi}}\Big(1 - \frac{r^2}{N}\Big)^{N/2-3/4}\,dr,$$

$$J_3 := \left|\int_{\lambda\sqrt{N}}^{\infty}\overline{\varphi_n}(r)\,dr\right| = |1 - \overline{\Phi_n}(\lambda\sqrt{N})|$$

and $\overline{\Phi_n}(x) = \Phi_n(x) + (1 - \Phi(x))/(16\,N^2)$. Substituting $u^2 = r^2/N$ we find

$$J_2 = \frac{\sqrt{N}}{\sqrt{2\pi}}\int_\lambda^1 \frac{u}{u}\Big(1 - u^2\Big)^{N/2-3/4}\,du \le \frac{\sqrt{N}}{\sqrt{2\pi}\,\lambda\,(N + 1/2)}\Big(1 - \lambda^2\Big)^{N/2+1/4}.$$

Using the second inequality of (16) to estimate $1 - \Phi(\lambda\sqrt{N})$ we find

$$J_3 \le \frac{2\,\varphi(\lambda\sqrt{N})\big(1 + 1/(16\,N^2)\big)}{\lambda\sqrt{N}\Big(1 + \sqrt{1 + 8/\big(\pi\,\lambda^2 t\big)}\Big)} + \frac{(\lambda\,\sqrt{N})^3\,\varphi(\lambda\sqrt{N})}{4\,N} + \frac{Q_2(\lambda\,\sqrt{N})|}{96\,N^2}.$$

Let now $0 < x \le \lambda\sqrt{N}$. To estimate J_1 we suppose $0 < r \le \lambda\sqrt{N}$ and define

$$a_1(r) = \frac{N}{2}\left[\ln\left(1-\frac{r^2}{N}\right)+\frac{r^2}{N}+\frac{r^4}{2\,N^2}+\frac{r^6}{3\,N^3}\right]$$
$$-\frac{3}{4}\left[\ln\left(1-\frac{r^2}{N}\right)+\frac{r^2}{N}+\frac{r^4}{2\,N^2}\right],$$
$$a_2(r) = -\,N^{-1}\big(r^4/4-3\,r^2/4\big) \quad \text{and} \quad a_3(r) = -\,N^{-2}\big(r^6/6-3\,r^4/8\big).$$

Then we have $\overline{\varphi_n}(r) = \varphi(r)\big(1+a_2(r)+a_2^2(r)/2+a_3(r)\big)$ and

$$\big(1-r^2/N\big)^{N/2-3/4} = e^{-r^2/2+a_1(r)+a_2(r)+a_3(r)}$$
$$= e^{-r^2/2}\Big[e^{a_2(r)+a_3(r)}\big(e^{a_1(r)}-1\big)+e^{a_2(r)}\big(e^{a_3(r)}-1-a_3(r)\big)$$
$$+e^{a_2(r)}\big(1+a_3(r)\big)\Big].$$

Using (41), $a_k^+ := \max(0,\,a_k)$ and $a_k^- := \max(0,\,-a_k)$, $k=1,2,3$, we find

$$J_1 \le \int_0^{\lambda\sqrt{N}} \varphi(r)\big|e^{a_1(r)+a_2(r)+a_3(r)} - \big(1+a_2(r)+a_2^2(r)/2+a_3(r)\big)\big|dr \le \sum\nolimits_{k=1}^4 J_{1,k},$$

where

$$J_{1,1} := \int_0^{\lambda\sqrt{N}} \varphi(r)e^{a_2(r)+a_3(r)}\big|e^{a_1(r)}-1\big|dr \le \int_0^{\lambda\sqrt{N}} \varphi(r)\big|a_1(r)\big|e^{a_2(r)+a_3(r)+a_1^+(r)}dr,$$
$$J_{1,2} := \int_0^{\lambda\sqrt{N}} \varphi(r)e^{a_2(r)}\big|e^{a_3(r)}-1-a_3(r)\big|\,dr \le \int_0^{\lambda\sqrt{N}} \varphi(r)\frac{a_3^2(r)}{2}\,e^{a_2(r)+a_3^+(r)}dr,$$
$$J_{1,3} := \int_0^{\lambda\sqrt{N}} \varphi(r)\Big|e^{a_2(r)}-1-a_2(r)-\frac{a_2^2(r)}{2}\Big|\,dr \le \int_0^{\lambda\sqrt{N}} \varphi(r)\frac{|a_2^3(r)|}{6}e^{a_2^+(r)}dr$$

and

$$J_{1,4} := \int_0^{\lambda\sqrt{N}} \varphi(r)\big|\big(e^{a_2(r)}-1\big)a_3(r)\big|\,dr \le \int_0^{\lambda\sqrt{N}} \varphi(r)\big|a_2(r)\,a_3(r)\big|e^{a_2^+(r)}dr.$$

Let $N \ge 4.5$ and $0 < r \le \lambda\sqrt{N}$. It follows from (39) with $y = r^2/N$ that

$$-\underline{a_1}(r) := -\left(\frac{r^8}{8\,N^3}+\frac{r^{10}}{10\,N^4\,(1-\lambda^2)}\right) \le a_1(r) \le \frac{r^6}{4\,N^3}$$
$$+\frac{3\,r^8}{16\,N^4\,(1-\lambda^2)} =: \overline{a_1}(r),$$

$$\int_0^s \varphi(r)\overline{a_1}(r)\,dr \le \int_0^s \varphi(r)\underline{a_1}(r)\,dr \quad \text{for } s \ge 1.7 \quad \text{and}$$
$$a_1^+(r) \le r^6/(4\,N^3) \le \lambda^6/4.$$

For $r > 0$ the functions $a_2(r)$ and $a_3(r)$ take both their only maximum at $r = \sqrt{3/2}$,

$$a_2(r) = \frac{3\,r^2 - r^4}{4\,t} \begin{cases} \le 0, & r \ge \sqrt{3} \\ > 0, & r < \sqrt{3} \end{cases} \quad \text{with} \quad a_2^+(r) \le \begin{cases} 0, & r \ge \sqrt{3} \\ 9/(16\,N), & r < \sqrt{3} \end{cases}$$

and

$$a_3(r) = \frac{9r^4 - 4r^6}{24\,t^2} \begin{cases} \le 0, & r \ge 3/2 \\ > 0, & r < 3/2 \end{cases} \quad \text{with} \quad a_3^+(r) \le \begin{cases} 0, & r \ge 3/2 \\ 9/(32N^2), & r < 3/2 \end{cases}.$$

Then we find with $e^{-a_k^-(r)} \le 1$, (17) and the moments $E(Y^4) = 3$, $E(Y^6) = 15$, $E(Y^8) = 105$, $E(Y^{10}) = 945$ and $E(Y^{12}) = 10{,}395$ if Y is standard normal distributed

$$J_{11} \le e^{9/(16\,N)+9/(32\,N^2)+\lambda^6/4} \int_0^{\lambda\sqrt{N}} \varphi(r)\left(\frac{r^8}{8N^3} + \frac{r^{10}}{10\,N^4\,(1-\lambda^2)}\right)dr$$
$$\le e^{9/(16\,N)+9/(32\,N^2)+\lambda^6/4}\left(\frac{105 - 2\,U_8(\lambda\sqrt{N})}{16\,N^3} + \frac{945 - 2\,U_{10}(\lambda\sqrt{N})}{20\,N^4\,(1-\lambda^2)}\right),$$
$$J_{12} \le \frac{e^{9/(16\cdot 4.5)+9/(32\cdot 4.5^2)}}{1{,}152\;N^4} \int_0^{\lambda\sqrt{N}} \varphi(r)\,\left(16\,r^{12} - 72\,r^{10} + 81\,r^8\right)\,dr$$
$$\le \frac{1.14899}{2{,}304\;N^4}\left(106{,}785 - 32\,U_{12}(\lambda\sqrt{N}) + 144\,U_{10}(\lambda\sqrt{N}) - 162\,U_8(\lambda\sqrt{N})\right),$$

with $a_2(r) \le 0$ only for $0 \le r \le \sqrt{3}$ and $\left(r^4 - 3\,r^2\right)^3 = r^{12} - 9r^{10} + 27r^8 - 27r^6$

$$J_{13} \le \frac{1}{384\,N^3}\Bigg(\int_0^{\lambda\sqrt{N}} \varphi(r)\left(r^4 - 3\,r^2\right)^3 dr + \left(1 + e^{9/(16\cdot 4.5)}\right)$$
$$\int_0^{\sqrt{3}} \varphi(r)\left(3\,r^2 - r^4\right)^3 dr\Bigg)$$
$$\le \frac{1}{384\,N^3}\Big(2{,}160 - U_{12}(\lambda\sqrt{N}) + 9\,U_{10}(\lambda\sqrt{N}) - 27\,U_8(\lambda\sqrt{N})$$
$$+27\,U_6(\lambda\sqrt{N}) + 2.937248\Big)$$

and with $a_2(r)\,a_3(r) = (96\,N^3)^{-1}\,(4r^{10} - 21\,r^8 + 27r^6) \le 0$ only for $3/2 \le r \le \sqrt{3}$

$$J_{14} \le \Big(\int_0^{\lambda\sqrt{N}} \varphi(r)\, a_2(r) a_3(r)\ dr + \big(e^{9/(16\cdot 4.5)} - 1\big) \int_0^{3/2} \varphi(r)\, a_2(r)\, a_3(r)\ dr$$
$$+ \big(1 + e^{9/(16\cdot 4.5)}\big) \int_{3/2}^{\sqrt{3}} \varphi(r) \big(-\ a_2(r)\, a_3(r)\big)\ dr \Big)$$
$$\le \frac{1}{96\, N^3} \Big(990 - 4\, U_{10}(\lambda\sqrt{N}) + 21\, U_8(\lambda\sqrt{N}) - 27\, U_6(\lambda\sqrt{N}) + 0.574299 \Big).$$

Hence, J_1 and also K_1 are estimated. Taking estimates (44)–(46) together, we obtain (18). □

Proof of Theorem 2 The first bound (19) follows immediately from Theorem 1 and $\sup_{x>0} |Q_2(x)| \le 14.758064$. To prove (20) we use (19) and Taylor expansion. As in the proof of Theorem 1 we may suppose $x > 0$. Here we have

$$F_n^*(x) = P(\sqrt{n-2}\, R \le x) = P\Big(\sqrt{n-2.5}\, R \le x\, \sqrt{1 - 1/(2n-4)}\Big) = F_n(y)$$

with $y = x\, \sqrt{1 - 1/(2n-4)}$. The bound (19) leads to

$$\sup_{y>0} \left| F_n(y)\ -\ \Phi(y) - \frac{y^3\, \varphi(y)}{4\,(n-2.5)} \right| \le \frac{B_n(\lambda)}{(n-2.5)^2}. \tag{47}$$

Put $M = 2\,(n-2) = 2\,n - 4$. Consider now the Taylor expansions

$$\Phi(y) = \Phi(x) - \varphi(x)\,(x-y) + \varphi'(z)\,(y-x)^2/2 \quad \text{with} \quad 0 < y < z < x,$$
$$\varphi(x)\,(x-y) = x\,\varphi(x)/(2M) + R_1(n) \quad \text{and} \quad x - y = x\ (1 - \sqrt{1 - 1/M}),$$

where

$$R_1(n) := \varphi(x)\big(x - y - x/(2M)\big) = x\,\varphi(x)\,\big(1 - \sqrt{1 - 1/M}\ - 1/(2M)\big)$$

and

$$R_2(n) := |\varphi'(z)|\,(y-x)^2/2 = z^3\,\varphi(z)\,\big(x/z\big)^2\,\big(1 - \sqrt{1 - 1/M}\ - 1/(2M)\big).$$

Formula (3.6.9) in Abramowitz and Stegun [1] implies

$$0 \le 1 - \sqrt{1 - 1/M}\ - 1/(2M) \le 1\big/\big(8\, M^2\,(1 - 1/M)\big) = 1\big/\big(8\, M\,(M-1)\big)$$

and

$$0 \le 1 - \sqrt{1 - 1/M} \le 1\big/\big(2\, M\,(1 - 1/M)\big) = 1\big/\big(2\,(M-1)\big)$$

Hence, with $(x/z)^2 \le (x/y)^2 = (1 - 1/M)^{-1}$ we obtain

$$R_1(n) \le \frac{e^{-1/2}}{\sqrt{2\pi}\; 8\,M^2\,(1-1/M)} \quad \text{and} \quad R_2(n) \le \frac{(3/e)^{3/2}}{\sqrt{2\pi}\; 8\,M^2\,(1-1/M)^2}. \tag{48}$$

Using $y = x\sqrt{1-1/M}$, $n-2.5 = (n-2)(1-1/M)$ and the Taylor expansion $\varphi(y) = \varphi(x) + \varphi'(z)\,(y-x)$ with $0 < y < z < x$, we find

$$\frac{y^3\,\varphi(x)}{4\,(n-5/2)} = \frac{x^3\,\varphi(x)}{4\,(n-2)}\sqrt{1-1/M} = \frac{x^3\,\varphi(x)}{4\,(n-2)} - R_3(n)$$

with

$$R_3(n) := \frac{x^3\,\varphi(x)}{2\,M}\left(1-\sqrt{1-1/M}\right) \le \frac{(3/e)^{3/2}}{\sqrt{2\pi}\; 4\,M^2\,(1-1/M)}. \tag{49}$$

It remains to estimate

$$\begin{aligned} R_4(n) &:= \frac{y^3\,\varphi'(z)\,(x-y)}{4\,(n-5/2)} = \frac{y^3\,z\,\varphi(z)\,x\left(1-\sqrt{1-1/M}\right)}{2(M-1)} \\ &\le \frac{z^5\,\varphi(z)\left(1-\sqrt{1-1/M}\right)}{\sqrt{1-1/M}\;2(M-1)} \le \frac{(5/e)^{5/2}}{4\,M^2\,(1-1/M)^{5/2}}. \end{aligned} \tag{50}$$

Taking (47)–(50) together we obtain (20). □

Proof of Theorem 3 Define $N = \sqrt{n-2.5}$ and $h(x) = \sqrt{N}\,\sin(x/\sqrt{N})$. Starting from (7), we have to prove (21). Considering (7) and that R is symmetric and $\sin(x)$ is an odd function, we may limit us to the case $x > 0$. In order to get smaller constants we use both Taylor expansions

$$\Phi(h(x)) = \Phi(x) + \varphi(x)\,(h(x)-x) + \varphi'(x)\,(h(x)-x)^2/2 + \varphi''(z)\,\big(h(x)-x\big)^3/6$$

or

$$\Phi(h(x)) = \Phi(x) + \varphi(x)\,(h(x)-x) + \varphi'(z)\,(h(x)-x)^2/2, \quad 0 < h(x) < z < x. \tag{51}$$

Using $\left|\sqrt{N}\,\sin(x/\sqrt{N}) - x + x^3\big/(6\,N)\right| \le x^5\big/(120\,N^2)$, we find

$$\varphi(x)\,(h(x)-x) = x^3\big/(6\,N) + S_1(n),$$

where

$$S_1(n) := \varphi(x)\left|\sqrt{N}\,\sin(x/\sqrt{N}) - x - \frac{x^3}{6\,N}\right| \le \frac{x^5\,\varphi(x)}{120\,N^2} \le \frac{(5/e)^{2.5}}{120\,\sqrt{2\pi}\,N^2} = \frac{0.015256}{N^2}.$$

With $\left|\sqrt{N}\,\sin(x/\sqrt{N}) - x\right| \leq x^3 \big/ (6\,N), 0 < x/z \leq x \big/ \big(\sqrt{N}\,\sin(x/\sqrt{N})\big) \leq \pi/2$ for $0 < x \leq \pi/2$ and having in mind $S_2(n) := \min\{S_{2a}(n),\ S_{2b}(n) + S_{2c}(n)\}$, where

$$S_{2a}(n) := \frac{|\varphi'(z)|}{2}\,(\sqrt{N}\,\sin(x/\sqrt{N}) - x)^2 \leq \frac{z\,\varphi(z)\,x^6}{72\,N^2} \leq \frac{(7/e)^{3.5}\,(\pi/2)^9}{72\,\sqrt{2\,\pi}\,N^2} = \frac{2.280916}{N^2}$$

or alternatively

$$S_{2b}(n) := \frac{|\varphi'(x)|}{2}\,(\sqrt{N}\,\sin(x/\sqrt{N}) - x)^2 \leq \frac{x^7\,\varphi(x)}{72\,N^2} \leq \frac{(7/e)^{3.5}}{72\,\sqrt{2\,\pi}\,N^2} = \frac{0.151842}{N^2}$$

and since $|z^{11} - z^9|\,\varphi(z)$ takes its maximum for $z = 3/\sqrt{2} + \sqrt{6}/2$

$$\begin{aligned} S_{2c}(n) &:= \frac{|\varphi''(z)|}{6}\left|\sqrt{N}\,\sin(x/\sqrt{N}) - x\right|^3 \leq \frac{|z^{11} - z^9|\,\varphi(z)\,(\pi/2)^9}{1{,}296\,N^3} \\ &\leq \frac{\big((3/\sqrt{2} + \sqrt{6}/2)^{11} - (3/\sqrt{2} + \sqrt{6}/2)^9\big)\,(\pi/2)^9}{1{,}296\,\sqrt{2\,\pi}\,\exp\{(3/\sqrt{2} + \sqrt{6}/2)^2/2\}\,N^3} = \frac{35.597236}{N^3}. \end{aligned}$$

Note that $S_{2b}(n) + S_{2c}(n) < S_{2a}(n)$ for $n \geq 20$.

Finally we define $m(x) := x^3\,\varphi(x)$ then we have

$$m(h(x)) = m(x) + m'(z)\big(h(x) - x\big) \quad \text{for} \quad 0 < h(x) < z < x.$$

Since $m'(x) = (3\,x^2 - x^4)\,\varphi(x)$ and the function $(z^7 - 3z^5)\,\varphi(z)$ takes its maximum at $z_{max} = \sqrt{5 + \sqrt{10}}$ we obtain

$$\begin{aligned} S_3(n) &:= \frac{|3z^2 - z^4|\,\varphi(z)}{4\,N}\,|\sqrt{N}\,\sin(x/\sqrt{N}) - x| \leq \frac{|3z^2 - z^4|\,\varphi(z)\,x^3}{24\,N^2} \\ &\leq \frac{\big(z_{max}^7 - 3\,z_{max}^5\big)\,\varphi(z_{max})\,(\pi/2)^3}{24\,N^2} = \frac{1.069085}{N^2} \end{aligned}$$

and (21) is proved. Changing $(\pi/2)^k$ by $(\pi/6)^k$ in the estimates of S_{2a}, S_{2c} and S_3, we find D_n^*. □

Proof of Theorem 4. Since the transformation T is assumed to be increasing, we get

$$P(S \leq x) = P(T(S) \leq T(x)).$$

Therefore, in order (26) holds it is enough to find the function T such that

$$\Phi(T(x)) = \Phi(x) + p_n(x)\,\varphi(x) + O(n^{-\alpha-1/2}).$$

Hence, by smoothness properties of $\Phi(x)$ we may take T given by (27). □

Proof of Theorem 5. Put $N = n - 2.5$ and $h(x) = \sqrt{N}\ F^{-1}(x/\sqrt{N})$, where

$$F^{-1}(y) = \frac{2}{\sqrt{3}}\ \frac{e^{\sqrt{3}\,y} - 1}{e^{\sqrt{3}\,y} + 1} \quad \text{for} \quad |y| \le \frac{\ln(7 + 4\sqrt{3})}{\sqrt{3}}$$

is the inverse function to $F(y)$, given in (29). Then we find by Theorem 2 as $n \to \infty$

$$P\Big(\sqrt{N}\,F(R)\ \le x\Big) = P\Big(\sqrt{N}\ R\ \le h(x)\Big) = \Phi\big(h(x)\big) + \frac{h^3(x)\,\varphi(h(x))}{4N} + O(n^{-2}).$$

Using (51) and $Z^{-1}(y) = y - y^3/4 + O(y^5)$ as $y \to 0$ we find in our case as $n \to \infty$

$$\Phi(h(x)) = \Phi(x) - x^3\varphi(x)/(4\,N) + O(n^{-2}) \text{ and } h^3(x)\varphi(h(x)) = x\,\varphi(x) + O(n^{-1}),$$

which lead to (30). With $F(y) = y + y^3/4 + O(|y|^7)$ as $y \to 0$ and similar calculations we find (31). □

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The Stein-Tikhomirov Method and Berry-Esseen Inequality for Sampling Sums from a Finite Population of Independent Random Variables

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Abstract We present a simplified version of the Stein-Tikhomirov method realized by defining a certain operator in class of twice differentiable characteristic functions. Using this method, we establish a criterion for the validity of a nonclassical central limit theorem in terms of characteristic functions, in obtaining of classical Berry-Esseen inequality for sampling sums from finite population of independent random variables.

Keywords Stein-Tikhomirov method • Distribution function • Characteristic function • Independent random variables • Berry-Esseen inequality • Sampling sums from finite population

Mathematics Subject Classification (2010): 60F05

1 The Stein-Tikhomirov Method and Nonclassical CLT

Suppose that $F(x)$ is an arbitrary distribution function and

$$\Phi(x) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{-u^2/2} du$$

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is the standard distribution function for the normal law. In [9] Stein proposed a universal method for estimating the quantity

$$\delta = \sup_x |F(x) - \Phi(x)|,$$

based on the following arguments. Suppose that $h(u)$ is a bounded measurable function on the line and

$$\Phi h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(u) e^{-u^2/2} du.$$

Consider the function $g(\cdot)$ which is a solution of the differential equation

$$g'(u) - ug(u) = h(u) - \Phi h. \tag{1}$$

Suppose that ζ is a random variable with distribution function

$$P(\zeta < x) = F(x).$$

Setting

$$h(u) = h_x(u) = I_{(-\infty, x)}(u)$$

in (1), we have

$$F(x) - \Phi(x) = E\left[g'(\zeta) - \zeta g(\zeta)\right]. \tag{2}$$

Thus, the problem of estimating δ can be reduced to that of estimating the difference of the expectations

$$\left|Eg'(\zeta) - E\zeta g(\zeta)\right|.$$

Also note that for the case in which the random variable ζ has normal distribution, the right-hand side of (2) vanishes. Using this method, Stein [9] obtained an estimate of the rate of convergence in the central limit theorem for stationary (in the narrow sense) sequences of random variables satisfying the strong mixing conditions (in the sense of Rosenblatt). Moreover, for the summands eighth-order moments must exits. In his paper, Stein stated his belief that his method is hardly related to that of characteristic functions.

In [10, 11] Tikhomirov refuted Stein's suggestion. He showed that a combination of Stein's ideas with the method of characteristic functions allows one to obtain the best possible estimates of the rate of convergence in the central limit theorem for sequences of weakly dependent random variables for less stringent conditions on the moments. He also used to best advantage the ideas [9] underling the proposed new method. The combination of methods our lined in [9, 10], later became known as the Stein-Tikhomirov method.

In the present paper, it will be shown that the arguments used in applying the Stein-Tikhomirov method can be considerably simplified. Thus will be demonstrated in the course of the proof of a nonclassical central limit theorem. Which can be called the generalized Lindeberg-Feller theorem.

Suppose that

$$X_{n1}, X_{n2}, \cdots$$

is a sequence of independent random variables constituting the scheme of a series of experiments and

$$S_n = X_{n1} + X_{n2} + \cdots, n = 1, 2, \cdots$$

with a possibly infinite number of terms in each sum. Set

$$EX_{nj} = 0, \quad EX_{nj}^2 = \sigma_{nj}^2, \quad j = 1, 2, \cdots$$

and

$$\sum_j \sigma_{nj}^2 = 1. \tag{3}$$

In what follows, condition (3) is assumed to be satisfied. As is well known, in the theory of summation of independent random variables an essential role is played by the condition of uniform infinite smallness of the summands

$$\lim_{n\to\infty} \sup_j P\left(\left|X_{nj}\right| \geq \varepsilon\right) = 0 \tag{4}$$

for any $\varepsilon > 0$.

The constraint (4) is needed if we want to make the limiting law for the distribution of the sum S_n insensitive to the behavior of individual summands. But in finding conditions for the conditions for the convergence of the sequence of distributions functions

$$F_n(x) = P(S_n < x)$$

for any given law it is not necessary to introduce constraints of type (4). Following Zolotarev, limit theorems making no use of condition (4) are said to be *nonclassical*. As was noted in the monograph "theory of summation of independent random variables", the ideas underlying the nonclassical approach go back to P.Lévy, who studied various versions of the central limit theorem.

In [7], Rotar' proved the following theorem, which is generalization of the classical Lindeberg-Feller theorem.

Theorem A. *In order that*

$$\sup_x |F_n(x) - \Phi(x)| \to 0$$

as $n \to \infty$, *it is necessary and sufficient that for any* $\varepsilon > 0$ *the following relation hold:*

$$R_n(\varepsilon) = \sum_i \int_{|x|>\varepsilon} |x| \left|F_{nj}(x) - \Phi_{nj}(x)\right| \to 0, \tag{5}$$

where

$$F_{nj}(x) = P\left(X_{nj} < x\right), \quad \Phi_{nj}(x) = \Phi\left(\frac{x}{\sigma_{nj}}\right).$$

Note that this version of Theorem A is not given in [7], but it can be obtained by combining Propositions 1 and 2 from [7].

The numerical characteristic $R_n(\varepsilon)$ defined in (5) is universal; it and its analogs have been used for some time in the "nonclassical" theory of summation of more general sequences of random variables (see, for example, [4], Chap. 5, Sect. 6).

Now consider the class of characteristic functions $f(t)$ given by

$$F = \left\{f(t) \,|\, f'(0) = 0, \;\; = -f''(0) = -\sigma^2 < \infty\right\}.$$

In the class F, we introduce the transformation (the Stein-Tikhomirov operator)

$$\Delta f(t) = f'(t) + t\sigma^2 f(t). \tag{6}$$

Obviously,

$$\Delta\left(e^{-t^2\sigma^2/2}\right) = 0, \tag{7}$$

i.e., the operator $\Delta(\cdot)$ "cancels" the normal characteristic function.

If we consider (6) as a differential equation to be solved for the initial condition $f(0) = 1$, then we obtain

$$f(t) - e^{-t^2\sigma^2/2} = e^{-t^2\sigma^2/2} \int_0^t \Delta(f(u))\, e^{u^2\sigma^2/2} du. \tag{8}$$

In relation (8), the sign of the variable of integration is identical with that of t and $|u| \leq |t|$. Relations (7) and (8) show that the expression $\Delta(f(t))$ characterizes the proximity of the distribution with characteristic function $f(t)$ to the normal law with mean 0 and variance σ^2.

It can be readily verified that the operator $\Delta(\cdot)$ possesses the following important property.

Lemma. *For characteristic functions $f(t)$ and $g(t)$ such that*

$$f'(0) = g'(0) = 0, \quad \max\left(\left|f''(0)\right|, \left|g''(0)\right|\right) < \infty$$

the following relation holds:

$$\Delta(f(t)\,g(t)) = f(t)\,\Delta(g(t)) + g(t)\,\Delta(f(t)). \tag{9}$$

It follows from this lemma that the operator $\Delta(\cdot)$ is the differentiation operator with respect to the product of characteristic functions.

Theorem 1. *In order that*

$$\sup_x |F_n(x) - \Phi(x)| \to 0$$

as $n \to \infty$, it is necessary and sufficient that for any $T > 0$ the following relation holds:

$$\sup_{|t| \le T} \sum_j \left|\Delta\left(f_{nj}(t)\right)\right| \to 0, \tag{10}$$

where $f_{nj}(\cdot)$ is the characteristic function corresponding to the distribution function $F_{nj}(x)$.

Proof. The proof of the sufficiency of condition (10) is simple enough. Indeed,

$$f_n(t) = Ee^{itS_n} = \prod_j f_{nj}(t)$$

and from relation (8) it follows that

$$\sup_{|t| \le T} \left| f_n(t) - e^{-t^2/2} \right| \le T \cdot \sup_{|t| \le T} |\Delta(f_n(t))| \tag{11}$$

for any $T > 0$.

Further, by (9) we have

$$\Delta(f_n(t)) = \sum_j \prod_{k \le j-1} f_{nk}(t) \Delta\left(f_{nj}(t)\right) \prod_{s \ge j+1} f_{ns}(t)$$

and, therefore,

$$|\Delta(f_n(t))| \le \sum_j \left|\Delta\left(f_{nj}(t)\right)\right|. \tag{12}$$

Relations (11) and (12) prove the necessity of condition (10) for the validity of the central limit theorem. To demonstrate the necessity of condition (10), let us prove that is not stronger than (5). Formally, this is sufficient, and the subsequent arguments will supply the necessary details. Set

$$\varphi_{nj}(t) = \int_{-\infty}^{\infty} e^{itx} d\Phi_{nj}(x) = \int_{-\infty}^{\infty} e^{itx} d\Phi\left(\frac{x}{\sigma_{nj}}\right).$$

Taking into account the fact that $\Delta\left(\varphi_{nj}(t)\right) = 0$ for any $j \geq 1$, we have

$$\sum_j \left|\Delta\left(f_{nj}(t)\right)\right| = \sum_j \left|\Delta\left(f_{nj}(t)\right) - \Delta\left(\varphi_{nj}(t)\right)\right| \leq \sum_j \left|f'_{nj}(t) - \varphi'_{nj}(t)\right|$$

$$+ |t| \sum_j \sigma^2_{nj} \left|f_{nj}(t) - \varphi_{nj}(t)\right| = \sum{}_1(t) + |t| \sum{}_2(t). \qquad (13)$$

Noting that

$$EX_{nj} = 0, \qquad \int_{-\infty}^{\infty} x^2 dF_{nj} = \int_{-\infty}^{\infty} x^2 d\Phi_{nj} = \sigma^2_{nj},$$

and integrating by parts, we obtain

$$\left|f'_{nj}(t) - \varphi'_{nj}(t)\right| = \left|\int_{-\infty}^{\infty} (ix)\left(e^{itx} - 1 - itx\right) d\left(F_{nj} - \Phi_{nj}\right)\right|$$

$$\leq \left|\int_{-\infty}^{\infty} \left(e^{itx} - 1 - itx\right)\left(F_{nj}(x) - \Phi_{nj}(x)\right) dx\right|$$

$$+ |t| \left|\int_{-\infty}^{\infty} (ix)\left(e^{itx} - 1\right)\left(F_{nj}(x) - \Phi_{nj}(x)\right) dx\right|.$$

Therefore

$$\sum{}_1(t) \leq t^2 \varepsilon \sum_i \int_{|x| \leq \varepsilon} |x| \left|F_{nj}(x) - \Phi_{nj}(x)\right| dx$$

$$+ \left(|t| + t^2\right) \sum_i \int_{|x| \leq \varepsilon} |x| \left|F_{nj}(x) - \Phi_{nj}(x)\right| dx$$

$$\leq t^2 \varepsilon \sum_i 2\sigma^2_{nj} + \left(|t| + t^2\right) R_n(\varepsilon) \leq 2\left(|t| + t^2\right)\left(\varepsilon + R_n(\varepsilon)\right). \qquad (14)$$

To derive (14), we use the following fact. If $F(x)$ is a distribution function with mean 0 and variance σ^2, then

$$\int_0^{\infty} u\left(1 - F(u) + F(-u)\right) du = \frac{\sigma^2}{2}.$$

It was established in [3] that

$$\sum\nolimits_2 (t) \leq 2 \left(t^2 + |t|^3\right) (\varepsilon + R_n (\varepsilon)) . \tag{15}$$

It follows from relations (13)–(15) that if condition (5) is satisfied, then for any $T > 0$ we have

$$\sup_{t \leq T} \sum_j \left| \Delta \left(f_{nj} (t) \right) \right| \to 0, \quad n \to \infty.$$

We can easily verify condition (10) using the following simple example of increasing sums of independent Bernoulli random variables as an illustration. Suppose that

$$Y_j = \begin{cases} 1 & \text{with} \quad \text{probability} \quad p_j, \\ 0 & \text{with} \quad \text{probability} \quad q_j = 1 - p_j. \end{cases}$$

Taking into account the fact that $MY_j = p_j$, $DX_j = p_j q_j$, we set

$$B_n^2 = \sum_{j=1}^n p_j q_j, \quad X_{nj} = \frac{Y_j - p_j}{B_n}, \quad S_n = \sum_{j=1}^n X_{nj}.$$

In the case considered, we have

$$f_{nj}(t) = E^{itX_{nj}} = p_j e^{itq_j/B_n} + q_j e^{-itp_j/B_n}.$$

Let us show that if $B_n \to \infty$, then condition (10) holds. Indeed, it is easy to see that

$$f_{nj}(t) = 1 - \frac{p_j q_j}{2B_n^2} t^2 + \frac{p_j q_j}{B_n^2} \varepsilon_n (t) , \tag{16}$$

$$f'_{nj}(t) = - \frac{p_j q_j}{B_n^2} t + \frac{p_j q_j}{B_n^2} \varepsilon'_n (t) , \tag{17}$$

where

$$\sup_{|t| \leq T} |\varepsilon_n(t)| = O\left(\frac{1}{B_n}\right), \quad n \to \infty,$$

for any T > 0.

It follows from relation (16) and (17) that, as $n \to \infty$, we have

$$\sup_{|t| \leq T} \sum_j \left| \Delta (f_{nj}(t)) \right| = O\left(\frac{1}{B_n}\right). \tag{18}$$

Obviously, for our sequence of simple random variables the direct verification of (5) or of the classical Lindeberg condition is more complicated that the estimates (18) obtained in this paper.

Remark 1. One can give more complicated examples of sequences of random variables for which the proof of the validity of the central limit theorem simplifies if the criterion (10) is used. Apparently, the present paper is the first paper in which the criterion for the convergtnce of the distribution of the sum S_n to the normal law is stated in terms of characteristic function of the summands.

Remark 2. In proving limit theorems for the distribution functions of sums of independent and weakly dependent random variables by the method of characteristic functions, on is mainly occupied with proving the fact that the characteristic function of these sums $f_n(t)$ does not vanish in a sufficiently large of neighborhood of the point $t = 0$. But there is no need for such a proof if we use the Stein-Tikhomirov method, this shows the advantage of this method over others.

Remark 3. Relation (8) and (11) show that the arguments used in the proof of the Theorem 1 allow us to obtain an estimate of the rate of convergence in the nonclassical case. Subsequent papers by this author will be concerned with exact statements and proofs for the corresponding assertion.

2 Berry-Esseen Inequality for Sampling Sums from Finite Population

Let $\{X_1, X_2, \dots, X_N\}$ be a population of independent random variables and S_n be a sampling sum of size n. The last means that the sum S_n consist from such random variables which hit in a sample of size n from the parent population. One can give the exact meaning to the formation of the sum S_n as follows. Let $I = (I_1, I_2, \dots, I_N)$ be an indicator random vectors such that $I_k = 0$ or $1 \quad (1 \le k \le N)$ and S_n contains the term X_k if and only if $I_k = 1$. Hence,

$$S_n = \sum_{k=1}^{N} I_k X_k.$$

It is assumed that I is independent from random variables $X_1, X_2, \dots, X_N$ and for every ordered sequence $i = (i_1, i_2, \dots, i_N)$ of n units and $N - n$ zeros

$$P\,(I = i) = \frac{1}{\binom{N}{n}} = \left(C_N^n\right)^{-1}.$$

We have $EI_k = \frac{n}{N} = f$ - the sampling ratio, and $EI_kI_i = \frac{n}{N} \cdot \frac{n-1}{N-1}$ for $k \neq i$. We introduce the moments $EX_k = \mu_k$, $EX_k^2 = \beta_k$ and then get

$$ES_n = \sum_{k=1}^{N} EI_kX_k = f\sum_{k=1}^{N}\mu_k,$$

$$ES_n^2 = \frac{n}{N}\sum_{k=1}^{N}\beta_k + \frac{n}{N}\cdot\frac{n-1}{N-1}\sum_{k\neq i}\mu_k\mu_i.$$

We will assume that (without loss of generality) the parent population of random variables has 0 mean and unit variance, i.e.

$$\sum_{k=1}^{N}\mu_k = 0, \quad \frac{1}{N}\sum_{k=1}^{N}\beta_k = 1. \tag{19}$$

Thus,

$$ES_n = 0, \quad DS_n = \text{var}S_n = n\left(1 - \frac{n-1}{N-1}\alpha^2\right), \quad \alpha^2 = \frac{1}{N}\sum_{k=1}^{N}\mu_k^2.$$

We prove that $S_n/\sqrt{n}$ has approximately normal distribution with 0 mean and variance $1 - f\alpha^2$, and also give an estimation of the remainder term. In addition, the obtained result is a generalization of the classical Berry-Esseen estimation in CLT (S_n is turned into usual sum of n independent random variables when $n = N$).

The special case $X_i = a_i = const$ is very important in statistical applications of sampling sums. This case was investigated in details be B. Rosen [6]. The convergence rate in CLT were studied by A. Bikelis [1] in the case $X_i = const$ and by B. von Bahr [3] for arbitrary population of independent random variables. In the present work the result of last paper is made more precise.

Set

$$E|X_k|^3 = \gamma_k, \quad L_N = \frac{1}{N}\sum_{k=1}^{N}\gamma_k, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du.$$

Theorem 2. *There exists an absolute positive constant C such that*

$$\sup_x \left| P\left(\frac{S_n}{\sqrt{n(1 - f\alpha^2)}} < x\right) - \Phi(x)\right| \le \frac{C \cdot L_N}{\sqrt{n}(1 - f\alpha^2)^{3/2}}.$$

Remark 4. In [12] $C = 60$ and it is involved less exact characteristic

$$\gamma = \max_{1\le k\le N} \gamma_k$$

instead of L_N.

Remark 5. Rather rough calculation shows that $C < 60$ in given theorem, but we note that the exact calculation of the constant C doesn't enter is our task.

Remark 6. If the set of random variables $(X_1, X_2, \ldots, X_N)$ doesn't satisfy the normalizing conditions (19), we can easily obtain a new set $\left(X_1', X_2', \ldots, X_N'\right)$ which satisfies (1), by a linear transformation. Application of the result of Theorem 2 to this new set of random variables gives, in terms of the original variables

$$\left| P\left(\frac{S_n - n\mu}{\sqrt{\frac{1}{n}\left[\frac{1}{N}\sum\limits_{k=1}^{N}\sigma_k^2 + \frac{1-f}{N}\sum\limits_{k=1}^{N}(\mu_k-\mu)^2\right]}} < x \right) - \Phi(x) \right| \le$$

$$\le \frac{C}{\sqrt{n}} \cdot \frac{L_N}{\left[\frac{1}{N}\sum\limits_{k=1}^{N}\sigma_k^2 + \frac{1-f}{N}\sum\limits_{k=1}^{N}(\mu_k-\mu)^2\right]^{3/2}},$$

where

$$\mu_k = EX_k, \quad \mu = \frac{1}{N}\sum_{k=1}^{N}\mu_k \quad \text{and} \quad \sigma_k^2 = \text{var}X_k.$$

Proof of the Theorem 2 is conducted by means of the Stein-Tikhomirov method above mentioned at the point 1. Notice that in the papers [2, 8] are demonstrated application of initial variant of Stein-Tikhomirov method for obtained of classical Berry-Esseen inequality in the case of usual sum from independent random variables (i.e. as $(N = n)$). Let ν be a random variable with uniform distribution on the set $\{1, 2, \ldots, N\}$ that is not independent neither from random variables $X_1, X_2, \ldots, X_N$ nor from indicator vector I and F_{IX} be a σ-algebra generated by random variables$\{I_1, I_2, \ldots, I_N, X_1, X_2, \ldots, X_N\}$.

Further we denote

$$\omega_n = \frac{N}{\sqrt{DS_n}} I_\nu X_\nu.$$

It is not difficult to see that

$$\bar{S}_n = E\left(\omega_n / F_{IX}\right) = \frac{S_n}{\sqrt{DS_n}}. \tag{20}$$

Set also

$$f_n(t) = Ee^{it\bar{S}_n}.$$

As it follows from the point 1, we must calculate the operator $\Delta\left(f_n\left(t\right)\right)$ by the formula (6).

By virtue of (20)

$$E\left(i\omega_n e^{it\bar{S}_n}\right) = E\left[E\left(i\omega_n e^{it\bar{S}_n}/F_{IX}\right)\right] = E\left[iE\left(\omega_n/F_{IX}\right)e^{it\bar{S}_n}\right] = E\left(i\bar{S}_n e^{it\bar{S}_n}\right).$$

Therefore,

$$f_n'(t) = E\left(i\bar{S}_n e^{it\bar{S}_n}\right) = E\left(i\omega_n e^{it\bar{S}_n}\right). \tag{21}$$

By direct calculation we can obtain the following equalities:

$$E\omega_n = E\left(E\left(\omega_n / F_{IX}\right)\right) = E\bar{S}_n = 0. \tag{22}$$

$$E\omega_n^2 = \frac{N}{1 - \frac{n-1}{N-1}\alpha^2} = \frac{n}{f\left(1 - \frac{n-1}{N-1}\alpha^2\right)}, \tag{23}$$

$$E\left|\omega_n^3\right| = \frac{N^2}{\sqrt{n}\left(1 - \frac{n-1}{N-1}\alpha^2\right)^{3/2}} \cdot L_N = \frac{n^2}{f^2 \cdot \sqrt{n}\left(1 - \frac{n-1}{N-1}\alpha^2\right)^{3/2}}, \tag{24}$$

Further, set

$$S_{n\nu} = \frac{1}{\sqrt{DS_n}} \sum_{i \neq \nu} I_i X_i.$$

By virtue of (21) we have

$$f_n'(t) = E\left(i\omega_n e^{itS_{n\nu}}\right) + E\left[i\omega_n\left(e^{it\bar{S}_n} - e^{itS_{n\nu}}\right)\right].$$

Since ω_n and $S_{n\nu}$ are independent on construction, we have

$$E\left(i\omega_n e^{itS_{n\nu}}\right) = E\left(e^{itS_{n\nu}}\right) E\left(i\omega_n\right) = 0.$$

Thus,

$$f_n'(t) = E\left[i\omega_n\left(e^{it\omega_n/N} - 1\right)\right] \cdot Ee^{itS_{n\nu}}. \tag{25}$$

In addition

$$Ee^{itS_{n\nu}} = Ee^{it\bar{S}_n} + E\left(e^{itS_{n\nu}} - e^{it\bar{S}_n}\right) = f_n(t) + E\left[e^{itS_{n\nu}}\left(1 - e^{it\omega_n/N}\right)\right]. \tag{26}$$

It follows from (25) and (26) that

$$f_n'(t) = E\left[i\omega_n\left(e^{it\omega_n/N} - 1\right)\right] f_n(t) + E\left[i\omega_n\left(e^{it\omega_n/N} - 1\right)\right] E\left(1 - e^{it\omega_n/N}\right) Ee^{itS_{n\nu}}. \tag{27}$$

Using the equalities (22)–(24) we can obtain the following estimates

$$\left|E\left[i\omega_n\left(e^{it\omega_n/N} - 1\right)\right] + t\right| \le \frac{t^2}{2} \frac{L_N}{\sqrt{n}(1 - f\alpha^2)^{3/2}}, \tag{28}$$

$$\left|E\left(1 - e^{it\omega_n/N}\right)\right| \le c_0 t^2 \frac{L_N}{\sqrt{n}(1 - f\alpha^2)^{3/2}}, \tag{29}$$

In what follows, the letter c_0 denotes different absolute constants.

Now, with regard to the inequalities (28) and (29), we can rewrite (27) in the form

$$f_n'(t) = A_n(t) f_n(t) + B_n(t) \tag{30}$$

where

$$A_n(t) = -t + \frac{\theta}{2} t^2 \bar{L}_N, \quad |B_n(t)| \le c_0 t^2 \, |f_{n\nu}(t)| \, \bar{L}_N,$$

$$f_{n\nu}(t) = E e^{itS_{n\nu}}, \quad |\theta| \le 1, \quad \bar{L}_N = \frac{L_N}{\sqrt{n}\,(1 - f\alpha^2)^{3/2}}.$$

We can consider the equality (14) as the differential equation that we must to solve under the initial condition $f_n(0) = 1$. Then we have

$$f_n(t) = \exp\left\{\int_0^t A_n(u) du\right\} + \int_0^t B_n(u) \exp\left\{\int_u^t A_n(s) ds\right\} du. \tag{31}$$

Further, we obtain

$$\int_0^t A_n(u) du = -\frac{t^2}{2} + \frac{\theta}{6} \bar{L}_N \, |t|^3, \tag{32}$$

$$\int_u^t A_n(s) ds = -\frac{t^2}{2} + \frac{u^2}{2} + a_n\,(t,u), \tag{33}$$

where

$$|a_n\,(t,u)| = \left|\theta \frac{\bar{L}_N}{2} \int_u^t s^2 ds\right| \le \frac{\bar{L}_N}{2} \, |t| \left(t^2 - u^2\right). \tag{34}$$

By direct calculation we obtain that

$$f_{n\nu}(t) = \sum_{j=1}^{N} E\left(e^{itS_{n\,\nu}}, \nu = j\right) = \frac{1}{N} \frac{1}{C_N^n} \sum_{j=1}^{N} \sum_{(r_1,\ldots,r_n)} \prod_{k=1}^{n}{}^{(j)} f_{r_k}\left(\frac{t}{\sqrt{DS_n}}\right), \tag{35}$$

where $f_j\,(t) = E e^{itX_j}$, $\prod^{(j)}$ means that in product $\prod_{k=1}^{n} f_k(t)$ the factor with index r_j is equal to 1 and the summation is produced on all samples $(r_1, \ldots, r_n)$ of size n.

By using the paper [12] and (35) we can prove that under $|t| \le \left(\bar{L}_N\right)^{-1/3}$

$$|f_{n\nu}\,(t)| \le e^{-t^2/3}. \tag{36}$$

From (31)–(34), (36) we obtain finally that under $|t| \leq \left(\bar{L}_N\right)^{-1/3}$

$$\left| f_n(t) - e^{-t^2/2} \right| \leq c_0 \bar{L}_N \, |t|^3 \, e^{-t^2/6}. \tag{37}$$

Further way of the proof is the same as the proof of the classical Berry-Esseen inequality for sums from independent random variables (see [5]).

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On One Inequality for Characteristic Functions

Nicko Gamkrelidze

Abstract This paper deals with an inequality for characteristic functions. This inequality (see (3) below) founds connection between "measure of almost normality" and characteristic functions. Also an analysis of accuracy in the local limit theorem and connection between the central limit and local limit theorem are given.

Keywords Characteristic functions • Limit theorems • Central limit theorem • Local limit theorem

Mathematics Subject Classification (2010): 60E10, 60E15

A good deal of probability theory consists of the study of limit theorems, because "in reality the epistemological value of the theory of probability is revealed only by limit theorems" (see [4]).

Important part of this area consists of the upper estimation of the rate of convergence in the central limit theorem. It is quite reasonable turn to the construction of the lower estimates. Unfortunately, many mathematicians working in the field of theory of limit theorems pay less attention to such a kind of problems.

This paper focuses around one inequality for characteristic function. It should be noted that this one don't demands from limit theorems. Taking account of this notion at first we introduce following

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Definition. An integer valued random variable ξ is said (A, B, λ) normal if there are some constants $A, B \geq 1, \lambda, (0 < \lambda < B)$ and integer $k(-\infty < k < \infty)$ such that

$$\sup_k \left| P(\xi = k) - (2\pi)^{-1/2} B^{-1} \exp\left\{-\frac{(k-A)^2}{2B^2}\right\} \right| \leq \frac{\lambda}{B}. \tag{1}$$

We will show that λ may be estimate from below by

$$\frac{1}{4\pi} \int\limits_{\delta \leq |t| < \pi} |f(t, \xi_k)|^2 dt \quad \text{where} \quad \delta > 0.$$

The exact formulation of this assertion will be given later (see Theorem 1 below). Denote by $\zeta = \xi - \xi'$ symmetrized random variable, where ξ and ξ' are independent and identically distributed with characteristic function $f(t, \zeta) = \left| f(t, \xi) \right|^2$

Proposition 1. *Let ξ be (A, B, λ) normal random variable, then for symmetrized random variable ζ we have:*

$$\sup_k \left| P(\zeta = k) - \frac{1}{2\sqrt{\pi} B} e^{-\frac{k^2}{4B^2}} \right| \leq \frac{\Lambda}{B}, \tag{2}$$

where $\Lambda = 2{,}01(\lambda + \frac{1}{2\sqrt{\pi}} e^{-\pi^2 B^2})$.

Starting from this assertion we get main result:

Theorem 1. *Let ξ be $(A, B, \lambda)t$ normal (1). Then for every integer $k \geq 1$.*

$$\frac{1}{4\pi} \int\limits_{\frac{2\pi}{2k+1} \leq |t| \leq \pi} |f(t, \xi)|^2 dt \leq \frac{2\Lambda}{B} + \frac{1}{2\sqrt{\pi} B} \left(1 - e^{-\frac{k^2}{4B^2}}\right). \tag{3}$$

Proof of Proposition 1. Denote by

$$Q(x) = \frac{1}{\sqrt{2\pi} B} e^{-\frac{x^2}{2}}.$$

Write

$$\begin{aligned} P(\zeta = k) &= \sum_{j=-\infty}^{\infty} P_\xi(k+j) P_\xi(j) \\ &= \sum_{j=-\infty}^{\infty} [P_\xi(k+j)(P_\xi(j) - Q(x_j))] + \sum_{j=-\infty}^{\infty} [P_\xi(k+j) - Q(x_{k+j})] Q(x_j) \\ &\quad + \sum_{j=-\infty}^{\infty} Q(x_{k+j}) Q(x_j). \end{aligned} \tag{4}$$

Denote by

$$S_1 := \sum_{j=-\infty}^{\infty} Q(x_j), \quad \text{and} \quad S_2 := \sum_{j=-\infty}^{\infty} Q(x_{k+j})Q(x_j).$$

We apply special case of Poisson summation formula (see [1] p. 629). By this formula for any real s and $t > 0$ we have

$$\frac{\sqrt{2\pi}}{\sqrt{t}} \sum_j \exp\left\{-\frac{1}{2t}(s+2j\pi)^2\right\} = \sum_{m=-\infty}^{\infty} e^{-\frac{m^2 t}{2}} \cos(ms)$$
$$= 1 + 2\sum_{m=1}^{\infty} e^{-\frac{m^2 t}{2}} \cos(ms). \tag{5}$$

Put $s = -2\pi A,\ t = 4\pi^2 B^2$ and write the left-hand side (5) as

$$S_1 = \frac{1}{\sqrt{2\pi}B} \sum_j \exp\left\{-\frac{1}{2B^2}(j-A)^2\right\}.$$

On the right-hand side (5) we get

$$1 + 2\sum_{m=1}^{\infty} e^{-2m^2\pi^2 B^2} \cos(m \cdot 2\pi A) \tag{6}$$

Since $B \geq 1$ we can write

$$2e^{-2\pi^2 B^2} \sum_{m=1}^{\infty} e^{-2(m^2-1)\pi^2 B^2} \leq 2e^{-2\pi^2 B^2}\left(1 + \sum_{m=2}^{\infty} (e^{-2\pi^2 B^2})^m\right)$$
$$\leq 2e^{-2\pi^2 B^2}\left(1 + \tfrac{e^{-4\pi^2}}{1-e^{-2\pi^2}}\right).$$

The evaluation $c := e^{-4\pi^2}/1 - e^{-2\pi^2}$ gives $c < 10^{-17}$. Consequently

$$S_1 = 1 + \Theta_1 2,01 e^{-2\pi^2 B^2}, \quad |\Theta_1| < 1. \tag{7}$$

In the same way we estimate S_2. At first consider j-th term of the sum S_2

$$\frac{1}{2\pi B^2} \exp\left\{-\frac{1}{2B^2}\Big[(k+j-A)^2 + (j-A)^2\Big]\right\}.$$

Note that

$$(k+j-A)^2 + (j-A)^2 = 2\Big(j - (A - k/2)\Big)^2 + \frac{k^2}{2}.$$

So

$$S_2 = \frac{1}{2\sqrt{\pi}B} e^{-\frac{k^2}{4B^2}} \sum_j \frac{1}{2\pi B} \exp\left\{ -\frac{1}{B^2}\Big(j-(a-k/2)\Big)^2 \right\} = \frac{1}{2\sqrt{\pi}B} e^{-\frac{k^2}{4B^2}} S_1', \tag{8}$$

where S_1' is the same as S_1 but with changing B to $B/\sqrt{2}$ and A to $A-\frac{k}{2}$.

According (8) we have

$$S_1' = 1 + \Theta_2 \cdot 2,01 e^{-\pi^2 B^2}, \qquad |\Theta_2| \le 1. \tag{9}$$

Taking in account (1) and (6)–(9) we receive the statement (2).

Remark 1. Applying Euler's summation formula or Yu. V. Prokhorov's Lemma from [6], to the estimation S_1 and S_2 we can write

$$\Lambda = 2\lambda + \frac{2}{\pi B}\left(1+\sqrt{2\pi}\lambda\right).$$

In preparation for the proof of main inequality (3) we are in need of

Lemma 1. *For any integer valued random variable η and integer $k \ge 1$*

$$\begin{aligned} \vartheta_k(\eta) :&= P(\eta=0) - \frac{1}{2k+1}\sum_{j=-k}^{k} P(\eta=j) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1-\frac{\sin((2k+1)t/2)}{(2k+1)\sin(t/2)}\right) f(t,\eta)dt. \end{aligned} \tag{10}$$

Proof of Lemma 1. By inversion formula we have

$$P(\eta=j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itj} f(t,\eta)dt$$

and

$$\sum_{j=-k}^{k} P(\eta=j) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{j=-k}^{k} e^{-itj}\right) f(t,\eta)dt.$$

Since

$$\sum_{j=-k}^{k} (e^{-it})^j = 1 + 2\sum_{j=1}^{k}\cos tj = \frac{\sin(2k+1)t/2}{\sin\frac{t}{2}}$$

this gives (10).

Lemma 2. *Let η be integer valued random variable with nonnegative characteristic function. Then for any integer $k \geq 1$*

$$\frac{1}{4\pi} \int\limits_{\frac{2\pi}{2k+1} \leq |t| \leq \pi} f(t, \eta) dt \leq \vartheta_k(\eta) \tag{11}$$

Proof of Lemma 2. It is obvious that for $|t| \leq \pi$

$$\left| \frac{\sin(2k+1)t/2}{(2k+1)\sin t/2} \right| \leq \frac{1}{(2k+1)|\sin t/2|}$$

and

$$|\sin \frac{t}{2}| \geq \frac{|t|}{\pi}.$$

Thus

$$\left| \frac{\sin(2k+1)\frac{t}{2}}{2k+1} \right| \leq \frac{1}{(2k+1)|\sin \frac{t}{2}|} \leq \frac{\pi}{(2k+1)|t|} \leq \frac{1}{2}, \tag{12}$$

where $|t| \geq \frac{2\pi}{2k+1}$. Therefore by (10) and (12) follows (11). This proves Lemma 2.

Proof of Theorem 1. It is enough to show that right-hand side of inequality (3) is correct. Since

$$P(\xi = 0) = \frac{1}{2\sqrt{\pi} B} + \frac{\Lambda}{B}$$

for $|j| \leq k$ we get

$$P(\xi = j) \geq \frac{1}{2\sqrt{\pi} B} e^{-\frac{j^2}{4B^2}} - \frac{\Lambda}{B} \geq \frac{1}{2\sqrt{\pi} B} e^{-\frac{k^2}{4B^2}} - \frac{\Lambda}{B},$$

which completes the proof (3).

