Limit Cycle for Composited Cellar Automata

Toshikazu Ishida[1] and Shuichi Inokuchi[2]

[1] Center for Fundamental Education, Kyushu Sangyo University, Fukuoka, 813-8503, Japan
tishida@ip.kyusan-u.ac.jp

[2] Faculty of Mathematics, Kyushu University, Fukuoka, 819-0395, Japan
inokuchi@math.kyushu-u.ac.jp

Abstract. We know that a few uniform cellular automata have maximum cycle lengths. However, there are many uniform cellular automata, and checking the cycles of all uniform cellular automata is impractical. In this paper, we define a cellular automaton by composition and show how its cycles are related.

1 Introduction

The study of cellular automata was initiated by von Neumann in the 1940s[6]. Cellular automata have cells on a lattice, and the states of their cells (configuration) are determined by a transition function that references the states of neighboring cells in the previous time step. When all cells evolve according to the same local transition function, the cellular automaton is called a uniform cellular automaton, otherwise, it is called a hybrid cellular automaton. Cellular automata have been developed by many researchers as computational models for simulating physical systems. For example, cellular automata with maximum cycle length have been used to make pseudo-random pattern generators [1]. By defining the transition function and cell size of uniform cellular automata well enough, cellular automata can be made to have long cycle lengths[11]. Moreover, the necessary and sufficient conditions for having a maximum cycle length have been shown for a hybrid cellular automata[2], and methods for finding it have been devised by Tezuka and Fushimi[8]. For uniform cellular automata, Matsumoto showed that five uniform cellular automata have a maximum cycle length[5]. However, there are many uniform cellular automata, and comparatively little is known about their maximum cycle lengths because checking the lengths of all of them would take so long.

Cellular automata have been defined on groups[3][7][12]. For instance, we introduced cellular automata on groups in [4]. A configuration is defined as being a function from the group into the set of states. Thus, a configuration is a way of attaching a state to each element of the group. There is a natural action of the group on the set of configurations, which is called the shift action. A cellular automaton is thus a self-mapping of the set of configurations defined from a system of local transition functions commuting with the shift.

G.C. Sirakoulis and S. Bandini (Eds.): ACRI 2012, LNCS 7495, pp. 32–41, 2012.

In this paper, we define a cellular automaton that is made from a composition of two cellular automata, and show how it is related to its composing automata. In fact, for two cellular automata CA_1, CA_2 with global transition functions F_1, F_2, we can define a composited cellular automaton CA that has a global transition function $F = F_1 \circ F_2$. Moreover, we show how CA_1, CA_2 and CA are associated.

2 Cellular Automata and Composited Cellular Automata

In this section, we shall review the definitions of cellular automata and compositions[4].

Definition 1. *Let G be a group. A cellular automaton on G is a triple $CA = (G, V, V')$, in which $V \subset G$ and $V' \subset 2^V$ are finite subsets of G. For V', we define a function $f : 2^V \to \{\phi, \{e\}\}$ by*

$$f(A) = \begin{cases} \phi & (A \notin V') \\ \{e\} & (A \in V') \end{cases}$$

and for all $X \in 2^G$ and a function $F : 2^G \to 2^G$ by $F(X) = \bigcup_{g \in G} gf(g^{-1}X \cap V)$. We call f a local transition function and F a global transition function.

The set 2^G is called the set of *configurations.*

Now let us define the operation $+$.

Definition 2. *For $X, Y \in 2^G$ and $A \in 2^V$, we define*

- $\phi + \phi = \phi$, $\phi + \{e\} = \{e\} + \phi = \{e\}$, $\{e\} + \{e\} = \phi$
- $X + Y = \bigcup_{g \in G} g((g^{-1}X \cap \{e\}) + (g^{-1}Y \cap \{e\}))$.

The following lemma holds for the operation $+$.

