Topological Perturbations and Their Effect on the Dynamics of Totalistic Cellular Automata

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Abstract. Although several studies addressed the dynamical properties of cellular automata (CAs) in general and the sensitivity to the initial condition from which they are evolved in particular, only minor attention has been paid to the interference between a CA's dynamics and its underlying topology, by which we refer to the whole of a CA's spatial entities and their interconnection. Nevertheless, some preliminary studies highlighted the importance of this issue. Henceforth, in contrast to the sensitivity to the initial conditions, which is frequently quantified by means of Lyapunov exponents, to this day no methodology is available for grasping this so-called topological sensitivity. Inspired by the concept of classical Lyapunov exponents, we elaborate on the machinery that is required to grasp the topological sensitivity of CAs, which consists of topological Lyapunov exponents and Jacobians. By relying on these concepts, the topological sensitivity of a family of 2-state irregular totalistic CAs is characterized.

1 Introduction

Ever since cellular automata (CAs) have been found capable of evolving striking spatio-temporal patterns in spite of their overly simple formulation, researchers in various branches of science have been desirous to comprehend their intriguing dynamics. For that purpose, several methods have been proposed during the last two decades [7,11,12,20], among which Lyapunov exponents are probably the most popular seen their successful application within the framework of continuous dynamical systems [8,9,15,16,17]. Notwithstanding several papers on Lyapunov exponents of 1-dimensional CAs stick to directional Lyapunov exponents [15], namely right and left exponents that quantify the rate with which perturbations or defects in such CAs propagate to right or left, respectively, recent studies have shown the strengths of direction-independent Lyapunov exponents [4,7]. The latter are preferred in case of higher-dimensional CAs because the directionality that is inherent to defect propagation in 1-dimensional CAs gets blurred if higher-dimensional CAs are at stake [4,5,6]. Moreover, it has been shown that an upper bound on non-directional Lyapunov exponents

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can be obtained by resorting to Jacobians whose elements constitute Boolean derivatives [18], and which express the sensitivity of CAs to their inputs [5,6].

Although several papers address the sensitivity of CAs to the initial conditions from which they evolve [4,5,6,8,17], it is rather surprising to notice that only a handful of papers address the interference between a CA's dynamical properties, on the one hand, and its topology, on the other hand, since the dynamics of a CA is inherently determined by both the states of its cells and its topology, *i.e.* the way its cells are interconnected. Indeed, only a few studies touch upon this topological sensitivity in the framework of CAs [10,13,14], but these largely discard its quantification. Nonetheless, the importance of the underlying topology has long been acknowledged for closely-related dynamical systems, such as neural networks, [1] and coupled-map lattices [2].

This stimulated us to formulate so-called topological Lyapunov exponents that measure the rate by which phase space trajectories diverge following the insertion of a topological perturbation, which may be envisaged as either the breaking up of the connectivity between two neighbouring cells, or as the establishment of a connection between two not yet neighbouring cells. Parallel to the framework that has been developed for classical Lyapunov exponents, we conceive topological Jacobians and derivatives to get a grip on the origin of the numerically obtained topological Lyapunov exponent for a given CA. Finally, we resort to these constructs for quantifying the topological sensitivity of a family of 2D 2-state irregular totalistic CAs.

In Sec. 2, we elaborate on the preliminaries that are indispensable for a clear understanding of the constructs that are presented in Sec. 3, and which allow for a quantification of the topological sensitivity of CAs. These constructs are employed to assess the topological sensitivity of a family of 2D 2-state irregular totalistic CAs.

2 Preliminaries

As a family of 2D totalistic CAs is considered throughout the remainder of this paper, we first state its definition after which we introduce a nomenclature that is applicable to the CA family at stake. It should be emphasized that this definition deviates from the original CA paradigm by von Neumann [19] because it is not restricted to hypercube tessellations.

Definition 1. *(Totalistic cellular automaton)*
A totalistic cellular automaton (CA) $\mathscr{C}$ can be represented as a quintuple

$$\mathscr{C} = \langle \mathcal{T}, S, s, N, \Omega \rangle \ ,$$

where

(i) $\mathcal{T}$ is a countably infinite tessellation of an n-dimensional Euclidean space $\mathbb{R}^n$, consisting of cells c_i, $i \in \mathbb{N}$.
(ii) S is a finite set of k states, here $S \subset \mathbb{N}$.