Remark 2. Sometimes it is more convenient in application less sharp estimation of (3):

$$\vartheta_k(\eta) \leq \frac{2\Lambda}{B} + \frac{k^2}{8\sqrt{\pi} B^3}.$$

Finally, let us consider a sequence of independent identically distributed random variables $S_n = \xi_1 + \cdots + \xi_n$ taking only integer values, $A_n = ES_n$, $B_n^2 = DS_n$ and let

$$\sup_m \left| P(S_n = m) - \frac{1}{\sqrt{2\pi} B_n} e^{-\frac{(m - A_n)^2}{2B_n^2}} \right| \leq \frac{\lambda_n}{B_n}.$$

Then

$$I_n = B_n \int\limits_{\frac{2\pi}{2k_n+1} \leq |t| \leq \pi} |f(t, S_n)|^2 dt \leq c\left(\frac{c_1}{B_n} + c_2 \lambda_n \right), \tag{13}$$

where c, c_1 and c_2 are absolute constants and k_n equal to the integer part of

$$B_n \sqrt{\frac{c_1}{B_n} + c_2\lambda_n} \quad \text{we can set} \quad c = 2 + \frac{1}{8\sqrt{\pi}}, \quad c_1 = \frac{2(1+\sqrt{2\pi})}{\pi}, \quad c_2 = 2.$$

This inequality (13) can be exploited in order to estimate the number of summands needed to achieve a given accuracy in the local limit theorem (l.l.t.) [2].

Moreover: Firstly, the inequality (13) presents necessary condition for the applicability of the l.l.t. Secondly, since conditions for the validity of the central limit theorem (c.l.t.), are well known the question arises naturally what has to be added to the conditions for the central limit theorem in order that the local theorems holds.

Yu.Prokhorov's hypothesis was that from (c.l.t.) asymptotically uniformly distributed (a.u.d.) and infinite negligibility property (i.n.p.) follows l.l.t. Unfortunately this is not so, we construct a sequence of a.u.d. independent random variables for which c.l.t. and i.n.p. holds and at the same time the l.l.t. fails to hold, because the necessary condition $I_n \to 0$ (when $n \to \infty$) is violated (3) (see [3, 5, 7]).

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On the Nonlinear Filtering Equations for Superprocesses in Random Environment

Bronius Grigelionis

Abstract In the paper we define the Dawson-Watanabe type superprocesses in random environment as solutions to the related martingale problems. An environment is modelled by a finite state time homogeneous Markov process with the given transition probability intensity matrix. A system of nonlinear stochastic equations is derived for a posteriori probabilities. Reduced system of linear equations is also obtained.

Keywords Covariance operator • Cylindrical martingale • Dawson-Watanabe superprocess • Martingale problem • Nonlinear filtering • Random environment • Reduced equation • Stochastic integral

Mathematics Subject Classification (2010): 60J70, 60K37

1 Introduction

Dawson-Watanebe type of superprocesses in R^d arise as the scaling limits of branching particle systems, which undergo near critical branching and Markov spatial motions, characterized via Prokorov's relative compactness criterion as unique solutions to the related martingale problems (see, e.g., [9]).

Restricting ourselves to the finite variance state depending branching mechanisms and diffusions with jumps spatial motions, we shall arrive to the following definition.

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Let

$$C_b(R^d) = \{f : R^d \to R^1, f \text{ is continuous and bounded}\},$$
$$C_b^2(R^d) = \{f : R^d \to R^1, f \text{ is } C^2 \text{ with bounded partials of order 2 or less}\},$$

equipped with the topology of uniform convergence on compact sets,

$$(Af)(x) = \sum_{j=1}^{d} b_j(x)\frac{\delta f}{\delta x_j}(x) + \frac{1}{2}\sum_{j,k=1}^{d} a_{jk}(x)\frac{\delta^2 f}{\delta x_j \delta x_k}(x) + \int\limits_{R^d\setminus\{x\}} \left[f(y) - f(x) - 1_{|x-y|\le 1}(y)\sum_{j=1}(y_j - x_j)\frac{\delta f}{\delta x_j}(x)\right]\Pi(x, dy), \quad x \in R^d,$$

with coefficients, satisfying the standard Lipschitz continuity and linear growth conditions,

$$A : C_b^2(R^d) \to C_b(R^d),$$

$M_F(R^d)$ be a space of finite measures on R^d, endowed with the topology of weak convergence, $\mu(f) = \int\limits_{R^d} f(x)\mu(dx)$, $\gamma : R^d \to [0, \infty)$ and $h_j : R^d \to R^1, j = 0, 1, \ldots, N$ are continuous bounded functions, 1_A is an indicator function.

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis, $\mathbb{F} = \{\mathscr{F}_t, t \ge 0\}$, $M_{loc}(\mathbb{P}, \mathbb{F})$ be the class of $(\mathbb{P}, \mathbb{F})$ – local martingales, $M_{loc}^c(\mathbb{P}, \mathbb{F})$ be the class of continuous $(\mathbb{P}, \mathbb{F})$ – local martingales. (For used terminology and notations from stochastic analysis see, e.g., [5, 7]).

Consider a stochastic process $\{(\theta_t, X_t),\ t \ge 0\}$, taking values in $\{0, 1, \ldots, N\} \times M_F(R^d)$, where $\{\theta_t, t \ge 0\}$ is a finite state time homogeneous $(\mathbb{P}, \mathbb{F})$ – Markov chain with the transition probability intensity matrix $\Lambda = (\Lambda(j, k))_{0\le j,k\le N}$ and $\{X_t, t \ge 0\}$ is an $\mathbb{F}$ – adapted $M_F(R^d)$ – valued process such that, for each $f \in C_b^2(R^d)$,

$$M_t(f) := X_t(f) - X_0(f) - \int_0^t X_s(Af + h_{\theta_s} f)ds, \quad t \ge 0,$$

is a continuous $(\mathbb{P}, \mathbb{F})$ – local martingale, satisfying

$$\langle M(f)\rangle_t = \int\limits_0^t X_s(\gamma f^2)ds, \quad t \ge 0.$$

The process $\{X_t, t \geq 0\}$ we call the superprocess in random environment with the branching variance function γ, the conditional drift functions h_j, $j = 0, 1, \ldots, N$, and the diffusion with jumps spatial motions, defined by the generator A. The environment is modelled by the Markov chain θ. Existence of such processes and properties, in a sense analogous to the classical diffusions, can be easily derived from [9], Theorem II.5.1 and [8], Proposition 3.1.

Observe, that $\{M_t, t \geq 0\}$ is a cylindrical local martingale taking values in $\widetilde{M}_F(R^d) := M_F(R^d) - M_F(R^d)$ with the covariance operator function $Q_t : C_b^2(R^d) \to \widetilde{M}_F(R^d)$ as

$$Q_t(f)(dx) = \gamma(x) f(x) X_t(dx), \quad t \geq 0.$$

Let

$$\mathscr{F}_t^X = \bigcap_{\varepsilon > 0} \sigma\{X_s, s \leq t + \varepsilon\}, \quad \mathbb{F}^X = \{\mathscr{F}_t^X, t \geq 0\}.$$

Following [4], in this paper we shall derive stochastic differential equations for the a posteriori probabilities

$$\pi_j(t) := \mathbb{P}\{\theta_t = j | \mathscr{F}_t^X\}, \quad j = 0, 1, \ldots, N, \ t \geq 0,$$

and their reduced form.

2 Nonlinear Filtering Equations for Superprocesses in Random Environment

For $f \in C_b^2(R^d)$, write

$$\overline{M}_t(f) = X_t(f) - X_0(f) - \int_0^t X_s \left(Af + \sum_{j=0}^N h_j \pi_j(s) f \right) ds, \quad t \geq 0.$$

Lemma 1 (cf. [2]). *For each $f \in C_b^2(R^d)$, $\overline{M}(f) \in M_{loc}^c(\mathbb{P}, \mathbb{F}^X)$ and*

$$\langle \overline{M}(f) \rangle_t = \int_0^t X_s(\gamma f^2) ds, \quad t \geq 0. \tag{1}$$

Proof. Taking

$$T_n = \inf \left\{ t \geq 0 : \int_0^t X_s(\gamma f^2) ds = n \right\}, \quad n = 1, 2, \ldots,$$

as the localizing sequence of stopping times and observing that, for $0 \leq s < t$,

$$\overline{M}_t(f) - \overline{M}_s(f) = M_t(f) - M_s(f) + \int_s^t X_u(h_{\theta_u} f - \\ - \sum_{j=0}^{N} h_j \pi_j(u) f) du, \tag{2}$$

we find that, for each $n \geq 1$, P-a.s.

$$E(\overline{M}_{t\Lambda T_n}(f) - \overline{M}_{s\Lambda T_n}(f) | \mathscr{F}_s^X) = E\left(\int_{s\Lambda T_n}^{t\Lambda T_n} X_u(h_{\theta_u} f - \\ - \sum_{j=0}^{N} h_j \pi_j(u) f) du | \mathscr{F}_s^X \right) = 0,$$

proving that $\overline{M}(f) \in M_{loc}^c(\mathbb{P}, \mathbb{F}^X)$.

In order to prove (1), let $s = t_0^{(\nu)} < t_1^{(\nu)} < \ldots < t_\nu^{(\nu)} = t$ and $\max_{1 \leq k \leq \nu} (t_k^{(\nu)} - t_{k-1}^{(\nu)}) \to 0$ as $\nu \to \infty$. From (2) we have

$$\sum_{k=1}^{\nu} \left(\overline{M}_{t_k^{(\nu)} \wedge T_n}(f) - \overline{M}_{t_{k-1}^{(\nu)} \wedge T_n}(f) \right)^2 - \sum_{k=1}^{\nu} \left(M_{t_k^{(\nu)} \wedge T_n}(f) - M_{t_{k-1}^{(\nu)} \wedge T_n}(f) \right)^2 = \\ = 2 \sum_{k=1}^{\nu} \left(M_{t_k^{(\nu)} \wedge T_n}(f) - M_{t_{k-1}^{(\nu)} \wedge T_n}(f) \right) \left(\int_{t_{k-1}^{(\nu)} \wedge T_n}^{t_k^{(\nu)} \wedge T_n} X_u(h_{\theta_u} f - \\ - \sum_{j=0}^{N} h_j \pi_j(u) f) du \right) + \sum_{k=1}^{\nu} \left(\int_{t_{k-1}^{(\nu)} \wedge T_n}^{t_k^{(\nu)} \wedge T_n} X_u(h_{\theta_u} f - \sum_{j=0}^{N} h_j \pi_j(u) f) du \right)^2 \tag{3}$$

From Theorem 2 of [6, p. 92], there exists a subsequence $\{\nu_r, r \geq 1\}$ such that the sums on the left hand side of (3) converge $\mathbb{P}$-a.s. to $\langle \overline{M}(f) \rangle_{t \wedge T_n} - \langle \overline{M}(f) \rangle_{s \wedge T_n} - \langle M(f) \rangle_{t \wedge T_n} + \langle M(f) \rangle_{s \wedge T_n}$ as $\nu_r \to \infty$. The sums on the right hand side of (3), obviously, converge to 0 $\mathbb{P}$-a.s. as $\nu \to \infty$, implying that $\mathbb{P}$-a.s.

$$\langle \overline{M}(f) \rangle_t = \langle M(f) \rangle_t = \int_0^t X_s(\gamma f^2) ds, \quad t \geq 0.$$

□

Let $\mathscr{P}(\mathbb{F}^X)$ be the σ-algebra of $\mathbb{F}^X$-predictable subsets of $[0,\infty)\times\Omega$.

Let $\Phi^2_{loc}(Q,\mathbb{P},\mathbb{F}^X)$ be the class of $\mathscr{P}(\mathbb{F}^X)\bigotimes\mathscr{B}(R^d)$-measurable functions $\varphi:[0,\infty)\times\Omega\times R^d\to R^1$ such that, for each $t\geq 0$, $\mathbb{P}$-a.s.

$$\int_0^t\int_{R^d}\varphi^2(s,\omega,x)\gamma(x)X_s(dx)ds<\infty.$$

We shall further assume that, for each $t>0$,

$$E\exp\left\{\frac{1}{2}\int_0^t X_s(H\gamma)ds\right\}<\infty, \tag{4}$$

where $H(x)=\max\limits_{0\leq j\leq N}h_j^2(x),\quad x\in R^d$.

Lemma 2. *Under the assumption (4), each $\overline{L}\in M_{loc}(\mathbb{P},\mathbb{F}^X)$ has a form:*

$$\overline{L}_t=\overline{L}_0+\int_0^t\int_{R^d}\varphi(s,x)\overline{M}(ds,dx) \tag{5}$$

for some $\varphi\in\Phi^2_{loc}(Q,\mathbb{P},\mathbb{F}^X)$, where $\overline{M}(ds,dx)$ means Ito's stochastic integral with respect to the cylindrical local martingale $\overline{M}(f),\ f\in C_b^2(R^d)$ (see [7, 8])

Proof. Define the probability measure $\widehat{\mathbb{P}}$ by means of the equalities

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}|_{\mathscr{F}_t^X}=\exp\left\{-\int_0^t\int_{R^d}\overline{h}(s,x)\overline{M}(ds,dx)-\right.$$

$$\left.-\frac{1}{2}\int_0^t\int_{R^d}\left(\overline{h}(s,x)\right)^2\gamma(x)X_s(dx)ds\right\},\quad t\geq 0, \tag{6}$$

where $\overline{h}(t,x)=\sum\limits_{j=0}^N h_j(x)\pi_j(t)$.

Because

$$\left(\overline{h}(t,x)\right)^2\leq H(x)\sum_{j=0}^N\pi_j(t)=H(x), \tag{7}$$

for each $t \geq 0$, $\mathbb{P}$-a.s.

$$\int_0^t \int_{R^d} \left(\overline{h}(s,x)\right)^2 \gamma(x) X_s(dx)ds < \infty,$$

i.e. $\overline{h} \in \Phi^2_{loc}(Q, \mathbb{P}, \mathbb{F}^X)$. From the other hand, from (4) and (7) it follows that, for each $t > 0$,

$$E \exp\left\{\frac{1}{2}\int_0^t \int_{R^d} \left(\overline{h}(s,x))^2 \gamma(x) X_s(dx)\right) ds\right\} \leq$$

$$E \exp\left\{\frac{1}{2}\int_0^t X_s(H\gamma)ds\right\} < \infty.$$

From the Novikov's criterion we find that the definition (6) is correct and from the Girsanov's type theorem (see, e.g., [5]) we derive that, for each $f \in C_b^2(R^d)$, $\widetilde{M}(f) \in M^c_{loc}(\widehat{\mathbb{P}}, \mathbb{F}^X)$ and $\widehat{\mathbb{P}}$-a.s.

$$\langle \widetilde{M}(f) \rangle_t = \int_0^t X_s(\gamma f^2)ds, \quad t \geq 0,$$

where

$$\widetilde{M}_t := X_t(f) - X_0(f) - \int_0^t X_s(Af)ds. \tag{8}$$

Applying the uniqueness theorem in [9] and the Jacod's theorem on predictable stochastic integral representation of local martingales in [5], we conclude the proof of Lemma 2. □

Theorem 1. *Under the assumption (4), for each $t \geq 0$, $\mathbb{P}$-a.s.*

$$\pi_j(t) = \pi_j(0) + \int_0^t \sum_{k=0}^N \Lambda(k,j)\pi_k(s)ds +$$

$$+ \int_0^t \int_{R^d} \pi_j(s) \left[h_j(x) - \sum_{k=0}^N h_k(x)\pi_k(s)\right] \overline{M}(ds,dx), \quad j = 0, 1, \ldots, N.$$

Proof. is based on the Lemma 2 and the properties of semimartingales, reducing the filtrations and changing probability measures (cf. [1,2,4]).

Observe that for any $g : \{0, 1, \ldots, N\} \to R^1$, $g(\theta_t),\ t \geq 0$, is a $(\mathbb{P}, \mathbb{F})$-special semimartingale such that

$$g(\theta_t) - g(\theta_0) - \int_0^t \sum_{k=0}^N g(k)\Lambda(\theta_s, k)ds, \quad t \geq 0, \tag{9}$$

is a $(\mathbb{P}, \mathbb{F})$-local martingale and

$$E\big(g(\theta_t)|\mathscr{F}_t^X\big) - E\big(g(\theta_0)|\mathscr{F}_0^X\big) - \int_0^t \sum_{j,k=0}^N g(k)\Lambda(j,k)\pi_j(s)ds, \quad t \geq 0, \tag{10}$$

is a $(\mathbb{P}, \mathbb{F}^X)$-local martingale.

Taking $g_j(k) = 1_{\{j\}}(k)$, $j, k = 0, 1, \ldots, N$, from (10) we have that $\overline{L}_j \in M_{loc}(\mathbb{P}, \mathbb{F}^X)$, $j = 0, 1, \ldots, N$, where

$$\overline{L}_j(t) := \pi_j(t) - \pi_j(0) - \int_0^t \sum_{k=0}^N \Lambda(k, j)\pi_k(s)ds, \quad t \geq 0.$$

From Lemma 2, for any $j = 0, 1, \ldots, N$, there exists $\varphi_j \in \Phi^2_{loc}(Q, \mathbb{P}, \mathbb{F}^X)$ such that

$$\overline{L}_j(t) = \overline{L}_j(0) + \int_0^t \int_{R^d} \varphi_j(s, x)\overline{M}(ds, dx), \quad t \geq 0.$$

It remains to identify that up to equivalence as elements of $\Phi^2_{loc}(Q, \mathbb{P}, \mathbb{F}^X)$

$$\varphi_j(s, x) = \pi_j(s)\left(h_j(x) - \sum_{k=0}^N h_k(x)\pi_k(s)\right).$$

The technical details, using Lemma 2, are standard (cf.[1,2,4]) and are omitted here. □

Remark 1. If A is a generator of a d-dimensional Lévy process, then we easily have that $A : C_b^2(R^d) \to C_b(R^d)$.

Now let us define the probability measure $\widetilde{\mathbb{P}}$ by means of the equalities

$$\left.\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathscr{F}_t} =: \widetilde{Z}_t, \quad t \geq 0,$$

where $\widetilde{Z}_t$, $t \geq 0$, solves the linear stochastic equation

$$\widetilde{Z}_t = 1 - \int_0^t \int_{R^d} \widetilde{Z}_s h_{\theta_s}(x) M(ds, dx), \quad t \geq 0,$$

i.e.

$$\widetilde{Z}_t = \exp\left\{-\int_0^t \int_{R^d} h_{\theta_s}(x) M(ds, dx) - \frac{1}{2}\int_0^t X_s(h_{\theta_s}^2 \gamma) ds\right\}, \quad t \geq 0.$$

If (4) is fulfilled, the definition is correct, because

$$|h_{\theta_t}(x)|^2 \leq H(x), \quad x \in R^d.$$

Lemma 3. *Under the assumption* (4)*,* $\widetilde{M}(f)$*,* $f \in C_b^2(R^d)$*, defined by* (8)*, is the cylindrical* $(\widetilde{\mathbb{P}}, \mathbb{F})$*-local martingale,*

$$\langle \widetilde{M}(f) \rangle_t = \int_0^t X_s(\gamma f^2) ds$$

and each $\widetilde{L} \in M_{loc}(\widetilde{\mathbb{P}}, \mathbb{F}^X)$ *has a form:*

$$\widetilde{L}_t = \widetilde{L}_0 + \int_0^t \int_{R^d} \widetilde{\varphi}(s, x) \widetilde{M}(ds, dx), \quad t \geq 0,$$

for some $\widetilde{\varphi} \in \Phi_{loc}^2(Q, \widetilde{\mathbb{P}}, \mathbb{F}^X)$.

Proof. Similarly to the proof of Lemma 2, the statement of Lemma 3 follows from Girsanov's type theorem, uniqueness theorem in [9] and the Jacod's theorem in [5]. □

Observe that the inverse density

$$Z_t = \frac{1}{\widetilde{Z}_t} = \exp\left\{\int_0^t \int_{R^d} h_{\theta_s}(x) \widetilde{M}(ds, dx) - \right.$$

$$\left. - \frac{1}{2}\int_0^t \int_{R^d} h_{\theta_s}^2(x) \gamma(x) X_s(dx) ds\right\}, \quad t \geq 0,$$

solves the equation

$$Z_t = 1 + \int_0^t \int_{R^d} Z_s h_{\theta_s}(x) \widetilde{M}(ds, dx), \quad t \geq 0. \tag{11}$$

Theorem 2 (cf. [3,4]). *If* (4) *is fulfilled, for each* $t \geq 0$

$$\pi_j(t) = \frac{\widetilde{\pi}_j(t)}{\sum_{k=0}^{N} \widetilde{\pi}_j(t)}, \quad t \geq 0, \; j = 0, 1, \ldots, N,$$

where $\widetilde{\pi}_j(t), \quad t \geq 0, \quad j = 0, 1, \ldots, N$*, solve the reduced nonlinear filtering equations:*

$$\widetilde{\pi}_j(t) = \widetilde{\pi}_j(0) + \int_0^t \sum_{k=0}^{N} \Lambda(k, j) \widetilde{\pi}_k(s) ds +$$

$$+ \int_0^t \int_{R^d} \widetilde{\pi}_j(s) h_j(x) \widetilde{M}(ds, dx), \quad t \geq 0, \; j = 0, 1, \ldots, N.$$

Proof. Let $\widetilde{E}$ be the mean value with respect to the probability measure $\widetilde{\mathbb{P}}$,

$$\widetilde{\pi}_j(t) := \widetilde{E}(Z_t 1_{\{\theta_t = j\}} | \mathscr{F}_t^X), \quad t \geq 0, j = 0, 1, \ldots, N.$$

From the Bayes formula

$$\widetilde{\pi} c_j(t) = \frac{\widetilde{E}(Z_t 1_{\{\theta_t = j\}} | \mathscr{F}_t^X)}{\widetilde{E}(Z_t | \mathscr{F}_t^X)} = \frac{\widetilde{\pi}_j(t)}{\widetilde{E}(Z_t \sum_{k=0}^{N} 1_{\{\theta_t = k\}} | \mathscr{F}_t^X)}$$

$$= \frac{\widetilde{\pi}_j(t)}{\sum_{k=0}^{N} \widetilde{\pi}_k(t)}, \quad t \geq 0, \; j = 0, 1, \ldots, N.$$

Because $Z_t, t \geq 0$, is a continuous $(\widetilde{\mathbb{P}}, \mathbb{F})$-martingale and

$$1_{\{\theta_t = j\}} - 1_{\{\theta_0 = j\}} - \int_0^t \Lambda(\theta_{s,j}) ds, \; j = 0, 1, \ldots, N, \tag{12}$$

are purely discontinuous $(\mathbb{P}, \mathbb{F})$ – martingales, then (12) define $(\widetilde{\mathbb{P}}, \mathbb{F})$-martingales also (see [5]).

From (11), (12) and Ito's formula we find that

$$1_{\{\theta_t = j\}} Z_t - 1_{\{\theta_0 = j\}} Z_0 - \int_0^t Z_s \Lambda(\theta_s, j) ds, \quad t \geq 0, \; j = 0, 1, \ldots, N,$$

are $(\widetilde{\mathbb{P}}, \mathbb{F})$-local martingales and, having in mind that

$$\Lambda(\theta_s, j) = \sum_{k=0}^{N} \Lambda(k, j) 1_{\{\theta_s = k\}},$$

$$\widetilde{L}_j(t) := \widetilde{E}(Z_t 1_{\{\theta_t = j\}} | \mathscr{F}_t^X)) - \widetilde{E}(Z_0 1_{\theta_0 = j} | \mathscr{F}_0^X) - \\ - \int_0^t \sum_{k=0}^{N} \Lambda(k, j) \widetilde{E}(Z_s 1_{\theta_s = k} | \mathscr{F}_s^X) ds, \quad t \geq 0, \quad j = 0, 1, \ldots, N,$$

are $(\widetilde{\mathbb{P}}, \mathbb{F}^X)$-local martingales. From Lemma 3, for any $j = 0, 1, \ldots, N$, there exists $\widetilde{\varphi}_j \in \Phi_{loc}^2(Q, \widetilde{\mathbb{P}}, \mathbb{F}^X)$ such that

$$\widetilde{L}_j(t) = \widetilde{L}_j(0) + \int_0^t \int_{R^d} \widetilde{\varphi}_j(s, x) \widetilde{M}(ds, dx), \quad t \geq 0.$$

It remains to identify that up to equivalence as elements of $\Phi_{loc}^2(Q, \widetilde{\mathbb{P}}, \mathbb{F}^X)$

$$\widetilde{\varphi}_j(s, x) = \widetilde{\pi}_j(s) h_j(x), \quad s \geq 0, \quad x \in R^d.$$

The technical details, using Lemma 3 are standard (cf. [1,2,4]) and are again omitted here. □

Example 1 *(change-point model).* Let

$$\theta_t = \begin{cases} 0, & \text{if } t < T, \\ 1, & \text{if } t \geq T, \end{cases}$$

where $P\{T > t\} = e^{-\Lambda t}$, $t \geq 0$, $\Lambda > 0$.

In this case $N = 1$, $\Lambda(0,0) = 0$, $\Lambda(1,0) = 0$, $\Lambda(0,1) = \Lambda$, $\Lambda(1,1) = -\Lambda$, $\pi_0(t) = 1 - \pi_1(t)$, $t \geq 0$.

Thus, the following equations hold true:

$$\pi_1(t) = \pi_1(0) + \Lambda \int_0^t (1 - 2\pi_1(s))ds +$$

$$+ \int_0^t \int_{R^d} \pi_1(s)(1 - \pi_1(s))(h_1(x) - h_0(x))\overline{M}(ds, dx), t \geq 0,$$

and

$$\widetilde{\pi_0}(t) = \widetilde{\pi_0}(0) + \int_0^t \int_{R^d} \widetilde{\pi_0}(s)h_0(x)\widetilde{M}(ds, dx), \quad t \geq 0,$$

$$\widetilde{\pi_1}(t) = \widetilde{\pi_1}(0) + \Lambda \int_0^t (\widetilde{\pi_0}(s) - \widetilde{\pi}_1(s))ds +$$

$$+ \int_0^t \int_{R^d} \widetilde{\pi}_1(s)h_1(x)\widetilde{M}(ds, dx), \quad t \geq 0.$$

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Upper Bounds for Bernstein Basis Functions

Vijay Gupta and Tengiz Shervashidze

Abstract From Markov's bounds for binomial coefficients (for which a short proof is given) upper bounds are derived for Bernstein basis functions of approximation operators and their maximum. Some related inequalities used in approximation theory and those for concentration functions are discussed.

Keywords Bernstein basis functions for approximation operators • Markov bounds for binomial coefficients • Zeng's upper bounds for binomial probabilities • Extension of upper bounds for binomial probabilities via discretization of the argument. Rogozin's and some other inequalities for concentration functions

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1 Markov's Bounds for Binomial Coefficients. Preliminaries

One can get upper bounds for Bernstein basis functions of approximation operators, i.e., binomial probabilities

$$b(k;n,p) = C_n^k p^k (1-p)^{n-k}, \quad p \in [0,1], \quad k = 0,1,\ldots,n,$$

using direct analytic or probabilistic methods.

First estimates of $b(k;n,p)$ can be found in "Ars Conjectandi" by J. Bernoulli, see [3] and commentary by Yu.V. Prokhorov "Law of Large Numbers and Estimates for Probabilities of Large Deviations" on pp. 116–155 in the same [3]. Using an additional argument together with one to obtain the Stirling formula Markov proved the double inequality for binomial coefficients C_n^k which we prefer to write in the form of bounds for $b(k;n,p)$ (see [12], pp. 72, 73 or formula (16) on p. 135 in above mentioned commentary in [3]; cf. formula (135) in Chap. IV "The rate of approximation of functions by linear positive operators" of [11]):

Theorem A. *Let* $n \geq 1$, $k \geq 1$, $n-k \geq 1$ *and* $p \in (0,1)$. *Then*

$$e^{\frac{1}{12n}-\frac{1}{12k}-\frac{1}{12(n-k)}} \sqrt{\frac{n}{2\pi k(n-k)}} \left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k}$$
$$< b(k;n,p) < \sqrt{\frac{n}{2\pi k(n-k)}} \left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k} =: \mathrm{Ma}(k;n,p). \quad (1)$$

Let us give a short proof of (1) with $1/(12n+1)$ instead of $1/(12n)$ in the exponent in the left-hand side.

Proof. The proof is based on the double inequality which refines Stirling asymptotics

$$(2\pi)^{1/2} n^{n+1/2} e^{-n+1/(12n+1)} < n! < (2\pi)^{1/2} n^{n+1/2} e^{-n+1/(12n)} \quad (2)$$

(see Feller's book [5], Chap. II, and Robbins' paper [15] referred therein).

Due to (2) we have

$$C_n^k = n!/[k!(n-k)!] < [n/(2\pi\, k(n-k))]^{1/2} n^n\, k^{-k}\, (n-k)^{-(n-k)}$$
$$\times \exp[1/(12n) - 1/(12k+1) - 1/(12(n-k)+1)]. \quad (3)$$

The nominator of the latter exponent equals to

$$(12k+1)(12(n-k)+1) - 12n(12n+2)$$
$$= 144[k(n-k) - (1/4)n^2] - 108n^2 - 12n + 1,$$

which is negative for each $n > 1$ and k. Multiplication of both sides of inequality (3) by $p^k(1-p)^{n-k}$ completes the proof of right-hand inequality of (1). Dealing with the left-hand inequality similarly we find that the exponent is negative, too, both in initial and weakened form.

From (1) immediately follows that for some p and n the binomial probabilities $b(np; n, p)$ is less than its De Moivre–Laplace asymptotic expression.

Corollary 1. (a) *For any rational* $p \in (0, 1)$ *and* n *such that* np *is an integer*

$$b(np; n, p) < \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{np(1-p)}} =: \mathrm{MoLa}(n, p). \quad (4)$$

(b) *Inequality* (4) *is valid for* $b(k_0(n); n, p)$ *with* $p = k_0(n)/n$ *for any integer* $k_0(n)$ *such that* $0 < k_0(n) < n$.
(c) *In both cases* (a) *and* (b) *inequality* (4) *holds for* $b(k; n, p)$ *with any* $k = 0, 1, \ldots, n$.
(d) *The constant* $\frac{1}{\sqrt{2\pi}}$ *in* (4) *is best possible.*

It is worth to mention that in the standard situation when for binomial probabilities Poisson's asymptotic formula is valid, i.e., $b(k; n, p) - \mathrm{Po}(k; np) \to 0$ as $n \to \infty$, $p \to 0$ and np remaining bounded, for any fixed $k \in N = \{0, 1, \ldots\}$ with $\mathrm{Po}(k; \lambda) = \lambda^k e^{-\lambda}/k!$, $\lambda > 0$, one can derive the following representations of $\mathrm{Ma}(k; n, p)$ as upper bounds for $b(k; n, p)$ and $b(n-k; n, p)$ for fixed k and $n-k$ respectively.

Corollary 2. *If* k *is fixed, then for* $n > k$

$$b(k; n, p) < \ell\mathrm{Ma}(k; n, p)$$
$$:= \mathrm{Po}(k; np) \sqrt{\frac{n}{n-k}} \frac{k!}{\sqrt{2\pi k}(k/e)^k} e^{np-k} \left(1 + \frac{k-np}{n-k}\right)^{n-k}. \quad (5)$$

If $l = n - k$ *is fixed, then for* $n > l$

$$b(l; n, p) = b(n-l; n, 1-p) < \ell\mathrm{Ma}(n-l; n, 1-p) =: \mathrm{rMa}(l; n, p). \quad (6)$$

The chain of results which has inspired our small contribution has began by the inequality established and used by Guo [7], to estimate the rate of convergence of the Durrmeyer operators for functions of bounded variation. His proof was based on the Berry–Esseen theorem; Guo obtained the inequality

$$b(k; n, p) \le \frac{C}{\sqrt{np(1-p)}}, \quad p \in (0, 1), \quad 0 \le k \le n,$$

with $C = 5/2$. In the year 1998, Zeng [17] has improved this bound having proved the following assertion.

Theorem B. *For a fixed $j \in N$ and*

$$C_j = ((j+1/2)^{j+1/2}/j!)e^{-(j+1/2)} \tag{7}$$

for all k, p such that $j \leq k \leq n-j$, $p \in (0,1)$, there holds

$$b(k;n,p) < \frac{C_j}{\sqrt{np(1-p)}} =: Z_j(n,p). \tag{8}$$

Moreover, the coefficient C_j is best possible (that is to say, for arbitrary $\varepsilon > 0$, it can not be replaced by $C_j - \varepsilon$), and the estimate order $n^{-1/2}$ is the optimal also.

The sequence of constants C_j decreases strictly and

$$\lim_{j\to\infty} C_j = \frac{1}{\sqrt{2\pi}}.$$

Hence for all $j \in N$, there holds

$$\frac{1}{\sqrt{2\pi}} < C_j \leq C_0 = \frac{1}{\sqrt{2e}}. \tag{9}$$

In particular, for $j = 0$ (8) reduces to

$$b(k;n,p) < \frac{1}{\sqrt{2enp(1-p)}} = Z_0(n,p), \quad p \in (0,1), \quad 0 \leq k \leq n. \tag{10}$$

Bastien and Rogalski solved in [2] a problem posed by V. Gupta in a private communication, having given there another proof that the upper bound (10) obtained by Zeng [17] is the optimum.

In the year 2001 Zeng and Zhao [18] have obtained the bound (4) for Bernstein basis functions (in fact assertions (b), (c) and (d) of our Corollary 1 of Theorem A from [11] and [3]).

In [1, 9] and [8] upper bound (10) is used to obtain the rate of convergence for Bernstein–Durrmeyer operators. Here we present the result of our collaboration to investigate the above mentioned problem concerning the optimal constant in the inequality (10).

Our first observation is that the inequalities given by Corollary 1 and Theorem B, namely relations (4) and (10) in fact are estimates for maximal probability of binomial distribution

$$b(n,p) = \max_{0\leq k\leq n} b(k;n,p).$$

It is well-known that due to De Moivre–Laplace local limit theorem, for $p \in (0,1)$ $b(n,p)$ is equivalent to

$$(2\pi np(1-p))^{-1/2}$$

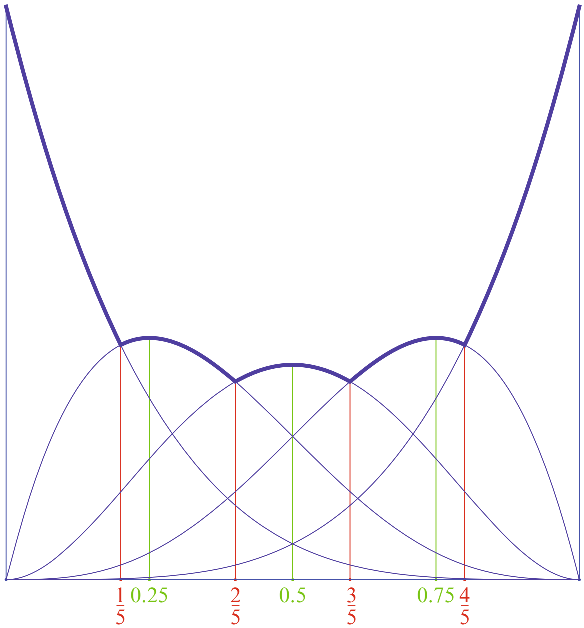

Fig. 1 Graphs of $b(k; n, p)$ as functions of $p \in [0, 1]$, $k = 0, \ldots, n$ for $n = 4$, their maxima and intersection points. In this figure, $b(n, p)$ is drawn by a *thick line*

as $n \to \infty$ (a nice proof is given in Feller's book [5], Chap. VII). It turns out that the latter expression is at the same time an upper bound for $b(n, p)$ for rational p and n such that np is an integer. The above equivalence shows that dependence on n and the constant in this upper bound are optimal. The fine structure of the system of modal binomial values $m = [(n+1)p]$, where $[\cdot]$ denotes the integer part, leads to an immediate upper bound for any n and p by substitution of p with the step function $p^* = m/n$; see Fig. 1 and a few useful facts concerning m, namely:

(a) The most probable value (or modal value or mode) m of the binomial distribution is defined by the inequality

$$(n+1)p - 1 < m \leq (n+1)p, \tag{11}$$

if $m = (n+1)p$, there are two modal values $b(m-1; n, p) = b(m; n, p)$.

(b) The suitable binomial probability is not greater than maximum of $b(m; n, p)$ in p attained at $p = p^* = m/n$, that is

$$b(m; n, p) \leq b(m; n, p^*). \tag{12}$$

2 Bounds for $b(n, p)$

The following proposition is in fact a reformulation of Corollary 1 for $b(n, p)$.

Proposition 1. *For any* $k_0 = k_0(n)$ *such that* $0 < k_0(n) < n$ *and* $p = k_0(n)/n$ *there holds*

$$b(n, k_0(n)/n) < \mathrm{MoLa}(n, p). \tag{13}$$

The estimate coefficient $\frac{1}{\sqrt{2\pi}}$ is the best possible.

In particular, for a constant rational probability $p, 0 < p < 1$, and n such that np is an integer, for $b(n, p) = b(np; n, p)$ inequality (13) holds true.

The right-hand side of inequality (12) is covered by Proposition 1. Thus we obtain

Proposition 2. *Define for* $0 < p < 1$ *the function* $p^* = p^*(p) = m/n$*, where* $m = m(p) = [(n + 1)p]$ *is the (maximal) mode of binomial distribution (*m/n *is equal to* 0 *on* $(0, 1/(n + 1))$*, to* $1/n$ *on* $[1/(n + 1), 2/(n + 1)), \ldots$ *and to* 1 *on* $[n/(n + 1), 1)$*). Then for any* n *and* $1/(n + 1) \leq p < n/(n + 1)$ *the inequality*

$$b(n, p) < (2\pi np^*(1 - p^*))^{-1/2} = \mathrm{MoLa}(n, p^*) \tag{14}$$

holds.

Proposition 3. *For any* $p \in [1/(n + 1), n/(n + 1))$ *we have*

$$b(n, p) < \mathrm{Ma}(m; n, p) = \frac{\left(\frac{p}{p^*}\right)^{np^*}\left(\frac{1-p}{1-p^*}\right)^{n(1-p^*)}}{\sqrt{2\pi np^*(1 - p^*)}}. \tag{15}$$

Let us now try to discuss whether Propositions 2 and 3 have some advantage in approximation theory compared with the curves

$$z_0(n, p) = 1 \vee Z_0(n, p) = 1 \vee (2enp(1 - p))^{-1/2}, p \in (0, 1),$$
$$z_1(n, p) = 1 \vee Z_1(n, p) = 1 \vee C_1(np(1 - p))^{-1/2}, p \in (0, 1)$$

(cf.(5) and (10); see (4) and (9) for C_1), which seem natural to be introduced as $b(n, p)$ does not exceed 1.

Denote

$$v(n, p) = 1 \vee \mathrm{MoLa}(n, p), \quad p \in (0, 1).$$

Our results make it meaningful to consider the function $v^*(n, p)$ as $v(n, p^*)$ which reduces the interval $(0, 1)$ for p to

$$1/(n + 1) \leq p < n/(n + 1);$$

out of this range lie the values of p for which $m = 0$ or $m = n$ which correspond to the values 0 and 1 for p^* excluded in the proposition. So we are motivated to

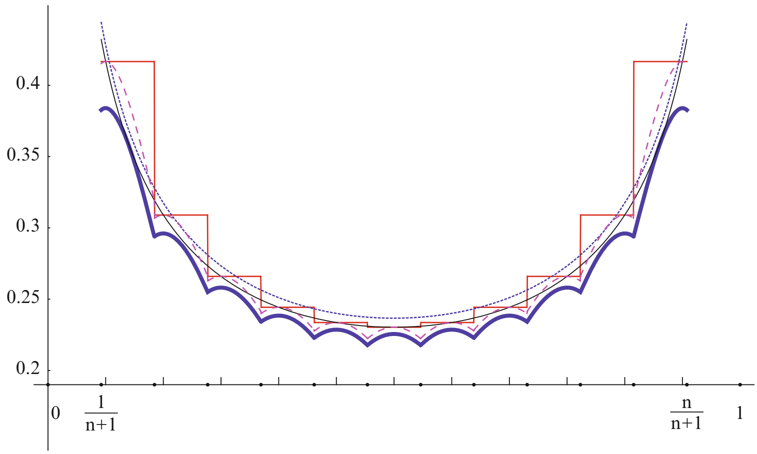

Fig. 2 Approximations of $b(n, p)$ ($\frac{1}{n+1} \leq p < \frac{n}{n+1}, n = 12$) *Thick line:* $b(n, p)$, *Dashed line:* $\mathrm{Ma}(m; n, p)$, *Step line:* $\mathrm{MoLa}(n, p^*)$, *Pointed line:* $Z_1(n, p)$, *Thin line:* $\mathrm{MoLa}(n, p)$

introduce probabilities $b(0; n, p)$ and $b(n; n, p)$ on corresponding intervals for p as extra summands into modified $v^*(n, p)$:

$$v^{**}(n, p) = v^*(n, p) + (1 - p)^n I_{(0,1/(n+1))}(p) + p^n I_{[n/(n+1),1)}(p),$$

where $I_E(p)$ stands for the indicator of a set E.

Figure 2 illustrates the fact that at least for p from some neighborhood of 1/2 the curves $z_0(n, p)$ and $z_1(n, p)$ lie over $v^{**}(n, p)$. In the same sense $\mathrm{Ma}(m; n, p)$ behaves much better.

In all the papers where Zeng's inequality (10) is used to obtain approximation estimations, see, e.g., [1, 8, 9], those will be evidently improved using inequalities (14) and (15).

As for each fixed k and l $b(k; n, k/n)$ and $b(n - l; n, 1 - l/n)$, according to Prokhorov's famous result (1953) [14], is better to treat via Poisson approximation than by normal one, this way may lead to better estimates useful for approximation theory.

Being motivated by this advantage for p close to 0 or 1, we tried to explore the following expression, using for $\mathrm{Ma}(k; n, p)$ the representations $\ell\,\mathrm{Ma}(k; n, p)$ for $k < n/2, 0 < p < 1/2$ and $r\,\mathrm{Ma}(k; n, p)$ for $k \geq n/2, 1/2 \leq p < 1$ (see relations (5) and (6)), each without two factors tending to one from three such ones:

$$\mathrm{Ma}^*(n, p) = I_{(0,1/2)}(p) \max_{0 \leq k < n/2} \sqrt{n/(n-k)}\, \mathrm{Po}(k; np) + I_{[1/2,1)}(p) \max_{n/2 \leq k < n} \sqrt{n/k}\, \mathrm{Po}(n - k; n(1 - p)).$$

Computer experiment shows that $\mathrm{Ma}^*(n, p)$ fits with $b(n, p)$ much better than $\mathrm{Ma}(m, n, p)$. This phenomenon is to be explained with theoretical argument.

An alternative way to construct estimates $b(n, p) = O(n^{-1/2})$ for Bernstein basis functions and similar ones for some other basis functions goes via inequalities for concentration functions of the sum S_n of the integer-valued i.i.d. random variables $\xi_1, \dots, \xi_n$, namely for maximal probabilities of such a sum. For example, Rogozin gave in [16] the estimate which implies that

$$\max_k P(S_n = k) \le c((1 - p_0)n)^{-1/2}, \tag{16}$$

where p_0 stands for the maximal probability of each summand and c is an absolute constant.

In the case of binomial distribution $p_0 = p \vee (1-p)$ and as $1-p_0 = p \wedge (1-p)$, we have $p(1-p) < 1-p_0$ in (0,1) and thus dependence on p in Rogozin's inequality turns out to be better. As for the constant c its comparison with De Moivre–Laplace asymptotic expression shows that $c \ge 1/\pi^{1/2}$. The upper bound 2π for this constant is available from [13] (the suitable inequality is wrongly reproduced in the Russian translation of [10]). A general explanation of optimality of the order $n^{-1/2}$ in bounds of type of (16) can be found in [4] (see also [10] and [6]).

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On Distribution of Zeros of Random Polynomials in Complex Plane

Ildar Ibragimov and Dmitry Zaporozhets

Abstract Let $G_n(z) = \xi_0 + \xi_1 z + \cdots + \xi_n z^n$ be a random polynomial with i.i.d. coefficients (real or complex). We show that the arguments of the roots of $G_n(z)$ are uniformly distributed in $[0, 2\pi]$ asymptotically as $n \to \infty$. We also prove that the condition $\mathbf{E} \ln(1 + |\xi_0|) < \infty$ is necessary and sufficient for the roots to asymptotically concentrate near the unit circumference.

Keywords Roots of random polynomial • Roots concentration • Random analytic function

Mathematics Subject Classification (2010): 60-XX, 30C15

1 Inroduction: Problem and Results

Let $\{\xi_k\}_{k=0}^{\infty}$ be a sequence of independent identically distributed real- or complex-valued random variables. It is always supposed that $\mathbf{P}(\xi_0 = 0) < 1$.

Consider the sequence of random polynomials

$$G_n(z) = \xi_0 + \xi_1 z + \cdots + \xi_{n-1} z^{n-1} + \xi_n z^n.$$

By $z_{1n}, \ldots, z_{nn}$ denote the zeros of G_n. It is not hard to show (see [1]) that there exists an indexing of the zeros such that for each $k = 1, \ldots, n$ the k-th zero z_{kn} is a one-valued random variable. For any measurable subset A of complex plain

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$\mathbb{C}$ put $N_n(A) = \#\{z_{kn} : z_{kn} \in A\}$. Then $N_n(A)/n$ is a probability measure on the plane (the empirical distribution of the zeros of G_n). For any a, b such that $0 \leqslant a < b \leqslant \infty$ put $R_n(a,b) = N_n(\{z : a \leqslant |z| \leqslant b\})$ and for any α, β such that $0 \leqslant \alpha < \beta \leqslant 2\pi$ put $S_n(\alpha, \beta) = N_n(\{z : \alpha \leqslant \arg z \leqslant \beta\})$. Thus R_n/n and S_n/n define the empiric distributions of $|z_{kn}|$ and $\arg z_{kn}$.

In this paper we study the limit distributions of N_n, R_n, S_n as $n \to \infty$.

The question of the distribution of the complex roots of G_n have been originated by Hammersley in [1]. The asymptotic study of R_n, S_n has been initiated by Shparo and Shur in [16]. To describe their results let us introduce the function

$$f(t) = \Big[\underbrace{\log^+ \log^+ \ldots \log^+ t}_{m+1}\Big]^{1+\varepsilon} \prod_{i=1}^{m} \underbrace{\log^+ \log^+ \ldots \log^+ t}_{i},$$

where $\log^+ s = \max(1, \log s)$. We assume that $\varepsilon > 0, m \in \mathbb{Z}^+$ and $f(t) = (\log^+ t)^{1+\varepsilon}$ for $m = 0$.

Shparo and Shur have proved in [16] that if

$$\mathbf{E}\, f(|\xi_0|) < \infty$$

for some $\varepsilon > 0, m \in \mathbb{Z}^+$, then for any $\delta \in (0,1)$ and α, β such that $0 \leqslant \alpha < \beta \leqslant 2\pi$

$$\frac{1}{n} R_n(1-\delta, 1+\delta) \xrightarrow{\mathbf{P}} 1, \quad n \to \infty,$$

$$\frac{1}{n} S_n(\alpha, \beta) \xrightarrow{\mathbf{P}} \frac{\beta - \alpha}{2\pi}, \quad n \to \infty.$$

The first relation means that under quite weak constraints imposed on the coefficients of a random polynomial, almost all its roots "concentrate uniformly" near the unit circumference with high probability; the second relation means that the arguments of the roots are asymptotically uniformly distributed.

Later Shepp and Vanderbei [15] and Ibragimov and Zeitouni [5] under additional conditions imposed on the coefficients of G_n got more precise asymptotic formulas for R_n.

What kind of further results could be expected? First let us note that if, e.g., $\mathbf{E}\,|\xi_0| < \infty$, then for $|z| < 1$

$$G_n(z) \to G(z) = \sum_{k=0}^{\infty} \xi_k z^k$$

as $n \to \infty$ a.s. The function $G(z)$ is analytical inside the unit disk $\{|z| < 1\}$. Therefore for any $\delta > 0$ it has only a finite number of zeros in the disk $\{|z| < 1 - \delta\}$. At the other hand, the average number of zeros in the domain $|z| > 1/(1-\delta)$

is the same (it could be shown if we consider the random polynomial $G(1/z)$). Thus one could expect that under sufficiently weak constraints imposed on the coefficients of a random polynomial the zeros concentrate near the unit circle $\Gamma = \{z : |z| = 1\}$ and a measure R_n/n converges to the delta measure at the point one. We may expect also from the consideration of symmetry that the arguments $\arg z_{kn}$ are asymptotically uniformly distributed. Below we give the conditions for these hypotheses to hold. We shall prove the following three theorems about the behavior of $N_n/n, R_n/n, S_n/n$.

For the sake of simplicity, we assume that $\mathbf{P}\{\xi_0 = 0\} = 0$. To treat the general case it is enough to study in the same way the behavior of the roots on the sets $\{\theta_n' = k, \theta_n'' = l\}$, where

$$\theta_n' = \max\{i = 0, \ldots, n \mid \xi_i \neq 0\}, \qquad \theta_n'' = \min\{j = 0, \ldots, n \mid \xi_j \neq 0\}.$$

Theorem A. *The sequence of the empirical distributions R_n/n converges to the delta measure at the point one almost surely if and only if*

$$\mathbf{E} \log(1 + |\xi_0|) < \infty. \tag{1}$$

In other words, (1) is necessary and sufficient condition for

$$\mathbf{P}\left\{\frac{1}{n} R_n(1-\delta, 1+\delta) \underset{n\to\infty}{\longrightarrow} 1\right\} = 1 \tag{2}$$

hold for any $\delta > 0$.

We shall also prove that if (1) does not hold then no limit distribution for $\{z_{nk}\}$ exist.

Theorem B. *Suppose the condition* (1) *holds. Then the empirical distribution N_n/n almost surely converges to the probability measure $N(\cdot) = \mu(\cdot \cap \Gamma)/(2\pi)$, where $\Gamma = \{z : |z| = 1\}$ and μ is the Lebesgue measure.*

Theorem C. *The empirical distribution S_n/n almost surely converges to the uniform distribution, i.e.,*

$$\mathbf{P}\left\{\frac{1}{n} S_n(\alpha, \beta) \underset{n\to\infty}{\longrightarrow} \frac{\beta - \alpha}{2\pi}\right\} = 1$$

for any α, β such that $0 \leqslant \alpha < \beta \leqslant 2\pi$.

Let us remark here that Theorem C does not require any additional conditions on the sequence $\{\xi_k\}$.

The next result is of crucial importance in the proof of Theorem C.

Theorem D. *Let $\{\eta_k\}_{k=0}^{\infty}$ be a sequence of independent identically distributed real-valued random variables. Put $g_n(x) = \sum_{k=0}^{n} \eta_k x^k$ and by M_n denote the number of real roots of the polynomial $g_n(x)$. Then*

$$\mathbf{P}\left\{\frac{M_n}{n}\underset{n\to\infty}{\longrightarrow}0\right\}=1,\quad \mathbf{E}\,M_n=o(n),\quad n\to\infty.$$

Theorem D is also of independent interest. In a number of papers it was shown that under weak conditions on the distribution of η_0 one has $\mathbf{E}\,M_n \sim c\times\log n$, $n\to\infty$ (see [2–4, 6, 9, 10]). L. Shepp proposed the following conjecture: for any distribution of η_0 there exist positive numbers c_1, c_2 such that $\mathbf{E}\,M_n \geqslant c_1\times\log n$ and $\mathbf{E}\,M_n \leqslant c_2\times\log n$ for all n. The first statement was disproved in [17, 18]. There was constructed a random polynomial $g_n(x)$ with $\mathbf{E}\,M_n < 1+\varepsilon$. It is still unknown if the second statement is true. However, Theorem D shows that an arbitrary random polynomial can not have too much real roots (see also [14]).

In fact, in the proof of Theorem C we shall use a slightly generalized version of Theorem D:

Theorem E. *For some integer r consider a set of r non-degenerate probability distributions. Let $\{\eta_k\}_{k=0}^{\infty}$ be a sequence of independent real-valued random variables with distributions from this set. As above, put $g_n(x)=\sum_{k=0}^{n}\eta_k x^k$ and by M_n denote the number of real roots of the polynomial $g_n(x)$. Then*

$$\mathbf{P}\left\{\frac{M_n}{n}\underset{n\to\infty}{\longrightarrow}0\right\}=1,\quad \mathbf{E}\,M_n=o(n), n\to\infty. \tag{3}$$

2 Proof of Theorem A

Let us establish the sufficiency of (1). Let it hold and fix $\delta\in(0,1)$. Prove that the radius of convergence of the series

$$G(z)=\sum_{k=0}^{\infty}\xi_k z^k \tag{4}$$

is equal to one with probability one.

Consider $\rho>0$ such that $\mathbf{P}\{|\xi_0|>\rho\}>0$. Using the Borel-Cantelli lemma we obtain that with probability one the sequence $\{\xi_k\}$ contains infinitely many ξ_k such that $|\xi_k|>\rho$. Therefore the radius of convergence of the series (4) does not exceed 1 almost surely.

On the other hand, for any non-negative random variable ζ

$$\sum_{k=1}^{\infty}\mathbf{P}\,(\zeta\geqslant k)\leqslant \mathbf{E}\,\zeta\leqslant 1+\sum_{k=1}^{\infty}\mathbf{P}\,(\zeta\geqslant k). \tag{5}$$

Therefore, it follows from (1) that

$$\sum_{k=1}^{\infty} \mathbf{P}\left(|\xi_k| \geqslant e^{\gamma k}\right) < \infty$$

for any positive constant γ. It follows from the Borel-Cantelli lemma that with probability one $|\xi_k| < e^{\gamma k}$ for all sufficiently large k. Thus, according to the Cauchy-Hadamard formula (see, e.g., [11]), the radius of convergence of the series (4) is at least 1 almost surely.

Hence with probability one $G(z)$ is an analytical function inside the unit ball $\{|z| < 1\}$. Therefore if $0 \leqslant a < b < 1$, then $R(a,b) < \infty$, where $R(a,b)$ denotes the number of the zeros of G inside the domain $\{z : a \leqslant |z| \leqslant b\}$. It follows from the Hurwitz theorem (see, e.g., [11]) that $R_n(0, 1-\delta) \leqslant R(0, 1-\delta/2)$ with probability one for all sufficiently large n. This implies

$$\mathbf{P}\left\{\frac{1}{n} R_n(0, 1-\delta) \underset{n\to\infty}{\longrightarrow} 0\right\} = 1.$$

In order to conclude the proof of (2) it remains to show that

$$\mathbf{P}\left\{\frac{1}{n} R_n(1+\delta, \infty) \underset{n\to\infty}{\longrightarrow} 0\right\} = 1.$$

In other words, we need to prove that $\mathbf{P}\{A\} = 0$, where A denotes the event that there exists $\varepsilon > 0$ such that

$$R_n(1+\delta, \infty) \geqslant \varepsilon n$$

holds for infinitely many values n.

By B denote the event that $G(z)$ is an analytical function inside the unit disk $\{|z| < 1\}$. For $m \in \mathbb{N}$ put

$$\zeta_m = \sup_{k \in \mathbb{Z}^+} |\xi_k e^{-k/m}|.$$

By C_m denote the event that $\zeta_m < \infty$. It was shown above that $\mathbf{P}\{B\} = \mathbf{P}\{C_m\} = 1$ for $m \in \mathbb{N}$. Therefore, to get $\mathbf{P}\{A\} = 0$, it is sufficient to show that $\mathbf{P}\{ABC_m\} = 0$ for some m.

Let us fix m. The exact value of it will be chosen later. Suppose the event ABC_m occurred. Index the roots of the polynomial $G_n(z)$ according to the order of magnitude of their absolute values:

$$|z_1| \leqslant |z_2| \leqslant \cdots \leqslant |z_n|.$$

Fix an arbitrary number $C > 1$ (an exact value will be chosen later). Consider indices i, j such that

$$|z_i| < 1 - \delta/C, \quad |z_{i+1}| \geqslant 1 - \delta/C,$$
$$|z_j| \leqslant 1 + \delta, \quad |z_{j+1}| > 1 + \delta.$$

If $|z_1| \geqslant 1 - \delta/C$, then $i = 0$; if $|z_n| \leqslant 1 + \delta$ then $j = n$.