Lemma 1. *Let X, Y, Z be elements of 2^G.*

1. $X + X = \phi$,
2. $X + Y = Y + X$,
3. $(X + Y) \cap Z = Z \cap (X + Y) = (X \cap Z) \cup (Y \cap Z)$,
4. $\forall g \in G, g(X + Y) = gX + gY$.

Now let us define the composition of cellular automata.

Definition 3. *For cellular automata $CA_1 = (G, V_1, V_1')$and $CA_2 = (G, V_2, V_2')$ on G, the cellular automaton $CA_1 \diamondsuit CA_2 = (G, V_1 \cdot V_2, V_1' \diamondsuit V_2')$ is defined by*

- $V_1 \cdot V_2 = \{v_1 v_2 \in G | v_1 \in V_1, v_2 \in V_2\}$
- $V_1' \diamondsuit V_2' = \{X \in 2^{V_1 \cdot V_2} | \{v \in V_1 | v^{-1}X \cap V_2 \in V_2'\} \in V_1'\}$

For $CA_1 \diamondsuit CA_2$, the following theorem hold [4].

Theorem 1. *For global transition functions* $F_{CA_1}, F_{CA_2}, F_{CA_1 \diamondsuit CA_2}$,

$$F_{CA_1} \circ F_{CA_2} = F_{CA_1 \diamondsuit CA_2}.$$

In the following, $CA_1 \diamondsuit CA_2$ is called a composited cellular automaton.

Definition 4. *Let* C *be a subset of* 2^G *and* F *be a global transition function of a cellular automaton on* G*. We define* $F^\infty(C)$ *by*

$$F^\infty(C) := \{c \in C | \exists n > 0 \ \ c = F^n(c)\}.$$

We call $c \in F^\infty(C)$ *an element of the limit cycle (LC) of* F*.*

Definition 5. *The local transition function* f *of a cellular automaton* $CA = (G, V, V')$ *is linear, if* $f(A + B) = f(A) + f(B)$ *for all* $A, B \in 2^V$*. So is* CA*. For the local transition function* f *of* CA *and* $A \in 2^V$*, if there exists a linear local transition function* q *satisfying* $f(A) = q(A) + \{e\}$*, then* f *is affine. So is* CA*.*

Lemma 2. *For all* $X, Y \in 2^G$*, if a cellular automaton* $CA = (G, V, V')$ *is linear, then* $F(X + Y) = F(X) + F(Y)$*.*

Theorem 2. *Let* f_1, f_2 *be local transition functions of* $CA_1 = (G, V_1, V_1')$ *and* $CA_2 = (G, V_2, V_2')$*. If* f_1, f_2 *are linear, then the local transition function* $f_1 \diamond f_2$ *of* $CA_1 \diamond CA_2$ *is linear.*

Definition 6. *If no cellular automaton* $CA_2 = (G, V_2, V_2')$ *satisfying* $F_{CA_1} = F_{CA_2}$ *and* $V_2 \subsetneqq V_1$ *exists, then* $CA_1 = (G, V_1, V_1')$ *is called the minimum cellular automaton.*

Definition 7. *For a cellular automaton* $CA = (G, V, V')$*, we define the cellular automaton* $CA_m = (G, V_m, V_m')$ *by*

- $V_m = \{v \in V | \{v\} \in V'\}$,
- $V_m' = \{A \subset V_m | A \in V'\}$.

Lemma 3. *For all* $X \in 2^G$*, if a cellular automaton* $CA = (G, V, V')$ *is linear, then*

$$X \cap V_m \in V_m' \Longleftrightarrow X \cap V \in V'.$$

Corollary 1. *If a cellular automaton* $CA = (G, V, V')$ *is linear, then* $F_{CA} = F_{CA_m}$*. Hence,* $CA \cong CA_m$*.*

Lemma 4. *If a cellular automaton* $CA = (G, V, V')$ *is linear, then the cellular automaton* CA_m *is a minimum.*

