Chapter 4
Financial Fuzzy Time Series Models Based on Ordered Fuzzy Numbers

Adam Marszałek and Tadeusz Burczyński

Abstract. The purpose of this chapter is to present an original concept of financial fuzzy time series models based on financial data in the form of Japanese Candlestick Charts. In this approach the Japanese Candlesticks are modeled using Ordered Fuzzy Numbers (OFN) called further Ordered Fuzzy Candlesticks (OFC). The use of ordered fuzzy numbers allows modeling uncertainty associated with financial data. Thanks to well-defined arithmetic of ordered fuzzy numbers, one can construct models of fuzzy time series, such as e.g. an autoregressive process, where all input values are OFC, while the coefficients and output values are arbitrary OFN, in the form of classical equations, without using rule-based systems. Finally, several applications of these models for modeling and forecasting selected financial time series are presented.

1 Introduction

It is hard to disagree with opinion that among all different sources of data, the financial market is the most uncertain. The main reason is the fact that huge amount of information is reflected in the financial market. What more, we can say that everything that happens in the world (e.g. in economy, politics) has an effect on quotations of financial instruments. On the other hand, how the information influence the market is decided by investors by taking a long or short position in the market.

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The investors can be simple divided into two groups. The first group of investors decides using fundamental analysis, while the second group decides on a basis technical analysis. Both groups must make a subjective assessment of macroeconomic factors and signals of technical analysis, respectively, so the human factor is a cause of uncertainty as well.

The second group of investors very often uses price charts analysis to make decisions. The price charts (e.g. Japanese Candlestick chart) are used to illustrate movements in the price of a financial instrument over time. Notice, that using the price chart, a large part of the information about the process is lost, e.g. using Japanese Candlestick chart with one hour frequency, for one hour, we know only four prices, while in this time the price must have changed hundreds of times.

In this paper we propose fuzzy logic (i.e. ordered fuzzy numbers), to model uncertainty associated with financial data and reduce the size of lost information. Further, we show how the concept (OFC) can be used to build models of financial time series.

2 Financial Data

In this work as a financial data we mean the quotations of financial instruments (e.g. stock prices or currency pair). Making investment decisions based on observation of each single quotation is very difficult or even impossible, when price changes tens times a minute.

In practice, quotations of financial instruments are represented using price charts [12]. The open-high-low-close chart (also OHLC chart, or simply bar chart) and Japanese Candlestick are most often used in technical analysis. Both types of charts are presented in Figs. 1 and 2, respectively.

Fig. 1 Open-High-Low-Close chart of EUR/USD, four hour frequency

Each bar represents the range of price movement over a given time interval. In both types of charts, bars are described by only four prices from given time period: first (open), highest, lowest and last (close) price at a given time interval. In addition, Japanese Candlestick has a body, whose color illustrates the relationship between

Fig. 2 Japanese Candlestick chart of EUR/USD, four hour frequency

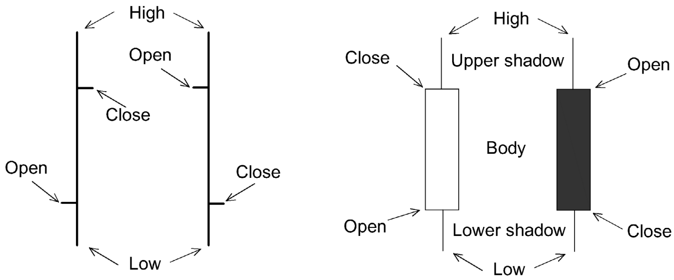

Fig. 3 The long and short OHLC bar, and long and short Japanese Candlestick

the opening and closing price. If the Candlestick closed higher than it opened, the body is white or unfilled, else the body is black. The formation of OHLC bar and Japanese Candlestick are shown in Fig. 3. More details about the Japanese Candlesticks and trading techniques based on them can be found in [13].

3 Ordered Fuzzy Numbers

One of many ways of uncertainty modeling is an approach based on fuzzy logic. Fuzzy data analysis requires also fuzzy arithmetic. Applications of classical fuzzy numbers (sets) [17, 18] or so-called (L,R)-numbers with two shape functions L and R [1] lead to some drawbacks that concern properties of fuzzy algebraic operations, as well as produce unexpected and uncontrollable results when using these operations in an iterative way [16, 17]. In the series of papers [4, 5, 6, 7, 8], W. Kosiński et al. introduced and developed main concepts of the space of ordered fuzzy numbers (OFN), whose arithmetic eliminates these drawbacks.

3.1 Definition of Ordered Fuzzy Number

The concept of membership functions has been weakened by requiring a mere *membership relation*. Consequently, *an ordered fuzzy number A* is identified with an ordered pair of continuous real functions defined on the interval $[0,1]$, i.e. $A=(f,g)$ with $f,g\colon [0,1]\to\mathbb{R}$.

