Chapter 7
An Application of Enhanced Knowledge Models to Fuzzy Time Series

Chung-Ming Own

Abstract. Knowledge is usually employed by domain experts to solve domain-specific problems. Huarng was the first to embed knowledge into forecasting fuzzy time series (2001). His model involved simple calculations and offers better prediction results once more supporting information has been supplied. On the other hand, Chen first proposed a high-order fuzzy time series model to overcome the drawback of existing fuzzy first-order forecasting models. Chen's model involved limited computing and came with higher accuracy than some other models. For this reason, the study is focused on these two types of models. The first model proposed here, which is referred to as a weighted model, aims to overcome the deficiency of the Huarng's model. Second, we propose another fuzzy time series model, called knowledge based high-order time series model, to deal with forecasting problems. This model aims to overcome the deficiency of the Chen's model, which depends strongly on highest-order fuzzy time series to eliminate ambiguities at forecasting and requires a vast memory for data storage. Experimental study of enrollment of University Alabama and the forecasting of a future's index show that the proposed models reflect fluctuations in fuzzy time series and provide forecast results that are more accurate than the ones obtained when using the to two referenced models.

Keywords: Fuzzy time series, knowledge model, domain specific knowledge.

1 Introduction

The forecasting of time series is crucial in daily life. It is used in forecasting the weather, earthquakes, stock fluctuations, and any phenomenon indexed by variables that change over time. Numerous investigations have solved the

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W. Pedrycz & S.-M. Chen (Eds.): Time Series Analysis, Model. & Applications, ISRL 47, pp. 139–175.
DOI: 10.1007/978-3-642-33439-9_7

associated problems by using the Moving Average, the Integrated Moving Average, and the Autoregressive Integrated Moving Average (Box & Jenkins, 1976; Janacek & Swift, 1993). Song and Chissom (1993) first defined fuzzy time series and modeled fuzzy relationships from historical data (Song & Chissom, 1979). The fuzzy time series is a novel concept that is used to solve forecasting problems that involve historical data with linguistic values. Song and Chissom (1993) used the fuzzy time series model to forecast enrollment at the University of Alabama and provided a step-by-step procedure. However, their method requires a large amount of computation time.

In reference to the time-invariant and time-variant models by Song and Chissom, Sullivan and Woodall proposed the time-invariant Markov model with linguistic labels of probability distribution (Sullivan & Woodall, 1994). Subsequently, Chen proposed a new fuzzy time series model that yielded excellent forecasting results (Chen, 1996). Chen's model simplified the complex computations of the models by Song and Chissom, and forecasted enrollment more accurately than other models. Hwang et al. (1998) proposed a method that focused on relation matrix computing for the variation between enrollments in the current year and those in past years. The Hwang et al. model was more efficient and simpler than most of the other models, although its accuracy was limited. Furthermore, Huarng (2001) solved the forecasting problem integrated the domain-specific knowledge into the fuzzy time series model. Knowledge is typically used by experts to solve domain-specific problems. In the Huarng model, available information was used to assist in the selection of proper fuzzy sets. Knowledge information can be used to solve the forecasting problem easily, and the resulting model outperformed previous models.

Chen (2002) presented a new fuzzy time series model called the high-order fuzzy time series to overcome disadvantages of current fuzzy forecasting models based on the first-order model. The Chen model came with excellent forecasting results. However, the disadvantage of the Chen model is its high dependence on high-order time series preprocessing. Additional methods have been presented to forecast Taiwan Futures Exchanges by using fuzzy forecasting techniques (Huarng & Yu, 2005, 2006a, 2006b, 2008, 2010; Huarng et al., 2007; Chen, 2008; Cheng, 2008). Leu et al. presented the distance-based fuzzy time series model to forecast exchange rates. Tanuwijaya and Chen (2009a, 2009b) also presented a clustering method to forecast enrollments at the University of Alabama. Jilani (2011) proposed a new particle swarm optimization-based multivariate fuzzy time series forecasting method. This model involves five factors with one main factor of interest. This study focused on applying swarm intelligence approaches to forecasting-related problems. Chen et al. (2012) proposed the equal frequency partitioning and fast Fourier transform algorithm to forecast stock prices in Taiwan. The results show the improving forecasting performance, and demonstrated an approach to enhance the efficiency of the fuzzy time-series.

This study proposes two enhanced models. The first model is a weighted model, which is an enhancement of the Huarng model. The proposed weighted model overcomes the disadvantage of the Huarng model, that is, the lack of an efficient measure of the significance of data in a series. Hence, the proposed model involves straightforward computation to defuzzify fuzzy forecasting with the support of knowledge and a weighting measure of the historical fuzzy sets. The second model is a high-order fuzzy time series model, which is an extension of the Chen model. The proposed high-order fuzzy time series model overcomes the disadvantage of the Chen model, which depends strongly on the derivation of highest-order fuzzy time series and requires large memory for data storage. Hence, this model has the advantage of a higher-order model and can apply the knowledge to eliminate the ambiguity in forecasting. An empirical analysis demonstrated that the two proposed fuzzy time series models can capture fluctuations in fuzzy time series and provide superior forecasting results than those coming from other models.

In this study, Section 2 briefly reviews basic concepts of fuzzy time series. In Section 3, we formulate the algorithms of the weighted knowledge and high-order models. Section 4 presents empirical analyses of enrollment and TAIFEX forecasts. Section 5 concludes this study.

2 Fuzzy Time Series

2.1 Basic Concept

Basic concepts related to fuzzy time series are reviewed below. U is the universe of discourse, $U = \{x_1, x_2 \cdots, x_k\}$. A fuzzy set A_i of U is defined as

$$A_i = \mu_{A_i}(x_1)/x_1 + \mu_{A_i}(x_2)/x_2 + \cdots + \mu_{A_i}(x_k)/x_k \,,$$

where μ_{A_i} is the membership function of A_i, $\mu_{A_i}: U \to [0,\ 1]$, and $\mu_{A_i}(x_j)$ represents the grade of membership of x_j in A_i, $\mu_{A_i}(x_j) \in [0,\ 1]$. The symbols "/" and "+" indicate the "separation" and "union" of elements in the universe of discourse U.

Definition 2.1. Let *Y*(*t*) (*t*=…, 0, 1, 2, …), a subset of R ($Y(t) \subseteq R$), be the universe of discourse in which fuzzy sets $u_i(t)$ $(i = 1,\ 2,\ \cdots)$ are defined. Assume that *F*(*t*) consists $\mu_i(t)$ (*i*=1,2, …); *F*(*t*) is called a fuzzy time series on *Y*(*t*).

From Definition 2.1, *F*(*t*) can be considered to be a linguistic variable and $u_i(t)$ $(i = 1,\ 2,\ \cdots)$ can be considered to be the possible linguistic values of *F*(*t*). The main difference between fuzzy time series and conventional time series is that the observations in the former are fuzzy sets and those of the latter are real numbers.

Definition 2.2. Suppose that *F*(*t*) is determined only by $F(t-1)$; then, there exists a fuzzy relationship $R(t-1,t)$ between *F*(*t*) and $F(t-1)$, such that

$$F(t) = F(t-1) \times R(t-1,t) \,,$$

where $\times$ is the composition operator. This relationship can also be represented as $F(t-1) \to F(t)$.

Definition 2.3. Let $F(t-1) = A_i$ and $F(t) = A_r$; a fuzzy relationship can be defined as $A_i \rightarrow A_r$. On the left-hand side of the fuzzy relationship, A_i, is called the current state of the relationship; A_r, on the right-hand side of the fuzzy relationship is called the next state of the relationship.

Definition 2.4. fuzzy relationships with the same current state can be further grouped in a combined fuzzy relationship called the grouped fuzzy relationship.

For example, some fuzzy relationships exist:

$$A_i \rightarrow A_{r_1},$$
$$A_i \rightarrow A_{r_2},$$
$$\ldots$$

These fuzzy relationships can be grouped together with the same current state and so

$$A_i \rightarrow A_{r_1}, A_{r_2}, \cdots$$

can be grouped together into a grouped fuzzy relationship.

Definition 2.5. According to Definition 2.3, if $F(t)$ is caused by more fuzzy sets, $F(t-n)$, $F(t-n+1)$, $\cdots$, and $F(t-1)$, then the fuzzy relationship can be represented as

$$A_{r_1},\ A_{r_2},\ \cdots, A_{r_n} \rightarrow A_j,$$

where $F(t-n) = A_{r_1}$, $F(t-n+1) = A_{r_2}$, $\cdots$, and $F(t-1) = A_{r_n}$. This relationship is called the nth-order fuzzy time series forecasting model. A_{r_1}, A_{r_2}, $\cdots$, and A_{r_n} are called as the current states of the time series, and A_j is called as the next state of the time series.

Accordingly, the above equation means "If the time series in the year $t-1$, $t-2$, $\cdots$, and $t-n$ are A_{r_1}, A_{r_2}, $\cdots$, and A_{r_n}, respectively, then that in the year t is A_j".

Definition 2.6. For any t, if $R(t-1,t)$ is independent of t, then $F(t)$ is called a time-invariant fuzzy time series. In contrast, if $R(t-1,t)$ is estimated from most recent observations, then $F(t)$ is called a time-variant fuzzy time series.

In this study, the models are all based on a time-invariant fuzzy time series.

2.2 Configuration of the Fuzzy Time Series

A pure fuzzy system generally comprises four parts: the fuzzifier, the fuzzy rule base, the fuzzy inference engine, and the defuzzifier. The fuzzifier is the input, which transforms a real-valued variable into a fuzzy set. The defuzzifier is the output, which transforms a fuzzy set into a real-valued variable. The fuzzy rule base represents the collection of fuzzy IF-THEN rules from human experts or domain knowledge. The fuzzy inference engine combines these fuzzy IF-THEN rules into a map from fuzzy sets in the input space, U, to fuzzy sets in the output space, V, according to the principles of fuzzy logic (Wang, 1997).

In (Song and Chissom 1993, 1994), Song and Chissom proposed both the time-variant and time-invariant models to forecast enrollments of the University of Alabama. Song and Chissom predicted fuzzy time series using historical data, and the model $F(t) = F(t-1) \times R(t-1,t)$. The procedures of time-variant models can be outlined as follows.