It is easily shown that if

$$|z| < \min\left(1, \frac{|\xi_0|}{n \times \max_{k=1,\dots,n} |\xi_k|}\right),$$

then

$$|\xi_0| > |\xi_1 z| + |\xi_2 z^2| + \cdots + |\xi_n z^n|.$$

Therefore such z can not be a zero of the polynomial G_n. Taking into account that the event C_m occurred, we obtain a lower bound for the absolute values of the zeros for all sufficiently large n:

$$|z_1| \geqslant \min\left(1, \frac{|\xi_0|}{n \times \max_{k=1,\dots,n} |\xi_k|}\right) \geqslant \frac{|\xi_0|}{n\zeta_m e^{n/m}} \geqslant |\xi_0|\zeta_m^{-1} e^{-2n/m}.$$

Therefore for any integer l satisfying $j + 1 \leqslant l \leqslant n$ and all sufficiently large n

$$\begin{aligned}|z_1 \dots z_l| &= |z_1 \dots z_i||z_{i+1} \dots z_j||z_{j+1} \dots z_l|\\ &\geqslant |\xi_0|^i \zeta_m^{-i} e^{-2ni/m} \left(1 - \frac{\delta}{C}\right)^{j-i} (1 + \delta)^{l-j}.\end{aligned}$$

Since A occurred, $n - j \geqslant n\varepsilon$ for infinitely many values of n. Therefore if l satisfies $n - \sqrt{n} \leqslant l \leqslant n$, then the inequalities $j + 1 \leqslant l \leqslant n$ and $l - j \geqslant n\varepsilon/2$ hold for infinitely many values of n. According to the Hurwitz theorem for all sufficiently large n we have $i \leqslant R_n(0, 1 - \delta/C) \leqslant R(0, 1 - \delta/(2C))$. Therefore for infinitely many values of n

$$|z_1 \dots z_l| \geqslant \left(\frac{|\xi_0|}{\zeta_m}\right)^{R(0,1-\delta/(2C))} e^{-2nR(0,1-\delta/(2C))/m} \left(1 - \frac{\delta}{C}\right)^n (1 + \delta)^{n\varepsilon/2}.$$

Choose now C large enough to yield

$$\left(1 - \frac{\delta}{C}\right)(1 + \delta)^{\varepsilon/2} > 1.$$

Furthermore, holding C constant choose m such that

$$b = e^{-2R(0,1-\delta/(2C))/m} \left(1 - \frac{\delta}{C}\right)(1 + \delta)^{\frac{\varepsilon}{2}} > 1.$$

Since

$$\left(\frac{|\xi_0|}{\zeta_m}\right)^{R(0,1-\delta/(2C))/n} \underset{n\to\infty}{\longrightarrow} 1,$$

there exists a random variable $a > 1$ such that for infinitely many values of n

$$|z_1 \dots z_l| \geqslant \left(\frac{|\xi_0|}{\zeta_m}\right)^{R(0,1-\delta/(2C))} b^n = \left(b\left(\frac{|\xi_0|}{\zeta_m}\right)^{R(0,1-\delta/(2C))/n}\right)^n \geqslant a^n.$$

On the other hand, it follows from $n - \sqrt{n} \leqslant l$ and Viéte's formula that

$$|z_{l+1} \dots z_n| \geqslant \binom{n}{n-\sqrt{n}}^{-1} |\sum_{i_1<\dots<i_{n-l}} z_{i_1} \dots z_{i_{n-l}}| = \binom{n}{n-\sqrt{n}}^{-1} \frac{|\xi_l|}{|\xi_n|}.$$

We combine these two inequalities to obtain for infinitely many values of n

$$\frac{|\xi_0|}{|\xi_n|} = |z_1 \dots z_n| \geqslant a^n \binom{n}{n-\sqrt{n}}^{-1} \frac{|\xi_l|}{|\xi_n|}$$

$$\geqslant c_1 a^n \frac{(\sqrt{n})^{\sqrt{n}+\frac{1}{2}}(n-\sqrt{n})^{n-\sqrt{n}+\frac{1}{2}}}{n^{n+\frac{1}{2}}} \frac{|\xi_l|}{|\xi_n|} \geqslant c_2 a^n (\sqrt{n})^{-\sqrt{n}} \left(1-\frac{1}{\sqrt{n}}\right)^n \frac{|\xi_l|}{|\xi_n|}$$

$$\geqslant c_3 \exp\left(n\log a - \frac{\sqrt{n}\log n}{2} - \sqrt{n}\right) \frac{|\xi_l|}{|\xi_n|} \geqslant e^{\alpha n} \frac{|\xi_l|}{|\xi_n|},$$

where α is a positive random variable. Multiplying left and right parts by $|\xi_n|$, we get

$$ABC_m \subset \bigcup_{i=1}^{\infty} D_i,$$

where D_i denotes the event that $|\xi_0| > e^{n/i} \max_{n-\sqrt{n}\leqslant l\leqslant n} |\xi_l|$ for infinitely many values of n.

To complete the proof it is sufficient to show that $\mathbf{P}\{D_i\} = 0$ for all $i \in \mathbb{N}$. Having in mind to apply the Borel-Cantelli lemma, let us introduce the following events:

$$H_{in} = \left\{|\xi_0| > e^{n/i} \max_{n-\sqrt{n}\leqslant l\leqslant n} |\xi_l|\right\}.$$

Considering $\theta > 0$ such that $\mathbf{P}\{|\xi_0| \leqslant \theta\} = F(\theta) < 1$, we have

$$H_{in} \subset \left\{|\xi_0| > \theta e^{n/i}\right\} \cup \left\{\max_{n-\sqrt{n}\leqslant l\leqslant n} |\xi_l| \leqslant \theta\right\},$$

consequently,

$$\sum_{n=1}^{\infty} \mathbf{P}\{H_{in}\} \leqslant \sum_{n=1}^{\infty} \mathbf{P}\{|\xi_0| > \theta e^{n/i}\} + \sum_{n=1}^{\infty} (F(\theta))^{\sqrt{n}} < \infty$$

and, according to the Borel-Cantelli lemma, $\mathbf{P}\{D_i\} = 0$.

We prove the implication (2)⇒(1) arguing by contradiction. Suppose (1) does not hold, i.e.,

$$\mathbf{E} \log(1 + |\xi_o|) = \infty.$$

It follows from (5) that

$$\sum_{n=1}^{\infty} \mathbf{P}(|\xi_n| \geqslant e^{\gamma n}) = \infty \tag{6}$$

for an arbitrary positive γ. For $k \in \mathbb{N}$ introduce an event F_k that $|\xi_n| \geqslant e^{kn}$ holds for infinitely many values of n. It follows from (6) and the Borel-Cantelli lemma that $\mathbf{P}\{F_k\} = 1$ and, consequently, $\mathbf{P}\{\cap_{k=1}^{\infty} F_k\} = 1$. This yields

$$\mathbf{P}\left\{\limsup_{n\to\infty} |\xi_n|^{1/n} = \infty\right\} = 1.$$

Therefore with probability one for infinitely many values of n

$$|\xi_n|^{1/n} > \max_{i=0,\dots,n-1} |\xi_i|^{1/i}, \quad |\xi_n|^{1/n} > \frac{3}{\varepsilon}, \quad |\xi_0| < 2^{n-1},$$

where $\varepsilon > 0$ is an arbitrary fixed value. Let us hold one of those n. Suppose $|z| \geqslant \varepsilon$. Then

$$\begin{aligned} &|\xi_0 + \xi_1 z + \dots + \xi_{n-1} z^{n-1}| \\ &\leqslant 2^{n-1} + |\xi_n z^n|^{1/n} + |\xi_n z^n|^{2/n} + \dots + |\xi_n z^n|^{(n-1)/n} \\ &= \frac{2^n}{2} - 1 + \frac{|\xi_n z^n| - 1}{|\xi_n^{1/n} z| - 1} \leqslant \frac{|\xi_n^{1/n} z|^n}{2} - 1 + \frac{|\xi_n z^n| - 1}{(3/\varepsilon) \times \varepsilon - 1} < |\xi_n z^n|. \end{aligned}$$

Thus with probability one for infinite number of values of n all the roots of the polynomial G_n are located inside the circle $\{z : |z| = \varepsilon\}$, where ε is an arbitrary positive constant. This means that (2) does not hold for any $\delta \in (0, 1)$.

3 Proof of Theorem B

The proof of Theorem B follows immediately from Theorems A and C. However, the additional assumption (1) significantly simplifies the proof.

Consider a set of sequences of reals

$$\{a_{11}\}, \{a_{12}, a_{22}\}, \dots, \{a_{1n}, a_{2n}, \dots a_{nn}\}, \dots,$$

where all $a_{jn} \in [0, 1]$. We say that $\{a_{jn}\}$ are uniformly distributed in [0, 1] if for any $0 \leqslant a < b \leqslant 1$

$$\lim_{n\to\infty} \frac{\#\{j \in \{1, 2, \cdots, n\} \,:\, a_{jn} \in [a, b]\}}{n} = b - a.$$

The definition is an insignificant generalization of the notion of uniformly distributed sequences (see, e.g., [7]). It is easy to see that the Weyl criterion (see Ibid.) continues to be valid in this case:

The set of sequences $\{a_{jn}, j = 1, \dots, n\}$, $n = 1, 2, \dots,$ *is uniformly distributed if and only if for all* $l = 1, 2, \dots$

$$\frac{1}{n} \sum_{j=1}^{n} e^{2\pi i l a_{jn}} \to 0, \quad n \to \infty.$$

Let $z_{jn} = r_{jn} e^{i\theta_{jn}}$ be a zero of $G_n(z)$, $r_{jn} = |z_{jn}|$, $\theta_{jn} = \arg z_{jn}$, $0 \leqslant \theta_{jn} < 2\pi$. The asymptotic uniform distribution of the arguments is equivalent to the statement that the set of sequences $\{\theta_{jn}/(2\pi)\}$ is uniformly distributed. Thus, according to Weyl's criterion, it is enough to show that for any $l = 1, 2, \dots$

$$\lim_{n} \frac{1}{n} \sum_{j=1}^{n} e^{il\theta_{jn}} = 0$$

with probability 1.

For the simplicity we assume that $\xi_0 \neq 0$. Consider the random polynomial

$$\tilde{G}_n(z) = \xi_n + \xi_{n-1} z + \dots + \xi_1 z^{n-1} + \xi_0 z^n.$$

Its roots are z_{kn}^{-1}. According to Newton's formulas (see, e.g., [8]),

$$\sum_{j=1}^{n} \frac{1}{z_{jn}^{l}} = \varphi_l \left(\frac{\xi_1}{\xi_0}, \dots \frac{\xi_l}{\xi_0} \right),$$

where $\varphi_l(x_1, \dots x_l)$ are polynomials which do not depend on n (for example, $\varphi_1(x) = -x$). It follows that

$$\frac{1}{n} \sum_{j=1}^{n} e^{-il\theta_{jn}} = \frac{1}{n} \sum_{j=1}^{n} e^{-il\theta_{jn}} \left(1 - \frac{1}{r_{jn}^{l}} \right) + \frac{\varphi_l}{n}. \tag{7}$$

As was shown in the proof of Theorem A, for $|z| < 1$ the polynomials $G_n(z)$ converge to the analytical function $G(z) = \sum_{k=0}^{\infty} \xi_k z^k$ with probability 1. Since $\xi_0 \neq 0$, the function $G(z)$ has no zeros inside a circle $\{z : |z| \leqslant \rho\}$, $\mathbf{P}\{\rho > 0\} = 1$. Hence for $n \geqslant N$, $\mathbf{P}\{N < \infty\}$, the polynomials $G_n(z)$ have no zeros inside $\{z : |z| \leqslant \rho\}$. Let $\gamma > 0$ be a positive number. It follows from (7) that

$$\left|\frac{1}{n}\sum_{j=1}^{n} e^{-il\theta_{jn}}\right| \leqslant (l+1)\frac{\gamma}{(1-\gamma)^l} + \frac{1}{n}\left(1+\frac{1}{\rho}\right)\#\{j : |r_{jn}-1| > \gamma, i = 1,\dots n\} + \frac{\varphi_l}{n}.$$

Theorem A implies that the second member on the right-hand side goes to zero as $n \to \infty$ with probability 1. Hence

$$\frac{1}{n}\sum_{j=1}^{n} e^{-il\theta jn} \to 0, \quad n \to \infty,$$

with probability 1 and the theorem follows.

4 Proof of Theorem C

Consider integer numbers p, q_1, q_2 such that $0 \leqslant q_1 < q_2 < p-1$. Put $\varphi_j = q_j/p$, $j = 1, 2$, and try to estimate $S_n = S_n(2\pi\varphi_1, 2\pi\varphi_2)$. Evidently $S_n = \lim_{R\to\infty} S_{nR}$, where S_{nR} is the number of zeros of $G_n(z)$ inside the domain $A_R = \{z : |z| \leqslant R,$ $2\pi\varphi_1 \leqslant \arg z \leqslant 2\pi\varphi_2\}$. It follows from the argument principle (see, e.g., [11]) that S_{nR} is equal to the change of the argument of $G_n(z)$ divided by 2π as z traverses the boundary of A_R. The boundary consists of the arc $\Gamma_R = \{z : |z| = R,$ $2\pi\varphi_1 \leqslant \arg z \leqslant 2\pi\varphi_2\}$ and two intervals $L_j = \{z : 0 \leqslant |z| \leqslant R, \arg z = \pi\varphi_j\}$, $j = 1, 2$. It can easily be checked that if R is sufficiently large, then the change of the argument as z traverses Γ_R is equal to $n(\varphi_2 - \varphi_1) + o(1)$ as $n \to \infty$. If z traverses a subinterval of L_j and the change of the argument of $G_n(z)$ is at least π, then the function $|G_n(z)|\cos(\arg G_n(z))$ has at least one root in this interval. It follows from Theorem E that with probability one the number of real roots of the polynomial

$$g_{n,j}(x) = \sum_{k=0}^{n} x^k \Re(\xi_k e^{2\pi i k\varphi_j}) = \sum_{k=0}^{n} x^k \eta_{k,j}$$

is $o(n)$ as $n \to \infty$. Thus the change of the argument of $G_n(z)$ as z traverses L_j is $o(n)$ as $n \to \infty$ and

$$\mathbf{P}\left\{\frac{1}{n}S_n(2\pi\varphi_1, 2\pi\varphi_2) = (\varphi_2 - \varphi_1) + o(1), \quad n \to \infty\right\} = 1.$$

The set of points of the form $\exp\{2\pi i q/p\}$ is dense in the unit circle $\{z : |z| = 1\}$. Therefore

$$\mathbf{P}\left\{\frac{1}{n}S_n(\alpha,\beta) \underset{n\to\infty}{\longrightarrow} \frac{\beta-\alpha}{2\pi}\right\} = 1$$

for any α, β such that $0 \leqslant \alpha < \beta \leqslant 2\pi$.

5 Proof of Theorem E

First we convert the problem of counting of real zeros of $g_n(x)$ to the problem of counting of sign changes in the sequence of the derivatives $\{g_n^{(j)}(1)\}_{j=0}^n$.

Let $\{a_j\}_{j=0}^n$ be a sequence of real numbers. By $Z(\{a_j\})$ denote the number of sign changes in the sequence $\{a_j\}$, which is defined as follows. First we exclude all zero members from the sequence. Then we count the number of the neighboring members of different signs.

For any polynomial $p(x)$ of degree n put $Z_p(x) = Z(\{p^{(j)}(x)\})$, i.e., the number of sign changes in the sequence $p(x), p'(x), \ldots, p^{(n)}(x)$.

Lemma 1 (Budan-Fourier Theorem). *Suppose $p(x)$ is a polynomial such that $p(a), p(b) \neq 0$ for some $a < b$. Then the number of the roots of $p(x)$ inside (a, b) does not exceed $Z_p(a) - Z_p(b)$. Moreover, the difference between $Z_p(a) - Z_p(b)$ and the number of the roots is an even number.*

Proof. See, e.g., [8]. □

Corollary 1. *The number of the roots of $p(x)$ inside $[1, \infty)$ does not exceed $Z_p(1)$.*

Proof. For all sufficiently large x the sign of $p^{(j)}(x)$ coincides with the sign of the leading coefficient. □

Corollary 2. *The function $Z_p(x)$ does not increase.*

Let us turn back to the random polynomial $g_n(x)$. Here and elsewhere we shall omit the index n when it can be done without ambiguity. By $M_n(a, b)$ denote the number of zeros of $g(x)$ inside the interval $[a, b]$.

First let us prove that

$$\mathbf{E}\, Z_g(1) = o(n), \quad n \to \infty. \tag{8}$$

Fix some $\varepsilon > 0$ and $\lambda \in (0, 1/2)$. Since the distributions of $\{\eta_j\}$ belong to a finite set, there exists $K = K(\varepsilon)$ such that

$$\sup_{j\in\mathbb{Z}^1} \mathbf{P}\{|\eta_j| \geqslant K\} \leqslant \varepsilon. \tag{9}$$

Let I be a subset of $\{0, 1, \dots, n\}$ consisting of indices j such that $|\eta_j| < K$ and $[\lambda n] \leqslant j \leqslant [(1-\lambda)n]$. Put

$$g_1(x) = \sum_{j \in I} \eta_j x^j, \quad g_2(x) = g(x) - g_1(x).$$

Let τ_k be the indicator of $\{|g_1^{(k)}(1)| \geqslant |g_2^{(k)}(1)|\}$ and χ_j be the indicator of $\{|\eta_j| \geqslant K\}$.

Lemma 2. *Let a_1, a_1, b_1, b_2 be real numbers. If $(a_1+a_2)(b_1+b_2) < 0$ and $a_2 b_2 \geqslant 0$, then either $|a_1| \geqslant |a_2|$ or $|b_1| \geqslant |b_2|$.*

Proof. The proof is trivial. □

It follows from Lemma 2 that

$$Z_g(1) = Z_{g_1+g_2}(1) \leqslant Z_{g_2}(1) + 2\sum_{j=0}^{n} \tau_j \leqslant Z_{g_2}(1) + 2\lambda n + 2 + 2 \sum_{j=[\lambda n]}^{[(1-\lambda)n]} \tau_j .$$

Owing to the monotonicity of the function $Z_{g_2}(x)$, one has

$$Z_{g_2}(1) \leqslant Z_{g_2}(0) \leqslant \sum_{j=0}^{n} \chi_j .$$

Hence,

$$Z_g(1) \leqslant 2\lambda n + 2 + \sum_{j=0}^{n} \chi_j + 2 \sum_{j=[\lambda n]}^{[(1-\lambda)n]} \tau_j . \tag{10}$$

Using (9) we have $\mathbf{E}\,\chi_j = \mathbf{P}\{|\eta_j| \geqslant K\} \leqslant \varepsilon$, therefore,

$$\mathbf{E}\, Z_g(1) \leqslant 2\lambda n + 2 + \varepsilon(n+1) + 2\mathbf{E} \sum_{j=[\lambda n]}^{[(1-\lambda)n]} \tau_j . \tag{11}$$

Let us now estimate the value $\mathbf{E}\,\tau_j$. Note that $g^{(k)}(x) = \sum_{l=k}^{n} \eta_l A_{k,l} x^{l-k}$, where $A_{k,l} = l(l-1)\cdots(l-k+1)$. Fix some integer k such that $\lambda n \leqslant k \leqslant (1-\lambda)n$. If $n-1 \geqslant j \geqslant k$, then

$$A_{k,j} \leqslant (1-\lambda) A_{k,j+1},$$

which implies

$$A_{k,j} \leqslant A_{k,[(1-\lambda)n]} (1-\lambda)^{[(1-\lambda)n]-j}$$

for $\lambda n \leqslant k \leqslant j \leqslant (1-\lambda)n$. Consequently,

$$|g_1^{(k)}(1)| = \Big| \sum_{j \in J, j \geqslant k} \eta_j A_{k,j} \Big|$$

$$\leqslant K A_{k,[(1-\lambda)n]} \sum_{j=0}^{[(1-\lambda)n]} (1-\lambda)^j \leqslant \frac{K}{\lambda} A_{k,[(1-\lambda)n]}.$$

This yields that

$$\mathbf{E}\,\tau_k = \mathbf{P}\left\{|g_1^{(k)}(1)| \geqslant |g_2^{(k)}(1)|\right\}$$

$$\leqslant \mathbf{P}\left\{|g_1^{(k)}(1)| \geqslant |g_1^{(k)}(1) + g_2^{(k)}(1)| - |g_1^{(k)}(1)|\right\}$$

$$= \mathbf{P}\left\{|g^{(k)}(1)| \leqslant 2|g_1^{(k)}(1)|\right\} \leqslant \mathbf{P}\left\{|g^{(k)}(1)| \leqslant \frac{2K}{\lambda} A_{k,[(1-\lambda)n]}\right\}.$$

For an arbitrary random variable X define the concentration function $Q(h; X)$ as follows:

$$Q(h; X) = \sup_{a \in \mathbb{R}^1} \mathbf{P}\{a \leqslant X \leqslant a + h\}.$$

If X, Y are independent random variables, then (see, e.g., [12])

$$Q(h; X + Y) \leqslant \min\left(Q(h; X), Q(h; Y)\right).$$

Therefore,

$$\mathbf{E}\,\tau_k \leqslant \mathbf{P}\left\{\frac{|g^{(k)}(1)|}{A_{k,[(1-\lambda)n]}} \leqslant \frac{2K}{\lambda}\right\} \tag{12}$$

$$\leqslant \mathbf{P}\left\{\frac{g^{(k)}(1)}{A_{k,[(1-\lambda)n]}} \leqslant \frac{2K}{\lambda}\right\} \leqslant Q\left(\frac{2K}{\lambda}; \frac{g^{(k)}(1)}{A_{k,[(1-\lambda)n]}}\right)$$

$$= Q\left(\frac{2K}{\lambda}; \sum_{j=k}^{n} \frac{A_{k,j}}{A_{k,[(1-\lambda)n]}} \eta_j\right) \leqslant Q\left(\frac{2K}{\lambda}; \sum_{j=[(1-\lambda)n]}^{n} \frac{A_{k,j}}{A_{k,[(1-\lambda)n]}} \eta_j\right).$$

To estimate the right-hand side of (12) we use the following result.

Lemma 3 (the Kolmogorov-Rogozin inequality). *Let $X_1, X_2, \ldots, X_n$ be independent random variables. Then for any $0 < h_j \leqslant h,\ j = 1, \ldots, n$,*

$$Q(h; X_1 + \cdots + X_n) \leqslant \frac{Ch}{\sqrt{\sum_{j=1}^{n} h_j^2 (1 - Q(h_j; X_j))}}, \tag{13}$$

where C is an absolute constant.

Proof. See [13]. □

Since the distributions of $\{\eta_j\}$ belong to a finite set, we get

$$\delta = \delta(\varepsilon, \lambda) = \inf_{j \in \mathbb{Z}^1} \left\{ 1 - Q\left(\frac{2K}{\lambda}; \eta_j \right) \right\} > 0.$$

Putting $h = h_j = 2K/\lambda$ in (13) and using (12), we obtain

$$\mathbf{E}\,\tau_k \leqslant C \left[\sum_{j=[(1-\lambda)n]}^{n} \left\{ 1 - Q\left(\frac{2K}{\lambda}; \frac{A_{k,j}}{A_{k,[(1-\lambda)n]}} \eta_j \right) \right\} \right]^{-1/2}$$
$$\leqslant C \left[\sum_{j=[(1-\lambda)n]}^{n} \left\{ 1 - Q\left(\frac{2K}{\lambda}; \eta_j \right) \right\} \right]^{-1/2} \leqslant \frac{C}{\sqrt{\delta \lambda n}}.$$

Combining this with (11), we have

$$\mathbf{E}\, Z_g(1) \leqslant 2\lambda n + 2 + \varepsilon(n+1) + \frac{2C}{\sqrt{\delta(\varepsilon, \lambda)\lambda}} n^{1/2}.$$

Since λ, ε are arbitrary positive numbers, we obtain (8), which together with the corollary from Lemma 1 implies

$$\mathbf{E}\, M_n(1, \infty) = o(n), \quad n \to \infty.$$

Considering the random polynomials $g(1/x)$ and $g(-x)$, it is possible to obtain similar estimates for $M_n(0,1)$ and $M_n(-\infty, 0)$. Thus the second part of (3) holds. To prove the first one, we estimate the probabilities of large deviations for the sums $\sum \chi_j$ and $\sum \tau_j$. The elementary considerations or the application of Bernstein inequalities (see, e.g., [12]) leads to

$$\mathbf{P} \left\{ \left| \sum_{j=0}^{n} \chi_j \right| > 2(n+1)\varepsilon \right\} \leqslant 2e^{-n\varepsilon/8}. \tag{14}$$

The analysis of the behavior of $\sum \tau_j$ is slightly more difficult.

Henceforth we shall use the following notation: for any positive functions f_1, f_2 we write $f_1 \ll f_2$, if there exists an absolute constant C such that $f_1 \leqslant C f_2$ in the domain of these functions.

Lemma 4. *There exists a constant c depending only on λ, ε and the distributions of $\{\eta_j\}$ such that*

$$\mathbf{E}\,\tau_k \leqslant c n^{-2}$$

for $\lambda n \leqslant k \leqslant (1-\lambda)n$.

Proof. As was shown in (12),

$$\mathbf{E}\,\tau_k \leqslant Q\left(\frac{2K}{\lambda}; \sum_{j=[(1-\lambda)n]}^{n} \frac{A_{k,j}}{A_{k,[(1-\lambda)n]}}\eta_j\right). \tag{15}$$

To estimate the concentration function in the right-hand side we use the result of Esseen (see, e.g., [12]). Let X be a random variable with a characteristic function $f(t)$. Then

$$Q(h;X) \ll \max\left(h, \frac{1}{T}\right)\int_{-T}^{T}|f(t)|\,dt$$

uniformly for all $T > 0$.

Putting $T = \lambda/(KA_{k,[(1-\lambda)n]})$ and applying (15) , we obtain

$$\mathbf{E}\,\tau_k \ll \frac{1}{T}\int_{-T}^{T}\prod_{j=[(1-\lambda)n]}^{n}|f_j(A_{kj}t)|\,dt,$$

where $f_j(t)$ is a characteristic function of η_j. Further,

$$\begin{aligned}
\mathbf{E}\,\tau_k &\ll \frac{1}{T}\int_{-T}^{T}\left[\prod_{j=[(1-\lambda)n]}^{n}|f_j(A_{kj}t)|^2\right]^{\frac{1}{2}}dt\\
&\ll \frac{1}{T}\int_{-T}^{T}\exp\left\{-\frac{1}{2}\sum_{j=[(1-\lambda)n]}^{n}\left(1-|f(A_{kj}t)|^2\right)\right\}dt\\
&= \frac{1}{T}\int_{-T}^{T}\exp\left\{-\frac{1}{2}\sum_{j=[(1-\lambda)n]}^{n}\int_{-\infty}^{\infty}\left[1-\cos(A_{kj}tx)\right]\mathscr{P}_j(dx)\right\}dt,
\end{aligned}$$

where $\mathscr{P}_j$ is a distribution of the symmetrized η_j, i.e., a distribution of $\eta_j - \eta_j'$, where η_j' is an independent copy of η_j.

There are at most r different distributions among $\{\mathscr{P}_j\}_{(1-\lambda)n\leqslant j\leqslant n}$. Therefore there exist a distribution $\mathscr{P}$ and a subset $J \subset \{j : (1-\lambda)n \leqslant j \leqslant n\}$ such that $|J| \geqslant n\lambda/r$ and $\mathscr{P}_j = \mathscr{P}$ for all $j \in J$. By $\sum'$ denote the summation taking over all indices such that $j \in J$. Thus,

$$\mathbf{E}\,\tau_k \ll \frac{1}{T}\int_{-T}^{T}\exp\left\{-\frac{1}{2}\sum_{j=[(1-\lambda)n]}^{n}{}' \int_{-\infty}^{\infty}\left[1-\cos(A_{kj}tx)\right]\mathscr{P}(dx)\right\}dt.$$

Choose $\delta > 0$ such that $\gamma = \mathscr{P}\{x : |x| > \delta\} > 0$. Since the integrands are non-negative, we get

$$\begin{aligned}\mathbf{E}\,\tau_k &\ll \frac{1}{T}\int_{-T}^{T}\exp\left\{-\frac{1}{2}\sum_{j=[(1-\lambda_r)n]}^{n}{}' \int_{|x|>\delta}\left[1-\cos(A_{kj}tx)\right]\,\mathscr{P}(dx)\right\}\\ &= \frac{1}{T}\int_{-T}^{T} e^{-\beta n+s(t)}\,dt,\end{aligned}$$

where $\lambda_r = \lambda(2r-1)/(2r)$, $\beta = |J \cap \{j\ :\ (1-\lambda_r)n \leqslant j \leqslant n\}|/(2n)$ and

$$s(t) = \frac{1}{2}\int_{|x|>\delta}\ \sum_{j=[(1-\lambda_r)n]}^{n}{}' \cos(A_{kj}tx)\,\mathscr{P}(dx).$$

Put $\alpha = \lambda\gamma/(4r)$ and consider $\Lambda_1 = \{t \in [-T,T]\ :\ |s(t)| < \alpha n/2\}$ and $\Lambda_2 = [-T,T] \setminus \Lambda_1$. Since $|J| \geqslant n\lambda/r$ and by the definition of β, we have $\beta \geqslant \alpha$. Therefore,

$$\mathbf{E}\,\tau_k \ll e^{-\alpha n/2} + \frac{\mu(\Lambda_2)}{T}, \tag{16}$$

where μ denotes the Lebesgue measure.

Let us estimate $\mu(\Lambda_2)$. It follows from Chebyshev's and Hölder's inequalities that

$$\mu(\Lambda_2) \leqslant \frac{16}{\alpha^4 n^4}\int_{-T}^{T}|s(t)|^4\,dt \leqslant \frac{1}{\alpha^4 n^4}\int_{|x|>\delta} d\mathscr{P}\int_{-T}^{T}\Big|\sum_{j=[(1-\lambda_r)n]}^{n}{}' \cos(A_{kj}tx)\Big|^4\,dt. \tag{17}$$

Put

$$S(x) = \int_{-T}^{T}\Big|\sum_{j=[(1-\lambda_r)n]}^{n}{}' \cos(A_{kj}tx)\Big|^4\,dt$$

and assume, for simplicity, that $r = 1$, i.e., $\lambda_r = \lambda/2$, $\sum = \sum'$ and the summation is taken over all j. The general case is considered in a similar way.

We have

$$\begin{aligned}S(x) = \int_{-T}^{T}\Bigg(&\sum_{j_1}\cos^4(A_{kj_1}tx) + \sum_{j_1\neq j_2}\cos^3(A_{kj_1}tx)\cos(A_{kj_2}tx) \qquad (18)\\ &+ \sum_{j_1\neq j_2}\cos^2(A_{kj_1}tx)\cos^2(A_{kj_2}tx)\\ &+ \sum_{j_1\neq j_2\neq j_3}\cos^2(A_{kj_1}tx)\cos(A_{kj_2}tx)\cos(A_{kj_3}tx)\\ &+ \sum_{j_1\neq j_2\neq j_3\neq j_4}\cos(A_{kj_1}tx)\cos(A_{kj_2}tx)\cos(A_{kj_3}tx)\cos(A_{kj_4}tx)\Bigg)\,dt.\end{aligned}$$

The first three summands in (18) are easily estimated as follows:

$$\Big|\int_{-T}^{T}\Big(\sum_{j_1}\cos^4(A_{kj_1}tx)+\sum_{j_1\neq j_2}\cos^3(A_{kj_1}tx)\cos(A_{kj_2}tx) \\ +\sum_{j_1\neq j_2}\cos^2(A_{kj_1}tx)\cos^2(A_{kj_2}tx)\Big)\,dt\Big|\ll Tn^2. \tag{19}$$

The next two summands have a common method of estimation. We consider only the last one. From the formula $\cos y=(e^{iy}+e^{-iy})/2$ it is easily shown that

$$\begin{aligned}&\Big|\int_{-T}^{T}\sum_{j_1\neq j_2\neq j_3\neq j_4}\cos(A_{kj_1}tx)\cos(A_{kj_2}tx)\cos(A_{kj_3}tx)\cos(A_{kj_4}tx)\,dt\Big| \\ &\quad\ll\sum_{j_1\neq j_2\neq j_3\neq j_4}\min\big(T,|x|^{-1}|\pm A_{kj_1}\pm A_{kj_2}\pm A_{kj_3}\pm A_{kj_4}|^{-1}\big) \\ &\quad\ll\sum_{j_1>j_2>j_3>j_4}\min\Big(T,|x|^{-1}A_{kj_1}^{-1}\Big|1-\frac{A_{kj_2}}{A_{kj_1}}-\frac{A_{kj_3}}{A_{kj_1}}-\frac{A_{kj_4}}{A_{kj_1}}\Big|^{-1}\Big),\end{aligned} \tag{20}$$

The summation in the middle term is taken over all possible combinations of signs.

Consider the partition of the index set

$$\{j=(j_1,j_2,j_3,j_4)\,:\,j_1>j_2>j_3>j_4\}=K_1\cup K_2,$$

where

$$K_1=\Big\{j\,:\,j_1-j_2\leqslant\frac{10}{\lambda},\ j_1-j_3\leqslant\frac{10}{\lambda}|\ln\lambda|\Big\}$$

and K_2 is the complement of K_1. Clearly, $|K_1|\ll n^2|\ln\lambda|/\lambda^2$. Therefore,

$$\sum_{j\in K_1}\min\Big(T,|x|^{-1}A_{kj_1}^{-1}\Big|1-\frac{A_{kj_2}}{A_{kj_1}}-\frac{A_{kj_3}}{A_{kj_1}}-\frac{A_{kj_4}}{A_{kj_1}}\Big|^{-1}\Big)\ll\frac{Tn^2|\ln\lambda|}{\lambda^2}. \tag{21}$$

Consider now

$$\sum_{j\in K_2}A_{kj_1}^{-1}\Big|1-\frac{A_{kj_2}}{A_{kj_1}}-\frac{A_{kj_3}}{A_{kj_1}}-\frac{A_{kj_4}}{A_{kj_1}}\Big|^{-1}.$$

Putting $p=j_1-j_2$, we have

$$\begin{aligned}\frac{A_{kj_2}}{A_{kj_1}}&=\frac{(j_1-p)\cdots(j_1-p-k+1)}{j_1\cdots(j_1-k+1)} \\ &=\Big(1-\frac{p}{j_1}\Big)\cdots\Big(1-\frac{p}{j_1-k+1}\Big)\leqslant\exp\Big\{-p\sum_{l=j_1-k+1}^{j_1}\frac{1}{l}\Big\}.\end{aligned}$$

Since for any natural l

$$\frac{1}{l} > \ln\left(1+\frac{1}{l}\right) = \ln(l+1) - \ln l,$$

we get

$$\sum_{l=j_1-k+1}^{j_1} \frac{1}{l} > \ln(j_1+1) - \ln(j_1-k+1) = -\ln\left(1-\frac{k}{j_1+1}\right).$$

Taking into account $\lambda n \leqslant k \leqslant (1-\lambda)n$ and $(1-\lambda/2)n \leqslant j_1 \leqslant n$ and using the inequality

$$-\ln(1-t) \geqslant t, \quad t \in [0,1],$$

we get

$$\sum_{l=j_1-k+1}^{j_1} \frac{1}{l} \geqslant \frac{\lambda n}{n+1} \geqslant \frac{1}{2}\lambda.$$

Therefore,

$$\frac{A_{kj_2}}{A_{kj_1}} \leqslant \exp\left\{-\frac{\lambda}{2}p\right\} = \exp\left\{-\frac{\lambda}{2}(j_1-j_2)\right\}. \tag{22}$$

If $j \in K_2$ and $j_1 - j_2 > 10/\lambda$, then

$$\frac{A_{kj_4}}{A_{kj_1}} \leqslant \frac{A_{kj_3}}{A_{kj_1}} \leqslant \frac{A_{kj_2}}{A_{kj_1}} \leqslant e^{-5} < \frac{1}{4},$$

which implies

$$1 - \frac{A_{kj_2}}{A_{kj_1}} - \frac{A_{kj_3}}{A_{kj_1}} - \frac{A_{kj_4}}{A_{kj_1}} \geqslant \frac{1}{4}. \tag{23}$$

Suppose now $j \in K_2$ and $j_1 - j_3 > 10|\ln\lambda|/\lambda$. Using (22) and $\lambda \in (0,1/2)$, we get

$$1 - \frac{A_{kj_2}}{A_{kj_1}} \geqslant 1 - e^{-\lambda/2} \geqslant \frac{\lambda}{2}\left(1-\frac{\lambda}{4}\right) \geqslant \frac{7}{16}\lambda.$$

Further, (22) also holds for j_3. Therefore,

$$\frac{A_{kj_4}}{A_{kj_1}} \leqslant \frac{A_{kj_3}}{A_{kj_1}} \leqslant \exp\left\{-\frac{\lambda}{2}(j_1-j_3)\right\} \leqslant \exp\left\{-\frac{10}{2}|\ln\lambda|\right\} \leqslant \lambda^5 \leqslant \frac{1}{16}\lambda.$$

Thus,

$$1 - \frac{A_{kj_2}}{A_{kj_1}} - \frac{A_{kj_3}}{A_{kj_1}} - \frac{A_{kj_4}}{A_{kj_1}} \geqslant \frac{5}{16}\lambda. \tag{24}$$

It follows from (23) and (24) that

$$\sum_{j\in K_2} A_{kj_1}^{-1}\left|1-\frac{A_{kj_2}}{A_{kj_1}}-\frac{A_{kj_3}}{A_{kj_1}}-\frac{A_{kj_4}}{A_{kj_1}}\right|^{-1} \ll \frac{1}{\lambda}\sum_j A_{kj_1}^{-1}.$$

Taking into account the structure of the index set $\{j\}$, we have

$$\sum_j A_{kj_1}^{-1} \leqslant \frac{(\lambda n)^4}{A_{k,[(1-\lambda/2)n]}},$$

consequently,

$$\sum_{j\in K_2} A_{kj_1}^{-1}\left|1-\frac{A_{kj_2}}{A_{kj_1}}-\frac{A_{kj_3}}{A_{kj_1}}-\frac{A_{kj_4}}{A_{kj_1}}\right|^{-1} \ll \frac{\lambda^3 n^4}{A_{k,[(1-\lambda/2)n]}}. \tag{25}$$

Combining (18)–(21) and (25), we obtain

$$S(x) \ll Tn^2 + \frac{Tn^2|\ln\lambda|}{\lambda^2} + \frac{\lambda^3 n^4}{|x|A_{k,[(1-\lambda/2)n]}}.$$

Applying this to (17), we get

$$\mu(\Lambda_2) \ll \frac{T}{\alpha^4 n^2} + \frac{T|\ln\lambda|}{\lambda^2\alpha^4 n^2} + \frac{\lambda^3}{\alpha^4\delta A_{k,[(1-\lambda/2)n]}}.$$

By (16),

$$\mathbf{E}\,\tau_k \ll e^{-\alpha n/2} + \frac{1}{\alpha^4 n^2} + \frac{|\ln\lambda|}{\lambda^2\alpha^4 n^2} + \frac{\lambda^3}{T\alpha^4\delta A_{k,[(1-\lambda/2)n]}}.$$

Recalling that $T = \lambda/(KA_{k,[(1-\lambda)n]})$, we obtain

$$\mathbf{E}\,\tau_k \ll e^{-\alpha n/2} + \frac{1}{\alpha^4 n^2} + \frac{|\ln\lambda|}{\lambda^2\alpha^4 n^2} + \frac{\lambda^2 K A_{k,[(1-\lambda)n]}}{\alpha^4\delta A_{k,[(1-\lambda/2)n]}}.$$

It follows from (22) that

$$\frac{A_{k,[(1-\lambda)n]}}{A_{k,[(1-\lambda/2)n]}} \leqslant e^{-\lambda^2 n/4}.$$

Thus,

$$\mathbf{E}\,\tau_k \ll e^{-\alpha n/2} + \frac{1}{\alpha^4 n^2} + \frac{|\ln\lambda|}{\lambda^2\alpha^4 n^2} + \frac{\lambda^2 K}{\alpha^4\delta} e^{-\lambda^2 n/4}.$$

Recalling that $\alpha = \gamma\lambda/4$, we obtain

$$\mathbf{E}\,\tau_k \ll e^{-\gamma\lambda n/8} + \frac{1}{\gamma^4\lambda^4 n^2} + \frac{|\ln\lambda|}{\gamma^4\lambda^6 n^2} + \frac{K}{\gamma^4\lambda^2\delta} e^{-\lambda^2 n/4}.$$

Since K is defined by ε and γ, δ are defined by the distributions of $\{\eta_j\}$, Lemma 4 is proved. □

Now we are ready to complete the proof of Theorem E. It follows from (10) that

$$M_n(1,\infty) \leqslant 2\lambda n + 2 + \sum_{j=0}^{n} \chi_j + 2 \sum_{j=[\lambda n]}^{[(1-\lambda)n]} \tau_j. \tag{26}$$

By Lemma 4 and Chebyshev's inequality,

$$\mathbf{P}\left\{\sum_{k=[\lambda n]}^{[(1-\lambda)n]} \tau_k > n^{3/4}\right\} \leqslant \frac{\sum_{j=[\lambda n]}^{[(1-\lambda)n]} \mathbf{E}\,\tau_k}{n^{3/4}} \leqslant c_1 n^{-5/4}. \tag{27}$$

Further, it follows from (14) that there exists a constant $c_2 > 0$ depending only on ε such that

$$\mathbf{P}\left\{\sum_{j=0}^{n} \chi_j > 2\varepsilon n\right\} \leqslant c_2 n^{-2}. \tag{28}$$

Combining (26)–(28), we get

$$\mathbf{P}\left\{M_n(1,\infty) > 2\lambda n + 2 + 2n^{3/4} + 2\varepsilon n\right\} \leqslant c_1 n^{-5/4} + c_2 n^{-2}.$$

Considering the random polynomials $g(1/x)$ and $g(-x)$, it is possible to obtain similar estimates for $M_n(0,1)$ and $M_n(-\infty,0)$. Thus there exist positive constants c_1', c_2' such that

$$\mathbf{P}\left\{M_n > 2\lambda n + 2 + 2n^{3/4} + 2\varepsilon n\right\} \leqslant c_1' n^{-5/4} + c_2' n^{-2}.$$

According to the Borel-Cantelli lemma, with probability one there exists only a finite number of n such that $M_n > 2\lambda n + 2 + 2n^{3/4} + 2\varepsilon n$. Since λ, ε are arbitrary small,

$$\mathbf{P}\left\{\frac{M_n}{n} \underset{n\to\infty}{\longrightarrow} 0\right\} = 1.$$

Theorem E is proved.

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Dependence and Interaction in Branching Processes

Peter Jagers and Fima C. Klebaner

Abstract Independence of reproducing individuals can be viewed as the very defining property of branching processes. It is crucial for the most famous results of the theory, the determination of the extinction probability and the dichotomy between extinction and exponential increase. In general processes, stabilisation of the age-distribution under growth follows, and indeed of the over-all population composition, and so do the many fine results of the area, like conditional stabilisation of the size of non-extinct subcritical processes. The last two decades have witnessed repeated attempts at treating branching processes with various kinds of dependence between individuals, ranging from local dependence between close relatives only to population size dependence. Of particular interest are very recent findings on processes that change from being supercritical to subcriticality at some threshold size, the carrying capacity of the habitat. We overview the development with an emphasis on these recent results.

Keywords Age-structure • General branching processes • Dependence • Carrying capacity

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1 Introduction

A drastic pedagogical example, illustrating the role of independence in branching processes is the "follow-the-generation-leader process". It also serves to demonstrate that non-linearity on the expectation level is not the same as dependence between individuals. It is defined like a Galton-Watson branching process, but without the requirement that individuals reproduce independently. Instead they all reproduce in the same manner in each generation, follow the leader of their generation as it were. The leaders of different generations have independent and identically distributed offspring numbers, say k with probability $p_k, 0 < p_0 < 1$. If the process is supercritical, $m = \sum kp_k > 1$, expectations, or for that sake the corresponding deterministically modelled population, will grow geometrically, like m^n. The actual population will, however, die out at the first instant the generation leader chooses to have no children. To a probabilist such phenomena will come as no surprise – from the point of view of prevailing deterministic population dynamics, based on differential operators, they may be illuminating.

If dependence, on the contrary, is local in the pedigree, so that e.g. only siblings may influence each other, the branching character remains. Indeed, as has been developed by Olofsson, [14] e.g., this situation can be reduced to a multi-type branching process. For single-type Galton-Watson processes with interacting siblings this is easily described: the whole sibship is turned into a "macro-individual". Different macro-individuals are independent but not identically distributed. Indeed, the sibship size matters. Thus, this number is the type of the multi-type process, and the rest goes by multi-type theory [16]. Results continue to hold for quite general branching processes. A broad investigation of various forms of dependence not destroying classical branching behaviour is contained in [8, 9].

More interesting are those dependence structures that result in new phenomena. One, and maybe the one of greatest importance, is that of population size dependence. Here the individual remains the initiator of reproduction, but the distribution of the latter is influenced by population size. This is straightforward in case of discrete time with non overlapping generations (the "Galton-Watson" case), less so for general processes, which evolve in continuous time and where individuals can give birth any time. Still, for quite (but not completely) general branching processes, those that are Markovian in the age structure, [6], this is feasible in terms of age-dependent intensities. Most literature concerns simpler, Galton-Watson and birth-and-death style, processes and populations where reproduction forces stabilise when population size grows to infinity, usually around a limiting critical reproduction. We shall focus upon populations with a finite so called carrying capacity, i.e. populations living in a habitat with a finite capacity, so that the population is supercritical while below this capacity and subcritical above [12].

2 No Bounded Population Can Persist

Branching processes are characterized by the fundamental dichotomy between extinction and exponential, Malthusian growth. As is well known Markov chains with zero as sole absorbing state and communication between states otherwise, exhibit a similar behaviour: extinction or else growth beyond all limits, albeit not necessarily at an exponential rate. The latter dichotomy holds much more generally, indeed for any natural closed population model, as seen from the following elementary but elegant theorem.

Theorem 1 (General Dichotomy). *Consider non-negative (not necessarily integer valued) random variables $X_1, X_2, \ldots$. Assume 0 absorbing (i.e. $X_n = 0 \Rightarrow X_{n+1} = 0$) and suppose that for any x there is a $\delta > 0$ such that $\mathbb{P}(\exists n; X_n = 0 | X_1, \ldots X_k) \geq \delta$, if only $X_k \leq x$. Then, with probability one, either there is an n such that all $X_k = 0$ for $k \geq n$ or $X_k \to \infty$, as $k \to \infty$. If $\mathbb{E}[X_n]$ remains bounded, it follows that X_n must turn zero, almost surely.*

An example of non-exponential population growth is the well-known linear increase in molecule number showing up in PCR, the polymerase chain reaction, mathematically a consequence of asymptotic criticality, as the number of molecules tends to infinity [11].

Proof. Let $D = \{\exists n; X_n = 0\}$ be the event of extinction. By Lévy's theorem on the convergence of conditional expectations with respect to increasing sigma-algebras, or more generally by martingale convergence,

$$\mathbb{P}(D | X_1, \ldots X_k) \to 1_D, k \to \infty,$$

since D is measurable with respect to the σ-algebra generated by all the $X_i, i = 1, 2, \ldots$. If the outcome is such that X_k does not tend to infinity, then it comes under some level x infinitely often. The conditional extinction probability on the left hand side exceeds δ, and hence so must 1_D. But $1_D > 0 \Rightarrow 1_D = 1$.

If thus any population with, say, a bounded expected size must die out, the question arises when and how this will occur, and if there may be a quasi-stationary plateau, persisting a long time before extinction. Another interesting question would be how size and composition distribution during such a quasi-stationary stage might relate to the stationary limit distributions exhibited by deterministic, differential equations based population dynamics.

3 General Branching Processes with Carrying Capacities

3.1 *Process Definition and Martingale Representation*

Consider a population of individuals with ages $(a^1, \ldots, a^z) = A$. A member individual of age a has a random life span with hazard rate $h_A(a)$. During life she gives birth with intensity $b_A(a)$, both rates dependent on the individual's age as well as the whole setup of ages. Finally, when she dies, she splits into a random number $Y(a)$ of off-spring with a distribution that may be influenced by the age a of the mother at death/splitting and beyond that by the number of individuals around and their ages A. Childbearing and life length may thus be affected by population size and age structure, but apart from this individuals live and reproduce independently of each other. We write $m_A(a) = \mathbb{E}[Y(a)]$ and $v_A(a)$ for the second moment, and generally suffix entities by age vectors to indicate present or starting age distribution. Thus, $\mathbb{P}_A$ and $\mathbb{E}_A$ indicate that the population started at time $t = 0$ not from one newborn ancestor but rather from z individuals, of ages $A = (a^1, \ldots, a^z)$, respectively. No index means start from a given age configuration.

It is convenient to look at the collection of ages A as a measure

$$A = \sum_{i=1}^{z} \delta_{a_i},$$

where δ_a denotes the point measure at a. If there were no deaths and no births, then the population would change only by ageing. When an individual dies, its point mass disappears and an offspring number of point masses appear at zero. Similarly, when an individual gives birth during life, a point mass appears at zero. Thus, population evolution in time is given by a measure-valued process. When a carrying capacity K is introduced, intensities will be influenced by the value of the latter, and a whole family of such processes, indexed by K, obtains.

Existence of such processes follow from the general Ulam-Harris construction, [7] or from general Markov theory, since these are measure-valued Markov processes. Following [12], we proceed to the generator and the integral representation, known as Dynkin's formula. (For any function f on $\mathbb{R}$

$$(f, A) = \int f(x)A(dx) = \sum_{i=1}^{z} f(a^i).$$

Theorem 2 ([10]). *For a bounded differentiable function F on $\mathbb{R}^+$ and a continuously differentiable function f on $\mathbb{R}^+$, the following limit exists*

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}_A\Big\{F((f, A_t)) - F((f, A))\Big\} = \mathscr{G}F((f, A)), \quad (1)$$

where

$$\mathscr{G}F((f, A)) = F'((f, A))(f', A) + \sum_{j=1}^{z} b_A(a^j)\{F(f(0) + (f, A)) - F((f, A))\}+$$

$$+ \sum_{j=1}^{z} h_A(a^j)\{\mathbb{E}_A[F(Y(a^j)f(0) + (f, A) - f(a^j))] - F((f, A))\}, \quad (2)$$

and $Y(a)$ denotes the number of children at death of a mother, dying at age a. Consequently, Dynkin's formula holds: for a bounded C^1 function F on $\mathbb{R}$ and a C^1 function on $\mathbb{R}^+$

$$F((f, A_t)) = F((f, A_0)) + \int_0^t \mathscr{G}F((f, A_s))ds + M_t^{F,f}, \tag{3}$$

where $M_t^{F,f}$ is a local martingale with predictable quadratic variation

$$\langle M^{F,f}, M^{F,f}\rangle_t = \int_0^t \mathscr{G}F^2((f, A_s))ds - 2\int_0^t F((f, A_s))\mathscr{G}F((f, A_s))ds.$$

As a corollary, the following representation was also obtained in [10]:

Theorem 3. *For a C^1 function f on $\mathbb{R}^+$*

$$(f, A_t) = (f, A_0) + \int_0^t (L_{A_s} f, A_s)ds + M_t^f, \tag{4}$$

where the linear operators L_A are defined by

$$L_A f = f' - h_A f + f(0)(b_A + h_A m_A), \tag{5}$$

and M_t^f is a local square integrable martingale with the sharp bracket given by

$$\langle M^f, M^f\rangle_t = \int_0^t \big(f^2(0)b_{A_s} + f^2(0)v_{A_s}^2 h_{A_s} + h_{A_s} f^2 - 2f(0)m_{A_s} h_{A_s} f, A_s\big)ds, \tag{6}$$

A further corollary gives the corresponding representation of the population size $Z_t = (1, A_t)$:

$$Z_t = Z_0 + \int_0^t (b_{A_s} + (m_{A_s} - 1)h_{A_s}, A_s)ds + M_t^1, \tag{7}$$

since $L_A 1 = b_A + h_A(m_A - 1)$.

Studies of populations by measure-valued Markov processes have been done in the past with various setups, see e.g. [1–3] Sect. 9.4, [4, 13, 15], and [17], which comes closest to the already quoted [10] and [12].

3.2 *Criticality and the Carrying Capacity*

The concepts of super-, sub-, and plain criticality are well known when there is no population dependence. In the Bellman-Harris case, when reproduction occurs only at death and independently of mother's age then, they are easily generalised to the present, more general set-up: criticality (at the age-configuration A) is determined by the offspring mean m_A, being greater, smaller or equal to 1 respectively. In the presence of births during lifetime the situation is more involved. In fact, from the martingale representation (7) reproduction is critical in the population-age-dependent case precisely when the criticality function

$$\chi_A = L_A 1 = b_A + (m_A - 1)h_A \tag{8}$$

satisfies $(\chi_A, A) = 0$. Super- or sub-criticality holds when (χ_A, A) is positive or negative, respectively. We recover the observation that without child-bearing during life ($b_A = 0$) but with Bellman-Harris splitting the usual criticality description persists but in general criticality at an age composition A is defined by (8).

A stronger criticality concept could be termed *strict*: A population process is strictly critical at A if and only if $\chi_A(a) = b_A(a) + h_A(a)(m_A(a) - 1) = 0$ identically in a. For Bellman-Harris type processes the two concepts obviously coincide.

Third, one could speak of *annealed* criticality at A, if the reproduction of an individual living in a non-population dependent branching population and having the reproduction parameters b_A, h_A, and m_A with A fixed throughout life is critical in the classical sense. In the Bellman-Harris case all three concepts coincide.

Following notation in population biology, we denote the carrying capacity by K. We think of it as a comparatively large number, such that reproduction is subcritical above and supercritical below the threshold level K, though this is a vague assertion until we make it clear what type of criticality we have in mind.

Provided that dependence on population composition is through the scaled population size $x = z/K$ only, it follows that any A with total mass K, or scaled mass 1, is a criticality point. In terms of the criticality function χ,

$$\chi_A = \chi_x, \quad \chi_1 = 0.$$

We refer to this as population size (or density) dependence and allow ourselves to index parameters analogously, b_x, h_x, m_x.

3.3 *Early Extinction*

Now consider a population size dependent process with carrying capacity K, starting at time $t = 0$ from z individuals. To ease notation we take them all as newborn. What are chances that population size will reach a vicinity of the carrying capacity before extinction? We write T for the time to extinction and T_d for the time the population first attains a size $\geq dK, 0 < d < 1$. Clearly (since such a population must die out eventually),

$$T < T_d \Rightarrow \forall t, Z_t < dK.$$

Now assume that reproduction decreases with increasing population and let units with a tilde denote entities pertaining to a not-population dependent branching process with the parameters b_d, h_d, m_d. Then

$$\mathbb{P}(T < T_d) \leq \mathbb{P}(\tilde{T} < \infty) = \tilde{q}^z,$$

where $\tilde{q}$ is the extinction probability of the not population dependent branching process. If $M_d > 1$ and V_d denote the mean and variance of the all-life reproduction of this latter process, we have by Haldane's inequality ([5], p. 125) that the probability of the original population never reaching dK is

$$\mathbb{P}(T < T_d) \leq \left(1 - \frac{2(M_d - 1)}{V_d + M_d(M_d - 1)}\right)^z,$$

quite small in typical cases (even if z is not excessively large). With a positive chance the population will thus reach a size or order K. Since it grows quicker than the process $\tilde{Z}_t$ while under the level dK, and the latter grows exponentially, we can conclude that this will occur after a time of order $\log K$.