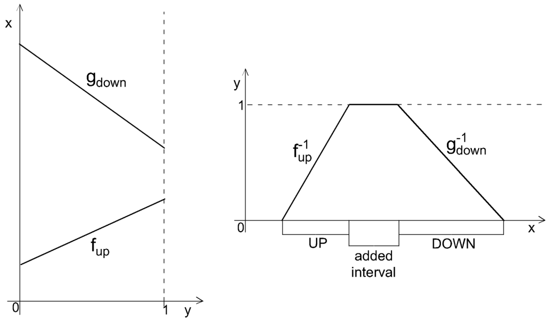

Fig. 4 Graphical interpretation of OFN and an OFN presented as fuzzy number in classical meaning

Functions f and g are called the *up* and *down*-parts of the fuzzy number A, respectively. The continuity of both parts implies their images are bounded intervals, say *UP* and *DOWN*, respectively. In general, the functions f and g need not be invertible, and only continuity is required. If we assume, however, that these functions are monotonous, i.e., invertible, and add the constant function of x on the interval $[1_A^-,1_A^+]$ with the value equal to 1, we might define the membership function

$$\mu(x)=\begin{cases} f^{-1}(x) & \text{if } x\in[f(0),f(1)],\\ g^{-1}(x) & \text{if } x\in[g(1),g(0)],\\ 1 & \text{if } x\in[1_A^-,1_A^+], \end{cases} \tag{1}$$

if f is increasing and g is decreasing, and such that $f\le g$ (pointwise). In this way, the obtained membership function $\mu(x),\ x\in\mathbb{R}$ represents a mathematical object which resembles a convex fuzzy number in the classical sense. The ordered fuzzy number and ordered fuzzy number as a fuzzy number in classical meaning are presented in Fig. 4.

Let us note that a pair of continuous functions (f,g) determines different ordered fuzzy number than the pair (g,f). It follows from the fact that we are dealing with an ordered pair of functions. In this way, we specified an extra feature to this object, named the *orientation*. In graphical interpretation of the ordered fuzzy number, orientation is presented by arrow. Depending on the orientation, the ordered fuzzy numbers can be divided into two types: *a positive orientation*, if the direction of ordered fuzzy number is consistent with the direction of the axis *Ox* and *a negative orientation*, if the direction of the ordered fuzzy number is opposite to the direction of the axis *Ox*, as shown in Fig. 5.

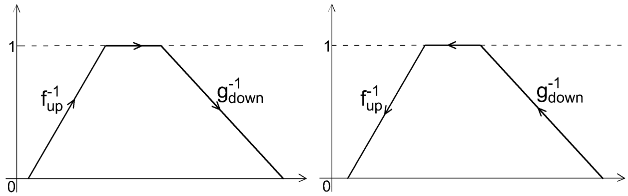

Fig. 5 Positively and negatively oriented OFN

3.2 Operations

The basic arithmetic operations on ordered fuzzy numbers are defined as the pair-wise operations of their elements.

Let $A = (f_A, g_A)$, $B = (f_B, g_B)$ and $C = (f_C, g_C)$ are mathematical objects called ordered fuzzy numbers. The sum $C = A + B$, subtraction $C = A - B$, product $C = A \cdot B$, and division $C = A \div B$ are defined by formula

$$f_C(y) = f_A(y) * f_B(y), \qquad g_C(y) = g_A(y) * g_B(y) \tag{2}$$

where $*$ works for $+$, $-$, $\cdot$ and $\div$, respectively, and where $C = A \div B$ is defined, if the functions $|f_B|$ and $|g_B|$ are bigger than zero. In a similar way, if we want to multiply an ordered fuzzy number A by a scalar $\lambda \in \mathbb{R}$, then the product $C = \lambda \cdot A$ is defined by formula

$$f_C(y) = \lambda \cdot f_A(y), \qquad g_C(y) = \lambda \cdot g_A(y) \tag{3}$$

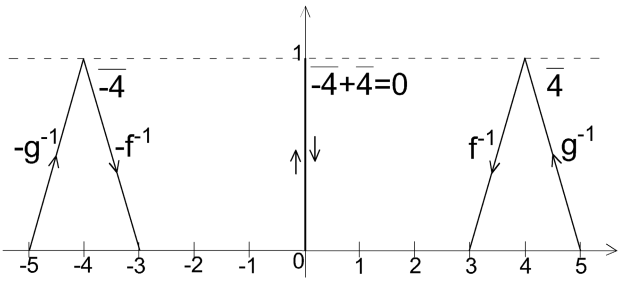

Fig. 6 Sum of two opposite ordered fuzzy numbers

Notice that the subtraction of B is the same as the addition of the opposite of B, i.e. the number $(-1) \cdot B$. If we will do $B + (-1) \cdot B$ we get a numeric zero, i.e., an ordered fuzzy number represented by the pair of constant functions equal to zero. In a similar way, the inverse $1/B$ of an ordered fuzzy number B is defined as an ordered fuzzy number such that the product $B \cdot (1/B)$ gives a number, i.e.,

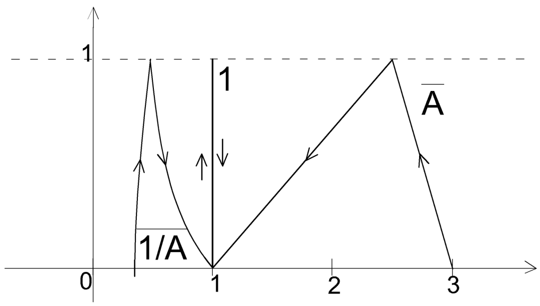

Fig. 7 Product of ordered fuzzy number and its inverse

an ordered fuzzy number represented by the pair of constant functions equal to one. This is presented in Figs. 6 and 7, respectively.

