Chapter 4
Asymptotic Methods in Statistics of Random Point Processes

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Abstract First we put together basic definitions and fundamental facts and results from the theory of (un)marked point processes defined on Euclidean spaces $\mathbb{R}^d$. We introduce the notion *random marked point process* together with the concept of Palm distributions in a rigorous way followed by the definitions of factorial moment and cumulant measures and characteristics related with them. In the second part we define a variety of estimators of second-order characteristics and other so-called *summary statistics* of stationary point processes based on observations on a "convex averaging sequence" of windows $\{W_n, n \in \mathbb{N}\}$. Although all these (mostly edge-corrected) estimators make sense for fixed bounded windows our main issue is to study their behaviour when W_n grows unboundedly as $n \to \infty$. The first problem of *large-domain statistics* is to find conditions ensuring strong or at least mean-square consistency as $n \to \infty$ under ergodicity or other mild mixing conditions put on the underlying point process. The third part contains weak convergence results obtained by exhausting strong mixing conditions or even m-dependence of spatial random fields generated by Poisson-based point processes. To illustrate the usefulness of asymptotic methods we give two Kolmogorov–Smirnov-type tests based on K-functions to check *complete spatial randomness* of a given point pattern in $\mathbb{R}^d$.

4.1 Marked Point Processes: An Introduction

First we present a rigorous definition of the marked point process on Euclidean spaces with marks in some Polish space and formulate an existence theorem for marked point processes based on their finite-dimensional distributions. Further, all essential notions and tools of point process theory such as factorial moment

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and cumulant measures with their densities and reduced versions as well as the machinery of Palm distributions in the marked and unmarked case are considered in detail.

4.1.1 Marked Point Processes: Definitions and Basic Facts

Point processes are mathematical models for irregular point patterns formed by randomly scattered points in some locally compact Hausdorff space. Throughout this chapter, this space will be the Euclidean space $\mathbb{R}^d$ of dimension $d \in \mathbb{N}$. In many applications to each point X_i of the pattern a further random element M_i, called *mark*, can be assigned which carries additional information and may take values in a rather general *mark space* M equipped with an appropriate σ-algebra $\mathcal{M}$. For example, for $d = 1$, the X_i's could be arrival times of customers and the M_i's their sojourn times in a queueing system and, for $d = 2$, one can interpret the X_i's as locations of trees in a forest with the associated random vectors M_i of stem diameter, stem height and distance to the nearest-neighbour tree. In this way we are led to the notion of a (random) *marked point process* which can be adequately modeled as random counting measure $\Psi_{\mathsf{M}}(\cdot)$ on Cartesian products $B \times L$, which gives the total number of points in a bounded subset B of $\mathbb{R}^d$ whose marks belong to a set of marks $L \subseteq \mathsf{M}$. To be precise, we need some further notation. Let $\mathcal{N}_{\mathsf{M}}$ denote the set of *locally finite counting measures* $\psi(\cdot)$ on the measurable product space $(\mathbb{R}^d \times \mathsf{M}, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{M})$, i.e. $\psi \in \mathcal{N}_{\mathsf{M}}$ is a σ-additive set function on $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{M}$ taking on non-negative integer values such that $\psi(B \times \mathsf{M}) < \infty$ for any bounded Borel set $B \in \mathcal{B}(\mathbb{R}^d)$. We then define $\mathfrak{N}_{\mathsf{M}}$ to be the smallest σ-algebra containing all sets $\{\psi \in \mathcal{N}_{\mathsf{M}} : \psi(B \times L) = n\}$ for $n \in \mathbb{N} \cup \{0\}$, any bounded $B \in \mathcal{B}(\mathbb{R}^d)$ and $L \in \mathcal{M}$. Finally, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a hypothetical probability space on which all subsequent random elements will be defined.

Definition 4.1. A $(\mathcal{F}, \mathfrak{N}_{\mathsf{M}})$-measurable mapping

$$\Psi_{\mathsf{M}} \mid (\Omega, \mathcal{F}, \mathbf{P}) \mapsto (\mathcal{N}_{\mathsf{M}}, \mathfrak{N}_{\mathsf{M}}), \qquad \Omega \ni \omega \mapsto \Psi_{\mathsf{M}}(\omega, \cdot) \in \mathcal{N}_{\mathsf{M}}$$

is said to be a (random) marked point process (briefly: MPP) on $\mathbb{R}^d$ with mark space $(\mathsf{M}, \mathcal{M})$. In other words, a MPP $\Psi_{\mathsf{M}}(\cdot)$ (ω will be mostly suppressed) is a random locally finite counting measure on $(\mathbb{R}^d \times \mathsf{M}, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{M})$.

The probability measure $P_{\mathsf{M}}(A) = \mathbf{P}(\{\omega \in \Omega : \Psi_{\mathsf{M}}(\omega, \cdot) \in A\})$ for $A \in \mathfrak{N}_{\mathsf{M}}$ induced on $(\mathcal{N}_{\mathsf{M}}, \mathfrak{N}_{\mathsf{M}})$ is called the *distribution* of Ψ_{M}—briefly expressed by $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$. Here and in what follows we put $\Psi(\cdot) = \Psi_{\mathsf{M}}(\cdot \times \mathsf{M})$ to denote the corresponding unmarked point process and write in general $\Psi \sim P$ to indicate point processes without marks. One often uses the notation

$$\Psi_{\mathsf{M}} = \sum_{i \geq 1} \delta_{(X_i, M_i)} \text{ or } \Psi = \sum_{i \geq 1} \delta_{X_i}\,, \tag{4.1}$$

where $\delta_x(A) = 1$ for $x \in A$ and $\delta_x(A) = 0$ for $x \notin A$ (Dirac measure). Note that due to the local finiteness of Ψ_{M} there are at most countably many atoms but each atom occurs with random (**P**-a.s. finite) multiplicity. The indexing in (4.1) does not need to be unique and the X_i's occur in the sums according to their multiplicity. In accordance with the general theory of random processes our next result formulates an analogue to Kolmogorov's extension theorem stating the existence of a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ in Definition 4.1 in case of Polish mark spaces M given the family of *finite-dimensional distributions* $\mathbf{P}(\Psi_{\mathsf{M}}(B_1 \times L_1) = n_1, \ldots, \Psi_{\mathsf{M}}(B_k \times L_k) = n_k)$.

Theorem 4.1. *Let* M *be a Polish space equipped with the corresponding Borel-*σ*-algebra* $\mathcal{B}(\mathsf{M})$ *generated by a family* $\mathcal{M}_0$ *of subsets in* M *such that* $\mathcal{S} = \left\{ \overset{d}{\underset{i=1}{\times}} [a_i, b_i) \times L : -\infty < a_i \le b_i < \infty,\ L \in \mathcal{M}_0 \right\}$ *is a semi-ring which generates the ring of bounded Borel sets in* $\mathbb{R}^d \times \mathsf{M}$ *and each bounded Borel set* $X \subset \mathbb{R}^d \times \mathsf{M}$ *can be covered by a finite sequence* $S_1, \ldots, S_m \in \mathcal{S}$.

For any finite sequence of pairwise disjoint $B_1 \times L_1, \ldots, B_k \times L_k \in \mathcal{S}$ *define the distribution*

$$p_{n_1,\ldots,n_k}(B_1 \times L_1, \ldots, B_k \times L_k) \text{ for } n_1, \ldots, n_k = 0, 1, 2, \ldots$$

of a k*-dimensional random vector with non-negative integer-valued components.*

Then there exists a unique probability measure P_{M} *on the measurable space* $(\mathcal{N}_{\mathsf{M}}, \mathfrak{N}_{\mathsf{M}})$ *with finite-dimensional distributions*

$$P_{\mathsf{M}}(\{\psi \in \mathcal{N}_{\mathsf{M}} : \psi(B_j \times L_j) = n_j,\ j = 1, \ldots, k\}) = p_{n_1,\ldots,n_k}(B_1 \times L_1, \ldots, B_k \times L_k)$$

for $n_1, \ldots, n_k \in \mathbb{N} \cup \{0\}$ *and any* $k \in \mathbb{N}$, *if the following conditions for the family of probabilities* $p_{n_1,\ldots,n_k}(B_1 \times L_1, \ldots, B_k \times L_k)$ *are satisfied:*

1. *Symmetry:*

$$p_{n_1,\ldots,n_k}(B_1 \times L_1, \ldots, B_k \times L_k) = p_{n_{\pi(1)},\ldots,n_{\pi(k)}}(B_{\pi(1)} \times L_{\pi(1)}, \ldots, B_{\pi(k)} \times L_{\pi(k)})$$

for any permutation $\pi : \{1, \ldots, k\} \mapsto \{1, \ldots, k\}$,

2. *Consistency:*

$$\sum_{n=0}^{\infty} p_{n_1,\ldots,n_{k-1},n}(B_1 \times L_1, \ldots, B_{k-1} \times L_{k-1}, B_k \times L_k) = p_{n_1,\ldots,n_{k-1}}(B_1 \times L_1, \ldots, B_{k-1} \times L_{k-1}),$$

3. *Additivity: If* $B_j \times L_j \cup \cdots \cup B_k \times L_k \in \mathcal{S}$, *then*

$$\sum_{\substack{n_j+\cdots+n_k=n\\ n_j,\ldots,n_k\geq 0}} p_{n_1,\ldots n_{j-1},n_j,\ldots,n_k}(B_1\times L_1,\ldots,B_{j-1}\times L_{j-1},B_j\times L_j,\ldots,B_k\times L_k)$$

$$= p_{n_1,\ldots,n_{j-1},n}(B_1\times L_1,\ldots,B_{j-1}\times L_{j-1},(B_j\times L_j\cup\cdots\cup B_k\times L_k))$$

for $j=1,\ldots,k$.