Theorem 4. *If reproduction decreases with population size, any population size* $dK, 0 < d < 1$ *is attained with positive probability within a time* $T_d = O(\log K)$*, as* $K \to \infty$*.*

3.4 *Lingering Around the Carrying Capacity*

So what happens if the population does not die out without approaching its carrying capacity? Ultimate extinction can not be avoided, but when will it happen and how will the process behave before it? To investigate this, we assume that the population is density dependent and strictly critical at the carrying capacity, $\chi_1 = 0$. Further, we assume a Lipschitz continuity in the neighbourhood of 1, $|\chi_x| \leq C|x - 1|$, for some constant C. Then:

Theorem 5. *Assume that* $X_0^K \to 1$ *in probability, as* $K \to \infty$*. Then the total population size scaled by the carrying capacity* $X_t^K = Z_t^K/K$ *converges in probability to 1, uniformly on any time interval* $[0, T], T > 0$*. In other words, for any* $\eta > 0$

$$\lim_{K\to\infty} P(\sup_{t\leq T} |X_t^K - 1| > \eta) = 0.$$

To prove the final result about remaining an exponential time around the carrying capacity, the existence of a moment generating function of the offspring number at splitting is required, and a condition that guarantees clear subcriticality not too far above K. Write $\phi_A(t)(a) := \mathbb{E}_A[e^{tY(a)}]$ for the moment generating function of the number $Y(a)$ of offspring at the splitting of an a-aged individual in a population of age-composition A.

Assumption. There exists a population size $V_K > K$ such that

$$(e^{1/K} - 1)b_A + (\phi_A(1/K)e^{-1/K} - 1)h_A = 0, \quad \text{when } |A| = V_K \quad \text{and} \tag{9}$$
$$(e^{1/K} - 1)b_A + (\phi_A(1/K)e^{-1/K} - 1)h_A \le 0, \quad \text{whenever } (1, A) > V_K.$$

Since the reproduction is subcritical for population sizes larger than K, such a number exists. What is needed, and guaranteed by this assumption is that it is not too far away from K. For example, when $b = 0$ and Y is a binary splitting with $\mathbb{P}(Y_A = 2) = K/(K + z)$, then V_K is determined from

$$\frac{z}{K+z}e^{-1/K} + \frac{K}{K+z}e^{1/K} = \mathbb{E}_z e^{(Y-1)/K} = 1.$$

Solving for z gives $V_K = e^{1/K}K$.

Theorem 6. *Let $X_0^K = 1$ and $\tau = \inf\{t : |X_t^K - 1| > \varepsilon\}$ for any $\varepsilon > 0$. Suppose that the previous assumptions hold and that the number of children possible at splitting is bounded by some constant. Then $\mathbb{E}\tau$ is exponentially large in K, i.e. for some positive constants C, c*

$$\mathbb{E}\tau > Ce^{cK}.$$

The proofs hinge upon the various martingale representations (3), (4), and (7); the reader is referred to [12]. For related results in a somewhat different setup, cf. [17].

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Testing Functional Connection between Two Random Variables

Gyula O.H. Katona

Dedicated to Professor Yuri Vasilyevich Prokhorov for his 80th birthday

Abstract Two discrete random variables, ξ and η are considered. The goal is to decide whether η is a function of ξ. A series of tests are performed, (ξ_i, η_i), $1 \le i \le m$, are independent experiences with the same distribution as (ξ, η). The hypothesis is declined if $\xi_i = \xi_j, \eta_i \ne \eta_j$ holds for some $i \ne j$. A condition is given on the character of convergence of the joint distribution of ξ and η ensuring the rejection of the hypothesis with a given limiting probability p.

Keywords Dependence testing • Sieve method

Mathematics Subject Classification (2010): 60F99

1 Introduction

Let ξ and η be two, not necessarily independent random variables. The goal of the present paper is to study the situation when one needs to decide if η is a (deterministic) function of ξ or not, by using many independent tests.

The probability of the event that $\xi = k$ and $\eta = \ell$ is $p_{k,\ell}$, the probability of ξ being k is $p_k = \sum_\ell p_{k\ell}$. Suppose that we have m tests. Let ξ_i (η_i) $(1 \le i \le m)$ be totally independent copies of ξ (η). We will study the probability $\Pr(\xi \to \eta, m)$ of

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the event that m experiments (mis)indicate that ξ (deterministically) determines η, that is, there are no i and j $(1 \le i, j \le m)$ such that $\xi_i = \xi_j, \eta_i \neq \eta_j$.

Of course, if η is really a function of ξ then $\Pr(\xi \to \eta, m) = 1$ for every m, otherwise it is a decreasing function of m. The most practical case is when the probabilities $p_{k,\ell}$ are constant. Then the probability $\Pr(\xi \to \eta, m)$ tends to 0 when $m \to \infty$. One could ask many questions in this case, for instance to study the rate of convergence as a function of the $p_{k,\ell}$'s, but we will be investigating another case, namely the one when the probabilities are very small.

In the rest of the paper a series of probability distributions will be considered, that is $p_{k,\ell}(n),\ \ p_k(n)$ where n tends to infinity. The number of possible values of ξ and η are finite, but this is also increasing with n. One can easily see that the smaller probabilities require a larger m to give a counter-example for the functional connection. Therefore m is also supposed to depend on n. For the sake of convenience we will not denote this dependence.

Heuristic form of Theorem 1. *If the probabilities uniformly decrease and m is increasing faster than*

$$\frac{1}{\sqrt{\sum_k p_k^2 - \sum_{k,\ell} p_{k,\ell}^2}}$$

then a counter-example shows, with large probability, that η is not a function of ξ. On the other hand, if m is increasing slower than the quantity above then the probability of a counter-example is nearly 0.

It is more convenient to use a logarithmic form in the precise formulation, this is why we introduce the following quantity:

$$H_2(\xi \to \eta) = -\log_2 \left(\sum_k p_k^2 - \sum_{k,\ell} p_{k,\ell}^2 \right). \tag{1}$$

Since the probabilities depend on n, the quantity $H_2(\xi \to \eta)$ will also do so (without denoting this dependence).

2 The Statement

Let $p(\xi, \eta, I)$ denote the probability of the event that the pair $(\xi_1, \eta_1), (\xi_2, \eta_2)$ gives a counter-example, that is, $\Pr(\xi_1 = \xi_2, \eta_1 \neq \eta_2)$.

Similarly $p(\xi, \eta, V)$ denotes the probability of the event that the triple (ξ_1, η_1), $(\xi_2, \eta_2), (\xi_3, \eta_3)$ gives two counter-examples in the following way: $\xi_1 = \xi_2 = \xi_3$, $\eta_1 \neq \eta_2 \neq \eta_3$.

Finally $p(\xi, \eta, N)$ is the probability of the event that the quadruple (ξ_1, η_1), $(\xi_2, \eta_2), (\xi_3, \eta_3), (\xi_4, \eta_4)$ gives three counter-examples forming a path: $\xi_1 = \xi_2 = \xi_3 = \xi_4$, $\eta_1 \neq \eta_2 \neq \eta_3 \neq \eta_4$.

Theorem 1. *Suppose that*

$$\frac{p(\xi, \eta, V)^2}{p(\xi, \eta, I)^3} \to 0$$

and

$$\frac{p(\xi, \eta, N)}{p(\xi, \eta, I)^2} \to 0$$

hold. Then

$$\Pr(\xi \to \eta, m) \to \begin{cases} 0 & \textit{if } 2\log_2 m - H_2(\xi \to \eta) \to +\infty, \\ e^{-2^{a-1}} & \textit{if } 2\log_2 m - H_2(\xi \to \eta) \to a, \\ 1 & \textit{if } 2\log_2 m - H_2(\xi \to \eta) \to -\infty. \end{cases}$$

The values $p(\xi, \eta, I)$, $p(\xi, \eta, V)$ and $p(\xi, \eta, N)$ will be expressed by the probabilities in the next section.

Motivations, consequences, related literature, and analysis of the conditions are postponed to the last section.

3 The Proofs

Lemma 1. $p(\xi, \eta, I) = \sum_k p_k^2 - \sum_{k,\ell} p_{k,\ell}^2$.

Proof. The left hand side is equal to $\Pr(\xi_u = \xi_v, \eta_u \neq \eta_v)$ by definition, what is equal to

$$\sum_k \sum_{\ell \neq \ell'} p_{k,\ell} p_{k,\ell'} = \sum_k \sum_{\ell,\ell'} p_{k,\ell} p_{k,\ell'} - \sum_k \sum_\ell p_{k,\ell}^2$$

$$= \sum_k \left(\sum_\ell p_{k,\ell} \right)^2 - \sum_{k,\ell} p_{k,\ell}^2 = \sum_k p_k^2 - \sum_{k,\ell} p_{k,\ell}^2$$

Observe that $H_2(\xi \to \eta) = -\log_2 p(\xi, \eta, I)$. □

Lemma 2.

$$p(\xi, \eta, V) = \sum_k p_k^3 - 2 \sum_{k,\ell} p_k p_{k,\ell}^2 + \sum_{k,\ell} p_{k,\ell}^3$$

Proof. Use the simple sieve for the "space" $\xi_1 = \xi_2 = \xi_3$.

$$p(\xi, \eta, \mathrm{V}) = \Pr(\xi_1 = \xi_2 = \xi_3, \eta_1 \neq \eta_2 \neq \eta_3) =$$

$$\Pr(\xi_1 = \xi_2 = \xi_3) - \Pr(\xi_1 = \xi_2 = \xi_3, \eta_1 = \eta_2) - \Pr(\xi_1 = \xi_2 = \xi_3, \eta_2 = \eta_3) +$$
$$\Pr(\xi_1 = \xi_2 = \xi_3, \eta_1 = \eta_2 = \eta_3) =$$
$$\Pr(\xi_1 = \xi_2 = \xi_3) - 2\Pr(\xi_1 = \xi_2 = \xi_3, \eta_1 = \eta_2) +$$
$$\Pr(\xi_1 = \xi_2 = \xi_3, \eta_1 = \eta_2 = \eta_3) =$$
$$\sum_k p_k^3 - 2\sum_k \Pr(\xi_3 = k)\Pr(\xi_1 = \xi_2 = k, \eta_1 = \eta_2) + \sum_{k,\ell} p_{k,\ell}^3$$

□

Lemma 3.

$$p(\xi, \eta, N) = \sum_k p_k^4 - 3\sum_{k,\ell} p_k^2 p_{k,\ell}^2 + 2\sum_{k,\ell} p_k p_{k,\ell}^3 + \sum_k \left(\sum_\ell p_{k,\ell}^2\right)^2 - \sum_{k,\ell} p_{k,\ell}^4.$$

The proof is analogous to that of Lemma 2. □

Let $C_{k,\ell}$ $(1 \le k, \ell \le m)$ be a partition of the set $\{1, 2, \ldots, m\}$, where some classes can be empty. The partition is denoted by $\mathscr{C}$. The vertex set of the graph $G(\mathscr{C})$ is $\{1, 2, \ldots, m\}$, two vertices x and y are joined by an edge if $x \in C_{k,\ell}$, $y \in C_{k,\ell'}$ holds for some $\ell \neq \ell'$. Define $C_k = \cup_\ell C_{k,\ell}$, and let $|C_k| = c_k$. The subgraph of $G(\mathscr{C})$ induced by C_k is called a component even in the case when it is an empty graph (that is $C_{k,\ell}$ are empty for all ℓ with one exception). Suppose that $|C_{k,1}| \ge |C_{k,2}| \ge \ldots \ge |C_{k,m}|$.

A subgraph consisting of vertex-disjoint edges of a graph is a *matching* in G. The vertex-disjoint union of a matching and one path consisting of two edges is called a *V-matching*. Finally, the vertex-disjoint union of a matching and one path consisting of three edges is an *N-matching*.

Lemma 4. *Let $G(\mathscr{C})$ be the graph defined above. Then*

$$\sum_{\substack{\text{matching of}\\ j \text{ edges}}} (-1)^j + 2 \sum_{\text{V-matching}} 1 + \sum_{\text{N-matching}} 1 \ge 0 \tag{2}$$

where the matchings, V-matchings and N-matchings are subgraphs of $G(\mathscr{C})$.

Proof. First suppose that $c_k > 2$ holds for at least one k. Let $\mathscr{M}_+$ ($\mathscr{M}_-$) denote the family of all matchings of $G(\mathscr{C})$ consisting of even (odd) number of edges. Furthermore, $\mathscr{V}$ and $\mathscr{N}$ denote the families of all V-matchings and N-matchings, respectively. We will give a mapping f from $\mathscr{M}_-$ to $\mathscr{M}_+ \cup \mathscr{V} \cup \mathscr{N}$.

Suppose that $M \in \mathscr{M}_-$ has two edges in one of the components. It is easy to see that $G(\mathscr{C})$ contains an edge joining endpoints of these edges. Add this edge to M. The so obtained set $f(M)$ of edges is in $\mathscr{N}$. If M contains at most one edge in every component and a C_k with $c_k > 2$ contains an edge e then add another edge to this component, having a common endpoint with e. The so obtained set $f(M)$ of edges is in $\mathscr{V}$. Finally, suppose that every component contains at most one edge of M, but the components C_k with $c_k > 2$ none. Then add an edge of such a C_k with the smallest index. The so obtained $f(M)$ contains an even number of edges, therefore $f(M) \in \mathscr{M}_+$ holds.

The mapping f is not an injection, but "almost". If $M' \in \mathscr{N}$ then the middle edge of the path is uniquely determined, $|f^-(M')| = 1$. On the other hand, if $M' \in \mathscr{V}$ then M' could be obtained in two different ways, therefore $|f^-(M')| \leq 2$. Finally, if $M' \in \mathscr{M}_+$ then the new edge can be only in the component having the smallest index k with $c_k > 2$. Then, $|f^-(M')| = 1$ holds, again. The mapping f indirectly associates a $+1$ term with every -1 on the left hand side of (1), since the terms associated with the V-matchings are doubled. This proves the inequality in this case.

The only remaining case is when $c_k = 2$ $(1 \leq k \leq r)$. If $|C_k| = |C_{k,\ell}|$ holds for some ℓ then this component contains no edge, it plays no role in (1). Therefore one can suppose that $|C_{k,1}| = |C_{k,2}| = 1$ holds for every k. Let the number of components with at least one edge be r. Then $G(\mathscr{C})$ has r vertex-disjoint edges. It contains neither a V-matching nor an N-matching. The number of matchings M of j edges in $G(\mathscr{C})$ is $\binom{r}{j}$ therefore the left hand side of (1) is

$$\sum_{j=0}^{r} \binom{r}{j} (-1)^j$$

which is 0 if $0 < r$ and 1 if $r = 0$. □

Lemma 5. *If $G(\mathscr{C})$ has at least one edge then*

$$\sum_{\substack{\text{matching of} \\ j \text{ edges}}} (-1)^j + 2 \sum_{V-\text{matching}} (-1) + \sum_{N-\text{matching}} (-1) \leq 0 \qquad (3)$$

where the matchings, V-matchings and N-matchings are subgraphs of $G(\mathscr{C})$.

Proof. The proof is analogous to the previous one. A mapping g can be defined from $\mathscr{M}_+$ to $\mathscr{M}_- \cup \mathscr{V} \cup \mathscr{N}$, basically in the same way as in the previous proof. □

Lemma 6.

$$1+\sum_{j=1}^{\lfloor \frac{m}{2}\rfloor}\frac{(-1)^j}{j!}\binom{m}{2}\binom{m-2}{2}\cdots\binom{m-2j+2}{2}2^{-jH_2(\xi\to\eta)}-$$

$$-\sum_{j=0}^{\lfloor \frac{m-3}{2}\rfloor}\frac{1}{j!}3\binom{m}{3}\binom{m-3}{2}\binom{m-5}{2}\cdots\binom{m-2j-1}{2}p(\xi,\eta,V)2^{-jH_2(\xi\to\eta)}-$$

$$-\sum_{j=0}^{\lfloor \frac{m-4}{2}\rfloor}\frac{1}{j!}12\binom{m}{4}\binom{m-4}{2}\binom{m-6}{2}\cdots\binom{m-2j-2}{2}p(\xi,\eta,N)2^{-jH_2(\xi\to\eta)}\le$$

$$\le \Pr(\xi\to\eta,m)\le \tag{4}$$

$$\le 1+\sum_{j=1}^{\lfloor \frac{m}{2}\rfloor}\frac{(-1)^j}{j!}\binom{m}{2}\binom{m-2}{2}\cdots\binom{m-2j+2}{2}2^{-jH_2(\xi\to\eta)}+$$

$$+\sum_{j=0}^{\lfloor \frac{m-3}{2}\rfloor}\frac{1}{j!}3\binom{m}{3}\binom{m-3}{2}\binom{m-5}{2}\cdots\binom{m-2j-1}{2}p(\xi,\eta,V)2^{-jH_2(\xi\to\eta)}+$$

$$+\sum_{j=0}^{\lfloor \frac{m-4}{2}\rfloor}\frac{1}{j!}12\binom{m}{4}\binom{m-4}{2}\binom{m-6}{2}\cdots\binom{m-2j-2}{2}p(\xi,\eta,N)2^{-jH_2(\xi\to\eta)}.$$

Proof. The random pairs (ξ_i,η_i) $(1\le i\le m)$ define a random partition on the set $\{1,2,\ldots,m\}$ in a natural way, by the equality of these pairs: $C_{k,\ell}=\{i:(\xi_i,\eta_i)=(k,\ell)\}$. Then $C_k=\cup_\ell C_{k,\ell}$ is the kth class in the partition defined by ξ's. The event that η seems to be functionally dependent on ξ, that is, there is no pair $(k,\ell),(k,\ell')(\ell\ne\ell')$ among the m outcomes is equivalent to the event that $G(\mathscr{C})$ has no edge, that is, $\Pr(\xi\to\eta,m)$ equals $\Pr(G(\mathscr{C})$is an empty graph). In other words,

$$\Pr(\xi\to\eta,m)=\Pr(G(\mathscr{C})\text{is an empty graph})+$$

$$\sum_{\mathscr{C}}0\cdot\Pr(\text{the pairs }(\xi_i,\eta_i)\text{ determine the partition }\mathscr{C}), \tag{5}$$

where the sum runs over all partitions with a non-empty $G(\mathscr{C})$. The elements $1,2,\ldots,m$ are of course numbered, but the classes are not.

The left hand side of (2) is 1 for the $\mathscr{C}$ with the empty $G(\mathscr{C})$, otherwise it is non-negative by Lemma 4. Therefore, replacing the weights of the probabilities by this left hand side, an upper bound is obtained for (5):

$$\sum_{\mathscr{C}} \left(\sum_{\substack{\text{matching of} \\ j \text{ edges}}} (-1)^j + 2 \sum_{V-\text{matching}} 1 + \sum_{N-\text{matching}} 1 \right) \cdot$$

$$\Pr(\text{the pairs } (\xi_i, \eta_i) \text{ determine the partition } \mathscr{C}) \tag{6}$$

where the matchings, V-matchings and N-matchings are subgraphs of $G(\mathscr{C})$ for the given $\mathscr{C}$. Break this sum into three parts and consider first the part

$$\sum_{\mathscr{C}} \sum_{\substack{\text{matching of} \\ j \text{ edges}}} (-1)^j \Pr(\text{the pairs } (\xi_i, \eta_i) \text{ determine the partition } \mathscr{C}) =$$

$$\sum_{\substack{\text{matching of} \\ j \text{ edges}}} (-1)^j \sum_{\mathscr{C}} \Pr(\text{the pairs } (\xi_i, \eta_i) \text{ determine the partition } \mathscr{C}). \tag{7}$$

The last sum is nothing else but the probability of the event that all the edges in the given matching M are in $\mathscr{C}$, that is,

$$\Pr(\forall \{u, v\} \in M \text{ the relations } \xi_u = \xi_v, \eta_u \neq \eta_v \text{ hold}).$$

Because of the independence, this is the jth power of $p(\xi, \eta, I)$ what is equal to

$$\sum_k p_k^2 - \sum_{k,\ell} p_{k,\ell}^2 = 2^{H_2(\xi \to \eta)} \tag{8}$$

by Lemma 1 and (1). We obtained

$$\sum_{\substack{\text{matching of} \\ j \text{ edges}}} (-1)^j 2^{-jH_2(\xi \to \eta)} \tag{9}$$

for (7).

The number of matchings consisting of j edges is

$$\frac{1}{j!} \binom{m}{2} \binom{m-2}{2} \cdots \binom{m-2j+2}{2}.$$

Using this in (9), a new form of (7) is obtained:

$$1 + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^j}{j!} \binom{m}{2} \binom{m-2}{2} \cdots \binom{m-2j+2}{2} 2^{-jH_2(\xi \to \eta)}$$

and this is the first row of the upper estimate in Lemma 6.

Now consider the second part of (6):

$$\sum_{\mathscr{C}} \sum_{V-\text{matching}} \Pr(\text{the pairs } (\xi_i, \eta_i) \text{ determine the partition } \mathscr{C}) =$$
$$\sum_{V-\text{matching}} \sum_{\mathscr{C}} \Pr(\text{the pairs } (\xi_i, \eta_i) \text{ determine the partition } \mathscr{C}). \tag{10}$$

The last sum is nothing else but the probability of the event that all the edges in the given V-matching V (containing $j+2$ edges) are in $\mathscr{C}$, that is,

$$\Pr(\forall \{u, v\} \in V \text{ the relations } \xi_u = \xi_v, \eta_u \neq \eta_v \text{ hold}).$$

Because of the independence, this is the jth power of (8) $(= 2^{-H_2(\xi \to \eta)})$ times $p(\xi, \eta, V)$ what is given in Lemma 2. The result for (10) is

$$\sum_{\substack{V-\text{matching of}\\ j+2 \text{ edges}}} \left(\sum_k p_k^3 - 2 \sum_{k,\ell} p_k p_{k,\ell}^2 + \sum_{k,\ell} p_{k,\ell}^3 \right) 2^{-jH_2(\xi \to \eta)}. \tag{11}$$

Since the number of V-matchings is

$$\sum_{j=0}^{\lfloor \frac{m-3}{2} \rfloor} \frac{1}{j!} 3 \binom{m}{3} \binom{m-3}{2} \binom{m-5}{2} \cdots \binom{m-2j-1}{2},$$

(11) leads to a new form of (10), giving the second row of the upper estimate of Lemma 6.

The third row can be obtained in an analogous way, the only difference is that $p(\xi, \eta, N)$ should be used rather than $p(\xi, \eta, V)$. This finishes the proof of the upper bound.

The proof of the lower bound is the same, but Lemma 5 should be the starting point rather than Lemma 4. □

Lemma 7. *If*

$$2 \log_2 m - H_2(\xi \to \eta) \to a \tag{12}$$

where a is a constant, independent on n and m $\to \infty$ *then*

$$1 + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^j}{j!} \binom{m}{2} \binom{m-2}{2} \cdots \binom{m-2j+2}{2} 2^{-jH_2(\xi \to \eta)} \tag{13}$$

tends to

$$e^{-2^{a-1}}.$$

Proof. The inequalities

$$\frac{(m-2j)^{2j}}{2^j} \leq \binom{m}{2}\binom{m-2}{2}\cdots\binom{m-2j+2}{2} \leq \frac{m^{2j}}{2^j}$$

lead to the following lower and upper estimates for (13):

$$-\sum_{j=1,3,\ldots,2j\leq m}\frac{1}{j!}\cdot\frac{m^{2j}}{2^j}2^{-jH_2(\xi\to\eta)}+1+\sum_{j=2,4,\ldots,2j\leq m}\frac{1}{j!}\cdot\frac{(m-2j)^{2j}}{2^j}2^{-jH_2(\xi\to\eta)}=$$
$$-\sum_{j=1,3,\ldots,2j\leq m}\frac{1}{j!}2^{j(2\log m-H_2(\xi\to\eta)-1)}+1+\sum_{j=2,4,\ldots,2j\leq m}\frac{1}{j!}2^{j(2\log(m-2j)-H_2(\xi\to\eta)-1)} \tag{14}$$

and

$$-\sum_{j=1,3,\ldots,2j\leq m}\frac{1}{j!}\cdot\frac{(m-2j)^{2j}}{2^j}2^{-jH_2(\xi\to\eta)}+1+\sum_{j=2,4,\ldots,2j\leq m}\frac{1}{j!}\cdot\frac{(m)^{2j}}{2^j}2^{-jH_2(\xi\to\eta)}=$$
$$-\sum_{j=1,3,\ldots,2j\leq m}\frac{1}{j!}2^{j(2\log(m-2j)-H_2(\xi\to\eta)-1)}+1+\sum_{j=2,4,\ldots,2j\leq m}\frac{1}{j!}2^{j(2\log m-H_2(\xi\to\eta)-1)} \tag{15}$$

Compare the members with $\log m$ and $\log(m-2j)$, respectively:

$$2^{j(2\log m-H_2(\xi\to\eta)-1)}-2^{j(2\log(m-2j)-H_2(\xi\to\eta)-1)}= \tag{16}$$
$$2^{j(2\log m-H_2(\xi\to\eta)-1)}\left(1-2^{2j(\log(m-2j)-\log m)}\right)=$$
$$2^{j(2\log m-H_2(\xi\to\eta)-1)}\left(1-\left(\frac{m-2j}{m}\right)^{2j}\right)=$$
$$=2^{j(2\log m-H_2(\xi\to\eta)-1)}\left(1-\left(1-\frac{2j}{m}\right)^{2j}\right).$$

Since $2j \leq m$, the last factor can be upperbounded using the Bernoulli inequality:

$$1-\left(1-\frac{2j}{m}\right)^{2j}\leq 2j\,\frac{2j}{m}=\frac{4j^2}{m}.$$

Hence

$$2^{j(2\log m-H_2(\xi\to\eta)-1)}\frac{4j^2}{m} \tag{17}$$

is an upper bound for (16).

Consider the total change in (14) if the terms with $\log(m-2j)$ are replaced by terms with $\log m$ and use (17).

$$\sum_{j=2,4,\dots,2j\leq m}\frac{1}{j!}2^{j(2\log m-H_2(\xi\to\eta)-1)}-\sum_{j=2,4,\dots,2j\leq m}\frac{1}{j!}2^{j(2\log(m-2j)-H_2(\xi\to\eta)-1)}\leq$$

$$\sum_{j=2,4,\dots,2j\leq m}\frac{1}{j!}2^{j(2\log m-H_2(\xi\to\eta)-1)}\frac{4j^2}{m}. \tag{18}$$

We need to show that this tends to 0 with n. Since $2\log m-H_2(\xi\to\eta)-1$ tends to $a-1$, there is a threshold n_1 such that $2\log m-H_2(\xi\to\eta)-1\leq a$ when $n>n_1$. Each term in (18) tends to 0, therefore the sum of the terms until $j\leq n_1$ will do so. In the terms with $j>n_1$ the expression $2\log m-H_2(\xi\to\eta)-1$ can be replaced by a without decreasing them. The value $\frac{1}{m}\frac{4j^2}{j!}2^{ja}$ is obtained as an upper bound for the jth term. Extend the sum with the odd terms and the large terms the following upper bound is obtained:

$$\frac{4}{m}\sum_{j=0}^{\infty}\frac{j^2}{j!}2^{ja}=\frac{4}{m}\left(2^{2(a+1)}e^{2^a}+2^{a+1}e^{2^a}\right)$$

which obviously tends to 0 with $n\to\infty$. This shows that $\log(m-2j)$ can be replaced by $\log m$ in (14) without changing its limit for $n\to\infty$. Then (14) becomes

$$e^{-2^{2\log m-H_2(\xi\to\eta)-1}}$$

which tends to $e^{-2^{a-1}}$. Therefore the lim inf of (13) is at least this much. Starting from (15), the same steps prove that that the lim sup of (13) cannot be more. This is really its limit. □

Lemma 8. *Suppose that $m\to\infty$, (12) and*

$$\frac{p(\xi,\eta,V)^2}{p(\xi,\eta,I)^3}\to 0 \tag{19}$$

hold. Then

$$\sum_{j=0}^{\lfloor\frac{m-3}{2}\rfloor}\frac{1}{j!}3\binom{m}{3}\binom{m-3}{2}\binom{m-5}{2}\cdots\binom{m-2j-1}{2}p(\xi,\eta,V)2^{-jH_2(\xi\to\eta)}\to 0.$$

Proof. It will be similar to that of Lemma 7. Start with the upper estimate

$$\binom{m-3}{2}\binom{m-5}{2}\cdots\binom{m-2j-1}{2} \le \frac{m^{2j}}{2^j}.$$

This leads to the following upper estimate for the investigated quantity:

$$3\binom{m}{3}p(\xi,\eta,\mathrm{V})\sum_{j=0}^{\infty}\frac{m^{2j}}{2^j\, j!}2^{-jH_2(\xi\to\eta)} =$$

$$3\binom{m}{3}p(\xi,\eta,\mathrm{V})\sum_{j=0}^{\infty}\frac{1}{j!}2^{j(2\log m-H_2(\xi\to\eta)-1)} =$$

$$3\binom{m}{3}p(\xi,\eta,\mathrm{V})e^{2^{2\log m-H_2(\xi\to\eta)-1}}.$$

Here the last factor tends to $e^{2^{a-1}}$ by (12), therefore we only have to show that

$$3\binom{m}{3}p(\xi,\eta,\mathrm{V}) \to 0. \tag{20}$$

Relation (12) implies

$$m^2\left(\sum_k p_k^2-\sum_{k,\ell}p_{k,\ell}^2\right)\to 2^a$$

and

$$m^3\left(\sum_k p_k^2-\sum_{k,\ell}p_{k,\ell}^2\right)^{\frac{3}{2}}\to 2^{\frac{3a}{2}}.$$

This convergence and the square root of (19) prove (20). □

Lemma 9. *Suppose that $m\to\infty$, (12) and*

$$\frac{p(\xi,\eta,N)}{p(\xi,\eta,I)^2}\to 0 \tag{21}$$

hold. Then

$$\sum_{j=0}^{\lfloor\frac{m-4}{2}\rfloor}\frac{1}{j!}12\binom{m}{4}\binom{m-4}{2}\binom{m-6}{2}\cdots\binom{m-2j-2}{2}p(\xi,\eta,N)2^{-jH_2(\xi\to\eta)}\to 0.$$

Proof. It is analogous to the previous one. □

Now the statement of the theorem is an easy consequence of Lemmas 6–9. □

4 Previous Work, Remarks, Future Work

Related earlier work. The problem in question was studied in the papers of Selivanov [8] and Mihailov and Selivanov [3]. They have proved limit theorems on the convergence of the quantity studied here to the Poisson and normal distributions, respectively.

Our motivation: database theory. Our primary motivation was database theory. A very simple model of a database is an $m \times n$ matrix, where the columns are representing the types of data (called *attributes*), say last name, first name, etc. while the data of one individual are in one row. A fundamental concept in the theory is the *functional dependency*. Let A be a set of columns, b one column. We say that b *functionally depends* on A if the individuals having the same data in the columns belonging to A have the same data in b. Shortly, the data in A uniquely determine the data in b. More precisely, the matrix has no two rows having the same entries in the columns in A and different in b. In notation $A \to b$. In most of the older works it is supposed that there is a "logical connection" among the data, so the functional dependencies are a priori given. Here we adopt the view that only those functional dependencies $A \to b$ exist which are determined by the given matrix.

Suppose that some probabilistic connections are a priori given among the data, that is a joint distribution

$$\Pr(\zeta_1 = u_1, \zeta_2 = u_2, \dots, \zeta_n = u_n)$$

is given among the n data in one row. (We might know or we might not know this distribution.) The choice of the rows is totally independent. Let ζ_A be the random vector with the components ζ_i for all $i \in A$. Of course, the distribution of a row determines the joint distribution of the pair ζ_A, ζ_b. For fixed n, A and b we could speak about the probability $\Pr(\zeta_A \to \zeta_b, m)$ of the event that the m actual rows indicate that $A \to b$. This situation leads to the problem only mentioned in the Introduction, but not considered in the present paper.

Now we describe our real motivating problem. Suppose that n is large, the m (it is a function of n) rows of the matrix are chosen following the given joint distribution. What are the sizes of A satisfying $A \to b$ for some b, that is, what are the typical sizes of the functional dependencies appearing in the matrix. It is intuitively clear that for small (say of fixed size) A this cannot happen (unless the distribution gives a functional dependency). The sizes of the A's showing $A \to b$ must increase by n. Then ζ_A as a vector of growing size has an increasing number of possible values, and their probabilities are typically decreasing. This is how we arrived to the model of the paper when the probabilities of ξ are decreasing with n. We will show in a forthcoming paper how to use the results of the present paper for the determination of the typical sizes of A's in a functional dependency $A \to b$ in a large database.

The special case when the ζ_i's are independent was considered in [2]. The method of the present paper is a generalization of that paper. Similar (but not identical) results using different methods can be found in [1]. Papers [6] and [7] contain somewhat related results on random databases.

On the conditions of the main theorem. The two conditions (19) and (21) are chosen by a very simple reason: the proof works under them. When are they satisfied? It is easy to see that if the probabilities "uniformly" tend to 0, ξ and η are "nearly independent" that is there are constants c, d, C, D such that

$$\frac{C}{n} < p_k < \frac{D}{n}, \qquad \frac{c}{n^2} < p_{k,\ell} < \frac{d}{n^2}$$

hold then (19) and (21) are satisfied. On the other hand, if one p_k does not tend to 0 then the conditions are not satisfied. More work is needed to find necessary and sufficient conditions for the probability distributions under which these conditions are true. We do not even know whether the two conditions are independent or not. Does (19) imply (21)?

Our function $H_2(\xi \to \eta)$, special cases. It is slightly related to the Rényi entropy of order 2 (see [4] and [5]):

$$H_2(\xi) = -\log_2 \sum_k p_k^2.$$

However our formula (1) is far from being a "conditional entropy" derived from the Rényi entropy.

If η is a function of ξ then there is a unique ℓ for which $p_{k,\ell}$ is non-zero, that is, $p_{k,\ell} - p_k$. Hence $p(\xi, \eta, I) = \sum_k p_k^2 - \sum_{k,\ell} p_{k,\ell}^2 = 0$ and $H_2(\xi \to \eta) = \infty$. The trivial statement $\Pr(\xi \to \eta, m) = 1$ in this case really follows from Theorem 1.

Suppose now that ξ and η are independent. Define $q_\ell = \sum_k p_{k,\ell}$. Also suppose that η is not "nearly one-valued" that is there is no ℓ for which q_ℓ is near to 1 for infinitely many n. More precisely we suppose that there is an ε such that $1 - \varepsilon > \sum_\ell q_\ell^2$ for large n's. Then

$$\sum_{k,\ell} p_{k,\ell}^2 = \sum_{k,\ell} p_k^2 q_\ell^2 = \sum_k p_k^2 \sum_\ell q_\ell^2$$

therefore

$$p(\xi, \eta, I) = \sum_k p_k^2 - \sum_{k,\ell} p_{k,\ell}^2 = \sum_k p_k^2 \left(1 - \sum_\ell q_\ell^2\right)$$

and

$$H_2(\xi \to \eta) = -\log_2\left(\sum_k p_k^2\right) - \log_2\left(1 - \sum_\ell q_\ell^2\right)$$

hold. The second term on the right hand side is upperbounded by $\log_2 \varepsilon$, while the first term tends to infinity by (19). Hence $H_2(\xi \to \eta)$ asymptotically depends only on ξ. By Theorem 1, the same is implied for $\Pr(\xi \to \eta, m)$ as it is expected in this case.

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The Symmetry Group of Gaussian States in $L^2(\mathbb{R}^n)$

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Abstract This is a continuation of the expository article by Parthasarathy (Commun Stoch Anal 4:143–160, 2010) with some new remarks. Let S_n denote the set of all Gaussian states in the complex Hilbert space $L^2(\mathbb{R}^n)$, K_n the convex set of all momentum and position covariance matrices of order $2n$ in Gaussian states and let $\mathscr{G}_n$ be the group of all unitary operators in $L^2(\mathbb{R}^n)$ conjugations by which leave S_n invariant. Here we prove the following results. K_n is a closed convex set for which a matrix S is an extreme point if and only if $S = \frac{1}{2}L^T L$ for some L in the symplectic group $Sp(2n, \mathbb{R})$. Every element in K_n is of the form $\frac{1}{4}(L^T L + M^T M)$ for some L, M in $Sp(2n, \mathbb{R})$. Every Gaussian state in $L^2(\mathbb{R}^n)$ can be purified to a Gaussian state in $L^2(\mathbb{R}^{2n})$. Any element U in the group $\mathscr{G}_n$ is of the form $U = \lambda W(\boldsymbol{\alpha})\Gamma(L)$ where λ is a complex scalar of modulus unity, $\boldsymbol{\alpha} \in \mathbb{C}^n$, $L \in Sp(2n, \mathbb{R})$, $W(\boldsymbol{\alpha})$ is the Weyl operator corresponding to $\boldsymbol{\alpha}$ and $\Gamma(L)$ is a unitary operator which implements the Bogolioubov automorphism of the Lie algebra generated by the canonical momentum and position observables induced by the symplectic linear transformation L.

Keywords Gaussian state • Momentum and position observables • Weyl operators • Symplectic group • Bogolioubov automorphism

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1 Introduction

In [4] we defined a quantum Gaussian state in $L^2(\mathbb{R}^n)$ as a state in which every real linear combination of the canonical momentum and position observables $p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n$ has a normal distribution on the real line. Such a state is uniquely determined by the expectation values of $p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n$ and their covariance matrix of order $2n$. A real positive definite matrix S of order $2n$ is the covariance matrix of the observables $p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n$ if and only if the matrix inequality

$$2S - i \;\; J \geq 0 \tag{1}$$

holds where

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \tag{2}$$

the right hand side being expressed in block notation with 0 and I being of order $n \times n$. We denote by K_n the set of all possible covariance matrices of the momentum and position observables in Gaussian states so that

$$K_n = \{ S \,|\, S \;\text{ is a real symmetric matrix of order } 2n \text{ and } 2S - iJ \geq 0\}\,. \tag{3}$$

Clearly, K_n is a closed convex set. Here we shall show that S is an extreme point of K_n if and only if $S = \frac{1}{2}L^T L$ for some matrix L in the real symplectic matrix group

$$Sp(2n, \mathbb{R}) = \left\{ L |\, L^T JL = J \right\} \tag{4}$$

with the superfix T indicating transpose. Furthermore, it turns out that every element S in K_n can be expressed as

$$S = \frac{1}{4}(L^T L + M^T M)$$

for some $L, M \in Sp(2n, \mathbb{R})$. This, in turn implies that any Gaussian state in $L^2(\mathbb{R}^n)$ can be purified to a pure Gaussian state in $L^2(\mathbb{R}^{2n})$.

Let $\boldsymbol{\alpha} \in (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in \mathbb{C}^n$, $L = ((\ell_{ij})) \in Sp(2n, \mathbb{R})$ and let $\alpha_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$. Define a new set of momentum and position observables $p'_1, p'_2, \ldots, p'_n; q'_1, q'_2, \ldots, q'_n$ by

$$p'_i = \sum_{j=1}^{n} \left\{ \ell_{ij}(p_j - x_j) + \ell_{in+j}(q_j - y_j) \right\},$$

$$q'_i = \sum_{j=1}^{n} \left\{ \ell_{n+i\,j}(p_j - x_j) + \ell_{n+i\;n+j}(q_j - y_j) \right\},$$

for $1 \leq i \leq n$. Here one takes linear combinations and their respective closures to obtain p_i', q_i' as selfadjoint operator observables. Then $p_1', p_2', \ldots, p_n'; q_1', q_2', \ldots, q_n'$ obey the canonical commutation relations and thanks to the Stone-von Neumann uniqueness theorem there exists a unitary operator $\Gamma(\boldsymbol{\alpha}, L)$ satisfying

$$p_i' = \Gamma(\boldsymbol{\alpha}, L)\ p_i\ \Gamma(\boldsymbol{\alpha}, L)^\dagger,$$
$$q_i' = \Gamma(\boldsymbol{\alpha}, L)\ q_i\ \Gamma(\boldsymbol{\alpha}, L)^\dagger$$

for all $1 \leq i \leq n$. Furthermore, such a $\Gamma(\boldsymbol{\alpha}, L)$ is unique upto a scalar multiple of modulus unity. The correspondence $(\boldsymbol{\alpha}, L) \to \Gamma(\boldsymbol{\alpha}, L)$ is a projective unitary and irreducible representation of the semidirect product group $\mathbb{C}^n$ Ⓢ $Sp(2n, \mathbb{R})$. Here any element L of $Sp(2n, \mathbb{R})$ acts on $\mathbb{C}^n$ real-linearly preserving the imaginary part of the scalar product. The operator $\Gamma(\boldsymbol{\alpha}, L)$ can be expressed as the product of $W(\boldsymbol{\alpha}) = \Gamma(\boldsymbol{\alpha}, I)$ and $\Gamma(L) = \Gamma(\mathbf{0}, L)$. Conjugations by $W(\boldsymbol{\alpha})$ implement translations of p_j, q_j by scalars whereas conjugations by $\Gamma(L)$ implement symplectic linear transformations by elements of $Sp(2n, \mathbb{R})$, which are the so-called Bogolioubov automorphisms of canonical commutation relations. In the last section we show that every unitary operator U in $L^2(\mathbb{R}^n)$, with the property that $U\rho U^\dagger$ is a Gaussian state whenever ρ is a Gaussian state, has the form $U = \lambda W(\boldsymbol{\alpha})\Gamma(L)$ for some scalar λ of modulus unity, a vector $\boldsymbol{\alpha}$ in $\mathbb{C}^n$ and a matrix L in the group $Sp(2n, \mathbb{R})$.

The following two natural problems that arise in the context of our note seem to be open. What is the most general unitary operator U in $L^2(\mathbb{R}^n)$ with the property that whenever $|\psi\rangle$ is a pure Gaussian state so is $U|\psi\rangle$? Secondly, what is the most general trace-preserving and completely positive linear map Λ on the ideal of trace-class operators on $L^2(\mathbb{R}^n)$ with the property that $\Lambda(\rho)$ is a Gaussian state whenever ρ is a Gaussian state?

2 Exponential Vectors, Weyl Operators, Second Quantization and the Quantum Fourier Transform

For any $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)^T$ in $\mathbb{C}^n$ define the associated *exponential vector* $e(\boldsymbol{z})$ in $L^2(\mathbb{R}^n)$ by

$$e(\boldsymbol{z})(\boldsymbol{x}) = (2\pi)^{-n/4} \exp \sum_{j=1}^{n} \left(z_j x_j - \frac{1}{2} z_j^2 - \frac{1}{4} x_j^2 \right). \tag{5}$$

Writing scalar products in the Dirac notation we have

$$\begin{aligned} \langle e(\boldsymbol{z}) | e(\boldsymbol{z}') \rangle &= \exp \langle \boldsymbol{z} | \boldsymbol{z}' \rangle \\ &= \exp \sum_{j=1}^{n} \bar{z}_j z_j'. \end{aligned} \tag{6}$$

The exponential vectors constitute a linearly independent and total set in the Hilbert space $L^2(\mathbb{R}^n)$. If U is a unitary matrix of order n then there exists a unique unitary $\Gamma(U)$ in $L^2(\mathbb{R}^n)$ satisfying

$$\Gamma(U)|e(z)\rangle = |e(Uz)\rangle \quad \forall\, z \in \mathbb{C}^n. \tag{7}$$

The operator $\Gamma(U)$ is called the *second quantization* of U. For any two unitary matrices U, V in the unitary group $\mathscr{U}(n)$ one has

$$\Gamma(U)\Gamma(V) = \Gamma(UV).$$

The correspondence $U \to \Gamma(U)$ is a strongly continuous unitary representation of the group $\mathscr{U}(n)$ of all unitary matrices of order n.

For any $\boldsymbol{\alpha} \in \mathbb{C}^n$ there is a unique unitary operator $W(\boldsymbol{\alpha})$ in $L^2(\mathbb{R}^n)$ satisfying

$$W(\boldsymbol{\alpha})\ |e(z)\rangle = e^{-\frac{1}{2}\|\boldsymbol{\alpha}\|^2 - \langle \boldsymbol{\alpha}|z\rangle}\ |e(z+\boldsymbol{\alpha})\rangle \quad \forall \quad z \in \mathbb{C}^n. \tag{8}$$

For any $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in $\mathbb{C}^n$ one has

$$W(\boldsymbol{\alpha})\ W(\boldsymbol{\beta}) = e^{-i\,\mathrm{Im}\langle \boldsymbol{\alpha}|\boldsymbol{\beta}\rangle}\ W(\boldsymbol{\alpha}+\boldsymbol{\beta}). \tag{9}$$

The correspondence $\boldsymbol{\alpha} \to W(\boldsymbol{\alpha})$ is a projective unitary and irreducible representation of the additive group $\mathbb{C}^n$. The operator $W(\boldsymbol{\alpha})$ is called the *Weyl operator* associated with $\boldsymbol{\alpha}$. As a consequence of (9) it follows that the map $t \to W(t\boldsymbol{\alpha})$, $t \in \mathbb{R}$ is a strongly continuous one parameter unitary group admitting a selfadjoint Stone generator $p(\boldsymbol{\alpha})$ such that

$$W(t\boldsymbol{\alpha}) = e^{-itp(\boldsymbol{\alpha})} \quad \forall \quad \boldsymbol{\alpha} \in \mathbb{C}^n. \tag{10}$$

Writing $\boldsymbol{e}_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^T$ with 1 in the j-th position,

$$p_j = 2^{-\frac{1}{2}}\ p(\boldsymbol{e}_j), \quad q_j = -2^{-\frac{1}{2}} p(i\boldsymbol{e}_j) \tag{11}$$

$$a_j = \frac{q_j + ip_j}{\sqrt{2}}, \quad a_j^\dagger = \frac{q_j - ip_j}{\sqrt{2}} \tag{12}$$

one obtains a realization of the momentum and position observables p_j, q_j, $1 \leq i \leq n$ obeying the canonical commutation relations (CCR)

$$[p_i, p_j] = 0, \quad [q_i, q_j] = 0, \quad [q_r, p_s] = i\delta_{rs}$$

and the adjoint pairs $a_j, a_j^\dagger$ of annihilation and creation operators satisfying

$$[a_i, a_j] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}$$

in appropriate domains. If we write

$$p_j^s = 2^{-\frac{1}{2}} p_j, \quad q_j^s = 2^{\frac{1}{2}} q_j$$

one obtains the canonical Schrödinger pairs of momentum and position observables in the form

$$\left(p_j^s \psi\right)(\boldsymbol{x}) = \frac{1}{i} \frac{\partial \psi}{\partial x_j}(\boldsymbol{x}), \left(q_j^s \psi\right)(\boldsymbol{x}) = x_j \ \psi(\boldsymbol{x})$$

in appropriate domains. We refer to [5] for more details.

We now introduce the sympletic group $Sp(2n, \mathbb{R})$ of real matrices of order $2n$ satisfying (4). Any element of this group is called a symplectic matrix. As described in [1,4], for any symplectic matrix L there exists a unitary operator $\Gamma(L)$ satisfying

$$\Gamma(L) \ W(\boldsymbol{\alpha}) \ \Gamma(L)^{\dagger} = W(\tilde{L}\boldsymbol{\alpha}) \quad \forall \quad \boldsymbol{\alpha} \in \mathbb{C}^n \tag{13}$$

where

$$\begin{bmatrix} \text{Re } \tilde{L}\boldsymbol{\alpha} \\ \text{Im } \tilde{L}\boldsymbol{\alpha} \end{bmatrix} = L \begin{bmatrix} \text{Re } \boldsymbol{\alpha} \\ \text{Im } \boldsymbol{\alpha} \end{bmatrix}. \tag{14}$$

Whenever the symplectic matrix L is also a real orthogonal matrix then $\tilde{L}$ is a unitary matrix and $\Gamma(L)$ coincides with the second quantization $\Gamma(\tilde{L})$ of $\tilde{L}$. Conversely, if U is a unitary matrix of order n, L_U is the matrix satisfying

$$L_U \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \text{Re } U(\boldsymbol{x} + i\,\boldsymbol{y}) \\ \text{Im } U(\boldsymbol{x} + i\,\boldsymbol{y}) \end{bmatrix}$$

then L_U is a symplectic and real orthogonal matrix of order $2n$ and $\Gamma(L_U) = \Gamma(U)$. Equations (13) and (10) imply that $\Gamma(L)$ implements the Bogolioubov automorphism determined by the symplectic matrix L through conjugation.

For any state ρ in $L^2(\mathbb{R}^n)$ its *quantum Fourier transform* $\hat{\rho}$ is defined to be the complex-valued function on $\mathbb{C}^n$ given by

$$\hat{\rho}(\boldsymbol{\alpha}) = \text{Tr } \rho \, W(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \mathbb{C}^n. \tag{15}$$

In [4] we have described a necessary and sufficient condition for a complex-valued function f on $\mathbb{C}^n$ to be the quantum Fourier transform of a state in $L^2(\mathbb{R}^n)$. Here we shall briefly describe an inversion formula for reconstructing ρ from $\hat{\rho}$. To this end we first observe that (15) is well defined whenever ρ is any trace-class operator in $L^2(\mathbb{R})$. Denote by $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively the ideals of trace-class and Hilbert-Schmidt operators in $L^2(\mathbb{R}^n)$. Then $\mathscr{F}_1 \subset \mathscr{F}_2$ and $\mathscr{F}_2$ is a Hilbert space with the inner product $\langle A|B \rangle = \text{Tr } A^{\dagger} B$. There is a natural isomorphism between $\mathscr{F}_2$ and $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$, which can, in turn, be identified with the Hilbert space of square integrable functions of two variables $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbb{R}^n$. We denote this isomorphism by $\mathscr{I}$ so that $\mathscr{I}(A)(\boldsymbol{x}, \boldsymbol{y})$ is a square integrable function of $(\boldsymbol{x}, \boldsymbol{y})$ for any $A \in \mathscr{F}_2$ and

$$\mathscr{I}(|e(\boldsymbol{u})\rangle\langle e(\bar{\boldsymbol{v}})|)(\boldsymbol{x}, \boldsymbol{y}) = e(\boldsymbol{u})(\boldsymbol{x}) e(\boldsymbol{v})(\boldsymbol{y}) \tag{16}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n$, $\bar{\boldsymbol{v}}$ denoting $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$. From (8) and (15) we have

$$\begin{aligned}(|e(\boldsymbol{u})\rangle\langle e(\bar{\boldsymbol{v}})|)^\wedge(\boldsymbol{\alpha}) &= \langle e(\bar{\boldsymbol{v}})\,|W(\boldsymbol{\alpha})|\, e(\boldsymbol{u})\rangle \\ &= \exp\left\{-\frac{1}{2}\|\boldsymbol{\alpha}\|^2 - \langle\boldsymbol{\alpha}|\boldsymbol{u}\rangle + \langle\bar{\boldsymbol{v}}|\boldsymbol{\alpha}\rangle + \langle\boldsymbol{v}|\boldsymbol{u}\rangle\right\}.\end{aligned}$$

Substituting $\boldsymbol{\alpha} = \boldsymbol{x} + i\boldsymbol{y}$ and using (5), the equation above, after some algebra, can be expressed as

$$(|e(\boldsymbol{u})\rangle\langle e(\bar{\boldsymbol{v}})|)^\wedge(\boldsymbol{x} + i\boldsymbol{y}) = (2\pi)^{n/2} e(\boldsymbol{u}')(\sqrt{2}\boldsymbol{x}) e(\boldsymbol{v}')(\sqrt{2}\boldsymbol{y}) \tag{17}$$

where

$$\begin{bmatrix}\boldsymbol{u}' \\ \boldsymbol{v}'\end{bmatrix} = U \begin{bmatrix}\boldsymbol{u} \\ \boldsymbol{v}\end{bmatrix},$$

$$U = 2^{-1/2}\begin{bmatrix}-I & I \\ iI & iI\end{bmatrix}. \tag{18}$$

Let $D_\theta, \theta > 0$ denote the unitary dilation operator in $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$ defined by

$$(D_\theta f)(\boldsymbol{x}, \boldsymbol{y}) = \theta^n f(\theta\boldsymbol{x}, \theta\boldsymbol{y}). \tag{19}$$

Then (17) can be expressed as

$$(|e(\boldsymbol{u})\rangle\langle e(\bar{\boldsymbol{v}}|)^\wedge(\boldsymbol{x} + i\,\boldsymbol{y}) = \pi^{n/2}\left\{D_{\sqrt{2}}\Gamma(U)e(\boldsymbol{u}) \otimes e(\boldsymbol{v})\right\}(\boldsymbol{x}, \boldsymbol{y})$$

where $\Gamma(U)$ is the second quantization operator in $L^2(\mathbb{R}^{2n})$ associated with the unitary matrix U in (18) of order $2n$. Since exponential vectors are total and $D_{\sqrt{2}}$ and $\Gamma(U)$ are unitary we can express the quantum Fourier transform $\rho \to \hat{\rho}(\boldsymbol{x} + i\boldsymbol{y})$ as

$$\hat{\rho} = \pi^{n/2} D_{\sqrt{2}}\ \Gamma(U)\ \mathscr{I}(\rho). \tag{20}$$

In particular, $\hat{\rho}(\boldsymbol{x} + i\,\boldsymbol{y})$ is a square integrable function of $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^n$ and

$$\rho = \pi^{-n/2}\ \mathscr{I}^{-1}\ \Gamma(U^\dagger)\ D_{2^{-1/2}}\ \hat{\rho} \tag{21}$$

is the required inversion formula for the quantum Fourier transform. It is a curious but an elementary fact that the eigenvalues of U in (18) are all 12th roots of unity and hence the unitary operators $\Gamma(U)$ and $\Gamma(U^\dagger)$ appearing in (20) and (21) have their 12-th powers equal to identity. This may be viewed as a quantum analogue of the classical fact that the fourth power of the unitary Fourier transform in $L^2(\mathbb{R}^n)$ is equal to identity.

3 Gaussian States and Their Covariance Matrices

We begin by choosing and fixing the canonical momentum and position observables $p_1, p_2, \ldots, p_n$; $q_1, q_2, \ldots, q_n$ as in equation (11) in terms of the Weyl operators. They obey the CCR. The closure of any real linear combination of the form $\sum_{j=1}^{n}(x_j p_j - y_j q_j)$ is selfadjoint and we denote the resulting observable by the same symbol. As in [4], for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, $\alpha_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$, the Weyl operator $W(\boldsymbol{\alpha})$ defined in Sect. 2 can be expressed as

$$W(\boldsymbol{\alpha}) = e^{-i\sqrt{2}\sum_{j=1}^{n}(x_j p_j - y_j q_j)}. \tag{22}$$

Sometimes it is useful to express $W(\boldsymbol{\alpha})$ in terms of the annihilation and creation operators defined by (12):

$$W(\boldsymbol{\alpha}) = e^{\sum_{j=1}^{n}(\alpha_j a_j^\dagger - \bar{\alpha}_j a_j)} \tag{23}$$

where the linear combination in the exponent is the closed version. A state ρ in $L^2(\mathbb{R})$ is said to be *Gaussian* if every observable of the form $\sum_{j=1}^{n}(x_j p_j - y_j q_j)$ has a normal distribution on the real line in the state ρ for $x_j, y_j \in \mathbb{R}$. From [4] we have the following theorem.

Theorem 1. *A state ρ in $L^2(\mathbb{R}^n)$ is Gaussian if and only if its quantum Fourier transform $\hat{\rho}$ is given by*

$$\begin{aligned}\hat{\rho}(\boldsymbol{\alpha}) &= \operatorname{Tr} \rho\, W(\boldsymbol{\alpha}) \\ &= \exp\left\{-i\sqrt{2}\left(\boldsymbol{\ell}^T \boldsymbol{x} - \boldsymbol{m}^T \boldsymbol{y}\right) - \left(\boldsymbol{x}^T, \boldsymbol{y}^T\right) S \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}\right\}\end{aligned} \tag{24}$$

for every $\boldsymbol{\alpha} = \boldsymbol{x} + i\boldsymbol{y}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ where $\boldsymbol{\ell}, \boldsymbol{m}$ are vectors in $\mathbb{R}^n$ and S is a real positive definite matrix of order $2n$ satisfying the matrix inequality $2S - iJ \geq 0$, with J as in (2).