Definition 8. *For a linear cellular automaton* $CA = (G, V, V')$*, we can form* $CA_m = (G, V_m, {V'}_m)$*. If* $\sharp V_m$ *is even, then* CA *is called even linear. If* $\sharp V_m$ *is odd, then* CA *is called odd linear. We assume the local transition function* f *of an affine cellular automaton* $CA = (G, V, V')$ *satisfies* $f(A) = q(A) + \{e\}$ $(\forall A \in 2\ V)$ *for the local transition function* q *of a linear cellular automaton* CA'*. If* CA' *is even linear, then* CA *is called even affine. Moreover, if* CA' *is odd linear, then* CA *is called odd affine.*

3 Commutativity Condition of Compositions

In this section, we discuss commutativity of transition functions. B. Voorhees proved that the set of all local transition functions commuting with given local transition functions is obtained by solving nonlinear Diophantine equations [9]. We state propositions for composited cellular automata and the commutativity conditions for linear and affine cellular automata.

First, we shall consider linear cellular automata. Two simple linear cellular automata commute as follows.

Proposition 1. *For cellular automata* $CA_1 = (G, V_1, V_1')$ *and* $CA_2 = (G, V_2, V_2')$, *the following hold.*

- *If* $V_1' = \emptyset$ *and* $\emptyset \notin V_2'$, *then* $CA_1 \diamond CA_2 = CA_2 \diamond CA_1$.
- *If* $\sharp V_1 = 1$ *and* $V_1' = \{V_1\}$, *then* $CA_1 \diamond CA_2 = CA_2 \diamond CA_1$ *for all* CA_2.

Lemma 5. *Let* $CA = (G, V, V')$ *be a minimum linear cellular automaton. For* $A \subset V$,

$$A \in V' \Longleftrightarrow \sharp A \text{ is odd.}$$

Using this lemma, we can prove the following theorem.

Theorem 3. *Let* $CA_1 = (G, V_1, V_1')$ *and* $CA_2 = (G, V_2, V_2')$ *be minimum linear cellular automata. If* G *is commutative for the composition,* $CA_1 \diamond CA_2 = CA_2 \diamond CA_1$.

Now let us consider affine cellular automata. Affine cellular automata are not linear.

Lemma 6. *For all* $A \in 2^V$, *we define* $\overline{A} = \bigcup_{v \in V} v((v^{-1}A \cap \{e\}) + \{e\})$.

1. *If* CA *is even linear, then* $f(\overline{A}) = f(A)$.
2. *If* CA *is odd linear, then* $\overline{f(A)} = f(\overline{A})$.

Let F_1 and F_2 be global transition functions of even linear and odd linear cellular automata, respectively. We define $\overline{X} = \bigcup_{g \in G} g((g^{-1}X \cap \{e\}) + \{e\})$. From this lemma and the definition of the global transition function, $F_1(\overline{X}) = F_1(X)$ and $\overline{F_2(X)} = F_2(\overline{X})$.

In the following, we define a cellular automaton CA_{rev} by $CA_{rev} = (G, V, V'), V = \{e\}, V' = \{\phi\}$. This cellular automaton corresponds to $\overline{@}$.

Lemma 7. *Let* CA_{even} *be even linear, and let* CA_{odd} *be odd linear. Then the following hold.*

- $CA_{even} \diamond CA_{rev} \cong CA_{even}$,
- $CA_{odd} \diamond CA_{rev} \cong CA_{rev} \diamond CA_{odd}$.

