# Chapter 6
# A Novel Choquet Integral Composition Forecasting Model for Time Series Data Based on Completed Extensional L-Measure

Hsiang-Chuan Liu

**Abstract.** In this study, based on the Choquet integral with respect to complete extensional L-measure and M-density, a novel composition forecasting model which composed the time series model , the exponential smoothing model and GM(1,1) forecasting model was proposed. For evaluating this improved composition forecasting model, an experiment with the data of the grain production in Jilin during 1952 to 2007 by using the sequential mean square error was conducted. Based on the M-density and N- density, the performances of Choquet integral composition forecasting model with the completed extensional L-measure, extensional L-measure, L-measure, Lambda-measure and P-measure, respectively, a ridge regression composition forecasting model and a multiple linear regression composition forecasting model and the traditional linear weighted composition forecasting model were compared. The experimental results showed that the Choquet integral composition forecasting model with respect to the completed extensional L-measure and M-density outperforms other ones. Furthermore, for each fuzzy measure, including the completed extensional L-measure, extensional L-measure, L-measure, Lambda-measure and P-measure, respectively, the Choquet integral composition forecasting model based on *M*-density is better than the one based on *N*-density.

**Keywords:** Choquet integral, composition forecasting model, *M*-density, completed extensional L-measure.

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# 1 Introduction

The composition forecasting model is first considered in the work of Bates and Granger (1969) [1]. They are now in a widespread use in many areas, especially in economic field. Zhang Wang and Gao (2008) [2] applied the linear composition forecasting model which composed the time series model, the second-order exponential smoothing model and GM(1,1) forecasting model in the Agricultural Economy Research, the GM(1,1) is one of the most frequently used grey forecasting model, it is a time series forecasting model, encompassing a group of differential equations adapted for parameter variance, rather than a first order differential equation [3-4]. In our previous works [5-9], we extended the work of Zhang, Wang, and Gao by proposing some nonlinear composition forecasting model which also composed the time series model, the second-order exponential smoothing model and GM(1,1) forecasting model by using the ridge regression model [5] and the theory of Choquet integral with respect to some fuzzy measures, including Sugeno's λ-measure [13], Zadeh's P-measure [14] and authors' fuzzy measures, L-measure, extensional L-measure and completed extensional L-measure [6-12]. Since the first two well-known fuzzy measures are univalent measures, each of them has just one feasible fuzzy measure satisfying the conditions of its own definition, but the others proposed by our previous works are multivalent fuzzy measures, all of them have infinitely feasible fuzzy measures satisfying the conditions of their own definition. The fuzzy measure based Choquet integral composition forecasting models are supervised methods, by comparing the mean square errors between the estimated values and the corresponding true values, each of our multivalent fuzzy measures based forecasting models has more opportunity to find the better feasible fuzzy measure, the performances of them are always better than the one based on the univalent fuzzy measures, λ-measure and P-measure. In addition, the author has proved that the P-measure is a special case of the L-measure [7], we know that all of the extended multivalent fuzzy measures of L-measure are at lest as good as their special case P-measure. However, the λ-measure is not a special case of the L-measure, so the improved L-measure, called extensional L-measure, was proposed to contain the λ-measure as a special case [7]. And then, all of the P-measure, λ-measure and L-measure are special cases of the extensional L-measure. However, the extensional L-measure does not attend the largest fuzzy measure B-measure, it is not a completed fuzzy measure, for overcoming this drawback, an improved extensional L-measure, called completed extensional L-measure was proposed, all of other above-mentioned fuzzy measures proposed are the special cases of it. The real data experiment showed that the extensional L-measure Choquet integral based composition forecasting model is the best one. On the other hand, all of above mentioned Choquet integral composition forecasting models with some different fuzzy measures are based on $N$-density. From the definition of Choquet integral and fuzzy measures, we know that the Choquet integral can be viewed as a function of its fuzzy measure, and the fuzzy measure can be viewed as a function of its fuzzy density function, therefore, the performance of any Choquet integral is predominate by its fuzzy measure, and the performance of any fuzzy measure is predominate by its fuzzy density function, in other words, the performance of any Choquet integral is predominate by its fuzzy density function. Since the older fuzzy density function *N*-density is based on the linear correlation coefficient, the new fuzzy density function *M*-density based on the mean square error is non-linear, the relations among the composition forecasting model and three given forecasting models are non-linear as well, hence, in the same Choquet integral with respect to the same fuzzy measure, the performance of the non-linear fuzzy density functions is always better than the linear fuzzy density functions.

In this paper, a novel fuzzy measure, called the completed extensional L-measure, and the new fuzzy density function, *M*-density, are considered. Based on the *M*-density and the proposed completed extensional L-measure, a novel composition forecasting model is also considered. For comparing the forecasting efficiency of two fuzzy densities *M*-density and *N*-density, is also considered.

## 2 The Composition Forecasting Model

In this paper, for evaluating the forecasting validation of forecasting model to sequential data, the sequential mean square error is used, its formal definition is listed as follows.