4. *Continuity: For any sequence of pairwise disjoint sets* $B_j^{(n)}\times L_j^{(n)}\in\mathcal{S}$ *for* $j=1,\ldots,k_n$ *with* $k_n\uparrow\infty$ *satisfying* $\bigcup_{j=1}^{k_n}(B_j^{(n)}\times L_j^{(n)})\downarrow\emptyset$, *it holds* $p_{0,\ldots,0}(B_1^{(n)}\times L_1^{(n)},\ldots,B_{k_n}^{(n)}\times L_{k_n}^{(n)})\uparrow 1$ *as* $n\to\infty$.
If $\mathsf{M}=\{m\}$ *consists of a single (or at most finitely many) element(s), then the latter condition can be replaced by*

$$\lim_{n\to\infty} p_0(\overset{j-1}{\underset{i=1}{\times}}[a_i,b_i)\times[x_j-\frac{1}{n},x_j)\times\overset{d}{\underset{i=j+1}{\times}}[a_i,b_i)\times\{m\})=1$$

$$\text{for all } (x_1,\ldots,x_d)^\top\in\mathbb{R}^d \text{ and } -\infty<a_i<b_i<\infty,\ i=1,\ldots,d\,.$$

This allows a canonical choice of the probability space $(\Omega,\mathcal{F},\mathbf{P})$ mapping Ψ_{M}:

$$\Omega:=\mathcal{N}_{\mathsf{M}},\quad \mathcal{F}:=\mathfrak{N}_{\mathsf{M}},\quad \mathbf{P}:=P_{\mathsf{M}},\quad \psi\mapsto\Psi_{\mathsf{M}}(\psi,\cdot):=\psi(\cdot)\ \text{(identical mapping)}.$$

Remark 4.1. There exists a metric τ in the set $\mathcal{N}_{\mathsf{M}}$ such that the metric space $(\mathcal{N}_{\mathsf{M}},\tau)$ is separable and complete whose Borel-σ-algebra $\mathcal{B}(\mathcal{N}_{\mathsf{M}})$ coincides with $\mathfrak{N}_{\mathsf{M}}$. This allows to introduce the notion of *weak convergence* of MPP's in the usual way, see [69].

To prove Theorem 4.1 one has only to reformulate the proof of a corresponding result for (unmarked) PP's on Polish spaces in [346]. Readers interested in more background of and a rigouros introduction to the theory of marked and unmarked PP's are referred to the two-volume monograph [140]. Less technical approaches combined with statistics of point processes are presented in [265, 489] and in the survey papers [22, 488]. In the following we reduce the rigour and in particular measurability questions will be not considered.

An advantage of the counting measure approach to point processes (in contrast to modelling with discrete random closed sets) consists in catching random multiplicities of the point or atoms which is important in several fields of application. For example, the description of batch arrivals in queueing systems or of end-points of edges in planar random tessellations requires multiplicities of points. On the other hand, for quite a few point processes in particular for the upmost occurring in stochastic geometry it is natural to assume that at most one point occurs in any $x\in\mathbb{R}^d$, more precisely

$$P_{\mathsf{M}}(\{\psi\in\mathcal{N}_{\mathsf{M}}:\psi(\{x\}\times\mathsf{M})\leq 1,\ \forall\, x\in\mathbb{R}^d\})=1\,.$$

MPP's $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ satisfying the later condition are called *simple*. In view of (4.1) we get $\mathrm{supp}(\Psi_{\mathsf{M}}) = \{(X_i, M_i) : i \in \mathbb{N}\}$ for the support set of a simple MPP Ψ_{M} which motivates the somewhat loose writing $\Psi_{\mathsf{M}} = \{(X_i, M_i) : i \in \mathbb{N}\}$. By the **P**-a.s. local boundedness of the counting measure Ψ_{M} its support set has no accumulation point in $\mathbb{R}^d$ and can therefore be considered as a discrete *random closed set*. The characterization of distributions of random closed sets Ξ by the family of hitting probabilities $\mathbf{P}(\Xi \cap K \neq \emptyset)$ for any compact set $K \subset \mathbb{R}^d$ leads to the following pendant to Theorem 4.1:

Theorem 4.2. *Let $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ be a simple MPP on $\mathbb{R}^d$ with Polish mark space M. Then the distribution P_{M} is completely determined by the void probabilities*

$$\mathbf{P}(\Psi_{\mathsf{M}}(X) = 0) = P_{\mathsf{M}}(\{\, \psi \in \mathcal{N}_{\mathsf{M}} : \psi(X) = 0 \,\})$$

for all compact $X \subset \mathbb{R}^d \times \mathsf{M}$.

For the sake of simplicity we shall consider only simple MPP's after the introductory Sect. 4.1.1.

The simplest numerical characteristic of a MPP $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ describes the mean number of points in bounded sets $B \in \mathcal{B}(\mathbb{R}^d)$ having marks in an arbitrary set $L \in \mathcal{B}(\mathsf{M})$. In this way we obtain the *intensity measure* Λ_{M} (on $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathsf{M})$) defined by

$$\Lambda_{\mathsf{M}}(B \times L) = \mathbf{E}\Big(\sum_{i \geq 1} \mathbf{1}(\,(X_i, M_i) \in B \times L\,)\Big) = \int\limits_{\mathcal{N}_{\mathsf{M}}} \psi(B \times L)\, P_{\mathsf{M}}(d\psi) \quad (4.2)$$

provided that $\Lambda(B) := \Lambda_{\mathsf{M}}(B \times \mathsf{M}) < \infty$ for any bounded $B \in \mathcal{B}(\mathbb{R}^d)$ expressing the local finiteness of Λ_{M}. By Theorem 4.1 we are now in a position to define the *marked Poisson process* $\Psi_{\mathsf{M}} \sim \Pi_{\Lambda_{\mathsf{M}}}$ with a given locally finite intensity measure Λ_{M}:

Definition 4.2. A marked Poisson process $\Psi_{\mathsf{M}} \sim \Pi_{\Lambda_{\mathsf{M}}}$ (more precisely its distribution) is completely determined by the following two conditions:

1. $\Psi_{\mathsf{M}}(B_1 \times L_1), \ldots, \Psi_{\mathsf{M}}(B_k \times L_k)$ are mutually independent random variables for pairwise disjoint $B_j \times L_j \in \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathsf{M})$ with bounded B_j for $j = 1, \ldots, k$ and $k \in \mathbb{N}$.
2. $\Psi_{\mathsf{M}}(B \times L)$ is Poisson distributed with mean $\Lambda_{\mathsf{M}}(B \times L)$ for any $B \times L \in \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathsf{M})$ with bounded B.

Remark 4.2. Since $\Lambda_{\mathsf{M}}(B \times L) \leq \Lambda(B)$ there exists (by the Radon–Nikodym theorem and disintegration arguments) a family $\{Q^x_{\mathsf{M}}, x \in \mathbb{R}^d\}$ of (regular) conditional distributions on $(\mathsf{M}, \mathcal{B}(\mathsf{M}))$ such that $\Lambda_{\mathsf{M}}(B \times L) = \int_{\mathbb{R}^d} Q^x_{\mathsf{M}}(L)\, \Lambda(dx)$, which justifies the interpretation $Q^x_{\mathsf{M}}(L) = \mathbf{P}(\, M_i \in L \mid X_i = x\,)$. It turns out that $\Psi_{\mathsf{M}} \sim \Pi_{\Lambda_{\mathsf{M}}}$ can be obtained from an *unmarked Poisson process* $\Psi \sim \Pi_{\Lambda}$ with *intensity measure* Λ by *location-dependent, independent marking*, that is, to

each atom X_i of Ψ located in $x \in \mathbb{R}^d$ the mark M_i is assigned according to the probability law Q^x_{M} independent of the location of all other atoms of Ψ and also independent of all other marks even those assigned to further atoms located in x (if $\Psi(\{x\}) > 1$).

Remark 4.3. A marked Poisson process $\Psi_{\mathsf{M}} \sim \Pi_{\Lambda_{\mathsf{M}}}$ is simple iff the intensity measure $\Lambda(\cdot)$ is *diffuse*, i.e. $\Lambda_{\mathsf{M}}(\{x\} \times \mathsf{M}) = 0$ for all $x \in \mathbb{R}^d$, see [346].

Remark 4.4. The conditions (1) and (2) in the above definition of $\Psi_{\mathsf{M}} \sim \Pi_{\Lambda_{\mathsf{M}}}$ can be expressed equivalently by means of the characteristic function of the random vector $(\Psi_{\mathsf{M}}(B_1 \times L_1), \ldots, \Psi_{\mathsf{M}}(B_k \times L_k))$ as follows: For any $k \in \mathbb{N}$, any pairwise disjoint $B_1 \times L_1, \ldots, B_k \times L_k \in \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathsf{M})$ and all $u_1, \ldots, u_k \in \mathbb{R}^d$

$$\mathbf{E} \exp\{\, i \sum_{j=1}^{k} u_j \, \Psi_{\mathsf{M}}(B_j \times L_j)\,\} = \prod_{j=1}^{k} \exp\{\Lambda_{\mathsf{M}}(B_j \times L_j)\,(\mathrm{e}^{i\,u_j} - 1)\}\,. \tag{4.3}$$

Next, we give an elementary explicit construction of an unmarked Poisson process $\Psi \sim \Pi_\Lambda$ with locally finite intensity measure $\Lambda(\cdot)$, see [102]. This construction is also the basis for simulations of Poisson processes in bounded Borel sets.