(iii) *The output function* $s : \mathcal{T} \times \mathbb{N} \rightarrow S$ *yields the state value of cell* c_i *at the* t*-th discrete time step, i.e.* $s(c_i, t)$.

(iv) *The neighborhood function* $N : \mathcal{T} \rightarrow \bigcup_{p=1}^{\infty} \mathcal{T}^p$ *maps every cell* c_i *to a finite sequence* $N(c_i) = \left(c_{i_j}\right)_{j=1}^{|N(c_i)|}$, *consisting of* $|N(c_i)|$ *distinct cells* c_{i_j}.

(v) *The transition function* $\Omega : \mathbb{N} \rightarrow S$ *governs the dynamics of each cell* c_i, *i.e.*

$$s(c_i, t+1) = \Omega\left(\sigma_i\right) ,$$

where $\sigma_i = \sum_{j=1}^{|N(c_i)|} s(c_{i_j}, t)$.

In order to identify every Ω that can be formulated for a given number of states k by means of an unique number, commonly referred to as a rule number, we set up a numbering convention. The rule number of a k-state, θ-sum irregular totalistic CA, denoted $R^{\mathcal{T}}_{\theta}$, can then be found from its base-k representation, containing $\theta + 1$ digits, $z_\theta\, z_{\theta-1} \cdots z_2\, z_1\, z_0$ as

$$R^{\mathcal{T}}_{\theta} = z_\theta\, k^\theta + z_{\theta-1}\, k^{\theta-1} + \ldots + z_2\, k^2 + z_1\, k^1 + z_0 \,, \tag{1}$$

where θ is an upper bound on σ_i such that $\Omega\left(\sigma_i\right) = \Omega(\theta)$ if $\sigma_i \geq \theta$ and $z_f \in \{0, 1, \ldots, k-1\}$ represents the state value assigned to c_i at the following time step if $\sigma_i = f$. This upper bound θ is introduced to overcome the unboundedness of the sum σ_i that naturally arises if CAs are built upon irregular tessellations of $\mathbb{R}^n$. A total of $k^{\theta+1}$ different rules can be enumerated for this family of irregular CAs. For reasons of brevity, we refer in the remainder of this paper to k-state, θ-sum irregular totalistic CAs as (k, θ) irregular totalistic CAs.

3 Quantifying Topological Sensitivity

In the remainder of this section, we consider both topological Lyapunov exponents and Jacobians for 2-state totalistic CAs, for which $S = \{0, 1\}$. The former are relied upon for measuring the rate by which two phase space trajectories diverge upon the introduction of a topological perturbation, while the latter characterizes the sensitivity of Ω to the underlying topology. By restricting the scope of this paper to totalistic CAs, of which the dynamics does not depend on the ordering imposed on $N(c_i)$, a topological perturbation may be simply contemplated either as the breaking up of the connectivity between two neighbouring cells, or as the establishment of a connection between two not yet neighbouring cells. Essentially, totalistic CAs enable one to investigate the consequences of such a true topological perturbation, whereas CAs of which the dynamics depends on the ordering of its neighbouring cells, such as elementary CAs, only allow for investigating the impact of substituting one of the neighbouring cells by an other cell. Hence, since our goal is to assess the effect of a true topological perturbation on the stability of CAs, we adhere to the family of totalistic CAs throughout the remainder of this paper. It should be emphasized that the measures introduced in the remainder of this section, may equally well be used for any kind of two-state discrete dynamical system that is based upon either a tessellation or a graph, which indicates the wide applicability of the proposed constructs.