***Definition 1.*** **Sequential Mean Square Error (SMSE) [9-10]**

If $\theta_{t+j}$ is the realized value of target variable at time $(t+j)$, $\hat{\theta}_{t+j|t}$ is the forecasted value of target variable at time $(t+j)$ based on training data set from time 1 to time *t*,

and
$$SMSE\left(\hat{\theta}_t^{(h)}\right) = \frac{1}{h}\sum_{j=1}^{h}\left(\hat{\theta}_{t+j|t+j-1} - \hat{\theta}_{t+j}\right)^2 \tag{1}$$

then $SMSE\left(\hat{\theta}_t^{(h)}\right)$ is called the sequential mean square error (SMSE) of the *h* forecasted values of target variable from time $(t+1)$ to time $(t+h)$ based on training data set from time 1 to time *t*. The composition forecasting model or combination forecasting model can be defined as follows.

***Definition 2.*** **Composition Forecasting Model [9-10]**

(i) Let $y_t$ be the realized value of target variable at time *t*.

(ii) Let $x_{t,1}, x_{t,2}, ..., x_{t,m}$ be a set of *m* competing predictors of $y_t$, $\hat{y}_t$ be a function *f* of $x_{t,1}, x_{t,2}, ..., x_{t,m}$ with some parameters, denoted as

$$\hat{y}_t = f\left(x_{t,1}, x_{t,2}, ..., x_{t,m}\right) \tag{2}$$

(iii) Let $x_{t+j|t,k}$ be the forecasted values of $y_t$ by competing predictor $k$ at time $(t+j)$ based on training data set from time 1 to time $t$, and for the same function $f$ as above,

Let
$$\hat{y}_{t+j|t} = f\left(x_{t+j,1}, x_{t+j,2}, \ldots, x_{t+j,m}\right) \tag{3}$$

(iv) Let
$$SMSE\left(\hat{y}_t^{(h)}\right) = \frac{1}{h}\sum_{j=1}^{h}\left(\hat{y}_{t+j|t+j-1} - y_{t+j}\right)^2 \tag{4}$$

$$SMSE\left(x_{t,k}^{(h)}\right) = \frac{1}{h}\sum_{j=1}^{h}\left(x_{t+j,k} - y_{t+j}\right)^2 \tag{5}$$

For current time $t$ and the future $h$ times, if

$$SMSE\left(\hat{y}_t^{(h)}\right) \le \min_{1\le k\le m} SMSE\left(x_{t,k}^{(h)}\right) \tag{6}$$

then $\hat{y}_t$ is called a composition forecasting model for the future $h$ times of $x_{t,1}, x_{t,2}, \ldots, x_{t,m}$ or, in brief, a composition forecasting model of $x_{t,1}, x_{t,2}, \ldots, x_{t,m}$.

***Definition 3.*** **Linear Combination Forecasting Model [9-10]**

For given parameters $\beta_k \in R, \sum_{k=1}^{m}\beta_k = 1$, let

$$\hat{y}_t = \sum_{k=1}^{m}\beta_k x_{t,k} \tag{7}$$

If $\hat{y}_t$ is a composite forecasting model of $x_{t,1}, x_{t,2}, \ldots, x_{t,m}$ then $\hat{y}_t$ is called a linear combination forecasting model or linear composition forecasting model, otherwise, it is called a non-linear combination forecasting model or non-linear composition forecasting model.

***Definition 4.*** **Ridge Regression Composition Forecasting Model [5,9,10]**

(i) Let $\underline{y}_t = \left(y_1, y_2, \ldots, y_t\right)^T$ be realized data vector of target variable from time 1 to time $t$, $\underline{x}_{t,k} = \left(x_{1,k}, x_{2,k}, \ldots, x_{t,k}\right)^T$ be a forecasted value vector of competing predictor $k$ of target variable $y_t$ from time 1 to time $t$.

(ii) Let $X_t$ be a forecasted value matrix of $m$ competing predictors of target variable $y_t$ from time 1 to time $t$.

(iii) Let
$$\hat{\underline{y}}_t = \left(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_t\right)^T \tag{8}$$

$$f\left(X_t\right)=f\left(\underline{x}_{t,1},\underline{x}_{t,2},\ldots,\underline{x}_{t,m}\right) \tag{9}$$

(iv) Let $$\underline{\beta}_t^{(r)}=\left(\beta_{t,1}^{(r)},\beta_{t,2}^{(r)},\ldots,\beta_{t,m}^{(r)}\right)^T=\left(X_t^TX_t+rI_m\right)^{-1}X_t^T\underline{y}_t \tag{10}$$

$$\underline{\hat{y}}_t=f\left(X_t\right)=X_t\underline{\beta}_t^{(r)} \tag{11}$$

Then $$\underline{\hat{y}}_{t+j|t}=f\left(X_{t+j}\right)=X_{t+j}\underline{\beta}_t^{(r)} \tag{12}$$

$$\begin{aligned}\hat{y}_{t+j|t}&=f\left(x_{t+j,1},x_{t+j,2},\ldots,x_{t+j,m}\right)\\&=\left[x_{t+j,1},x_{t+j,2},\ldots,x_{t+j,m}\right]\underline{\beta}_t^{(r)}=\sum_{k=1}^{m}\beta_{t,k}^{(r)}x_{t+j,k}\end{aligned} \tag{13}$$

For current time $t$ and the future $h$ times, if

$$SMSE\left(\hat{y}_t^{(h)}\right)\le\min_{1\le k\le m}SMSE\left(x_{t,k}^{(h)}\right) \tag{14}$$