Let $\{K_m,\ m \in \mathbb{N}\}$ be a partition of $\mathbb{R}^d$ into bounded Borel sets. Consider an array of independent random variables $\{\tau_m, X_{mj}\}_{m,j\in\mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$ such that τ_m and X_{mj} take values in $\mathbb{Z}_+$ and $\mathbb{R}^d$ respectively, namely,

1. τ_m is Poisson distributed with mean $\Lambda(K_m)$ (briefly $\tau_m \sim \mathrm{Pois}(\Lambda(K_m))$).
2. $\mathbf{P}(X_{mj} \in C) = \begin{cases} \Lambda(C \cap K_m)/\Lambda(K_m), & \Lambda(K_m) \neq 0, \\ 0, & \text{otherwise} \end{cases}$ for any $C \in \mathcal{B}(\mathbb{R}^d)$.

Here $Y \sim \mathrm{Pois}(0)$ means that $Y \equiv 0$. Clearly, $\mathbf{P}(X_{mj} \in C) = \mathbf{P}(X_{mj} \in C \cap K_m)$ for $m, j \in \mathbb{N}$. Note that the random variables X_{mj} are uniformly distributed on K_m if $\Lambda(K_m)$ is a multiple of $\nu_d(K_m)$.

For any $B \in \mathcal{B}(\mathbb{R}^d)$ and $m \in \mathbb{N}$ put

$$\Psi_m(B) := \sum_{j=1}^{\tau_m} \mathbf{1}(X_{mj} \in B) \quad \text{and} \quad \Psi(B) := \sum_{m=1}^{\infty} \Psi_m(B)\,. \tag{4.4}$$

Obviously, $\Psi_m(B)$ is a random variable for each $m \in \mathbb{N}$ and any $B \in \mathcal{B}(\mathbb{R}^d)$ such that

$$\mathbf{P}(\,0 \le \Psi_m(B) \le \tau_m < \infty\,) = 1 \quad \text{for any} \quad B \in \mathcal{B}(\mathbb{R}^d)\,,\ m \in \mathbb{N}\,.$$

Moreover, it turns out that $\Psi(\cdot)$ is a locally finite random counting measure.

Theorem 4.3. *Let Λ be a locally finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. For any partition of $\mathbb{R}^d$ into bounded Borel sets K_m, $m \in \mathbb{N}$, the family of non-negative,*

integer-valued random variables $\{\Psi(B), B \in \mathcal{B}(\mathbb{R}^d)\}$ *introduced in* (4.4) *defines an unmarked Poisson process with intensity measure* $\Lambda(\cdot)$.

Proof. For any $m, k \in \mathbb{N}$ consider pairwise disjoint Borel sets $B_1, \ldots, B_k \subset K_m$. Then $\Psi(B_r) = \Psi_m(B_r)$ for $r = 1, \ldots, k$ and it follows that

$$\begin{aligned}
&\mathbf{E}\exp\{i\,(u_1\,\Psi_m(B_1) + \ldots + u_k\,\Psi_m(B_k))\} \\
&= \sum_{n=0}^{\infty} \mathbf{E}\Big(\exp\Big\{i\,\sum_{j=1}^{n}\big(u_1\,\mathbf{1}(X_{mj} \in B_1) + \ldots + u_k\,\mathbf{1}(X_{mj} \in B_k)\big)\Big\}\mathbf{1}(\tau_m = n)\Big) \\
&= \sum_{n=0}^{\infty} \varphi_n(u_1, \ldots, u_k)\frac{\Lambda(K_m)^n}{n!}\mathrm{e}^{-\Lambda(K_m)}, \qquad (4.5)
\end{aligned}$$

where $u_1, \ldots, u_k \in \mathbb{R}^1$ and $\varphi_n(u_1, \ldots, u_k)$ is the characteristic function of the random vector $Y_n = (Y_{n1}, \ldots, Y_{nk})$ with components $Y_{nr} = \sum_{j=1}^{n} \mathbf{1}(X_{mj} \in B_r)$ for $r = 1, \ldots, k$.

By setting $Y_{n0} = \sum_{j=1}^{n} \mathbf{1}(X_{mj} \in B_0)$ for $B_0 = K_m \setminus \cup_{r=1}^{n} B_r$ we get a multinomially distributed random vector $(Y_{n0}, Y_{n1}, \ldots, Y_{nk})$ with success probabilities $p_r = \mathbf{P}(X_{mj} \in B_r) = \Lambda(B_r)/\Lambda(K_m)$, $r = 0, 1, \ldots, k$, i.e.,

$$\mathbf{P}(Y_{n0} = l_0, Y_{n1} = l_1, \ldots, Y_{nk} = l_k) = \frac{n!}{l_0!\,l_1!\ldots l_k!}\,p_0^{l_0}\,p_1^{l_1}\ldots p_k^{l_k}$$

for $l_0, l_1, \ldots, l_k \geq 0$ and $\sum_{r=0}^{k} l_r = n$. Hence, $\zeta_n(u_0, u_1, \ldots, u_k) = (\,p_0\,\mathrm{e}^{i\,u_0} + p_1\,\mathrm{e}^{i\,u_1} + \ldots + p_k\,\mathrm{e}^{i\,u_k}\,)^n, u_0, \ldots, u_k \in \mathbb{R}^1$ is the characteristic function of $(Y_{n0}, Y_{n1}, \ldots, Y_{nk})$ and we obtain

$$\varphi_n(u_1, \ldots, u_k) = \zeta_n(0, u_1, \ldots, u_k) = (\,p_0 + p_1\,\mathrm{e}^{i\,u_1} + \ldots + p_k\,\mathrm{e}^{i\,u_k}\,)^n.$$

Using (4.5) and the latter relation together with $p_0 = 1 - p_1 - \ldots - p_k$ we may write

$$\begin{aligned}
\mathbf{E}\exp\{i\,\sum_{r=1}^{k} u_r\,\Psi_m(B_r)\} &= \mathrm{e}^{-\Lambda(K_m)}\sum_{n=0}^{\infty}\frac{1}{n!}\,[\,\Lambda(K_m)(p_0 + p_1\,\mathrm{e}^{i\,u_1} + \ldots + p_k\,\mathrm{e}^{i\,u_k})\,]^n \\
&= \exp\{\Lambda(K_m)(p_1(\mathrm{e}^{i\,u_1} - 1) + \ldots + p_k(\mathrm{e}^{i\,u_k} - 1))\} \\
&= \prod_{r=1}^{k}\exp\{\Lambda(B_r)(\mathrm{e}^{i\,u_r} - 1)\}\,.
\end{aligned}$$

This, however, is just (4.3) (rewritten for the unmarked case) for pairwise disjoint Borel sets $B_1, \ldots, B_k \subset K_m$.

Now, consider pairwise disjoint bounded $B_1, \dots, B_k \in \mathcal{B}(\mathbb{R}^d)$ for $k \in \mathbb{N}$ so that

$$(\Psi(B_1), \dots, \Psi(B_k)) = \sum_{m=1}^{\infty} (\Psi_m(B_1 \cap K_m), \dots, \Psi_m(B_k \cap K_m)) .$$

Obviously, $(\Psi_m(B_1 \cap K_m), \dots, \Psi_m(B_k \cap K_m))_{m\in\mathbb{N}}$ forms a sequence of independent random vectors with independent components as we have proved above. Since $\Psi(B_r) = \sum_{m=1}^{\infty} \Psi_m(B_r \cap K_m)$ and the summands are nonnegative, it follows that

$$\mathbf{E}\Psi(B_r) = \sum_{m=1}^{\infty} \mathbf{E}\Psi_m(B_r \cap K_m) = \sum_{m=1}^{\infty} \Lambda(B_r \cap K_m) = \Lambda(B_r) < \infty$$

implying $\mathbf{P}(\, \Psi(B_r) < \infty \,) = 1$ for $r = 1, \dots, k$, and

$$\Big(\sum_{m=1}^{n} \Psi_m(B_1 \cap K_m), \dots, \sum_{m=1}^{n} \Psi_m(B_k \cap K_m) \Big) \overset{\mathbf{P}-\text{a.s.}}{\underset{n\to\infty}{\longrightarrow}} (\Psi(B_1), \dots, \Psi(B_k)) .$$

Since, for each fixed $n \in \mathbb{N}$, the sums at the left-hand side of the latter relation are mutually independent, it follows that the limits $\Psi(B_1), \dots, \Psi(B_k)$ are independent as well. Finally, in view of the fact that

$$\sum_{m=1}^{n} \Psi_m(B_r \cap K_m) \sim \mathrm{Pois}\Big(\sum_{m=1}^{n} \Lambda(B_r \cap K_m) \Big)$$

we conclude that $\Psi(B_r) \sim \mathrm{Pois}(\Lambda(B_r))$ for $r = 1, \dots, k$, which completes the proof of Theorem 4.3. □

Next, we introduce the translation- and rotation operator T_x resp. $R_O \mid \mathcal{N}_{\mathsf{M}} \mapsto \mathcal{N}_{\mathsf{M}}$ by

$$(T_x\psi)(\, B \times L \,) := \psi(\, (B + x) \times L \,) \quad \text{resp.} \quad (R_O\psi)(\, B \times L \,) := \psi(\, O\, B \times L \,)$$

for $\psi \in \mathcal{N}_{\mathsf{M}}$, $x \in \mathbb{R}^d$ and $O \in \mathrm{SO}_d$ = group of orthogonal $d \times d$-matrices with determinant equal to 1. Then the MPP $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ is said to be (strictly) *stationary* or *homogeneous* if, for all $x \in \mathbb{R}^d$ and bounded Borel sets $B_1, \dots, B_k$,

$$T_x\, \Psi_{\mathsf{M}} \overset{d}{=} \Psi_{\mathsf{M}} \quad \Longleftrightarrow \quad (\Psi_{\mathsf{M}}((B_j + x) \times L_j))_{j=1}^{k} \overset{d}{=} (\Psi_{\mathsf{M}}(B_j \times L_j))_{j=1}^{k} \,,\ k \in \mathbb{N},$$

and *isotropic* if, for all $O \in \mathrm{SO}_d$ and bounded Borel sets $B_1, \dots, B_k$,

$$R_O\, \Psi_{\mathsf{M}} \overset{d}{=} \Psi_{\mathsf{M}} \quad \Longleftrightarrow \quad (\Psi_{\mathsf{M}}(OB_j \times L_j))_{j=1}^{k} \overset{d}{=} (\Psi_{\mathsf{M}}(B_j \times L_j))_{j=1}^{k} \,,\ k \in \mathbb{N}.$$

A MPP is said to be *motion-invariant* if it is both stationary and isotropic.