3.1 Topological Lyapunov Exponents

In line with the definition of classical Lyapunov exponents that express a CA's sensitivity to the initial condition from which it evolves [4,7], we can contemplate topological Lyapunov exponents that express the sensitivity of a CA to topological perturbations, and hence to its topology. By resorting to a graph representation of a totalistic CA $\mathscr{C} = \langle \mathcal{T}, S, s, N, \Omega \rangle$, which boils down to identifying $\mathcal{T}$ as the graph's vertex set V_G and by setting $\{c_i, c_j\} \in D_G$ if $c_j \in N(c_i)$, where D_G is the graph's edge set, such perturbations may be envisaged as either the deletion or establishment of an edge in an undirected graph G. As such, we can consider a CA $\mathscr{C} = \langle G, S, s, \Omega \rangle$ that is built upon the original topology embodied in G and the one built upon a perturbed topology G^*, *i.e.* $\mathscr{C}^* = \langle G^*, S, s^*, \Omega \rangle$. The sensitivity of a given transition function Ω to the underlying topology can then be assessed by tracking the number of cells c_i for which $s(c_i, t) \neq s^*(c_i, t)$, further referred to as defects, that emerge during the evolution $\mathscr{C}$ and $\mathscr{C}^*$. If we denote the number of defects at the t-th time step during the evolution of a CA as ϵ_t, the maximum topological Lyapunov exponent (MTLE) of a 2-state totalistic CA can be defined as

$$\lambda_\tau = \lim_{t \to \infty} \frac{1}{t-1} \log\left(\frac{\epsilon_t}{\epsilon_1}\right) . \tag{2}$$

Clearly, since $\mathscr{C}$ and $\mathscr{C}^*$ evolve from the same s_0, it holds that $\epsilon_0 = 0$. Yet, ϵ_1 is possibly strictly positive as a topological perturbation may give rise to either zero, one or two defective cells at $t = 1$. Naturally, in every subsequent time step additional defects may be introduced by the topological perturbation, which must be tracked separately since multiple defects can cancel each other due to the utter discrete nature of the dynamical systems at stake. It should be emphasized that practical considerations restrict the assessment of λ_τ to finite T, and, likewise, to finite tessellations $\mathcal{T}^*$.

Based upon the numerically calculated λ_τ, we have a means to identify a CA as topologically insensitive if $\epsilon_t = 0$ for all $t \geq t^*$ such that $\lambda_\tau = -\infty$, where t^* represents the number of time steps constituting the CA's transient period, or as topologically sensitive if $\lambda_\tau > 0$ since a positive MTLE entails exponentially diverging phase space trajectories as a consequence of the topological perturbation. Clearly, the topological sensitivity becomes more pronounced as λ_τ increases. Some CAs might evolve towards $\lambda_\tau = 0$, which entails that $\epsilon_1 \, t^{-m} \leq \epsilon_t \leq \epsilon_1 \, t^m$ where $m \in \mathbb{N}$ and means that polynomial growth of the number defects will give rise to a zero MTLE as the number of time steps becomes infinitely large. In the remainder of this paper, such CAs will also be referred to as topologically insensitive.

3.2 Topological Jacobians

Similarly to the study of classical Lyapunov exponents of a CA $\mathscr{C}$, which relies on a statistical measure $\bar{\mu}_\alpha$ that quantifies the sensitivity of Ω to its inputs using a Jacobian matrix J^α of which the elements are Boolean derivatives (see Eq. (13) in [4]), we can set up topological Jacobians to encode the sensitivity of Ω to the topology upon which it is based. Therefore, let us first define the topological derivative of Ω with respect to the independent variable c_j as

$$\frac{\partial\Omega\big(\tilde{s}(N(c_i),t)\big)}{\partial c_j} = \begin{cases} \Omega\big(\tilde{s}(N(c_i),t)\big) \oplus \Omega\big(\tilde{s}(N^{-j}(c_i),t)\big) \text{ , if } c_j \in N(c_i)\,, \\ \Omega\big(\tilde{s}(N(c_i),t)\big) \oplus \Omega\big(\tilde{s}(N^{+j}(c_i),t)\big) \text{ , if } c_j \notin N(c_i)\,, \end{cases} \quad (3)$$