And ridge coefficient $r=0$ then $\hat{y}_t$ is called a multiple linear regression combination forecasting model of $x_{t,1},x_{t,2},\ldots,x_{t,m}$. If formula (14) is satisfied and $r>0$, then $\hat{y}_t$ is called a ridge regression composition forecasting model of $x_{t,1},x_{t,2},\ldots,x_{t,m}$. Note that Hoerl, Kenard, and Baldwin (1975) suggested that the ridge coefficient of ridge regression is

$$r=\frac{m\hat{\sigma}^2}{\underline{\beta}_t^T\underline{\beta}},\quad \hat{\sigma}^2=\frac{1}{t}\sum_{i=1}^{t}\left(y_i-\hat{y}_t\right)^2 \tag{15}$$

# 3 Choquet Integral Composition Forecasting Model

## *3.1 Fuzzy Measures [6-13]*

***Definition 5.*** **Fuzzy Measure [6-13]**
A fuzzy measure $\mu$ on a finite set $X$ is a set function $\mu:2^X\to[0,1]$ satisfying the following axioms:

$$\mu(\phi)=0,\mu\left(X\right)=1 \qquad \text{(boundary conditions)} \tag{16}$$

$$A\subseteq B\Rightarrow\mu\left(A\right)\le\mu\left(B\right) \qquad \text{(monotonicity)} \tag{17}$$

## 3.2 Fuzzy Density Function [6-10]

***Definition 6.*** **Fuzzy Density Function, Density [6-10]**
(i) A fuzzy density function of a fuzzy measure $\mu$ on a finite set $X$ is a function $d: X \to [0,1]$ satisfying:

$$d(x) = \mu(\{x\}), x \in X \tag{18}$$

$d(x)$ is called the density of singleton $x$.
(ii) A fuzzy density function is called a normalized fuzzy density function or a density if it satisfying

$$\sum_{x \in X} d(x) = 1 \tag{19}$$

***Definition 7.*** **Standard Fuzzy Measure [6-10]**
A fuzzy measure is called a standard fuzzy measure, if its fuzzy density function is a normalized fuzzy density function.

***Definition 8.*** ***N*-density [8-10]**
Let $\mu$ be a fuzzy measure on a finite set $X = \{x_1, x_2, ..., x_n\}$, $y_i$ be global response of subject $i$ and $f_i(x_j)$ be the evaluation of subject $i$ for singleton $x_j$, satisfying:

$$0 < f_i(x_j) < 1, i = 1, 2, ..., N, \quad j = 1, 2, ..., n \tag{20}$$

If

$$d_N(x_j) = \frac{r(f(x_j))}{\sum_{j=1}^{n} r(f(x_j))}, j = 1, 2, ..., n \tag{21}$$

Where $r(f(x_j))$ is the linear regression coefficient of $y_i$ on $f(x_j)$ satisfying

$$r(f(x_j)) = \frac{S_{y,x_j}}{S_y S_{x_j}} \geq 0 \tag{22}$$

$$S_y^2 = \frac{1}{N} \sum_{i=1}^{N} \left( y_i - \frac{1}{N} \sum_{i=1}^{N} y_i \right)^2 \tag{23}$$

$$S_{x_j}^2 = \frac{1}{N} \sum_{i=1}^{N} \left[ f_i(x_j) - \frac{1}{N} \sum_{i=1}^{N} f_i(x_j) \right]^2 \tag{24}$$

$$S_{y,x_j} = \frac{1}{N}\sum_{i=1}^{N}\left(y_i - \frac{1}{N}\sum_{i=1}^{N} y_i\right)\left[f_i\left(x_j\right) - \frac{1}{N}\sum_{i=1}^{N} f_i\left(x_j\right)\right] \quad (25)$$

then the function $d_N : X \to [0,1]$ satisfying $\mu(\{x\}) = d_N(x), \forall x \in X$ is a fuzzy density function, called $N$-density of $\mu$.

Note that

(i) $N$-density is a normalized fuzzy density function.

(ii) $N$-density is a linear fuzzy density function based on linear correlation coefficients

### 3.3 M-Density [10]

We know that any linear function can be viewed as a special case of some corresponding non-linear function, In this paper, a non-linear fuzzy density function based on Mean Square Error, denoted $M$-density, is proposed, its formal definition is introduced as follows:

***Definition 9.*** ***M*-density**

Let $\mu$ be a fuzzy measure on a finite set $X = \{x_1, x_2, ..., x_n\}$, $y_i$ be global response of subject $i$ and $f_i(x_j)$ be the evaluation of subject $i$ for singleton $x_j$, satisfying:

$$0 < f_i\left(x_j\right) < 1,\ i = 1,2,...,N,\ \ j = 1,2,...,n \quad (26)$$

If

$$d_M\left(x_j\right) = \frac{\left[MSE\left(x_j\right)\right]^{-1}}{\sum_{j=1}^{n}\left[MSE\left(x_j\right)\right]^{-1}},\ j = 1,2,...,n \quad (27)$$

Where

$$MSE\left(x_j\right) = \frac{1}{N}\sum_{i=1}^{N}\left(y_i - f_i\left(x_j\right)\right)^2 \quad (28)$$

then the function $d_M : X \to [0,1]$ satisfying $\mu(\{x\}) = d_M(x), \forall x \in X$ is a fuzzy density function, and called $M$-density of $\mu$.

