Limit Cycle Structure for Block-Sequential Threshold Systems

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Abstract. This paper analyzes the possible limit set structures for the standard threshold block-sequential finite dynamical systems. As a special case of their work on Neural Networks (generalized threshold functions), Goles and Olivos (1981 [2]) showed that for the single block case (parallel update) one may only have fixed points and 2-cycles as ω-limit sets. Barrett et al (2006 [1]), but also Goles et al (1990 [3]) as a special case, proved that for the case with n blocks (sequential update) the only ω-limit sets are fixed points. This paper generalizes and unifies these results to standard threshold systems with block-sequential update schemes.

Keywords: graph dynamical systems, finite dynamical system, automata networks, neural networks, block-sequential, cellular automata, threshold function, periodic orbit, limit cycle, sequential, parallel (37B99,68Q80).

1 Introduction

This paper analyzes the structure of limit sets for finite dynamical systems (also called automata networks), see for example [2, 3, 5–9], where each vertex function is a standard threshold function over the domain $\{0, 1\}$. In [2] it is demonstrated, as a special case of a more general result on neural networks (generalized threshold functions), that for the parallel update scheme standard threshold dynamical systems may only exhibit fixed points and 2-cycles as limit sets. It is shown in for example [1] that under the sequential update scheme the only limit sets are fixed points. Here we extend these results to block sequential update schemes.

Following the notation of Serre, let X denote a simple graph with vertex set $\mathrm{v}[X] = \{1, 2, 3, \ldots, n\}$, and write S_X for the set of permutations over $\mathrm{v}[X]$. We refer to the elements of a partition $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ of $\mathrm{v}[X]$ as *blocks* and write $S_{\mathcal{B}}$ for the set of permutations of $\mathcal{B}$. We say that a block $B \in \mathcal{B}$ is *non-trivial* if it induces a connected subgraph. Clearly, any block can be decomposed into non-trivial blocks. The main result can now be stated as follows:

Theorem 1. *Let X be a simple graph and $\mathcal{B}$ a block partition of $\mathrm{v}[X]$. If the largest non-trivial block of $\mathcal{B}$ has size at most three, then any block-sequential threshold finite dynamical system with update sequence $\pi \in S_{\mathcal{B}}$ only has fixed points as limit sets. If the largest nontrivial block has size at least four, then a block-sequential threshold finite dynamical system may have periodic orbits of length at least two.*

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We provide an explicit example of a 2-cycle in a threshold finite dynamical system (FDS) where the maximal non-trivial block size is four and where there are multiple blocks. In the remainder of this paper we first introduce the necessary terminology. The proof, which is based on a potential function argument, is then presented in Section 3 before we finish by discussing generalizations in Section 4.

2 Terminology

Let X be a simple graph as above, and assign a state $x_v \in K = \{0, 1\}$ to each vertex $v \in \mathrm{v}[X]$. Here we refer to x_v as a *vertex state* and $x = (x_1, x_2, \dots, x_n)$ as a *system state*. Whenever it is clear from the context we will simply say state for either case. Let $n[v]$ denote the sorted sequence of vertices from the 1-neighborhood of v in X, and let $x[v]$ denote the corresponding restriction of x to $n[v]$. Denoting the degree of v by $d(v)$, each vertex is assigned a *vertex function* $f_v \colon K^{d(v)+1} \longrightarrow K$. The function f_v is used to map the vertex state at time t to $t+1$, that is, $x_v(t)$ to $x_v(t+1)$, taking $x[v]$ (at time t) as input.

Using the *parallel update* we obtain the finite dynamical system map $F \colon K^n \longrightarrow K^n$ given by

$$F(x_1, \dots, x_n) = (f_1(x[1]), \dots, f_n(x[n])) \ .$$

For a sequential application of the maps f_v, it is convenient to introduce the X-local maps $F_v \colon K^n \longrightarrow K^n$ given by

$$F_v(x_1, \dots, x_n) = (x_1, \dots, x_{v-1}, f_v(x[v]), x_{v+1}, \dots, x_n) \ .$$

For a sequential update given by the permutation (or order) $\pi = (\pi(1), \pi(2), \dots, \pi(n)) \in S_X$ we obtain the finite dynamical system map $F_\pi \colon K^n \longrightarrow K^n$ given by the composition

$$F_\pi = F_{\pi(n)} \circ F_{\pi(n-1)} \circ \cdots \circ F_{\pi(1)} \ .$$

A block-sequential update scheme generalizes both maps above. Let $\mathcal{B} = \{B_1, \dots, B_m\}$ be a block partition as above. The map $F_{B_k} \colon K^n \longrightarrow K^n$ is given by

$$\big(F_{B_k}(x)\big)_v = \begin{cases} f_v(x[v]), & \text{if } v \in B_k \text{ and,} \\ x_v, & \text{otherwise.} \end{cases}$$