where $\tilde{s}(N(c_i),t) = \big(s(c_{i_j},t)\big)_{j=1}^{|N(c_i)|}$, $N^{-j}(c_i) = N(c_i) \setminus \{c_j\}$, $N^{+j}(c_i) = N(c_i) \cup \{c_j\}$ and $\oplus$ is the addition modulo 2 operator. By restricting this paper to 2-state totalistic CA, $\frac{\partial\Omega}{\partial c_j}$ can be either one or zero, depending on whether or not the computation of $s(c_i,t+1)$ is affected by perturbing the connectivity between c_i and c_j. It should be stressed that the formalization of topological derivatives becomes much more intricate in case of CAs for which the transition function depends on the ordering of the neighbours because a cell can then only be excluded from the neighbourhood if it is replaced by an other one. We refer the reader to [3] for more details on this issue.

Using these topological derivatives, we can construct a $|\mathcal{T}^*| \times |\mathcal{T}^*|$ topological Jacobian matrix $J^\mathcal{T}$ with elements

$$J^\mathcal{T}_{ij} = \frac{\partial s(c_i,t+1)}{\partial c_j}\,, \quad (4)$$

which can then be exploited to obtain a mean-field estimate of the proportion of connections between c_i and any other $c_j \in \mathcal{T}^*$ that affects the determination of $s(c_i,t+1)$, and which is given by

$$\mu_\mathcal{T}(t) = \frac{1}{|\mathcal{T}^*|}\sum_{c_i}\left(\frac{1}{|N(c_i)|}\sum_{c_j\in N(c_i)} J^\mathcal{T}_{ij} + \frac{1}{|\mathcal{T}^*| - |N(c_i)|}\sum_{c_j\notin N(c_i)} J^\mathcal{T}_{ij}\right)\,. \quad (5)$$

In this equation, the first term represents the contribution from the topological connections $(c_i,c_j) \in D_G$ that alter $s(c_i,t+1)$ if broken, whereas the second term is the contribution from topological connections (c_i,c_k), where $(c_i,c_k) \notin D_G$, which alter $s(c_i,t+1)$ if c_k belongs to $N(c_i)$. Clearly, $\mu_\mathcal{T}(t) = 1$ if and only if $J^\mathcal{T}_{ij} = 1$ for all $c_i\,, c_j$ in $\mathcal{T}^*$, whereas $\mu_\mathcal{T}(t) = 0$ can occur if and only if the determination of $s(c_i,t+1)$ is completely independent from the CA's underlying topology. In order to get an idea of $\mu_\mathcal{T}(t)$ throughout a CA's evolution, we consider its geometric mean after a large number of time steps T

$$\bar{\mu}_\mathcal{T} = \left(\prod_{t=1}^{T}\mu_\mathcal{T}(t)\right)^{T^{-1}}\,, \quad (6)$$

which yields that higher values of $\bar{\mu}_\mathcal{T}$ indicate a higher sensitivity to topological perturbations.

4 Topological Sensitivity of $(2, 6)$ Irregular Totalistic CA

4.1 Simulation Setup

In this section we consider the CAs within the family of $(2, 6)$ irregular totalistic CAs, for which $S = \{0, 1\}$. We opt to focus on this family because it allows for assessing the impact of a true topological perturbation, its set of states complies with the main presumption underlying Section 3, namely $S = \{0, 1\}$, and finally, its upper bound $\theta = 6$ is such that it does not interfere with the CAs' stability as pointed out in [4]. All together, 128 distinct rules can be listed for this CA family in accordance with the numbering convention presented in Section 2. The results presented in this section were obtained numerically for $T = 500$, since by then λ_τ, $\bar{\mu}_\tau$ and $\bar{\mu}_\alpha$ showed convergence in the sense that an increase of the number of time steps did not significantly alter the numerically assessed values. Furthermore, periodic boundary conditions were applied in order to minimize boundary effects owing to the finiteness of $\mathcal{T}^*$, which consisted of 675 irregular cells covering a unit square, and were generated from random seeds in $[0, 1]^2$ using a Voronoi tessellation. Further, it should be stressed that the values of λ_τ represent averages obtained over an ensemble $E_\tau = \{{}_eG^* \mid e = 1, \ldots, 8\}$ of eight different topological perturbations, while $\bar{\mu}_\tau$ and $\bar{\mu}_\alpha$ are calculated from an ensemble of $E_\alpha = \{{}_e s_0 \mid e = 1, \ldots, 8\}$ of eight different initial conditions. Hereafter, ${}_e\lambda_\tau$ denotes the MTLE found for the e-th member of the ensemble E_τ. Figure 1 depicts the exemplary Voronoi tessellation used throughout this paper, on which an exemplary initial condition is superimposed.