Proof. We refer to the proof of Theorem 3.1 in [4]. □

We remark that $\boldsymbol{\ell}, \boldsymbol{m}$ and S in (24) are defined by the equations

$$\boldsymbol{\ell}^T \boldsymbol{x} - \boldsymbol{m}^T \boldsymbol{y} = \operatorname{Tr}\, \rho \sum_{j=1}^{n}(x_j p_j - y_j q_j)$$

$$(\boldsymbol{x}^T, \boldsymbol{y}^T) S \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \operatorname{Tr}\, \rho\ X^2 - (\operatorname{Tr}\, \rho X)^2,\ X = \sum_{j=1}^{n}(x_j p_j - y_j q_j).$$

It is clear that ℓ_j is the expectation value of p_j, m_j is the expectation value of q_j and S is the covariance matrix of $p_1, p_2, \ldots, p_n; -q_1, -q_2, \ldots, -q_n$ in the state ρ defined by (24). By a slight abuse of language we call S the covariance matrix of the Gaussian state ρ. All such Gaussian covariance matrices constitute the convex set K_n defined already in (3). We shall now investigate some properties of this convex set.

Proposition 1 (Williamson's normal form [1]). *Let A be any real strictly positive definite matrix of order $2n$. Then there exists a unique diagonal matrix D of order n with diagonal entries $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ and a symplectic matrix M in $Sp(2n, \mathbb{R})$ such that*

$$A = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M. \tag{25}$$

Proof. Define

$$B = A^{1/2} \, J \, A^{1/2}$$

where J is given by (2). Then B is a real skew symmetric matrix of full rank. Hence its eigenvalues, inclusive of multiplicity, can be arranged as $\pm i d_1, \pm i d_2, \ldots, \pm i d_n$ where $d_1 \geq d_2 \geq \cdots \geq d_n > 0$. Define $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, i.e., the diagonal matrix with d_i as the ii-th entry for $1 \leq i \leq n$. Then there exists a real orthogonal matrix Γ of order $2n$ such that

$$\Gamma^T \, B \, \Gamma = \begin{bmatrix} 0 & -D \\ D & 0 \end{bmatrix}.$$

Define

$$L = A^{1/2} \, \Gamma \begin{bmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{bmatrix}.$$

Then $L^T J L = J$ and

$$A = L \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} L^T.$$

Putting $M = L^T$ we obtain (25).

To prove the uniqueness of D, suppose that $D' = \mathrm{diag}(d_1', d_2', \ldots, d_n')$ with $d_1' \geq d_2' \geq \cdots \geq d_n' > 0$ and M' is another symplectic matrix of order $2n$ such that

$$A = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M = M'^T \begin{bmatrix} D' & 0 \\ 0 & D' \end{bmatrix} M'.$$

Putting $N = MM'^{-1}$ we get a symplectic N such that

$$N^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} N = \begin{bmatrix} D' & 0 \\ 0 & D' \end{bmatrix}.$$

Substituting $N^T = JN^{-1}J^{-1}$ we get

$$N^{-1}\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} N = \begin{bmatrix} 0 & D' \\ -D' & 0 \end{bmatrix}.$$

Identifying the eigenvalues on both sides we get $D = D'$ □

Theorem 2. *A real positive definite matrix S is in K_n if and only if there exists a diagonal matrix $D = diag(d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq \frac{1}{2}$ and a symplectic matrix $M \in Sp(2n, \mathbb{R})$ such that*

$$S = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M. \tag{26}$$

In particular,

$$det\, S = \prod_i^n d_j^2 \geq 4^{-n}. \tag{27}$$

Proof. Let S be a real strictly positive definite matrix in K_n. From (3) we have $S \geq \frac{i}{2}J$ and therefore, for any $L \in Sp(2n, \mathbb{R})$,

$$L^T\, S\, L \geq \frac{i}{2}J. \tag{28}$$

Using Proposition 1 choose L so that

$$L^T\, S\, L = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$$

where $D = \text{diag}(d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \cdots \geq d_n > 0$. Now (28) implies

$$\begin{bmatrix} D & \frac{i}{2}I \\ -\frac{i}{2}I & D \end{bmatrix} \geq 0.$$

The minor of second order in the left hand side arising from the jj, $j\, n+j$, $n+jj$, $n+j\; n+j$ entries is $d_j^2 - \frac{1}{4} \geq 0$. Choosing $L = M^{-1}$ we obtain (26) and (27). Now we drop the assumption of strict positive definiteness on S. From the definition of K_n in (3) it follows that for any $S \in K_n$ one has $S + \varepsilon I \in K_n$ for every $\varepsilon > 0$. Since $S + \varepsilon I$ is strictly positive definite $\det S + \varepsilon I \geq 4^{-n}$ $\forall\, \varepsilon > 0$. Letting $\varepsilon \to 0$ we see that (27) holds and S is strictly positive definite.

To prove the converse statement, let us consider an arbitrary diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq \frac{1}{2}$. Clearly

$$2\begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} - i \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \geq 0,$$

and hence for any $M \in Sp(2n, \mathbb{R})$

$$2M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M - i \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \geq 0.$$

In other words,

$$M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M \in K_n \qquad M \in Sp(2n, \mathbb{R}).$$

Finally, the uniqueness of the parameters $d_1 \geq d_2 \geq \cdots \geq d_n \geq \frac{1}{2}$ in the theorem is a consequence of Proposition 1. □

We now prove an elementary lemma on diagonal matrices before the statement of our next result on the convex set K_n.

Lemma 1. *Let $D \geq I$ be a positive diagonal matrix of order n. Then there exist positive diagonal matrices D_1, D_2 such that*

$$D = \frac{1}{2}(D_1 + D_2) = \frac{1}{2}(D_1^{-1} + D_2^{-1}).$$

Proof. We write $D_2 = D_1 X$ and solve for D_1 and X so that

$$2D = D_1(I + X) = D_1^{-1}(I + X^{-1}),$$

D_1 and X being diagonal. Eliminating D_1 we get the equation

$$(I + X)(I + X^{-1}) = 4D^2$$

which reduces to the quadratic equation

$$X^2 + (2 - 4D^2)X + I = 0.$$

Solving for X we do get a positive diagonal matrix solution

$$X = I + 2(D^2 - 1) + 2D(D^2 - I)^{1/2}.$$

Writing

$$D_1 = 2D(I + X)^{-1}, \quad D_2 = D_1 X$$

we get D_1, D_2 satisfying the required property. □

Theorem 3. *A real positive definite matrix S of order $2n$ belongs to K_n if and only if there exist symplectic matrices L, M such that*

$$S = \frac{1}{4}(L^T L + M^T M).$$

Furthermore, S is an extreme point of K_n if and only if $S = \frac{1}{2} L^T L$ for some symplectic matrix L.

Proof. Let $S \in K_n$. By Theorem 2 we express S as

$$S = N^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} N \tag{29}$$

where N is symplectic and $D = \text{diag}(d_1, d_2, \dots, d_n)$, $d_1 \geq d_2 \geq \cdots \geq d_n \geq \frac{1}{2}$. Thus $2D \geq I$ and by Lemma 1 there exist diagonal matrices $D_1 > 0$, $D_2 > 0$ such that

$$2D = \frac{1}{2}(D_1 + D_2) = \frac{1}{2}(D_1^{-1} + D_2^{-1}).$$

We rewrite (29) as

$$S = \frac{1}{4} N^T \left(\begin{bmatrix} D_1 & 0 \\ 0 & D_1^{-1} \end{bmatrix} + \begin{bmatrix} D_2 & 0 \\ 0 & D_2^{-1} \end{bmatrix} \right) N.$$

Putting

$$L = \begin{bmatrix} D_1^{1/2} & 0 \\ 0 & D_1^{-1/2} \end{bmatrix} N, \quad M = \begin{bmatrix} D_2^{1/2} & 0 \\ 0 & D_2^{-1/2} \end{bmatrix}$$

we have

$$S = \frac{1}{4}(L^T L + M^T M).$$

Since $\begin{bmatrix} D_i^{1/2} & 0 \\ 0 & D_i^{-1/2} \end{bmatrix}$, $i = 1, 2$ are symplectic it follows that L and M are symplectic. This proves the only if part of the first half of the theorem.

Since

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - i \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \geq 0$$

multiplication by L^T on the left and L on the right shows that $L^T L - iJ \geq 0$ for any symplectic L. Hence $\frac{1}{2} L^T L \in K_n \ \forall \ L \in Sp(2n, \mathbb{R})$. Since K_n is convex, $\frac{1}{4}(L^T L + M^T M) \in K_n$, completing the proof of the first part.

The first part also shows that for an element S of K_n to be extremal it is necessary that $S = \frac{1}{2} L^T L$ for some symplectic L. To prove sufficiency, suppose there exist $L \in Sp(2n, \mathbb{R})$, $S_1, S_2 \in K_n$ such that

$$\frac{1}{2} L^T L = \frac{1}{2}(S_1 + S_2).$$

By the first part of the theorem there exist $L_j \in Sp(2n, \mathbb{R})$ such that

$$L^T L = \frac{1}{4} \sum_{j=1}^{4} L_j^T L_j \tag{30}$$

where $S_1 = \frac{1}{4}(L_1^T L_1 + L_2^T L_2)$, $S_2 = \frac{1}{4}(L_3^T L_3 + L_4^T L_4)$. Left multiplication by $(L^T)^{-1}$ and right multiplication by L^{-1} on both sides of (30) yields

$$I = \frac{1}{4} \sum_{j=1}^{4} M_j \tag{31}$$

where

$$M_j = (L^T)^{-1} L_j^T L_j L^{-1}.$$

Each M_j is symplectic and positive definite. Multiplying by J on both sides of (31) we get

$$\begin{aligned} J &= \frac{1}{4} \sum_{j=1}^{4} M_j J \\ &= \frac{1}{4} \sum_{j=1}^{4} M_j J M_j M_j^{-1} \\ &= \frac{1}{4} J \sum_{j=1}^{4} M_j^{-1}. \end{aligned}$$

Thus

$$I = \frac{1}{4} \sum_{j=1}^{4} M_j = \frac{1}{4} \sum_{j=1}^{4} M_j^{-1} = \frac{1}{4} \sum_{j=1}^{4} \frac{1}{2}(M_j + M_j^{-1}),$$

which implies

$$\sum_{j=1}^{4} \left(M_j^{1/2} - M_j^{-1/2} \right)^2 = 0,$$

or

$$M_j = I \quad \forall \quad 1 \le j \le 4$$

Thus

$$L_j^T L_j = L^T L \quad \forall \quad j$$

and $S_1 = S_2$. This completes the proof of sufficiency. □

Corollary 1. *Let S_1, S_2 be extreme points of K_n satisfying the inequality $S_1 \ge S_2$. Then $S_1 = S_2$.*

Proof. By Theorem 3 there exist $L_i \in Sp(2n, \mathbb{R})$ such that $S_i = \frac{1}{2} L_i^T L_i$, $i = 1, 2$. Note that $M = L_2 L_1^{-1}$ is symplectic and the fact that $S_1 \ge S_2$ can be expressed as $M^T M \le I$. Thus the eigenvalues of $M^T M$ lie in the interval (0, 1] but their product is equal to $(\det M)^2 = 1$. This is possible only if all the eigenvalues are unity, i.e., $M^T M = I$. This at once implies $L_1^T L_1 = L_2^T L_2$. □

Using the Williamson's normal form of the covariance matrix and the transformation properties of Gaussian states in Sect. 3 of [4] we shall now derive a formula for the density operator of a general Gaussian state. As in [4] denote by $\rho_g(\boldsymbol{\ell}, \boldsymbol{m}, S)$ the Gaussian state in $L^2(\mathbb{R}^n)$ with the quantum Fourier transform

$$\rho_g(\boldsymbol{\ell}, \boldsymbol{m}, S)^\wedge(z) = \exp -i\sqrt{2}(\boldsymbol{\ell}^T \boldsymbol{x} - \boldsymbol{m}^T \boldsymbol{y}) - (\boldsymbol{x}^T \boldsymbol{y}^T) S \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}, z = \boldsymbol{x} + i\boldsymbol{y}$$

where $\boldsymbol{\ell}, \boldsymbol{m} \in \mathbb{R}^n$ and S has the Williamson's normal form

$$S = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M$$

with $M \in Sp(2n, \mathbb{R})$, $D = \text{diag}(d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \cdots \geq d_n \geq \frac{1}{2}$. From Corollary 3.3 of [4] we have

$$W\left(\frac{\boldsymbol{m} + i\boldsymbol{\ell}}{\sqrt{2}}\right)^\dagger \rho_g(\boldsymbol{\ell}, \boldsymbol{m}, S) W\left(\frac{\boldsymbol{m} + i\boldsymbol{\ell}}{\sqrt{2}}\right) = \rho_g(\boldsymbol{0}, \boldsymbol{0}, S)$$

and Corollary 3.5 of [4] implies

$$\rho_g(\boldsymbol{0}, \boldsymbol{0}, S) = \Gamma(M)^{-1} \rho_g\left(\boldsymbol{0}, \boldsymbol{0}, \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}\right) \Gamma(M).$$

Since $\begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$ is a diagonal covariance matrix

$$\rho_g\left(\boldsymbol{0}, \boldsymbol{0}, \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}\right) = \bigotimes_{j=1}^{n} \rho_g(0.0, d_j I_2)$$

where the j-th component in the right hand side is the Gaussian state in $L^2(\mathbb{R})$ with means 0 and covariance matrix $d_j I_2$, I_2 denoting the identity matrix of order 2. If $d_j = \frac{1}{2}$ we have

$$\rho_g\left(0, 0, \frac{1}{2} I_2\right) = |e(0)\rangle\langle e(0)| \text{ in } L^2(\mathbb{R}).$$

If $d_j > 1/2$, writing $d_j = \frac{1}{2}\coth\frac{1}{2}s_j$, one has

$$\begin{aligned} \rho_g(0, 0, d_j I_2) &= (1 - e^{-s_j}) e^{-s_j a^\dagger a} \\ &= 2\sinh\frac{1}{2}s_j \; e^{-\frac{1}{2}s_j(p^2+q^2)} \text{ in } L^2(\mathbb{R}) \end{aligned}$$

with $a, a^\dagger, p, q$ denoting the operator of annihilation, creation, momentum and position respectively in $L^2(\mathbb{R})$. We now identify $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R})^{\otimes^n}$ and combine the reductions done above to conclude the following:

Theorem 4. *Let $\rho_g(\boldsymbol{\ell}, \boldsymbol{m}, S)$ be the Gaussian state in $L^2(\mathbb{R}^n)$ with mean momentum and position vectors $\boldsymbol{\ell}, \boldsymbol{m}$ respectively and covariance matrix S with Williamson's normal form*

$$S = M^T \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} M, \quad M \in Sp(2n, \mathbb{R}),$$

$D = diag(d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \cdots \geq d_m > d_{m+1} = d_{m+2} = \cdots = d_n = \frac{1}{2}$, $d_j = \frac{1}{2}coth\frac{1}{2}s_j$, $1 \leq j \leq m$, $s_j > 0$. Then

$$\rho_g(\boldsymbol{\ell}, \boldsymbol{m}, S) = W\left(\frac{\boldsymbol{m} + i\boldsymbol{\ell}}{\sqrt{2}}\right) \Gamma(M)^{-1} \prod_{j=1}^{m} (1 - e^{-s_j}) \times$$

$$e^{-\sum_{j=1}^{m} s_j a_j^\dagger a_j} \otimes (|e(0)\rangle \times \langle e(0)|)^{\otimes^{n-m}} \Gamma(M) W \left(\frac{\boldsymbol{m} + i\boldsymbol{\ell}}{\sqrt{2}}\right)^{-1} \quad (32)$$

where $W(\cdot)$ denotes Weyl operator, $\Gamma(M)$ is the unitary operator implementing the Bogolioubov automorphism of CCR corresponding to the symplectic linear transformation M and $|e(0)\rangle$ denotes the exponential vector corresponding to 0 *in any copy of $L^2(\mathbb{R})$.*

Proof. Immediate from the discussion preceding the statement of the theorem. □

Corollary 2. *The wave function of the most general pure Gaussian state in$L^2(\mathbb{R}^n)$ is of the form*

$$|\psi\rangle = W(\boldsymbol{\alpha})\Gamma(U)\ |e_{\lambda_1}\rangle|e_{\lambda_2}\rangle \cdots |e_{\lambda_n}\rangle$$

where

$$e_\lambda(x) = (2\pi)^{-1/4}\lambda^{-1/2} \exp -4^{-1}\lambda^{-2}x^2, \quad x \in \mathbb{R}, \lambda > 0,$$

$\boldsymbol{\alpha} \in \mathbb{C}^n$, U is a unitary matrix of order n, $W(\boldsymbol{\alpha})$ is the Weyl operator associated with $\boldsymbol{\alpha}$, $\Gamma(U)$ is the second quantization unitary operator associated with U and λ_j, $1 \leq j \leq n$ are positive scalars.

Proof. Since the number operator $a^\dagger a$ has spectrum $\{0, 1, 2, \ldots\}$ it follows from Theorem 4 that $\rho_g(\boldsymbol{\ell}, \boldsymbol{m}, S)$ is pure if and only if $m = 0$ in (32). This implies that the corresponding wave function $|\psi\rangle$ can be expressed as

$$|\psi\rangle = W(\boldsymbol{\alpha})\Gamma(M)^{-1}(|e(0)\rangle)^{\otimes^n} \quad (33)$$

where $M \in Sp(2n, \mathbb{R})$ and $\boldsymbol{\alpha} = \frac{\boldsymbol{m}+i\boldsymbol{\ell}}{\sqrt{2}}$. The covariance matrix of this pure Gaussian state is $\frac{1}{2}M^T M$. The symplectic matrix M has the decomposition [1]

$$M = V_1 \begin{bmatrix} D & 0 \\ 0 & D^{-1} \end{bmatrix} V_2$$

where V_1 and V_2 are real orthogonal as well as symplectic and D is a positive diagonal matrix of order n. Thus

$$\begin{aligned} M^T M &= V_2^T \begin{bmatrix} D^2 & 0 \\ 0 & D^{-2} \end{bmatrix} V_2 \\ &= N^T N \end{aligned}$$

where

$$N = \begin{bmatrix} D & 0 \\ 0 & D^{-1} \end{bmatrix} V_2.$$

Since the covariance matrix of $|\psi\rangle$ in (33) can also be written as $\frac{1}{2} N^T N$, modulo a scalar multiple of modulus unity $|\psi\rangle$ can also be expressed as

$$|\psi\rangle = W(\boldsymbol{\alpha}) \Gamma(V_2)^{-1} \Gamma \left(\begin{bmatrix} D^{-1} & 0 \\ 0 & D \end{bmatrix} \right) |e(0)\rangle^{\otimes^n}. \tag{34}$$

If U is the complex unitary matrix of order n satisfying

$$\begin{aligned} U(\boldsymbol{x} + i\,\boldsymbol{y}) &= \boldsymbol{x}' + i\,\boldsymbol{y}', \\ \begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \end{bmatrix} &= V_2^T \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \quad \forall \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \end{aligned}$$

and $D^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ we can express (34) as

$$\begin{aligned} |\psi\rangle &= W(\boldsymbol{\alpha}) \Gamma(U) \left\{ \bigotimes_{j=1}^{n} \quad \Gamma \left(\begin{bmatrix} \lambda_j & 0 \\ 0 & \lambda_j^{-1} \end{bmatrix} \right) |e(0)\rangle \right\} \\ &= W(\boldsymbol{\alpha}) \Gamma(U)\, |e_{\lambda_1}\rangle |e_{\lambda_2}\rangle \cdots |e_{\lambda_n}\rangle \end{aligned}$$

where we have identified $L^2(\mathbb{R}^n)$ with $L^2(\mathbb{R})^{\otimes^n}$.

We conclude this section with a result on the purification of Gaussian states. □

Theorem 5. *Let ρ be a mixed Gaussian state in $L^2(\mathbb{R}^n)$. Then there exists a pure Gaussian state $|\psi\rangle$ in $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$ such that*

$$\rho = Tr_2\ U\ |\psi\rangle\langle\psi|\, U^\dagger$$

for some unitary operator U in $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$ with Tr_2 denoting the relative trace over the second copy of $L^2(\mathbb{R}^n)$.

Proof. First we remark that by a Gaussian state in $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$ we mean it by the canonical identification of this product Hilbert space with $L^2(\mathbb{R}^{2n})$. Let $\rho = \rho_g(\boldsymbol{\ell}, \boldsymbol{m}, S)$ where by Theorem 3 we can express

$$S = \frac{1}{4}\left(L_1^T L_1 + L_2^T L_2\right), \quad L_1, L_2 \in Sp(2n, \mathbb{R}).$$

Now consider the pure Gaussian states,

$$|\psi_{L_i}\rangle = \Gamma(L_i)^{-1}\ |e(\mathbf{0})\rangle, \quad i = 1, 2$$

in $L^2(\mathbb{R}^n)$ and the second quantization unitary operator Γ_0 satisfying

$$\Gamma_0\ e(\boldsymbol{u} \oplus \boldsymbol{v}) = e\left(\frac{\boldsymbol{u}+\boldsymbol{v}}{\sqrt{2}} \oplus \frac{\boldsymbol{u}-\boldsymbol{v}}{\sqrt{2}}\right) \quad \forall\ \boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n$$

in $L^2(\mathbb{R}^{2n})$ identified with $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$, so that

$$e(\boldsymbol{u} \oplus \boldsymbol{v}) = e(\boldsymbol{u}) \otimes e(\boldsymbol{v}).$$

Then by Proposition 3.11 of [4] we have

$$\mathrm{Tr}_2\ \Gamma_0\left(|\psi_{L_1}\rangle\langle\psi_{L_1}| \otimes |\psi_{L_2}\rangle\langle\psi_{L_2}|\right) \Gamma_0^{\dagger} = \rho_g(\mathbf{0}, \mathbf{0}, S).$$

If $\boldsymbol{\alpha} = \frac{\boldsymbol{m}+i\boldsymbol{\ell}}{\sqrt{2}}$ we have

$$W(\boldsymbol{\alpha})\rho_g(\mathbf{0}, \mathbf{0}, S)W(\boldsymbol{\alpha})^{\dagger} = \rho_g(\boldsymbol{\ell}, \boldsymbol{m}, S).$$

Putting

$$U = (W(\boldsymbol{\alpha}) \otimes I)\, \Gamma_0 \left(\Gamma(L_1)^{-1} \otimes \Gamma(L_2)^{-1}\right)$$

we get

$$\rho_g\left(\boldsymbol{\ell}, \boldsymbol{m}, S\right) = \mathrm{Tr}_2\ U\ |e(\mathbf{0}) \otimes e(\mathbf{0})\rangle\langle e(\mathbf{0}) \otimes e(\mathbf{0})|\ U^{\dagger}$$

where $|e(\mathbf{0})\rangle$ is the exponential vector in $L^2(\mathbb{R}^n)$. □

For a more comprehensive account of quantum Fourier transform and its applications to the systematic study of various properties of Gaussian states in terms of their means and covariance matrices we refer to Chap. V of [3].

4 The Symmetry Group of the Set of Gaussian States

Let S_n denote the set of all Gaussian states in $L^2(\mathbb{R})$. We say that a unitary operator U in $L^2(\mathbb{R}^n)$ is a *Gaussian symmetry* if, for any $\rho \in S_n$, the state $U\rho U^{\dagger}$ is also in S_n. All such Gaussian symmetries constitute a group $\mathcal{G}_n$. If $\boldsymbol{\alpha} \in \mathbb{C}^n$ and $L \in Sp(2n, \mathbb{R})$ then the associated Weyl operator $W(\boldsymbol{\alpha})$ and the unitary operator $\Gamma(L)$ implementing the Bogolioubov automorphism of CCR corresponding to L are in $\mathscr{G}_n$ (See Corollary 3.5 in [4].) The aim of this section is to show that any element U in $\mathscr{G}_n$ is of the form $\lambda W(\boldsymbol{\alpha})\Gamma(L)$ where λ is a complex scalar of modulus unity, $\boldsymbol{\alpha} \in \mathbb{C}^n$ and $L \in Sp(2n,\mathbb{R})$. This settles a question raised in [4].

We begin with a result on a special Gaussian state.

Theorem 6. *Let $s_1 > s_2 > \cdots > s_n > 0$ be irrational numbers which are linearly independent over the field Q of rationals and let*

$$\rho_s = \rho_g(\mathbf{0},\mathbf{0},S) = \prod_{j=1}^{n}(1-e^{-s_j})e^{-\sum_{j=1}^{n} s_j a_j^\dagger a_j}$$

be the Gaussian state in $L^2(\mathbb{R}^n)$ with zero position and momentum mean vectors and covariance matrix

$$S = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}, \quad D = diag(d_1, d_2, \ldots, d_n)$$

with $d_j = \frac{1}{2}coth\frac{1}{2}s_j$. Then a unitary operator U in $L^2(\mathbb{R}^n)$ has the property that $U\rho_s U^\dagger$ is a Gaussian state if and only if, for some $\boldsymbol{\alpha} \in \mathbb{C}^n$, $L \in Sp(2n,\mathbb{R})$ and a complex-valued function β of modulus unity on $\mathbb{Z}_+^n$

$$U = W(\boldsymbol{\alpha})\Gamma(L)\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots, a_n^\dagger a_n) \tag{35}$$

where $\mathbb{Z}_+ = \{0,1,2,\ldots\}$.

Proof. Sufficiency is immediate from Corollaries 3.3 and 3.5 of [4]. To prove necessity assume that

$$U\rho_s U^\dagger = \rho_g(\boldsymbol{\ell}, \boldsymbol{m}, S') \tag{36}$$

Since $a^\dagger a$ in $L^2(\mathbb{R})$ has spectrum $\mathbb{Z}_+$ and each eigenvalue k has multiplicity one [2] it follows that the selfadjoint positive operator $\sum_{j=1}^{n} s_j a_j^\dagger a_j$, being a sum of commuting self adjoint operators $s_j a_j^\dagger a_j$, $1 \le j \le n$ has spectrum $\left\{ \sum_{j=1}^{n} s_j k_j \,\middle|\, k_j \in \mathbb{Z}_+ \ \forall\ j \right\}$ with each eigenvalue of multiplicity one thanks to the assumption on $\{s_j, 1 \le j \le n\}$. Since ρ_s and $U\rho_s U^{-1}$ have the same set of eigenvalues and same multiplicities it follows from Theorem 4 that

$$U\rho_s U^{-1} = W(z)\Gamma(M)^{-1}\rho_t\Gamma(M)W(z)^{-1} \tag{37}$$

where $z \in \mathbb{C}^n$, $M \in Sp(2n,\mathbb{R})$, $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)^T$ and

$$\rho_t = \prod_{j=1}^{n}(1-e^{-t_j})e^{-\sum_{j=1}^{n} t_j a_j^\dagger a_j}.$$

Since the maximum eigenvalues of $\rho_{\boldsymbol{s}}$ and $\rho_{\boldsymbol{t}}$ are same it follows that

$$\prod(1-e^{-s_j}) = \prod(1-e^{-t_j}).$$

Since the spectra of $\rho_{\boldsymbol{s}}$ and $\rho_{\boldsymbol{t}}$ are same it follows that

$$\left\{ \sum_{j=1}^{n} s_j k_j \middle| k_j \in \mathbb{Z}_+ \quad \forall\ j \right\} = \left\{ \sum_{j=1}^{n} t_j k_j \middle| k_j \in \mathbb{Z}_+ \quad \forall\ j \right\}.$$

Choosing $\boldsymbol{k} = (0,0,\ldots,0,1,0,\ldots,0)^T$ with 1 in the k-th position we conclude the existence of matrices A, B of order $n \times n$ and entries in $\mathbb{Z}_+$ such that

$$\boldsymbol{t} = A\boldsymbol{s}, \quad \boldsymbol{s} = B\boldsymbol{t}$$

so that $BA\boldsymbol{s} = \boldsymbol{s}$. The rationally linear independence of the s_j's implies $BA = I$. This is possible only if A and $B = A^{-1}$ are both permutation matrices.

Putting $V = \Gamma(M)W(\boldsymbol{z})^\dagger U$ we have from (37)

$$V\rho_{\boldsymbol{s}} = \rho_{\boldsymbol{t}} V.$$

Denote by $|\boldsymbol{k}\rangle$ the vector satisfying

$$a_j^\dagger a_j\ |\boldsymbol{k}\rangle = k_j\ |\boldsymbol{k}\rangle$$

where $|\boldsymbol{k}\rangle = |k_1\rangle|k_2\rangle\cdots|k_n\rangle$. Then

$$\begin{aligned} V\rho_{\boldsymbol{s}}\ |\boldsymbol{k}\rangle &= \prod_{j=1}^{n}(1-e^{-s_j})e^{-\sum s_j k_j}\ V\ |\boldsymbol{k}\rangle \\ &= \rho_{\boldsymbol{t}}\ V|\boldsymbol{k}\rangle, \quad \boldsymbol{k} \in \mathbb{Z}_+^n. \end{aligned}$$

Thus $V\,|\boldsymbol{k}\rangle$ is an eigenvector for $\rho_{\boldsymbol{t}}$ corresponding to the eigenvalue

$$\begin{aligned} \prod(1-e^{-s_j})e^{-\boldsymbol{s}^T\boldsymbol{k}} &= \prod_{j=1}^{n}(1-e^{-t_j})e^{-\boldsymbol{t}^T B^T \boldsymbol{k}} \\ &= \prod_{j=1}^{n}(1-e^{-t_j})e^{-\boldsymbol{t}^T A\boldsymbol{k}}. \end{aligned}$$

Hence there exists a scalar $\beta(\boldsymbol{k})$ of modulus unity such that

$$\begin{aligned} V\ |\boldsymbol{k}\rangle &= \beta(\boldsymbol{k})\ |A\boldsymbol{k}\rangle \\ &= \Gamma(A)\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots, a_n^\dagger a_n)\,|\boldsymbol{k}\rangle\ \ \forall\ \ \boldsymbol{k} \in \mathbb{Z}_+^n. \end{aligned}$$

where $\Gamma(A)$ is the second quantization of the permutation unitary matrix A acting in $\mathbb{C}^n$. Thus

$$U = W(z)\Gamma(M)^\dagger \Gamma(A)\beta(a_1^\dagger a_1, a_2^\dagger a_2, \dots, a_n^\dagger a_n).$$

which completes the proof. □

Theorem 7. *A unitary operator U in $L^2(\mathbb{R}^n)$ is a Gaussian symmetry if and only if there exist a scalar λ of modulus unity, a vector $\boldsymbol{\alpha}$ in $\mathbb{C}^n$ and a symplectic matrix $L \in Sp(2n, \mathbb{R})$ such that*

$$U = \lambda W(\boldsymbol{\alpha})\Gamma(L)$$

where $W(\boldsymbol{\alpha})$ is the Weyl operator associated with $\boldsymbol{\alpha}$ and $\Gamma(L)$ is a unitary operator implementing the Bogolioubov automorphism of CCR corresponding to L.

Proof. The if part is already contained in Corollaries 3.3 and 3.5 of [4]. In order to prove the only if part we may, in view of Theorem 6, assume that $U = \beta(a_1^\dagger a_1, a_2^\dagger a_2, \dots, a_n^\dagger a_n)$ where β is a function of modulus unity on $\mathbb{Z}_+^n$. If such a U is a Gaussian symmetry then, for any pure Gaussian state $|\psi\rangle$, $U|\psi\rangle$ is also a pure Gaussian state. We choose

$$|\psi\rangle = e^{-\frac{1}{2}\|\boldsymbol{u}\|^2}|e(\boldsymbol{u})\rangle = W(\boldsymbol{u})|e(\boldsymbol{0})\rangle$$

where $\boldsymbol{u} = (u_1, u_2, \dots, u_n)^T \in \mathbb{C}^n$ with $u_j \neq 0 \;\forall\; j$. By our assumption

$$|\psi'\rangle = e^{-\frac{1}{2}\|\boldsymbol{u}\|^2}\beta(a_1^\dagger a_1, a_2^\dagger a_2, \dots, a_n^\dagger a_n)|e(\boldsymbol{u})\rangle \tag{38}$$

is also a pure Gaussian state. By Corollary 2, $\exists \boldsymbol{\alpha} \in \mathbb{C}^n$, a unitary matrix A of order n and $\lambda_j > 0$, $1 \leq j \leq n$ such that

$$|\psi'\rangle = W(\boldsymbol{\alpha})\Gamma(A)\,|e_{\lambda_1}\rangle|e_{\lambda_2}\rangle\cdots|e_{\lambda_n}\rangle. \tag{39}$$

Using (38) and (39) we shall evaluate the function $f(z) = \langle\psi'|e(z)\rangle$ in two different ways. From (38) we have

$$\begin{aligned} f(z) &= e^{-\frac{1}{2}\|\boldsymbol{u}\|^2}\langle e(\boldsymbol{u})\left|\bar{\beta}(a_1^\dagger a_1, a_2^\dagger a_2, \dots, a_n^\dagger a_n)\right| e(z)\rangle \\ &= e^{-\frac{1}{2}\|\boldsymbol{u}\|^2}\sum_{z\in\mathbb{Z}_+^n}\frac{\bar{\beta}(k_1, k_2, \dots, k_n)}{k_1!k_2!\dots k_n!}(\bar{u}_1 z_1)^{k_1}\dots(\bar{u}_n z_n)^{k_n}\,|k_1 k_2\cdots k_n\rangle \end{aligned} \tag{40}$$

where $|k_1 k_2 \cdots k_n\rangle = |k_1\rangle|k_2\rangle\cdots|k_n\rangle$ and $|e(z)\rangle = \sum_{k\in\mathbb{Z}_+}\frac{z^k}{\sqrt{k!}}|k\rangle$ for $z \in \mathbb{C}$.

Since $|\beta(\boldsymbol{k})| = 1$, (40) implies

$$|f(z)| \leq \exp\left\{-\frac{1}{2}\|\boldsymbol{u}\|^2 + \sum_{j=1}^{n}\;|u_j|\;|z_j|\right\}. \tag{41}$$

From the definition of $|e_\lambda\rangle$ in Corollary 2 and the exponential vector $|e(z)\rangle$ in $L^2(\mathbb{R})$ one has

$$\langle e_\lambda | e(z)\rangle = \sqrt{\frac{2\lambda}{1+\lambda^2}}\,\exp\,\frac{1}{2}\left(\frac{\lambda^2-1}{\lambda^2+1}\right)z^2, \quad \lambda > 0, \quad z \in \mathbb{C}.$$

This together with (39) implies

$$\begin{aligned} f(\boldsymbol{z}) &= \langle e_{\lambda_1}\otimes e_{\lambda_2}\otimes\cdots\otimes e_{\lambda_n}\mid \Gamma(A^{-1})W(-\boldsymbol{\alpha})e(\boldsymbol{z})\rangle \\ &= e^{\langle\boldsymbol{\alpha}|\boldsymbol{z}\rangle-\frac{1}{2}\|\boldsymbol{\alpha}\|^2}\langle e_{\lambda_1}\otimes e_{\lambda_2}\otimes\cdots\otimes e_{\lambda_n}\mid e(A^{-1}(\boldsymbol{z}+\boldsymbol{\alpha}))\rangle \end{aligned}$$

which is a nonzero scalar multiple of the exponential of a polynomial of degree 2 in $z_1, z_2, \ldots, z_n$ except when all the λ_j's are equal to unity. This would contradict the inequality (40) except when $\lambda_j = 1\ \forall\ j$. Thus $\lambda_j = 1\ \forall\ j$ and (39) reduces to

$$\begin{aligned} |\psi'\rangle &= W(\boldsymbol{\alpha})\Gamma(A)\,|e(\boldsymbol{0})\rangle \\ &= e^{-\frac{1}{2}\|\boldsymbol{\alpha}\|^2}\,|e(\boldsymbol{\alpha})\rangle. \end{aligned}$$

Now (38) implies

$$\begin{aligned} &\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots, a_n^\dagger a_n)\,|e(\boldsymbol{u})\rangle \\ &= e^{\frac{1}{2}(\|\boldsymbol{u}\|^2-\|\boldsymbol{\alpha}\|^2)}\,|e(\boldsymbol{\alpha})\rangle, \end{aligned}$$

or

$$\sum_{\boldsymbol{k}\in\mathbb{Z}_+^n}\frac{u_1^{k_1}u_2^{k_2}\ldots u_n^{k_n}}{\sqrt{k_1!}\cdots\sqrt{k_n!}}\beta(k_1,k_2,\ldots,k_n)\,|k_1k_2\ldots k_n\rangle$$

$$= e^{\frac{1}{2}(\|\boldsymbol{u}\|^2-\|\boldsymbol{\alpha}\|^2)}\sum\frac{\alpha_1^{k_1}\alpha_2^{k_2}\ldots\alpha_n^{k_n}}{\sqrt{k_1!}\cdots\sqrt{k_n!}}\,|k_1k_2\ldots k_n\rangle.$$

Thus

$$\beta(k_1,k_2,\ldots,k_n) = e^{\frac{1}{2}(\|\boldsymbol{u}\|^2-\|\boldsymbol{\alpha}\|^2)}\left(\frac{\alpha_1}{u_1}\right)^{k_1}\cdots\left(\frac{\alpha_n}{u_n}\right)^{k_n}.$$

Since $|\beta(\boldsymbol{k})| = 1$ and $u_j \neq 0\,\forall\,j$ it follows that $|\frac{\alpha_j}{u_j}| = 1$ and

$$\beta(\boldsymbol{k}) = e^{i\sum_{j=1}^{n}\theta_j k_j} \quad \forall\ \boldsymbol{k}\in\mathbb{Z}_+^n$$

where θ_j's are real. Thus $\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots, a_n^\dagger a_n) = \Gamma(D)$, the second quantization of the diagonal unitary matrix $D = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$. This completes the proof. □

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Solution of the Optimal Stopping Problem for One-Dimensional Diffusion Based on a Modification of the Payoff Function

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Abstract A problem of optimal stopping for one-dimensional time-homogeneous regular diffusion with the infinite horizon is considered. The diffusion takes values in a finite or infinite interval $]a,b[$. The points a and b may be either natural or absorbing or reflecting. The diffusion may have a partial reflection at a finite number of points. A discounting and a cost of observation are allowed. Both can depend on the state of the diffusion. The payoff function $g(z)$ is bounded on any interval $[c,d]$, where $a < c < d < b$, and twice differentiable with the exception of a finite (may be empty) set of points, where the functions $g(z)$ and $g'(z)$ may have a discontinuities of the first kind. Let L be an infinitesimal generator of diffusion which includes the terms corresponding to the discounting and the cost of observation. We assume that the set $\{z : Lg(z) > 0\}$ consists of a finite number of intervals. For such problem we propose a procedure of constructing the value function in a finite number of steps. The procedure is based on a fact that on intervals where $Lg(z) > 0$ and in neighborhoods of points of partial reflections, points of discontinuities, and points a or b in case of reflection, one can modify the payoff function preserving the value function. Many examples are considered.

Keywords Markov chain • Markov process • One-dimensional diffusion • Optimal stopping • Elimination algorithm

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1 Introduction

We consider a time-homogenous strong Markov process $Z = (Z_t)_{t\geq 0}$ with values in $X \bigcup e$, where $(X, \mathscr{B})$ is a measurable space, and e is an absorbing state. The time may be discrete or continuous. We assume that Z is defined on some filtered probabilistic space and that the following measurable functions are given:

$\rho(z) \geq 0$ – killing intensity; $g(z)$ – payoff function, $g(e) = 0$; $c(z)$ – cost of observations, $c(e) = 0$.

In the continuous time we consider the functional

$$V(z, \tau) = E_z\left[g(Z_\tau) - \int_0^\tau c(Z_s)ds\right], \tag{1}$$

where τ is an arbitrary stopping time with respect to the given filtration. In the discrete time instead of the integral one has a sum from 0 to $\tau - 1$.

We assume that the expectation is defined for any τ. The optimal stopping problem consists in a maximization of the functional (1). This problem is equivalent to the problem where instead of absorbtion one has in the continuous time the functional

$$V(z, \tau) = \bar{E}_z\left[g(Z_\tau)e^{-\int_0^\tau \rho(Z_u)du} ds - \int_0^\tau c(Z_s)e^{-\int_0^s \rho(Z_u)du} ds\right],$$

where $\bar{E}$ corresponds to the process without absorbtion. In the discrete time instead of $\exp(-\int_0^t \rho(Z_u)du)$ one has $\prod_{u=0}^{t-1}(1 - \rho(Z_u))$.

The aim of this paper is to present a procedure for constructing the value function

$$V(z) = \sup_\tau V(z, \tau).$$

There are numerous papers devoted to the optimal stopping problems. We mention only some, related to our approach. The general theory of optimal stopping and some methods to obtain the value function can be found in Shiryaev [19], Peskir and Shiryaev [13]. Dayanik and Karatzas [3] reduce optimal stopping of one-dimensional diffusion to the optimal stopping of Brownian motion, Salminen [18] uses Martin's boundaries for solution of optimal stopping problem, Bronstein et al. [2] consider one-dimensional diffusion on a halfline with discounting dependent on state and piecewise constant nondecreasing payoff function.

The main point of the proposed approach is the following simple lemma. Let $C \in \mathscr{B}$ and $\tau_C = \inf\{t : t \geq 0,\ \ Z_t \notin C\}$ be a stopping time. Define

$$g_C(z) = V(z, \tau_C) = E_z\left[g(Z_{\tau_C}) - \int_0^{\tau_C} c(Z_s)ds\right].$$

By the definition $g_C(z) = g(z)$ if $z \notin C$. The definition for the discrete time is similar.

Lemma 1. *If $g_C(z) > g(z)$ for all $z \in C$, then C belongs to the continuation set and the problem of optimal stopping with the payoff function $g_C(z)$ has the same value function as the problem with the payoff function $g(z)$.*

Proof. Let $V_C(z)$ be the value function in the problem with the payoff function $g_C(z)$. It follows from $g_C(z) \geq g(z)$ that $V_C(z) \geq V(z)$. On the other side for any τ one can define $\tau' = \inf\{t : t \geq \tau, Z_t \notin C\}$. Then by the strong Markov property

$$V(z,\tau') = E_z\left[-\int_0^\tau c(Z_s)ds + E_{z_\tau}\left[g(Z_{\tau'}) - \int_\tau^{\tau'} c(Z_s)ds\right]\right]$$
$$= E_z\left[-\int_0^\tau c(Z_s)ds + g_C(z_\tau)\right] = V_C(z,\tau),$$

and consequently $V(z) \geq V_C(z)$. The proof for the discrete time is similar. □

We say that a function $f(z)$ is a modification of the payoff function $g(z)$ (or is a modified payoff function) if there exists $C \in \mathscr{B}$ such that $f(z) = g_C(z)$ and $g_C(z) > g(z)$ for all $z \in C$. It follows from Lemma 1 that the optimal stopping problems with the payoff function $g(z)$ and the modified payoff function $g_C(z)$ have the same value function.

The question is how to find such a set C. In Sect. 2 we discuss very shortly the case of the discrete time. In Sect. 3 we formulate the procedure for one-dimensional diffusion. The proofs of all lemmas from Sect. 3 are given in Sect. 6. In Sect. 4 all examples from [3] are considered from the point of view of modification of the payoff function. In Sect. 5 some other examples are considered.

2 Discrete Time

The case of the discrete time was considered in [15]. It was assumed that functions $g(z),\ c(z)$ are bounded and there exists $n_0 > 0$ such that $\mathbf{P}_z\{Z_{n_0} = e\} \geq 1-\beta > 0$ for all $z \in X$. Let T be the revaluation operator, i.e. $Tf(z) = -c(z) + E_z f(Z_1)$. The following statements were proved.

(a) If $C = \{z :\ Tg(z) > g(z)\}$ is empty, then $V(z) = g(z)$,
(b) If $C = \{z :\ Tg(z) > g(z)\}$ is not empty, then $g_C(z)$ is a modification of the payoff function $g(z)$.

Let $g_0(z) = g(z),\ \ C_1 = \{z : Tg(z) > g(z)\}$,

$$C_{k+1} = \{z : Tg_{C_k}(z) > g(z)\} = C_k \bigcup \{z : Tg_{C_k}(z) > g_{C_k}(z)\}$$

and $g_k(z) = g_{C_k}(z)$, $k \geq 1$. The respective sequence of the modified payoff functions $g_k(z)$ is nondecreasing. There are two possibilities.

(1) There exists k_0 such that the set $\{z : Tg_{k_0}(z) > g(z)\}$ is empty. This always holds if X is finite. If such k_0 exists then $g_k(z) = g_{k_0}(z)$ for all $k \geq k_0$, $g_{k_0}(z)$ coincides with the value function and the set C_{k_0} coincides with the continuation set.
(2) If such k_0 does not exists then the sequence $g_k(z)$ converges to the value function, and the sequence C_k converges to the continuation set.

Sonin [21–23] was the first who used the set $\{z : Tg(z) > g(z)\}$ to solve the optimal stopping problem. For the case of finite state space X he proposed to eliminate the states from this set and to consider a new Markov chain, with a new reduced state space and new transition probabilities. These probabilities coincide with the distribution of the initial chain at the time of the first return to the new state space. They can be simply recalculated from the old ones. In the case of a finite number of states after finite number of steps we obtain the new chain and the new state space for which the reward for stopping – which equals to the payoff function – is greater than or equal to the expected reward for doing one more step for all points. In such situation the stopping set coincides with the final state space and the value function coincides with the reward for the instant stopping. After that the value functions corresponding to the previous chains can be restored sequentially.

The possibilities of generalization of Sonin's elimination algoriphm to the countable case in some situations were discussed in [23]. A generalization to a special case with not necessary countable state space was considered in [17]. In [14] the algorithm was modified to study the case of an arbitrary state space.

The modification of the payoff function was proposed in [15]. The algorithms which are close to the modification procedure in discrete time were elaborated in [7, 8] and [11].

3 One-dimensional Diffusion

We consider a time-homogeneous strong Markov process $Z = (Z_t)_{t\geq 0}$ with values in $X \bigcup e$, where e is an absorbing state and $X =]a, b[$, $-\infty \leq a < b \leq +\infty$. We assume that the following measurable functions are given:

$\sigma(z) \geq 0$ – diffusion coefficient, $m(z)$ – drift coefficient, $\rho(z) \geq 0$ – killing intensity.

We assume that

(I) $\forall z \in]a, b[\ \exists \varepsilon > 0 : \int_{z-\varepsilon}^{z+\varepsilon} \left(\frac{1 + |m(u)|}{\sigma^2(u)} + \rho(u) \right) du < \infty$, which implies that the diffusion is regular (see for example [3]);
(II) There exists a finite (possibly empty) set A^0 inside $]a, b[$ where the diffusion has a partial reflection, i.e. there exists a function $\alpha(z)$ such that

$$-1 < \alpha(z) < 1, \;\; \alpha(z) \neq 0 \text{ for } z \in A^0, \;\; \alpha(z) = 0 \text{ for } z \notin A^0,$$

and

$$\mathbf{P}_z[Z_t > z] \to \frac{1+\alpha(z)}{2} \text{ as } t \to 0;$$

(III) Each point a and b is either natural (it can not be reached during the finite time) or reflecting or adsorbing.

We assume also that the following measurable functions are given:

$$g(z) \text{ – payoff function, } \; g(e) = 0; \quad c(z) \text{ – cost of observation, } \; c(e) = 0.$$

Additional assumptions on the function $g(z)$ will be given later.
As above we define a functional

$$V(z, \tau) = E_z \left[g(Z_\tau) - \int\limits_0^\tau c(Z_s) ds \right],$$

and the value function $V(z) = \sup\limits_\tau V(z, \tau)$.

For any interval $]c, \, d[$ such that $a < c < d < b$ let us define the stopping time

$$\tau_{]c,\,d[} = \inf\{t : \; t \geq 0, \;\; Z_t \notin]c, \, d[\}$$

and the respective function

$$g_{]c,\,d[}(z) = V\left(z, \tau_{]c,\,d[}\right).$$

Obviously $g_{]c,\,d[}(z) = g(z)$ for $z \notin \,]c, \, d[$. Let us define also

$$g_{]a,\,d[}(z) = \limsup_{c \downarrow a} g_{]c,\,d[}(z), \qquad g_{]c,\,b[}(z) = \limsup_{d \uparrow b} g_{]c,\,d[}(z),$$

We define the following two operators:

$$Lf(z) := \frac{\sigma^2(z)}{2} \frac{d^2}{dz^2} f(z) + m(z) \frac{d}{dz} f(z) - \rho(z) f(z) - c(z),$$

$$L_1 f(z) := (1 + \alpha(z)) f'_+(z) - (1 - \alpha(z)) f'_-(z),$$

where $\alpha(z)$ is defined in II) and $f'_-(z)$ is the left and $f'_+(z)$ is the right derivative of the function $f(z)$.

The following statement is well known for the case without partial reflection (see, for example, [13] Sect. 7). The case of partial reflection can be found in [5, 11].

Statement 1. *A function $f(z)$ is continuous, twice differentiable for $z \in]c,d[$, $a < c < d < b, z \notin A$, and satisfies on $]c,d[$ the relations*

$$Lf(z) = 0 \text{ for } z \in]c,d[,\ z \notin A_0, \qquad L_1 f(z) = 0 \text{ for } z \in]c,d[\bigcap A_0, \tag{2}$$

and boundary conditions $f(c) = g(c)$, $f(d) = g(d)$, if and only if $f(z) = g_{]c,d[}(z)$ on $]c,d[$.

If a is a reflecting point, then $f(z)$ is continuous, twice differentiable for $z \in]a,d[$, $a < d < b, z \notin A$, and satisfies on $]a,d[$ the relations (2) with $c = a$ and boundary conditions $f'_+(a) = 0$, $f(d) = g(d)$ if and only if $f(z) = g_{[a,d[}(z)$ on $]a,d[$.

If b is a reflecting point, then $f(z)$ is continuous, twice differentiable for $z \in]c,b[$, $a < c < b$, $z \notin A$, and satisfies on $]c,b[$, the relations (2) with $d = b$ and boundary conditions $f'_-(b) = 0$, $f(c) = g(c)$ if and only if $f(z) = g_{]c,b]}(z)$ on $]c,b[$.

Now we are ready to formulate the rest of the assumed properties of $g(z)$. In what follows we assume that $g(z) \in \mathscr{C}$, where $\mathscr{C}$ is the set of functions $f(z)$ satisfying the following properties.

(1) $f(z)$ is bounded $\forall\ ]c,d[$, $a < c < d < b$; $f''(z)$ exists and is finite and continuous on $]a,b[$ with exception of a finite (possibly empty) set A^1; $f(z)$ and $f'(z)$ have left and right limits at points from $A^0 \bigcup A^1$.
(2) If a is natural then $f(a)$ is not defined, if a is reflecting or absorbing then $f(a)$ is finite. Analogously for the point b.
(3) The set of points where $Lf(z) > 0$ is either empty or consists of a finite number of intervals. Denote by A^2 the set of the endpoints of these intervals.

Let $A = A^0 \bigcup A^1 \bigcup A^2 = \{z_1, \ldots, z_k\}$, where $a = z_0 < z_1 < \ldots < z_k < z_{k+1} = b$. In what follows we consider only functions from $\mathscr{C}$.

Below we present a procedure of a sequential modification of the payoff function for the model satisfying conditions (I)–(III) with $g(z) \in \mathscr{C}$. On each step we modify the payoff function. By the definition it means that the modified payoff function also belongs to $\mathscr{C}$ and the optimal stopping problem with the modified payoff function has the same value function as the initial problem. After finite number of steps we obtain the new payoff function which satisfies the conditions of Theorem 1 below, and, according to this theorem, coincides with the value function. This procedure is based on Lemma 1 and the following theorems and lemmas. Below we assume that $g(z_i) \geq \max(g(z_i+), g(z_i-))$ for all $i = 0, \ldots, k+1$. At the end of this section we shall show that in the case $g(z_i) < \max(g(z_i+), g(z_i-))$ for some i, the value function is the same as for the case of equality.

The following theorem describes a situation when the value function coincides with the payoff function and consequently the optimal stopping time identically equals to zero. The proof for the case without partial reflection is standard and can be found, for example, in [13]. The proof for the case with partial reflection follows the same scheme and uses the results from [11].

Theorem 1. *Let* (a) *the function* $g(z)$ *be continuous*;

(b) *the set where* $Lg(z) > 0$ *be empty* ; (c) $L_1 g(z) \le 0$ *for all* $z \in A$;
(d1) *if a is reflecting then* $g'_+(a) \le 0$; (d2) *if b is reflecting then* $-g'_-(b) \le 0$.
Then $V(z) = g(z)$.

In the procedure that we describe below we shall modify sequentially the payoff function in such a way, that it will satisfy conditions (a), (b), (c), (d1), (d2) of Theorem 1.

The following lemma shows that one always can modify the payoff function to make it continuous, i.e. satisfying condition (a) of Theorem 1.

Lemma 2. *If* $1 \le i \le k$ *and* $g(z_i) > g(z_i+)$ *then there exists* $\varepsilon \in]z_i, z_{i+1}[$ *such that* $g_{]z_i,\varepsilon[}(z) > g(z)$ *for* $z \in]z_i, \varepsilon[$ (see Fig. 1a). *A similar statement is true for the case* $g(z_i) > g(z_i-)$ *and for points a and b in case they are absorbing.*

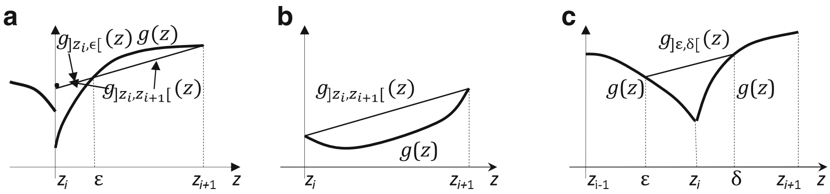


Fig. 1 Illustration of Lemmas 2, 3, and 4. (**a**) $g(z_i) > g(z_i-) > g(z_i+)$, (**b**) $Lg(z) > 0$ for $z \in]z_i, z_{i+1}[$, and (**c**) $L_1 g(z_i) > 0$

To construct the continuous modification we need no more then $2k + 2$ steps, corresponding to the modification at the left and the right neighborhood of the points z_i, $i = 1, \ldots, k$, and at the points a and b. The set A corresponding to modified payoff function consists of no more then $3k + 2$ points.

The next lemma shows that the continuous payoff function can be modified in such a way, that it will satisfy the condition (b) of Theorem 1.

Lemma 3. *If* $g(z)$ *is continuous,* $Lg(z) > 0$ *for* $z \in]z_i, z_{i+1}[$ *and some* $0 \le i < k$, *then* $g_{]z_i,z_{i+1}[}(z)$ *is a modification of* $g(z)$ (see Fig. 1b). *Moreover* $Lg_{]z_i,z_{i+1}[}(z) = 0$ *for* $z \in]z_i, z_{i+1}[$ *and*

$$g'_{]z_i,z_{i+1}[+}(z_i) > g'_+(z_i), \quad g'_{]z_i,z_{i+1}[-}(z_{i+1}) < g'_-(z_{i+1}).$$

We shall consider now the payoff functions which are continuous and such that application of the operator L gives nonpositive values. In this case in the neighborhood of the points, where an application of the operator L_1 is positive, the payoff function can be modified.