Step 1. Specify the universe of discourse U in which fuzzy sets will be defined;
Step 2. Partition the universe of discourse U into the even length intervals;
Step 3. Define the fuzzy sets on U;
Step 4. Fuzzify the input data x_{t-1} to $F(t-1)$;
Step 5. Forecast by the model $F(t) = F(t-1) \times R(t-1,t)$, and use the past w years data as a relationship;
Step 6. Defuzzify the output.

The main difference between the Song and Chissom's time-invariant and time-variant models is that the relationship $R(t,t-1)$ of the former must be established by all the historical data, whereas that of the latter must be determined only by some of the historical data. Figure 1 is shown the configuration of the fuzzy system to emphasize the distinguishing features of Song and Chissom's models. The fuzzifier is used to map the input to the fuzzy set $F(t)$ (corresponding to Steps 1-4). The fuzzy rule base is established based on all possible relationships. The fuzzy inference engine is used to compute by the model $F(t) = F(t-1) \times R(t-1,t)$ (corresponding to Step 5). The defuzzifier is used to transform the resulting fuzzy sets into a real-valued variable y (corresponding to Step 6).

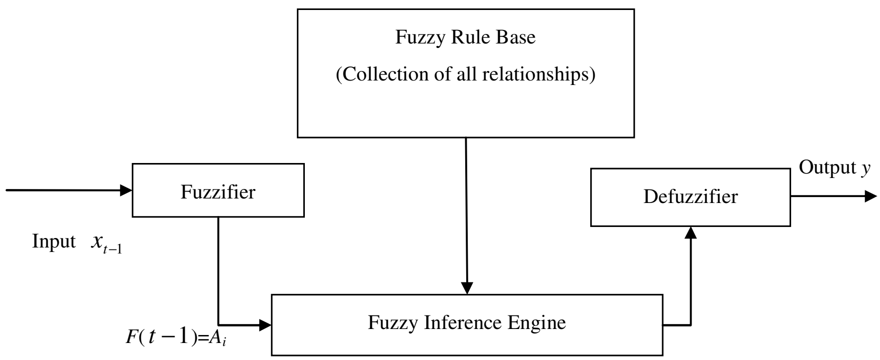

Fig. 1 The configuration of Song and Chissom's Model

The derivation of Song and Chissom's model was very tedious, and the matrix composition required a large amount of computation time. Chen proposed a model that involved straightforward knowledge reasoning to simplify the calculations in

Song and Chissom's model (Song and Chissom 1993). Chen's model not only applied simplified arithmetic operations rather than complicated max-min composition operations, but also provided more accurately forecasts than the other models. The procedures of Chen's model can be outlined as follows.

Step 1. Specify the universe of discourse U in which fuzzy sets will be defined;
Step 2. Partition the universe of discourse U into the even length intervals;
Step 3. Define the fuzzy sets on U, and fuzzify the data;
Step 4. Establish the fuzzy relationships into a group;
Step 5. Forecast;
Step 6. Defuzzify the output by using the arithmetic average.

Figure 2 shows the configuration of Chen's system. The fuzzifier is the same as in Song and Chissom's model (corresponding to Steps 1-4). In the fuzzy rule base, the important feature of Chen's model is that the fuzzy relationships are selected and summarized here (corresponding to Step 5). These computations are simple and straightforward. The defuzzifier takes an arithmetic average operation to derive the result (corresponding to Step 6). For more detail, refer to the (Song and Chissom 1993).

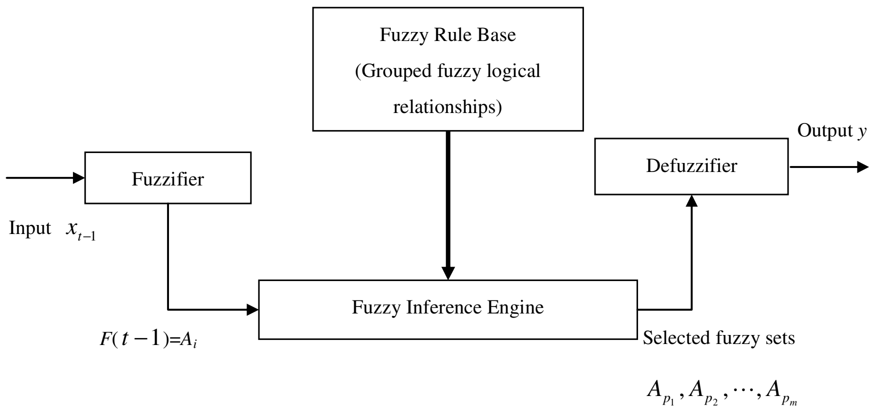

Fig. 2 The configuration of the Chen's model

Huarng's model is introduced below (Huarng 2001). Huarng improved forecasting by incorporating domain-specific knowledge into Chen's model. Experts usually apply knowledge in solving domain-specific problems. Accordingly, domain-specific knowledge is used to help to obtain the proper fuzzy sets during the forecasting. His model was easy to calculate and provided better forecasts as more supporting information was used. The procedures of Huarng's model can be outlined as follows.

Step 1. Specify the universe of discourse U in which fuzzy sets will be defined;
Step 2. Partition the universe of discourse U into the even length intervals;
Step 3. Define the fuzzy sets on U, and fuzzify the data;

Step 4. Establish the fuzzy relationships into a group;
Step 5. Forecast with knowledge assistance;
Step 6. Defuzzify the output by using the arithmetic average.

The inference engine uses two rule bases; one is the same as Chen's model; grouped fuzzy relationships (corresponding to Step 5). The other is the base of domain-specific knowledge. All the other parts are the same as Chen's model. Figure 3 is depicted the configuration of Huarng's model.

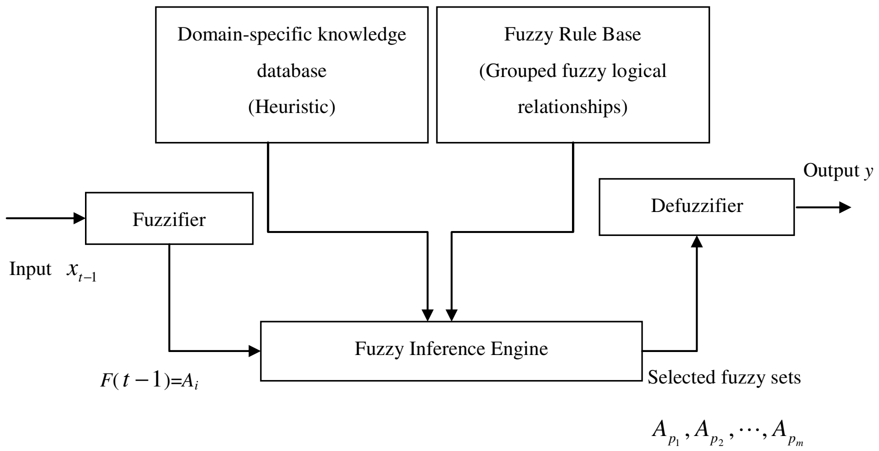

Fig. 3 The configuration of the Huarng's model

Chen (2002) proposed the high-order fuzzy time series model to improve the forecasting accuracy of his model in 1996. This new model can overcome the deficiency of the first-order fuzzy time series, which is inefficiently to eliminate the ambiguity in the forecasting. The procedure of Chen's model is outlined as follows.

Step 1. Specify the universe of discourse U in which fuzzy sets will be defined, and partition the universe of discourse U into the even length intervals;
Step 2. Define the fuzzy sets on U;
Step 3. Fuzzify the input data;
Step 4. Establish the high-order fuzzy relationship groups;
Step 5. Forecast by selecting the appropriate nth-order fuzzy relationship;
Step 6. Defuzzify the output with the elements in the nth-order fuzzy relationship.

Because this model is highly depended on the establishment of high-order fuzzy relationship, the time complexity is O(p), where p denotes the number of grouped fuzzy relationship.

3 The Proposed Model

3.1 Weighted Knowledge Model

This model improves the forecasts based on the Huarng model. First, the Huarng model was selected because it is easy to calculate. The Huarng model has the advantage of the straightforward model by Chen. Second, the Huarng model yielded superior forecasts compared to other models. Third, the Huarng model used problem-specific knowledge by using an extra information base to guide the search. However, the disadvantage of the Huarng model is its lack of an efficient measure of the significance of each fuzzy set in fuzzy relationships; that is, every fuzzy set in a grouped fuzzy relationship has the resembling trajectories in the Huarng model. This is reflected by the use of the arithmetic average in the defuzzifier. The significance of fuzzy sets can be stressed by various measures; that is, defuzzification varies according to the observed information. The proposed model is based on the weighted measure of historical information and the frequencies of the fuzzy sets to adjust their ratios. Hence, this study considered the support of weighted measures and knowledge for the proposed model, which is introduced in the following paragraphs.

Step 1. **Define the universe of discourse and partition the intervals.** According to the problem domain, the universe of discourse U can be determined. Then, let the universe of discourse be partitioned into intervals $u_1 = [a_1,\ a_2],\ \ u_2 = [a_1,\ a_2],\ \cdots,\ \ ,u_n = [a_n,\ a_{n+1}]$ of even length, where u_i is the ith divided interval. The midpoints of these intervals are $m_1,\ m_2,\ \cdots, m_n$, respectively.

Step 2. **Define the fuzzy sets and fuzzify the data.** Subsequently, let $A_1,\ A_2,\ \cdots,\ A_n$ be fuzzy sets, all of which are labeled by possible linguistic values. For example, linguistic values can be applied as fuzzy sets; A_1=(not many), A_2= (not too many), A_3=(many), A_4=(many many), A_5=(very many), A_6=(too many), A_7= (too many many). Hence, A_i is defined on as

$$A_i = \mu_{A_i}(u_1)/u_1 + \mu_{A_i}(u_2)/u_2 + \cdots + \mu_{A_i}(u_n)/u_n\ , \tag{1}$$

where u_i is the interval expressed as an element of the fuzzy set, $i = 1,\ 2,\ \cdots,\ n$. $\mu_{A_i}(u_j)$ states the degree to which u_j belongs to A_i, and $\mu_{A_i}(u_j) \in [0,1]$, Then, the historical data are fuzzified by the intervals and expressed in the forms of linguistic values. Note that Eq. (1) uses interval u_j as an equation element.