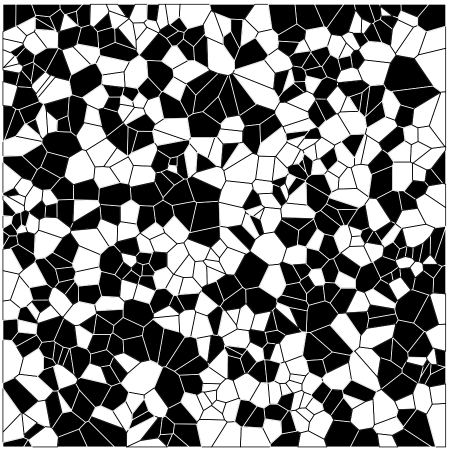

Fig. 1. Exemplary Voronoi tessellation used throughout this paper on which an exemplary initial condition is superimposed where cells with states zero and one, are coloured white and black, respectively

4.2 Simulation Results

In order to verify whether the numerically obtained MTLE converges to a steady value as the number of time steps upon which its assessment is based increases, Fig. 2 depicts the MTLE versus the number of time steps upon which the evaluation of Eq. (2) is based for rules 49, 65 and 83. The plots in this figure clearly illustrate that the MTLE steadily converges to a limit value as t becomes increasingly large. Moreover, it shows that the limit value is approached already quite closely after not more than 200 time steps, which indicates that the convergence is not caused by saturation effects that might come into play due to the finiteness of the underlying Voronoi tessellation and the predefined number of time steps. The consistency of the MTLE over E_τ's elements is supported by the small standard deviation σ_{λ_τ} observed on λ_τ for all but a few totalistic rules (Fig. 3(a)). Similar observations apply to $\bar{\mu}_\tau$ (Fig. 3(b)). More pronounced σ_{λ_τ} (*i.e.* $\sigma_{\lambda_\tau} > 0.01$) are sometimes found for rules leading to $\lambda_\tau = -\infty$ for several members of the ensemble E_τ.