### 3.4 Classification of Fuzzy Measures [6-10]

***Definition 10.*** Additive measure, sub-additive measure and supper- additive measure

(i) A fuzzy measure $\mu$ is called an sub-additive measure, if

$$\forall A, B \subset X, A \cap B = \phi \Rightarrow g_\mu\left(A \cup B\right) < g_\mu\left(A\right) + g_\mu\left(B\right) \quad (29)$$

(ii) A fuzzy measure $\mu$ is called an additive measure, if

$$\forall A, B \subset X, A \cap B = \phi \Rightarrow g_{\mu}(A \cup B) = g_{\mu}(A) + g_{\mu}(B) \tag{30}$$

(iii) A fuzzy measure $\mu$ is called a supper-additive measure, if

$$\forall A, B \subset X, A \cap B = \phi \Rightarrow g_{\mu}(A \cup B) > g_{\mu}(A) + g_{\mu}(B) \tag{31}$$

(iv) A fuzzy measure is called a mixed fuzzy measure, if is not a Additive measure, sub-additive measure and supper- additive measure.

***Theorem 1.*** Let *d* be a given fuzzy density function of an additive measure, A-measure, then its measure function $g_A : 2^X \to [0,1]$ satisfies

$$\forall E \subset X \Rightarrow g_A(E) = \sum_{x \in E} d(x) \tag{32}$$

## 3.4 λ-Measure [13]

***Definition 10.*** **λ-measure [13]**
For a given fuzzy density function *d* on a finite set *X*, $|X| = n$, a measure is called $\lambda$-measure, if its measure function, $g_\lambda : 2^X \to [0,1]$, satisfying:

(i)
$$g_\lambda(\phi) = 0,\ g_\lambda(X) = 1 \tag{33}$$

(ii)
$$\begin{aligned} & A, B \in 2^X, A \cap B = \phi, A \cup B \neq X \\ & \Rightarrow g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B) \end{aligned} \tag{34}$$

(iii)
$$\prod_{i=1}^{n} \left[1 + \lambda d(x_i)\right] = \lambda + 1 > 0,\ \ d(x_i) = g_\lambda(\{x_i\}) \tag{35}$$

***Theorem 2.*** Let *d* be a given fuzzy density function on a finite set *X*, $|X| = n$,
Under the condition of $\lambda$-measure, the equation (35) determines the parameter $\lambda$ uniquely:

(i) $\sum_{x \in X} d(x) > 1 \Rightarrow \lambda < 0$, $\lambda$-measure is a sub-additive measure (36)

(ii) $\sum_{x \in X} d(x) = 1 \Rightarrow \lambda = 0$, $\lambda$-measure is an additive measure (37)

(iii) $\sum_{x \in X} d(x) < 1 \Rightarrow \lambda > 0$, $\lambda$-measure is a supper-additive measure (38)

Note that

(i) $\lambda$-measure has just one feasible fuzzy measure satisfies the conditions of its own definition.

(ii) In equation (35), the value of $d(x_i)$ is decided first, and then to find the solution of the measure parameter $\lambda$, and $\prod_{i=1}^{n}\left[1+\lambda d(x_i)\right]$ can be viewed as a function of its fuzzy density $d(x_i)$. Therefore, we can say that $\lambda$-measure is predominate by its fuzzy density function.

(iii) $\lambda$-measure can not be a mixed fuzzy measure.

## 3.5 P-Measure [14]

***Definition 11.*** **P-measure [14]**
For a given fuzzy density function $d$ on a finite set $X$, $|X|=n$, a measure is called P-measure, if its measure function, $g_P : 2^X \to [0,1]$, satisfying:

(i) $$g_P(\phi)=0,\ g_P(X)=1 \tag{39}$$

(ii) $$\forall A\in 2^X \Rightarrow g_P(A)=\max_{x\in A} d(x)=\max_{x\in A} g_P(\{x\}) \tag{40}$$

***Theorem 3.*** **P-measure is always a sub-additive measure [6-10]**

Note that since the maximum of any finite set is unique, hence, P-measure has just one feasible fuzzy measure satisfies the conditions of its own definition.

## 3.6 Multivalent Fuzzy Measure [6-10]

***Definition 12.*** **Univalent fuzzy measure, multivalent fuzzy measure [4-8]**
A fuzzy measure is called a univalent fuzzy measure, if it has just one feasible fuzzy measure satisfies the conditions of its own definition, otherwise, it is called a multivalent fuzzy measure.

Note that both λ-measure and P-measure are univalent fuzzy measures.

## 3.7 L-Measure [6-10]

In my previous work **[4],** a multivalent fuzzy measure was proposed, which is called L-measure, since my last name is Liu. Its formal definition is as follows

***Definition 13.*** **L-measure [6-10]**
For a given fuzzy density function $d$ on a finite set $X$, $|X|=n$, a measure is called L-measure, if its measure function, $g_L : 2^X \to [0,1]$, satisfying:

(i) $$g_L(\phi)=0,\ g_L(X)=1 \tag{41}$$

(ii) $$L\in[0,\infty), X\neq A\subset X \Rightarrow g_L=\max_{x\in A} d(x)+\frac{(|A|-1)L\sum_{x\in A} d(x)\left[1-\max_{x\in A} d(x)\right]}{\left[n-|A|+L(|A|-1)\right]\sum_{x\in X} d(x)} \tag{42}$$

***Theorem 4***. Important Properties of L-measure [6]

(i) For any $\mathrm{L}\in[0,\infty)$, L-measure is a multivalent fuzzy measure, in other words, L-measure has infinite fuzzy measure solutions.