The existence of neutral elements of addition and multiplication is the most important advantage for our further consideration. This fact causes that not always the result of an arithmetic operation is a fuzzy number with a larger support. This allows to build fuzzy models based on ordered fuzzy numbers in the form of the classical equations without losing the accuracy.

3.3 Defuzzification of Ordered Fuzzy Number

Let $\mathcal{O}$ be a universe of all ordered fuzzy numbers. $\mathcal{O}$ can be identified with $\mathcal{C}^0([0,1]) \times \mathcal{C}^0([0,1])$, hence the space $\mathcal{O}$ is a Banach space [7]. A class of defuzzification operators of ordered fuzzy numbers can be defined, as a linear and continuous functionals on the Banach space $\mathcal{O}$, thanks to the general representation theorem (of Banach-Kakutami-Riesz) they are uniquely determined by a pair of Radon measures (ν_1, ν_2) on $[0,1]$, as

$$Def(A) = \int_0^1 f_A \mathrm{d}\nu_1 + \int_0^1 g_A \mathrm{d}\nu_2 \tag{4}$$

where $Def(A)$ is the value of a defuzzification operator at the ordered fuzzy number $A = (f_A, g_A)$.

The above formula gives a continuum of defuzzification operators, both linear and nonlinear, which map ordered fuzzy numbers into reals. For example, the standard defuzzification procedure in terms of the area under membership relation can be defined. It is realized by a linear combinations of two Lebesgue measures of $[0,1]$.

4 Ordered Fuzzy Candlesticks

The aim of our research is to find a new tool for modeling of financial data. We want to make it so easy as classical Japanese Candlesticks for observing by investors. At the same time to allow for modeling of uncertainty associated with financial data and also keep more information about the prices than Japanese Candlestick. Ordered fuzzy numbers presented previously, in a simple way, satisfy our requirements.

Generally, in this approach, further as Japanese Candlestick is identified with ordered fuzzy number and it is called *Ordered Fuzzy Candlestick* (OFC). The general idea is presented in Fig. 8. Notice, that the orientation of the ordered fuzzy number shows whether the ordered fuzzy candlestick is long or short. While the information about movements in the price are contained in the shape of the f and g functions. In the following sections we will show how the ordered fuzzy candlestick can be constructed.

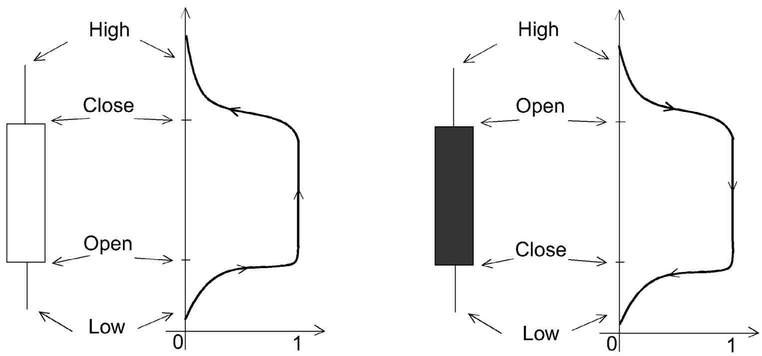

Fig. 8 The Japanese Candlesticks presented as a ordered fuzzy numbers

4.1 Proposal of Global Definition of Ordered Fuzzy Candlestick

Let $\{X_t \colon t \in T\}$ be a given time series and $T = \{1, 2, \dots, n\}$. The ordered fuzzy candlestick is defined as an ordered fuzzy number $C = (f, g)$ which satisfies the following properties 1 - 4 or 5 - 8.

Long Candlestick

1. $X_1 \leq X_n$
2. $f \colon [0,1] \to \mathbb{R}$ is continuous and increasing on $[0,1]$
3. $g \colon [0,1] \to \mathbb{R}$ is continuous and decreasing on $[0,1]$
4. $S_1 < S_2,\ f(1) = S_1,\ f(0) = \min_{t \in T} X_t - C_1,\ g(1) = S_2$ and $g(0)$ such that

$$\frac{\int_0^1 g(y)\mathrm{d}y - S_2}{A} = \frac{S_1 - \int_0^1 f(y)\mathrm{d}y}{B} \tag{5}$$

Short Candlestick

5. $X_1 > X_n$
6. $f\colon [0,1] \to \mathbb{R}$ is continuous and decreasing on $[0,1]$
7. $g\colon [0,1] \to \mathbb{R}$ is continuous and increasing on $[0,1]$
8. $S_1 < S_2,\ f(1) = S_2,\ f(0) = \max_{t\in T} X_t + C_2,\ g(1) = S_1$ and $g(0)$ such that

$$\frac{\int_0^1 f(y)\mathrm{d}y - S_2}{A} = \frac{S_1 - \int_0^1 g(y)\mathrm{d}y}{B} \tag{6}$$

The center of ordered fuzzy candlestick (i.e. added interval) is designated by parameters $S_1, S_2 \in \left[\min_{t\in T}, \max_{t\in T}\right]$, while C_1 and C_2 are arbitrary nonnegative real numbers. The parameters A and B are positive real numbers, and together with equations (5) and (6) determine the relationship between the function f and g. A selection of parameters are discussed in greater detail in the next section.