The stationarity of $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ implies the shift-invariance of the locally finite measure $\Lambda_{\mathsf{M}}((\cdot) \times L)$ for any fixed $L \in \mathcal{B}(\mathsf{M})$ provided the *intensity*

$$\lambda := \mathbf{E}\Psi_{\mathsf{M}}([0,1]^d \times \mathsf{M})$$

of the unmarked point process $\Psi \sim P$ exists. This entails that the intensity measure $\Lambda_{\mathsf{M}}(B \times L)$ is a multiple of the Lebesgue-measure $v_d(B)$ which can be rewritten as

$$\Lambda_{\mathsf{M}}(B \times L) = \lambda\, v_d(B)\, Q^o_{\mathsf{M}}(L) \;\text{ for }\; B \in \mathcal{B}(\mathbb{R}^d),\; L \in \mathcal{B}(\mathsf{M})\,, \tag{4.6}$$

where Q^o_{M} is called the *distribution of the typical mark* or plainly the *mark distribution*.

4.1.2 Higher-Order Moment Measures and Palm Distributions

From now on we suppose that the MPP $\Psi_{\mathsf{M}} = \sum_{i\geq 1} \delta_{(X_i,M_i)} \sim P_{\mathsf{M}}$ is simple. Under the additional assumption $\mathbf{E}((\Psi_{\mathsf{M}}(B \times \mathsf{M}))^k) < \infty$ for some integer $k \geq 2$ and any bounded Borel set $B \subset \mathbb{R}^d$ we define the *factorial moment measure* $\alpha^{(k)}_{\mathsf{M}}$ on $\mathcal{B}((\mathbb{R}^d \times \mathsf{M})^k)$ by

$$\alpha^{(k)}_{\mathsf{M}}(\overset{k}{\underset{j=1}{\times}}(B_j \times L_j)) := \mathbf{E}\Big(\sum_{i_1,\ldots,i_k\geq 1}^{\neq} \prod_{j=1}^{k} \mathbf{1}((X_{i_j}, M_{i_j}) \in B_j \times L_j)\Big)$$

which is dominated by the kth-order factorial moment measure

$$\alpha^{(k)}(\overset{k}{\underset{j=1}{\times}} B_j) = \alpha^{(k)}_{\mathsf{M}}(\overset{k}{\underset{j=1}{\times}}(B_j \times \mathsf{M}))$$

of the unmarked simple point process $\Psi = \sum_{i\geq 1} \delta_{X_i} \sim P$. Note that the sum $\sum^{\neq}$ stretches over pairwise distinct indices indicated under the sum sign. If the sum is taken over all k-tuples of indices we get the (ordinary) kth-order moment measure.

For any fixed $L_1, \ldots, L_k \in \mathcal{B}(\mathsf{M})$ we obtain as Radon–Nikodym derivative a family of distributions $Q^{x_1,\ldots,x_k}_{\mathsf{M}}(L_1 \times \cdots \times L_k)$ (for $\alpha^{(k)}$-almost every $(x_1, \ldots, x_k)^\top \in (\mathbb{R}^d)^k$) satisfying

$$\alpha^{(k)}_{\mathsf{M}}(\overset{k}{\underset{j=1}{\times}}(B_j \times L_j)) = \int_{B_1} \cdots \int_{B_k} Q^{x_1,\ldots,x_k}_{\mathsf{M}}(\overset{k}{\underset{j=1}{\times}} L_j)\, \alpha^{(k)}(d(x_1, \ldots, x_k))\,,$$

where the integrand is interpreted as (regular) conditional distribution

$$Q^{x_1,\ldots,x_k}_{\mathsf{M}}(\overset{k}{\underset{j=1}{\times}} L_j) = \mathbf{P}(M_1 \in L_1, \ldots, M_k \in L_k \mid X_1 = x_1, \ldots, X_k = x_k)\,.$$

Definition 4.3. The stochastic kernel $Q_{\mathsf{M}}^{x_1,\dots,x_k}(A)$ (which is defined to be $\mathcal{B}(\mathbb{R}^{dk})$-measurable in $(x_1,\dots,x_k)^\top \in (\mathbb{R}^d)^k$ and a probability measure in $A \in \mathcal{B}(\mathsf{M}^k)$) is called *$k$th-order* or *$k$-point Palm mark distribution* of the MPP $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$.

If $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ is stationary then the factorial moment measures $\alpha_{\mathsf{M}}^{(k)}$ as well as $\alpha^{(k)}$ are invariant under diagonal shifts, i.e.,

$$\alpha_{\mathsf{M}}^{(k)}\big(\bigtimes_{j=1}^{k}((B_j+x)\times L_j)\big) = \alpha_{\mathsf{M}}^{(k)}\big(\bigtimes_{j=1}^{k}(B_j\times L_j)\big) \text{ for all } x\in\mathbb{R}^d,$$

which in turn implies $Q_{\mathsf{M}}^{x_1+x,\dots,x_k+x}(A) = Q_{\mathsf{M}}^{x_1,\dots,x_k}(A)$ for all $x \in \mathbb{R}^d$. By disintegration with respect to the Lebesgue-measure ν_d (see [140, Vol. II] for the more details) we can introduce so-called *reduced kth-order factorial moment measures* $\alpha_{\mathsf{M},red}^{(k)}$ and $\alpha_{red}^{(k)}$ by

$$\alpha_{\mathsf{M}}^{(k)}\big(\bigtimes_{j=1}^{k}(B_j\times L_j)\big) = \lambda\int_{B_1}\alpha_{\mathsf{M},red}^{(k)}\big(L_1\times\bigtimes_{j=2}^{k}((B_j-x)\times L_j)\big)\,dx,$$

where $\lambda > 0$ stands for the intensity of $\Psi \sim P$ which already occurred in (4.6). Putting $\alpha_{red}^{(k)}\big(\bigtimes_{j=2}^{k}B_j\big) = \alpha_{\mathsf{M},red}^{(k)}\big(\mathsf{M}\times\bigtimes_{j=2}^{k}((B_j-x)\times\mathsf{M})\big)$ we obtain an analogous relation between $\alpha^{(k)}$ and $\alpha_{red}^{(k)}$ for the unmarked PP $\Psi\sim P$. Rewriting this relation as integrals over indicator functions we are led by algebraic induction to

$$\begin{aligned}&\int_{(\mathbb{R}^d)^k} f(x_1,x_2,\dots,x_k)\alpha^{(k)}\big(d(x_1,x_2,\dots,x_k)\big)\\ &\quad=\lambda\int_{\mathbb{R}^d}\int_{(\mathbb{R}^d)^{k-1}} f(x_1,x_2+x_1,\dots,x_k+x_1)\alpha_{red}^{(k)}\big(d(x_2,\dots,x_k)\big)\,dx_1.\end{aligned}$$

for any non-negative $\mathcal{B}(\mathbb{R}^{dk})$-measurable function $f:(\mathbb{R}^d)^k\mapsto\mathbb{R}^1$. Setting $f(x_1,x_2,\dots,x_k) = \mathbf{1}(x_1\in B)\,\mathbf{1}((x_2-x_1,\dots,x_k-x_1)\in C)$ for an arbitrary bounded set $C\in\mathcal{B}(\mathbb{R}^{d(k-1)})$ and any $B\in\mathcal{B}(\mathbb{R}^d)$ with $\nu_d(B)=1$, for example $B=[0,1]^d$, we arrive at the formula

$$\begin{aligned}&\alpha_{red}^{(k)}(C)\\ &=\frac{1}{\lambda}\int_{(\mathbb{R}^d)^k}\mathbf{1}(x_1\in B)\,\mathbf{1}((x_2-x_1,\dots,x_k-x_1)\in C)\,\alpha^{(k)}\big(d(x_1,x_2,\dots,x_k)\big)\end{aligned}\tag{4.7}$$

which again confirms that $\alpha_{red}^{(k)}$ is a locally-finite measure on $(\mathbb{R}^{d(k-1)},\mathcal{B}(\mathbb{R}^{d(k-1)}))$. Below it will be shown that $\alpha_{red}^{(k)}$ coincides with the $(k-1)$st-order factorial moment

measure with respect to the reduced Palm distribution. It should be mentioned that both factorial moment measures $\alpha^{(k)}$ and $\alpha^{(k)}_{red}$ are symmetric in their components and the reduction is possible in any component of $\alpha^{(k)}$ yielding $\alpha^{(k)}_{red}(-C) = \alpha^{(k)}_{red}(C)$. Furthermore, the above reduction and therefore the definition of $\alpha^{(k)}_{red}$ needs only the diagonal shift-invariance of $\alpha^{(k)}$ (and of the intensity measure Λ) and not the shift-invariance of the finite-dimensional distributions of all orders and even not of the kth order.

