Behavior of Social Dynamical Models II: Clustering for Some Multitype Particle Systems with Confidence Threshold

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Abstract. We generalize the clustering theorem by Lanchier (2012) on the infinite one-dimensional integer lattice $\mathbb{Z}$ for the constrained voter model and the two-feature two-trait Axelrod model to multitype biased models with confidence threshold. Types are represented by a connected graph Γ, and dynamics is described as follows. At independent exponential times for each site of type i, one of the neighboring sites is chosen randomly, and its type j is adopted if i, j are adjacent on Γ. Starting from a product measure with positive type densities, the clustering theorem dictates that fluctuation and clustering occurs, i.e., each site changes type at arbitrary large times and looking at a finite interval consensus is reached asymptotically with probability 1, if there is one or two vertices of Γ adjacent to all other vertices but each other. Additionally, we propose a simple definition of clustering on a finite set, in which case one can apply the clustering theorem that justifies known previous claims.

Keywords: Multitype biased voter models, Axelrod model, confidence threshold, fluctuation phenomena.

1 Introduction

One of the most popular and interesting social dynamical models is the model of Axelrod for the evolution of cultural domains [2]. It is formulated as a stochastic spatial model, where each site is characterized by f features and each feature by q possible traits. Two assumptions are employed in the description of the dynamics. Pairs of neighboring sites interact at rate equal to the number of features they share (homophily assumption), and the one site adopts a feature

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of its neighboring site they do not share (social influence assumption). After more than a decade of interdisciplinary research primarily by computer simulation and mean-field approximation [3], Nicolas Lanchier [9,10] with Jason Schweinsberg [11] and the second author [12] has recently achieved analytical findings in one-dimensional lattices. The infinite model clusters to a monopolar configuration (consensus is reached) whenever $q = 2$ [10,11], and the finite model converges to a highly fragmented configuration for $f \leq cq$ where the slope satisfies the equation $e^{-c} = c$ [10] (see also, [12, Introduction]). For the same parameter region as in the latter result or if $f = 2$ and $q \geq 3$ each site of the infinite model fixates to a final cultural type with probability 1 [12].

In the first installment, the second author showed fixation in symmetric cyclic particle systems [14]. In this article, we examine the behavior of social dynamical models with respect to fluctuation and clustering for alternatives of the Axelrod model, which generalize systems presented in [16,10,1] and also appear as discrete analogues of certain models with continuous types [9].

The investigated dynamics is described as continuous-time Markov processes $(\xi_t)_{t\geq 0}$ with state space $\{0, 1, \cdots, N-1\}^{\mathbb{Z}}$, where $\mathbb{Z}$ represents the one dimensional integers. Types are represented by a connected graph Γ with vertex set $V(\Gamma)$ of cardinality $\#V(\Gamma) = N$, and edge set $E(\Gamma)$. Let $d^{\Gamma}_{z,w}$ denote the distance of two vertices z, w of a graph Γ. The initial configuration (state) is ξ according to a product measure with positive type densities. From then on, for each site x, type $\xi(x)$ independently becomes of the type $\xi(y)$ at an exponential rate proportional to the number of *neighbors* y that satisfy $d^{\mathbb{Z}}_{x,y} = 1$, provided that the *weight* of edge $\{x, y\}$ is equal to 1, that is, $d^{\Gamma}_{\xi(x),\xi(y)} = 1$. For each site x, the transition rule is formally written as

$$\xi(x) \to c \quad \text{at rate} \quad \#\{y \in \mathbb{Z} : \ d^{\mathbb{Z}}_{x,y} = 1, \ d^{\Gamma}_{\xi(x),\xi(y)} = 1, \ \xi(y) = c\} \ . \qquad (1)$$

For example, suppose that graph Γ is the hypercube Q_f with 2^f vertices. If $f = 1$, rates (1) describe linear dynamics of a voter model, where transitions occur at rate proportional to the number of neighboring sites with a different type [7,4]. If $f = 2$, the four-type system with rates (1) coincides with the two-feature two-trait Axelrod model, which alike the voter model fluctuates and clusters [10]. For $f > 1$, rates (1) describe conditionally attractive dynamics on bounded confidence. A central problem is to determine the phase transition from fluctuation to fixation in the asymptotic limit of time, and full qualitative or asymptotically sharp results are valuable. The exhibited qualitative behaviors are formally defined as follows.