Step 3. **Establish and group fuzzy relationships.** According to the definition of relationship in Definition 2.3, the fuzzy relationship can be determined. For instance,

$$A_j \to A_r,$$
$$A_j \to A_s,$$
$$A_j \to A_t,$$
$$\cdots,$$
$$A_m \to A_r,$$
$$A_m \to A_q,$$
$$\ldots$$

According to Definition 2.4, the fuzzy relationships can be grouped by the same origin:

$$A_j \to A_r, A_s, A_t,$$
$$A_m \to A_r, A_q,$$
$$\ldots$$

Step 4. **Measure the frequency of fuzzy sets shown in the fuzzy relationships.** Subsequently, according to the fuzzy relationships obtained in Step 3, calculates the frequency of each fuzzy set shown in the fuzzy relationships. For expressive simplicity, the frequency for fuzzy set A_i is denoted as f_i.

Example 3.1: Suppose that the fuzzy relationships calculated from the data set are obtained below:

$$A_r \to A_{r_1},$$
$$A_s \to A_{r_3},$$
$$A_r \to A_{r_1},$$
$$A_r \to A_{r_1},$$
$$A_r \to A_{r_2}.$$

Hence, fuzzy set A_{r_1} shown in relationship $A_r \to A_{r_1}$ occurs three times, denoted as f_{r_1} =3. Fuzzy set A_{r_2} shown in relationship $A_r \to A_{r_2}$ occurs once, denoted as f_{r_2} =1. Fuzzy set A_{r_3} shown in relationship $A_r \to A_{r_3}$ occurs once, denoted as f_{r_3} =1.

Step 5. **Introduce knowledge and establish selection strategy.** In this proposed model, knowledge is used to guide the selection of proper fuzzy sets. Concerning knowledge in this study, changes in time series are used as a variable. According to the changes, the trend in selection strategy specifies the difference between times to an increase, a decrease or no change, and the symbolization of trend, α is set to 1, -1 and 0, respectively. The trend for an increase is used as a trigger to select whose fuzzy sets they have higher ranking in the grouped fuzzy relationship. On the contrary, the trend for a decrease is used to select whose fuzzy sets they have lower ranking. The trend for no change means to select the current state of the grouped fuzzy relationship. The selection strategy is introduced as follows.

Consider all the fuzzy sets $A_1, A_2, \cdots, A_n$ are ordered in accordance. That is, $A_1, A_2, \cdots, A_n$ are fuzzy sets on intervals $[a_1, a_2], [a_2, a_3], \cdots, [a_n, a_{n+1}]$, respectively, where $a_1 < a_2 < \cdots < a_{n+1}$. Suppose that $F(t-1) = A_i$ and the grouped fuzzy relationship of A_i is $A_i \to A_{i_1}, A_{i_2}, \cdots, A_{i_k}, \cdots, A_{i_\ell}$, where $i_1 < i_2 < \cdots < i_k < \cdots < i_\ell$.

Definition 3.1. Accordingly, all the fuzzy sets are partitioned into two parts; high and low parts. High part includes the fuzzy sets in high ranking; on the contrary, low part includes the fuzzy sets in low ranking. If $i_{k-1} < i < i_k$, then fuzzy sets in the low part are $\{A_{i_1}, A_{i_2}, \cdots, A_{i_{k-1}}\}$, and fuzzy sets in the high part are $\{A_{i_k}, A_{i_{k+1}}, \cdots, A_{i_\ell}\}$. Otherwise, If $i = i_k$, then fuzzy sets in the low part are $\{A_{i_1}, A_{i_2}, \cdots, A_{i_k}\}$, and fuzzy sets in the high part are $\{A_{i_k}, A_{i_{k+1}}, \cdots, A_{i_\ell}\}$. Note that, if the low/high part is empty set, then the low/high part needs to include their current stat of the relationship for instead, that is A_i.

Hence, the fuzzy sets in the high part mean to be selected in higher ranking, and fuzzy sets in the low part mean to be selected in lower ranking. The selection strategy of fuzzy sets is: If the trend of time series leads to an increase, the fuzzy sets in the high part are all selected, else if the trend of time series leads to a decrease, the fuzzy sets in the low part are all selected, else if the trend of time series leads to no change, the existing state of affairs would be preferred, the origin fuzzy set A_i is selected.

Step 6. **Establish the weighted function**. Suppose that the grouped fuzzy relationship of A_i is $A_i \to A_{i_1}, A_{i_2}, ..., A_{i_j}, ..., A_{i_k}, ..., A_{i_m}$, where $j < k < \ell < m$. For the purpose to stress the significance of fuzzy sets, the weighting of each fuzzy set is applied as the factor during the forecasting.

Definition 3.2. The weighting of the fuzzy set A_{i_j} in the low part is computed as the probability of frequency defined by

$$l_{i_j} = \frac{f_{i_j}}{f_{i_1} + f_{i_2} + \cdots + f_{i_Q}}, \tag{2}$$

where $Q = k-1$, if $i_{k-1} < i < i_k$, and $Q = k$, if $i = i_k$. f_{i_j} is denoted as the frequency of fuzzy set A_{i_j}. On the other hand, the weighting of the fuzzy set A_{i_ℓ} in the high part is computed as the probability of frequency defined by

$$l_{i_j} = \frac{f_{i_\ell}}{f_{i_k} + \cdots + f_{i_m}}. \tag{3}$$

Hence, for each grouped fuzzy relationships, the weighted function is established as follows.

$$h_i(\alpha)|_{\alpha=-1,0,1}=$$
$$\frac{\alpha(\alpha-1)}{2}\sum_{j=i_1,\cdots,i_Q} l_j m_j,+\frac{2(1+\alpha)(1-\alpha)}{2}m_i+\frac{\alpha(\alpha+1)}{2}\sum_{j=i_k,\cdots,i_m} q_j m_j \tag{4}$$

where $Q=k-1$, if $i_{k-1}<i<i_k$, and $Q=k$, if $i=i_k$. Note that $h_i(\alpha)$ is derived corresponding to the original fuzzy set A_i of each grouped fuzzy relationships, $i=1,\cdots,n$. α is the variable to select the proper fuzzy sets derived from Step 5. m_j is the midpoint of the interval u_j where the maximum membership value of fuzzy set A_j occurs.

Example 3.2: Consider the same problem as in Example 3.1, in which five fuzzy relationships include. The grouped fuzzy relationships are

$$A_r \rightarrow A_{r_1}, A_{r_2},\ (\text{where } f_{r_1}=3,\ f_{r_2}=1)$$
$$A_s \rightarrow A_{r_3}.\ (\text{where } f_{r_3}=1)$$

Hence, the weighted functions are

$$h_r(\alpha)|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}m_r+\frac{2(1+\alpha)(1-\alpha)}{2}m_r+\frac{\alpha(\alpha+1)}{2}\left(\frac{3}{4}m_{r_1}+\frac{1}{4}m_{r_2}\right),$$

$$h_m(\alpha)|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}m_{r_3}+\frac{2(1+\alpha)(1-\alpha)}{2}m_s+\frac{\alpha(\alpha+1)}{2}m_s,$$

where the ranking of fuzzy sets are $r<r_1<r_2$ and $r_3<s$.

Step 7: **Calculate the forecasted outputs.** Subsequently, suppose that input x_{t-1} in time $t-1$ is fuzzified to $F(t-1)=A_i$, the calculations are carried out as follows. Because the fuzzy set is A_i, then the corresponding weighted function $h_i(\alpha)$ is selected, $i=1,\cdots,n$. In this study, the difference between time $t-1$ and t leads to an increase, a decrease or no change, denoted as parameter $\alpha=1$, -1 and 0, respectively. Accordingly, the output is derived as $h_i(\alpha)|_{\alpha=-1,0,1}$.

According to the configuration of fuzzy systems, the weighted function involves the parts of fuzzy inference engine and defuzzifier. Figure 4 presents the configuration of the weighted model.

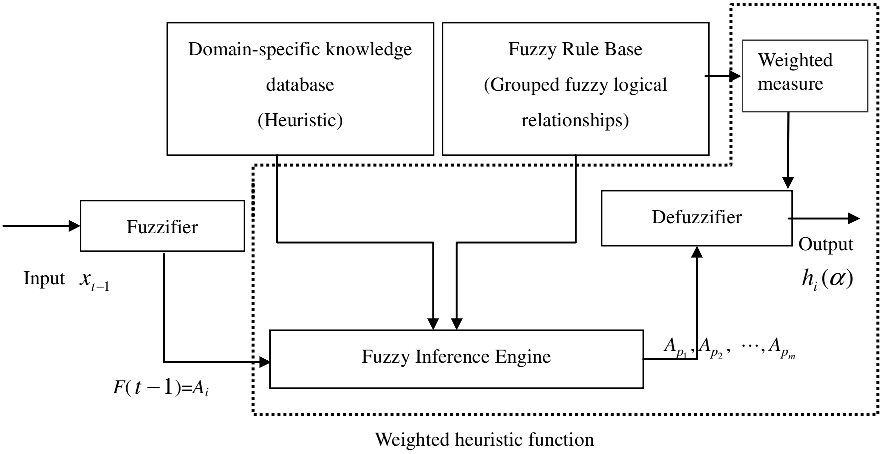

Fig. 4 The configuration of the weighted model

3.2 *High-Order Model*

This study aimed to overcome the deficiency of the Chen model (2002), which is strongly dependent on the derivation of highest-order fuzzy time series and requires a large amount of memory. In other words, according to Chen's definition, if an ambiguity occurs in the *i*th-order fuzzy relationship groups, the model seeks a higher order, such as (*i*+1)th-order fuzzy relationship, to perform the forecast. The highest-order fuzzy relationship must be computed before the model can conduct forecasts. Thus, the model requires a large amount of memory to derive the fuzzy relationships from the lowest order to the highest order.

Knowledge was applied to the high-order fuzzy time series model to eliminate the computation "bottleneck." In the domain of expert systems, knowledge is typically considered guides that can be used by domain experts to solve domain-specific problems (Russell & Norvig, 1995). Based on Huarng's assumption (2001), knowledge is used to guide the search for suitable fuzzy sets appropriate for forecasting indices. Therefore, this study enhanced Chen's model by integrating knowledge with high-order fuzzy time series to eliminate ambiguities in forecasting. Thus, the proposed model can be restricted to lower-order fuzzy time series to achieve acceptable forecast accuracy and required memory.