(ii) L-measure is an increasing function on L.

(iii) If $\mathrm{L}=0$ then L-measure is just the P-measure.

(iv) L-measure may be a mixed fuzzy measure

Note that

(i) P-measure is a special case of L-measure

(ii) L-measure does not contain additive measure and λ-measure, in other words, additive measure and λ-measure are not special cases of L-measure.

## 3.8 Extensional L-Measure [7]

For overcoming the drawback of L-measure, an improving multivalent fuzzy measure which containing additive measure and λ-measure., called extensional L-measure, was proposed by my next previous paper [7], Its formal definition is as follows;

***Definition 14.*** **Extensional L-measure, $\mathbf{L_E}$-measure [7]**

For a given fuzzy density function *d* on a finite set *X*, $|X|=n$, a measure is called extensional L-measure, if its measure function, $g_{L_E}:2^X\to[0,1]$, satisfying:

(i) $$g_{L_E}(\phi)=0,\ g_{L_E}(X)=1 \tag{43}$$

(ii) $$L\in[-1,\infty), A\subset X$$
$$\Rightarrow g_{L_E}(A)=\begin{cases}(1+L)\sum_{x\in A} d(x)-L\max_{x\in A} d(x) & ,L\in[-1,0]\\ \sum_{x\in A} d(x)+\dfrac{(|A|-1)L\sum_{x\in A} d(x)\left[1-\sum_{x\in A} d(x)\right]}{\left[n-|A|+L(|A|-1)\right]\sum_{x\in X} d(x)} & ,L\in(0,\infty)\end{cases} \tag{44}$$

***Theorem 5.*** Important Properties of $L_E$ –measure [7]

(i) For any $L \in [-1, \infty)$, $L_E$ -measure is a multivalent fuzzy measure, in other words, $L_E$-measure has infinite fuzzy measure solutions.
(ii) $L_E$-measure is an increasing function on L.
(iii) if $L = -1$ then $L_E$-measure is just the P-measure.
(iv) if $L = 0$ then $L_E$-measure is just the additive measure.
(v) if $L = 0$ and $\sum_{x \in X} d(x) = 1$, then $L_E$-measure is just the λ-measure.
(vi) if -1< L< 0 then $L_E$-measure is a supper-additive measure.
(vii) if L> 0 then $L_E$-measure is a sub-additive measure

Note that additive measure, λ-measure and P-measure are two special cases of $L_E$-measure.

## 3.9 B-Measure [7]

For considering to extend the extensional L-measure, a special fuzzy measure was proposed by my previous work as below;

***Definition 15.*** **B-measure [7]**
For a given fuzzy density function *d*, a B-measure, $g_B$, is a measure on a finite set $X, |X| = n$, satisfying:

$$\forall A \subset X \Rightarrow g_B(A) = \begin{cases} \sum_{x \in A} d(x) & if\ |A| \le 1 \\ 1 & if\ |A| > 1 \end{cases} \tag{45}$$

***Theorem 6.*** Any B-measure is a supper-additive measure.

## 3.10 Comparison of Two Fuzzy Measures [7-10]

***Definition 16.*** Comparison of two fuzzy measures [7-10]
For a given fuzzy density function, $d(x)$, on a finite set, X, let $\mu_1$ and $\mu_2$ be two fuzzy measures on X,

(i) If $g_{\mu_1}(A) = g_{\mu_2}(A), \forall A \subset X,$, then we say that $\mu_1$-measure is equal to $\mu_2$-measure, denoted as

$$\mu_1 - measure = \mu_2 - masure \tag{46}$$

(ii) If $g_{\mu_1}(A) < g_{\mu_2}(A), \forall A \subset X, 1 < |A| < |X|$ then we say that $\mu_1$-measure is less than $\mu_2$-measure, or $\mu_2$-measure is larger than $\mu_1$-measure, denoted as

$$\mu_1 - measure < \mu_2 - masure \tag{47}$$

(iii) If $g_{\mu_1}(A) \le g_{\mu_2}(A), \forall A \subset X, 1 < |A| < |X|$, then we say that $\mu_1$-measure is not larger than $\mu_2$-measure, or $\mu_2$-measure is not smaller than $\mu_1$-measure, denoted as

$$\mu_1 - measure \le \mu_2 - masure \tag{48}$$

.

***Theorem 7.*** For any given fuzzy density function, if $\mu - measure$ is a fuzzy measure, then we have

$$P - measure \le \mu as - meaure \le B - measure \tag{49}$$

In other words, for any given fuzzy density function, the P-measure is the smallest fuzzy measure, and the B-measure is the largest fuzzy measure.