Proof. For $X \in 2^G$,

1.

$$
\begin{aligned}
F\overline{(X)} &= \bigcup_{g\in G} gf\overline{(g^{-1}(X)\cap V)} \\
&= \bigcup_{g\in G} gf(g^{-1}(X)\cap V) \\
&= F(X).
\end{aligned}
$$

2.

$$
\begin{aligned}
\overline{F(X)} &= \bigcup_{g\in G} g\overline{f(g^{-1}(X)\cap V)} \\
&= \bigcup_{g\in G} gf\overline{(g^{-1}(X)\cap V)} \\
&= F\overline{(X)}.
\end{aligned}
$$

□

Lemma 8. *If $CA_{rev} \diamond CA_2 \cong CA_2 \diamond CA_{rev}$ and $CA_1 \diamond CA_2 \cong CA_2 \diamond CA_1$, then $CA_1 \diamond CA_{rev} \diamond CA_2 \cong CA_2 \diamond CA_1 \diamond CA_{rev}$.*

The above leads us to the following theorem.

Theorem 4. *Let CA_1, CA_2 be even affine cellular automata. Then $CA_1 \diamond CA_2 \cong CA_2 \diamond CA_1$.*

Proof. Let F_1, F_2 be global transition functions of CA_1 CA_2. Then there are global transition functions F_1', F_2' of even linear cellular automata such that $F_1 = \overline{F_1'}$ and $F_2 = \overline{F_2'}$. For all $X \in 2^G$,

$$F_1F_2(X) = \overline{F_1'\overline{\overline{F_2'(X)}}} = \overline{F_1'F_2'(X)} = \overline{F_2'F_1'(X)} = \overline{F_2'\overline{\overline{F_1'(X)}}} = F_2F_1(X).$$

□

Theorem 5. *Let cellular automata CA_1, CA_2 be odd affine and let a cellular automaton CA_3 be odd linear. Then $CA_1 \diamond CA_2 \cong CA_2 \diamond CA_1$ and $CA_1 \diamond CA_3 \cong CA_3 \diamond CA_1$.*

Corollary 2. *For $CA_1 = (G, V_1, V_1')ACA_2 = (G, V_2, V_2')$, if $V_1' = 2^{V_1}$ and $V_2 \in V_2'$, $CA_1 \diamond CA_2 = CA_2 \diamond CA_1$.*

4 Cycles of Composited Cellular Automata

In this section, we discuss the circumstances under which a limit cycle (LC) exists and the cycles for composited cellular automata.

In the following, we assume cellular automata compositions are commutative. Let $CA = (G, V, V')$ be a composited cellular automaton of $CA_1 = (G, V_1, V_1')$ and $CA_2 = (G, V_2, V_2')$. Thus CA satisfies $CA = CA_1 \diamond CA_2 = CA_2 \diamond CA_1$ and the global transition function F is defined as $F = F_1 \circ F_2 = F_2 \circ F_1$.

Lemma 9. *Following hold.*

- $F(C) \subseteq F_1(C)$,
- $F(C) \subseteq F_2(C)$.

$c \in C - F(C)$ is called a configuration of the Garden of Eden (GOE). This lemma show $C - F_1(C) \cup C - F(C) \subseteq C - F(C)$. Therefore if $c \in C$ is a configuration of GOE of F_1 or F_2, then c is a configuration of GOE of F.

Lemma 10. *For the set of configurations of LC, the following lemma holds.*

1. $F_1^{\infty}(C) \cap F_2^{\infty}(C) \subset F^{\infty}(C)$,
2. $F^{\infty}(C) \subset F_1^{\infty}(C) \cup F_2^{\infty}(C)$.

Proof.
(1) Let c be a configuration of $F_1^{\infty}(C) \cap F_2^{\infty}(C)$. Then there exist $n_1, n_2 > 0$ that satisfy $c = F_1^{n_1}(c) = F_2^{n_2}(c)$. Thus, $F^{n_1 \times n_2}(c) = (F_1 \circ F_2)^{n_1 \times n_2}(c) = F_1^{n_1 \times n_2}(F_2^{n_1 \times n_2}(c)) = c$ by $F_1 \circ F_2 = F_2 \circ F_1$. Therefore, $c \in F^{\infty}(C)$.
(2) Let c be a configuration of $F^{\infty}(C)$. Then, there exists $n > 0$ that satisfies $c = F^n(c)$. Hence there exists an integer m that satisfies $n \times m > \sharp C$. Therefore $c = F^{n \times m}(c) = F_1^{n \times m}(F_2^{n \times m}(c))(= F_2^{n \times m}(F_1^{n \times m}(c)))$ and $c \in F_1^{\infty}(C)(c \in F_2^{\infty}(C))$. (For $t > \sharp C$ and $\forall c \in C, F^n(c) \in F^{\infty}(C)$.) $\square$

This lemma means that if $c \in C$ is a configuration of LC of F_1 and LC of F_2, then c is a configuration of LC of F.