Step 1: **Define the universe of discourse and partition the intervals.** According to the problem domain, the universe of discourse U can be determined. Then, let the universe of discourse be partitioned into intervals $u_1 = [a_1, a_2], \ u_2 = [a_1, a_2], \ \cdots, \ u_n = [a_n, a_{n+1}]$ of even length, where u_i is the *i*th divided interval. The midpoints of these intervals are symbolized as $m_1, m_2, \cdots, m_n$, respectively.

Step 2. **Define the fuzzy sets and fuzzify the data.** Subsequently, let $A_1, A_2, \cdots, A_n$ be fuzzy sets, all of which are labeled by possible linguistic values. For example, linguistic values can be applied as fuzzy sets; A_1=(not many), A_2= (not too many), A_3=(many), A_4=(many many), A_5=(very many), A_6=(too many), A_7= (too many many). Hence, A_i is defined on as

$$A_i = \mu_{A_i}(u_1)/u_1 + \mu_{A_i}(u_2)/u_2 + \cdots + \mu_{A_i}(u_n)/u_n, \tag{5}$$

where u_i is the interval expressed as an element of the fuzzy set, $i = 1, 2, \cdots, n$. $\mu_{A_i}(u_j)$ states the degree to which u_j belongs to A_i, and $\mu_{A_i}(u_j) \in [0,1]$. Then, the historical data are fuzzified by the intervals and expressed in the forms of linguistic values. Note that Eq. (5) uses the interval u_j as an element.

Step 3. **Establish and group the *n*th-order fuzzy relationships.** The *n*th-order fuzzy relationships are established based on the fuzzified historical time series. Besides, if there are ambiguities, these fuzzy relationships are grouped together according to Definition 2.4.

Step 4. **Introduce knowledge and establish the knowledge function.** Changes in time series (the trend) are used as the knowledge to specify the difference between times to an increase, a decrease or no change. For instance, the input in the year $t-1$ is x_{t-1} and year t is x_t. Then the trend leads to an increase, if $x_t - x_{t-1} > 0$, denoted as $\alpha = 1$ for the simplification. The trend leads to a decrease, if $x_t - x_{t-1} < 0$, denoted as $\alpha = 0$. The trend leads to no change, if $x_t - x_{t-1} = 0$, denoted as $\alpha = -1$.

Consider all the fuzzy sets $A_1, A_2, \cdots, A_n$ are well ordered. That is, $A_1, A_2, \cdots, A_n$ are fuzzy sets on intervals $[a_1, a_2], [a_2, a_3], \cdots, [a_n, a_{n+1}]$, respectively, where $a_1 < a_2 < \cdots < a_{n+1}$. Suppose that there is a certain ambiguity in the *i*th-order fuzzy relationship, and they are grouped as

$A_{r_1}, A_{r_2}, \cdots, A_{r_i} \to A_{j_1}, A_{j_2}, \cdots, A_{j_k}, \cdots, A_{j_\ell}$,

where $j_1 < j_2 < \cdots < j_k < \cdots < j_\ell$.

According to fuzzy set A_{r_i} at the right most side of the current states, all the fuzzy sets in the next states of the grouped fuzzy relationship are partitioned into two parts; high and low parts. If $j_{k-1} < r_i < j_k$, then fuzzy sets in the low part are $\{A_{j_1}, A_{j_2}, \cdots, A_{j_{k-1}}\}$, and fuzzy sets in the high part are $\{A_{j_k}, A_{j_{k+1}}, \cdots, A_{j_\ell}\}$. Otherwise, if $r_i = j_k$, then fuzzy sets in the low part are $\{A_{j_1}, A_{j_2}, \cdots, A_{j_k}\}$, and fuzzy sets in the high part are $\{A_{j_k}, A_{j_{k+1}}, \cdots, A_{j_\ell}\}$.

Hence, the fuzzy sets in the high part mean to be selected in higher ranking, and fuzzy sets in the low part mean to be selected in lower ranking. The selection strategy is: If the trend of time series leads to an increase, the fuzzy sets in the high part are all selected, else if the trend of time series leads to a decrease, the fuzzy sets in the low part are all selected, else if the trend of time series leads to no change, the existing current states are selected. According to the selection strategy, in this proposed model, the knowledge function accepts the relevant trend α, the fuzzy set of the right most current states and grouped fuzzy relationships as parameters. The knowledge function is established as follows:

$$h(\alpha; A_{r_i}; A_{j_1}, A_{j_2}, \cdots, A_{j_\ell}) =$$
$$\begin{cases} \dfrac{(1-\alpha)}{k-1} \displaystyle\sum_{g=j_1,\cdots,j_{k-1}} m_g + \dfrac{\alpha}{(\ell-k+1)} \displaystyle\sum_{g=j_k,\cdots,j_\ell} m_g, & \text{if } \alpha = 0 \text{ or } 1, \text{and } j_{k-1} < r_i < j_k, \\ \dfrac{(1-\alpha)}{k} \displaystyle\sum_{g=j_1,\cdots,j_k} m_g + \dfrac{\alpha}{(\ell-k+1)} \displaystyle\sum_{g=j_k,\cdots,j_\ell} m_g, & \text{if } \alpha = 0 \text{ or } 1, \text{and } r_i = j_k, \\ \dfrac{1}{i} \displaystyle\sum_{g=r_1, r_2,\cdots,r_i} m_g, & \text{if } \alpha = -1. \end{cases} \quad (6)$$

Note that α is the variable. m_{j_k} is the midpoint of the interval μ_{j_k} where the maximum membership value of fuzzy set A_{j_k} occurs.

Step 5. **Calculate the forecasted outputs.** The calculations are implemented as follows.

(1) If the *i*th-order fuzzified history time series for time *t* are $A_{r_1}, A_{r_2}, \cdots, A_{r_i}$, where $i \geq 2$, and there is the following fuzzy relationship in the *i*th grouped order fuzzy relationships shown as follows:

$$A_{r_1}, A_{r_2}, \cdots, A_{r_i} \rightarrow A_j .$$

The forecasted fuzzy set at time *t* is A_j, and the forecasting result is m_j, it is the midpoint of the interval u_j where the maximum membership value of fuzzy set A_j occurs.

(2) If the *i*th-order fuzzified history time series for time *t* are $A_{r_1}, A_{r_2}, \cdots, A_{r_i}$, where $i \geq 2$, and there is the following fuzzy relationship in the *i*th grouped order fuzzy relationships shown as follows:

$$A_{r_1}, A_{r_2}, \cdots, A_{r_i} \rightarrow A_{j_1}, A_{j_2}, \cdots, A_{j_k}, \cdots, A_{j_\ell} ,$$

where $j_1 < j_2 < \cdots < j_k < \cdots < j_\ell$. Then, the function is applied to eliminate the ambiguity and obtain the forecasting result, $h(\alpha; A_{r_i}; A_{j_1}, A_{j_2}, \cdots, A_{j_\ell})$, where the difference between time *t*-1 and *t* leads to an increase, a decrease or no change, denoted as parameter α=1, 0 and -1, respectively. Accordingly, the output is derived as $h(\alpha; A_{r_i}; A_{j_1}, A_{j_2}, \cdots, A_{j_\ell}) |_{\alpha=1,\, 0,\, -1}$.

4 Forecasting Experiment

4.1 Forecasting Enrollment with Weighted Knowledge Model

The proposed weighted knowledge model was applied for effective forecasting of university enrollments. Enrollments from 1971 to 1992 at the University of Alabama were already forecast in a series of experiments. The forecasting of enrollments using the weighted model is detailed in the following paragraphs.

Step 1. As in Table 1, the historical data on enrollments of the University of Alabama yields U=[13000, 20000]. The universe of the discourse is divided into seven equally long intervals u_1, u_2, …, u_7 with length 1000, where u_1=[13000, 14000], u_2=[14000, 15000], u_3=[15000, 16000], u_4=[16000, 17000], u_5=[17000, 18000], u_6=[18000, 19000], u_7=[19000, 20000].

Step 2. The enrollments of the University of Alabama can be represented as seven fuzzy sets A_i (i=1, 2, …, 7). The linguistic values are A_1=(not many), A_2=(not too many), A_3=(many), A_4=(many many), A_5=(very many), A_6=(too many) , A_7=(too many many). Each A_i (i=1, 2, …, 7) is defined as follows.

$$A_1=1/u_1 + 0.5/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7,$$

$$A_2=0.5/u_1 + 1/u_2 + 0.5/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7,$$

$$A_3=0/u_1 + 0.5/u_2 + 1/u_3 + 0.5/u_4 + 0/u_5 + 0/u_6 + 0/u_7,$$

$$A_4=0/u_1 + 0/u_2 + 0.5/u_3 + 1/u_4 + 0.5/u_5 + 0/u_6 + 0/u_7,$$

$$A_5=0/u_1 + 0/u_2 + 0/u_3 + 0.5/u_4 + 1/u_5 + 0.5/u_6 + 0/u_7,$$

$$A_6=0/u_1 + 0/u_2 + 0\ /u_3 + 0/u_4 + 0.5/u_5 + 1/u_6 + 0.5/u_7,$$

$$A_7=0/u_1 + 0/u_2 + 0\ /u_3 + 0/u_4 + 0/u_5 + 0.5/u_6 + 1/u_7.$$

Table 1 lists the corresponding fuzzy enrollment A_i.

Step 3. The fuzzy relationships are established and grouped. Table 2 lists the fuzzy relationships derived from Table 1. Table 3 lists the grouped fuzzy relationships.

Step 4. Subsequently, the frequency of the fuzzy set in each fuzzy relationship is calculated and recorded in the appendix of Table A-1.

Step 5. The existence knowledge regarding the trend of increase or decrease in university enrollment is referred from the Huarng in [1]. This trend of increase or decrease is used as a guide in selecting the proper fuzzy sets for forecasting enrollment. The increase, unchanged, decrease of trends are symbolized as $\alpha = 1$, -1 or 0, respectively.