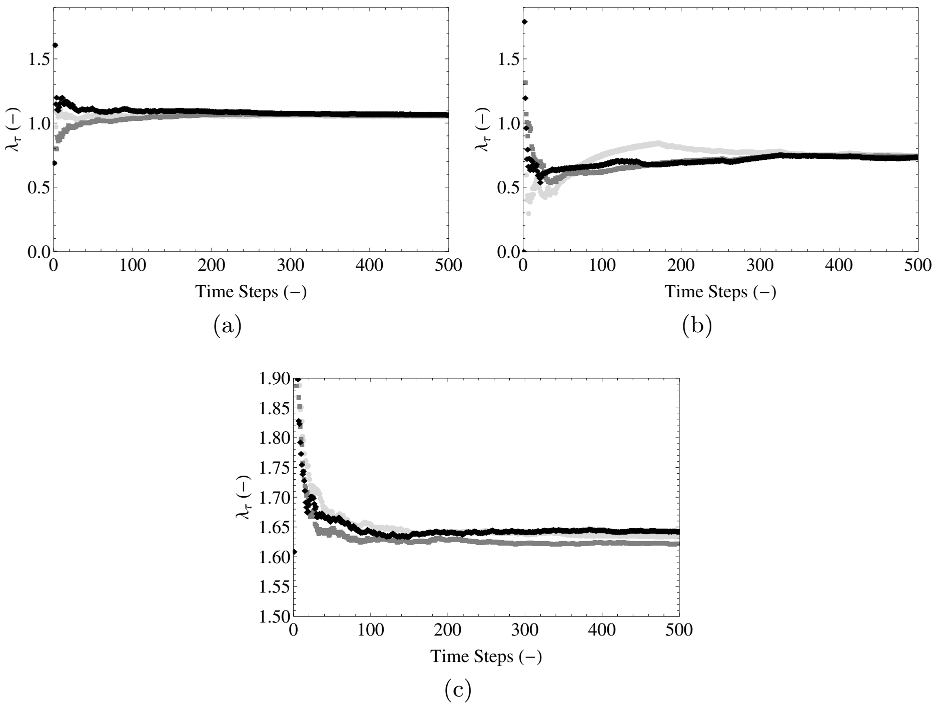

Fig. 2. MTLE obtained for three members of the ensemble of topological perturbations $E_\tau = \{_e s_0^* \mid e = 1, \ldots, 8\}$ versus the number of time steps for rules (a) 49, (b) 65 and (c) 83

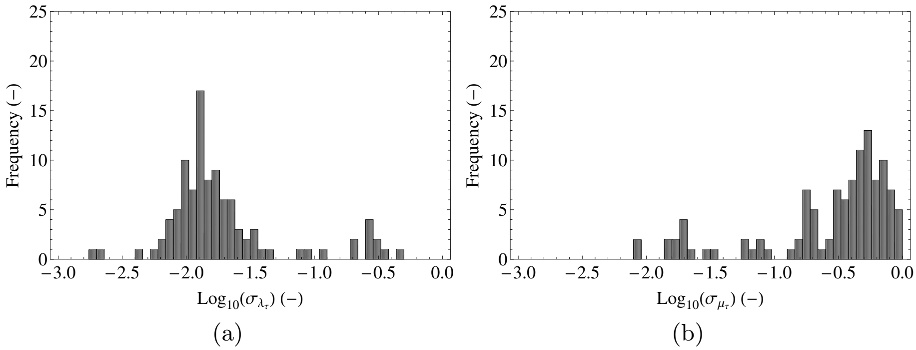

Fig. 3. Frequency distribution of the standard deviation (base 10 logarithm) of (a) λ_τ (σ_{λ_τ}), and (b) $\bar{\mu}_\tau$ ($\sigma_{\bar{\mu}_\tau}$) calculated over the ensemble of topological perturbations $E_\tau = \{{}_e s_0^* \mid e = 1, \ldots, 8\}$ and an ensemble of initial conditions $E_\alpha = \{{}_e s_0 \mid e = 1, \ldots, 8\}$, respectively

It can be understood that the MTLE depends on both $\bar{\mu}_\tau$ and $\bar{\mu}_\alpha$ as the former gives a means to express the probability with which defects are introduced during consecutive time steps of the CA's evolution due to a topological perturbation, whereas the latter indicates how easy these newly emerged defects can propagate throughout the tessellation or graph at stake, Figure 4 depicts the MTLE versus the geometric mean of the proportion of non-zero entries in both J^α and J^τ of the $(2,6)$ totalistic CAs for which $\lambda_\tau \neq -\infty$ for at least one member of the ensemble E_τ. Rules giving rise to $\lambda_\tau = -\infty$ for at least one member of the ensemble E_τ are indicated with cuboid markers. This figure clearly shows that the topological sensitivity of $(2,6)$ totalistic CAs is more pronounced as either $\bar{\mu}_\tau$ or $\bar{\mu}_\alpha$ increases, and λ_τ is highest if both Jacobian-based constructs approach one, which is to be expected since the rate with which defects are introduced due to a topological perturbation as well as the rate with which such defects propagate is maximal if $\bar{\mu}_\tau$ and $\bar{\mu}_\alpha$ approach 1. Similar findings have been made recently with regard to the topological sensitivity of elementary CAs and $(2,7)$ totalistic CAs [3]. All together, 20 rules within the family of $(2,6)$ totalistic CAs give rise to ${}_e\lambda_\tau = -\infty$ for all members of the ensemble E_τ and 31 other rules lead to ${}_e\lambda_\tau = -\infty$ for at least one and at most all but one member of E_τ. Consequently, among the investigated rules, 77 rules are topologically sensitive irrespective of the imposed topological perturbation, which entails that significant discrepancies will arise between the evolved spatio-temporal patterns if their evolution is based upon a different topology. Consequently, our findings support the claim of earlier preliminary studies, in that topological perturbations might have a severe impact on the dynamics, and hence the stability, of CAs [3,10,13,14].