From the commutativity of compositions, the following lemma holds.

Lemma 11. *1. If $c \in F_1^{\infty}(C)$, then $F_2(c) \in F_1^{\infty}(C)$.*
2. If $c \in F^{\infty}(C)$, then $F_1(c) \in F^{\infty}(C)$.

Corollary 3.

$$(C - F_1^{\infty}(C)) \cap (C - F_2^{\infty}(C)) \subseteq (C - F^{\infty}(C)).$$

If the configuration c is not an element of the set of configurations LC of F_1 or F_2, then this corollary guarantees that c is not an element of LC of F.

Let us discuss the cycles of each transition function of cellular automata and composited cellular automaton.

Lemma 12. *For any $c \in C$, if there exists integers $n_1, n_2 > 0$ satisfying $c = F_1^{n_1}(c) = F_2^{n_2}(c)$, then $c = F^{LCM(n_1,n_2)}(c)$.*

In the following, we define $c \in F_1^{\infty}(C) \cap F_2^{\infty}(C)$, $C_1 = \{F_1^t(c) | t \geq 0\}$, $C_2 = \{F_2^t(c) | t \geq 0\}$, $\sharp C_1 = n_1$, $\sharp C_2 = n_2$, $\sharp(C_1 \cap C_2) = m$.

Lemma 13. *$m | n_1$ and $m | n_2$ hold.*

Proof. We will show the proof of $m|n_1$. Assume $m \not| n_1$. Let the integer t be $t = min\{t' > 0 | F_1^{t'}(c) \in C_2\}$. Then, $c \in C_2, F_1^t(c) \in C_2$ and $F_1^{t'}(c) \notin C_2$ for $1 < t' < t$. Thus, $F_1^s(c) \in C_2$ and there exist $s > 0, s' > 0$ that satisfy $F_1^{s+s'}(c) \in C_2 \wedge F_1^{s+t'}(c) \notin C_2 \ \ (1 \geq t' < s') \wedge t \neq s'$. Let $c = (F_2^k F_1^s)(c)$ for any integer k.

1. If $t > s'$,

$$\begin{aligned} C_2 &\not\ni F_1^{s'}(c) \\ &= F_1^{s'}(F_2^k F_1^s)(c) \\ &= F_2^k(F_1^{s+s'}(c)). \end{aligned}$$

   By $F_1^{s+s'}(c) \in C_2$, this runs counter to our assumption.
2. We can apply the same method as above to $t < s'$.

Therefore, $m|n_1$. □

Corollary 4. *We have $F_1^{\frac{n_1}{m}}(C_2) = C_2$ and $F_2^{\frac{n_2}{m}}(C_1) = C_1$.*

- *For $0 < t < \frac{n_1}{m}$, $F_1^t(C_2) \neq C_2$.*
- *For $0 < t < \frac{n_2}{m}$, $F_1^t(C_1) \neq C_1$.*

Theorem 6. *If $C_1 \cap C_2 = \{c\}$, then*

$$\min\{t|t > 0, c = F^t(c)\} = LCM(n_1, n_2).$$

Proof. The fact $c = F^{LCM(n_1,n_2)}(c)$ is well defined. $C_1 \cap C_2 = \{c\}$ implies that $C_1 \neq F_2^t(C_1)$ for $t > 0$ satisfying $n_1 \not| t$. Thus, $c \neq F^t(c)$ $(1 \leq t < LCM(n_1, n_2))$. Therefore, $\min\{t > 0|c = F^t(c)\} = LCM(n_1, n_2)$. □