Table 1 Enrollment data sets

Years	Enrollments	Fuzzy Set	Years	Enrollments	Fuzzy Set
1971	13055	A_1	1972	13563	A_1
1973	13867	A_1	1974	14696	A_2
1975	15460	A_3	1976	15311	A_3
1977	15603	A_3	1978	15861	A_3
1979	16807	A_4	1980	16919	A_4
1981	16388	A_4	1982	15433	A_3
1983	15497	A_3	1984	15145	A_3
1985	15163	A_3	1986	15984	A_3
1987	16859	A_4	1988	18150	A_6
1989	18970	A_6	1990	19328	A_7
1991	19337	A_7	1992	18876	A_6
1993	18909	A_6	1994	18707	A_6
1995	18561	A_6	1996	17572	A_5
1997	17877	A_5	1998	17929	A_5
1999	18267	A_6	2000	18859	A_6
2001	18735	A_6	2002	19181	A_7
2003	19828	A_7	2004	20512	A_8

Table 2 Enrollment of fuzzy relationships

$A_1 \to A_1, \quad A_1 \to A_2$

$A_2 \to A_3$

$A_3 \to A_3, \quad A_3 \to A_4$

$A_4 \to A_3, \quad A_4 \to A_4, \quad A_4 \to A_6$

$A_6 \to A_6, \quad A_6 \to A_7$

$A_7 \to A_6, \quad A_7 \to A_7$

Table 3 Grouped fuzzy relationships

$A_1 \to A_1, A_2$
$A_2 \to A_3$
$A_3 \to A_3, A_4$
$A_4 \to A_3, A_4, A_6$
$A_6 \to A_6, A_7$
$A_7 \to A_6, A_7$

Hence, according to Definition 3.1, the selection strategy of fuzzy sets is: if the annual trend in university enrollment leads to an increase, then the fuzzy sets in the high part are all selected. Conversely, if the annual trend in university enrollment leads to a decrease, then the fuzzy sets in the low part are all selected, or the annual trend in university enrollment leads to no change, then the original fuzzy sets is selected.

Step 6. According to the grouped fuzzy relationships in Table 3, the corresponding weighted knowledge functions can be established and are listed as follows:

$$h_1(\alpha)\,|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}m_1+\frac{2(1+\alpha)(1-\alpha)}{2}m_1+\frac{\alpha(\alpha+1)}{2}(\frac{2}{3}m_1+\frac{1}{3}m_2),$$

$$h_2(\alpha)\,|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}m_2+\frac{2(1+\alpha)(1-\alpha)}{2}m_2+\frac{\alpha(\alpha+1)}{2}m_3,$$

$$h_3(\alpha)\,|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}m_3+\frac{2(1-\alpha)(1+\alpha)}{2}m_3+\frac{\alpha(\alpha+1)}{2}(\frac{7}{9}m_3+\frac{2}{9}m_4),$$

$$h_4(\alpha)\,|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}(\frac{1}{3}m_3+\frac{2}{3}m_4)+\frac{2(1-\alpha)(1+\alpha)}{2}m_4+\frac{\alpha(\alpha+1)}{2}(\frac{2}{3}m_4+\frac{1}{3}m_6)$$

$$h_6(\alpha)\,|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}m_6+\frac{2(1-\alpha)(1+\alpha)}{2}m_6+\frac{\alpha(\alpha+1)}{2}(\frac{1}{2}m_6+\frac{1}{2}m_7),$$

$$h_7(\alpha)\,|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}(\frac{1}{2}m_6+\frac{1}{2}m_7)+\frac{2(1-\alpha)(1+\alpha)}{2}m_7+\frac{\alpha(\alpha+1)}{2}m_7,$$

where m_k is the midpoint of the interval u_k, and $m_1=13500$, $m_2=14500$, m_3=15500, m_4=16500, m_5=17500, m_6=18500 and m_7=19500.

Step 7: Subsequently, suppose that input x_{t-1} in the year $t-1$ is fuzzified to $F(t-1)=A_i$, the corresponding weighted function $h_i(\alpha)$ is selected.

Accordingly, the output is derived as $h_i(\alpha)|_{\alpha=-1,0,1}$. The following examples are used to demonstrate the procedure of selecting the corresponding weighted function and using the knowledge to derive the forecasts.

[years 1972, 1973, 1974]: The enrollment in 1971 was 13055(A_1), in 1972 was 13563(A_1) and in 1973 was 13867(A_1). While forecasting 1972, the grouped fuzzy relationship of A_1 is $A_1 \rightarrow A_1, A_2$, so the weighted function $h_1(\alpha)$ is selected. Suppose that the knowledge points to an increase for the enrollment forecasts in 1972. Hence, α is set to 1. The forecast in 1972 is

$$h_1(\alpha)|_{\alpha=1} = \frac{2}{3}m_1 + \frac{1}{3}m_2 = 13833\,.$$

That is, the enrollment forecast for the year 1972 is 13833. However, the actual enrollment in 1972 was 13522. Therefore, the forecasting error is 1.99%. The main goal in this paper is to minimize the forecasting error. Meanwhile, the trend in 1973 and 1974 leads the knowledge to an increase. Hence, the forecasts for 1973 and 1974 are both 13833.

[year 1975]: The enrollment in 1974 was 14696 (A_2). The weighted function $h_2(\alpha)$ is determined). Meanwhile, suppose that the knowledge points to an increase for the enrollment forecast in 1975, so α=1. Therefore, the weighted function is

$$h_2(\alpha)|_{\alpha=1} = m_3 = 15500\,.$$

That is, the forecast for 1975 is 15500.

[year 1976]: The enrollment of 1975 was 15460(A_3). The weighted function $h_3(\alpha)$ is selected. Meanwhile, suppose that the knowledge points to a decrease for the enrollment forecast in 1976, so α=-1. Therefore, the weighted function is

$$h_3(\alpha)|_{\alpha=-1} = m_3 = 15500\,.$$

That is, the forecast for 1976 is 15500.

[years 1977, 1978, 1979]: The enrollment of 1976 was 15311(A_3), 1977 was 15603(A_3), and 1978 was 15861(A_3). The grouped fuzzy relationship of A_3 is $A_3 \rightarrow A_3, A_4$, so the weighted function $h_3(\alpha)$ is selected for forecasting year 1977. Meanwhile, suppose that the knowledge points to an increase for the enrollment forecast in 1977, so α=1. The forecast for 1977 is

$$h_3(\alpha)|_{\alpha=1} = \frac{7}{9}m_3 + \frac{1}{9}m_4 = 15722.$$

Meanwhile, the enrollment trends in 1978 and 1979 both lead the knowledge to an increase. Hence, the forecasts for 1978 and 1979 are both 15722.

[year 1980]: The enrollment of 1979 was 16807(A_4). The weighted function $h_4(\alpha)$ is selected. Meanwhile, suppose that the knowledge points to an increase for the enrollment forecast in 1980, so α=1. Hence, the weighted function is

$$h_4(\alpha)|_{\alpha=1} = \frac{2}{3}m_4 + \frac{1}{3}m_6 = 17167.$$

That is, the forecast for the year 1980 is 17167.

Table 4 shows all of the remaining forecasts, and compares various studies of fuzzy time series used for forecasting enrollments. Suppose that knowledge is available. Empirical analysis yields average forecasting errors of 3.22% and 4.38% by Song and Chissom's two models, respectively (1993, 1994), 3.11% by Chen's model (2002), and 2.45% by Huarng's model (2001). This proposed weighted model, however, has an error of 2.24%. In enrollment forecasting, the proposed model outperforms the others.

Table 4 Comparison of enrollment forecasting

Years	Enrollment	Song-I [5]	Song-II [11]	Chen [7]	Huarng [1]	This proposed model
1971	13055					
1972	13563	14000		14000	14000	13833
1973	13867	14000		14000	14000	13833
1974	14696	14000		14000	14000	13833
1975	15460	15500	14700	15500	15500	15500
1976	15311	16000	14800	16000	15500	15500
1977	15603	16000	15400	16000	16000	15722
1978	15861	16000	15500	16000	16000	15722
1979	16807	16000	15500	16000	16000	15722
1980	16919	16813	16800	16833	17500	17167
1981	16388	16813	16200	16833	16000	16167
1982	15433	16789	16400	16833	16000	16167
1983	15497	16000	16800	16000	16000	15500
1984	15145	16000	16400	16000	15500	15500
1985	15163	16000	15500	16000	16000	15500
1986	15984	16000	15500	16000	16000	15722
1987	16859	16000	15500	16000	16000	15722

Table 4 (*continued*)

1988	18150	16813	16800	16833	17500	17167
1989	18970	19000	19300	19000	19000	19000
1990	19328	19000	17800	19000	19000	19000
1991	19337	19000	19300	19000	19500	19500
1992	18876	No forecasting	19600	19000	19000	19000
		3.22%	4.38%	3.11%	2.45%	2.24%

4.2 Robust Forecasting with the Memorizing Capability

In the empirical case, the enrollments for weighted measure and performance forecasting were derived from the same years, and prior knowledge was constructed by analyzing the obtained information. This is referred to as the memorizing capability. Consequently, another robust capability was considered. To evaluate robustness, the weighted measure and performance forecasting must originate from different sources. Therefore, the enrollments of fuzzy relationships were grouped and analyzed from 1971 to 1992 at the University of Alabama, and robustness was tested according to the enrollments from 1993 to 2004. The authors used current knowledge on the annual increase or decrease in university enrollment. This trend of increase, decrease, or no change was used as a guide to select the proper fuzzy sets for forecasting enrollment. For example, forecasting the enrollment in the year t was dependent on the difference between years $t-2$ and $t-1$. The positive difference led to an increase, and $\alpha = 1$. Conversely, the negative difference led to a decrease, and $\alpha = -1$. A difference of less than 100 led to no change, and $\alpha = 0$.

Enrollment forecasting using the weighted model proceeded as described. Table 5 shows various studies on fuzzy time series used for forecasting enrollment. Empirical analysis yielded average forecasting errors of 2.99% when using the Chen model, 2.61% when using the Huarng model, and 2.31% when using the proposed model. Therefore, the proposed model outperformed the other models in robust forecasting.