This fact gives rise to consider weaker notions of kth-order (moment) stationarity, see Sect. 4.2.1 for $k = 2$.

A rather technical, but useful tool in expressing and studying dependences between distant parts of a stationary point pattern is based on so-called (factorial) cumulant measures of both marked and unmarked PP's. The origins of these characteristics can be traced back up to the beginning of the systematic study of random processes, random fields and particle configurations in statistical mechanics. In probability theory cumulants of random variables or vectors are defined by logarithmic derivatives of the corresponding moment-generating or characteristic functions. Along this line cumulant measures of point processes are defined by mixed higher-order partial derivatives of the logarithm of the probability generating functional of the point process, see [140, Chap. 9]. This approach is the background of the following

Definition 4.4. For any fixed $L_1, \dots, L_k \in \mathcal{B}(\mathsf{M})$ and bounded $B_1, \dots, B_k \in \mathcal{B}(\mathbb{R}^d)$ we define the *kth-order factorial cumulant measure* $\gamma^{(k)}_{\mathsf{M}}$ of the MPP $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ by

$$\gamma^{(k)}_{\mathsf{M}}\Big(\bigtimes_{j=1}^{k}(B_j \times L_j)\Big) := \sum_{j=1}^{k}(-1)^{j-1}(j-1)! \sum_{\substack{K_1 \cup \dots \cup K_j \\ = \{1,\dots,k\}}} \prod_{r=1}^{j} \alpha^{(|K_r|)}_{\mathsf{M}}\Big(\bigtimes_{s \in K_r}(B_s \times L_s)\Big),$$

where the sum $\sum_{K_1 \cup \dots \cup K_j = \{1,\dots,k\}}$ is taken over all partitions of the set $\{1, \dots, k\}$ into j non-empty sets $K_1, \dots, K_j$ and $|K_r|$ denotes the cardinality of K_r.

In general, $\gamma^{(k)}_{\mathsf{M}}$ is a locally finite *signed measure* on $\mathcal{B}((\mathbb{R}^d \times \mathsf{M})^k)$ which in case of a stationary MPP $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ can also be reduced in analogy to $\alpha^{(k)}_{\mathsf{M}}$ which leads to

$$\gamma^{(k)}_{\mathsf{M}}\Big(\bigtimes_{j=1}^{k}(B_j \times L_j)\Big) = \lambda \int_{B_1} \gamma^{(k)}_{\mathsf{M},red}\Big(L_1 \times \bigtimes_{j=2}^{k}((B_j - x) \times L_j)\Big)\, dx$$

By setting in the latter formula $L_1 = \dots = L_k = \mathsf{M}$ we obtain the corresponding relationship between the kth-order factorial cumulant measure $\gamma^{(k)}$ and the reduced kth-order factorial cumulant measure $\gamma^{(k)}_{red}$ of the unmarked point process $\Psi \sim P$. In the special case $k = 2$ we have

$$\gamma^{(2)}_{\mathsf{M}}(B_1 \times L_1 \times B_2 \times L_2) = \alpha^{(2)}_{\mathsf{M}}(B_1 \times L_1 \times B_2 \times L_2) - \Lambda_{\mathsf{M}}(B_1 \times L_1)\Lambda_{\mathsf{M}}(B_2 \times L_2)$$

$$\gamma^{(2)}_{\mathsf{M},red}(L_1 \times B_2 \times L_2) = \alpha^{(2)}_{\mathsf{M},red}(L_1 \times B_2 \times L_2) - \lambda\, Q^o_{\mathsf{M}}(L_1)\, Q^o_{\mathsf{M}}(L_2)\, \nu_d(B_2)\,.$$

Finally, if in addition $\alpha^{(k)}$ is absolutely continuous with respect to the Lebesgue measure ν_{dk} then the *kth-order product density* $\rho^{(k)} : \mathbb{R}^{d(k-1)} \mapsto [0, \infty]$ and the *kth-order cumulant density* $c^{(k)} : \mathbb{R}^{d(k-1)} \mapsto [-\infty, \infty]$ exist such that

$$\alpha^{(k)}_{red}(C) = \int_C \rho^{(k)}(x)\, dx \;\text{ and }\; \gamma^{(k)}_{red}(C) = \int_C c^{(k)}(x)\, dx \;\text{ for }\; C \in \mathcal{B}(\mathbb{R}^{d(k-1)})\,.$$

The interpretation of $\rho^{(k)}$ as density of k-point subsets of stationary point configurations is as follows:

$$\mathbf{P}(\Psi(dx_1) = 1, \ldots, \Psi(dx_{k-1}) = 1 \,|\, \Psi(\{o\}) = 1) = \rho^{(k)}(x_1, \ldots, x_{k-1})\, dx_1 \cdots dx_{k-1}\,.$$

and in a similar way $c^{(2)}(x)\, dx = \mathbf{P}(\Psi(dx) = 1 \mid \Psi(\{o\}) = 1) - \mathbf{P}(\Psi(dx) = 1)$ for any $x \neq o$.

Note that in statistical physics $\rho^{(k)}(x_1, \ldots, x_{k-1})$ and $c^{(k)}(x_1, \ldots, x_{k-1})$ are frequently used under the name *kth-order correlation function* resp. *kth-order truncated correlation function*.

In some cases under slight additional assumptions the knowledge of the (factorial) moment—or cumulant measures and their densities of any order determine the distribution of point processes uniquely. So far this moment problem for point processes is not completely solved. Another longstanding question which is still unanswered to the best of the author's knowledge is: Which properties of a locally-finite measure on $(\mathbb{R}^{dk}, \mathcal{B}(\mathbb{R}^{dk}))$ are sufficient and necessary for being a kth-order (factorial) moment measure of some unmarked point process $\Psi \sim P$?

In the simplest case of a Poisson process with given intensity measure we have the following characterization:

Theorem 4.4. *A MPP $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ with intensity measure Λ_{M} is a marked Poisson process, i.e. $P_{\mathsf{M}} = \Pi_{\Lambda_{\mathsf{M}}}$, iff*

$$\alpha^{(k)}_{\mathsf{M}} = \Lambda_{\mathsf{M}} \times \cdots \times \Lambda_{\mathsf{M}} \;\textit{ or equivalently }\; \gamma^{(k)}_{\mathsf{M}} \equiv 0 \;\textit{ for any }\; k \geq 2\,.$$

For stationary unmarked PP's with intensity λ this means $\rho^{(k)}(x) = \lambda^{k-1}$ or equivalently $c^{(k)}(x) \equiv 0$ for all $x \in \mathbb{R}^{d(k-1)}$ and any $k \geq 2$.

If the marks are real-valued it is natural to consider the higher-order mixed moments and mixed cumulants between marks conditional on their locations. For a MPP with mark space $\mathsf{M} = \mathbb{R}$ and k-point Palm mark distribution $Q^{x_1,\ldots,x_k}_{\mathsf{M}}$ let us define

$$m_{p_1,\ldots,p_k}(x_1,\ldots,x_k) = \int_{\mathbb{R}^k} m_1^{p_1}\cdots m_k^{p_k}\, Q_{\mathsf{M}}^{x_1,\ldots,x_k}(d(m_1,\ldots,m_k)) \text{ for } p_1,\ldots,p_k \in \mathbb{N}\,.$$

For stationary MPP the function $c_{mm}(x) = m_{1,1}(o,x)$ for $x \neq o$ has been introduced by D. Stoyan in 1984 in order to describe spatial correlations of marks by means of the *mark correlation function* $k_{mm}(r) = c_{mm}(x)/\mu^2$ for $\|x\| = r > 0$, where $\mu = \int_{\mathbb{R}^1} m Q_{\mathsf{M}}^o(dm)$ denotes the mean value of the typical mark, see [265, Chap. 5.3], for more details on the use and [438] for a thorough discussion of this function. Kernel-type estimators of the function $c_{mm}(x)$ and their asymptotic properties including consistency and asymptotic normality have been studied in [233] by imposing total variation conditions on higher-order reduced cumulant measures $\gamma_{\mathsf{M},red}^{(k)}$.

Finally, we give a short introduction to general Palm distributions of (marked) point processes. We first consider a simple stationary unmarked point process $\Psi \sim P$ with positive and finite intensity $\lambda = \mathbf{E}\Psi([0,1]^d)$. Let us define the product measure $\mu^!$ on $(\mathbb{R}^d \times \mathcal{N}, \mathcal{B}(\mathbb{R}^d) \otimes \mathfrak{N})$ by

$$\mu^!(B \times A) = \frac{1}{\lambda} \int_{\mathcal{N}} \sum_{x \in s(\psi)} \mathbf{1}(x \in B)\, \mathbf{1}(T_x\psi - \delta_o \in A)\; P(d\psi)$$

for bounded $B \in \mathcal{B}(\mathbb{R}^d)$ and $A \in \mathfrak{N}$, where the exclamation mark indicates that the atom of $T_x\psi$ in the origin, i.e., the atom of ψ in $x \in \mathbb{R}^d$, is removed from each counting measure $\psi \in \mathcal{N}$; $s(\psi)$ is shorthand for the support $\mathrm{supp}(\psi) = \{x \in \mathbb{R}^d : \psi(\{x\}) > 0\}$.