(ξ_t) *fixates* if there exists a random (possibly deterministic) limiting configuration ξ_∞ such that for each x,

$$\lim_{t\to\infty} P[\xi_t(x) = \xi_\infty(x)] = 1 \ . \qquad (2)$$

(ξ_t) *fluctuates* if for each x,

$$P[\xi_t(x) \text{ changes at arbitrarily large times } t] = 1 \ . \qquad (3)$$

(ξ_t) *clusters* if for each x, y,

$$\lim_{t \to \infty} P[\xi_t(x) = \xi_t(y)] = 1 \ . \tag{4}$$

The eccentricity of a vertex in a connected graph is the maximum distance from it to any other vertex, the center of the graph is the set of all vertices of minimum eccentricity, and a peripheral vertex has eccentricity equal to the diameter of the graph, which is the maximum eccentricity of any vertex in the graph. Starting from any product measure with positive type densities, as the diameter of graph of types Γ increases, by increasing the number of types N and adding accordingly vertices and edges on Γ, edges in $\mathbb{Z} + 1/2$ with types that cannot interact with each other are more probable at $t = 0$. Furthermore, it is more likely that most types will have no neighbors to interact with for $t > 0$, so that the system fixates. Definition (2) does not a priori exclude the more sophisticated regime of clustering, and particularly, fixation of the examined systems corresponds to convergence to a highly fragmented configuration. Depending on the interaction mechanism and the dimension of the integer lattice, fluctuation may be accompanied with clustering, which is the case in systems with rates (1).

In this article, we present a generalization of [10, Theorem 1] formulated in Theorem 1 below for the clustering of systems with many types represented by an arbitrary connected graph Γ. Our motivation, and excuse at the same time, is to attack arbitrary multitype particle systems with confidence threshold to the hopes of understanding asymptotic behavior with respect to graph theoretic properties of the structure of types. For asymptotic results which consider the structure of the social network in systems with continuous types in the interval $[0, 1]$, see [9]. We believe that part (i) of Theorem 1 is asymptotically sharp, and we mention that it seems more potent than part (ii) if the center of Γ is a strict subset of the full graph (for all confidence values different from the diameter of Γ), while the converse seems to hold otherwise. The previous statement is explained in Section 2, where additionally simulations are conducted and corollaries of Theorem 1 are obtained. In Section 3 we sketch a proof Theorem 1. In Section 4 we provide applications for finite systems.

Theorem 1 (Generalized Lanchier's Theorem). *Consider a voter model with $N \geq 3$ types and confidence 1 as follows. Each type is in the vertex set $V(\Gamma)$ of a connected graph Γ, and two types can interact if they are adjacent in Γ. Starting from a product measure on $V(\Gamma)^{\mathbb{Z}}$ with positive type densities, fluctuation and clustering occurs if (i) there is a vertex of Γ adjacent to all other vertices, or (ii) there are two vertices of Γ adjacent to all other vertices but each other.*

2 Discussion

In this Section we discuss Theorem 1 by providing corollaries and conducting simulations for special cases. Our discussion is accommodated by the consideration of a conditional convergent interaction (see, rates (5)).

The substitution in rates (1) of graph Γ with graph Γ^ε, where Γ^ε is induced from the original graph by linking each two vertices within distance ε, defines a certain multitype particle system with confidence parameter. If $\Gamma = Q_f$ is the hypercube with 2^f vertices, the following corollary of Theorem 1(ii) holds, which was given a sketch of proof up to fluctuation in [1].

Corollary 1. *The hypercubic particle system with 2^f types represented by hypercube Q_f and confidence ε, starting from a product measure on $V(Q_f)^{\mathbb{Z}}$ with positive type densities, fluctuates and clusters if $\varepsilon \geq f-1$.*

The substitution in rates (1) of graph Γ with graph P_N^ε, where P_N^ε is induced from path P_N with N vertices by linking each two vertices within distance ε, defines a certain constrained voter model with confidence parameter and implies the following corollary of Theorem 1(i).