Table 5 Comparison of enrollment forecasting with robustness (1993 ~ 2004)

	Chen's Model [7]	Huarng's model [1]	This proposed model
Average error	2.99%	2.61%	2.31%

4.3 Forecasting TAIFEX with Weighted Model

Forecasting of the Taiwan Futures Exchange (TAIFEX) was used to demonstrate the advantages of the proposed model (Huarng, 2001). The Taiwan Stock Exchange Capitalization-Weighted Stock Index (TAIEX) was used as knowledge to evaluate the trend over a number of days. In other words, any two consecutive days in the TAIEX reflect gains or losses in the stock market. The TAIFEX and the TAIEX are highly related; therefore, differences between consecutive days in the TAIEX were used as knowledge to forecast the TAIFEX.

In forecasting the TAIFEX, the data range from August 3 to September 30 1988. Forecasting proceeds as follows.

Step 1. From the historical data in Table 6, U=[6100, 7700] is derived. Then, the universe of the discourse is divided into 16 equally long intervals u_1, u_2, …, u_{16} of length 100, where u_1=[6100, 6200], u_2=[6200, 6300], u_3=[6300, 6400], u_4=[6400, 6500], u_5=[6500, 6600], u_6=[6600, 6700], u_7=[6700, 6800], u_8=[6800, 6900], u_9=[6900, 7000], u_{10}=[7000, 7100], u_{11}=[7100, 7200], u_{12}=[7200, 7300], u_{13}=[7300, 7400], u_{14}=[7400, 7500], u_{15}=[7500, 7600], u_{16}=[7600, 7700].

Step 2. In this case, the linguistic variable "TAIFEX" which can be represented as 16 fuzzy sets; A_i (i=1, 2, ⋯, 16). The linguistic values are A_1=(lowest), A_2=(very very very low), A_3=(very very low), A_4=(very low), A_5=(low), A_6=(quite low) , A_7=(low medium), A_8=(medium), A_9=(quite medium), A_{10}=(medium high), A_{11}=(quite high), A_{12}=(high), A_{13}=(very high), A_{14}=(very very high), A_{15}=(very very very high), A_{16}=(highest). Each A_i (i=1, 2, ⋯, 16) is defined in the Table 7.

Table 6 The fuzzy sets of "TAIFEX"

A_1= $1/u_1 + 0.5/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0/u_{12} + 0/u_{13} + 0/u_{14} + 0/u_{15} + 0/u_{16}$

A_2= $0.5/u_1 + 1/u_2 + 0.5/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0/u_{12} + 0/u_{13} + 0/u_{14} + 0/u_{15} + 0/u_{16}$

A_3= $0/u_1 + 0.5/u_2 + 1/u_3 + 0.5/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0/u_{12} + 0/u_{13} + 0/u_{14} + 0/u_{15} + 0/u_{16}$

A_4= $0/u_1 + 0/u_2 + 0.5/u_3 + 1/u_4 + 0.5/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0/u_{12} + 0/u_{13} + 0/u_{14} + 0/u_{15} + 0/u_{16}$

Table 6 (*continued*)

$A_{14}= 0/u_1 + 0/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0.5/u_{12} + 1/u_{13} + 0.5/u_{14} + 0/u_{15} + 0/u_{16}$

$A_{15}= 0/u_1 + 0/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0/u_{12} + 0.5/u_{13} + 1/u_{14} + 0.5/u_{15} + 0/u_{16}$

$A_{16}= 0/u_1 + 0/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0/u_{12} + 0/u_{13} + 0.5/u_{14} + 1/u_{15} + 0.5/u_{16}$

The data set of TAIFEX and corresponding fuzzy sets are shown in the appendix of Table A-2.

Step 3. The fuzzy relationships are established and grouped in Table 7.

Step 4. The frequencies of fuzzy sets in each fuzzy relationship are calculated, and shown in Table 8.

Step 5. Daily changes in the TAIEX are used as knowledge to select the proper fuzzy sets for forecasting (listed in Table 8). Hence, the trend specifies increase, decrease or no change. Then the variable in the weighted function is represented as $\alpha=1$ for an increase, $\alpha=-1$ for a decrease, and $\alpha=0$ for no change.

Table 7 Grouped TAIFEX fuzzy relationship

$A_1 \rightarrow$	$A_2 \rightarrow$
$A_3 \rightarrow$	$A_4 \rightarrow A_2 A_4 A_6$
$A_5 \rightarrow A_4$	$A_6 \rightarrow A_7$
$A_7 \rightarrow A_5, A_7, A_8, A_9$	$A_8 \rightarrow A_6, A_7, A_8, A_9, A_{10}$
$A_9 \rightarrow A_7, A_8, A_9$	$A_{10} \rightarrow A_8$
$A_{11} \rightarrow$	$A_{12} \rightarrow A_7, A_8, A_9$
$A_{13} \rightarrow A_{12}, A_{13}$	$A_{14} \rightarrow A_{14}, A_{15}$
$A_{15} \rightarrow A_{13}, A_{14}, A_{15}$	$A_{16} \rightarrow$

Table 8 Frequency of fuzz sets in TAIFEX relationship

$A_4 \to A_6$	1	$A_4 \to A_4$	1
$A_4 \to A_2$	1	$A_5 \to A_4$	1
$A_6 \to A_7$	2	$A_7 \to A_9$	1
$A_7 \to A_8$	2	$A_7 \to A_7$	4
$A_7 \to A_5$	1	$A_8 \to A_{10}$	1
$A_8 \to A_9$	1	$A_8 \to A_8$	4
$A_8 \to A_7$	2	$A_8 \to A_6$	1
$A_9 \to A_9$	2	$A_9 \to A_8$	2
$A_9 \to A_7$	1	$A_{10} \to A_8$	1
$A_{12} \to A_{13}$	1	$A_{12} \to A_{12}$	4
$A_{12} \to A_9$	1	$A_{13} \to A_{13}$	3
$A_{13} \to A_{12}$	2	$A_{14} \to A_{15}$	1
$A_{14} \to A_{14}$	1	$A_{15} \to A_{14}$	1
$A_{15} \to A_{13}$	1	$A_{15} \to A_{15}$	1

Hence, according to Definition 3.1, the selection strategy of fuzzy sets is: if the trend in TAIEX leads to an increase, then the fuzzy sets in the high part are all selected. Conversely, if the trend in TAIEX leads to a decrease, then the fuzzy sets in the low part are all selected. Otherwise, if the trend in TAIEX leads to no change, then the origin fuzzy set is selected.

Step 6. According to the grouped fuzzy relationships in Table 8, the corresponding weighted functions can be established and are listed as follows.

$$h_4(\alpha)\,|_{\alpha=-1,0,1} = \frac{\alpha(\alpha-1)}{2}\left(\frac{1}{2}m_2 + \frac{1}{2}m_4\right) + \frac{2(1-\alpha)(1+\alpha)}{2}m_4 + \frac{\alpha(\alpha+1)}{2}\left(\frac{1}{2}m_4 + \frac{1}{2}\alpha m_6\right),$$

$$h_5(\alpha)\,|_{\alpha=-1,0,1} = \frac{\alpha(\alpha-1)}{2}m_4 + \frac{2(1-\alpha)(1+\alpha)}{2}m_5 + \frac{\alpha(\alpha+1)}{2}m_5\,,$$

$$h_6(\alpha)\,|_{\alpha=-1,0,1} = \frac{\alpha(\alpha-1)}{2}m_6 + \frac{2(1-\alpha)(1+\alpha)}{2}m_6 + \frac{\alpha(\alpha+1)}{2}m_7\,,$$

$$h_7(\alpha)|_{\alpha=-1,0,1}=$$

$$\frac{\alpha(\alpha-1)}{2}(\frac{1}{5}m_5+\frac{4}{5}m_7)+\frac{2(1-\alpha)(1+\alpha)}{2}m_7+\frac{\alpha(\alpha+1)}{2}(\frac{4}{7}m_7+\frac{2}{7}m_8+\frac{1}{7}m_9),$$

$$h_8(\alpha)|_{\alpha=-1,0,1}$$

$$=\frac{\alpha(\alpha-1)}{2}(\frac{1}{7}m_6+\frac{2}{7}m_7+\frac{4}{7}m_8)+\frac{2(1-\alpha)(1+\alpha)}{2}m_8+\frac{\alpha(\alpha+1)}{2}(\frac{4}{6}m_8+\frac{1}{6}m_9+\frac{1}{6}m_{10}),$$

$$h_9(\alpha)|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}(\frac{1}{5}m_7+\frac{2}{5}m_8+\frac{2}{5}m_9)+\frac{2(1-\alpha)(1+\alpha)}{2}m_9+\frac{\alpha(\alpha+1)}{2}m_9,$$

$$h_{10}(\alpha)|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}m_8+\frac{2(1-\alpha)(1+\alpha)}{2}m_{10}+\frac{\alpha(\alpha+1)}{2}m_{10},$$

$$h_{12}(\alpha)|_{\alpha=-1,0,1}=$$

$$\frac{\alpha(\alpha-1)}{2}(\frac{1}{5}m_9+\frac{4}{5}m_{12})+\frac{2(1-\alpha)(1+\alpha)}{2}m_{12}+\frac{\alpha(\alpha+1)}{2}(\frac{4}{5}m_{12}+\frac{1}{5}m_{13}),$$

$$h_{13}(\alpha)|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}(\frac{2}{5}m_{12}+\frac{3}{5}m_{13})+\frac{2(1-\alpha)(1+\alpha)}{2}m_{13}+\frac{\alpha(\alpha+1)}{2}m_{13},$$

$$h_{14}(\alpha)|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}m_{14}+\frac{2(1-\alpha)(1+\alpha)}{2}m_{14}+\frac{\alpha(\alpha+1)}{2}(\frac{1}{2}m_{14}+\frac{1}{2}m_{15})$$

$$h_{15}(\alpha)|_{\alpha=-1,0,1}=\frac{\alpha(\alpha-1)}{2}(\frac{1}{3}m_{13}+\frac{1}{3}m_{14}+\frac{1}{3}m_{15})+\frac{2(1-\alpha)(1+\alpha)}{2}m_{15}+\frac{\alpha(\alpha+1)}{2}m_{15}.$$

That is, m_k is the midpoint of the interval u_k, and m_1 =6150, m_2 =6250, m_3 =6350, m_4 = 6450, m_5 =6550, m_6 =6650, m_7 =6750, m_8 =6850, m_9 =6950, m_{10} =7050, m_{11} = 7150, m_{12} =7250, m_{13} =7350, m_{14} =7450, m_{15} =7550, m_{16} =7650, respectively.