By the stationarity of Ψ, that is $P \circ T_x = P$ for any $x \in \mathbb{R}^d$ combined with standard arguments from measure theory it is easily seen that

1. $\mu^!(\,(B + x) \times A\,) = \mu^!(\,B \times A\,)$ for any $x \in \mathbb{R}^d$.
2. $P^{o\,!}(A) := \mu^!(\,[0,1]^d \times A\,)$ for $A \in \mathfrak{N}$ is a probability measure on $(\mathcal{N}, \mathfrak{N})$.

which is concentrated on the subset $\mathcal{N}^o = \{\psi \in \mathcal{N} : \psi(\{o\}) = 0\}$ of counting measures having no atom in the origin o and called the *reduced Palm distribution* of $\Psi \sim P$. As an immediate consequence of (1) and (2) we obtain the factorization

$$\mu^!(B \times A) = \nu_d(B)\; P^{o\,!}(A) \text{ for any fixed } B \times A \in \mathcal{B}(\mathbb{R}^d) \times \mathfrak{N}$$

which in turn implies, by algebraic induction, the *Campbell–Mecke formula*—also known as *refined Campbell theorem*

$$\int_{\mathcal{N}} \int_{\mathbb{R}^d} f(x, T_x\psi - \delta_o)\, \psi(dx)\, P(d\psi) = \lambda \int_{\mathbb{R}^d} \int_{\mathcal{N}^o} f(x,\psi)\, P^{o\,!}(d\psi)\, dx \quad (4.8)$$

for any non-negative $\mathcal{B}(\mathbb{R}^d) \otimes \mathfrak{N}$-measurable function $f : \mathbb{R}^d \times \mathcal{N} \mapsto \mathbb{R}^1$. This formula connects the stationary distribution P and the reduced Palm distribution

$P^{o\,!}$ in a one-to-one correspondence. $P^{o\,!}(A)$ can be interpreted (justified in a rigorous sense by limit theorems, see [346]) as probability that $\Psi - \delta_o \in A$ conditional on the null event $\Psi(\{o\}) > 0$. Loosely speaking, $P^{o\,!}$ describes the stationary point pattern by an observer sitting in a "typical atom" shifted in the origin.

To describe the distributional properties of stationary PP's it is often more effective to use $P^{o\,!}$ rather than P, for example in case of *recurrent*, *semi-Markov*- or *infinitely divisible* PP's, see for example [346].

A crucial result in this direction is Slivnyak's characterization of homogeneous Poisson processes:

Theorem 4.5. *A stationary (unmarked) PP $\Psi \sim P$ on $\mathbb{R}^d$ with intensity $0 < \lambda < \infty$ is a Poisson process, i.e. $P = \Pi_\lambda$ iff $P = P^{o\,!}$.*

As announced above we apply (4.8) to prove that, for any $k \geq 2$, the kth-order reduced factorial moment measure $\alpha_{red}^{(k)}$ is nothing else but the $(k-1)$st-order factorial moment measure w.r.t the reduced Palm distribution, formally written:

$$\int\limits_{(\mathbb{R}^d)^{k-1}} f(x_2,\ldots,x_k)\,\alpha_{red}^{(k)}(d(x_2,\ldots,x_k)) = \int\limits_{\mathcal{N}^o} \sum_{x_2,\ldots,x_k\in s(\psi)}^{\neq} f(x_2,\ldots,x_k)\,P^{o\,!}(d\psi)$$

for any non-negative Borel-measurable function f on $\mathbb{R}^{d(k-1)}$. For notational ease we check this only for $k = 2$. From (4.7) and the very definition of $\alpha^{(2)}$ we get for bounded $B, C \in \mathcal{B}(\mathbb{R}^d)$ with $\nu_d(B) = 1$ that

$$\begin{aligned}\alpha_{red}^{(2)}(C) &= \frac{1}{\lambda}\int\limits_{\mathcal{N}} \sum_{x,y\in s(\psi)}^{\neq} \mathbf{1}(x\in B)\,\mathbf{1}(y-x\in C)\,P(d\psi) \qquad (4.9)\\ &= \frac{1}{\lambda}\int\limits_{\mathcal{N}} \sum_{x\in s(\psi)} \mathbf{1}(x\in B)\,(T_x\psi-\delta_o)(C)\,P(d\psi) = \int\limits_{\mathcal{N}^o} \psi(C)\,P^{o\,!}(d\psi)\,.\end{aligned}$$

Quite similarly, we can define reduced Palm distributions $P_L^{o\,!}$ for simple stationary MPP's with respect to any fixed mark set $L \in \mathsf{M}$ with $Q_{\mathsf{M}}^o(L) > 0$. For this we have to replace λ by $\Lambda_{\mathsf{M}}([0,1]^d \times L) = \lambda\, Q_{\mathsf{M}}^o(L)$ which leads to the following extension of (4.8):

$$\lambda\, Q_{\mathsf{M}}^o(L)\int\limits_{\mathbb{R}^d}\int\limits_{\mathcal{N}_L^o} f(x,\psi)P_L^{o\,!}(d\psi)dx = \int\limits_{\mathcal{N}_{\mathsf{M}}}\int\limits_{\mathbb{R}^d\times L} f(x,T_x\psi-\delta_{(o,m)})\psi(d(x,m))P_{\mathsf{M}}(d\psi)$$

for any non-negative, measurable function f on $\mathbb{R}^d \times \mathcal{N}$, where $\mathcal{N}_L^o = \{\psi \in \mathcal{N}_{\mathsf{M}} : \psi(\{o\}\times L) = 0\}$.

To include mark sets L and, in particular single marks m, with Q_{M}^o-measure zero, we make use of the Radon–Nikodym derivative of the so-called *reduced Campbell measure* $C_{\mathsf{M}}^!$ defined by

$$C^{!}_{\mathsf{M}}(B\times L\times A)=\int\limits_{\mathcal{N}_{\mathsf{M}}}\int\limits_{\mathbb{R}^d\times\mathsf{M}}\mathbf{1}((x,m)\in B\times L)\mathbf{1}(\psi-\delta_{(x,m)}\in A)\psi(d(x,m))\,P_{\mathsf{M}}(d\psi)$$

with respect to the intensity measure $\Lambda_{\mathsf{M}}(B\times L)=\lambda\,\nu_d(B)\,Q^{o}_{\mathsf{M}}(L)$. The corresponding Radon–Nikodym density $P_m^{x\,!}(A)$ is called the *reduced Palm distribution* of $\Psi_{\mathsf{M}}\sim P_{\mathsf{M}}$ with respect to (x,m) and can be heuristically interpreted as conditional probability that $\Psi-\delta_{(x,m)}\in A$ given a marked point at x with mark m. This interpretation remains also valid for non-stationary MPP's and can even be generalized in an appropriate way to *k-point reduced Palm distributions* $P^{x_1,\ldots,x_k\,!}_{m_1,\ldots,m_k}(A)$ of $A\in\mathfrak{N}_{\mathsf{M}}$ with respect to $(x_1,m_1),\ldots,(x_k,m_k)$ with pairwise distinct $x_1,\ldots,x_k$.

In the stationary case we get $P^{x_1,x_2\ldots,x_k\,!}_{m_1,\ldots,m_k}(A)=P^{x_1-x_j,\ldots,o,\ldots,x_k-x_j\,!}_{m_1,\ldots,m_j,\ldots,m_k}(T_{-x_j}A)$ for each $j=1,\ldots,k$ (due to the intrinsic symmetry), which, for $k=1$, yields the Campbell–Mecke-type formula

$$\begin{aligned}\mathbf{E}\Big(\sum_{i\ge1}f(X_i,M_i,T_{X_i}\Psi-\delta_{(o,M_i)})\Big)&=\int\limits_{\mathcal{N}_{\mathsf{M}}}\int\limits_{\mathbb{R}^d\times\mathsf{M}}f(x,m,T_x\psi-\delta_{(o,m)})\psi(d(x,m))P_{\mathsf{M}}(d\psi)\\&=\lambda\int\limits_{\mathbb{R}^d}\int\limits_{\mathsf{M}}\int\limits_{\mathcal{N}_{\mathsf{M}}}f(x,m,\psi)P_m^{o\,!}(d\psi)Q^{o}_{\mathsf{M}}(dm)dx\end{aligned}\tag{4.10}$$

for any non-negative measurable function f on $\mathbb{R}^d\times\mathsf{M}\times\mathcal{N}_{\mathsf{M}}$. Furthermore, this formula can be extended for $k\ge2$ to the following relationship involving the k-point reduced Palm distribution, k-point Palm mark distribution and the k-order reduced factorial moment measure introduced at the beginning of Sect. 4.1.2:

$$\begin{aligned}&\mathbf{E}\Big(\sum_{i_1,i_2,\ldots,i_k\ge1}^{\neq}f_k(X_{i_1},M_{i_1},X_{i_2},M_{i_2},\ldots,X_{i_k},M_{i_k},T_{X_{i_1}}\Psi-\delta_{(o,M_{i_1})}-\sum_{j=2}^{k}\delta_{(X_{i_j},M_{i_j})})\Big)\\&=\lambda\int\limits_{(\mathbb{R}^d\times\mathsf{M})^k}\int\limits_{\mathcal{N}_{\mathsf{M}}}f_k(x_1,m_1,x_2,m_2,\ldots,x_k,m_k,\psi)\,P^{o,x_2,\ldots,x_k\,!}_{m_1,m_2,\ldots,m_k}(d\psi)\\&\quad\times\alpha^{(k)}_{\mathsf{M},red}(d(m_1,x_2,m_2,\ldots,x_k,m_k))\,dx_1\\&=\lambda\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^{d(k-1)}}\int\limits_{\mathsf{M}^k}\int\limits_{\mathcal{N}_{\mathsf{M}}}f_k(x_1,m_1,x_2,m_2,\ldots,x_k,m_k,\psi)\,P^{o,x_2,\ldots,x_k\,!}_{m_1,m_2,\ldots,m_k}(d\psi)\\&\quad\times Q^{o,x_2,\ldots,x_k}_{\mathsf{M}}(d(m_1,m_2,\ldots,m_k))\,\alpha^{(k)}_{red}(d(x_2,\ldots,x_k))\,dx_1\end{aligned}$$

for any non-negative measurable function f_k on $(\mathbb{R}^d\times\mathsf{M})^k\times\mathcal{N}_{\mathsf{M}}$.