Corollary 2. *The constrained voter model with $N \geq 3$ types represented by path P_N and confidence ε, starting from a product measure on $V(P_N)^{\mathbb{Z}}$ with positive type densities, fluctuates and clusters if $N \leq 2\varepsilon + 1$.*

Similarly, if $\Gamma = C_N$ is the N-cycle, one can define a symmetric cyclic particle system [14] with arbitrary confidence threshold. In addition, we define a *convergent transition rule* for certain particle systems with confidence threshold, and for each site x

$$\xi(x) \to c \quad \text{at rate} \quad \#\{y \in \mathbb{Z} : \, d^{\mathbb{Z}}_{x,y} = 1, \; 0 < d^{\Gamma}_{\xi(x),\xi(y)} \leq \varepsilon, \; d^{\Gamma}_{c,\xi(y)} = d^{\Gamma}_{\xi(x),\xi(y)} - 1\} \tag{5}$$

where c, depending on graph Γ, may be random (uniformly chosen among all possible vertices of Γ which satisfy rates (5)). The confidence parameter ε is positive with maximum value equal to the diameter $d(\Gamma) = \max_{z,w \in V(\Gamma)} d^{\Gamma}_{z,w}$.

Figure 1 compares conditional convergent and conditional attractive interactions on bounded confidence, the former following rates (5) for a graph Γ, and the latter following rates (1) for induced graph Γ^ε. This comparison is with respect to the mean size of clusters at absorption versus the confidence parameter (hundred-site torus, ensemble size 10^4). Small cluster sizes compared to the size of the torus correspond to highly fragmented configurations, while cluster sizes that equal the size of the torus correspond to consensus. For models with rates (1) all evidence so far is that, these two regimes match in an infinite setting the behaviors of fixation and fluctuation accompanied with clustering, respectively. Although there may be exceptions, a similar statement holds according to the rigorous findings for certain systems with a more sophisticated rates than (5) which include the assumption of homophily as well - viz., the model of Axelrod.

Paying attention on models with conditional attractive interactions on confidence, Figure 1(blue circle marks) shows the mean cluster size at absorption versus confidence in a fifteen-type constrained voter model. The center of graph P_N^ε is a strict subset of the full graph for all nontrivial values of confidence that do not produce a linear multitype voter model $\varepsilon \neq d(P_N) = N-1$. In particular,

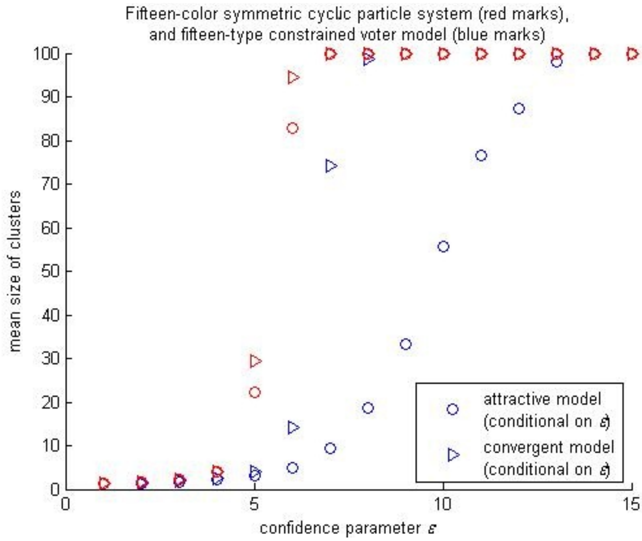

Fig. 1. Mean size of clusters at absorption versus confidence parameter ε in models with convergent or attractive interactions, conditional on bounded confidence: the constrained voter model with confidence threshold (blue marks) and the symmetric cyclic particle system with confidence threshold (red marks). A hundred-site torus, and a 10^4 ensemble was used.

if $N = 15$ and $\varepsilon = (N-1)/2 = 7$, Theorem 1(i) implies clustering on the infinite lattice, which is clearly more potent than Theorem 1(ii) in the sense that part (i) of the theorem provides clustering while part (ii) does not. Comparing with the data for conditional convergent interactions on $\varepsilon = 7$ (blue triangle marks), there seems like a huge contradiction between the qualitative behavior of the two finite models, since the mean cluster size is a lot smaller in the former case, while clustering is indicated in the latter case. However, as Nicolas Lanchier spoke it "spatial simulations are usually difficult to interpret" [10]. In our particular case, no contradiction between the qualitative behaviors of two finite models with the same parameters is somewhat suggested in simulations by the not atypical two spatial scales at absorption, Figure 2. To further strengthen this view, one needs to consider larger lattices in simulations, or prove it analytically as in the next Section.