4.2 *Parameters of Ordered Fuzzy Candlesticks*

Let $\{X_t : t \in T\}$ be a given time series and $T = \{1, 2, \ldots, n\}$.

Parameters S_1 and S_2
For to designate the center of the ordered fuzzy candlestick, we can use the average of time series X_t. There are many types of average, the most popular ones are

Simple Average

$$SA = \frac{1}{n}(X_1 + X_2 + \cdots + X_n) \tag{7}$$

Linear Weighted Average

$$LWA = \frac{X_1 + 2X_2 + \cdots + nX_n}{1 + 2 + \cdots + n} \tag{8}$$

Exponential Average

$$EA = \frac{(1-\alpha)^{n-1}X_1 + (1-\alpha)^{n-2}X_2 + \cdots + (1-\alpha)X_{n-1} + X_n}{(1-\alpha)^{n-1} + (1-\alpha)^{n-2} + \cdots + (1-\alpha) + 1}, \quad \alpha = \frac{2}{n+1} \tag{9}$$

Consequently we propose the following

$$S_1, S_2 \in \{SA, LWA, EA\} \text{ such that } S_1 \leq S_2$$

Determination of parameters A and B
For parameters A and B the following formula is proposed

$$A = 1 + S^{+S_2} \quad \text{and} \quad B = 1 + S^{-S_1}$$

where S^{+S_2} and S^{-S_1} means that one of the sums from numerator in the formulas (7), (8) or (9), calculated only for $X_t \geq S_2$ and $X_t \leq S_1$, respectively. These parameters shows how much the movement is concentrated above and below parameters S_1 and S_2, respectively. If formula (8) or (9) is selected then we assume that the more recent time series values are more important than the past ones, which is a natural assumption in financial processes.

Parameters C_1 and C_2
The parameters C_1 and C_2 are defined as a standard deviation of X_t

$$C_1 = C_2 = \sigma_{X_t}$$

4.3 Special Types of Ordered Fuzzy Candlesticks

In this section, some simple types of ordered fuzzy candlesticks are presented.

Trapezoid OFC
Suppose that f and g are linear functions in form

$$f(y) = (f(1) - f(0))y + f(0) \tag{10}$$

$$g(y) = (g(1) - g(0))y + g(0) \tag{11}$$

then the ordered fuzzy candlestick $C = (f, g)$ is called *a trapezoid OFC*, especially if $S_1 = S_2$ then also can be called *a Triangular OFC*.

Let X_t be a given time series. Suppose that $X_1 \leq X_n$ then we have

$$f(y) = (S_1 - \min X_t + C_1)y + \min X_t - C_1 \tag{12}$$

$$g(y) = (S_2 - g(0))y + g(0) \tag{13}$$

where

$$g(0) = \frac{A}{B}(S_1 - \min X_t + C_1) + S_2 \tag{14}$$

Whereas if $X_1 > X_n$ then we have

$$f(y) = (S_2 - \max X_t + C_2)y + \max X_t + C_2 \quad (15)$$

$$g(y) = (S_1 - g(0))y + g(0) \quad (16)$$

where

$$g(0) = \frac{B}{A}(S_2 - \max X_t - C_2) + S_1 \quad (17)$$

Gaussian OFC

The ordered fuzzy candlestick $C = (f,g)$ where the membership relation has a shape similar to the Gaussian function is called *a Gaussian OFC*. It means that f and g are given by functions

$$f(y) = f(z) = \sigma_f \sqrt{-2\ln(z)} + m_f \quad (18)$$

$$g(y) = g(z) = \sigma_g \sqrt{-2\ln(z)} + m_g \quad (19)$$

where e.g. $z = 0.99y + 0.01$.

Let X_t be a given time series. Suppose that $X_1 \leq X_n$ then we have

$$f(z) = \sigma_f \sqrt{-2\ln(z)} + m_f \text{ where } m_f = S_1,\ \sigma_f = \frac{\min X_t - C_1 - S_1}{\sqrt{-2\ln(0.01)}} \leq 0 \quad (20)$$

$$g(z) = \sigma_g \sqrt{-2\ln(z)} + m_g \text{ where } m_g = S_2,\ \sigma_g = -\frac{A}{B}\sigma_f \quad (21)$$

Whereas if $X_1 > X_n$ then we have

$$f(z) = \sigma_f \sqrt{-2\ln(z)} + m_f \text{ where } m_f = S_2,\ \sigma_f = \frac{\max X_t + C_1 - S_2}{\sqrt{-2\ln(0.01)}} \geq 0 \quad (22)$$

$$g(z) = \sigma_g \sqrt{-2\ln(z)} + m_g \text{ where } m_g = S_1,\ \sigma_g = -\frac{B}{A}\sigma_f \quad (23)$$

4.4 *Experimental Studies*

Let X_t be a given time series of quotations of EUR/USD for the 1-hour period ending 09.01.2011 at 7pm (236 ticks). The time series X_t and its histogram are presented in Fig. 9. For time series X_t we have $X_0 = X_{235} = 1.2894$, so this Japanese Candlestick has no body. It is so-called Doji Candlestick. Assume that $S_1 = EA = 1.28972$, $S_2 = SA = 1.28986$ and $C_1 = C_2 = \sigma_{X_t} = 2.23e^{-7}$. The exponential average was used in the calculation of parameters A and B of the ordered fuzzy candlesticks, so we have $A = 60.32825$ and $B = 70.83852$. The classical Japanese Candlestick, Trapezoid OFC and Gaussian OFC for time series X_t are presented in Fig. 10.