4.1.3 Different Types of Marking and Some Examples

In the following we distinguish three types of MPP's $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ by the dependences between marks at distinct points in $\mathbb{R}^d$, by interactions between separated parts of the point pattern and, last but not least, by cross correlations between the whole point pattern and the whole mark field. In the most general case, only the family of k-point Palm mark distributions seems to be appropriate to describe such complicated structure of dependences.

1. Independently Marked (Stationary) Point Processes

Given an unmarked (not necessarily stationary) PP $\Psi = \sum_{i\geq 1} \delta_{X_i} \sim P$ on $\mathbb{R}^d$ and a stochastic kernel $Q(x, L)$, $x \in \mathbb{R}^d, L \in \mathcal{B}(\mathsf{M})$ we assign to an atom X_i located at x the mark $M_i \sim Q(x, \cdot)$ independently of $\Psi - \delta_{X_i}$ and of any other mark M_j, $j \neq i$. The resulting MPP $\Psi_{\mathsf{M}} = \sum_{i\geq 1} \delta_{(X_i, M_i)}$ is said to be derived from Ψ by *location-dependent independent marking*. We obtain for the intensity measure and the k-point Palm mark distribution

$$\Lambda_{\mathsf{M}}(B \times L) = \int_B Q(x, L)\, \Lambda(dx) \quad \text{resp.} \quad Q_{\mathsf{M}}^{x_1,\ldots,x_k}\Big(\overset{k}{\underset{i=1}{\times}} L_i\Big) = \prod_{i=1}^{k} Q(x_i, L_i), \quad k \geq 1,$$

where Λ denotes the intensity measure of Ψ.

Note that the MPP $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ is stationary iff $\Psi \sim P$ is stationary and independent of an i.i.d. sequence of marks $\{M_i,\ i \geq 1\}$ with a common distribution $Q(\cdot)$—the (mark) distribution of the typical mark M_0.

2. Geostatistically or Weakly Independently Marked Point Processes

Let unmarked PP $\Psi = \sum_{i\geq 1} \delta_{X_i} \sim P$ on $\mathbb{R}^d$ be stochastically independent of a random field $\{M(x),\ x \in \mathbb{R}^d\}$ taking values in the measurable mark space $(\mathsf{M}, \mathcal{B}(\mathsf{M}))$. To each atom X_i we assign the mark $M_i = M(X_i)$ for $i \geq 1$. In this way the k-point Palm mark distribution coincide with the finite-dimensional distributions of the mark field, that is,

$$Q_{\mathsf{M}}^{x_1,\ldots,x_k}(L_1 \times \cdots \times L_k) = \mathbf{P}(M(x_1) \in L_1, \ldots, M(x_k) \in L_k) \text{ for all } k \in \mathbb{N}.$$

Note that the MPP $\Psi_{\mathsf{M}} \sim P_{\mathsf{M}}$ is stationary iff both the point process $\Psi \sim P$ and the random field are (strictly) stationary. In case of real-valued marks (stationary) Gaussian random fields $M(x)$ with some covariance function, see Definition 9.10, or shot-noise fields $M_g(x) = \sum_{i\geq 1} g(x - \xi_i)$ with some response function

$g : \mathbb{R}^d \mapsto \mathbb{R}^1$ and a (stationary Poisson) point process $\{\xi_i,\ i \geq 1\}$ chosen independently of Ψ (see Sect. 9.2.5 for more details) are suitable examples for mark fields.

3. General Dependently Marked Point Processes

In this case the locations X_i of the marked atoms and their associated marks M_i may depend on each other and, in addition, there are intrinsic interactions within the point field $\{X_i\}$ as well as within the mark field $\{M_i\}$. This means that the k-point Palm mark distribution $Q_{\mathsf{M}}^{x_1,\ldots,x_k}$ must be considered as an conditional distribution, in particular, $Q_{\mathsf{M}}^{o,x}(L \times \mathsf{M})$ does not coincide with $Q_{\mathsf{M}}^{o}(L)$.

Examples

1. Germ-Grain Processes: Germ-Grain Models

A stationary independently MPP $\Psi_{\mathsf{M}} = \{(X_i, \Xi_i),\ i \geq 1\}$ on $\mathbb{R}^d$ with mark space $\mathsf{M} = \mathcal{K}^d$ (= space of all non-empty compact sets in $\mathbb{R}^d$ equipped with the Hausdorff metric) is called *germ-grain process* or *particle process* driven by the PP $\Psi = \{X_i,\ i \geq 1\}$ of germs and the *typical grain* $\Xi_0 \sim Q$. The associated random set $\Xi = \bigcup_{i\geq 1}(\Xi_i + X_i)$ is called *germ-grain model*. Note that in general Ξ need not to be closed (**P**-a.s.). The condition

$$\sum_{i\geq 1} \mathbf{P}(\Xi_0 \cap (K - X_i) \neq \emptyset) < \infty \quad \mathbf{P}-\text{a.s. for all } K \in \mathcal{K} \tag{4.11}$$

is sufficient to ensure the **P**-a.s.-closedness of Ξ, see [229]. The most important and best studied germ-grain model is the *Poisson-grain model* (also called *Boolean model*) driven by a Poisson process $\Psi \sim \Pi_\Lambda$ of germs $\{X_i,\ i \geq 1\}$, see for example [366, 489] for more details.

2. Poisson-Cluster Processes

If the typical grain $\Xi_0 = \{Y_1, \ldots, Y_{N_0}\}$ is a **P**-a.s. finite random point set satisfying (4.11) then the discrete random closed set $\Xi = \bigcup_{i\geq 1}\{Y_1^{(i)} + X_i, \ldots, Y_{N_i}^{(i)} + X_i\}$ coincides with the support of a random locally finite counting measure Ψ_{cl} and is called a *cluster point process* with the PP $\Psi_c = \{X_i,\ i \geq 1\}$ of cluster centres and the typical cluster $\{Y_1, \ldots, Y_{N_0}\}$. Factorial moment and cumulant measures of any order can be expressed in terms of the corresponding measures of Ψ and the finite PP $\sum_{i=1}^{N_0} \delta_{Y_i}$, see for example [227]. In case of a (stationary) Poisson cluster centre process Ψ_c we get a (stationary) Poisson-cluster process Ψ_{cl}, see Sect. 3.1.4 for more details. In particular, if $\Psi \sim \Pi_{\lambda_c v_d}$ and the random number

N_0 with probability generating function $g_0(z)$ is independent of the i.i.d. sequence of random vectors $Y_1, Y_2, \ldots$ in $\mathbb{R}^d$ with common density function f we obtain a so-called *Neyman–Scott process* $\Psi_{\rm cl}$ with intensity $\lambda_{\rm cl} = \lambda_c \, \mathbf{E}N_0$, second-order product density $\rho^{(2)}(x) = c^{(2)}(x) + \lambda_{\rm cl}$ and its kth-order cumulant density for $k \geq 2$ takes on the form

$$c^{(k)}(x_1, \ldots, x_{k-1}) = \frac{g_0^{(k)}(1)}{\mathbf{E}N_0} \int\limits_{\mathbb{R}^d} f(y)\, f(y + x_1) \cdots f(y + x_{k-1})\, dy \ . \quad (4.12)$$

Compare Definition 3.11 for its special case.

3. Doubly Stochastic Poisson Processes

Now, let Λ be a (stationary) random measure on $\mathbb{R}^d$, see for example [140] for details. The new unmarked PP $\Psi_\Lambda \sim P_\Lambda$ defined by the finite-dimensional distributions

$$\mathbf{P}(\Psi_\Lambda(B_1) = n_1, \ldots, \Psi_\Lambda(B_k) = n_k) = \mathbf{E}\Big(\prod_{i=1}^{k} \frac{\Lambda^{n_i}(B_i)}{n_i!} \, \mathrm{e}^{-\Lambda(B_i)} \Big)$$

for any disjoint bounded $B_1, \ldots, B_k \in \mathcal{B}_0(\mathbb{R}^d)$ and any $n_1, \ldots, n_k \in \mathbb{N} \cup \{0\}$, is called *doubly stochastic Poisson (or Cox) process* with driving measure $\Lambda(\cdot)$, compare Definition 3.7. In the special case $\Lambda(\cdot) = \lambda \, \nu_d((\cdot) \cap \Xi)$, where Ξ is a (stationary) random closed set, for example a Boolean model, the (stationary) PP Ψ_Λ (called *interrupted Poisson process*) is considered as a Poisson process restricted on the (hidden) realizations of Ξ. The factorial moment and cumulant measures of Ψ_Λ are expressible in terms of the corresponding measures of random driving measure Λ, see for example [289].

4.2 Point Process Statistics in Large Domains

Statistics of stationary point processes is mostly based on a single observation of some point pattern in a sufficiently large domain which is assumed to extend unboundedly in all directions. We demonstrate this concept of asymptotic spatial statistics for several second-order characteristics of point processes including different types of K-functions, product densities and the pair correlation function. Variants of Brillinger-type mixing are considered to obtain consistency and asymptotic normality of the estimators.