Furthermore, the center of graph C_N^{ε} is the full graph for all N, ε. If N is even, Theorem 1(ii) is more potent than Theorem 1(i) in the sense that part (ii) of the theorem provides an asymptotically sharper condition $\varepsilon \geq N/2 - 1$ than the one provided by part (i) $\varepsilon = N/2$. Moreover, if N is odd and $\varepsilon \geq d(C_N)$, Theorem 1 implies clustering of the infinite model. In particular, if $N = 15$ and $\varepsilon \geq d(C_{15}) = 7$, based on the mean cluster sizes of Figure 1 the condition $\varepsilon \geq 7$ does not seem asymptotically sharp, since clustering is highly indicated for $\varepsilon = 6$, and less clearly for $\varepsilon = 5$. Based on such simulations, we conjecture that the one-dimensional symmetric cyclic particle system with confidence threshold fixates if $\varepsilon < N/3$, and fluctuates and clusters if $\varepsilon > N/3$. In writing the previous two paragraphs, we had the faith that infinite and finite models with the same parameters should exhibit no essential difference in their behavior, which we attempt to strengthen in Section 4, after the proof of generalized Lanchier Theorem in the next Section.

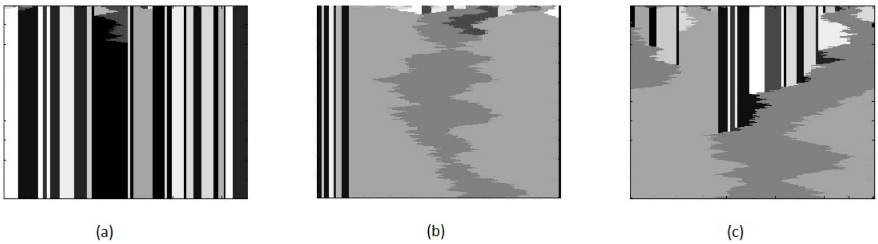

Fig. 2. Three not atypical realizations of a fifteen-type constrained voter model with confidence $\varepsilon = 7$ using a random cellular automaton with double clock [1]. Fixation to a fragmented configuration (a), two spatial scales near equilibrium (b), convergence towards consensus (c). Time is running from top to bottom of the page.

3 Proof of Theorem 1

Reference [10] starts from the two-feature two-trait Axelrod model, and employs a coupling observed by Vázquez and Redner [16]. One recovers the voter model from the two-feature two-trait Axelrod model by identifying the cultural types that have no feature in common. Using fluctuation and clustering of the two-type voter model, Lanchier showed clustering of a four-type Axelrod model. Then, the constrained voter model with three types represented by the path graph P_3 clusters as well, since the mean size of clusters is stochastically larger in the latter case.

The following proof of Theorem 1 briefly reviews and generalizes by the mapping of [8] steps within the proof of the first theorem in [10]. We work in opposite, starting from the constrained voter model (ζ_t) with rates (1) and types represented by path P_3. As a side note, the non-spatial models in references [8] and [17] deal independently with the same system, and arrive at the same result through a different proof.

The three-type (ζ_t) can be graphically constructed on space-time lattice $\mathbb{Z} \times [0, \infty)$ following a versatile technique by Harris [6], which is applicable for any dimension d of the multidimensional integers. At $t = 0$, label independently the sites of $\mathbb{Z}$ with random types $0, 1, 2$ according to a product measure μ with positive type densities $\mu(\zeta(x) = i) = \theta_i > 0$ and $\theta_0 + \theta_1 + \theta_2 = 1$. The types are hierarchically labeled, that is, for two vertices u, v in P_3, $d^{P_3}_{u,v} = |u - v|$.

For $t > 0$, assign independent Poisson processes with parameter 1 $\{T_n^x, n \geq 1\}$ for each site x, together with independent sequences of i.i.d. fair coin tosses $\{U_n^x, n \geq 1\}$ ($P(U_n^x = 1) = \frac{1}{2}$, $P(U_n^x = -1) = \frac{1}{2}$). At each arrival time T_n^x, allocate a directed edge that is called arrow yx from $y = x + U_n^x$ to x, which has the metaphorical meaning that the voter at x at time T_n^x considers the opinion of a random neighboring voter y. The voter at head x of *an active arrow* yx assumes the type/opinion of the voter at tail y provided that $d^{P_3}_{\zeta(y),\zeta(x)} = 1$. An arrow yx is inert and induces no change at head x, if $d^{P_3}_{\zeta(y),\zeta(x)} \neq 1$. To distinguish the two kinds of arrows, one can mark inert arrows, for instance, with an 'x'. Active arrows and fixed-site increasing time segments give rise to directed paths, which connect different points in $\mathbb{Z} \times [0, \infty)$. Important in this construction is the concept of *an active path*, which is a directed path that does not coincide with the head of an active arrow. For each point (x, t) there is always a unique $(z, 0)$, such that there is an active path from $(z, 0)$ to (x, t) and z is called the *ancestor of x at time t*, $\alpha_t(x) = z$. Using this construction, (ζ_t) is defined inductively for each x by writing $\zeta_t(x) = \zeta_0(\alpha_t(x))$.