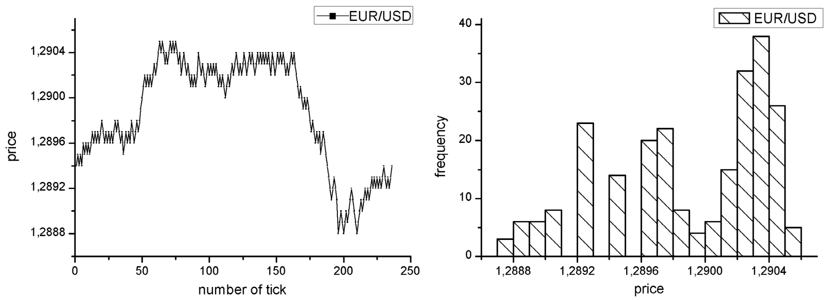

Fig. 9 The tick chart and histogram of EUR/USD

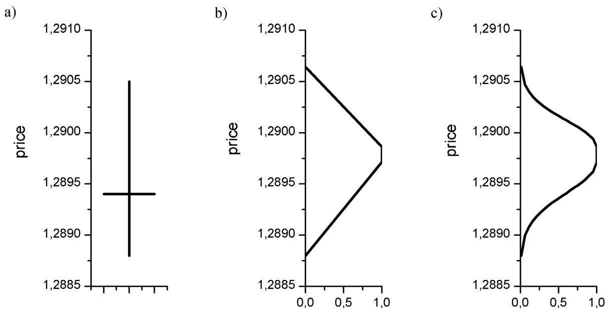

Fig. 10 Types of Candlesticks for time series X_t: a) classical Japanese Candlestick, b) Trapezoidal OFC, c) Gaussian OFC

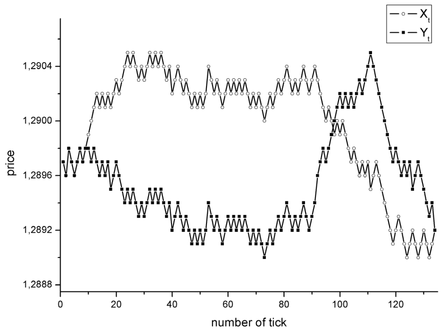

Fig. 11 The tick charts of the time series X_t and Y_t

Now, the two different time series X_t and Y_t are presented in Fig. 11. Both have the same Japanese Candlestick (see Fig. 12a), because the main prices (i.e. OHLC) are the same. However, the ordered fuzzy candlesticks for time series X_t and Y_t presented in Figs. 12b and 12c are different. Therefore, we can conclude that the ordered fuzzy candlesticks effectively contain more information than classical Japanese Candlesticks.

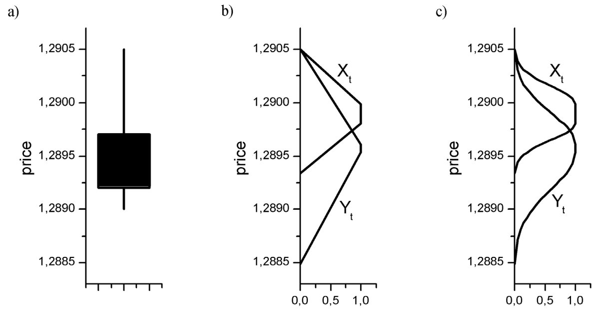

Fig. 12 Types of Candlesticks for time series X_t and Y_t: a) Classical Japanese Candlestick, b) Trapezoid OFC, c) Gaussian OFC

5 An Application of Ordered Fuzzy Candlesticks

5.1 Ordered Fuzzy Simple Moving Average (OFSMA(s))

Ordered fuzzy candlesticks can be used e.g. to construct a fuzzy version of simple technical indicators (i.e. a indicators that require only arithmetic operations such as addition, subtraction and multiplication by a scalar). The Simple Moving Average is presented as an example of technical indicator.

The classical Simple Moving Average with order s at a time period t is given by formula

$$SMA_t(S) = \frac{1}{s}(X_t + X_{t-1} + \cdots + X_{t-s+1}) \tag{24}$$

where X_t is the observation (real) at a time period t (e.g. closing prices) [12].

Now, the Ordered Fuzzy Simple Moving Average with order s at a time period t is also given by formula (24) but the observations X_t are OFC, i.e.

$$OFSMA_t(S) = \frac{1}{s}(\bar{X}_t + \bar{X}_{t-1} + \cdots + \bar{X}_{t-s+1}) \tag{25}$$

where $\bar{X}_t$ is the ordered fuzzy candlestick at a time period t. The process of fuzzification of the other simple technical indicators can be done by analogy.