## *3.11 Completed Fuzzy Measure*

***Definition 17.*** Completed fuzzy measure [8]
If the measure function of a multivalent fuzzy measure has continuously infinite fuzzy measure solutions, and both P-measure and $B$-measure are its limit fuzzy measure solutions, then this multivalent fuzzy measure is called a completed fuzzy measure.

Note that both the L –measure and $L_E$ –measure are not completed fuzzy measures, since

$\lim_{L\to\infty} \frac{(|A|-1)L\sum_{x\in A} d(x)}{\left[n-|A|+(|A|-1)L\right]\sum_{x\in X} d(x)} = \frac{\sum_{x\in A} d(x)}{\sum_{x\in X} d(x)} \ne 1$, the B-measure is not a limit fuzzy measure of the L –measure and $L_E$ –measure

## *3.12 Completed Extensional L-Measure*

***Definition 18.*** Completed extensional L-measue, $L_{CE}$ –measure
For a given fuzzy density function $d$ on a finite set $X$, $|X| = n$, a measure is called extensional L-measure, if its measure function, $g_{L_{CE}} : 2^X \to [0,1]$, satisfying:

(i) $$g_{L_{CE}}(\phi) = 0, \; g_{L_{CE}}(X) = 1 \tag{50}$$

$L \in [-1, \infty), A \subset X$

(ii)
$$\Rightarrow g_{L_{CE}}(A) = \begin{cases} (1+L)\sum_{x \in A} d(x) - L \max_{x \in A} d(x) & , L \in [-1, 0] \\ \sum_{x \in A} d(x) + \dfrac{(|A|-1)L\sum_{x \in A} d(x)\left[1 - \sum_{x \in A} d(x)\right]}{\left[n-|A|\right]\sum_{x \in X} d(x) + L(|A|-1)\sum_{x \in A} d(x)} & , L \in (0, \infty) \end{cases} \quad (51)$$

***Theorem 7.*** Important Properties of $L_{CE}$ –measure [7]

(i) For any $L \in [-1, \infty)$, $L_{CE}$ -measure is a multivalent fuzzy measure, in other words, $L_{CE}$ -measure has infinite fuzzy measure solutions.
(ii) $L_{CE}$ -measure is an increasing function on L.
(iii) if $L = -1$ then $L_{CE}$ -measure is just the P-measure.
(iv) if $L = 0$ then $L_{CE}$ -measure is just the additive measure.
(v) if $L = 0$ and $\sum_{x \in X} d(x) = 1$, then $L_{CE}$ -measure is just the λ-measure.
(vi) if $-1 < L < 0$ then $L_{CE}$ -measure is a sub-additive measure.
(vii) if $L > 0$ then $L_{CE}$ -measure is a supper-additive measure
(viii) $L \to \infty$ then $L_{CE}$ -measure is a B- measure
(ix) $L_{CE}$ -measure is a completed fuzzy measure.

Note that additive measure, λ-measure, P-measure and B-measure are special cases of $L_{CE}$ -measure.

## *3.13 Choquet Integral*

***Definition 19.*** **Choquet Integral [9-10]**

Let $\mu$ be a fuzzy measure on a finite set $X = \{x_1, x_2, ..., x_m\}$. The Choquet integral of $f_i : X \to R_+$ with respect to $\mu$ for individual *i* is denoted by

$$\int_C f_i d\mu = \sum_{j=1}^{m} \left[ f_i\left(x_{(j)}\right) - f_i\left(x_{(j-1)}\right) \right] \mu\left(A^i_{(j)}\right), \; i = 1, 2, ..., N \quad (52)$$

where $f_i\left(x_{(0)}\right) = 0$, $f_i\left(x_{(j)}\right)$ indicates that the indices have been permuted so that

$$0 \le f_i\left(x_{(1)}\right) \le f_i\left(x_{(2)}\right) \le ... \le f_i\left(x_{(m)}\right), \; A_{(j)} = \left\{x_{(j)}, x_{(j+1)}, ..., x_{(m)}\right\} \quad (53)$$

Note that from Definition 19, for given integrand $f_i : X \to R_+$, the Choquet integral can be viewed as a function of the fuzzy measure $\mu$ -measure, in other words, the value of Choquet integral is predominate by its fuzzy measure.

**Theorem 8.** If a λ-measure is a standard fuzzy measure on $X=\{x_1,x_2,...,x_m\}$, and $d:X\to[0,1]$ is its fuzzy density function, then the Choquet integral of $f_i:X\to R_+$ with respect to λ for individual $i$ satisfying

$$\int_C f_i d\lambda=\sum_{j=1}^{m}d\left(x_j\right)f_i\left(x_j\right),\ i=1,2,...,N \tag{54}$$