Theorem 7. *Let m be $\sharp(C_1 \cap C_2) = m > 1$ and let t be $t = min\{t' > 0|F_1^{\frac{n_1}{m}}(c) = F_2^{\frac{n_2}{m}t'}(c)\}$.*

$$min\{i|F^i(c) = c\} = LCM(\frac{n_1 + n_2 t}{m}, n_2) \times \frac{n_1}{(n_1 + n_2 t)}.$$

Proof. By $F^{\frac{n_1}{m}}(c) = F_2^{\frac{n_1}{m}}(F_1^{\frac{n_1}{m}}(c)) = F_2^{\frac{n_1+n_2 t}{m}}(c)$,

$$\begin{aligned} &= F^{LCM(\frac{n_1+n_2 t}{m}, n_2) \times \frac{n_1}{(n_1+n_2 t)}}(c) \\ &= F^{LCM(\frac{n_1+n_2 t}{m}, n_2) \times \frac{m}{n_1+n_2 t} \times \frac{n_1}{m}}(c) \\ &= F_2^{LCM(\frac{n_1+n_2 t}{m}, n_2)}(c) \\ &= c. \end{aligned}$$

We assume there exists k satisfying $F^k(c) = c$ and $0 < k < LCM(\frac{n_1+n_2 t}{m}, n_2) \times \frac{n_1}{(n_1+n_2 t)}$. By corollary 4, k is a multiple of $\frac{n_1}{m}$. Let k be $k = \frac{n_1}{m}h$.

$$c = F^{\frac{n_1}{m}h} = F_2^{\frac{n_1+n_2 t}{m}h}(c).$$

Thus $\frac{n_1+n_2 t}{m}h$ must be a multiple of n_2. We have $LCM(\frac{n_1+n_2 t}{m}, n_2) < \frac{n_1+n_2 t}{m}h$. Thus $LCM(\frac{n_1+n_2 t}{m}, n_2) \times \frac{n_1}{(n_1+n_2 t)} < \frac{n_1}{m}h = k$. Then this is in conflict with the assumption. Therefore $min\{i|F^i(c) = c\} = LCM(\frac{n_1+n_2 t}{m}, n_2) \times \frac{n_1}{(n_1+n_2 t)}$. □

Corollary 5. *Let $\sharp C_1 = \sharp C_2 = n$ and $\sharp(C_1 \cap C_2) = m > 1$. Then let $t = min\{t' > 0|F_1^{\frac{n}{m}}(c) = F_2^{\frac{n}{m}t'}(c)\}$.*

$$min\{i > 0|F^i(c) = c\} = \frac{n}{m} \times \frac{LCM(m, t+1)}{t+1}.$$

5 Examples of Composited Cellular Automata

In this section, we present examples of compositions of one-dimensional two-state cellular automata that have periodic boundary conditions. We express the local transition functions by their Wolfram number [10]. We represent a configuration as a binary number and show it as a decimal number.

Let us being with an example in which the cycle length of composited cellular automaton is lowest common multiple of the cycle lengths of each cellular automaton. Figure 1, 2 and 3 correspond to CA90(5), CA240(5) and the composited cellular automaton. For the configuration $c = 9$, $C_1 = \{6, 9, 15\}, C_2 = \{5, 9, 10, 18, 20\}$ and $\min\{t > 0 | F^t(9) = 9\} = 15$.