Notice, if A and B are Trapezoidal (Gaussian) ordered fuzzy candlesticks and $\lambda \in \mathbb{R}$ then ordered fuzzy candlesticks $C = A + B$, $C = A - B$ and $C = \lambda \cdot A$ are Trapezoidal (Gaussian) OFC as well. Moreover, if their functions are in the form of following expressions

$$\phi(y; a, b) = ay + b, \quad \text{for Trapezoid OFC} \tag{26}$$

$$\psi(y; \sigma, m) = \psi(z; \sigma, m) = \sigma\sqrt{-2\ln(z)} + m, \quad \text{for Gaussian OFC} \tag{27}$$

then we have

$$\begin{aligned}\phi(y;a_1,b_1) \pm \phi(y;a_2,b_2) &= \phi(y;a_1 \pm a_2, b_1 \pm b_2)\\ \psi(y;\sigma_1,m_1) \pm \psi(y;\sigma_2,m_2) &= \psi(y;\sigma_1 \pm \sigma_2, m_1 \pm m_2)\end{aligned} \tag{28}$$

$$\begin{aligned}\lambda \cdot \phi(y;a,b) &= \phi(y;\lambda \cdot a, \lambda \cdot b)\\ \lambda \cdot \psi(y;\sigma,m) &= \psi(y;\lambda \cdot \sigma, \lambda \cdot m)\end{aligned} \tag{29}$$

This causes that the numerical implementation of these operations is much simpler.

Empirical Results

The practical case study was performed on data from FOREX market. The data covering the period of 93 hours from 5pm of 09.01.2011 till 2pm of 14.01.2011 of quotations of EUR/USD. The data set included 65376 ticks and is presented in Fig. 13. The classical Japanese Candlestick chart of 1 hour frequency for the set data is shown in Fig. 14. The result of fuzzification of each Candlestick by Gaussian OFC is presented in Fig. 15 by a triangle symbols. The triangles correspond to the value of the function f and g for values 0, 0.5 and 1. Moreover, if an OFC is long then the triangles are pointing straight up, otherwise down.

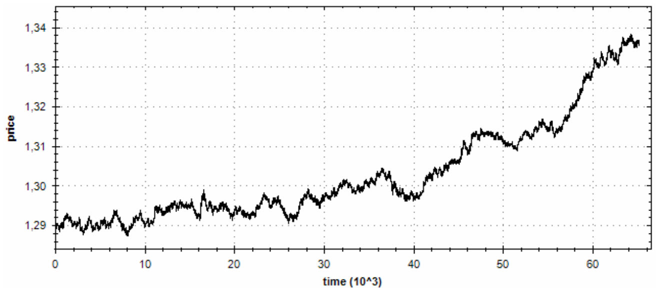

Fig. 13 Tick chart of the data set

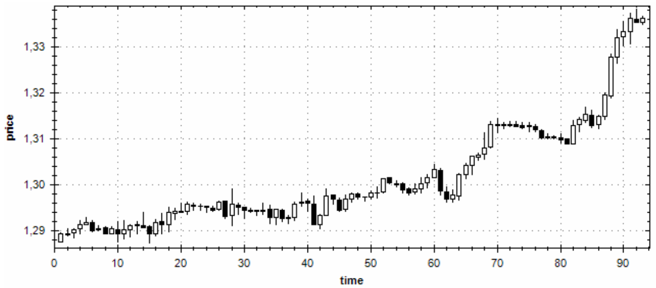

Fig. 14 Tick chart and Japanese Candlestick chart of the data set

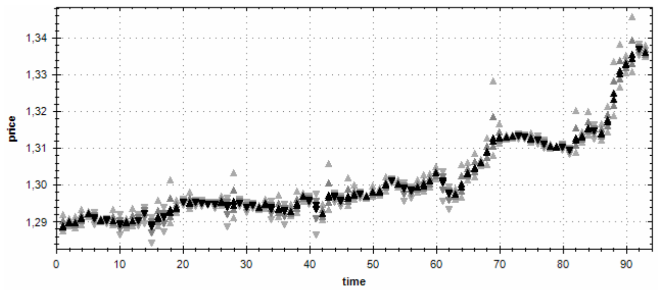

Fig. 15 The ordered fuzzy candlestick chart of the data

Fig. 16 shows results of realization of classical (line with xcross symbol) and ordered fuzzy (triangle symbols) simple moving average with order equal to 7 for the data set. Fig. 16 also shows the ordered fuzzy simple moving average defuzzification by center of gravity operator (line with circle symbol). In technical analysis the moving average indicator usually is used to define the current trend. Notice that the ordered fuzzy moving average determines the current trend by orientation of ordered fuzzy candlesticks, if orientation is positive then trend is long else trend is short.

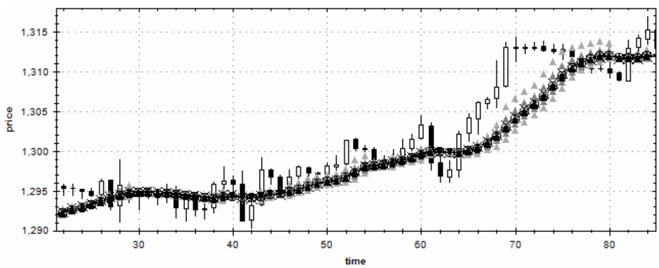

Fig. 16 The Japanese Candlestick chart of the data set, realization of a classical and ordered fuzzy simple moving average

5.2 *Ordered Fuzzy Autoregressive Model*

In a similar way as it is shown in the previous section we construct fuzzy financial time series models based on ordered fuzzy numbers and candlesticks. In this section, the autoregressive process is presented as an example.