## 3.14 Choquet Integral Composition Forecasting Model

***Definition 20.*** **Choquet Integral Composition Forecasting Model [8]**
(i) Let $y_t$ be the realized value of target variable at time *t*,
(ii) Let $X=\{x_1,x_2,...,x_m\}$ be the set of m competing predictors,
(iii) Let $f_t:X\to R_+$, $f_t\left(x_1\right),f_t\left(x_2\right),...,f_t\left(x_m\right)$ be $m$ forecasting values of $y_t$ by competing predictors $x_1,x_2,...,x_m$ at time *t*.
If $\mu$ is a fuzzy measure on $X$, $\alpha,\beta\in R$ satisfying

$$\left(\hat{\alpha},\hat{\beta}\right)=\arg\min_{\alpha,\beta}\left[\sum_{t=1}^{N}\left(y_i-\alpha-\beta\int_C f_t dg_\mu\right)\right] \tag{55}$$

$$\hat{\alpha}=\frac{1}{N}\sum_{t=1}^{N}y_t-\hat{\beta}\frac{1}{N}\sum_{t=1}^{N}\int f_t dg_\mu\ ,\ \ \hat{\beta}=\frac{S_{yf}}{S_{ff}} \tag{56}$$

$$\hat{\alpha}=\frac{1}{N}\sum_{t=1}^{N}y_t-\hat{\beta}\frac{1}{N}\sum_{t=1}^{N}\int f_t dg_\mu \tag{57}$$

$$S_{yf}=\frac{\sum_{t=1}^{N}\left[y_i-\frac{1}{N}\sum_{t=1}^{N}y_t\right]\left[\int f_t dg_\mu-\frac{1}{N}\sum_{t=1}^{N}\int f_t dg_\mu\right]}{N-1} \tag{58}$$

then $\hat{y}_t=\hat{\alpha}+\hat{\beta}\int f_t dg_\mu, t=1,2,...,N$ is called the Choquet integral regression composition forecasting estimator of $y_t$, and this model is also called the Choquet integral regression composition forecasting model with respect to $\mu$-measure.

***Theorem 9.*** If a λ-measure is a standard fuzzy measure then Choquet integral regression composition forecasting model with respect to λ-measure is just a linear combination forecasting model.

## 4 Experiments and Results

A real data of the grain production with 3 kinds of forecasted values of the time series model, the exponential smoothing model and GM(1,1) forecasting model, respectively, in Jilin during 1952 to 2007 from the paper of Zhang, Wang and Gao [2],was listed in Table 2. For evaluating the proposed new density based composition forecasting model, an experiment with the above-mentioned data by using sequential mean square error was conducted.

We arrange the first 50 years grain production and their 3 kinds of forecasted values as the training set and the rest data as the forecasting set. And the following *N*-density and *M*-density of all fuzzy measures were used

$$N\text{-density:} \qquad \{0.3331, \quad 0.3343, \quad 0.3326\} \tag{59}$$

$$M\text{-density:} \qquad \{0.2770, \quad 0.3813, \quad 0.3417\} \tag{60}$$

The performances of Choquet integral composition forecasting model with extensional L-measure, L-measure, λ-measure and P-measure, respectively, a ridge regression composition forecasting model and a multiple linear regression composition forecasting model and the traditional linear weighted composition forecasting model were compared. The result is listed in Table 1.

**Table 1** SMSEs of 2 densities for 7 composition forecasting models

| Composition forecasting Models | | SMSE | |
|---|---|---|---|
| | | *N*-density | *M*-density |
| Choquet integral regression | $L_{CE}$-measure | 13149.64 | 13217.31 |
| | $L_E$-measure | 13939.84 | 13398.29 |
| | L-measure | 14147.83 | 13751.60 |
| | λ-measure | 21576.38 | 19831.86 |
| | P-measure | 16734.88 | 16465.98 |
| Ridge regression | | 18041.92 | |
| Multiple linear regression | | 24438.29 | |

Table 1 shows that the *M*-density based Choquet integral composition forecasting model with respect to $L_{CE}$-measure outperforms other composition forecasting models. Furthermore, for each fuzzy measure, including the $L_{CE}$-measure, $L_E$-measure, L-measure, λ-measure and P-measure, the *M*-density based Choquet integral composition forecasting model is better than the *N*-density based.

## 5 Conclusion

In this paper, a new density, *M*-density, was proposed. Based on *M*-density, a novel composition forecasting model was also proposed. For comparing the forecasting efficiency of this new density with the well-known density, *N*-density, a real data experiment was conducted. The performances of Choquet integral composition forecasting model with the completed extensional L-measure, extensional L-measure, λ-measure and P-measure, by using *M*-density and *N*-density, respectively, a ridge regression composition forecasting model and a multiple linear regression composition forecasting model and the traditional linear weighted composition forecasting model were compared. Experimental result showed that for each fuzzy measure, including the $L_{CE}$-measure, $L_E$-measure, L-measure, λ-measure and P-measure, the *M*-density based Choquet integral composition forecasting model is better than the *N*-density based, and the *M*-density based Choquet integral composition forecasting model outperforms all of other composition forecasting models.