Next, let us show an example in which the cycle length of a composited cellular automaton is the maximum cycle length for a linear cellular automaton

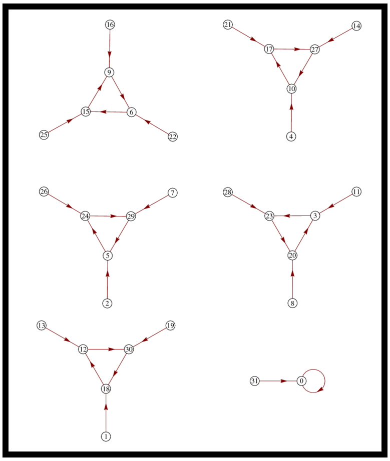

Fig. 1. CA90(5)

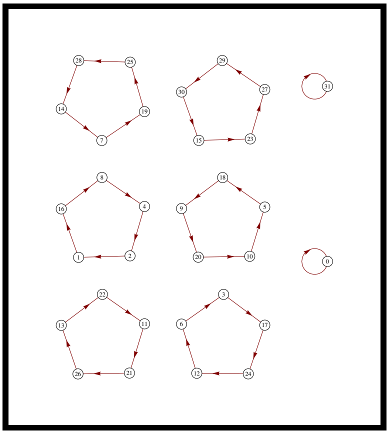

Fig. 2. CA240(5)

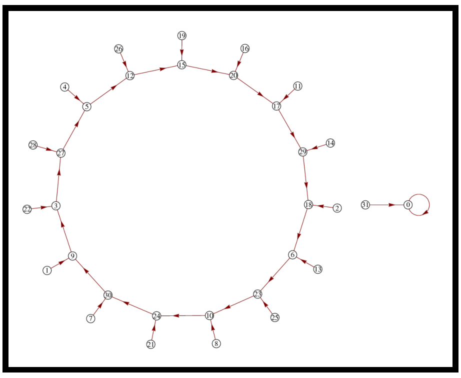

Fig. 3. CA90(5) $\times$ CA240(5)

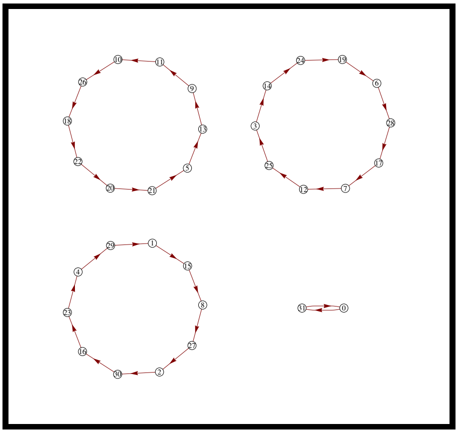

Fig. 4. CA15(5)

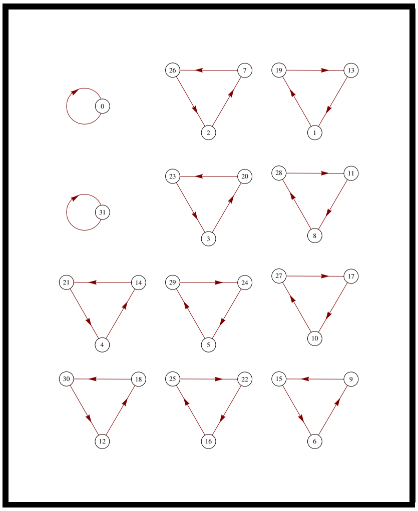

Fig. 5. CA150(5)