An classical autoregressive model $(AR(p))$ is one where the current value of a variable, depends upon only the values that the variable took in previous periods plus an error term [15]. The presented approach, an ordered fuzzy autoregressive model of order p, denoted as $OFAR(p)$, in natural way is fully fuzzy $AR(p)$ and can be expressed as

$$\bar{X}_t = \bar{\alpha}_0 + \sum_{i=1}^{p} \bar{\alpha}_i \bar{X}_{t-i} + \bar{\varepsilon}_t \tag{30}$$

where $\bar{X}_{t-i}$ are the ordered fuzzy candlesticks at a time period t, $\bar{\alpha}_i$ are fuzzy coefficients given by arbitrary ordered fuzzy numbers and $\bar{\varepsilon}_t$ is an error term.

Fuzzy Coefficients and Their Estimation

The assumption that fuzzy coefficients of *OFAR* are arbitrary ordered fuzzy numbers requires the ability to approximate all possible shapes of functions f and g, and the ability to perform arithmetic operations on them.

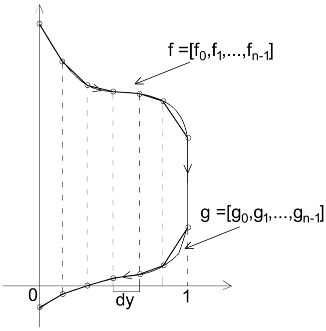

Fig. 17 Discretization of ordered fuzzy numbers

Furthermore, if we multiply the functions of the same class (e.g. linear) we get a function of another class, so the output of *OFAR* must be represented by arbitrary ordered fuzzy numbers as well. The simplest of solutions, discretization and approximation using the linear function is proposed by us and presented in Fig. 17. Then the arithmetic operations are performed on individual points.

The Least Squares Method is proposed for estimation fuzzy parameters $\bar{\alpha}_i$ in $OFAR(p)$ model. Rearranging the terms in (30) we obtain

$$\bar{\varepsilon}_t = \bar{X}_t - \left(\bar{\alpha}_0 + \sum_{i=1}^{p} \bar{\alpha}_i \bar{X}_{t-i} \right) \tag{31}$$

From a least-square perspective, the problem of estimation then becomes

$$\min \sum_t \bar{\varepsilon}_t^2 = \min \sum_t \left(\bar{X}_t - \bar{\alpha}_0 - \sum_{i=1}^{p} \bar{\alpha}_i \bar{X}_{t-i} \right)^2 \tag{32}$$

However, the error term $\bar{\varepsilon}_t$ is the ordered fuzzy number so we do not know what equation (32) mean. Therefore, the least-square method is defined using a distance measure. The measure of the distance between two ordered fuzzy numbers is expressed by formula

$$d(A,B) = d\left((f_A, g_A), (f_B, g_B)\right) = \|f_A - f_B\|_{L^2} + \|g_A - g_B\|_{L^2} \tag{33}$$

where $\|\cdot\|$ is a metric induced by the L^2-norm. Hence, the least-square method for $OFAR(p)$ is to minimize the following objective function

$$E = \sum_{t} d\left(\bar{X}_t, \bar{\alpha}_0 + \sum_{i=1}^{p} \bar{\alpha}_i \bar{X}_{t-i}\right) \tag{34}$$

So-defined function does not guarantee that received coefficients will be ordered fuzzy numbers, so we have to control coefficients in the course of estimation.

Empirical Results

For the case study the empirical data was the same as in Section 5.1. but was divided into two sets, the first 80 candlesticks are used for estimation, while the next 13 candlesticks are used to evaluate the quality of prediction. The empirical results of several types of ordered fuzzy autoregressive processes are presented.

Model 1

First, in Fig. 18 we can see the realization of a classical autoregressive process with order 4, where the variables X_t are selected prices. On the left side we can see $AR(4)$ of close prices, while on the right side we can see $AR(4)$ of average of OHLC prices. Estimation of $AR(4)$ processes was performed in statistical applications Eviews. We can notice, that the ordered fuzzy autoregressive process is natural generalization of the classical autoregressive process in the space of ordered fuzzy numbers. Assume that all coefficients and input values are numbers (i.e. ordered fuzzy numbers, where functions f and g are equal and constant), then the processes $OFAR$ and AR are equivalent (i.e. give the same results). For the set data, it is presented in Fig. 19.

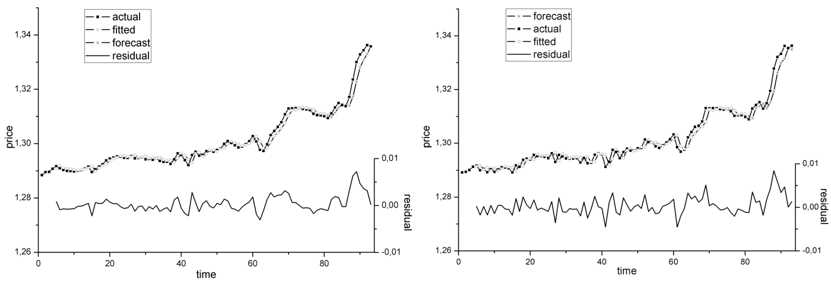

Fig. 18 Realization and static forecast of classical $AR(4)$ processes of close prices and average of OHLC prices, respectively

Model 2

Now, assume that the coefficients still are numbers, while input values are ordered fuzzy candlesticks. Then $OFAR$ can be identified with the vector autoregressive model (VAR) and we can use Eviews for estimation coefficients. The realization $OFAR(4)$ are presented in Fig. 20. In Fig. 20 are shown also defuzzification values of $OFAR(4)$ received by the center of gravity operator (black line).