Lemma 4. *Let $g(z)$ be continuous, $Lg(z) \le 0$ for $z \notin A$, and $L_1 g(z_i) > 0$ for some $i = 1, \dots, k$. Then* (see Fig. 1c)*:*

(a) *There exist $\varepsilon_1 \in]z_{i-1}, z_i[$ and $\delta_1 \in]z_i, z_{i+1}[$, such that $g_{]\varepsilon,\delta[}(z)$ is a modification of $g(z)$ for any $\varepsilon \in [\varepsilon_1, z_i[$ and any $\delta \in]z_i, \delta_1]$, and, moreover, $L_1 g_{]\varepsilon,\delta[}(\varepsilon) > 0$, $L_1 g_{]\varepsilon,\delta[}(\delta) > 0$ for any $\varepsilon \in]\varepsilon_1, z_i[$ and any $\delta \in]z_i, \delta_1[$.*

(b) *If $Lg(z) = 0$ for $z \in]\varepsilon_2, z_i[$, and some $\varepsilon_2 \in [z_{i-1}, z_i[$, then one can take $\varepsilon_1 = \varepsilon_2$ in statement* (a)*. Moreover, $g'_{]\varepsilon_2,\delta[+}(\varepsilon_2) > g'_+(\varepsilon_2)$ A similar statement is true for the interval $]z_i, \delta_2[$.*

If $g_{]\varepsilon,\delta[}(z) > g(z)$ for $z \in]\varepsilon, \delta[$ and at least one of inequalities $L_1 g_{]\varepsilon,\delta[}(\varepsilon) > 0$, $L_1 g_{]\varepsilon,\delta[}(\delta) > 0$ holds, then applying Lemma 4 to the point ε or/and δ one can see that the interval $]\varepsilon, \delta[$ where $g_{]\varepsilon,\delta[}(z) > g(z)$ can be enlarged.

In the next theorem we consider the payoff function which satisfies conditions (a) and (b) of Theorem 1 on $]a, b[$, condition (c) on $]a, z_i[$ for same i, and such that $L_1 g(z_i) > 0$. We prove the existence of a modification, that satisfies condition (c) on $]a, z_{i+1}[$. For this aim it is convenient to define the function $\bar{g}_{]c,d[}(z)$ as follows: $\bar{g}_{]c,d[}(z) = g_{]c,d[}(z)$ for $z \in]c, d[$, $\bar{g}_{]c,d[}(z)$ satisfies (2) on $]a, b[$. The functions $\bar{g}_{[a,d[}(z)$ and $\bar{g}_{]c,a]}(z)$ are defined similarly.

Theorem 2. *Let $g(z)$ be continuous; $Lg(z) \le 0$ for $z \notin A$, and there exist $z_i \in A$ such that $L_1 g(z_i) > 0$ and $L_1 g(z_j) \le 0$ for all j, $1 \le j \le i-1$. Then there exist unique $c_* \in [a, z_i[$, $d_* \in]z_i, z_{i+1}]$, and $c_*^1 \in [a, c_*]$, $d_*^1 \in [d_*, z_{i+1}]$, such that $g_{]c_*, d_*[}(z)$ is a modification of $g(z)$* (see Fig. 2) *and*

$$\bar{g}_{]c_*, d_*[}(z) > g(z) \;\; for \;\; z \in]a, c_*^1[\; \bigcup \;]c_*, d_*[\; \bigcup \;]d_*^1, z_{i+1}[,$$

$$\bar{g}_{]c_*, d_*[}(z) = g(z), \;\; so \; that \;\; Lg(z) = 0, \;\; L_1 g(z) = 0 \;\; for \;\; z \in]c_*^1, c_*] \; \bigcup [d_*, d_*^1[.$$

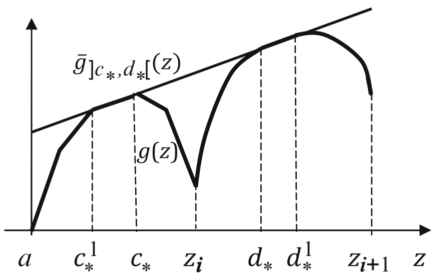


Fig. 2 Illustration of Theorem 2

To prove Theorem 2 and describe properties of the interval $]c_*, d_*[$, which help to find the values c_* and d_*, we shall formulate the following two lemmas. Lemma 5 describes a positional relationship between a payoff function, that satisfies conditions (a), (b) and (c) of Theorem 1 on some interval $]c,d[\subset [a,b]$ and an arbitrary solution of (2). It appears that the situation is the same as for the Brownian motion, i.e. for the concave function and the set of liner functions.

Lemma 5. *Let the functions $f(z), g(z)$ be continuous; $Lg(z) \leq 0$, $L_1 g(z) \leq 0$ for all $z \in]c,d[\subset [a,b]$, $Lf(z) = 0$, $L_1 f(z) = 0$ for all $z \in]a,b[$. Then either $f(z) > g(z)$ for all $z \in]c,d[$; or there exist $c_1, d_1 \in [c,d]$ such that $c_1 \leq d_1$, $f(z) > g(z)$ for all $z \in [c,d] \setminus [c_1,d_1]$, and either $f(z) = g(z)$ for all $z \in]c_1,d_1[\subset]c,d[$; or $f(z) < g(z)$ for all $z \in]c_1,d_1[$, and in this case if $c_1 < d_1$ and $g(c_1) = f(c_1)$ then $f'_+(c_1) < g'_+(c_1)$, if $c_1 < d_1$ and $g(d_1) = f(d_1)$ then $f'_-(d_1) > g'_-(d_1)$.*

It is convenient to define a generalized tangent line at point h, i.e. the function $g_h(z)$, $z \in]a,b[$, depending on a parameter $h \in [a,b]$, such that $g_h(h) = g(h)$, $g'_h(h) = g'(h)$ and $g_h(z)$ satisfies (2) on $]a,b[$. We assume that $g_a(z) = g_{h+}(z)$, $g_h(z) = g_{h-}(z)$ if these limits exist, and if $h \in A$ then two functions are defined: $g_{h+}(z)$ and $g_{h-}(z)$. Lemma 6 describes the behavior of the generalized tangent line as a function of h for any fixed z.

Lemma 6. *Let function $g(z)$ be continuous; $Lg(z) \leq 0$, $L_1 g(z) \leq 0$ for all $z \in]c,d[$. Then for each fixed $z \in]a,b[$ function $g_h(z)$ nonincreases in h for $h < z$ and nondecreases in h for $h > z$* (see Fig. 3).

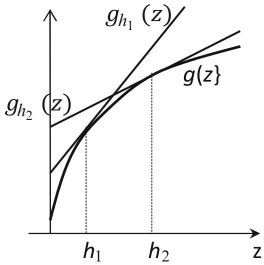


Fig. 3 Illustration of Lemma 6

Proof of Theorem 2 . Theorem 2 is a simple consequence of Lemmas 5 and 6. Indeed, for $h \in [z_i, z_{i+1}]$ consider $g_h(z)$ as a function $f(z)$ in Lemma 5. Since for any fixed $z \in]a, z_i]$, the function $g_h(z)$ nondecreases in h, we obtain that $d_1(h)$ from Lemma 5 nonincreases in h and, by Lemma 4, $z_i - d_1(h)$ is positive and small if $h - z_i$ is small. Respectively, $c_1(h)$ from Lemma 5 nondecreases in h. Consequently, either there exists $d_* \in]z_i, z_{i+1}]$ such that $d_1(d_*) = c_*$, $c_1(d_*) = c_*^1$, or $d_* = z_{i+1}$ and we shall consider functions $f^t(z)$ such that $f^t(z_{i+1}) = g(z_{i+1})$, $f^{t\prime}(z_{i+1}) = t$, $t < g'_-(z_{i+1})$, and $f^t(z)$ satisfies (2) on $]a, z_{i+1}[$. Decreasing t we obtain c_* and c_*^1.

The next theorem describes properties of the interval $]c_*, d_*[$, which help to find the values c_* and d_*.

Theorem 3. *Under conditions of Theorem 2 the following properties hold:*

(a) *If* $g(z) = g_{]c,z_i[}(z)$ *for* $z \in]c, z_i[$ *for some* $c \in [a, z_i[$, *then* $c_* \le c$;
if $Lg(z) = 0$ *for* $z \in]z_i, d[$ *for some* $d \in]z_i, z_{i+1}[$, *then* $d_* \ge d$;

(b) *The following relations hold:*

(b1) $L_1 g_{]c,\, d_*[}(c) < 0$ *for* $c \in]a, c_*^1[$, $\quad L_1 g_{]c,\, d_*[}(c) > 0$ *for* $c \in]c_*, z_i[$,
$L_1 g_{]c,\, d_*[}(c) = 0$ *for* $c \in]c_*^1, c_*[$; (see Fig. 4a)

(b2) $L_1 g_{]c_*,\, d[}(d) > 0$ *for* $d \in]z_i, d_*[$, $\quad L_1 g_{]c_*,\, d[}(d) < 0$ *for* $d_*^1 \le d < z_{i+1}$,
$L_1 g_{]c_*,\, d[}(d) = 0$ *for* $d \in [d_*, d_*^1[$; (see Fig. 4b)

(c) *For each fixed* $z < d$ *function* $g_d(z)$ *nondecreases in* d. *If* $d_* < z_{i+1}$ *then:*

(c1) *If* $g_{z_i+}(a+) \ge g(a+)$ *then* $g(z) > g_{z_i+}(z)$ *for all* $z \in]a, z_i[$ *and* $c_* > a$;

(c2) *In the opposite case for any* $c \in]a, z_i[$ *there exists* $d_c \le d_*$ *such that* $g_{d_c}(c) = g(c)$, *so that* $g_{d_c}(z) = g_{]c,d_c[}(z)$ *for* $z \in]c, d_c[$. *Moreover,* $L_1 g_{]c,\, d_c[}(c) < 0$ *for* $c \in]a, c_*^1[$ *and* $L_1 g_{]c,\, d_c[}(c) > 0$ *for* $c \in]c_*, z_i[$.

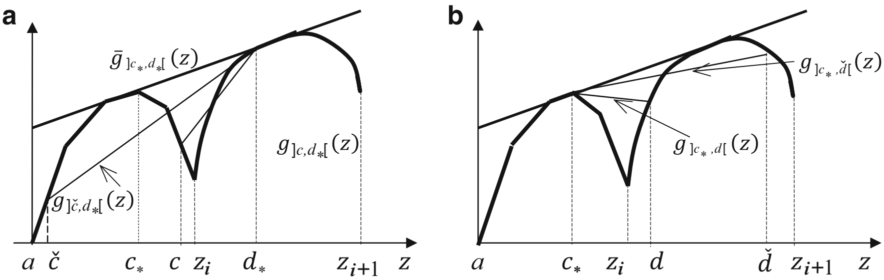


Fig. 4 Illustration of Theorem 3. (**a**) $L_1 g_{]\check c,d_*[}(\check c) < 0$, $L_1 g_{]c,d_*[}(c) > 0$ and (**b**) $L_1 g_{]c_*,d[}(d) > 0$, $L_1 g_{]c_*,\check d[}(\check d) < 0$

Remark 1. Statement (b) is an analog of the well-known smooth fitting conditions.

Proof of Theorem 3. The statement (a) follows from statement (b) of Lemma 4. Statements (b) and (c) are direct consequences of Lemmas 5 and 6.

To check the properties (d) of Theorem 2 we shall consider now points a, b.

Lemma 7. *Let* $g(z)$ *be continuous,* $Lg(z) \le 0$ *for all* $z \notin A$ *and* $L_1 g(z) \le 0$ *for all* $z \in A$. *If* a *is reflecting and* $g'_+(a) > 0$ *then:*

(a) *There exist unique* $d_* \in]a, b]$ *and* $d_*^1 \in [d_*, b]$ *such that* (see Fig. 5)

$$\bar g_{[a,d_*[}(z) > g(z) \text{ for } z \in [a, d_*[\bigcup]d_*^1, b[;\ \bar g_{[a,d_*[}(z) = g(z), \text{ for } z \in [d_*, d_*^1[;$$

(b) *If* $g(z) = g_{]a,d[}(z)$ *for some* d *then* $d_* \geq d$*;*
(c) *The following relations hold:*

$$L_1 g_{[a\,d[}(d) > 0 \text{ for } d \in]a, d_*[, \quad L_1 g_{[a,\,d[}(d) = 0 \text{ for } d \in [d_*, d_*^1[,$$

$$L_1 g_{[a,\,d_*^1[}(d_*^1) \leq 0, \quad L_1 g_{[a,\,d[}(d) < 0 \text{ for } d_*^1 \leq d < b.$$

Similar statements hold for the point b *if* $-g'_-(b) > 0$.

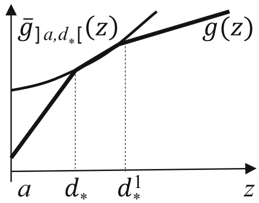


Fig. 5 Illustration of Lemma 7

Now we can formulate the procedure.

At the first stage using Lemma 2 after no more than $2k + 2$ steps we obtain a modification $g_1(z)$ which satisfies the condition (a) of Theorem 1.

At the second stage using Lemma 3 after no more than k steps we obtain the modification $g_2(z)$ of $g_1(z)$, which satisfies the conditions (a) and (b) of Theorem 1. The set A_2 corresponding to $g_2(z)$ coincides with the set A_1 corresponding to $g_1(z)$.

At the third stage we use Theorem 2. On each step the number of points, where the result of application of operator L_1 to the new payoff function is positive, decreases. So, after no more than $3k + 2$ steps we obtain the modification $g_3(z)$ of $g_2(z)$ which satisfies the conditions (a), (b), and (c) of Theorem 1.

At the fourth stage we use Lemma 7. As a result we obtain the modified payoff function which satisfies all conditions of Theorem 1 and therefore coincides with the value function.

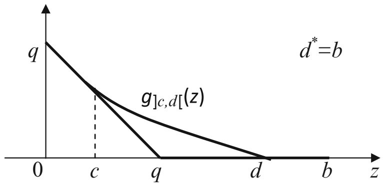
Remark 2. Let $\tilde{A} = \{i \,:\, g(z_i) < \max(g(z_i+), g(z_i-))$. Then the value function for the payoff function $g(z)$ is the same as the value function for the payoff function $\tilde{g}(z)$, where $\tilde{g}(z) = g(z)$ for $z \notin \tilde{A}$, and $\tilde{g}(z) = \max(g(z_i+), g(z_i-))$ for $z \in \tilde{A}$. To show this it suffices for example in the case $g(z_i+) \geq g(z_i-)$ to construct the value function for the payoff function $g_C(z)$, where $C = \bigcup_{i \in \tilde{A}}]z_i, \varepsilon_i[, \ \varepsilon_i < z_{i+1}$ and then take $\varepsilon_i \downarrow z_i$.

4 Eleven Examples from Dayanik and Karatzas [3]

In this section we consider the same examples which were considered in the paper [3]. In all examples the set A^0 is empty and only in Example 9 the function $c(z)$ differs from identical zero.

Example 1 *(*Karatzas and Wang [9]). Pricing an "Up–and–out" barrier put–option of American type: geometric Brownian motion Z_t on $]0,\ b]$ with parameters (r, σ), killing intensity r, absorbtion at b and $g(z) = (q - z)^+,\ q < b$ (see Fig. 6).

Fig. 6 Examples 1 and 2



For this problem the operator L has a form

$$Lf(z) := \left(\sigma^2 z^2/2\right) f''(z) + rzf'(z) - rf(z). \tag{3}$$

Since $Lg(z) = -rq < 0$ for $0 < z < q$ and $Lg(z) = 0$ for $q < z < b$, we are under conditions of Theorem 2, where the set A consists of one point $z_1 = q$ with $L_1 g(q) = g'_+(q) - g'_-(q) = 1 > 0$. It follows from Lemma 1 and Theorem 2 that instead of $g(z)$ one can consider $g_{]c_*,d_*[}(z)$, where according to statement (a) of Theorem 3 $d_* = b$, and according to statement (b1) of Theorem 3 either the equation

$$g'_{]c,\,b[+}(c) = -1 \; (= g'(c)) \tag{4}$$

has a unique root in $]0, q[$ and c_* coincides with this root, or $c_* = 0$ and in this case $g'_{+[c,\,b[}(c) > -1$ for $0 < c < q$. It follows from (3) and Statement 1 that

$$g_{]c,b[}(z) = (q - c)\frac{b\,(b/z)^{\beta} - z}{b\,(b/c)^{\beta} - c}, \qquad z \in [c, b], \tag{5}$$

where $\beta = 2r/\sigma^2$. Hence

$$g'_{]c,b[+}(c) = -(q - c)\frac{1 + \beta\,(b/c)^{1+\beta}}{b\,(b/c)^{\beta} - c}\,, \tag{6}$$

and $g'_{]c,b[+}(c) \to -\infty$, i. e. the Eq. (4) has a unique root c_* in $]0, q[$. By (6) Eq. (4) can be rewritten as $\beta q - c(1 + \beta) + q\,(c/b)^{1+\beta} = 0$. The respective function $g_{]c_*,b[}(z)$ (see (5)) satisfies conditions of Theorem 1 and therefore coincides with the value function.

Example 2 (Dayanik and Karatzas [3]). Pricing an "Up–and–out" barrier put–option of American type under the Constant–Elasticity–of–Variance (CEV model). It means that the stock price dynamics are described according to the CEV model, $dS_t = rS_t dt + \sigma S_t^{1-\alpha} dB_t$, $S_0 \in]0, b[$, for some $\alpha \in]0, 1[$, with killing intensity r, absorbtion at b, and $g(z) = (q - z)^+$, $q < b$, $c(z) \equiv 0$ (see Fig. 6).

For this problem the operator L has a form

$$Lf(z) := \left(\sigma^2 z^{2(1-\alpha)}/2\right) f''(z) + rzf'(z) - rf(z). \tag{7}$$

Since $Lg(z) = -rq < 0$ for $0 < z < q$ and $Lg(z) = 0$ for $q < z < b$, we are under conditions of Theorem 2, where the set A consists of one point $z_1 = q$ with $L_1 g(q) = g'_+(q) - g'_-(q) = 1 > 0$. It follows from Lemma 1 and Theorem 2 that instead of $g(z)$ one can consider $g_{]c*,d*[}(z)$, where according to statement (a) of Theorem 3 $d_* = b$, and according to statement (b1) of Theorem 3 either the equation

$$g'_{]c,\,b[+}(c) = -1 \;(= g'(c))$$

has a unique root in $]0, q[$ and c_* coincides with this root, or $c_* = 0$ and in this case $\lim_{c\to 0} g'_{]c,\,b[+}(c) \geq -1$. It follows from (7) and Statement 1 that

$$g_{]c,b[}(z) = (q-c)\frac{z\int\limits_z^b \frac{1}{v^2} e^{-\frac{r}{\alpha\sigma^2}v^{2\alpha}}\,dv}{c\int\limits_c^b \frac{1}{v^2} e^{-\frac{r}{\alpha\sigma^2}v^{2\alpha}}\,dv}, \quad z \in [c, b] \quad 0 < c < q.$$

Differentiating and integrating by parts we obtain for $z \in]c, b[$

$$g'_{]c,b[}(z) = -\frac{(q-c)}{c\int\limits_c^b \frac{1}{v^2} e^{-\frac{r}{\alpha\sigma^2}v^{2\alpha}}\,dv}\left[\frac{1}{b^2}e^{-\frac{r}{\alpha\sigma^2}b^{2\alpha}} + \int\limits_z^b \frac{r}{\alpha\sigma^2}v^{2(\alpha-1)}e^{-\frac{r}{\alpha\sigma^2}v^{2\alpha}}\,dv\right], \quad (8)$$

$$\lim_{c\downarrow 0} g'_{]c,b[+}(c) = -qA(b, \alpha, r, \sigma),$$

where

$$A(b, \alpha, r, \sigma) = \left[\frac{1}{b^2}e^{-\frac{r}{\alpha\sigma^2}b^{2\alpha}} + \int\limits_z^b \frac{r}{\alpha\sigma^2}v^{2(1-\alpha)}e^{-\frac{r}{\alpha\sigma^2}v^{2\alpha}}\,dv\right].$$

Now we can apply statement (b1) of Theorem 3. If $qA(d, \alpha, r, \sigma) \leq 1$ (it is possible only for $1/2 < \alpha \leq 1$) then $c_* = 0$. If $qA(d, \alpha, r, \sigma) > 1$ then c_* satisfies $g'_{]c_*,b[-}(c_*) = -1$ (see (8)). The respective function $g_{]c_*,b[}(z)$ (see (8)) satisfies conditions of Theorem 1 and therefore coincides with the value function.

Example 3 *(*Broadie and Detemple [1]). Pricing an American Capped Call Option on Dividend–Paying Assets: geometric Brownian motion Z_t on $]0, \infty[$ with parameters $(r - \gamma, \sigma)$, killing intensity r and $g(z) = (\min[l, z] - K)^+$, dividend rate $\gamma \geq 0$, strike price $K \geq 0$ and the cap $l > K$ (see Fig. 7).

For this problem the operator L has a form

$$Lf(z) := \left(\sigma^2 z^2/2\right) f''(z) + (r-\gamma)zf'(z) - rf(z), \quad (9)$$

and $Lg(z) = 0$ for $0 < z < K$, $Lg(z) = -\gamma z + rK$ for $K < z < l$, $Lg(z) = -r(l - K)$ for $z > l$.

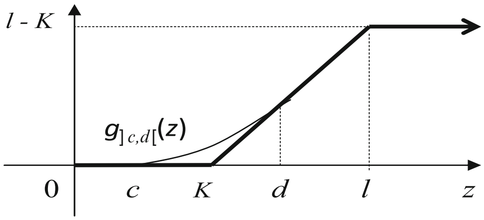


Fig. 7 Example 3

Let $d_1 = \max[K, \min[l, Kr/\gamma]]$. So, the set A consists of three or two points: K, d_1, l. According to Lemmas 1 and 3 we can change $g(z)$ to $g_{]K,d_1[}(z)$. With the payoff function $g_{]K,d_1[}(z)$ we are under conditions of Theorem 2. According to Statement 1, $L_1 g_{]K,d_1[}(K) = g'_{]K,d_1[+}(K) \geq g'_+(K) = 1$, and we can apply statement (a) of Theorem 3 to $z_1 = K$ and change $g_{]K,d_1[}(z)$ to $g_{]0,d_1[}(z)$.

With the payoff function $g_{]0,d_1[}(z)$ we are under conditions of Theorem 2, where the set A_1 consists of one point d_1, and according to Lemma 3, $L_1 g_{]0,d_1[}(d_1) > 0$. It follows from Lemma 1 and Theorem 2 that instead of $g_{]0,d_1[}(z)$ one can consider $g_{]0,d_*[}(z)$, where according to statements (b2) of Theorem 3 either the equation $g'_{]0,\,d[-}(d) = 1\ (= g'(c))$ has a unique root in $]d_1, l[$ and d_* coincides with this root, or $d_* = l$ and in this case $g'_{]0,\,d[-}(d) < 1$ for $d \in]d_1, l[$.

Let $\varphi(k) := \sigma^2\kappa^2 - \left(\sigma^2 - 2(r-\gamma)\right)\kappa - 2r$ and $\kappa_+ > \kappa_-$ be the solutions of the equation $\varphi(k) = 0$. It follows from $\varphi(0) < 0, \varphi(1) < 0$ that $\kappa_+ > 1$ and $\kappa_- < 0$. It follows from (9) and Statement 1 that

$$g_{]0,d[}(z) = (d-K)(z/d)^{\kappa_+} \text{ for } z \in]0,d], \quad d \in]K,l], \tag{10}$$

so that $g'_{]0,d[-}(d) = \kappa_+ \dfrac{d-K}{d}$. The equation $g'_{]0,d[-}(d) = 1\ (= g'(d))$ has a root d_* on $]K, l[$ iff $d_* = K\dfrac{\kappa_+}{\kappa_+ - 1} < l$. According to Theorem 2 we can change $g_{]0,d_1[}(z)$ to $g_{]0,d_*[}(z)$ where $d_* = \min\left[l, K\dfrac{\kappa_+}{\kappa_+ - 1}\right]$. The function $g_{]0,d_*[}(z)$ (see (10)) satisfies conditions of Theorem 1 and therefore coincides with the value function.

Example 4 *(*Guo and Shepp [4]). Options for Risk–Averse Investors: geometric Brownian motion Z_t on $]0,\infty]$ with parameters (m,σ), killing intensity ρ and $g(z) = \max[l, z]$ (see Fig. 8).

For this problem the operator L has a form

$$Lf(z) := \left(\sigma^2 z^2/2\right) f''(z) + mzf'(z) - \rho f(z), \tag{11}$$

and $Lg(z) = -\rho l$ for $z \in]0, l[$, $Lg(z) = (m-\rho)z$ for $z > l$.

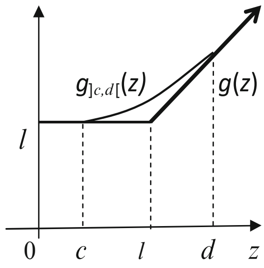


Fig. 8 Example 4

Let $\varphi(\kappa) = \sigma^2\kappa^2 - (\sigma^2 - 2m)\,\kappa - 2\rho$ and $\kappa_+ > 0$, $\kappa_- < 0$ be the solutions of the equation $\varphi(\kappa) = 0$. According to (11) and Statement 1

$$g_{]c,d[}(z) = l\,\frac{(z/d)^{\kappa_-} - (z/d)^{\kappa_+}}{(c/d)^{\kappa_-} - (c/d)^{\kappa_+}} + d\,\frac{(z/c)^{\kappa_+} - (z/c)^{\kappa_-}}{(d/c)^{\kappa_+} - (d/c)^{\kappa_-}} \text{ for } c < z < d\,, \tag{12}$$

where $0 < c \le l < d < +\infty$.

If $m > \rho$ then $Lg(z) = (m - \rho)z > 0$ and according to Lemmas 1 and 3 $V(z) \ge g_{]l,d[}(z)$ for any $d > l$. Since $g_{]l,d[}(z) \to +\infty$ as $d \to +\infty$ for any $z > l$ we obtain that in this case $V(z) = +\infty$.

The case $m = \rho$ will be considered in Example 5.

Let now $m<\rho$. Then $Lg(z)\le 0$ for all $z \in]0,\infty[$, $z \ne l$, and we are under conditions of Theorem 2, where A consists of one point $z_1 = l$. It follows from here and (12) that $\lim_{c\downarrow 0} g'_{]c,d[-}(c) = -\infty$ for any $d > l$, and $\lim_{d\downarrow +\infty} g'_{]c,d[-}(d) = \kappa_+ > 1$ for any $0 < c < l$, and hence, according to statement (b) of Theorem 3 there exists a unique solution $\{c_*, d_*\}$ of the system $L_1 g_{]c,d[}(c) = 0$, $L_1 g_{]c,d[}(d) = 0$. We can rewrite this (see (12)) as follows:

$$g_{]c_*,d_*[}(z) = \frac{l}{\kappa_+ - \kappa_-}\left[\kappa_+\left(\frac{z}{c_*}\right)^{\kappa_-} - \kappa_-\left(\frac{z}{c_*}\right)^{\kappa_+}\right] \text{ for } c_* < z < d_*, \tag{13}$$

where

$$c_* = l\left(1 - \frac{1}{\kappa_+}\right)^{\frac{-\kappa_-}{\kappa_+ - \kappa_-}}\left(1 - \frac{1}{\kappa_-}\right)^{\frac{1-\kappa_+ + \kappa_-}{\kappa_+ - \kappa_-}}, \quad d_* = c_*\left(\frac{\kappa_+(1-\kappa_-)}{(\kappa_+ - 1)(-\kappa_-)}\right)^{\frac{1}{\kappa_+ - \kappa_-}}.$$

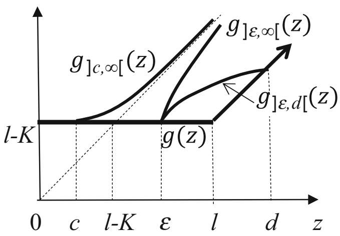
The function $g_{]c_*,d_*[}(z)$ (see (13)) satisfies conditions of Theorem 1 and therefore coincides with the value function.

Example 5 *(*Guo and Shepp [4]). Another "Exotic" Option: geometric Brownian motion Z_t on the half-line $]0,\infty[$ with parameters (m,σ), killing intensity m and $g(z) = (\max[l,z] - K)^+$ with $0 < K < l$ (see Fig. 9).

For this problem the operator L has a form

$$Lf(z) := \frac{\sigma^2 z^2}{2} f''(z) + mzf'(z) - mf(z). \tag{14}$$

Fig. 9 Example 5



Since $Lg(z) = mK > 0$ for $z > l$, $\quad Lg(z) = -m(l - K) < 0$ for $0 < z < l$, we are under conditions of Theorem 2, where the set A consists of one point $z_1 = l$ with $L_1 g(l) = g'_+(l) - g'_-(l) = 1 > 0$. It follows from Lemma 1 and Theorem 2 that instead of $g(z)$ one can consider $g_{]c_*,d_*[}(z)$, where according to statement (a) of Theorem 3 $d_* = \infty$, and according to statement (b1) of Theorem 3 either the equation $g'_{]c,\infty[+}(c) = 0\ (= g'(c))$ has a unique root in $]0, \infty[$ and c_* coincides with this root, or $c_* = 0$ and then $g'_{]c,\,\infty[+}(c) < 0$ for all $c \in]0, \infty[$. It follows from (14) and Statement 1 that

$$g_{]c,d[}(z) = (l - K)\frac{(z/d)^{-2m/\sigma^2} - (z/d)}{(c/d)^{-2m/\sigma^2} - (c/d)} + d\,\frac{(z/c) - (z/c)^{-2m/\sigma^2}}{(d/c) - (d/c)^{-2m/\sigma^2}}\,, \quad z \in [c, d]\,,$$

where $0 < c < l < d < \infty$, and hence

$$g_{]c,\infty[}(z) = z + (l - K - c)\left(\frac{z}{c}\right)^{-2m/\sigma^2} \quad \text{for } z \geq c. \tag{15}$$

So, $g'_{]c,\,\infty[+}(c) = 1 - 2m\dfrac{l - K - c}{c\sigma^2}$. Since $g'_{]c,\,\infty[+}(c) \to -\infty$ as $c \to 0$, the equation $g'_{]c,\,\infty[+}(c) = 0$ has a unique root $c_* = \dfrac{2m}{2m + \sigma^2}(l - K)$ in $]0, l[$. The function $g_{]c_*,\infty[}(z)$ (see (15)) satisfies conditions of Theorem 1 and therefore coincides with the value function. Note, that there is no optimal τ in this problem.

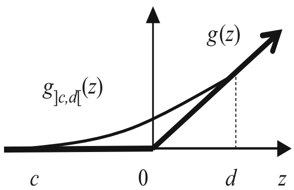
Example 6 (An Example of Taylor [24]). Brownian motion Z_t on $]-\infty, \infty[$ with parameters $(m, 1)$, killing intensity ρ and $g(z) = z^+$ (see Fig. 10).

For this problem the operator L has a form

$$Lf(z) = (1/2)f''(z) + mf'(z) - \rho f(z). \tag{16}$$

It follows from (16) that $Lg(z) = 0$ for $z < 0$, $Lg(z) = m - z\rho > 0$ for $z \in]0, m\rho[$, $Lg(z) = m - z\rho < 0$ for $z > m\rho$. We can apply Lemmas 3 and 1 and change $g(z)$ to $g_{]0,m/\rho[}(z)$. With the payoff function $g_{]0,m/\rho[}(z)$ we are under conditions of Theorem 2, where A consists of two points: $z_1 = 0$, $z_2 = m/\rho$, with $L_1 g_{]0,m/\rho[}(0) > 0$, $L_1 g_{]0,m/\rho[}(m/\rho) > 0$. Using statement (a) of Theorem 3 and Lemma 1 we can change $g_{]0,m/\rho[}(z)$ to $g_{]-\infty,m/\rho[}(z)$. With the payoff function $g_{]-\infty,m/\rho[}(z)$ we are

Fig. 10 Example 6



under conditions of Theorem 2, where A consists of one point: $z_1 = m/\rho$, with $L_1 g_{]-\infty,m/\rho[}(m/\rho) > 0$. Using statement (a) of Theorem 3 and Lemma 1 we can change $g_{]-\infty,m/\rho[}(z)$ to $g_{]-\infty,c*[}(z)$, where according to statement (b2) of Theorem 3 either the equation $g'_{]-\infty,\,d[-}(d) = 1\ (= g'(d))$ has a unique root in $]0,\infty[$ and d_* coincides with this root, or $d_* = +\infty$ and in this case $g'_{]-\infty,\,d[-}(d) < 1$ for all $d > 0$.

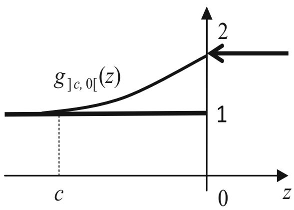
It follows from (16) and Statement 1 that

$$g_{]c,d[}(z) = d\,\frac{e^{(z-c)\gamma_+} - e^{(z-c)\gamma_-}}{e^{(d-c)\gamma_+} - e^{(d-c)\gamma_-}}\ ,\ z \in [c,d]\,,$$

where $c < 0 < d$, and $\gamma_+ > 0,\ \gamma_- < 0$ are solutions of $\gamma^2 + 2m\gamma - 2\rho = 0$. Hence $g_{]-\infty,d[}(z) = de^{(z-d)\gamma_+}$ and $g'_{]-\infty,d[-}(d) = d\gamma_+$. The equation $g'_{]-\infty,d[-}(d) = 1$ has a unique root $d_* = 1/\gamma_+$ on $]0,\infty[$. The function $g_{]-\infty,d_*[}(z)$ satisfies conditions of Theorem 1 and therefore coincides with the value function.

Example 7 (Salminen [18] (see Fig. 11)). Brownian motion Z_t with parameters $(m,1)$, killing intensity ρ and $g(z) = 1$ for $z \le 0$ and $g(z) = 2$ for $z > 0$.

Fig. 11 Example 7



According to Remark 2 we have the same value function if we set $g(0) = 2$. By Lemma 2 instead of $g(z)$ one can consider $g_{]c,0[}(z)$, where $c < 0$, $|c|$ is small enough. For this problem the operator L has a form

$$Lf(z) := f''(z) + mf'(z) - \rho f(z). \tag{17}$$

Since $Lg_{]c,0[}(z) \le 0$ for all z, we are under conditions of Theorem 2 with $k = 2$, $z_1 = c,\ z_2 = 0,\ i = 1$. It follows from Lemma 1 and Theorem 2 that instead of $g(z)$ one can consider $g_{]c_*,d_*[}(z)$, where according to statement (a) of Theorem 3 $d_* = 0$, and according to statements (b1) of Theorem 3 either the equation $g'_{]c,\,0[+}(c) = 0$

has a unique root in $]-\infty,0[$ and c_* coincides with this root, or $c_* = -\infty$ and in this case $g'_{]c,0[+}(c) > 0$ for all $c \in]-\infty,0[$. It follows from (17) and Statement 1 that

$$g_{]c,0[}(z) = \frac{(1-2e^{c\gamma_+})\,e^{z\gamma_-} - (1-2e^{c\gamma_-})\,e^{z\gamma_+}}{e^{c\gamma_-} - e^{c\gamma_+}}, \quad z \in [c,0], \quad c < 0\,, \tag{18}$$

where $\gamma_+>0$, $\gamma_-<0$, are solutions of $\gamma^2 - m\gamma - \rho = 0$. It follows from (18) that $\left(1-e^{c(\gamma_+-\gamma_-)}\right) g'_{]c,0[}(c) = f(c)$ where $f(c) = \gamma_-\,(1-2e^{c\gamma_+}) + \gamma_+ e^{c\gamma_+}\,(2-e^{-c\gamma_-})$ and, hence, $\lim_{c\downarrow-\infty} g'_{]c,0[}(c) = \gamma_- < 0$. So, there exists a unique c^* such that $f(c^*) = 0$. The function $g_{]c_*,0[}(z)$ satisfies conditions of Theorem 1 and therefore coincides with the value function.

Example 8 (Dayanik and Karatzas [3]). Standard Brownian motion Z_t on $]0,\infty[$ with parameters (0, 1), killing intensity ρ, absorbtion at $z = 0$ and $g(z) = z^p$ (see Fig. 12).

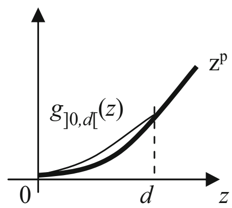


Fig. 12 Example 8

For this problem the operator L has a form

$$Lf(z) = (1/2)\,f''(z) - \rho f(z)\,, \tag{19}$$

and $Lg(z) = z^{p-2}\left(p(p-1) - \rho z^2\right)$.

If $p \le 1$ then $Lg(z) < 0$ for all $z > 0$ and therefore $\tau^* \equiv 0$.

If $p > 1$ then $Lg(z) > 0$ for $0 \le z < d_1 := \sqrt{p(p-1)/\rho}$. According to Lemmas 1 and 3 we can change $g(z)$ to $g_{]0,d_1[}(z)$, where $L_1 g_{]0,d_1[}(d_1) > 0$. With the payoff function $g_{]0,d_1[}(z)$ we are under conditions of Theorem 2, with $k = 1, z_1 = d_1$. It follows from Lemma 1 and Theorem 2 that instead of $g_{]0,d_1[}(z)$ one can consider $g_{]0,d_*[}(z)$, where according to statement (b2) of Theorem 3 either the equation $L_1 g_{]0,\,d[}(d) := pd^{\,p-1} - g'_{]0,\,d[-}(d) = 0$ has a unique root d_* in $]d_1,\infty[$ and d_* coincides with this root, or $d_* = \infty$ and in this case $L_1 g_{]0,\,d[}(d) > 0$ for all $d \ge d_1$. It follows from (19) and Statement 1 that

$$g_{]0,d[}(z) = \frac{e^{z\sqrt{2\rho}} - e^{-z\sqrt{2\rho}}}{e^{d\sqrt{2\rho}} - e^{-d\sqrt{2\rho}}}\,d^{\,p} \quad \text{for } z \in [0,d]. \tag{20}$$

Hence $L_1 g_{]0,d[}(d) = pd^{p-1} - \dfrac{1+e^{-2d\sqrt{2\rho}}}{1-e^{-2d\sqrt{2\rho}}} d^p \sqrt{2\rho}$. Since $L_1 g_{]0,d[} \to -\infty$ as $d \to \infty$ we have that d^* is finite and satisfies $e^{-2d\sqrt{2\rho}} = \dfrac{p - d\sqrt{2\rho}}{p + d\sqrt{2\rho}}$. The function $g_{]0,d*[}(z)$ (see (20)) satisfies conditions of Theorem 1 and therefore coincides with the value function.

Example 9 *(*Karatzas and Ocone [8]). Brownian motion Z_t on $]0,\infty[$ with parameters $(-m, 1)$, killing intensity ρ, absorbtion at $z = 0$, and $g(z) = -\delta z^2$, $c(z) = z^2$ $V(z) := \sup_{\tau \geq 0} E_z \left[-\delta Z_\tau^2 - \int\limits_0^\tau Z_t^2 dt \right]$.

Let $V(z, m, \rho, \delta)$ be the value function corresponding to the problem. Considering the process $\tilde{Z}_t = m Z_{t/m^2}$ we obtain that

$$V(z, m, \rho, \delta) = \frac{1}{m^4} V\left(mz, 1, \frac{\rho}{m^2}, \delta m^2\right).$$

So, in what follows we shall assume without restriction of generality that $m = 1$.

For the problem with $m = 1$ the operator L has a form

$$Lf(z) := (1/2)\, f''(z) - f'(z) - \rho f(z) - z^2, \tag{21}$$

and $Lg(z) = \delta\varphi(z)$ where $\varphi(z) = xz^2 + 2z - 1,\ x = \rho - \dfrac{1}{\delta} \in]-\infty, \rho[$.

If $x \leq -1$ then $\varphi(z) \leq 0$ for all z and by Theorem 1 $\tau^* \equiv 0$ and $V(z) = g(z)$.

If $-1 < x < \rho$ then there exist $\gamma_1 = \dfrac{-1+\sqrt{1+x}}{x}$ and $\gamma_2 = \dfrac{-1-\sqrt{1+x}}{x}$ such that $\varphi(z)$ change sign at these points.

It remains to consider two cases: **(1)** $0 \leq x < \rho$, **(2)** $-1 < x < 0$.

(1) $0 \leq x < \rho$. In this case $\varphi(z) < 0$ for $z \in [0, \gamma_1[$ and $\varphi(z) > 0$ for $z > \gamma_1$. By Lemma 1 and Lemma 3 we can change $g(z)$ to $g_{]\gamma_1,\infty[}(z)$ with $L_1 g_{]\gamma_1,\infty[}(\gamma_1) > 0$. It follows from Theorem 2 and Lemma 1 with $A = \{z_1 = \gamma_1\}$, that there exists $c_* \in [0, \gamma_1[$ such that $V(z) = g_{]c_*,\infty[}(z)$, where either $L_1 g_{]c,\infty[}(c) > 0$ for all $c \in]0, \gamma_1[$ and in this case $c_* = 0$, or c_* is a unique root of $L_1 g_{]c,\infty[}(c) = 0$ (see Fig. 13a).

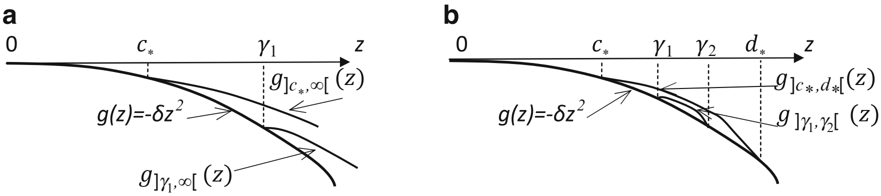


Fig. 13 Example 9 (**a**) $0 \leq \rho - \frac{1}{\delta} < \rho$ and (**b**) $-1 \leq \rho - \frac{1}{\delta} < 0$

(2) $-1 < x < 0$. In this case $\varphi(z) > 0$ for $z \in]\gamma_1, \gamma_2[$, $\varphi(z) < 0$ for $z \notin]\gamma_1, \gamma_2[$. Applying Lemma 3 and Lemma 1 to $g(z)$, then Theorem 2 and Lemma 1 to $g_{]\gamma_1,\gamma_2[}(z)$ at first to the point γ_1 and then to the point γ_2, and then Theorem 1, we obtain that $V(z) = g_{]c_*,d_*[}(z)$ for some $c_* \in [0, \gamma_1[$ and $d_* \in]\gamma_2, \infty]$ (see Fig. 13b).

We shall find now c_* and d_* for both cases **(1)** and **(2)**.

Let $P(z) = -\frac{1}{\rho}z^2 + \frac{2}{\rho^2}z - \frac{1}{\rho^2} - \frac{2}{\rho^3}$, $\quad P_1(z) = g(z) - P(z)$, and $\kappa_+ > 0, \kappa_- < 0$ be the roots of the equation $\kappa^2 - 2\kappa - 2\rho = 0$. Then $LP(z) = 0$, and according to Statement 1 and (21)

$$g_{]c,d[}(z) = P(z) + P_1(d)\frac{e^{\kappa_+(z-c)} - e^{\kappa_-(z-c)}}{e^{\kappa_+(d-c)} - e^{\kappa_-(d-c)}} + P_1(c)\frac{e^{\kappa_-(z-d)} - e^{\kappa_+(z-d)}}{e^{\kappa_-(c-d)} - e^{\kappa_+(c-d)}}, \quad z \in]c, d[. \tag{22}$$

(1) $0 \le x < \rho$. It follows from (22) that in this case

$$g_{]c,\infty[}(z) = P(z) + P_1(c)e^{\kappa_-(z-c)} \text{ for } z > c, \; c \in]0, \gamma_1[, \tag{23}$$

and hence $L_1 g_{]c,\infty[}(c) = -P_1'(c) + \kappa_- P_1(c) := -P_-(c)$. It is simple to check that

$$-\rho^3 P_-(0) = 2\rho + \left(1 - \sqrt{1+2\rho}\right)(2+\rho) < 0.$$

Due to statement (b1) of Theorem 3, the equation $\kappa_- P_1(c) = P_1'(c)$ has a unique positive root, c_* coincides with this root and $V(z)$ is given by (23) with $c = c_*$.

(2) $-1 < x < 0$. It follows from (22) that in this case

$$\lim_{d\to\infty} \frac{L_1 g_{]c,d[}(d)}{P_1(d)} = \lim_{d\to\infty} \frac{g'(z) - g'_{]c,d[-}(d)}{P_1(d)} = -\kappa_+ \text{ as } d \to \infty.$$

Since $P_1(z)/z^2 \to \frac{1}{\rho} - \delta > 0$, using statement (b2) of Theorem 3 we have that $d^* < \infty$. To find c_* and d_* we shall use statement (c) of Theorem 3. It is convenient to define $P_\pm(z) = P_1'(z) - \kappa_\pm P_1(z)$. Let

$$f(d,z) = P(z) + \frac{P_-(d)}{\kappa_+ - \kappa_-}e^{-\kappa_+(d-z)} - \frac{P_+(d)}{\kappa_+ - \kappa_-}e^{-\kappa_-(d-z)}. \tag{24}$$

Since $f(d,z)$ satisfies (21), $f(d,d) = g(d)$, $\frac{\partial f}{\partial z}(d,d) = g'(d)$, we obtain that $f(d,z)$ concides with the generalized tangent line $g_d(z)$ and if $g_d(c) = g(c)$ for some $c < d$ then $g_{]c,d[}(z) = f(d,z)$ for $z \in]c, d[$.

According to statement (c) of Theorem 3, if $f(\gamma_2, 0) > 0$ then $c_* > 0$ and, due to statement (b) of Theorem 3, c_* and d_* are the unique solution of the system

$$g(c) = f(d, c), \quad L_1 g_{]c,d[}(c) = 0. \tag{25}$$

The first equality can be rewritten in the form

$$P_+(d)e^{-\kappa_-(d-c)} = P_-(d)e^{-\kappa_+(d-c)} - (\kappa_+ - \kappa_-)P_1(c)\,. \tag{26}$$

Using first (24) and then (26) we obtain

$$\begin{aligned} L_1 g_{]c,d[}(c) &= \frac{\partial f}{\partial z}(d,c) - g'(c) = \frac{\kappa_+ P_-(d)e^{-\kappa_+(d-c)} - \kappa_- P_+(d)e^{-\kappa_-(d-c)}}{\kappa_+ - \kappa_-} - P_1'(c) \\ &= P_-(d)e^{-\kappa_+(d-c)} - P_-(c). \end{aligned} \tag{27}$$

Using (27) and substituting equality $P_-(d)e^{-\kappa_+(d-c)} = P_-(c)$ into (26) we obtain that the system (25) for c_*, d_* can be rewritten in the form

$$P_-(d)e^{-\kappa_+(d-c)} = P_-(c), \quad P_+(d)e^{-\kappa_-(d-c)} = P_+(c). \tag{28}$$

If $f(\gamma_2, 0) \leq 0$ then, according to statement (c1) of Theorem 3, for each $c \in [0, \gamma 2[$ there exists $d = d_c \in]\gamma_2, \infty[$, such that equality (26) holds. For these c and d_c equality (27) holds also.

If $P_-(d_0)e^{-\kappa_+(d_0)} \geq P_-(0)$, where d_0 is defined from (26) with $c = 0$, then due to statement (c2) of Theorem 3, $c_* = 0$ and $d_* = d_0$. If $P_-(d_0)e^{-\kappa_+(d_0)} < P_-(0)$ then $c_* > 0$, and c_*, d_* are defined from (28).

Example 10 *(*Dayanik and Karatzas [3]). An optimal stopping problem for a mean-reverting diffusion: A diffusion on $]0, \infty[$ with $m(z) = z\mu(\alpha - z)$, $\sigma(z) = z\sigma$ for some $\alpha, \mu, \sigma > 0$, with killing intensity ρ, $g(z) = (z - K)^+$, $K > 0$ (see Fig. 14).

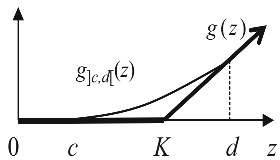


Fig. 14 Example 10

For this problem the operator L has a form

$$Lf(z) := \left(\sigma^2 z^2/2\right) f''(z) + z\mu(\alpha - z) f'(z) - \rho f(z), \tag{29}$$

and $Lg(z) := \varphi(z) = -\mu z^2 + (\alpha\mu - \rho)z + \rho K$ for $z > K$, $Lg(z) = 0$ for $0 < z < K$. Let γ be the positive root of the equation $\varphi(z) = 0$. Then $Lg(z) < 0$ for $z > \max[K, \gamma]$, $Lg(z) > 0$ for $K < z < \max[K, \gamma]$.

Let $d_1 = \max[K, \gamma]$. So, the set A consists of two or one point: K, d_1. According to Lemmas 1 and 3 we can change $g(z)$ to $g_{]K,d_1[}(z)$. With the payoff function $g_{]K,d_1[}(z)$ we are under conditions of Theorem 2. According to Statement 1 $L_1 g_{]K,d_1[}(K) = g'_{]K,d_1[+}(K) \geq g'_+(K) = 1$ and we can apply statement (a) of Theorem 3 to $z_1 = K$ and change $g_{]K,d_1[}(z)$ to $g_{]0,d_1[}(z)$.

With the payoff function $g_{]0,d_1[}(z)$ we are under conditions of Theorem 2, where the set A_1 consists of one point d_1, and according to Statement 1, $L_1 g_{]0,d_1[}(d_1) > 0$. It follows from Lemma 1 and Theorem 2 that instead of $g_{]0,d_1[}(z)$ one can consider $g_{]0,d_*[}(z)$, where according to statements (b2) of Theorem 3 either the equation $g'_{]0,\,d[-}(d) = 1 \ (= g'(d))$ has a unique root in $]d_1, \infty[$ and d_* coincides with this root, or $d_* = \infty$ and in this case $g'_{]0,\,d[-}(d) < 1$ for $d \in]d_1, \infty[$.

It is easy to check that the function $\varphi(a, b, u) = \displaystyle\int_0^1 e^{tu} t^{a-1}(1-t)^b dt$ satisfies the equation

$$u\frac{d^2}{dt^2}\varphi(a,b,u) + (a+b+1-u)\frac{d}{dt}\varphi(a,b,u) - a\varphi(a,b,u) = 0 \tag{30}$$

(it suffices to differentiate under the sign of the integral, substitute the results in (30) and then integrate by parts the term $\displaystyle\int_0^1 t^a(1-t)^{b+1} de^{tu}$). Let $\theta_+ > 0,\ \theta_- < 0$, be the roots of the equation $\theta(\theta-1) + \alpha\dfrac{2\mu}{\sigma^2}\theta - \dfrac{2\rho}{\sigma^2} = 0$. It follows from (30) that

$$\psi(z) = \left(z\frac{2\mu}{\sigma^2}\right)^{\theta_+} \varphi\left(\theta_+, -\theta_-, z\frac{2\mu}{\sigma^2}\right) = \left(z\frac{2\mu}{\sigma^2}\right)^{\theta_+} \int_0^1 e^{tz\frac{2\mu}{\sigma^2}} t^{\theta_+ -1}(1-t)^{-\theta_-} dt \tag{31}$$

satisfies the equation $L\psi(z) = 0$ (see (29)) with the boundary condition $\psi(0) = 0$. Hence, according to Statement 1,

$$g_{]0,d[}(z) = (d-K)\psi(z)/\psi(d) \ \text{ for } d > K,\ z \in [0, d]\,. \tag{32}$$

It is easy to check that $g'_{]0,d[-}(d) = (d-K)\psi'(d)/\psi(d) > 1$ if d is large enough, and $d_* < \infty$ is defined as the unique root of the equation $(d-K)\psi'(d)/\psi(d) = 1$. The function $g_{]0,d_*[}(z)$ (see (31) and (32)) satisfies conditions of Theorem 1 and therefore coincides with the value function.

Example 11 *(*Oksendal and Reikvam [12]). Brownian motion Z_t on $]-\infty, \infty[$ with parameters (0, 1), killing intensity ρ, and $g(z) = 1$ for $z \leq 0$; $g(z) = 1 - hz$ for $0 < z < a$; $g(z) = 1 - ha > 0$ for $z \geq a$ (see Fig. 15).

For this problem the operator L has a form

$$Lf(z) = (1/2)f''(z) - \rho f(z). \tag{33}$$

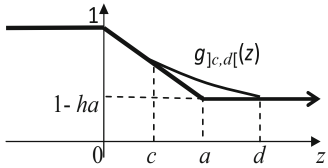


Fig. 15 Example 11

Since $Lg(z) < 0$ for all z, we are under conditions of Theorem 2, where A consists of two points: $z_1 = 0$ with $L_1 g(q) = g'_+(0) - g'_-(0) = -h < 0$, and $z_2 = a$ with $L_1 g(a) = g'_+(a) - g'_-(a) = h > 0$, i.e. $i = 2$. It is evident that $g_{]0,d[}(z) < 1$ for any $z \in]0, d]$, $d > 0$. Hence $L_1 g_{]0,d[}(0) \geq 0$ for any $d \geq a$. It follows from Lemma 1 and Theorem 2 that instead of $g(z)$ one can consider $g_{]c_*,d_*[}(z)$, where $c_* \in [0, a[$, $d_* \in]a, \infty]$. It follows from (33) and Statement 1 that

$$g_{]c,d[}(z) = \frac{g(c)\sinh((d-z)\mu) + g(d)\sinh((z-c)\mu)}{\sinh((d-c)\mu)} \quad \text{for } z \in [0, d], \tag{34}$$

where $\mu = \sqrt{2\rho}$, so that

$$g'_{]c,d[-}(d) = \mu\frac{g(d)\cosh((d-c)\mu) - g(c)}{\sinh((d-c)\mu)}, \tag{35}$$

$$g'_{]c,d[+}(c) = \mu\frac{g(d) - g(c)\cosh((d-c)\mu)}{\sinh((d-c)\mu)}. \tag{36}$$

The relations (35)–(36) yield that $L_1 g_{]c,d[}(d) = -g'_{]c,d[-}(d) \to -(1-ha) < 0$ for any c, and $L_1 g_{]c,d[}c) = -g'_{]c,d[+}(c) < 0$ for $c < 0$. Hence, due to statement (b) of Theorem 3, d_* is finite and $c_* \in [1, a[$.

Consider now for $c \in [1, a[$ the function

$$f(c, z) = (1-ha)\cosh((d(c) - z)\mu), \tag{37}$$

where $d(c) > c$ is chosen from the condition $f(c, c) = g(c) = 1 - hc$, so that

$$d(c) = c + \ln\frac{1 - hc + \sqrt{h(a-c)(2-h(a-c))}}{1-ha}. \tag{38}$$

It follows from (33) and Statement 1 that $g_{]c,d(c)[}(z) = f(c, z)$ for $1 \leq c \leq z \leq d(c)$, and therefore, using (37) and equality $f(c, c) = 1 - hc$ we obtain

$$g'_{]c,d(c)[+}(c) = \mu(1-ha)\sqrt{(\cosh((d(c)-c)\mu))^2 - 1} = \mu\sqrt{(1-hc)^2 - (1-ha)^2},$$

and hence $\lim_{c\downarrow 0} L_1 g'_{]c,d(c)[+}(c) = \sqrt{2\rho(1-(1-ha)^2)} - h$.

If $\rho \leq \dfrac{h}{a(2-ha)}$ then $\lim_{c\downarrow 0} L_1 g'_{]c,d(c)[+}(c) \geq 0$, and according to statement (c) of Theorem 3 we have $c_* = 0$. If $\rho > \dfrac{h}{a(2-ha)}$ then $\lim_{c\downarrow 0} L_1 g'_{]c,d(c)[+}(c) < 0$, and according to statement (c) of Theorem 3 the value c_* is defined from the conditions $f'(c,c) = -h, f(c,c) = 1 - hc$, so that $c_* = h^{-1}\left(1 + \sqrt{(1-ha)^2 + h/(2\rho)}\right)$. The value d_* is defined from (38) for $c = c_*$.

The respective function $g_{]c*,b[}(z)$, where $g_{]c*,b[}(z) = f(c^*, z)$ for $c_* \leq z \leq d_*$ (see (37)), satisfies conditions of Theorem 1 and therefore coincides with the value function.

It is written in [3]: "Oksendal and Reikvam show that the value function of an optimal stopping problem is the unique viscosity solution to the relevant variational inequalities under suitable assumptions. Later, they use their results to solve the problem by further assuming that $\frac{4\rho a}{1+2\rho a^2} < c$. The origin turns out to be one of the boundaries of the optimal continuation region and since the reward function $g(z)$ is not differentiable at the origin, using smooth-fit principle would not give the solution. They also point out that the solution could not have been found by using the verification lemma in Brekke and Oksendal (1998), either".