The block-sequential map $F_{\mathcal{B}} \colon K^n \longrightarrow K^n$ is defined by

$$F_{\mathcal{B}} = F_{B_m} \circ F_{B_{m-1}} \circ \cdots \circ F_{B_1} \ .$$

Regardless of the choice of update scheme, we write $\mathrm{Per}(F)$ and $\mathrm{Fix}(F)$ for the set of periodic points and fixed points of $F \colon K^n \longrightarrow K^n$, respectively. Of course, $\mathrm{Fix}(F) \subset \mathrm{Per}(F)$.

Define $\sigma_m \colon K^m \longrightarrow \mathbb{N}$ by $\sigma_m(x_1, \dots, x_m) = |\{i \mid x_i = 1\}|$. The focus of this paper is on *standard threshold vertex functions*. The standard threshold function $\boldsymbol{t}_{k,m} \colon K^m \longrightarrow K$ is defined by

$$\boldsymbol{t}_{k,m}(x_1, \dots, x_m) = \begin{cases} 1, & \sigma_m(x_1, \dots, x_m) \geq k \ , \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

A finite dynamical system map is a threshold system if each of its vertex functions is a threshold function. The threshold need not be the same for all vertices.

We remark that the generalized threshold function $f\colon \{0,1\}^n \longrightarrow \{0,1\}$ of neural networks (see [2]) is defined by

$$f(x_1,\ldots,x_n) = \begin{cases} 1, & \text{if } \sum_{j=1}^{n} a_{ij}x_j < \theta_i \\ 0, & \text{otherwise} \end{cases},$$

where $\theta = (\theta_1, \ldots \theta_n) \in R^n$ and $A = (a_{ij})_{i,j=1}^n$ is a real symmetric matrix. The case considered in this paper additionally follows by restricting the a_{ij}'s to be either 0 or 1.

Example: The following example illustrates the concepts. As graph, take $X = \text{Circle}_4$ as shown in Figure 1. In this case we have $n[4] = (1,3,4)$, $x[4] = (x_1, x_3, x_4)$ and $F_4(x) = (x_1, x_2, x_3, f_4(x_1, x_3, x_4))$. Taking $x = (1,0,1,0)$ and threshold-2 vertex functions, we see that with the parallel update scheme $F(x) = (0,1,0,1)$, whereas with sequential update and sequence $\pi = (1,2,3,4)$ we have $F_\pi(x) = (0,0,0,0)$. Using the block partition $\mathcal{B} = \{B_1 = \{2,4\}, B_2 = \{1,3\}\}$ and update sequence $\pi' = (B_1, B_2)$ we get $F_{\pi'}(x) = (1,1,1,1)$. For the map F the state x is on a 2-cycle, but is not periodic under either F_π or $F_{\pi'}$.

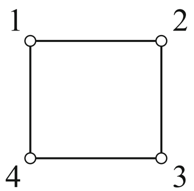

Fig. 1. The graph Circle_4

Note that for the synchronous update in this example we have a case where block-size 4 yields a 2-cycle. Later, we give another example of this where the update is neither parallel or sequential, but is instead block-sequential with three blocks.

3 Main Result

In this section we present the proof of the main result. The technique is an extension of the threshold function argument used in [1] and that was developed further in [4].

Proof (Theorem 1). For $v \in \mathrm{v}[X]$ let $T_1(v)$ denote the threshold value for vertex v. Also, let $T_0(v)$ denote the smallest number of states in $x[v]$ that must be zero to ensure that x_v is mapped to zero. Clearly, we have the relation $\big(d(v)+1\big) - T_0(v) = T_1(v) - 1$, or $T_0(v) + T_1(v) = d(v) + 2$. We next introduce the vertex potential function

$$P(x,v) = \begin{cases} T_1(v), & x_v = 1 \\ T_0(v), & x_v = 0 \end{cases}$$

and the edge potential function

$$P(x, e = \{v, v'\}) = \begin{cases} 1, & x_v \neq x_{v'} \\ 0, & \text{otherwise.} \end{cases}$$

We combine these and define the potential function $P\colon K^n \longrightarrow \mathbb{N}$ by

$$P(x) = \sum_{v \in \mathrm{v}[X]} P(x,v) + \sum_{e \in \mathrm{e}[X]} P(x,e)\ . \tag{2}$$

Clearly, there exist an integer $M \geq 0$ such that $0 \leq P(x) \leq M$ for all $x \in K^n$. We write $n_i(x,v)$ for the number of neighbors of v with state $x_v = i$ with $i = 0, 1$. We then have $n_0(x,v) + n_1(x,v) = d(v)$. In the following we set $x' = F_v(x)$.