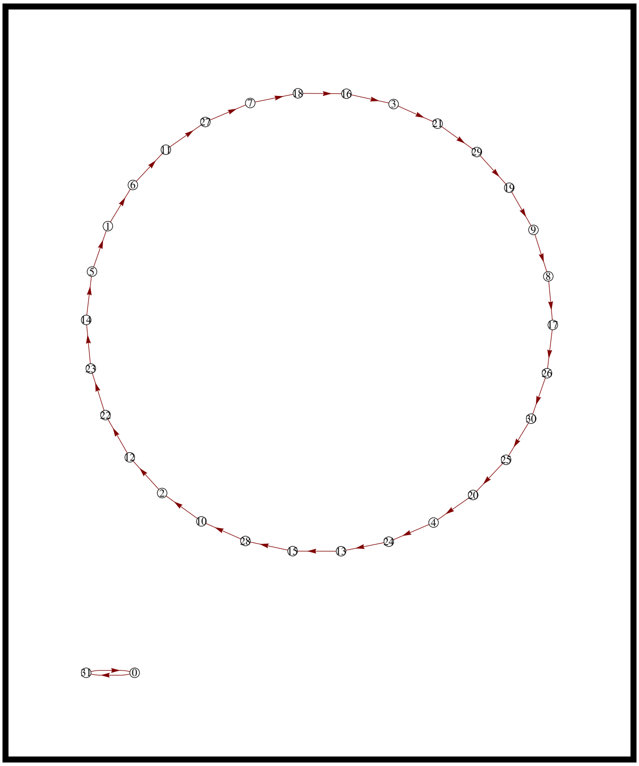

Fig. 6. CA15(5) $\times$ CA150(5)

and a non-linear cellular automaton. Figure 4, 5 and 6 correspond to CA15(5), CA150(5) and the composited cellular automaton. For the configuration $c = 6$, $C_1 = \{3, 6, 7, 12, 14, 17, 19, 24, 25, 28\}, C_2 = \{6, 9, 15\}$ and $\min\{t > 0 | F^t(6) = 6\} = 30$.

6 Conclusion

In this paper, we discussed the commutativity conditions of composition and behavior of composited cellular automata. We presented the commutativity conditions of compositions of linear cellular automata and affine cellular automata. In addition, we showed the relations of cycles of cellular automata and their composited cellular automaton. We presented that a cellular automaton made by composition of cellular automata has a maximum cycle length.

In the future, we will study more commutativity conditions of compositions and the behaviors of all cellular automata. In addition, we would like to show a systematic way to define cellular automata with maximum cycle lengths.

References

1. Pal Chaudhuri, P., Chowdhury, D.R., Nandi, S., Chattopadhyay, S.: Additive Cellular Automata, Theory and Applications, vol. 1. John Wiley & Sons (1997)
2. Elspas, B.: The Theory of Autonomous Linear Sequential Networks Circuit Theory. IRE Transactions on Issue Date 6(1), 45–60 (1959)
3. Fujio, M.: XOR2 = 90 - graded algebra structure of the boolean algebra of local transition rules. Papers Res. Inst. Math. Sci. 1599, 97–102 (2008)
4. Ito, T., Fujio, M., Inokuchi, S., Mizoguchi, Y.: Composition, union and division of cellular automata on groups. In: Proc. of the 16th International Workshop on Cellular Automata and Discrete Complex Systems, Automata 2010, pp. 255–264 (2010)
5. Matsumoto, M.: Simple Cellular Automata as Pseudorandom m-Sequence Generators for Bulit-In Self-Tes. ACM Transactions on Modeling and Computer Simulation 8(1), 31–42 (1988)
6. von Neumann, J.: Theory of self-reproducing automata. Univ. of Illinois Press (1966)
7. Pries, W., Thanailakis, A., Card, H.: Group properties of cellular automata and VLSI applications. IEEE Trans. on Computers C-35(12), 1013–1024 (1986)
8. Tezuka, S., Fushimi, M.: A method of designing cellular automata as pseudorandom number generators for built-in self-test for VLSI. Finite Fields: Theory, Applications, and Algorithms 168, 363–367 (1994)
9. Voorhees, B.: Commutation of Cellular Automata Rules. Complex Systems 7, 309–325 (1993)
10. Wolfram, S.: Statistical Mechanics of Cellular Automata. Reviews of Modern Physics 55, 601–644 (1983)
11. Wolfram, S.: Random Sequence Generation by Cellular Automata. Advances in Applied Mathematics 7, 123–169 (1986)
12. Yukita, S.: Dynamics of Cellular Automata on Groups. IEICE Trans. Inf. & Syst. E82-D(10), 1316–1323 (1999)