In our construction we do not need the notion of viscosity solution, variational inequalities, smooth-fit principle, and verification theorems. We just use Lemma 1 and Theorems 2 and 3 for the constructive construction of the value function.

5 Other Examples

Example 12. Geometric Brownian motion Z_t on $[1; \infty]$ with parameters $(-m, \sigma)$, killing intensity ρ, reflection at $z = 1$ and functional $E_z[x_\tau]$. i.e. $g(z) = z$ (see Fig. 16).

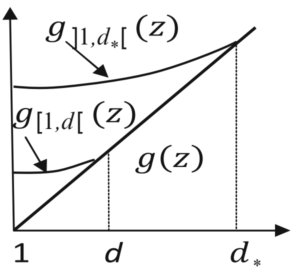


Fig. 16 Example 12

This example corresponds to the Russian option (see [13], Sect. 26).

For this problem the operator L has a form

$$Lf(z) := \left(\sigma^2 z^2/2\right) f''(z) - mzf'(z) - \rho f(z). \tag{39}$$

Since $Lg(z) = -(m + \rho)z < 0$, we are under conditions of Lemma 7. According to this Lemma instead of $g(z)$ we can consider $g_{[1,d_*[}(z)$, where either the equation $L_1 g_{[1,d[}(d) = 0$ has a unique root in $]1, \infty[$ and d_* coincides with this root, or $d_* = \infty$ and in this case $L_1 g_{[1,d[}(d) \geq 0$ for all $d > 1$.

Let $\kappa_+ > 1$ and $\kappa_- < 0$ be the solutions of $\sigma^2\kappa^2 - \left(\sigma^2 + 2m\right)\kappa - 2\rho = 0$. Due to Statement 1 and (39)

$$g_{[1,d)}(z) = \frac{d\ (\kappa_+ z^{\kappa_-} - \kappa_- z^{\kappa_+})}{\kappa_+ d^{\kappa_-} - \kappa_- d^{\kappa_+}} \text{ for } z \in [1, d)\,,$$

and hence $L_1 g_{[1,d)}(d) = 1 - \kappa_+\kappa_- \dfrac{1 - d^{\kappa_+ - \kappa_-}}{\kappa_+ - \kappa_- d^{\kappa_+ - \kappa_-}} \to 1 - \kappa_+ < 0$ as $d \to \infty$. From here we obtain that d_* is finite and coincides with the unique solution of the equation $L_1 g_{[1,d)}(d) = 0$, i.e. $d_* = \left(\dfrac{\kappa_-(1 - \kappa_+)}{\kappa_+(1 - \kappa_-)}\right)^{1/(\kappa_+ - \kappa_-)}$. The function $g_{[1,d_*[}(z)$ satisfies conditions of Theorem 1 and therefore coincides with the value function.

Example 13. The same process and the same functional as in Example *9*, but with $\rho = 0$. Just as in the Example *9* we shall assume without restriction of generality that $m = 1$. Note , that $E\left[\int_0^t Z_s^2 ds\right] \to \infty$ as $t \to \infty$ in this case and it is impossible to change the reward function, as it was done in [3] for $\rho > 0$.

For this problem with $m = 1$ the operator L has a form

$$Lf(z) := (1/2)\, f''(z) - f'(z) - z^2, \tag{40}$$

and $Lg(z) = -\delta + 2\delta z - z^2$.

If $\delta \leq 1$ then $Lg(z) \leq 0$ for all z and by Theorem 1 $\tau^* \equiv 0$ and $V(z) = g(z)$.

If $\delta > 1$ then $Lg(z) > 0$ for $z \in]\gamma_-, \gamma_+[$ and $Lg(z) < 0$ for $z \notin [\gamma_-, \gamma_+]$, where $\gamma_\pm = \delta \pm \sqrt{\delta^2 - \delta}$. Applying Lemmas 3 and 1 to $g(z)$, then Theorems 2 and 3 and Lemma 1 to $g_{]\gamma_-,\gamma_+[}(z)$ at first to the point γ_- and then to the point γ_+, and then Theorem 1, we obtain that $V(z) = g_{]c_*,d_*[}(z)$ for some $c_* \in [0, \gamma_-[$ and $d_* \in]\gamma_+, \infty]$ (see Fig. 13b).

We shall find now c_* and d_*. Let

$$P(z) = -\frac{1}{6}z(z^2+3z+3), \quad P_1(z) = g(z) - P(z) = \frac{1}{6}z(2z^2+3(1-2\delta)z+3). \tag{41}$$

Then $LP(z) = 0$, and according to Statement 1 and (40)

$$g_{]c,d[}(z) = P(z) + P_1(d)\frac{z - ce^{2(z-c)}}{d - ce^{2(d-c)}} + P_1(c)\frac{z - de^{2(z-d)}}{c - de^{2(c-d)}},\ \ z \in]c, d[. \tag{42}$$

It follows from (42) that

$$\lim_{d\to\infty}\frac{L_1 g_{]c,d[}(d)}{P_1(d)}=\lim_{d\to\infty}\frac{g'(d)-g'_{]c,d[-}(d)}{P_1(d)}=-2 \text{ as } d\to\infty\,.$$

Since $P_1(z)/z^3 \to \dfrac{1}{6} > 0$, using statement (b2) of Theorem 3 we get that $d^* < \infty$. To find c_* and d_* we shall use statement (c) of Theorem 3. It is convenient to define two functions

$$P_-(z)=\frac{zP_1'(z)-P_1(z)}{2z-1}=\frac{z^2}{6(2z-1)}(4z+3(1-2\delta)), \tag{43}$$

$$P_+(z)=\frac{P_1'(z)-2P_1(z)}{2z-1}=\frac{1}{6(2z-1)}(-4z^3+12\delta z^2-12\delta z+3). \tag{44}$$

Let

$$f(d,z)=P(z)+P_-(d)e^{2(z-d)}-P_+(d)z. \tag{45}$$

Since $f(d,z)$ satisfies (40), $f(d,d)=g(d)$, $\dfrac{\partial f}{\partial z}(d,d)=g'(d)$, we obtain that $g_d(z)=f(d,z)$ and if $g_d(c)=g(c)$ for some $c<d$ then $g_{]c,d[}(z)=f(d,z)$ for $z\in]c,\,d[$.

We shall show now that $c_*>0$. Indeed, according to (45)

$$f(\gamma_+,0)=P_-(\gamma_+)e^{-2\gamma_+}=\frac{\gamma_+^2}{6}e^{-2\gamma_+}\left[4\left(\delta+\sqrt{\delta^2-\delta}\right)-3(2\delta-1)\right].$$

If $\delta>\delta_1=\dfrac{1+\sqrt{23}}{6}\approx 1{,}0486$ then $g_{\gamma_+}(0)=f(\gamma_+,0)>0$ and, according to statement (c1) of Theorem 3, $c_*>0$.

Consider now the case $\delta\in]1,\delta_1[$, Then $g_{\gamma_+}(z)=f(\gamma_+,0)\le 0$, and according to statement (c2) of Theorem 3 there exists $d_0\ge\gamma_+$ such that $f(d_0,0)=g(0)=0$. It follows from (45) and (43) that

$$P_-(d_0)=0, \text{ so that } d_0=\frac{3}{4}(2\delta-1). \tag{46}$$

Consider $L_1=\lim_{c\downarrow 0} g_{]c,d_0[}(c)=\dfrac{\partial}{\partial z}f(d_0,0)$. It follows from (45) and (43), first equality in (46) and (44), and the second equality in (46) that

$$\begin{aligned}2(2d_0-1)L_1&=-1-2d_0-2P_+(d_0)=2(2\delta-1)d_0-4\delta d_0^2+\frac{4}{3}d_0^3\\&=\frac{8}{3}d_0^2-4\delta d_0^2+\frac{4}{3}d_0^3=\frac{d_0^2}{8}(5-6\delta)<0 \text{ for } \delta\in]1,\delta_1[.\end{aligned} \tag{47}$$

It follows from statement (c2) of Theorem 3 that $c_*>0$.

Due to statement (b) of Theorem 3, c_* and d_* are the unique solution of the system

$$g(c) = f(d,c), \quad L_1 g_{]c,d[}(c) = 0. \tag{48}$$

The first equality can be rewritten in the form

$$P_-(d)e^{2(c-d)} = cP_+(d) + P_1(c)\,. \tag{49}$$

Using sequentially (45), (49), and (45) we obtain

$$\begin{aligned} L_1 g_{]c,d[}(c) &= \frac{\partial f}{\partial z}(d,c) - g'(c) = 2P_-(d)e^{2(c-d)} - P_+(d) - P_1'(c) \\ &= (2c-1)(P_+(d) - P_+(c)). \end{aligned} \tag{50}$$

Using (50), second equality in (48), substituting equality $P_+(d) = P_-(c)$ into (49), and using (44) we obtain that the system (48) for c_*, d_* can be rewritten in the form

$$P_+(d)e^{-2d} = P_+(c)e^{-2c}, \quad P_-(d) = P_-(c).$$

Example 14. The diffusion is the same as in Example *10*. The payoff function equals $g(z) = (\psi(z) - K)^+$, where $\psi(z)$ is defined in (31).

For this problem the operator L is given by (29) and $Lg(z) = 0$ for $0 < z < d_K$, $Lg(z) = \rho K > 0$ for $z > d_K$, where d_K is the unique root of the equation $\psi(z) = K$. According to Lemmas 3 and 1 instead of $g(z)$ we can consider $g_{]d_K,\infty[}(z)$. With the payoff function $g_{]d_K,\infty[}(z)$ we are under conditions of Theorem 2, where the set A_1 consists of one point d_K, and according to statement (a) of Theorem 3 and Lemma 1 instead of $g_{]d_K,\infty[}(z)$ we can consider $g_{]0,\infty[}(z)$.

It follows from Statement 1 that $g_{]0,d[}(z) = \dfrac{(\psi(d) - K)\psi(z)}{\psi(d)}$ for $z \in [0,d]$, $d > d_K$ and hence $g_{]0,\infty[}(z) \equiv \psi(z)$. The function $\psi(z)$ satisfies conditions of Theorem 1 and therefore coincides with the value function. Note that there is no optimal stopping time in this problem.

Example 15 *(*Presman [16]). We consider a standard Wiener process w_t with an initial point $z \in (-\infty, +\infty)$ and a functional $E_z[e^{-\rho\tau} g(w_\tau)]$. Such problem is equivalent to the problem with functional $\tilde{E}_z[g(\tilde{w}_\tau)]$, where $\tilde{w}_t$ is a standard Wiener process with a killing intensity ρ. Without restriction of generality we take $\rho = 1/2$.

The differential operator corresponding to this process is

$$Lf(z) = (1/2)f''(z) - (1/2)f(z)\,. \tag{51}$$

We shall consider the case, when $g(0) = 0$, $Lg(z) < 0$ for $z \neq 0$, $g'_+(0) = h > 0$, $g'_-(0) = f < 0$, and $\lim_{d\to\infty}(g'(d) - g(d)) < 0$, $\lim_{c\to-\infty}(g(c) - g'(c)) < 0$. The payoff function $g(z) = fz$ for $z \leq 0$ and $g(z) = hz$ for $z \geq 0$ satisfies these relations.

Under these assumptions we are under conditions of Theorem 2 were A consists of one point $z_1 = 0$ with $L_1 g(0) = h - f > 0$ and according to this theorem and Lemma 1 there exist $c_* \in [-\infty, 0[,\ d_* \in]0, \infty]$ such that instead of $g(z)$ we can consider $g_{]c_*,d_*[}(z)$. Moreover the function $g_{]c_*,d_*[}(z)$ satisfies the conditions of Theorem 1 and hence $V(z) = g_{]c_*,d_*[}(z)$. So, we need just to construct the values c_*, d_*.

It follows from (51) and Statement 1 that

$$g_{]c,d[}(z) = g(d)\frac{\sinh(z-c)}{\sinh(d-c)} + g(c)\frac{\sinh(d-z)}{\sinh(d-c)} \quad \text{for } z \in (c,d). \tag{52}$$

From here and our assumptions we obtain that

$$\lim_{d\to\infty} L_1 g_{]c,d[}(d) = \lim_{d\to\infty}(g'(d) - g'_{]c,d[-}(d)) = \lim_{d\to\infty}(g'(d) - g(d)) < 0$$

and similarly $\lim_{c\to-\infty} L_1 g_{(c,d)}(c) < 0$. It follows from here and statement (b) of Theorem 3 that in our case the values c_* and d_* are finite and are the roots of the system of equations $L_1 g_{]c,d[}(d) = 0$, $L_1 g_{]c,d[}(c) = 0$, which can be written in the form

$$g(d) - g(c)\cosh(d-c) = g'(c)\sinh(d-c)\,, \tag{53}$$

$$g(d)\cosh(d-c) - g(c) = g'(d)\sinh(d-c)\,. \tag{54}$$

The inequalities (53)–(54) can be considered as linear equations with respect to $\cosh(d-c)$ and $\sinh(d-c)$. So, there are two excluding each other possibilities.

(a) $g(d)g'(c) + g(c)g'(d) = 0$.

It follows from (53)–(54) that in this case $g(d_*) = g(c_*)$, $g'(d_*) = -g'(c_*)$ and

$$g(d_*)(1 - \cosh(d_* - c_*)) = -g'(d_*)\sinh(d_* - c_*). \tag{55}$$

(b) $g(d)g'(c) + g(c)g'(d) = 0$.

In this case the system (53)–(54) can be rewritten as

$$g(d)g'(d) + g(c)g'(c) = (g(d)g'(c) + g(c)g'(d))\cosh(d-c)\,, \tag{56}$$

$$g^2(d) - g^2(c) = (g(d)g'(c) + g(c)g'(d))\ \sinh(d-c)\,, \tag{57}$$

and therefore $g(d_*) \neq g(c_*)$. Using the equality $\cosh^2(x) - \sinh^2(x) = 1$ the system (56)–(57) can be rewritten as

$$(g(d)g'(d) + g(c)g'(c))^2 - (g^2(d) - g^2(c))^2 = (g(d)g'(c) + g(c)g'(d))^2\,, \tag{58}$$

$$g^2(d) - g^2(c) = (g(d)g'(c) + g(c)g'(d))\ \sinh(d-c)\,. \tag{59}$$

Equation (58) can be represented in the form

$$\left(g^2(d)-g^2(c)\right)\left[\left(g^2(d)-g^2(c)\right)-\left((g'(d))^2-(g'(c))^2\right)\right]=0\,. \qquad (60)$$

So, in case (b) the system (53)–(54) is equivalent to

$$g^2(d)-g^2(c)=(g'(d))^2-(g'(c))^2 \;=(g(d)g'(c)+g(c)g'(d))\,\sinh(d-c)\,. \quad (61)$$

Consider now the case $g(z)=hz$ for $z<0$, $g(z)=fz$ for $z<0$.

If $h=-f$ then, due to the symmetry, $c_*=-d_*$ and we are under conditions (a), and in accordance with the equality (55), d_* is the root of the equation $d_*(\cosh(2d_*-1)=\sinh(2d_*)$, which does not depend on the specific value of f.

If $h\neq -f$, the solution $hd=fc$ of (60) contradicts to (59). So the optimal values c^*,d^* are the roots of the system

$$h^2d^2-f^2c^2=h^2-f^2=hf(d+c)\sinh(d-c)\,, \qquad (62)$$

Solving the first equation in (62) with respect to c and substituting the result into the second we obtain the equation with respect to d^* which has a unique positive solution.

Example 16. Brownian motion Z_t on the interval $[a,b], a<0<b$, with the absorbtion at points a and b, and a partial reflection with a coefficient α, $0<|\alpha|<1$, at the point 0, and the functional $E_z\left[g(Z_\tau)\right]$, where $g(z)$ is a twice differentiable function defined for all $z\geq a$ and such that $g''(z)<0$. The function $g(z)=\sqrt{z-a_1}$ with $a_1\leq a$ satisfies this conditions.

For this problem the operator L has a form $Lf(z)=f''(z)$. Since $Lg(z)<0$, $L_1g(0)=2\alpha g'(0)$ we have that if $\alpha g'(0)\leq 0$, then, by Theorem 1, $\tau_*=0$ and $V(z)=g(z)$.

If $\alpha g'(0)>0$, we are under conditions of Theorem 2, where the set A consists of one point $z_1=0$. It follows from Theorem 2 and Lemma 1 that we can change $g(z)$ to $g_{]c_*,d_*[}(z)$, $c_*\in[a,0[$, $d_*\in]0,b]$, which satisfies conditions of Theorem 1 and hence $V(z)=g_{]c_*,d_*[}(z)$. The values c_* and d_* can be found from Theorem 3 using function

$$g_d(z)=\begin{cases} g(d)+(z-d)g'(d) & \text{for } z>0,\\ g(d)-dg'(d)+z\dfrac{1+\alpha}{1-\alpha}g'(d) & \text{for } z<0.\end{cases} \qquad (63)$$

$$g^c(z)=\begin{cases} g(c)+(z-c)g'(c) & \text{for } z<0,\\ g(c)-cg'(d)+z\dfrac{1-\alpha}{1+\alpha}g'(c) & \text{for } z>0.\end{cases} \qquad (64)$$

Consider the case $g(z)=\sqrt{z-a_1}$. Then for $c\in]a_1,0[$ the equation $g_{d_c}(c)=g(c)$ can be written as $\sqrt{d_c-a_1}-\dfrac{d_c}{2\sqrt{d_c-a_1}}+\dfrac{(1+\alpha)c}{(1-\alpha)2\sqrt{d_c-a_1}}=\sqrt{c-a_1}$, or as

$$(d_c-a_1)(1-\alpha)+2a_1\alpha+(c-a_1)(1+\alpha)=2(1-\alpha)\sqrt{(c-a_1)(d_c-a_1)}. \quad (65)$$

The equation $g'_{d_c}(c)=g'(c)$ can be written as

$$\frac{1+\alpha}{(1-\alpha)2\sqrt{(d_c-a_1)}}=\frac{1}{2\sqrt{c-a_1}}. \quad (66)$$

Substituting d_c-a_1 from (66) into (65) we obtain that the system (65)–(66) has a unique solution $\bar{c}=\frac{2\alpha}{1+\alpha}a_1,\ \bar{d}=-\frac{2\alpha}{1+\alpha}a_1$. If $a\le\bar{c},\ b\ge\bar{d}$ then $c_*=\bar{c},\ d_*=\bar{d}$.

Let $\bar{c}<a<0$. It is simple to check that then $c_*=a$, and: if $b\ge d_a$ then $d_*=d_a$; if $0<b\le d_a$ then $d_*=b$; where d_a is defined from (65) with $c=a$.

Let $a_1<a\le\bar{c},\ b<\bar{d}$. It is simple to check that then $d_*=b$, and: if $c_b\le a<\bar{c}$ then $c_*=a$; if $a_1\le a\le c_b$ then $c_*=c_b$; where c_b is defined from

$$(c_b-a_1)(1+\alpha)-2a_1\alpha+(b-a_1)(1-\alpha)=2(1+\alpha)\sqrt{(c_b-a_1)(b-a_1)}. \quad (67)$$

6 Proofs

Proof of Lemma 2. Consider $g_{]z_i,z_{i+1}[}(z)$. If $g_{]z_i,z_{i+1}[}(z)>g(z)$ for all $z\in]z_i,z_{i+1}[$ then the lemma is proved. Otherwise, by the continuity of $g(z)$ and $g_{]z_i,z_{i+1}[}(z)$ on $]z_i,z_{i+1}[$ there exists $\varepsilon\in]z_i,z_{i+1}[$ such that $g_{]z_i,z_{i+1}[}(z)>g(z)$ for all $z\in]z_i,\varepsilon[$ and $g_{]z_i,z_{i+1}[}(\varepsilon)=g(\varepsilon)$. But due to Statement 1, $g_{]z_i,z_{i+1}[}(z)=g_{]z_i,\varepsilon[}(z)$ for all $z\in]z_i,\varepsilon[$.

Proof of Lemma 3. Set $Lg(z)=\tilde{c}(z),\ h(z)=g_{]z_i,z_{i+1}[}(z)-g(z)$. Then $h(z)$ satisfies on $]z_i,z_{i+1}[$ the equation

$$\frac{\sigma^2(z)}{2}\frac{d^2}{dz^2}h(z)+m(z)\frac{d}{dz}h(z)-\rho(z)h(z)+\tilde{c}(z)=0 \quad (68)$$

with boundary conditions $h(z_i)=h(z_{i+1})=0$ and therefore, due to Statement 1, $h(z)=E_z\left[\int_0^{\tau_{]z_i,z_{i+1}[}}\tilde{c}(Z_s)ds\right]$. If $z\in]z_i,z_{i+1}[$ then $Z_s\in]z_i,z_{i+1}[$ for $0\le s<\tau_{]z_i,z_{i+1}[}$.

Since $\tilde{c}(z)>0$ for $z\in]z_i,z_{i+1}[$ we get that $h(z)>0$. Due to Statement 1, $Lg_{]z_i,z_{i+1}[}(z)=0$.

Set $\bar{z}_i=(z_i+z_{i+1})/2,\ h_1(z)=g_{]z_i,\bar{z}_i[}(z)-g(z)$. Just as above we get that $h_1(z)>0$ for $z\in]z_i,\bar{z}_i[,\ h_1(z_i)=0$, and therefore $h'_{1+}(z_i)\ge0$. The functions $h(z)$ and $h_1(z)$ satisfy on $]z_i,\bar{z}_i[$ the same Eq. (68), and $h(z_i)=h_1(z_i)=0,\ h(\bar{z}_i)>h_1(\bar{z}_i)=0$. Hence $h'_+(z_i)>h'_{1+}(z_i)$. Since $h'_{1+}(z_i)\ge0$ we get that $h'_+(z_i)>0$. This is equivalent to $g'_{]z_i,z_{i+1}[+}(z_i)-g'_+(z_i)>0$. The proof for the point z_{i+1} is similar.

Remark 3. Let $Lg(z) \le 0$ on $]z_i, z_{i+1}[$ for some $z_i \in A$. Then either $g_{]z_i,z_{i+1}[}(z) = g(z)$ for all $z \in]z_i, z_{i+1}[$ or there exists $w \in]z_i, z_{i+1}[$ such that $g_{]z_i,z_{i+1}[}(w) \ne g(w)$. It is evident from the proof of Lemma 3 that the result of Lemma 3 is valid for $-g(z)$ on $]z_i, z_{i+1}[$, and so in the second case $g_{]z_i,z_{i+1}[}(z) < g(z)$ for all $z \in]z_i, z_{i+1}[$ and the inequalities for the derivatives are also valid.

Proof of Lemma 4. It follows from the equalities

$$g_{]\varepsilon,\delta[}(\varepsilon) = g(\varepsilon), \quad g(\varepsilon) = g(z_i) - g'_{-}(z_i)(z_i - \varepsilon) + o(|\varepsilon - z_i|),$$

$$g_{]\varepsilon,\delta[}(\varepsilon) = g_{]\varepsilon,\delta[}(z_i) - g'_{-]\varepsilon,\delta[}(z_i)(z_i - \varepsilon) + o(|\varepsilon - z_i|)$$

that

$$g_{]\varepsilon,\delta[}(z_i) - g(z_i) = \left(g'_{]\varepsilon,\delta[-}(z_i) - g'_{-}(z_i)\right)(z_i - \varepsilon) + o(|\varepsilon - z_i|).$$

Similarly

$$g_{]\varepsilon,\delta[}(z_i) - g(z_i) = -\left(g'_{]\varepsilon,\delta[+}(z_i) - g'_{+}(z_i)\right)(\delta - z_i) + o(|\delta - z_i|).$$

Multiplying the first of this equalities by $(1-\alpha)(\delta - z_i)$, the second one by $(1+\alpha)(z_i - \varepsilon)$, adding the result and taking into account that $L_1 g_{]\varepsilon,\delta[}(z_i) = 0$ we get

$$(\varepsilon + \delta + \alpha(z_i - \varepsilon + z_i - \delta))(g_{]\varepsilon,\delta[}(z_i) - g(z_i)) = (z_i - \varepsilon)(\delta - z_i)(L_1 g(z_i) + o(1)).$$

It follows from here and $L_1 g(z_i) > 0$, that there exist ε_1 and δ_1 such that

$$g_{]\varepsilon,\delta[}(z_i) - g(z_i) > 0 \quad \text{for all} \quad \varepsilon \in]\varepsilon_1, z_i[, \ \delta \in]z_i, \delta_1[. \tag{69}$$

From the last inequality just as in the proof of Lemma 3 one obtains that $g_{]\varepsilon,\delta[}(z) - g(z) > 0$ for $z \in]\varepsilon, \delta[$. From the definition of L_1 and (69) follows that

$$L_1 g_{]\varepsilon,\delta[}(\varepsilon) = g'_{]\varepsilon,\delta[+}(\varepsilon) - g'(\varepsilon) = \frac{g_{]\varepsilon,\delta[}(z_i) - g(z_i)}{z_i - \varepsilon} + o(1) > 0.$$

The proof for the point δ is similar. It proves statement (a).

Let ε and δ are from statement (a). The function $g'_{]c,\delta[+}(c) - g'_{+}(c)$ is continuous on $[\varepsilon_2, z_i[$ and positive on $[\varepsilon, z_i]$. Then it is either positive for all $c \in [\varepsilon_2, z_i]$ or there exists c_1 such that it equals to 0 at c_1. If such a point exists then the functions $g_{]c,\delta[}(z)$ and $g(c)$ identically equal on $[c, z_i]$, since they satisfies the same differential equation and coincide at point c together with derivatives. It contradicts to the fact that $g_{]c,\delta[}(z_i) > g(z_i)$, $g'_{]c,\delta[-}(\delta) > g'_{-}(\delta)$.

Proof of Lemma 5. It suffices to consider three cases.

(a) There exist $u, v, w \in [c, d]$ such that $u < w < v$ and $f(u) = g(u)$, $f(v) = g(v)$, $f(w) \ne g(w)$.
(b) There exist $u, v \in [c, d]$ such that $u < v$ and $f(z) = g(z)$ for all $z \in [u, v]$

(c) There exists $u \in [c,d]$ such that $f(u) = g(u)$, $f(z) \neq g(z)$ for all $z \in [c,d] \backslash \{u\}$.

Let us show that under conditions a) the following inequalities hold.

$$f(z) < g(z) \text{ for all } z \in]u,v[, \quad f'_+(u) < g'_+(u), \quad f'_-(v) > g'_-(v). \tag{70}$$

Indeed, let $J = \{j : z_j \in]u,v[\}$, $i = \min\{j : j \in J\}$, $l = \max\{j : j \in J\}$. Let $g_1(z)$ be obtained from $g(z)$ by application of Remark *3* to $]u,v[$ if $J = \emptyset$, or by sequential application of this remark to intervals $]u,z_i[$, $]z_l,v[$, and $]z_j,z_{j+1}[$ if $j, j+1 \in J$. The function $g_1(z)$ satisfies $Lg_1(z) = 0$ for all $z \in]u,v[$, $z \notin A$.

Let $g_2(z)$ be obtained from $g_1(z)$ by application of statement (b) of Lemma 3 to points z_j, $j \in J$. The function $g_2(z)$ satisfies $Lg_2(z) = 0$ for all $z \in]u,v[$, $z \notin A$, and $L_1 g_2(z) = 0$ for all z_j, $j \in J$. Hence $g_2(z) = f(z)$. Since $f(w) \neq g(w)$ at least on one step we obtain a strong inequality. It proves (70).

It follows from the inequalities for the derivatives in (70) that if $u \neq c$ $(v \neq d)$ then $f(z) > g(z)$ in the left neighborhood of u (in the right neighborhood of u). The existence of $w_1 \in [c,u[$ (of $w_1 \in [c,u[$) such that $f(w_1) = g(w_1)$ would contradict to (70). It completes the proof of Lemma 5 for the case (a), where we can take $c_1 = u$, $d_1 = v$.

The proofs for the cases (b) and (c) are even simpler, and we omit them.

Proof of Lemma 6. Applying Lemma 5 to functions $g(z)$, $g_h(z)$ and interval $]c,d[$ we obtain that if $g(\tilde{z}) \neq g_d(\tilde{z})$ for some $\tilde{z} \in]c,d[$ then $g(z) < g_h(z)$ for all $z \in]c,d[$ such that $g(z) \neq g_h(z)$.

Let $h_1 > h > z$, $g_h(z) \neq g_{h_1}(z)$ and $\tilde{g}(z) = g_h(z)$ for $z \in]a,h[$, $\tilde{g}(z) = g(z)$ for $z \in]h,b[$. Applying Lemma 5 to functions $\tilde{g}(z)$, $g_{h_1}(z)$ and interval $]a,b[$ we obtain that $g_h(z) < g_{h_1}(z)$ for all $z < h$. The proof for the case $h_1 < h < z$ is similar.

Proof of Lemma 7. This lemma follows from Lemma 5. Indeed, it suffices to take $\bar{g}_{[a,d[}(z)$ as $f(z)$ in Lemma 5. If $d \leq d_*$, then d coincides with c_1 in Lemma 5.

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Part II
Interviews

The Times of Yuri Vasilyevich Prokhorov

Friedrich Götze and Willem R. van Zwet

Abstract Yuri Vasilyevich Prokhorov is the *eminence grise* of Russian probability theory. Every one of us interested in probability or asymptotic statistics has come across his celebrated weak compactness theorem at one time or another. He was interviewed earlier by Larry Shepp (Stat Sci 7: 123–130, 1992). That interview dealt largely with his impressive career and scientific work, his international contacts and the issue of discrimination in the Soviet Union.

The world has changed considerably in the intervening years and our knowledge and perspective of the past has developed accordingly. It seemed natural to us to talk once more to the man who lived through these turbulent times as the intellectual heir of Kolmogorov, and as one who was in a position to observe the inner workings of the powerful Soviet (later Russian) Academy of Sciences, the Steklov Mathematical Institute in Moscow, and the activities of his many colleagues throughout the country and elsewhere. This interview took place between November 13 and 28, 2006 at Bielefeld University. As it was more like a friendly three-way conversation than a formal interview, we did not identify the two interviewers, but merely indicated the person who asked a question as "interviewer".

A Promising Young Man

Interviewer: Yuri Vasilyevich, we are happy that you have agreed to a second interview after 15 years. First of all, let us encourage readers to read Larry Shepp's interview first, as we shall try to keep overlap at a minimum. However, we do need to

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start with a brief review of your early years. You were born in Moscow on December 15, 1929 and after the war started, the family moved to the town of Chistopol, about 800 km East of Moscow in August 1941. I think that at this point you want to make a correction of the earlier interview.

Yu.P.: Yes, Larry mentioned that Chistopol is on the Volga, but it is not. It is on the Kama River. I realize that not many people will be interested in the exact location of Chistopol, but I like to get such details right. *Little foxes destroy the vineyards*.[1]

Interviewer: You returned to Moscow in August 1943. Presumably, Moscow was a much safer place then.

Yu. P.: Before we left in August 1941 the air defence was far from perfect and I saw lots of bombed buildings. In the fall of 1941 it became much more effective and even though the German army came very close to Moscow, there was less damage. At night aerostats were launched. They were attached to the ground with wires that were almost invisible for the pilots and could destroy a plane when it hit them. I later learned that Kolmogorov had recommended randomizing the location as well as the height of each aerostat every night which made this threat completely unpredictable. When we were back in Moscow I saw the aerostats, but I only remember that they were handled by very pretty girls in nice uniforms.

Interviewer: In Chistopol on Kama you passed through grades 5 through 8 in 2 years and after your return to Moscow you repeated this performance by passing through grades 9 and 10 in a year. This gave you a high school diploma in the summer of 1944 at age 14 and after spending a brief period at the Bauman High School (now Moscow Technical University) to study engineering, you walked into the office of the Dean of the Faculty of Mechanics and Mathematics of Moscow University. Why and what happened?

Yu.P.: I had decided that I wanted to study mathematics, so in the winter of 1944–1945 I went to the Dean's office. Nowadays it is difficult to see the Dean, even for full professors. But I, a 14-year old boy, just walked in from the street and told the secretary I would like to speak to the Dean. Dean Golubev had a background in mathematics and gave courses at one of the military academies with the rank of general-major. He was thin and wore a pince-nez. He also wore an overcoat because it was cold at the university. He asked me very politely what I wanted. I replied that I would like to be a student of mathematics. 'Maybe next year. You are young' he replied. I asked him to give me permission to take exams for first-year students. During the winter you could take exams in analytic geometry, algebra, analysis and English. He wrote something on a small piece of paper that would allow me to take these exams. I was graded "good" once and "excellent" three times, so I again asked the Dean to be admitted. In May 1945 I formally transferred to Moscow University.

[1] Song of Solomon 2.15.

Interviewer: What was life like as a freshman?

Yu.P.: At the university it was cold. When I arrived in the morning I took a chair and with that chair and my bag and coat I went from classroom to classroom: *Omnia mea mecum porto*.[2] It was dangerous to leave a chair just for a moment because another student would take it. The large lecture halls of Moscow University were constructed in the eighteenth century and there were two small lamps on the ceiling. I had weak eyes, even in childhood, and by the time I was 16, one eye was −6 and the other −7. I had difficulty reading what was written on the blackboard. But there was also an advantage. At age 16 I received a certificate that I was a soldier of the second kind, meaning that I didn't have to serve in the army. Still my eyesight kept bothering me throughout my life. In the 1980s it suddenly became much worse. The doctors had no explanation.

In contrast to some other departments and institutions, students in the department of mechanics and mathematics did not get a deferment from military service during 1944-46. As a result, when I was admitted in 1945 there were almost no male students. In mathematics there were maybe 15–20 boys and about 100 girls. The boys were either too young for the army – like me – or they were wounded veterans who had been discharged.

Interviewer: What about professors? Where they still around in the middle of the war?

Yu. P.: Of course not all of them, but we were certainly kept busy. There were 16 or 17 seminars and special courses, and as students we competed who would attend the most. I myself took two courses in differential geometry from Fennikov and Rashevsky, and one in simplectic geometry from Bakhvalov. I gave a talk on boundary properties in a seminar on analytic functions, attended a seminar of Gelfond on number theory and lectures of Novikov on mathematical logic. You can't say we were not broadly educated!

Then in 1947 the great demobilization sharply increased the number of male students and for the girls it was of course a great opportunity to get married. These were the boys born in 1922–1923 and among them were very clever people. There was Bolshev going from the air force into statistics, Mischenko, student of Pontryagin on optimal control, and many others. Quite a few of them later became members of the Academy. It was an exciting time with teachers like Kolmogorov, Petrovskii, Stepanov, Nemytsky, Keldysh and many others.

Interviewer: Please tell us how you first came to know Kolmogorov.

Yu. P.: Kolmogorov gave me a topic for a seminar talk on the strong law of large numbers. Then he invited Sevastyanov and me to his apartment in Moscow on December 15, 1947. When he asked me 'How old are you?' I told him 'Today is my birthday, and I'm now 18 years old'. He then wanted to know whether I liked

[2] *I carry all of my possessions with me*. Seneca, Epistulae Morales 9.18–19.

skiing, and I said I did, whereupon he invited me to his house in Komarovka. So since 1947 I think I saw Kolmogorov for 15 or 17 years practically every day. It was an excellent education. He taught me mathematics, but also other important matters such as making things happen and getting them organized. Over the years he left more and more things to me and people got accustomed to view me as a kind of deputy of Kolmogorov. He also asked me to check his book with Gnedenko on limit theorems, and as a young man, I was mighty proud that they thanked me for the many corrections.

Interviewer: Did this early acquaintance also make you decide to choose probability theory as your main area?

Yu. P.: Yes, at first I was fascinated by number theory. Partly complex variables as well. But after Kolmogorov's lectures I decided that I would devote whatever abilities I had, to the study of probability theory.

I finished in June 1949 and moved to the Division of Probability of the Steklov Institute. It was a small division, but with Kolmogorov, Smirnov, Khinchin, Sevastyanov, what more could you want? After some time I became a member of the scientific council of the Steklov Institute and I am still at the Institute after 57 years.

During this entire period I have also been teaching at Moscow University. In 1952 I gave my first special course on limit theorems and among my audience were Hasminskii, Borovkov, Zolotarev and Yushkevich. I remember that when I started this course, Kolmogorov told me to be dignified and modest! In 1957 I was appointed full professor in the Department of Mathematics. At Moscow University there is a special cloakroom for professors with an attendant watching over their overcoats. He probably took me for a student and refused to take my coat. Luckily I spotted the rector of the university, academician Petrovsky and asked him "Ivan Georgievich, please tell this gentleman that I'm entitled to leave my coat here". The rector said "Yuri Vasilyevich is young but he is a professor at the university. Please help him".

In 1970 I moved from the Department of Mathematics to become the Chair of Mathematical Statistics in the new Department of Computational Mathematics and Cybernetics, where I have remained ever since.

Adventures with Geologists

Interviewer: For a moment let us change to a somewhat lighter note. You and I both like to drink vodka, in civilized quantities of course. You once told me that you learned to drink with the miners in the Ural Mountains. This sounds exciting.

Yu. P.: Yes, Kolmogorov once told me that when he was young, he was drinking with sailors, but at the same time he objected to my drinking. One summer day a young geologist showed up at the Steklov Institute His colleagues said he was an excellent geologist, but with a peculiar hobby of studying Cramér's book on mathematical statistics all day. When I spoke to him, he told me about granite, which is something I never saw outside Red Square. He then proposed that I should join a geologist party going to Central Kazakhstan. So I spent the summer of 1961 in Kazakhstan and the next summer in the Polar Ural, where there is no sunset during the summer. In 1963 we were near the Chinese border South of Chita, and in 1965, just after my return from the Berkeley Symposium, my friends and I went to Kamchatka. So in three weeks time, I saw the Pacific from both sides.

For me it was a different life. In these geologist parties, it doesn't matter who you are, what titles and degrees you have; you should just be a man of good character. When we left for Central Kazakhstan, a small ceremony was arranged, complete with herring and spiritus vini (alcoholic spirits) of 95 %. These spirits are a special story. It is dangerous to breathe while drinking and you should drink some water immediately afterwards, because otherwise you ruin your throat. The geologists at the table were pleasantly surprised that a Prof. Dr. of Mathematics did pass this test, and quite frankly, so was I!

Incidentally, these high percentage spirits are transported to distant places in Russia, because water to dilute it is available everywhere and it makes no sense to carry it around. Diluted to 55 % it is called spirit for drinking.

By the way, this reminds me of another story involving herring and vodka. In Holland you have these herring stalls in the street, where they clean and sell raw salted herring. The Dutch grab the fish by the tail and eat it on the spot. So one day Ildar Ibragimov and I were in Leiden and I happened to have some vodka with me. Ildar told me that his wife would never allow him to drink vodka in the street, but if she would see me do it, it would probably be all right. So he suggested that we improve on this Dutch custom by having some vodka with our raw herring, and we did.

Interviewer: Wasn't there an anti-alcohol campaign during Gorbachev's rule, much like prohibition American style?

Yu. P.: Yes, but I don't think it was Gorbachev's idea. In a TV documentary he said that he used to drink in the old days and was still drinking today, but now the doctors recommended that he should only drink vodka. They say it was Gorbachev's deputy Ligachev who was the main figure in this anti-alcohol campaign. The campaign cost the state 72 billion roubles, which went into the so-called grey economy and was later used to buy factories and oil fields. It was a heavy blow for the state, but the party was against alcohol! But where is this party now? We do know where the alcohol is: on the shops' shelves! Ligachev claimed that in 1988 they saw that this campaign was becoming so dangerous for the USSR that they stopped it, but at that time they didn't have enough power to stop the dissolution of the USSR itself! So in the Crimea they destroyed centuries old vineyards with exceptional types of grapes. It was really very stupid. It was just one of those initiatives which sound convincing, but in practical realization only lead to disaster. The leaders may have one idea, but at the local level many other things will play a role. As the proverb says: *'People are ready to burn down my house to get fried eggs for themselves'*.

There was another attempt this year, but Putin and the Russian prime-minister Fradkov were strongly opposed and tried to find out who was responsible for artificially limiting the supply of alcohol. This was because many people died or became very ill drinking bad alcohol produced by nobody knows whom. The price of the usual bottle of vodka is around 100 roubles, but you can buy an illegal bottle for 20 roubles and end up in hospital.

Interviewer: I understood that while they were trying to ban alcohol, there was a sudden shortage of sugar.

Yu. P.: Certainly, because people tried to produce alcohol themselves. The car factory ZIL produced a device which looked like a small case made of stainless steel. You carry this case with you, add sugar and water and after some time you are supposed to get good vodka. In Georgia for centuries they added some apricots or other fruit to give it a special flavour.

When Roosevelt became president of the USA, a delegation of American workers visited the USSR and Stalin met with them. They asked Stalin why there is no prohibition in the USSR. Stalin replied that (1) it will not stop the use of alcohol and (2) we need the 800,000 golden roubles for the industrialization of the country. The reason I know this for a fact is that is was a compulsory subject for study when I was a student.

The Academy of Sciences

Interviewer: Let us return to more serious academic matters. In 1966 you became a corresponding member of the Soviet Academy of Sciences.

Yu. P.: Actually I was proposed for this in 1964, but failed to be elected. In the Division of Mathematics of the Academy some people thought that I was one of these overly abstract mathematicians they called ‘abstractionists’ and for whom they had no use. Two years later I received strong support from the Minister of Geology who wrote that I did some important work with geologists, and apparently this changed their minds. Also Jerzy Neyman spoke to Bernstein and asked him to support me in the election. At the same time Neyman was thinking of nominating me for election in the National Academy of Sciences in the United States. He sent me the necessary forms to fill out, but I was timid enough to tell the Director of the Steklov Institute about this. He made a grimace as if he just tasted something very unsavoury. I was to be his deputy for 16 years and I decided not to go ahead with this.

What I hadn’t realized was that becoming a corresponding member was not only an honour, but would also turn into a demanding job. The secretary of the Mathematics Division was Bogoljubov and upon my election I was appointed as his deputy. The division secretary took part in the work of the Presidium of the Academy, and in Bogoljubov’s absence this task fell to me. At the time Bogoljubov was the head of the Institute of Nuclear Research in Dubna which is a very pleasant place about 100 km from Moscow. He had a dacha there, so he was absent from Moscow quite often. As his deputy this gave me a chance to see the work of the powerful presidium at first hand. I took my task very seriously and tried to advance the interest of mathematics and mathematicians, while avoiding any action that would benefit me personally. People noticed this and repaid me with their trust. For my part I learned how important problems involving large sums of money are solved by competent administration.

The Mathematics Division itself is also quite interesting but complicated. There is a permanent struggle between various subgroups. If there is a position for a new member, it is often difficult to reach a compromise on whom to propose. If there is no winner after three rounds of voting, then the position is lost and goes to another division. Of course the physicists are much better organized. They wait patiently until the mathematicians loose the position and immediately propose a physicist.

The heavy workload had a price and in 1968 I suffered a heart attack. I spent about a month in hospital and more or less completed my recovery by sports activities. During the winter after this I went skiing four times in the mountains at home and in France. In 1972 I became an Academician.

As early as 1954, before the International Congress of Mathematicians in Amsterdam, Kolmogorov said to me "Yura, I decided to leave probability and move to another subject, say, ergodic problems in celestial mechanics."[3] He suggested beautiful problems to his students such as the three body problem. From then on he sent people to me if something should be decided concerning probability theory. When the Division of Mathematics of the Academy appointed a committee for probability with Skorokhod, Koroljuk, Petrov, Ibragimov, Borovkov and many others as members, Kolmogorov insisted that I should chair this committee. Once a year we met in a suburb of Moscow and discussed matters of common interest, concerning publications and dissertations, for instance. I would be sitting to the right of Kolmogorov who was, of course, still the leader. But in this way people got accustomed to talk to me about any problems they had, so that Kolmogorov would not be bothered except in very rare cases. Of course this was exactly what Kolmogorov had in mind! So after he became seriously ill and then passed away, there was no break and it was a very smooth transition period when the leadership came to me. As always, I tried to convince people rather than to press them.

Interviewer: In Western Europe we admired the Soviet Academy for taking an independent position, as it did, for instance, in the Sakharov case. As you obviously know the Academy inside out, let us discuss in how far the Academy is really independent of political interference and pressure.

[3]Kolmogorov developed a method to avoid the problem of 'small denominators' in the series describing the mutual pertubation of the movement of celestial bodies, which subsequently led to the famous KAM-Theory (Komogorov-Arnol'd-Moser).

Yu. P.: The Academy is definitely an independent organization. In Soviet times some members of the Central Committee wanted to be elected as Academicians. A gentleman from Belorussia – in fact the President of the Belorussian Academy – even proposed to elect Brezhnev as a member of the Academy in the Division of Process Control. He argued that Brezhnev controlled the entire state! Sometimes such candidates passed the first step in the appropriate division, but the General Assembly usually voted against them. These votes are secret.

However, it so happens that while we are having this interview, there is a movement to turn the Academy into a State organization. The President of the Academy would no longer be elected by the members, but appointed by the President of the country. This movement started just 6 months ago and it is very powerful. Of course the Academy wishes to remain independent and as far as I understand, Putin is also against these plans. In this uncertain situation he has allowed the Academy to postpone the election of the next president for 2 years. So the present president Osipov will remain in office for 2 years after his 5-year tenure is finished. What happens after that, nobody knows.

Interviewer: But who are proposing these changes? People in the government?

Yu. P.: Partly. But I think the initiative comes from business people. You see, the Academy owns a lot of very valuable real estate. It owns areas in the centre of Moscow, in very fashionable places, and also outside of Moscow. Of course developers would like to replace small buildings such as the Mathematical Institute and the Institute of Physics by 3, 4, or 5 star hotels, and maybe, casinos and brothels. Several attempts were already made to get hold of this property of the Academy, but so far they failed. But if the Academy would be a State organization, the government could do whatever it wanted. Today there is one government, tomorrow another; today one president, tomorrow someone else. So the Academy would be subject to all political oscillations.[4] In Soviet times, nobody approached this matter in this way. The Central Committee and the Government had other means to pressure the Academy, but not like this. Some people believe that the Church should be part of the State, but the Church hierarchy said that there is no need for this at all and that it would be harmful for the Church. The same is true for the Academy.

Interviewer: Please tell us how the Academy reacted to the political pressure to expel Sakharov.

Yu. P.: Before I do that, I would like to tell you how Sakharov became an Academician. In 1953 two outstanding physicists Kurchatov who was officially the Head of the atomic project, and Tamm came to the general meeting of the Division of Mathematics and Physics (it was still a joint division at the time). They appealed to the members to believe them that Sakharov should be an Academician, even

[4]At the time of publication of this interview, it seems that this threat to the independence of the Academy has been temporarily (?) lifted. The economic crisis has had the healthy effect that right now few people in Moscow are interested in building five star hotels, casinos, or even brothels.

though they could not explain why. Of course there were rumours among scientists and some people – including Kolmogorov – had some idea of what Sakharov was doing. Sakharov had three golden stars of Hero of Socialist Labour and the times when these stars were awarded coincided more or less with the dates of successful experiments with a nuclear weapon. So Sakharov became an Academician very early.

When the Academy came under pressure to expel Sakharov for his political views, Academy President Alexandrov simply replied that it would be impossible to get the necessary 2/3 of the votes for this. Then Sakharov was sent in exile to Nizhny Novgorod where he remained under house arrest. He was later brought back to Moscow and to power by Gorbachev who supported him.

The case of Sakharov was not at all exceptional. During the entire Soviet period hardly anyone was expelled from the Academy. Of course the Academy could not protect its members against other problems with the regime, especially in the days of Stalin. Nina Solomonovna Stern was in exile in Kazakhstan at about the time of the Jewish doctors' case, an alleged plot to murder the political and military leadership of the Soviet Union. When she returned to Moscow and asked to be reinstated in the Academy, she was told 'Please go to your bank and check your account. Your salary as an Academician was paid during the years of your absence'. The geneticist Nikolai Vavilov, whose brother Sergei was president of the Academy, died in prison but he was not expelled from the Academy. So in a sense, Academy membership is like royalty or nobility: You can be beheaded, but you are beheaded while being a king or a duke. Those who were in power found this difficult to understand. They remove people in very high positions from the Party with the stroke of a pen. But the Academy is not a party!

By the way, it was not exceptional in those days that one family member would hold an important position while another would be in prison or executed. Kaganovich was among the six people closest to Stalin when his brother was shot. Molotov's wife Polina Zhemchuzhnaya and the wife of Kalinin were in prison. There were TV documentaries "Wives of the Kremlin" about this. When people from the NKVD came to arrest the wife of Semyon Budenny, Hero of the Soviet Union, he drew his famous sabre and threatened to kill them. They had to give up and when Stalin heard about this, he roared with laughter and ordered them not to repeat this effort. Similarly an attempt of the NKVD to arrest Voroshilov's wife was stopped when he threatened to shoot them. There is a – strictly fictional – film called 'The wife of Stalin' where Stalin tells his Russian wife 'All of my friends have married Jewish girls. These girls are very clever and obey their husbands. And you love only yourself and not me'.

Interviewer: I guess there is some historical truth in this, in the sense that quite a few revolutionary leaders had Jewish wives?

Yu. P.: Yes, definitely. Earlier under Lenin quite a few of the leaders themselves were Jewish too and you can find Jews in leading positions in the KGB throughout the Soviet period. In Lenin's time, they often changed their names. Trotsky was in fact Bronstein and Kamenev was Rosenfeld. There is a cynical anecdote of a Jew who applied to have his name changed to Ivanov. Some time later he wanted his name changed to Petrov and when asked why he didn't change it to Petrov right away, he explained 'If someone asks me what my former name is, I can now say Ivanov'. Of course the worst anti-Semitism occurred near the end of Stalin's rule, but we'll speak about that later.

Interviewer: Didn't the Academy also have honorary and foreign members?

Yu. P.: Before the revolution honorary members were, say, members of the family of the Czar. Konstantin Romanov, the brother of Nikolai II even became President of the Academy. He published translations of Shakespeare into Russian under the pen name KR. Also ruling heads of other countries, such as Emperor Wilhelm I, who was elected in the same year as Gauss. Poisson became a member during the centenary celebration of the Academy in 1824.

During the Soviet years honorary members were also elected and the last three were Gamaleya who was a specialist in biology and medicine, Stalin and Molotov. Gamaleya died in the forties and Stalin in 1953, which left Molotov as the only honorary member. The Presidium didn't know what to do about this and at last decided to abolish the notion of honorary member altogether, so that Molotov automatically lost this title. Stalin used to call Molotov jokingly 'cast-iron but' for his capacity to sit in his chair and work for many hours without stopping. Molotov lived up to this nickname and proved to be remarkably tough. He became a member of the Central Committee of the Party in 1921 and died in 1986 at the age of 96. A few years later the names of Gamaleya, Stalin and Molotov were discretely deleted from the list of past and present Academy members. Perhaps we should now consider restoring the notion of honorary academician. There are obvious advantages and disadvantages!

In recent years strange new Academies appeared, such as the Academy of TV Arts, the Academy of Journalists, etc. The main reason seems to be that the members of these Academies sign their letters as Academician Ivanov or Academician Petrov without mentioning to which Academy this refers. The same thing happened in Stalin's time, but Stalin himself decreed that the word Academician could only be used for members of the Academy of Sciences of the USSR. In practical matters like this, Stalin was often quite sensible. He spent much time in prison and in exile in far away places, so perhaps he knew real life much better than most people. He was a professional revolutionary, unlike the next generations of bureaucrats whose life was largely restricted by the walls of their offices.

Interviewer: Let us look at the public image of the Academy. It has always been a highly prestigious institution and Academicians commanded a great deal of respect in the Soviet Union.

Yu.P.: When Kolmogorov died his obituary was published in the two central Soviet newspapers Pravda and Izvestiya and signed by all members of the Politburo, members of the Government and the Academy. They wrote that Soviet science should be proud that such a scientist worked in our country. Kolmogorov received a golden star of Hero of Socialist Labour and seven orders of Lenin, which was the highest distinction in the USSR.

Interviewer: Obviously Kolmogorov was exceptional, but why were science and scientists so important?

Yu.P.: People understood, especially after the war, the important contributions of science to our national defence. Stalin met with Kurchatov, the head of the atomic project, and at the end of their conversation Stalin said: "Scientists are people who do not ask much for themselves. Please ask them what they need for more effective research". Kurchatov collected opinions and soon large settlements for members of the Academy with houses imported from Finland appeared in Mozzhinka, Abramtsevo and Lutsino. The scientists received these houses free of charge. Vinogradov said: "I have a hectare of land with my own river and my own mushrooms". Not only physicists and chemists who participated in the nuclear programme were rewarded, but scientists in other fields received similar privileges.

Interviewer: And what is the situation now? Are Academicians still very important people?

Yu.P.: I would not say so. When I was young, Kolmogorov told me: "Yura, keep in mind that each member of the Academy equals one research institute!" This is no longer so. Major defence programs have come to a stop and we don't hear much about the aerospace program anymore.

Interviewer: Science could also contribute to the strategic industrial development of the country. Doesn't the government care about this?

Yu.P.: Nowadays we buy everything abroad: Modern technical development, computers, and so on. Even the Ministry of Defence buys the computers they need, say, in Germany, our main trade partner. Presumably they check a PC to see whether some additional hardware was put inside. In Soviet time we tried to make everything ourselves. That was not wise. Now we sell oil, gas and electricity to other countries and buy what we need. The wisdom of this is not clear either.

An effort was made to develop a modern computer at Zelenograd not far from Moscow. Some kind of Silicon Valley. Khruschev supported the project, but Brezhnev did not give real support and the project collapsed. Such things do not only depend on the scientists, but also on the political leadership.

Discrimination

Interviewer: In the earlier interview with Shepp the extent of discrimination of Jews in the Soviet Union was discussed at some length. Then and above, you argued that Jews played an important role in the early Soviet regime and that evidence of serious systematic discrimination in the sciences before World War II would be difficult to find. You also noted that the worst anti-Semitism occurred near the end of the Stalin era. Without reopening this discussion, would you mind elaborating on the last point?

Yu.P.: In the interview it says World War I, but I obviously meant World War II.

All right, a few words about Stalin's last years. When the State of Israel was founded in 1948 it was immediately recognized by the Soviet Union. Especially Stalin himself supported this. He thought that Israel could perhaps be drawn into the sphere of influence of the Soviet Union, rather than the United States. When Golda Meir became the first Israeli ambassador to the USSR in 1948-49, she naturally met Soviet leaders and also spoke (in Yiddish) with Molotov's wife. Golda Meir told Stalin 'You have a Jewish Autonomic Republic in the Far East of the Soviet Union. Why not give the Crimea to the Jewish people?' Stalin probably didn't react, but after she left he said 'I don't like the idea of having American warships in the Black Sea'. Some time later he had apparently given up the idea of this approach to Israel, because in the last few years of his life he started a vicious anti-Semitic campaign that culminated in hundreds of arrests after an alleged plot of Jewish doctors to kill members of the Soviet elite had been 'uncovered'. Three weeks after Stalin's death in 1953, Beria insisted that this campaign should be stopped and all people connected with the affair be released from prison. So altogether this was a brief but serious outburst of anti-Semitism.

I also remember a discussion between Shepp and Statulevicius when Shepp insisted that during World War II, Jews in Lithuania suffered much more from the local population than from the Germans. After twenty minutes Statulevicius agreed. However, this is not directly relevant for discrimination in science. More to the point is my experience in choosing the delegation of probabilists for the Congress in Nice. It required a lot of effort but I succeeded to include Gikhman in the delegation. Then people said: "He is not an ordinary Jew. He is a probabilistic Jew."

Finally, let me dig up a curious fact from Czarist times. There was a so-called settlement line surrounding an area in Belorussia and the Ukraine that Jews were not allowed to enter. There were exceptions for midwives, important merchants and people with higher education who could cross this line. In the first edition of the Soviet Encyclopaedia you can read that Sergei Bernstein could not get higher education in Russia before the revolution. He went to Paris and returned as a person with higher education and could live in Russia wherever he wanted.

Interviewer: I imagine that Khruschev put a definite end to the Stalin era. But how does the average Russian feel about this?