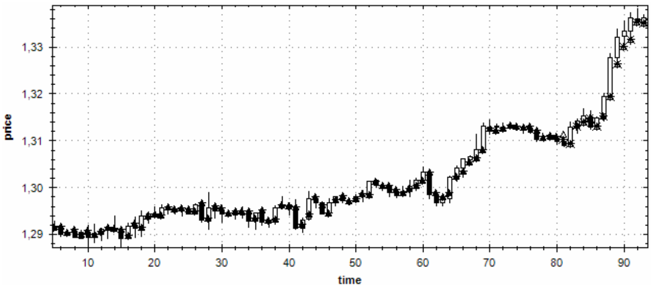

Fig. 19 Realization and static forecast of $OFAR(4)$ (triangle symbols) and classical $AR(4)$ (plus and star symbols)

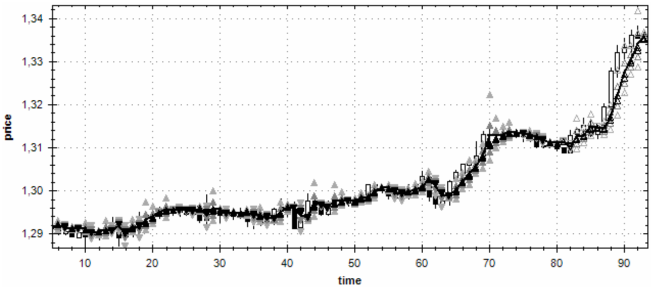

Fig. 20 Realization and static forecast of $OFAR(4)$ with assumption from Model 2 (triangle symbols) with Gaussian OFC and defuzzification values of $OFAR(4)$ (black line)

Model 3
Finally, assume that the coefficients are ordered fuzzy numbers and input values are ordered fuzzy candlesticks. In this case the realization of $OFAR(4)$ are presented in Fig. 21. In Fig. 21 are shown also defuzzification values of $OFAR(4)$ (black line).

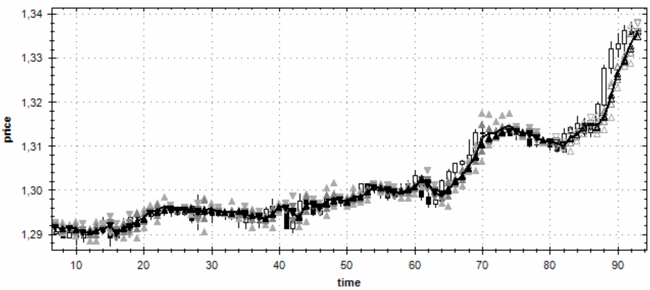

Fig. 21 Realization and static forecast of $OFAR(4)$ with assumption from Model 3 (triangle symbols) with Gaussian OFC and defuzzification values of $OFAR(4)$ (black line)

5.3 Ordered Fuzzy Single-Period Simple Return

In financial studies more often used are returns, instead of prices. Return series are easier to handle than price series because they have more attractive statistical properties and for average investors they form a complete and scale-free summary of the investment opportunity [15]. Using the concept of ordered fuzzy candlestick the fuzzy return series can be defined in a natural way.

Let $\bar{X}_t$ be a ordered fuzzy time series (time series of OFC) given by time series of prices. Then ordered fuzzy time series of one period return is defined by following formula

$$\bar{R}_t = \frac{\bar{X}_t - \bar{X}_{t-1}}{\bar{X}_{t-1}} \tag{35}$$

Empirical Results

For the case study we take the time series of ordered fuzzy Gaussian Candlestick obtained in section 5.1 (see Fig. 15). The time series of ordered fuzzy simple return is presented in Fig. 22.

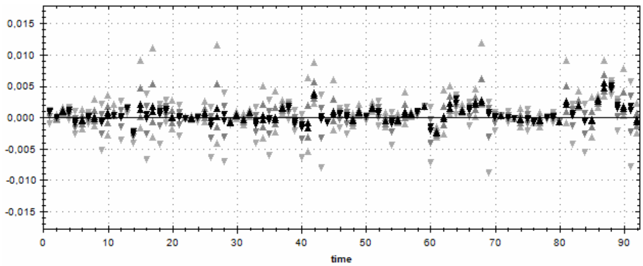

Fig. 22 Ordered fuzzy one period return series

6 Conclusions

The novel approach to financial time series modeling based on ordered fuzzy numbers is presented in this chapter. We described the representation of financial data using concept of the ordered fuzzy candlestick. The ordered fuzzy candlestick keeps more information about the prices than the classical Japanese Candlestick. Moreover, the proposed approach enables to build the fuzzy financial time series models in the simple form of classical equations. It allows to reduce the size of models compared to models based on fuzzy rule-based systems. It is too early to evaluate the usefulness of ordered fuzzy candlesticks in financial engineering, however one can expect that this approach to fuzzy modeling based on ordered fuzzy numbers will bring a new quality. Furthermore, the time series of ordered fuzzy return presented in section 5.3 can be used in the most interesting area of financial modeling, i.e. modeling of volatility. Results of further experiments to validate this approach will be reported on in the future.

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