In [1] it is shown that whenever $x' \neq x$ we have $P(x') < P(x)$, which clearly implies that sequential FDS maps only have fixed points as limit sets. This covers the case where the maximal non-trivial block size is 1. In the following we prove that the same holds when the non-trivial blocks sizes are less than 4.

For a vertex state transition from x to x' where x_v is mapped from 0 to 1 by F_v we must have that $n_1(x,v) \geq T_1(v)$ or $T_1(v) - n_1(v) \leq 0$. Similarly, for the transition where x_v is mapped from 1 to 0 we must have that $n_1(x,v) + 1 \leq T_1(v) - 1$ so that $n_1(x,v) - T_1(v) \leq -2$.

In the argument to follow, we will first consider block-size 2 before handling block-size 3. For a block B of size $|B| = 2$ we may limit our consideration to the case where all elements $v \in B$ change their state in the transition $x \mapsto x'$. If one or more of the states do not change, we are effectively working with a smaller block-size.

When determining the difference in potential ΔP when a block B is updated by F_B we may also limit our attention to the vertices in B and their incident edges since all other terms in the potential function P are the same before and after. However, if we simply add $P_v(x) = P(x,v) + \sum_{e=\{v,v'\}} P(x,e)$ for the elements $v \in B$ we may over-count the potential of all common edges in the block. However, by the previous remark that all states in the block must change, this over-counting in edge-potential is precisely the same for $P(x)$ and $P'(x)$. Consequently, we may disregard this without any consequence and simply add up $P_v(x)$ for each vertex v in the block.

To determine the potential change $\Delta P_v = P_v(x') - P_v(x)$ at vertex v, assume that $v \in B$ is adjacent to $\beta - 1$ other vertices in B, and assume that of these, α are in state 1 in x. It follows that the remaining $\beta - \alpha - 1$ other vertices in B adjacent to v have state 0. Since, all states are inverted, we conclude that in x' we have α adjacent vertices in state 0 and $\beta - \alpha - 1$ in state 1.

We first consider the transition where x_v is mapped from 0 to 1 in which case $n_1(x, v) \geq T_1(v)$:

$$\begin{aligned}
\Delta P_v &= T_1(v) + n_0(x', v) - [T_0(v) + n_1(x, v)] \\
&= T_1(v) + [n_0(x, v) + \alpha - (\beta - \alpha - 1)] - [T_0 + n_1(x, v)] \\
&= T_1(v) + d(v) - n_1(x, v) + 2\alpha - \beta + 1 - [d(v) + 2 - T_1(v) + n_1(x, v)] \\
&= 2(T_1(v) - n_1(x, v)) + 2\alpha - \beta - 1 \\
&\leq 2\alpha - \beta - 1 \ .
\end{aligned}$$

Similarly, if x_v is mapped from 1 to 0, and therefore $n_1(x, v) - T_1(v) \leq -2$, we have

$$\begin{aligned}
\Delta P_v &= T_0(v) + n_1(x', v) - [T_1(v) + n_0(x, v)] \\
&= T_0(v) + [n_1(x, v) + (\beta - \alpha - 1) - \alpha] - [T_1 + n_0(x, v)] \\
&= d(v) + 2 - T_1(v) + n_1(x, v) + \beta - 2\alpha - 1) - [T_1(v) + d(v) - n_1(x, v)] \\
&= 2(n_1(x, v) - T_1(v)) + \beta - 2\alpha + 1 \\
&\leq 2(-2) + \beta - 2\alpha + 1 \\
&= \beta - 2\alpha - 3 \ .
\end{aligned}$$

Block-size 2**.** For any block $B = \{v, v'\}$ and state x for which $x' = F_B(x) \neq x$ we have $P(x') < P(x)$.

If $\{v, v'\}$ is not an edge, we are effectively in the block-size 1 case and the statement is known to hold. Assume therefore that v and v' are connected. By symmetry, there are three cases to consider: (a) $(0, 0) \mapsto (1, 1)$, (b) $(1, 1) \mapsto (0, 0)$, and (c) $(1, 0) \mapsto (0, 1)$. In all three cases we have $\beta(v) = \beta(v') = 2$. For case (a) we have $\alpha(v) = \alpha(v') = 0$, so $\Delta P \leq 2(2 \cdot 0 - 2 - 1) = -6$. Similarly, for case (b) we have $\alpha(v) = \alpha(v') = 1$ so that $\Delta P \leq 2(2 - 2 \cdot 1 - 3) = -6$. Finally, for case (c) we have $\alpha(v) = 0$ and $\alpha(v') = 1$, so $\Delta P \leq -1 + (-1) = -2$, so in all cases we have $\Delta P < 0$.