Yu.P.: When Khruschev gave his speech against the Cult of Personality at the XX'th Party Congress in 1956, the delegates were invited to the Kremlin at 2 a.m. and the speech continued until 6 a.m.. They then went back to their homes and hotels. Two days later the full text of the speech was published in the United States. Two months later, the speech was read at the Steklov Institute, first to Party members, then to members of the Scientific Council, and later to the others. The speech was never published in the Soviet Union. Many people were not ready for these things, or for the many changes that would follow. They are tired. For ordinary people it is important to know what the world will be like tomorrow and next year. Repeated major changes make people uncertain and their lives very difficult.

Interviewer: Later Jewish people who applied for visa allowing them to emigrate to Israel had very serious problems.

Yu P.: There were very different periods. The first reaction was very negative. It was considered unpatriotic behaviour that should be punished. It was proposed that these Jews should pay large sums of money for the education they received in the USSR. Some Western journalists commented tongue-in-cheek that it would be more profitable for the USSR to educate more Jews than to develop industry and agriculture. Another idea was that the scientific councils that awarded degrees of candidate and doctor should cancel the degrees of these Jews. However, these councils operate by secret voting, so this invention of the administration never worked either. I know this for a fact because I did some work for the Supreme Committee on Attestation.

The number of applicants peaked in the late seventies and many of these people had great difficulties. Under Gorbachev the situation changed. It became much easier to go abroad for all kinds of people, including those who wanted to emigrate.

How to Predict Who Could Travel

Interviewer: For a number of years during the Soviet period I wrestled with the problem how to invite conference speakers from the Soviet Union who would have something interesting to say and would actually show up. There were usually lots of suggestions for interesting speakers, but they never came when invited. For me the practical solution of this problem was to talk to Albert Shiryaev who would discuss this back home with the appropriate people and would then suggest some excellent speakers who actually came. However, I'm still curious to know how it could be predicted who would get permission for foreign travel.

Yu.P.: You know, Francis Bacon wrote in one of his essays that the State never gives credit to the unmarried.[5] Because most of those who went to the enemy, or at least to the United States, were single! But in Soviet times there was a professor of chemistry who went to Canada and didn't come back. People from the Foreign Department told Academician Semyonov, the Nobel Prize laureate for his branching processes in physics, that they had recommended that it was better not to send this gentleman abroad. Why? He had three wives. Yes, consecutively, but this is also not good! So it is better not to send him. When he remained in Canada there was some

[5] *Unmarried men are best friends, best masters, best servants, but not always best subjects, for they are light to run away, and almost all fugitives are of this condition.* Of Marriage and Single Life, Francis Bacon Essays (1601).

scandal. These things came in waves and after such an incident the rules would be tightened. For example, at one point these limitations on foreign travel were weakened, but then a Deputy Minister of Aviation didn't come back from France and immediately the rules were tightened. But after some time there was a generally known procedure. First you went abroad to one of the people's democratic republics: Hungary, or Poland, etc. Poland was a little suspicious. At first you go abroad in company, to a conference, not alone. Next you would go to a developing country like India and if there are still no objections, you may be able to go to Britain or the US. But some colleagues of mine were told that if they wanted to go to the US, then they'd better join the Party. They did, but for me that would go several steps too far!

Interviewer: But it can't be as simple as that.

Yu.P.: Of course there is more to it than this. Even today there is a problem if you have ever handled classified information. When you apply for a foreign passport you have to fill out a form which contains the question whether you have the right to read classified documents. A colleague of mine did have this right, but never used it. He decided to write NO on the form, but was punished for giving false information. A friend of mine, who worked for 3 years on a nuclear project, was not allowed to go abroad for 18 years. Through the Division of Mathematics of the Academy I got involved in an attempt to straighten this out because Bogoljubov thought that I could find the proper arguments even though I knew nothing about the case. Much later I noticed that all buses in Bielefeld have a quotation from the bible right next to the driver that explains Bogoljubov's strategy. It reads: "*Verlaß dich auf den Herrn von ganzen Herzen*" but omits the second part of the quote that continues with: "*und verlaß dich nicht auf deinen Verstand.*"[6]

The problem with classified information is that in advising the authorities about a proposal to send someone abroad, I can't say that they should let this person go because he didn't have access to classified information. This is also classified information that I'm not supposed to know. Once when Bogoljubov was on vacation and I was acting Head of the Mathematics Division, they did not allow Dobrushin to go abroad because he once worked for some applied institute that objected to this trip. There was nothing I could do about it.

Some people joined defence institutes where they do classified work on purpose. They may be able to get an apartment in Moscow or double their salary if they agree to get stars on their shoulders. Later they find out that there is also a downside to this.

Interviewer: I remember that at an ISI Session many years ago someone who was obviously not a scientist, spoke about Lenin's thoughts about statistics, or something like that. How could that happen?

[6] *Trust in the Lord with all your heart and do not rely on your own insight.* Proverbs of Solomon 3.5.

Yu P.: Most likely they invited a delegation without mentioning all of the names of the members. Just to take part in the meeting. Of course, people with administrative power would take advantage of this and push some other candidate off the list.

Interviewer: I understand that refusing a scientist permission for foreign travel was also used as punishment for breaking the rules.

Yu P.: Certainly! In the late sixties a group of 99 scientists, mostly mathematicians, signed a letter of protest to the Ministry of Health concerning a mathematician who had been put in a psychiatric clinic without a proper investigation. Unfortunately, the letter was read in a BBC radio transmission before it had reached the Ministry. This infuriated the government and the 99 were denied permission to go abroad for many years.

Interviewer: One final question about foreign travel of a different kind. Jerzy Neyman invited Kolmogorov to visit Berkeley and it was announced many times that he would come. However, it never happened. I don't suppose that he wouldn't get permission. Do you know what the reason was?

Yu P.: Kolmogorov's last flight was in the fall of 1954 when we attended a very interesting meeting on probability and mathematical statistics organized in Berlin by Gnedenko, who was at that time at Humboldt University. On the way there was one stop at Warsaw or Prague and Kolmogorov asked the pilot to go down as slowly as possible because his ears were hurting from the pressure change as the plane went down. After that he never travelled by air again. Of course he could travel in Europe by train, but he still wanted to visit the United States. He asked the Foreign Department of the Academy to let him go by boat. But they refused because it was too much trouble for them with visa and so on. He tried a number of times and thought he would succeed eventually, which is why he never gave up plans to come to Berkeley but never made it either.

Interviewer: This sounds like the same problem that Lucien Le Cam suffered. In his case it was cured by the simple trick of puting tiny tubes in his eardrums to release the pressure. They used to do that with children with frequent ear infections.

Meetings Abroad: ICM and Berkeley Symposium

Interviewero: Your travel abroad followed the pattern you have just described: First you went to the meeting organized by Gnedenko in East Berlin in 1954 when Kolmogorov's ear problems manifested themselves. In 1956 you went to the first information theory meeting in Prague and in 1958 you came to Western Europe when you attended the International Congress of Mathematicians (ICM) in Edinburgh. Please tell us about these ICM's and the International Mathematical Union (IMU), the body that organizes these congresses every four years. Relations between IMU and the Soviet Union had been somewhat stormy in the past.

Yu.P.: Yes, in 1950 in Cambridge, Mass. in the United States, nobody from the Soviet Union attended. In Amsterdam in 1954, after the death of Stalin, Kolmogorov, Alexandrov, Nikol'sky and Panov attended. Kolmogorov gave the opening lecture. The congress was held at the famous concert hall *Concertgebouw* in Amsterdam, and as a true lover of classical music Kolmogorov began his lecture by saying: "I know this auditorium as the place where Mengelberg conducted his orchestra". The closing lecture was given by John von Neumann.

Interviewer: We knew Kolmogorov, Alexandrov and Nikol'sky. But who was Panov?

Yu.P.: I believe Panov was an expert in computing problems. But perhaps he was just accompanying the others.

Interviewer: . . . and keeping an eye on them, I suppose.

Yu.P.: Maybe. There is an interesting sequel to the fact that Kolmogorov and von Neumann were the only two plenary speakers in Amsterdam. At one time we asked Springer to publish a translation of Khinchin's collected papers. We were thinking of the blue series where Pontryagin's collected papers were published. Doob was greatly in favour of this plan but Springer refused. The collected papers of Kolmogorov did not appear in this series either. The reason for that was that Springer sent a young gentleman to negotiate with Kolmogorov about publishing his collected papers, but he didn't seem to know who Kolmogorov was and didn't approach his errand very tactfully. This was right after the Amsterdam congress and by way of explanation Kolmogorov said "Right now I'm sharing first and second place in the world with somebody else". When the young man returned to Heidelberg with this message and his lack of success, he was immediately sent back to Moscow with instructions to agree with everything Kolmogorov asked for, but it was too late. The rights went to another publisher.

Interviewer: This was not at all the style of Julius Springer, the founder of Springer, who was a personal friend of people like Einstein and Hilbert!

Yu.Pr.: Right. And Kolmogorov told me that when Felix Klein was editor of Math. Annalen, he also contacted authors very often himself.

In 1958 in Edinburgh many more people from the USSR attended the ICM. There was even a 'tourist group' for young people including three probabilists: Statulevicius, Sirazhdinov and me. This opportunity to meet people like Feller and Kac was very important for me.

In 1970 in Nice I had more responsibilities. I already mentioned that it was hard work to get Gikhman permission to attend, but together with Pontryagin, Yablonskii, Mushelishvili and Lavrentiev who was vice-president of IMU at the time, I also belonged to the official Soviet delegation that took part in the meeting of the general assembly of IMU. In the years that followed I served on the committee to select invited speakers and on the committee for the Fields medal. Next I was elected vice-president of IMU for the period 1978–1982. It was a very difficult period because the next ICM had been scheduled for 1982 in Warsaw, but this was the time of Solidarnost and the situation in Poland was very unstable. The possibility of moving the ICM to another venue was discussed. During a meeting of the executive committee of IMU in Paris, Czeslaw Olech of the Mathematical Institute of the Polish Academy of Sciences and I argued that the ICM had only been cancelled during both World Wars and that there was no reason to do so again. The final decision was to organize the ICM in Warsaw, but 1 year later in 1983. The French supported their catholic brethren in Poland and I promised to send 300 participants to Warsaw. Indeed, the ICM did take place in Warsaw in 1983, Faddeev became vice-president and later president of IMU.

Interviewer: In 1945 Jerzy Neyman organized the (first) Berkeley Symposium on Mathematical Statistics and Probability. Five more were to follow at 5 year intervals until the sixth and final symposium in 1970. Neyman's idea was to make researchers all over the world partners in the development of probability and statistics. During the fifth symposium in 1965 we were sitting in a lecture room listening to one of the speakers when the door of the room opened and you quietly walked into the room with Linnik and Yaglom. For the first time a delegation from the Soviet Union had arrived and people started to applaud. The speaker was standing with his back to you and probably thought he had said something sensational.

Yu.Pr.: Yes, everybody congratulated Neyman because the Russian probabilists were finally taking part for the first time and were having their papers published in the Proceedings of the Berkeley Symposium. For us the symposium was a unique opportunity for meeting and getting personally acquainted with so many interesting people. Five one-hour lectures a day with coffee breaks and dinner also gave you a chance to find out what these people were doing. For me this was a very important thing for the rest of my life. Then there was a 3–4 day break around Independence Day and Linnik and I went to Asilomar.

Interviewer: There was also an excursion to Stinson Beach and I believe that you and I were the only people foolish enough to try and go for a swim in the ocean. We didn't realize that the water over there comes straight from Alaska. I was quite skinny in those days and almost froze.

Yu.Pr.: Linnik was never one to waste time on such things, so on the day we arrived in Berkeley he bought a small typewriter and started to write some paper. Yaglom and I knocked on the door of his room and when he opened the door he was wearing his pants and a heavy mackintosh and in the corner of the room we saw a large carton of milk and a straw.

Interviewer: He was probably working very vigorously on one of these very long and complicated papers of his.

Yu.Pr.: Certainly. He was very surprised that in a discussion of one of his papers in the Mathematical Reviews the reviewer remarked at the end that there was a gap in the proof of lemma 64. So 63 lemmas were fine, but there was a problem with lemma 64. I said "Yuri Vladimirovich, please look …", but he said "No, no. This gap can easily be repaired".

Let me take this opportunity to tell you a bit more about Linnik because he was such an interesting character. We can return to the Berkeley Symposium in a minute. Linnik's father was a Ukrainian astronomer and also an Academician. His mother was half Tartar, half Jewish. Linnik used to say: "They can not have anything against me because I'm baptised, my children are baptised and I'm good at arithmetic".

Interviewer: Didn't he also have three wives?

Yu.P.: No, according to him he had five wives, but the first, third and fifth coincided.

Linnik started out in number theory and for his work he received a Stalin prize and became a corresponding member of the Academy in 1953. In the meantime he had published his first paper in probability in 1947, possibly because Vinogradov and Bernstein insisted that he should move to this new domain. Speaking about his work in probability he said later: "I introduced heavy theory of functions of complex variables in probability". If you read his papers you see that this is true. Before him, nobody would have thought of using Lindelöf's principle or the properties of zeros of entire functions, etc. in probability.

Interviewer: The problems that Linnik left are very hard.

Yu.Pr.: I once asked Wassily Hoeffding about the work of Linnik. He smiled and said: "Hmmm. Too much analysis".

Interviewer: That is typical for Hoeffding, who was the master of simplicity. He never stopped until he could prove his results with the absolute minimum of technical tools.

Yu.Pr.: Linnik was really in functional analysis. He never gave a probability course for mathematicians at Leningrad University. He said: "I do not like these sigma-algebras. I have Romanovsky and Sudakov. They know".

Interviewer: He didn't like Lusin-type analysis.

Yu.Pr.: Linnik had an office in LOMI, the Leningrad branch of the Steklov Insitute. He liked people who knew several foreign languages. He knew perhaps ten languages: Bulgarian, Polish, etc. On his desk there was a bell-button he could push. If he pushed it once, it meant that Kagan should come; twice Romanovsky or Sudakov; three times Mitrofanova, the lady who worked with him. Linnik also tried to protect Kagan who had serious problems after applying for visa for Israel.

In Moscow nobody checks when people come to work and leave. Your results are what counts. In LOMI people were afraid that Linnik would become director. Every day he arrived at 9 a.m. and left at 6 p.m. and people felt sure that he would expect the same from them.

Interviewer: Somewhere I read a story that Kolmogorov wrote his famous *Grundbegriffe der Wahrscheinlichkeitstheorie* because he needed money to have the roof of his "dacha" in Komarovka repaired. The person who wrote this quoted this jokingly as a major triumph of kapitalism! However, you told me that this story is really about Linnik's roof.

Yu.P.: Well, Linnik once told me: "I have no money now. I gave 150,000 roubles to my father-in-law for building my "dacha". He assured me that this would be enough, but now the money is gone and only a part of the first floor is finished. So I wrote a book about the method of least squares". Writing a book was perhaps the only way to get a large amount of money at once, because the publisher pays in advance.

Interviewer: Thank you for sketching an interesting and very clear picture of Linnik. Now let's go back to the Berkeley Symposium, or rather to your travel through the US after this.

Yu.Pr.: We had quite a few invitations after the symposium, but not enough time to go everywhere. Linnik, Yaglom and I paid a brief visit to Lukacs at Catholic University in Washington DC, and went to Columbia University and Cornell. At Columbia we met Herbert Robbins. When I asked whether he wrote *'What is mathematics'* with Courant, he replied "Yes, I am the famous Robbins". He was very friendly to us. I met Robbins again in 1991. Of course he had grown older, but he was still very critical about his colleagues.

Interviewer: That sounds like Robbins all right! I understand his relationship with Courant was far from perfect. He was also very cynical, but I really liked him.

Yu.Pr: Before we went to Cornell, Linnik said to me: "Yuri, please write to Wolfowitz that we need some money to visit Cornell". I did and we received a brief reply: "Dear Yuri, Your travel expenses will be more than met".

I had met Jacob Wolfowitz before, together with Gnedenko. On that occasion he said to us: "I have a car with me, I can bring you to your hotel. It is inconvenient for me. But I can bring you to the hotel".

In 1963 Wolfowitz attended a conference in Tbilisi. On the way there we had dinner in hotel Ukraina. Wolfowitz checked his coat in the cloakroom, but when we left he told us he had lost the ticket with his number on it. I asked him whether he could identify his coat and he did. I explained the situation and gave 5 roubles to the attendant. At the time this was not an entirely trivial amount, but the problem was solved. Afterwards Yaglom told me: "He did not loose the ticket, but he just wanted to see what would happen".

Interviewer: It sounds as if you were the subject of some psychological experiment.

Yu.Pr.: At Cornell we also met Kiefer and Hunt. Hunt told me he would very much like to visit Moscow. I asked him whom he would be interested in seeing. He said: "Vladimir Kutz" who was the world champion 10,000 meter runner at the time.

The World Meeting at Tashkent

Interviewer: You said earlier that Kolmogorov not only taught you mathematics, but also how to make things happen and getting them organized. This is indeed a very difficult art and the Bernoulli Society World Meeting in Tashkent in 1986 must have been one of your masterpieces! It was the first meeting where large numbers of mathematical statisticians and probabilists from all over the world got together to discuss their science at a very high level.

I presume it is easier and nicer to organize such a meeting in a Republic away from Moscow, but on the other hand the enormous distance between Moscow and Tashkent must have posed some problems too.

Yu.P.: For me it was a real struggle to organize this congress. Some people in mathematics were jealous and tried to prevent the congress from being held, or at least to create some difficulties for its organization. Luckily I got some support from Marchuk who was at that time Head of the Committee on Science and Technology and a member of government. Also V. Kotel'nikov, first vice president of the Academy supported me and my colleagues.

The local organizer in Tashkent was S. Kh. Sirazhdinov. At the recommendation of Romanovsky he spent 2 years from 1949 to 1951 at the Steklov Institute in Moscow to complete his Doctor of Science thesis with Kolmogorov as advisor. After that Kolmogorov offered him a position in the Department of Mathematics of Moscow University which he held for three more years. In Uzbekistan there was some resentment of people who told Kolmogorov that they would also like to work in Moscow. But life in Moscow is very different from that in Tashkent in many aspects such as climate and food, so Sirazhdinov decided to return to Tashkent. After his return he soon became Director of the Mathematical Institute named after Romanovsky, then he was elected Academician, next Vice-President of the Academy of Uzbekistan and Rector of the University of Tashkent. He also held many other positions. Clearly we had a very powerful friend in Tashkent.

You ask about the advantage of organizing a meeting like this in one of the Republics. In Moscow perhaps the President of the Academy could get in touch directly with the people in government who make the decisions, but even for him this would probably not be easy. In the Republics the lines of command were much shorter. Sirazhdinov could very easily make an appointment with the first secretary or the prime-minister of the Republic Uzbekistan. Moreover, important large congresses increased the prestige of the Republic. Shirazhdinov and I visited the prime-minister and gave him a conference bag. In return he provided financial support for the conference and made 50–70 cars with drivers available for the invited speakers. Also you and I, David Kendall and David Cox lived in the residence where apartment #1 was used by Brezhnev when he came to Tashkent.

Interviewer: Yes, I vividly remember my apartment in the residence. What impressed me most were a large wall safe, a gigantic TV set and Persian carpets even in the bathroom. Also the setting in a park was very pleasant. When I asked whether I needed some kind of identification to get past the armed guards at the gate, I was told this wasn't necessary because they would recognize my face!

Yu.P.: This meeting was inconceivable in Moscow or Leningrad. So we preferred the Republics. And we are not the only ones: physicists and specialists in mechanics prefer to arrange conferences there. Also the Soviet-Japanese (later Russian-Japanese) meetings were held in Khabarovsk, Tashkent, Tbilisi, and Kiev. In all cases the reasons were the same.

Let me mention an interesting occurrence during the congress. Mark Freidlin was asked to present an invited Section report. I was in another building when he presented his paper. People came to me to tell me that he invited everyone to Israel. This did not seem possible and Freidlin's invitation seemed to have upset some people. I have my own opinion about this matter, but during a conversation with Freidlin that followed, I noticed something interesting. While we spoke, other people were standing far away from us, not approaching, but watching what we were doing. You see, this is not a new feature of Russian people. In the beginning of the nineteenth century Lermontov wrote in the introduction of 'Hero of our time' that when people see two diplomats having a friendly conversation during a dinner, they believe that each of them is betraying his country.[7] They do not understand polite behaviour, independent of one's point of view.

Interviewer: You also handed out a number of beautiful Bernoulli medals to people who worked hard to make the meeting a success.

Yu.P.: Yes, A. Holevo found a very good specialist who made the medal. Through my geologist friend I also obtained a small box made of malachite which I presented to Mr Drescher, the representative of Springer Verlag who arranged the book exhibition. He left all the books brought for the exhibition in Tashkent.

We also had help from an unexpected quarter. At one point we had to wrap up several hundreds of books and nobody had the time and energy for this chore. One of my colleagues found some children who were willing to do this for a small reward.

Interviewer: Let me add one personal memory of this meeting. The Bernoulli Society has a tradition to organize a mid-week excursion for all participants during their meetings, and some other societies now do the same. I have participated in hikes, watched various performances, sailed down the Nile, but the excursion (by airplane) to Samarkand was a lifetime experience.

[7] *Our public is like the person from the sticks who, overhearing a conversation between two diplomats belonging to hostile courts, becomes convinced that each is being false to his own government for the sake of a tender mutual friendship.* Lermontov, M. (1840). A Hero of Our Time, Author's preface to the 2nd Edition.

Applied Statistics

Interviewer: Yuri, there seems to be a feeling that applied statistics was not doing very well in the USSR. With a few notable exceptions, very little applied work of high quality became known to the outside world. What can you tell us about that?

Yu.P.: After the Tashkent conference the First Secretary of the Communist Party in Uzbekistan invited several participants. Among them was David Kendall who said that statistics in the USSR was in a bad state. One can think of several reasons for this.

When statistical data were collected, they were often kept secret. For example, the number of people having a certain blood type, or suffering from a certain disease, constituted strategic data and were therefore not made available. But even when statistical data were available, there was still the question how useful they were. People like Khruschev did not believe in the limitation of science. According to him any problem could be solved if we wanted to. However, statistics is useful only in societies in a more or less stable situation. When the situation changes from day to day and from month to month, it is generally impossible to predict the future, with or without the use of statistics. The future with Jeltsin was one, that with Putin another. With Putin's successors it may again be different. Another example: it was announced on TV that at least 20 % of all medicine is fake. Aspirin you buy may not be aspirin at all. So why study the properties of a drug when it is simpler and cheaper to sell fake stuff?

There were other periods when the Academy was heavily pressed to make some statistical recommendations. When Abraham Wald's work became known after the war, it was clear that this could produce considerable savings in finding the defective items in mass production. Academy president M. Keldysh spoke to Kolmogorov and insisted that he start investigations in this area. Students of Kolmogorov such as Sevastyanov, Bolshev and Sirazhdinov went to major factories in Moscow such as Frazer which produced drills and made some recommendations. From time to time the necessary statistical data were collected, but when our mathematicians went to the factories themselves they saw in a rather obvious way that on Mondays the percentage defectives was much higher after the workers had spent their Sundays drinking vodka. So statistical quality control is not much help, if you can't enforce a minimum amount of discipline. Some time later Kolmogorov and Bolshev studied the toxic effects of medicines and reported to the Ministry of Health, but unfortunately the results disappeared.

Interviewer: I believe that in the automobile industry they call these unsatisfactory products "Monday morning cars". I think that you are basically saying that statistics makes little sense if the rules of the game are not clear, or at least not being followed. You are absolutely right. But was quality control not enforced for important products for which correct performance is really critical?

Yu.P.: Yes, in the production of shells to be used against tanks, they did use quality control because of the major risks involved. They would develop their own standards, or use those of other countries, for instance Japan where quality control was far advanced.

Interviewer: Let me try to suggest another reason why applied statistics may not have been easy. Suppose you are the director of a factory that produces tractors. The 5-years plan says that you will produce 10,000 tractors. A statistician shows up and wants to know how many tractors you have produced. Of course you say 10,000 because otherwise you'll be in trouble. Statistics based on such 'data' is of course nonsense.

Yu.P.: Yes, the violation of the plan had very bad consequences for those who violated it.

There is yet one more circumstance that contributed to the unsatisfactory development of applied statistics, and perhaps applied mathematics in general. During the war and many years after, many mathematicians were involved in applied research on problems related to our national defence such as atomic research, and later to the aerospace program. Kolmogorov, for instance, investigated procedures for artillery fire and found that it is not a good idea to try and hit the target with every shot. An artificially larger dispersion around the target is usually better. Lawrentjev invented cumulative anti-tank shells. Bogoljubov, Vladimirov, Tikhonov and others were active participants in the atomic program. Before he became president of the Academy, Keldysh was closely connected with the space project and computed the trajectory of a space device travelling around the moon to get information about the back of the moon. Chentsov did important research on an aerospace problem and also on the problem of detecting underground explosions.

At the time I graduated in the late 1940s mathematicians had no problem finding jobs. Some people continued their education as PhD students, but others started working in defence related institutes. Recruiters for these institutes were sitting behind a table in one of the large auditoriums of the old university building in the centre of Moscow and asked students: "How would you like an interesting job within 5 min from a subway station?". That was about all the information they provided. When you accepted the offer, you found yourself at one of a number of very serious institutes which – for instance – did not produce rockets, but computed their trajectories.

Interviewer: I would imagine that cybernetics of Norbert Wiener could play a useful role here, but I understand the party didn't like it.

Yu.P.: There were different views concerning cybernetics: one from the Marxist theorists and another from the Defence Ministry. The defence people were more enthusiastic! Two popular books at that time were Wiener's *Extrapolation of random processes'* which was called the yellow peril because of its bright yellow cover, and volume I of Feller's *Introduction to Probability Theory and Its Applications* which gave solutions to many problems for discrete random variables, that were unknown to specialists in Moscow.

So applied mathematicians were quite active in those days, but then times changed. Officially, the Russian Federation had no more enemies and therefore no need to develop weapons. Jeltsin said that our rockets were not directed at any one particular place, so presumably their directions were uniformly distributed over the half-sphere. Defence spending went down sharply and funding for the aerospace program also decreased considerably. Of course applied research suffered accordingly.

Interviewer: Finally there was the problem of applied research that went against the official views in the USSR. A prominent example is Kolmogorov's 1940 paper *On a new confirmation of Mendel's laws* in which he showed with elementary statistical methods that the same data used by Lysenko's proponents to discredit Mendel's theory, were in fact an impressive confirmation of the theory. This was not without risk as Lysenko's ideas on inheritability of acquired characteristics were supposed to revolutionize Soviet agriculture and were strongly supported by Party leaders. For all of these reasons, it is hardly surprising that applied statistics wasn't too popular.

Yu.P.: It is interesting to note that when the Russian translation of volume I of Feller's *Introduction to Probability Theory and Its Applications* appeared, the part dealing with the theory of Mendel and Morgan was omitted. Kolmogorov agreed because he figured that it was a choice between loosing 1–2 % of the text, and not publishing the book in Russian at all. He was convinced that the complete text would be published eventually, and the second Russian edition already proved him right.

Kolmogorov

Interviewer: Toward the end of this interview you said: "I tried to tell you what I know about mathematical life in the Soviet Union". You most certainly have, and if you have made one thing clear, it is the role of the towering figure of Kolmogorov in this history. It seems almost impossible that one person could influence matters to such an extent. When he was in Moscow, Tuesday through Thursday and part of Friday, people wanted to talk to him all of the time. However he had a perfect way of recovering from this kind of stress during the remainder of the week.

Yu.Pr: His wife said: "Andrei, in Moscow you have no time to eat dinner with a phone permanently in your hand". But in the house that he and Alexandrov owned in Komarovka it was a different world. Here mathematics, culture and physical exercise played the major role.

There was gymnastic exercise and it was a strictly regulated life. At 8 a.m. coffee, milk, cheese and black bread.. At 1 p.m. lunch, consisting of half a litre of fresh milk from a neighbouring farm, white bread and jam. After lunch skiing, and in summertime some other exercise. At 5 p.m dinner. The colour of the table cloth changed every time, so there were three colours during the day. At 7 p.m.

tea, and after tea Kolmogorov invited students for scientific conversation. But that could also happen at other moments, for example during skiing, so the students had better be well prepared! After that there was time to listen to classical music or to read. Reading was often Russian classics, but for Alexandrov's students it was compulsory to read *Faust* in German. To me it all was like an oasis, like an island with a very organized life and a friendly atmosphere.

Interviewer: The people who came to Komarovka, were of course mainly from the Moscow school of probability. Leningrad was far away. However, I understand that Ildar Ibragimov was an exception and spent much time in Kolmogorov's house in Komarovka.

Yu.Pr.: Yes, Ibragimov was perhaps the only representative of the Leningrad school of probability who spent much time in Kolmogorov's house. He was between the Leningrad and the Moscow school of probability. The two of them also made long trips to South Osetia.

Interviewer: I'm happy to hear that the house in Komarovka is still being used, so perhaps we shall meet there one day! Yuri, we thank you for taking the trouble to give us this extensive interview.

A Conversation with Yuri Vasilyevich Prokhorov*

Larry Shepp

Abstract Yuri Vasilyevich Prokhorov was born on December 15, 1929. He graduated from Moscow University in 1949 and worked at the Mathematical Institute of the Academy of Sciences from 1952, and as a Professor on the faculty of Moscow University since 1957. He became a corresponding member of the Academy in 1966 and an Academician in 1972. He received the Lenin Prize in 1970. The basic directions of his research are the theory of probability. He developed asymptotic methods in the theory of probability. In the area of the classical limit theorems, he studied the conditions of applicability of the strong law of large numbers and the so-called local limit theorems for sums of independent random variables. He proposed new methods for studying limit theorems for random processes; these methods were based on studying the convergence of measures in function space. He applied these methods to establish the limiting transition from discrete processes to continuous ones. He found (in 1953 and 1956) necessary and sufficient conditions for weak convergence in function spaces. He has several papers on mathematical statistics, on queuing theory and also on the theory of stochastic control. This conversation

*This interview took place in September, 1990, at the end of the Soviet Union. Some of my questions to Yuri Vasilevich were sharply worded, and some readers have told me that my questions were disrespectful of him.

I regret this and want to apologize; in fact I have always had great respect for Yuri Vasilevich, both as a mathematician and as a human being. My questions were intended to reveal to readers what the atmosphere was like in the Stekhlov Mathematical Institute and in Moscow State University, where individuals were under pressure from the authorities to discourage gatherings and also to display hostility towards Jews. Despite this atmosphere, in my opinion and to the best of my knowledge, Yuri Vasilevich followed a very high standard of fairness and non-discrimination, when doing this was both difficult and dangerous.

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took place at the Steklov Institute in early September 1990. It was taped in Russian and translated by Abram Kagan. The final version was edited by Ingram Olkin.

The Early Years

Shepp: Yuri Vasilyevich, you are the only full member (Academician) of the USSR Academy of Sciences whose field is probability and statistics. Please draw the main lines of your biography, talking about the main events in your life since your birth. I know that you are a member of the Scientific Council of Steklov Mathematical Institute. What are the other positions at the Academy you kept in the past and keep now?

Prokhorov: I was born in Moscow on December 15, 1929. My parents also lived in Moscow, and it seems that earlier ancestors were also Moscovites. I went to school, and in the summer of 1941 when the war began the family was evacuated to a small town of Chistopol on the Volga River (about 300 miles east of Moscow), not far from Kazan. We Larry lived there for 2 years, and in 1943 came back to Moscow.

When we left for Chistopol, I finished 4 years of school. While in evacuation I had much time, and in 2 years studied the curriculum of 4 years so that I came to Moscow as a student of the 8th year. Also, in Moscow, I finished the 2-year curriculum in 1 year, and in 1944 graduated from high school.

Like my father, I wanted to become an engineer, and I first entered the Higher Technical College named after Bauman (actually, a Technical University). There, I took a class in mathematical analysis of Professor Adolph Pavlovich Yuškevič, renowned in particular by his works in the history of mathematics. Pretty soon I understood that my primary interests were in mathematics. I began taking classes at Moscow University, first as an external student and in the next year transferred to the university. My main interests at the time were in analysis and number theory, and the first seminar I attended was that of Professor Alexander Gel'fond in elementary number theory, without any theory of analytic functions.

But in the fall of 1946 Kolmogorov started, for the first time at Moscow University, a course entitled "Supplementary Chapter of Analysis". Actually, the course contained foundations of functional analysis, measure theory and theory of orthogonal series. It was a big and serious course. When I took this course – and I attended all the classes and took notes – I decided at once that it would be my field.

Shepp: Was there any special subject in Kolmogorov's course you liked most?

Prokhorov: Yes, measure theory. Simultaneously with this course, Kolmogorov began another one in probability, along the line of his book "Basic Concepts of Probability Theory". The next year Kolmogorov had a seminar in probability that I attended. Thus, my fate turned out to be tied to probability theory. Kolmogorov saw that I knew analysis and had an interest in set theoretical problems.

Shepp: What year was this?

Prokhorov: It was the fall of 1946 and spring of 1947. I was in my 3rd and 4th years at the university.

Shepp: Where were you during the war?

Prokhorov: During the war? In 1944 I entered Bauman College and in the spring of 1945, when the war was approaching its end, I transferred to the university. I was only 16 at the time. This was the way I came to Kolmogorov's seminar. The seminar was very small at that time. Its participants were A.M. Obukhov, A.S. Monin, E.B. Dynkin and B.A. Sevastyanov, who was already working in the theory of branching processes. It was a small group and the seminar increased significantly later, in a few years, about the time when V.M. Zolotarev, R.L. Dobrusin and A.A. Yuškevič (unior) were finishing their studies at the university.

Shepp: And afterwards you got a Candidate of Science (Ph.D) degree?

Prokhorov: My first paper on the strong law of large numbers was a success. It was my diploma (M.Sc.) work. My Candidate of Science work dealt with local limit theorems.

Shepp: When did all this happen?

Prokhorov: I got my Candidate degree in 1952 and at the same time changed the topic of my research. Again, under Kolmogorov's influence, I began to study distributions in functional spaces. In 1956, I wrote a dissertation on this subject for a Doctor of Science degree.

Shepp: Oh, it is your very well-known work!

Prokhorov: Yes, a larger part of it was published in our journal, *Probability Theory and Its Applications*, but a part has never been published.

Shepp: The paper was also a great success in the West. And afterwards? Please describe your career in general lines. What positions have you had?

Prokhorov: As to formal positions, in 1966 I was elected a corresponding member, and in 1972 an Academician of the USSR Academy. For many years I was a Vice-Secretary of the Mathematics Department of the Academy. This was from 1966 through 1989.

Shepp: A very long period.

Prokhorov: Such positions at the Academy were permanent at that time. Now it is different. Recently, changes have been voted for, and since the end of 1989 N.N. Bogoljubov is no longer the Academician Secretary of the Mathematics Department. Other people at the Academy were replaced as well. For 18 years I was also a Deputy Director of the Steklov Institute.

Shepp: What positions do you hold now?

Prokhorov: Now, within the Academy, I am only a member of the Bureau of the Mathematics Department. It is a relatively small position; I hold no other positions.

Shepp: Aren't you a member of the Academy Presidium?

Prokhorov: I have never been one. I used to attend meetings of the Presidium and take part in its activities in my capacity as Bogolyubov's Deputy. Bogolyubov, then Academician Secretary of the Mathematics Department, was often out of Moscow, and on those occasions I took part in the Presidium's activities. I also held some positions related to international mathematical bodies. Of them, the most significant was that of Vice- President of the International Mathematical Union that I occupied from 1978 to 1982.

International Contacts

Shepp: Yuri Vasilyevich, you and your colleagues here, in the Soviet Union, have had for many years contacts with probabilists from abroad. I shall go through the list with several names and ask you to share with me and future readers your personal reflections on meetings and talks to these people. I shall begin with Joe Doob.

Prokhorov: I have known him personally since his visit to Moscow. I think it happened in the fall of 1963 when Doob spent a few days here, and almost all of those days we spent together. I had known his work long before; as a student I had studied his papers that later became a part of his monograph on stochastic processes.

Shepp: On martingales?

Prokhorov: Yes. I had very good relations with Doob, and after 1963 we corresponded for some time. However, we did not meet any more.

Shepp: Other probabilists? Maybe you can tell something about their relations with you or other Soviet colleagues.

Prokhorov: I remember that such people as Will Feller were vividly interested in our results.

Shepp: Probably you know that Feller was my teacher. Go on, please.

Prokhorov: The first time I met Feller was at the International Congress of Mathematicians in Edinburgh in 1958. I made a closer acquaintance with him during the Fifth Berkeley Symposium in 1965 when I had an opportunity to spend a long time with him. He came to give a talk and afterwards we had a long walk together. Besides his original results, I highly appreciated his excellent two volumes on probability. Together with my students, we prepared the Russian translation of the second edition of the books. In the foreword, I had an opportunity to express my gratitude to both the author and his remarkable book. I think that for many more years to come it will be highly useful for all those who work in probability.

Shepp: I completely agree with you. What about Mark Kac?

Prokhorov: I have known him as well. The first time I met him was in Edinburgh, and later in Berkeley we again met each other, and probably elsewhere. I remember that we got along rather well. I took the initiative to translate into Russian his small, but very well-written book on statistical independence.

Shepp: May I ask you to compare his style and achievements with those of people mentioned previously? I am trying to, get a general idea of how you personally estimate different achievements. Certainly, you do not have to answer the question.

Prokhorov: In his work, other aspects were more important. (Thinks.) I can tell you that I read with interest the papers of Kac and found them very useful.

Shepp: Maybe, his approach was different? More concrete?

Prokhorov: Yes, more concrete, if you like. It is difficult to find the proper words.

Shepp: How about Harald Cramér?

Prokhorov: Cramér was well known in our country for his two books: the small book *Random Variables and Probability Distributions* from Cambridge Tracts and his larger work *Mathematical Methods of Statistics*. These were translated into Russian on Kolmogorov's initiative. By the way, the translation of the latter gave an impetus to creating Russian statistical terminology. At that time, many English statistical terms had no Russian analogs. Kolmogorov should be credited for changing this.

Cramér came to the Soviet Union in 1963 to attend the All-Union Conference on Probability and Statistics, in Tbilisi, near the Turkish border. Yu. V. Linnik and I met him in Moscow, and we spent much time together in Tbilisi. Actually, I saw Cramér before when he visited the USSR in 1955 or 1956 (this was his first visit to our country), as he remembered. But, at that time, I had not been introduced to him.

Shepp: Did Cramér have close contacts with some of your colleagues?

Prokhorov: Judging from his recollections, *Fifty Years in Probability*, published in the *Annals of Statistics*, he was closer in scientific interests to Kolmogorov, A.M. Yaglom and J.A. Rozanov, since they all worked in the field of random processes.

Shepp: Carl-Gustav Essén?

Prokhorov: I should say that his memoir of 1945 was studied here by practically everyone who works in the field of limit theorems. By the way, it is now on my desk, and I am rereading it. A significant part of the memoir, its main theorems, were included into the well-known monograph by B.V. Gnedenko and A.N. Kolmogorov. His subsequent papers, although shorter, were always noted here with great interest.

Shepp: Paul Lévy?

Prokhorov: As far back as in Kolmogorov's seminar, I began to study Lévy's monograph *Théorie de l'Addition des Variables Aléatoires* and many times returned to it. I have never met Paul Lévy; however, for a short time, we corresponded. Once, I asked him to recommend my paper for publication in *Comptes Rendus*. His monograph and the later *Concrete Problems of Functional Analysis* were very useful.

Shepp: May I ask if you know about Kolmogorov's relations with his colleagues, especially with Paul Lévy?

Prokhorov: What I know for sure is that Kolmogorov corresponded with Paul Lévy, and some of Kolmogorov's theorems were contained in his letters to Lévy.

Shepp: Kyosi Itô?

Prokhorov: I have met him, in particular, at the Soviet-Japanese symposia on probability and statistics. I never went to Japan, but Itô came here.

Shepp: I think there were many Soviet-Japanese symposia.

Prokhorov: Yes, we have had many. It seems that the first time I met Itô was in Berkeley around 1965. That visit to Berkeley was extremely useful, since during the 18 or 20 days that we spent there we met with many colleagues. It was an exceptional opportunity.

Shepp: Norbert Wiener?

Prokhorov: I have never met him.

Shepp: Maybe 'you can say something about Kolmogorov's meetings with Wiener?

Prokhorov: I have to say that during Wiener's visit to the Soviet Union, it was after WWII (I think Wiener visited the Soviet Union only once), Kolmogorov and Wiener did not meet. However, one can read Kolmogorov's article in the *Soviet Encyclopedia* entitled "*Norbert Wiener*" and will find it very interesting. Kolmogorov liked writing biographical articles. He was very proud of his article about Hilbert in the same *Soviet Encyclopedia*; it is a short article, but Kolmogorov prepared it for a long period. He also wrote about Wiener. I have heard that there was a discussion among mathematicians, at least, of the priority question relating to their work on stationary processes. I think that everything Kolmogorov wanted to say about the subject he said in the article "*Norbert Wiener*".

Shepp: Monroe Donsker?

Prokhorov: I have met him. We were working independently in almost parallel ways on the invariance principle. I began with studying the wellknown paper by Paul Erdös and Mark Kac related to the invariance principle. It contained a special case of it. Donsker and I were advancing on almost parallel courses, although by different methods.

Shepp: Henry McKean?

Prokhorov: I have never been acquainted with him.

Shepp: Frank Spitzer?

Prokhorov: I became acquainted with Spitzer in 1965, when I visited Cornell University on my way back from Berkeley to Moscow. In Ithaca we rode a canoe and almost immediately I fell into the water. Later, Spitzer visited the Soviet Union. I liked his book *Principles of Random Walk* very much and suggested a Russian translation of it; my students later translated it. Afterwards, I did not meet Spitzer any more, unfortunately, but always followed his work.

Shepp: Jerzy Neyman?

Prokhorov: (Laughs.)

Shepp: Why are you laughing?

Prokhorov: The thing is that I probably met Jerzy Neyman more often than the other people you mentioned. We met the first time in Berkeley in 1965 and then during his multiple visits to Moscow. Practically every time he came to Moscow, I had opportunities for long talks with him and attended his seminars. He was always very interested in Soviet life, both scientific and everyday. He knew Russian culture and spoke fluent Russian. By the way, he supported me when I was nominated to the Soviet Academy.

Shepp: He was a foreign member of the Soviet Academy, wasn't he?

Prokhorov: No, he wasn't, but he wrote a personal letter on my behalf when I was nominated as a corresponding member. I know that he was discussing my nomination with Sergei Natanovitch Bernstein and supported me.

Shepp: Did Neyman meet with your colleagues?

Prokhorov: Neyman used to spend much time with Kolmogorov. In particular, Neyman's works on rain stimulation were continued in Kolmogorov's laboratory at Moscow University. Some of Neyman's other work was continued at the Mathematical Institute. Neyman had good connections with many people here.

Shepp: Has anyone in the Soviet Union had any contacts with Karl or Egon Pearson?

Prokhorov: To the best of my knowledge, no.

Shepp: Ronald Fisher?

Prokhorov: It is possible that some of the older generation here could have corresponded with him, but I don't know about it.

Shepp: Kendall?

Prokhorov: Maurice Kendall?

Shepp: Both Maurice and David.

Prokhorov: Maurice Kendall's books were translated into Russian, as well as R.A. Fisher's. The monograph *Statistical Methods for Researchers* was published here several times. There was a paper by S.N. Bernstein containing a discussion of Fisher's viewpoint on confidence probabilities. Kolmogorov highly praised Fisher's works on mathematical genetics, and the last time he quoted them was in 1969 in Oberwolfach at a small conference on branching processes. Kendall's multivolume book was translated into Russian on Kolmogorov's suggestion, who praised it.

As for David Kendall, Kolmogorov knew him personally and, on a number of occasions, praised his works. David Kendall was one of those foreign scholars who, like Cramér, attended the All-Union Conference on Probability in Statistics held in Tbilisi in 1963. Actually, it was our first conference attended by our colleagues from abroad: Harald Cramér, David Kendall, Murray Rosenblatt and Jack Wolfowitz. Maybe, I forgot some; there were not that many foreigners there, but they were renowned scholars.

Shepp: Were there other scholars from abroad who had good contacts with you or other Soviet colleagues?

Prokhorov: I think we have talked about most of them, although I may have forgotten a few names.

Scientific Work

Shepp: May I ask you what you consider your main scientific or administrative achievements? I know that you have contributed much, and I ask you to describe in a few words what you consider most important.

Prokhorov: Certainly, my most successful work was on the applications of functional-analytic methods to limit theorems.

Shepp: Yes, no doubt. That paper of 1956 has been a tremendous success!

Prokhorov: In the years that followed, I returned to the subject, although in shorter papers. This is my principal contribution if we speak about mathematics. As for the administrative sphere, my greatest success may well be the organization of the 1st Congress of the Bernoulli Society in Tashkent. It required a lot of effort, and I made maximum use of all the positions I had at the Academy at the time to arrange many things related to the Congress.

Shepp: I heard that the Congress was a success, although I could not attend it.

Prokhorov: It was mainly organizational, administrative work, and I did use all my administrative possibilities in order for the Congress to take place.

Statistics in the Soviet Union

Shepp: My next question is why mathematical statistics in the Soviet Union has developed so slowly if you agree with such an assessment of the situation with statistics.

Prokhorov: Yes, I do. Both we and our colleagues from abroad see this situation. After the Tashkent Congress, when we were discussing its scientific results, David Kendall noticed a backwardness of statistics in the Soviet Union. I think the explanation is that here there has not existed a demand for serious statistical research compared with, say, those in the U.S. or England. After we learned about Abraham Wald's work and became interested in statistical acceptance control, Kolmogorov – with his students – began to work in the field and wrote a few papers. But the thing is that statistical acceptance quality control is aimed at the well-organized manufacturer, and very often the need here was not in implementing statistical control but in arranging the elementary order. Now, I think we are approaching the time when the government or its institutions have become interested in reliable statistics, and it will result in a demand for statistical researchers. As for the present situation in the Soviet Union, there is not a single statistics department. All the statisticians at universities, if there are any, come from mathematics departments.

Shepp: I think I have seen somewhere here the sign on a door, "Department of Statistics."

Prokhorov: It means a chair and not a department in your understanding of the word. Usually, it is a small unit, maybe five persons.

Shepp: Do you think glasnost will eventually help in developing statistics here?

Prokhorov: I think it may help. For example, our weekly *Arguments and Facts*, with its huge circulation, publishes in almost every issue statistical data, such as survey results. Readers are gradually becoming accustomed to statistical data.

Shepp: I think that now it is possible to describe everyday life through statistical data, certainly in newspapers. Changing direction a bit, may I ask you to describe changes at Steklov Institute after Vinogradov's death, if there are any.

Prokhorov: I can tell you that since Bogolyubov has become the Director, the Institute has hired some people who did not work here before, as for example, V. I. Arnol'd.

Shepp: I believe these changes are for the better. Is the process going on?

Prokhorov: I think so, yes.

Electronic Mail

Shepp: What do you think about the offer, now under discussion, to provide the Steklov Institute with the equipment for electronic mail correspondence?

Prokhorov: It will make postal connections with other countries easier and should be welcome.

Shepp: This offer came in a package along with the idea that the equipment (computer, modem, etc.) for E-mail correspondence be allowed for use also by mathematicians and not affiliated with the Steklov Institute, say, by members of the Moscow Mathematical Society. What is your opinion about the free access to the E-mail terminal installed at the Institute?

Prokhorov: The following is an example. The Institute has a very good mathematical library, and many mathematicians working at the university prefer to use our library for borrowing books and journals, since our library receives them earlier, and some journals can be found only at the institute's library. As a rule they are not refused. I think that if we can get something else that we can share with our colleagues working elsewhere, we shall do it.

On Discrimination

Shepp: I think it will also be good. Now I would like to pursue another direction and ask you to tell us the story about the group of students at Moscow University to which you belonged. I have heard about it from many people, but maybe you would like to add details.

Prokhorov: Yes. The story lasted for a short time, but was very instructional like many similar stories that happened at the time. Let us hope now that the times have changed and such stories are no longer possible.

Shepp: But what happened at that old time? Can you and do you want to tell us the story? I am sure that practically none of the readers of Statistical Science have ever heard about it.

Prokhorov: The story was very simple. A group of students, some of them war veterans and serious people, met at participants' homes.

Shepp: And discussed…?

Prokhorov: As I understand, there was nothing criminal there, from the participants' viewpoint. Among the participants there were serious people, war veterans and party members. Maybe, on some occasions, we showed thoughtlessness. For example, we promised each other to be together in the years to come and never be separated. But once in the form of a joke, all of these wishes were written down as a document. At that time, those things should not have been done, and even the most experienced members of the group did not understand it. This resulted in a rather severe punishment: participants were expelled from the university, and also the party members from the party. Similar things used to happen in later times, for example, in 1956 when comparatively innocent – according to present-day standards – students' actions were promptly and severely condemned. Such an incident happened at the Mathematics Department in 1956, I think.

Shepp: Why did the KGB act so promptly and uncompromisingly?

Prokhorov: The story of our group developed as follows. It was openly discussed within the party and Komsomol (young communist league) organizations. The investigation lasted for several days. A big meeting of students of the Mathematics Department took place, and professors also attended the meeting. A general accusation aimed at all members of the group was that they had formed an organization opposed to Komsomol. The accusation was based on the discovery of the origins of an organization in the meeting's record. It was a general accusation against all. Moreover, an additional accusation not directly connected, was charged against Jewish members of the group, namely because of Jewish nationalism.

Shepp: I didn't know about the second accusation, although I heard that members of the group were Jews.

Prokhorov: Yes, there were. They were accused also of Jewish nationalism.

Shepp: (Joking) You weren't among them were you?

Prokhorov: I don't know how serious these accusations were, but the words "Jewish nationalism" were spoken at the meeting. Recall the time, it was 1949. It was the time when any nationalism, Jewish in particular, was persecuted.

Shepp: Thanks for this clarification. I guess that Soviet science is falling behind. I cannot judge Soviet sciences as a whole, nor the whole of mathematics, so I am speaking mainly about probability. It seems to me that the Soviet school of probability, which under Kolmogorov and even earlier (before the revolution, and later in the twenties, thirties and forties) was a world leader, is gradually falling behind. I would like to know if at this point you agree with me, that this falling behind has resulted, to a certain degree, from discrimination. I appreciate that you already mentioned the discrimination based on the fifth paragraph. (In standard Soviet questionnaires, the fifth paragraph asks for the nationality, e.g., Russian, Ukrainian, Jewish, etc.) Certainly, there also existed discrimination based on political grounds and on some other grounds that I don't know. In any case, it was part of the academician I. M. Vinogradov's policy. Do you think some energetic actions should be taken in order to correct the situation inherited from Vinogradov?

Prokhorov: Your question turned out very long, actually consisting of two parts. The first concerns the falling behind of the Soviet probability, although relative. When I came to the Institute, the head of its Probability Department was

Kolmogorov. Bernstein worked here, although in another department, and Khinchin (A.Ya. Hinčin) and N.V. Smirnov worked in Kolmogorov's department. Earlier, Slutsky (E.E. Slutskii) also worked in the department, but by the time I came to the Institute, he had passed away. Certainly now the department, however good, has not reached that level. It is a small piece of the overall picture, but it reveals the general situation. We are facing the serious problem of how not to lose what we have inherited from our predecessors but to preserve and multiply it. A similar problem is faced by the son who inherited his father's business. He has to behave properly to push the business upward, not let it go down.

The second part of your question concerns the Mathematical Institute. One should distinguish different periods of its activities. If you address the prewar or even World War I years and look for discrimination on the basis of nationality, you will see that one of the most active researchers was Lazar Aronovitch Lusternick, for example, and the scientific secretary of the institute (i.e., actually the closest aid of Vinogradov was Alexander Lvovitch Seagel). Thus, up to a certain period, the situation looked normal.

Shepp: There was no discrimination?

Prokhorov: In any case, it was impossible to detect it. If we try to detect discrimination by statistical methods, it could be found in the postwar period.

Shepp: How do you think the situation should be changed?

Prokhorov: I think the events are now developing in such a way that the problem will be resolved automatically.

Shepp: I think we owe much to the Russian and Soviet schools of probability and have to help it in overcoming its lagging position. Personally, I am trying to provide the Mathematical Institute with the equipment for E-mail correspondence. As I have understood you, you support the idea of getting the equipment and even installing it in your department. I am glad to see that you support the idea of getting the E-mail equipment for the Institute and look to the future with optimism.

Prokhorov: The final decision will be made by the director. The scientific council also votes.

Shepp: Maybe you want to add something else concerning other topics of the interview.

Prokhorov: Yes, I would like to add the name of C.R. Rao to the list of foreign colleagues who collaborated with us. I've met him several times, was his guest in India and hosted him during his visits to our country.

Shepp: Do you collaborate with India now?

Prokhorov: Yes, in particular, there is an agreement including probability and statistics.

Shepp: Certainly, I have known of Rao's close ties to Soviet colleagues but somehow missed his name. I'm extremely glad that you recalled Rao's significant contribution to strengthening the cooperation of Soviet and foreign scholars. Do you want to make other comments?

Prokhorov: I would like to add that the now arising opportunities for personal contacts, visits to and from other countries that have become by and large more free than before will contribute much to the advancement of mathematical science and especially to the progress of younger mathematicians. Imagine that there was time after WWII when even correspondence was practically prohibited. I remember that Kolmogorov neither wrote nor received letters from abroad, and it was in a sharp contrast to the intensive correspondence before the war. Here, as probably in other countries, we face the problem of selecting able young students and directing them to probability and especially mathematical statistics. The problem is not simple at all, since at mathematics departments there is a strong competition for capable students and the probability is high that they will choose other fields of mathematics, more modern, in a sense. If an able student enters a mathematics department, the odds are high that the student chooses modern algebra or geometry rather than probability. When Kolmogorov was alive, his personality alone attracted many strong students.

Shepp: Maybe a part of the problem is also in that probability in the Soviet Union is falling behind?

Prokhorov: One more reason is evident, but somehow we have not mentioned it. All the great Russian and Soviet probabilists, starting with Chebyshev, Markov, Lyapunnov and then Bernstein, Kolmogorov, Khinchin, Linnik, were all mathematicians of broad profiles. They were not only probabilists, but knew much more. We are losing this feature of breadth and together with it connections of probability with other areas of mathematics are being lost. Maybe a similar picture can be seen elsewhere, but certainly probability does not benefit from it. We are in a difficult situation. On one side, we have to understand applications, in particular, of statistics since nobody else will do it. For example, the first papers by A.N. Shiryaev on disorder were directed toward practical applications (by the way, the very first paper was joint with Kolmogorov). Some very good papers on statistical quality control were written by Kolmogorov. Those and similar research are a probabilist's task. On the other hand, probabilists here have to keep the level of their science of probability sufficiently high. In a sense, they are carrying a double burden.

Shepp: I am glad you mentioned Shiryaev's papers on change-points. I have read them with interest and found them extremely useful.

Prokhorov: Yes, they were very good. After WWII, some papers by Kolmogorov on fire control were published in the Proceedings of Steklov Institute. Probably, they were written during the war, but like some papers by Wald, were not published at that time. Certainly, applications are important, and we have to deal with them. But look, we have fewer people working in probability and statistics than the United States.

Shepp: This happens despite the fact that probability and statistics are important for applications?

Prokhorov: Yes. When we are looking for speakers at congresses, conferences, etc., we often find this task difficult, and time and again choose the same people.

Shepp: We also have problems. Yuri Vasilyevich, thank you very much for a pleasant conversation. I wish you all good wishes in everything. We have known each other for many years, and I was glad to have this opportunity to interview you for *Statistical Science*.

Prokhorov: Thank you very much for the opportunity to give an interview for *Statistical Science*. It is a rare opportunity. Actually, it is the first interview in my life, and I ask you and future readers to excuse me in advance for all its shortcomings. Maybe, on working together on the final text, we'll be able to improve it and make it interesting and pleasant reading.

Shepp: For me, it is also the first experience as an interviewer. Thank you very much.