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# Appendix

**Table 2** SMSEs of 2 densities for 6 composition forecasting models

| Years | Y | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|---|
| 1952 | 613.20 | 490.67 | 518.60 | 399.51 | 472.45 |
| 1953 | 561.45 | 549.73 | 570.84 | 414.09 | 511.35 |
| 1954 | 530.95 | 542.83 | 586.41 | 429.20 | 524.94 |
| 1955 | 556.53 | 549.57 | 584.31 | 444.86 | 530.10 |
| 1956 | 493.64 | 582.69 | 591.12 | 461.09 | 542.51 |
| 1957 | 429.35 | 598.64 | 570.80 | 477.91 | 538.81 |
| 1958 | 528.84 | 610.69 | 531.14 | 495.35 | 524.37 |
| 1959 | 526.60 | 633.88 | 540.11 | 513.43 | 537.85 |
| 1960 | 394.70 | 655.07 | 544.78 | 532.16 | 549.04 |
| 1961 | 398.55 | 672.97 | 497.45 | 551.58 | 531.58 |
| 1962 | 437.16 | 694.53 | 465.26 | 571.71 | 523.02 |
| 1963 | 501.67 | 617.26 | 457.04 | 592.57 | 519.94 |
| 1964 | 491.80 | 738.99 | 475.53 | 614.19 | 547.93 |
| 1965 | 525.10 | 761.94 | 484.57 | 636.61 | 563.02 |
| 1966 | 597.60 | 786.18 | 503.23 | 659.84 | 583.82 |
| 1967 | 647.74 | 810.67 | 543.38 | 683.91 | 616.78 |
| 1968 | 622.15 | 835.87 | 589.95 | 708.87 | 653.65 |
| 1969 | 498.70 | 862.13 | 612.17 | 734.74 | 677.54 |
| 1970 | 738.80 | 889.12 | 580.22 | 761.55 | 672.02 |
| 1971 | 713.05 | 916.86 | 647.93 | 789.33 | 721.78 |

**Table 2** (*continued*)

| | | | | | |
|---|---|---|---|---|---|
| 1972 | 556.99 | 945.56 | 684.75 | 818.14 | 754.99 |
| 1973 | 783.00 | 975.15 | 650.45 | 847.99 | 749.50 |
| 1974 | 858.15 | 1005.63 | 711.01 | 878.93 | 796.71 |
| 1975 | 906.50 | 1037.08 | 780.79 | 911.01 | 849.50 |
| 1976 | 755.50 | 1069.53 | 846.59 | 944.25 | 900.60 |
| 1977 | 728.35 | 1102.98 | 833.96 | 978.70 | 909.05 |
| 1978 | 914.70 | 1137.47 | 813.29 | 1014.40 | 913.61 |
| 1979 | 903.34 | 1173.05 | 867.92 | 1051.40 | 960.20 |
| 1980 | 859.60 | 1209.74 | 900.44 | 1089.80 | 995.23 |
| 1981 | 921.91 | 1247.58 | 905.13 | 1129.60 | 1015.54 |
| 1982 | 1000.04 | 1286.60 | 930.45 | 1170.80 | 1047.82 |
| 1983 | 1477.98 | 1326.84 | 976.21 | 1213.50 | 1092.01 |
| 1984 | 1634.46 | 1368.34 | 1187.28 | 1257.80 | 1227.91 |
| 1985 | 1225.26 | 1411.14 | 1391.77 | 1303.70 | 1360.87 |
| 1986 | 1397.71 | 1455.27 | 1376.80 | 1351.30 | 1373.74 |
| 1987 | 1675.81 | 1500.79 | 1428.01 | 1400.60 | 1423.79 |
| 1988 | 1693.25 | 1547.73 | 1565.57 | 1451.70 | 1522.16 |
| 1989 | 1351.29 | 1596.14 | 1664.67 | 1504.60 | 1600.13 |
| 1990 | 2046.52 | 1646.06 | 1600.61 | 1559.50 | 1589.13 |
| 1991 | 1898.87 | 1697.54 | 1814.80 | 1616.50 | 1732.23 |
| 1992 | 1840.31 | 1750.64 | 1904.70 | 1675.40 | 1807.73 |
| 1993 | 1900.90 | 1805.39 | 1940.83 | 1736.60 | 1854.61 |
| 1994 | 2015.70 | 1861.86 | 1984.40 | 1799.90 | 1906.49 |
| 1995 | 1992.40 | 1920.09 | 2053.78 | 1865.60 | 1973.62 |
| 1996 | 2326.60 | 1980.15 | 2089.00 | 1933.70 | 2022.96 |
| 1997 | 1808.30 | 2042.08 | 2235.53 | 2004.30 | 2134.67 |
| 1998 | 2506.00 | 2105.95 | 2137.39 | 2077.40 | 2112.74 |
| 1999 | 2305.60 | 2171.82 | 2328.13 | 2153.20 | 2251.02 |
| 2000 | 1638.00 | 2239.75 | 2381.33 | 2231.80 | 2314.77 |
| 2001 | 1953.40 | 2309.80 | 2161.43 | 2313.20 | 2229.36 |
| 2002 | 2214.80 | 2382.04 | 2123.07 | 2397.60 | 2245.19 |

**Table 2** (*continued*)

| | | | | | |
|---|---|---|---|---|---|
| 2003 | 2259.60 | 2456.54 | 2192.19 | 2485.10 | 2321.52 |
| 2004 | 2510.00 | 2533.38 | 2254.69 | 2575.80 | 2395.57 |
| 2005 | 2581.21 | 2612.61 | 2390.11 | 2669.80 | 2511.18 |
| 2006 | 2720.00 | 2694.33 | 2508.40 | 2767.20 | 2618.82 |
| 2007 | 2454.00 | 2778.60 | 2560.38 | 2831.50 | 2677.95 |

Y: realized value of target variable
$X_1$: Fitting value of time series model
$X_2$: Fitting value of exponential smoothing model
$X_3$: Fitting value of GM(1,1) model
$X_4$: Fitting value of composition forecasting model