Looking at a finite interval $A \subset \mathbb{Z}$, the configuration at time t is then determined by the process (α_t) that keeps track of the ancestor of each x in A. To compute configuration $\zeta_t(x)$ one has to invert all arrows and follow backwards in time the active path from (x, t) to $(\alpha_t(x), 0)$. Note that while computing $\zeta_t(x)$ backwards, to avoid following an inert arrow at a given time, one needs to have constructed the process up to this time. Therefore, the construction of (α_t) depends on the initial configuration, which differs from the construction of dual paths in the voter model.

However, if $\zeta_t(x) = 1$, all arrows in backward computations are followed, and the process can be constructed regardless of the initial configuration. Thus, a connection with coalescing random walks can be exploited as in the voter model. In this case, (α_t) is a dual process of (ζ_t) defined exclusively for type 1 as a system of coalescing symmetric random walks, which start from $A \subset \mathbb{Z}$. By well know results in linear particle systems, the density of type 1 is preserved for any dimension d of multidimensional integers, and in one and two dimensions $d = 1, 2$ clustering occurs for type 1 with positive probability

$$\lim_{t \to \infty} P[\zeta_t(x) = \zeta_t(y) = 1] = \theta_1 > 0 \ . \tag{6}$$

The obtained duality with coalescing random walks for type 1 suggests a particular mapping of the types. We consider an imbedded Markov process (i_t) within (ζ_t), which cannot distinguish types that are different from type 1. If we identify types u such that $u \neq 1$, (i_t) is the two-type voter model, which fluctuates on $\mathbb{Z}$ owing to known results. It is crucial that the voter model fluctuates for rather general initial configurations [15, Remark]. Therefore, each site of the three-type process fluctuates between type 1 and type 0 or 2. This idea is applicable for any connected Γ, if there is a vertex of Γ adjacent to all other vertices. In this case, the N-type process clusters for a central vertex of Γ, and each site fluctuates between this central vertex and one of the remaining vertices of Γ, which shows fluctuation in Theorem 1(i).

Fluctuation in Theorem 1(ii) can be proved by the mapping of Itoh, Mallows, and Shepp [8, Section 3], which obtains quantitative results for the asymptotic distributions of constrained models with more than three types from the distributions of the three-type model. We employ the mapping of [8] to obtain a qualitative result. If there are two vertices u, v of Γ at distance 1 from all other vertices but each other, one can identify all $j \neq u, v$. By the first part of the theorem, the imbedded three-type system within the N-type system fluctuates. Therefore, each site of the N-type system fluctuates among types u, v and one of the types $j \neq u, v$.

Following [10], clustering for all three types is a result of the key facts of fluctuation and clustering for type 1, together with the analysis of the evolution of weights of edges $\{x, x+1\}$ using the edge process (e_t) for each x

$$e_t(x) \;=\; d^{P_3}_{\zeta_t(x),\zeta_t(x+1)} \;,$$

which keeps track of the type distances along the edges of $\mathbb{Z}$ rather than the types of the sites. We say that edge x_+ is vacant, active, blockade (resp.) at time t, if $e_t(x) = 0, 1, 2$ (resp.). Following the motion of an active path, an active edge jumps to one of two nearest neighbor vacant edges with equal probability, unless the nearest neighbor edge is blockade or active in which case a collision occurs. Taking into account symmetry, all possible transitions of edge pairs are collisions of two active edges, which annihilate $(1,1) \to (0,0)$ or annihilate thus creating a blockade $(1,1) \to (0,2)$, and jumps of an active edge to a nearest neighbor vacant edge $(1,0) \to (0,1)$ or to a nearest neighbor blockade $(1,2) \to (0,1)$. The edge pair transitions show that active edges cannot be created, and that they evolve as a system of annihilating symmetric random walks. Moreover, clustering for type 1 implies almost sure extinction of active edges