Step 7. Subsequently, suppose that input x_{t-1} in date $t-1$ is fuzzified to $F(t-1)=A_i$, the corresponding weighted function $h_i(\alpha)$ is selected. Accordingly, the output is derived as $h_i(\alpha)|_{\alpha=-1,0,1}$. The following examples are used to demonstrate the procedure of selecting the corresponding weighted function and using the knowledge to derive the forecasts.

[1998/8/4]: The fuzzy set of 1998/8/3 is A_{15} (TAIFEX was 7552). The proper weighted function $h_{15}(\alpha)$ is selected. The TAIEX was 7599 on 1998/8/3 and 7593 on 1998/8/4. The difference between these two days is -6, the trend leads the knowledge to a decrease, so α=-1. Hence, the forecast for 1998/8/4 is

$$h_{15}(\alpha)\,|_{\alpha=-1} = \frac{1}{3}m_{13} + \frac{1}{3}m_{14} + \frac{1}{3}m_{15} = 7450\,.$$

[1998/8/6]: The fuzzy set of 1998/8/5 is A_{14} (TAIFEX was 7486). The proper weighted function $h_{14}(\alpha)$ is selected. The TAIEX were 7500 on 1998/8/3 and 7472 on 1998/8/6. The difference between these two days is -28, the trend leads the knowledge to a decrease, so α=-1. Hence, the forecast for 1998/8/6 is

$$h_{14}(\alpha)\,|_{\alpha=-1} = m_{14} = 7450\,.$$

[1998/8/7]: The fuzzy set of 1998/8/6 is A_{14} (TAIFEX was 7462). The proper weighted function $h_{14}(\alpha)$ is selected. The TAIEX were 7472 on 1998/8/6 and 7530 on 1998/8/7. The difference between these two days is 58, the trend leads the knowledge to an increase, so α=1. Hence, the forecast for 1998/8/7 is

$$h_{14}(\alpha)\,|_{\alpha=1} = \frac{1}{2}m_{14} + \frac{1}{2}m_{15} = 7500\,.$$

[1998/8/10]: The fuzzy set of 1998/8/7 is A_{15} (TAIFEX was 7530). The proper weighted function $h_{15}(\alpha)$ is selected. The TAIEX were 7530 on 1998/8/7 and 77372 on 1998/8/10. The difference between these two days is -158, the trend leads knowledge to a decrease, so α=-1. The forecast for 1998/8/10 is

$$h_{A_{15}}(\alpha)\,|_{\alpha=-1} = \frac{1}{3}m_{13} + \frac{1}{3}m_{14} + \frac{1}{3}m_{15} = 7450\,.$$

Table A-3 in the appendix shows all of the remaining forecasts.

Table 9 compares various studies of fuzzy time series used to forecast TAIFEX. From left to right, the columns in Table 10 present the forecasts by Chen (2002), by Huarng's knowledge models (2001), by this proposed weighted model. The average forecast errors are 1.05%, 1.06%, 0.94%, respectively. Clearly, the proposed model outperforms Chen's model and Huarng's model.

Table 9 Comparison of TAIFEX forecasts

Data Set (1998)	Index	Chen [7]	Huarng [1]	This Proposed Model
8/3	7552			
8/4	7560	7450	7450	7450
8/5	7487	7450	7450	7450
8/6	7462	7500	7450	7450
8/7	7515	7500	7500	7500

Table 9 (*continued*)

8/10	7365	7450	7450	7450
8/11	7360	7300	7350	7350
8/12	7330	7300	7300	7310
8/13	7291	7300	7350	7350
8/14	7320	7183.33	7100	7190
8/15	7300	7300	7350	7350
8/17	7219	7300	7300	7310
8/18	7220	7183.33	7100	7190
8/19	7285	7183.33	7300	7270
8/20	7274	7183.33	7100	7190
8/21	7225	7183.33	7100	7190
8/24	6955	7183.33	7100	7190
8/25	6949	6850	6850	6870
8/26	6790	6850	6850	6870
8/27	6835	6775	6650	6710
8/28	6695	6850	6750	6792.857
8/29	6728	6750	6750	6750
8/31	6566	6775	6650	6710
9/1	6409	6450	6450	6450
9/2	6430	6450	6550	6550
9/3	6200	6450	6350	6350
9/4	6403.2	6450	6250	6250
9/5	6697.5	6450	6550	6550
9/7	6722.3	6750	6750	6750
9/8	6859.4	6775	6850	6807.143
9/9	6769.6	6850	6750	6792.857
9/10	6709.75	6775	6650	6710
9/11	6726.5	6775	6850	6807.143

Table 9 (*continued*)

9/14	6774.55	6775	6850	6807.143
9/15	6762	6775	6650	6710
9/16	6952.75	6775	6850	6807.143
9/17	6906	6850	6950	6950
9/18	6842	6850	6850	6870
9/19	7039	6850	6950	6900
9/21	6861	6850	6850	6850
9/22	6926	6850	6950	6900
9/23	6852	6850	6850	6870
9/24	6890	6850	6950	6900
9/25	6871	6850	6850	6838.889
9/28	6840	6850	6750	6792.857
9/29	6806	6850	6750	6792.857
9/30	6787	6850	6750	6792.857
		1.05%	1.06%	0.94%

4.4 Forecasting TAIFEX with High-Order Model

According to the previous definition of TAIFEX, the author proposed the empirical analysis of knowledge second order model as follows. The data range from August 3 to September 30, 1988.

Step 1. From the historical data in Table 8, U=[6100, 7700] is derived. Then, the universe of the discourse is divided into 16 equally long intervals u_1, u_2, …, u_{16} of length 100, where u_1=[6100, 6200], u_2=[6200, 6300], u_3=[6300, 6400], u_4=[6400, 6500], u_5=[6500, 6600], u_6=[6600, 6700], u_7=[6700, 6800], u_8=[6800, 6900], u_9=[6900, 7000], u_{10}=[7000, 7100], u_{11}=[7100, 7200], u_{12}=[7200, 7300], u_{13}=[7300, 7400], u_{14}=[7400, 7500], u_{15}=[7500, 7600], u_{16}=[7600, 7700].

Step 2. In this case, the linguistic variable "TAIFEX" which can be represented as 16 fuzzy sets; A_i (i=1, 2, …,16). The linguistic values are A_1=(lowest), A_2=(very very very low), A_3=(very very low), A_4=(very low), A_5=(low), A_6=(quite low) , A_7=(low medium), A_8=(medium), A_9=(quite medium), A_{10}=(medium high),

A_{11}=(quite high), A_{12}=(high), A_{13}=(very high), A_{14}=(very very high), A_{15}=(very very very high), A_{16}=(highest). Each A_i (i=1, 2, …, 16) is defined as follows.

$A_1= 1/u_1 + 0.5/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0/u_{12} + 0/u_{13} + 0/u_{14} + 0/u_{15} + 0/u_{16}$,

$A_2= 0.5/u_1 + 1/u_2 + 0.5/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0/u_{12} + 0/u_{13} + 0/u_{14} + 0/u_{15} + 0/u_{16}$,

$A_3= 0/u_1 + 0.5/u_2 + 1/u_3 + 0.5/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0/u_{12} + 0/u_{13} + 0/u_{14} + 0/u_{15} + 0/u_{16}$,

$A_{14}= 0/u_1 + 0/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0.5/u_{12} + 1/u_{13} + 0.5/u_{14} + 0/u_{15} + 0/u_{16}$,

$A_{15}= 0/u_1 + 0/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0/u_{12} + 0.5/u_{13} + 1/u_{14} + 0.5/u_{15} + 0/u_{16}$,

$A_{16}= 0/u_1 + 0/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6 + 0/u_7 + 0/u_8 + 0/u_9 + 0/u_{10} + 0/u_{11} + 0/u_{12} + 0/u_{13} + 0.5/u_{14} + 1/u_{15} + 0.5/u_{16}$.

The data set of TAIFEX and corresponding fuzzy sets are shown in Table 8.

Step 3. The second order fuzzy relationships are established and grouped in Table 10.

Step 4. Daily changes in the TAIEX are used as the knowledge to select the proper fuzzy sets for forecasting (listed in the appendix of Table A-4). Hence, the trend specifies increase, decrease or no change. Then the variable in the weighted function is represented as α=1 for an increase, α=0 for a decrease, and α=-1 for no change.

Hence, if there are ambiguities in the fuzzy relationships, then the selection strategy is: if the trend in TAIEX leads to an increase, then the fuzzy sets in the high part are all selected. Otherwise, if the trend in TAIEX leads to a decrease, then the fuzzy sets in the low part are all selected. Otherwise, if the trend in TAIEX leads to no change, then the origin fuzzy set is selected.

Table 10 Second order grouped TAIFEX fuzzy relationship

$A_{15}, A_{15} \rightarrow A_{14}$
$A_{15}, A_{14} \rightarrow A_{14}$
$A_{15}, A_{13} \rightarrow A_{13}$
$A_{14}, A_{15} \rightarrow A_{13}$
$A_{14}, A_{14} \rightarrow A_{15}$
$A_{13}, A_{13} \rightarrow A_{12}, A_{13}$
$A_{13}, A_{12} \rightarrow A_{12}, A_{13}$
$A_{12}, A_{13} \rightarrow A_{13}$
$A_{12}, A_{12} \rightarrow A_{9}, A_{12}$
$A_{12}, A_{9} \rightarrow A_{9}$
$A_{10}, A_{8} \rightarrow A_{9}$
$A_{4}, A_{4} \rightarrow A_{2}$
$A_{4}, A_{2} \rightarrow A_{4}$
$A_{2}, A_{4} \rightarrow A_{6}$

Accordingly, let the grouped fuzzy relationship for forecasting $TAIFEX_t$ (TAIFEX at time t) be $A_{r_1}, A_{r_2} \rightarrow A_{j_1}, A_{j_2}, \cdots$. The knowledge function is set as $h(\alpha; A_{r_2}; A_{j_1}, A_{j_2}, \cdots)|_{\alpha=1,\,0,\,-1}$.