Further, this figure indicates that the majority of the depicted data points seemingly lies on a smooth trend surface such that it is to be expected that there exists an upper bound on λ_τ that is determined by $\bar{\mu}_\tau$ and $\bar{\mu}_\alpha$, which is in line with the existence of an upper bound on the classical Lyapunov exponent of CAs that depends on $\bar{\mu}_\alpha$ [4]. For comprehensiveness, it should be remarked that the data points which deviate from the overall trend are largely located in the region where both $\bar{\mu}_\tau$ and $\bar{\mu}_\alpha$ are relatively low, and often originate from rules that either give rise to ${}_e\lambda_\tau = -\infty$ for some members of the ensemble E_τ or for which the standard deviation among ${}_e\lambda_\tau$ is relatively high as opposed to the data points from which a trend can be inferred.

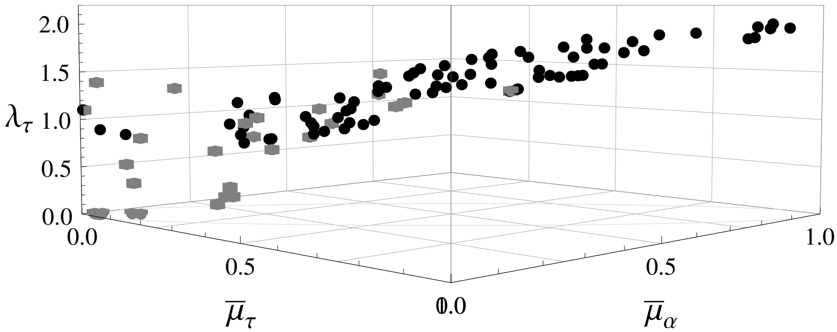

Fig. 4. Maximum topological Lyapunov exponent (λ_τ) versus the geometric mean of the proportion of non-zero entries in both J_α ($\bar{\mu}_\alpha$) and J_τ ($\bar{\mu}_\tau$) after 500 time steps, starting from a random initial condition. Results are averages calculated over an ensemble of different topological perturbations $E_\tau = \{{}_eG^* \mid e = 1, \ldots, 8\}$ for λ_τ and over an ensemble of initial conditions $E_\alpha = \{{}_es_0 \mid e = 1, \ldots, 8\}$ for both $\bar{\mu}_\alpha$ and $\bar{\mu}_\tau$, and are only shown for those $(2, 6)$ irregular totalistic CAs for which $\lambda_\tau \neq -\infty$ for all members of E_τ (spheres), and for rules giving rise to $\lambda_\tau = -\infty$ for at most all but one member of E_τ (cuboids)

Actually, a closer inspection of Fig. 4 reveals that there exists an ellipsoidal region in the $\bar{\mu}_\alpha\,\bar{\mu}_\tau$ plane, which encloses all topologically sensitive rules within the CA family at stake. Hence, only rules for which $(\bar{\mu}_\alpha, \bar{\mu}_\tau)$ are located in this region might be sensitive to topological perturbations and, as a consequence of the ellipsoidal shape of the concerned region, a lower $\bar{\mu}_\tau$ may to some extent be compensated by a higher $\bar{\mu}_\alpha$ so that the CA can still evolve diverging phase space trajectories, and vice versa.

5 Conclusions and Further Work

Following the formulation of Lyapunov exponents for assessing the sensitivity of CAs to the initial condition, we defined topological Lyapunov exponents for quantifying the topological sensitivity of CAs, which, to this day, deserved only minor attention notwithstanding the dynamics of a CA is explicitly determined by the interconnection between its cells. Further, these topological Lyapunov exponents were compared with a quantity expressing the sensitivity of a CA's transition function to topological perturbations. Simulations suggest that the topological Lyapunov exponent is upper bounded somehow by both this construct and a construct quantifying the sensitivity of a CA's transition function to its inputs, but further research should aim at retrieving the functional relationship defining this upper bound.

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