$$\lim_{t\to\infty} P[e_t(x) = 1] = 0 \;. \tag{7}$$

Letting $0 < s < t < \infty$, where s is large, on the one hand, the probability of a blockade at time t that has been created after time s is at most ϵ, for all small $\epsilon > 0$ (as a consequence of (7), and the fact that a blockade can only be created by the annihilation of two active edges). On the other hand, the probability of a blockade at time t that has been created by time s fixed is at most ϵ, for some $t > s$ (as a consequence of fluctuation). Then, the combination of the previous two estimates implies almost sure extinction of blockades

$$\lim_{t\to\infty} P[e_t(x) = 2] = 0 \;. \tag{8}$$

By (7) and (8), the three-type process clusters (4) for all types and each x, y:

$$\lim_{t\to\infty} P[\zeta_t(x) \neq \zeta_t(y)] \leq \lim_{t\to\infty} P[e_t(x) \neq 0] = 0 \;.$$

In any N-type process, if a vertex of Γ is adjacent to all other vertices, clustering occurs for this central vertex with positive probability. In addition, all possible transitions of edge pairs are as in the three-type process (it suffices that in both processes the initial type densities are positive). As previously, the combination of clustering for a particular type and the already established fluctuation of the N-type process implies clustering (4) in Theorem 1(i).

In any N-type process, if there are two vertices u, v of Γ adjacent to all other vertices but each other, then the mapping of [8] identifies all $j \neq u, v$. As already shown, the imbedded three-type process (ζ_t) clusters with probability 1. It is crucial that clustering of (ζ_t) occurs for any positive initial densities. Therefore, the N-type process clusters for types u, v with positive probability. The combination of clustering for particular types and the already established fluctuation of the N-type process implies clustering (4) in Theorem 1(ii).

4 Applications

In this section, we provide applications of Theorem 1, which are grounded on a simple definition and justify previous claims of fluctuation until absorption of certain finite systems and their generalizations in the present article.

Any stationary distribution of a finite system with rates (1) is supported on the set of absorbing states. Apparently, definition (2) implies that any finite system fixates with probability 1 for any number of types and confidence threshold, which contradicts observations from simulations that a model seems to exhibit different qualitative behaviors for complementary ranges of its parameters. We clarify this discrepancy and justify the understanding in [1], by proposing a definition of fluctuation on a finite set, which is also applicable for the systems in [16] and [9], and seems to be applicable for all systems with rates (1) that fluctuate and cluster according to Theorem 1.

The definition of fluctuation on a finite set is slightly more involved than definition (3). Suppose that the process (ξ_t) has rates (1) on finite connected $G \subset \mathbb{Z}$ for an induced graph Γ^ε. Then, define the process (g_t) on $G \cup \{l, r\}$, which starts from finite configuration g that is obtained from ξ by adding two peripheral sites on G, $l = \min\{x \in V(G)\} - 1$ and $r = \max\{x \in V(G)\} + 1$, and evolves as (ξ_t) except that all arrows towards the leftmost site l and the rightmost site r of initial types $g(l)$ and $g(r)$, respectively, are deleted.

We say that, (ξ_t) *fluctuates until absorption on* G if for each site x different from l and r, conditional on the event that $g(l)$ and $g(r)$ have distance ε,

$$P\big[g_t(x) \text{ changes at arbitrary times } t \mid x \neq l, r,\ d^{\Gamma}_{g(l),g(r)} = \varepsilon\big] = 1 \ . \qquad (9)$$

Definition (9) has the interpretation that, by deactivating all arrows towards the end sites of a finite interval with types that can change one another, the definition of fluctuation assimilates the idea of definition (3) of the infinite system.

Suppose that Γ is a tree graph. Under the imposed conditions by definition (9), if Theorem 1 holds for the infinite model, then for the finite model with the same parameters there is large time t such that the ancestor of each site $x \neq l, r$ is either l or r and the edge of the two domains with types $\xi(l)$ and $\xi(r)$ is a symmetric random walk with jump rate 1/2 which bounces at the end sites of the interval. Therefore, the following application holds.

Application 1. *Consider a voter model with $N \geq 3$ types, confidence ε, and rates (1) for an induced graph Γ^ε where Γ is a tree. Starting from a product measure on $V(\Gamma)^{\mathbb{Z}}$ with positive type densities, if the infinite model fluctuates and clusters according to Theorem 1, then the finite model fluctuates until absorption (which may be either consensus or a fragmented configuration).*

Considering more applications, the graph of types Γ can be a hypercube or a cycle graph, whence the underlaid claim of Application 1 holds.

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