Step 5. Subsequently, the second-order forecasting process of TAIFEX $F(t)$ is carried out by the fuzzified input of $F(t\text{-}2)$ and $F(t\text{-}1)$. Some examples below are used to illustrate the forecasting process.

[1998/8/5]: The TAIFEX in 1998/8/3 and 1998/8/4 were 7552 (A_{15}) and 7560 (A_{15}). According to the list of second order fuzzy relationship in Table 13, the current states " A_{15}, A_{15} " mapping to the suitable fuzzy relationship is $A_{15}, A_{15} \rightarrow A_{14}$. Because the maximum membership value of the fuzzy set A_{14} occurs at the interval u_{14}, then the midpoint of the interval u_{14} is 7450. Thus, the forecasted TAIFEX of 1998/8/5 is equal to 7450.

[1998/8/6]: The TAIFEX in 1998/8/4 and 1998/8/5 were 7560 (A_{15}) and 7487 (A_{14}). According to the list of second order fuzzy relationship in Table 13, the current states " A_{15}, A_{14} " mapping to the suitable fuzzy relationship is $A_{15}, A_{14} \rightarrow A_{14}$. Because the maximum membership value of the fuzzy set A_{14} occurs at the interval u_{14}, then the midpoint of the interval u_{14} is 7450. Thus, the forecasted TAIFEX of 1998/8/5 is equal to 7450

[1998/8/12]: The TAIFEX in 1998/8/10 and 1998/8/11 were 7365 (A_{13}) and 7360 (A_{13}). According to the list of second order fuzzy relationship in Table 13, the current states " A_{13}, A_{13} " mapping to the suitable fuzzy relationships is $A_{13}, A_{13} \rightarrow A_{12}, A_{13}$. It means that there is an ambiguity. The TAIEX were 7384 on 1998/8/11 and 7352 on 1998/8/12, respectively. The difference between these TAIEX was -32, the trend is positive and $\alpha = -1$. Hence, the knowledge function is $h(\alpha; A_{13}; A_{12}, A_{13})\,|_{\alpha=-1} = (m_{12} + m_{13})/2 = 7300$, where m_{13} is the midpoint of the interval u_{13}.

Table 15 compares various studies of fuzzy time series used to forecast TAIFEX. Mean square errors (*MSE*s) are taken as forecasting errors:

$$MSE = \frac{\sum_{i=1}^{n}(acturall\_TAIFEX - forecasted\_TAIFEX)^2}{n},$$

where i represents the year. From left to right, the columns in Table 15 present the forecasts by Chen's model (Chen 1996), by Huarng's two-variable model and by his three-variable knowledge model (Huarng 2001). The *MSE*s are 9668.94, 7856.5 and 5437.38, respectively.

Table 15 A comparison of the MSE of previous models

	Chen's model (1996)	Huarng's two variable model	Huarng's three variable model
MSE	9668.94	7856.50	5437.58

Table 16 compares the *MSE*s of Chen's restricted model and the proposed knowledge high-order fuzzy time series model. That is to say, Chen's later model (2002) is restricted in lower-order fuzzy time series and lacking the ability to handle the ambiguity well. The averaging operation in Chen's model (1996) is applied to assist to eliminate the ambiguity in Chen's later model (2002) to compare the performance obtained using fuzzy time series of different orders.

Table 16 A comparison of the MSE of Chen's model and this proposed model by using different orders fuzzy time series

Second-order		Third-order		Fourth-order		Fifth-order	
Chen	Knowledge	Chen	Knowledge	Chen	Knowledge	Chen	Knowledge
5900.64	4109.09	3209.98	3052.19	1999.09	1830.2	864.64	864.64

From the left to right, the columns in Table 16 present the *MSE* of Chen second-order model, the knowledge second-order model, Chen third-order model, the knowledge third-order model, Chen fourth-order model, the knowledge fourth-order model, Chen fifth-order model and the knowledge fifth-order model, respectively. The *MSE*s are 5900.64, 4109.09, 3209.98, 3052.19, 1999.09, 1830.2, 864.64 and 864.64, respectively. Obviously, the forecasting accuracy is better than that of Chen's model of the same order. Therefore, the knowledge high-order fuzzy time series model represents an improvement over the Chen's model.

5 Conclusions

Most fuzzy time series models are independent of a specific domain. Among these models, the Chen model uses the simple and straightforward method to find the best forecasting results. In the field of expert systems, experts typically consider knowledge to solve domain-specific problems. Hence, Huarng enhanced the Chen model by integrating knowledge. The weighted model overcomes the disadvantage of the Huarng model, that is, a lack of an efficient measure of the significance of the knowledge. The first proposed model was based on the weighted measure of the fuzzy sets, which differs from the arithmetic average in the traditional defuzzifier. The significance of the derived fuzzy sets was considered in the defuzzification phase. The knowledge model is proposed to forecast time series based on the high-order fuzzy time series and domain-specific knowledge. The proposed model overcomes the deficiency of the Chen model, which is strongly dependent on the highest-order fuzzy time series and requires a large amount of memory.

The results showed that the weighted models can reflect fluctuations in fuzzy time series and provide superior overall forecasting results compared to previous models. The forecasts of university enrollment and the futures index show that domain-specific knowledge can be used with ease to assist forecasting. The efficient measure of the significance of fuzzy relationships provides additional information for improving forecasts. Empirical analysis showed that the proposed high-order model yielded more accurate forecasts than the Chen model when using the same orders. Therefore, the knowledge high-order fuzzy time series model offers the advantages of high-order time series forecasting and the elimination of ambiguity; that is, the forecasting model can be restricted to the acceptable-order fuzzy time series to reduce the amount of memory and computation time required.

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APPENDIX

Table A-1 Frequency of relationships

Fuzzy relationship	Frequency	Fuzzy relationship	Frequency
$A_1 \to A_1$	2	$A_1 \to A_2$	1
$A_2 \to A_3$	1	$A_3 \to A_3$	7
$A_3 \to A_4$	2	$A_4 \to A_3$	1
$A_4 \to A_4$	2	$A_4 \to A_6$	1
$A_6 \to A_6$	1	$A_6 \to A_7$	1
$A_7 \to A_6$	1	$A_7 \to A_7$	1

Table A-2 TAIFEX data set

Date (1998)	Index	Fuzzy Set	Date (1998)	Index	Fuzzy Set
8/3	7552	A_{15}	8/4	7560	A_{15}
8/5	7487	A_{14}	8/6	7462	A_{14}
8/7	7515	A_{15}	8/10	7365	A_{13}
8/11	7360	A_{13}	8/12	7330	A_{13}
8/13	7291	A_{12}	8/14	7320	A_{13}
8/15	7300	A_{13}	8/17	7219	A_{12}
8/18	7220	A_{12}	8/19	7285	A_{12}
8/20	7274	A_{12}	8/21	7225	A_{12}
8/24	6955	A_9	8/25	6949	A_9
8/26	6790	A_7	8/27	6835	A_8
8/28	6695	A_6	8/29	6728	A_7
8/31	6566	A_5	9/1	6409	A_4
9/2	6430	A_4	9/3	6200	A_2
9/4	6403.2	A_4	9/5	6697.5	A_6
9/7	6722.3	A_7	9/8	6859.4	A_8
9/9	6769.6	A_7	9/10	6709.75	A_7

Table A-2 (*continued*)

9/11	6726.5	A_7	9/14	6774.55	A_7
9/15	6762	A_7	9/16	6952.75	A_9
9/17	6906	A_9	9/18	6842	A_8
9/19	7039	A_{10}	9/21	6861	A_8
9/22	6926	A_9	9/23	6852	A_8
9/24	6890	A_8	9/25	6871	A_8
9/28	6840	A_8	9/29	6806	A_8
9/30	6787	A_7			

Table A-3 TAIEX data set

Date (1998)	Index	Difference	Date (1998)	Index	Difference
8/3	7599		8/4	7593	-6
8/5	7500	-93	8/6	7472	-28
8/7	7530	58	8/10	7372	-158
8/11	7384	12	8/12	7352	-32
8/13	7363	11	8/14	7348	-15
8/15	7372	24	8/17	7274	-98
8/18	7182	-92	8/19	7293	111
8/20	7271	-22	8/21	7213	-58
8/24	6958	-255	8/25	6908	-50
8/26	6814	-94	8/27	6813	-1
8/28	6724	-89	8/29	6736	12
8/31	6550	-186	9/1	6335	-215
9/2	6472	137	9/3	6251	-221
9/4	6463	212	9/5	6756	293
9/7	6801	45	9/8	6942	141
9/9	6895	-47	9/10	6804	-91
9/11	6842	38	9/14	6860	18

Table A-3 (*continued*)

9/15	6858	-2	9/16	6973	115
9/17	7001	28	9/18	6962	-39
9/19	7150	188	9/21	7029	-121
9/22	7034	5	9/23	6962	-72
9/24	6980	18	9/25	6980	0
9/28	6911	-69	9/29	6885	-26
9/30	6834	-51			

Table A-4 TAIEX data set

Date (1998)	Index	Difference
8/3	7599	
8/4	7593	-6
8/5	7500	-93
8/6	7472	-28
8/7	7530	58
8/10	7372	-158
8/11	7384	12
8/12	7352	-32
8/13	7363	11
8/14	7348	-15
8/15	7372	24
8/17	7274	-98
8/18	7182	-92
8/19	7293	111
8/20	7271	-22
8/21	7213	-58
8/24	6958	-255
8/25	6908	-50

Table A-4 (*continued*)

8/26	6814	-94
8/27	6813	-1
8/28	6724	-89
8/29	6736	12
8/31	6550	-186
9/1	6335	-215
9/2	6472	137
9/3	6251	-221
9/4	6463	212
9/5	6756	293
9/7	6801	45
9/8	6942	141
9/9	6895	-47
9/10	6804	-91
9/11	6842	38
9/14	6860	18
9/15	6858	-2
9/16	6973	115
9/17	7001	28
9/18	6962	-39
9/19	7150	188
9/21	7029	-121
9/22	7034	5
9/23	6962	-72
9/24	6980	18
9/25	6980	0
9/28	6911	-69
9/29	6885	-26
9/30	6834	-51