Block-size 3**.** For any block $B = \{u, v, w\}$ and state x for which $x' = F_B(x) \neq x$ we have $P(x') < P(x)$.

Again we may assume that B is non-trivial and that x_u, x_v and x_w are all mapped non-identically since all other possibilities reduce to the block-size 1 or block-size 2 cases. There are two possibilities for the subgraph induced by B: (i) the 3-line with with edges $\{u, v\}$ and $\{v, w\}$ and (ii) the 3-cycle.

Case (i): the induced subgraph of $B = \{u, v, w\}$ is a 3-line. There are eight transitions to consider, but by symmetry, it follows that $(1, 1, 0) \mapsto (0, 0, 1)$ has the same potential change as $(0, 1, 1) \mapsto (1, 0, 0)$ and similarly for $(1, 0, 0) \mapsto (0, 1, 1)$ and $(0, 0, 1) \mapsto (1, 1, 0)$. We write α and β as vectors, so that in this case $\beta = (2, 3, 2)$. This gives us the following cases listed in Tab. 1.

Case (ii): the induced subgraph of $B = \{u, v, w\}$ is a 3-circle. In this case, symmetry implies that there are four cases to consider: $(0, 0, 0) \mapsto (1, 1, 1)$, $(1, 0, 0) \mapsto (0, 1, 1)$, $(1, 1, 0) \mapsto (0, 0, 1)$ and $(1, 1, 1) \mapsto (0, 0, 0)$. Here $\beta = (3, 3, 3)$ with cases summarized in Table 2.

Table 1. Potential changes for case (i) where block size is 3

Transition	α	Potential change
$(1,1,1) \mapsto (0,0,0)$	$(1,2,1)$	$\Delta P \leq -3-4-3=-10$
$(0,0,1) \mapsto (1,1,0)$	$(0,1,0)$	$\Delta P \leq -3-2-1=-6$
$(1,1,0) \mapsto (0,0,1)$	$(1,1,1)$	$\Delta P \leq -3-2-1=-6$
$(0,1,0) \mapsto (1,0,1)$	$(1,0,1)$	$\Delta P \leq -1-0-1=-2$
$(1,0,1) \mapsto (0,1,0)$	$(0,2,0)$	$\Delta P \leq -1-0-1=-2$
$(0,0,0) \mapsto (1,1,1)$	$(0,0,0)$	$\Delta P \leq -3-4-3=-10$

Table 2. Potential changes for case (ii) where block size is 3

Transition	α	Potential change
$(1,1,1) \mapsto (0,0,0)$	$(2,2,2)$	$\Delta P \leq -4-4-4=-12$
$(0,1,1) \mapsto (1,0,0)$	$(2,1,1)$	$\Delta P \leq \quad 0-2-2=-4$
$(0,0,1) \mapsto (1,1,0)$	$(1,1,0)$	$\Delta P \leq -2-2-0=-4$
$(0,0,0) \mapsto (1,1,1)$	$(0,0,0)$	$\Delta P \leq -4-4-4=-12$

It follows that whenever the maximal non-trivial block size in $\mathcal{B}$ is at most three, the block sequential threshold map may only have fixed points as limit cycles as claimed.

Block-size 4**.** For this case it is possible to construct systems with 2-cycles. Specifically, let X be the graph displayed in Figure 2. Let the blocks be $B_1 = \{0\}$, $B_2 = \{1,2,3,4\}$ and $B_3 = \{5\}$. The state x obtained by assigning 0 to the even vertices and 1 to the odd vertices is clearly periodic with period 2 for any permutation update sequence of B_1, B_2 and B_3.

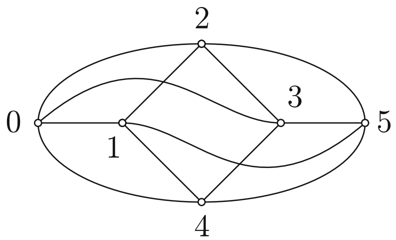

Fig. 2. A graph where threshold finite dynamical system maps with block size 4 can have periodic orbits of size ≥ 2

4 Summary and Open Questions

We note that extending the results above to the generalized threshold functions is non-trivial and will require additional constraints on the matrix A, see [3].

Here we did not address the question of what is the maximal periodic orbit size when the maximal non-trivial block size b falls in the range $4 \leq b \leq n-1$. It seems plausible that it is bounded by 2, but we have no proof for this at the moment. We close with this as a conjecture and challenge the reader to settle it.

Conjecture 1. The periodic orbits of any block sequential threshold system have length at most 2.

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