The philosophy of large-domain spatial statistics is as follows: Let there be given a single realization of a random point pattern or a more general random

set in a sufficiently large sampling window $W_n \subset \mathbb{R}^d$, which is thought to expand in all directions as $n \to \infty$. Further, we assume that there is an adequate model describing the spatial random structure whose distribution is at least shift-invariant (stationary) and sometimes additionally even isotropic. Then only using the information drawn from the available observation in W_n we define empirical counterparts (estimators) of those parameters and non-parametric characteristics which reflect essential properties of our model. To study the asymptotic behaviour of the estimators such as weak or strong consistency and the existence of limit distributions (after suitable centering and scaling) we let W_n increase unboundedly which requires additional weak dependence conditions. Throughout we assume that $\{W_n, n \in \mathbb{N}\}$ is a *convex averaging sequence*, that is,

1. W_n is bounded, compact, convex and $W_n \subseteq W_{n+1}$ for $n \in \mathbb{N}$.
2. $r(W_n) := \sup\{r > 0 : B_r(x) \subseteq W_n \text{ for some } x \in W_n\} \uparrow \infty$.

The second property means that W_n expands unboundedly in all directions and is equivalent to $v_{d-1}(\partial W_n)/v_d(W_n) \underset{n\to\infty}{\longrightarrow} 0$ as immediate consequence of the geometric inequality

$$\frac{1}{r(W_n)} \le \frac{v_{d-1}(\partial W_n)}{v_d(W_n)} \le \frac{d}{r(W_n)}, \tag{4.13}$$

see [237].

Exercise 4.1. Show that

$$v_d(W_n \setminus (W_n \ominus B_r(o))) = \int_0^r v_{d-1}(\partial(W_n \ominus B_s(o)))\, ds \le r\, v_{d-1}(\partial W_n)$$

for $0 \le r \le r(W_n)$ from which, together with $v_d(W_n \ominus B_{r(W_n)}(o)) = 0$, the l.h.s. of (4.13) immediately follows. The r.h.s. of (4.13) results from an inequality by J.M. Wills, see [519].

From the mathematical view point it is sometimes more convenient to consider rectangles $W_n = \times_{i=1}^d [0, a_i^{(n)}]$ with $a_i^{(n)} \uparrow \infty$ for $i = 1, \ldots, d$ or blown up sets $W_n = n\,W$, where $W \subset \mathbb{R}^d$ is a fixed convex body containing the origin o as inner point.

4.2.1 Empirical K-Functions and Other Summary Statistics of Stationary PP's

Second-order statistical analysis of spatial point patterns is perhaps the most important branch in point process statistics comparable with the spectral density

estimation in time series analysis. We assume that the simple unmarked PP $\Psi = \sum_{i\geq 1}\delta_{X_i}$ has finite second moments, i.e. $\mathbf{E}\Psi^2(B) < \infty$ for all bounded $B \in \mathcal{B}(\mathbb{R}^d)$, and is strictly or at least weakly stationary.

Weak (or second-order) stationarity of an unmarked PP $\Psi \sim P$ requires only the shift-invariance of the first- and second-order moment measures, i.e. $\Lambda(B_1 + x) = \Lambda(B_1)$ and $\alpha^{(2)}((B_1 + x) \times (B_2 + x)) = \alpha^{(2)}(B_1 \times B_2)$ for any bounded $B_1, B_2 \in \mathcal{B}(\mathbb{R}^d))$ and all $x \in \mathbb{R}^d$. Obviously, strictly stationary point processes having finite second moments are weakly stationary. Further note that the reduced second factorial moment measure $\alpha^{(2)}_{red}(\cdot)$ is well-defined also under weak stationarity but it can not be expressed as first-order moment measure w.r.t. $P^{o!}$ as in (4.9), see [140] for more details. In what follows we assume strict stationarity. By applying the Palm and reduction machinery sketched in Sect. 4.1.2 we can describe the first and second moment properties by the intensity $\lambda = \mathbf{E}\Psi([0,1]^d)$ and the reduced second factorial moment measure $\alpha^{(2)}_{red}(\cdot)$ defined by (4.7) for $k = 2$ resp. (4.9) as first moment measure with respect to the Palm distribution $P^{o!}$ in case of strict stationarity. If Ψ is additionally strictly or at least weakly isotropic, i.e. $R_O\alpha^{(2)}_{red} = \alpha^{(2)}_{red}$ for $O \in SO_d$, then it suffices to know the function $\alpha^{(2)}_{red}(B_r(o))$ for $r \geq 0$. In [424] B. Ripley introduced the *K-function*

$$K(r) := \frac{1}{\lambda}\,\alpha^{(2)}_{red}\big(B_r(o)\big) = \frac{1}{\lambda^2}\,\mathbf{E}\Big(\sum_{i\geq 1}\mathbf{1}(X_i \in [0,1]^d)\,\Psi(B_r(X_i)\setminus\{X_i\})\Big) \tag{4.14}$$

for $r \geq 0$ as basic summary characteristic for the second-order analysis of motion-invariant PP's, see also [22] or [265, Chap. 4.3] for more details and historical background. From (4.9) we see that $\lambda\,K(r)$ coincides with conditional expectation $\mathbf{E}\big(\Psi\big(B_r(\{o\})\setminus\{o\}\big) \mid \Psi(\{o\}) = 1\big)$ giving the mean number of points within the Euclidean distance r from the typical point (which is not counted). If Ψ is a homogeneous Poisson process with intensity λ, then, by Slivnyak's theorem (see Theorem 4.5 in Sect. 4.1.2), $\alpha^{(2)}_{red}(\cdot) = \mathbf{E}\Psi(\cdot) = \lambda\,\nu_d(\cdot)$ and hence we get

$$K(r) = \omega_d\,r^d \;\text{ with }\; \omega_d := \nu_d(B_1(o)) = \frac{\pi^{d/2}}{\Gamma\big(\frac{d}{2}+1\big)}\,. \tag{4.15}$$

For better visualization of the Poisson property by a linear function the so-called *L-function* $L(r) := \big(K(r)/\omega_d\big)^{1/d}$ is sometimes preferred instead of the K-function. Both the K- and L-function represent the same information, but they cannot completely characterize the distribution of a (motion-invariant) PP. In other words, there are different point processes having the same K-function. Further note that an explicit description of the family of K-functions does not exist so far. Nevertheless, the K-function and its empirical variants, see below, are used to check point process hypotheses when the K-function of the null hypothesis is known (or generated by simulation on a finite interval $[0, r_0]$), see Sect. 3.3.3 and Figs. 3.47–3.48.

In particular, the simple parabola-shape of the K-function (4.15) facilitates to check the property of *complete spatial randomness* (briefly *CSR*) of a given point pattern. Lemma 3.12 shows the connection between CSR and the Poisson property shows the following

It contains the interpretation of the Poisson point process in statistical mechanics as particle configuration, for example molecules in "ideal gases", modelled as *grand canonical ensemble*, where neither attraction nor repulsion forces between particles occur. Lemma 3.12 also reveals an easy way to simulate homogeneous Poisson processes in bounded domains, see Algorithm 3.6 in Sect. 3.2.2.

Since the K-function is also used to analyze (second-order) stationary, non-isotropic PP's we introduce two generalized versions of Ripley's K-function (4.14). First, the Euclidean d-ball $B_r(o)$ in (4.14) is replaced by $r\,B$, where $B \subset \mathbb{R}^d$ is a compact, convex, centrally symmetric set containing o as inner point. Such set B is called *structuring element* in image analysis and coincides with the unit ball $\{x \in \mathbb{R}^d : \mathrm{N}_B(x) \le 1\}$ generated by a unique norm $\mathrm{N}_B(\cdot)$ on $\mathbb{R}^d$. Let $K_B(r)$ denote the analogue to (4.14) which equals $v_d(B)\,r^d$ if $\Psi \sim \Pi_{\lambda v_d}$. In case of a Neyman–Scott process we obtain from (4.12) that

$$K_B(r) = v_d(B)\,r^d + \frac{\mathbf{E}N_0(N_0-1)}{\lambda_c\,(\mathbf{E}N_0)^2} \int\limits_{r\,B} f_s(x)\,dx \;\text{ with }\; f_s(x) = \int\limits_{\mathbb{R}^d} f(y)\,f(y+x)\,\mathrm{d}y\,.$$

A second generalization of (4.14) is the *multiparameter K-function*, see [231], defined by

$$K(r_1,\ldots,r_d) := \frac{1}{\lambda}\,\alpha^{(2)}_{red}\Big(\bigtimes_{k=1}^{d}[-r_k,r_k]\Big) \;\text{ for }\; r_1,\ldots,r_d \ge 0\,,$$

which contains the same information as the centrally symmetric measure $\alpha^{(2)}_{red}(\cdot)$. For stationary Poisson processes we get

$$K(r_1,\ldots,r_d) = v_d\Big(\bigtimes_{k=1}^{d}[-r_k,r_k]\Big) = 2^d\,r_1\cdot\ldots\cdot r_d \;\text{ for }\; r_1,\ldots,r_d \ge 0\,.$$

We next define three slightly different non-parametric estimators of the function $\lambda^2\,K_B(r)$ (briefly called *empirical K-functions*):

$$\widehat{\big(\lambda^2 K_B\big)}_{n,1}(r) := \frac{1}{v_d(W_n)} \sum_{i\ge 1} \mathbf{1}(X_i \in W_n)\,(\Psi - \delta_{X_i})(r\,B + X_i)\,,$$

$$\widehat{\big(\lambda^2 K_B\big)}_{n,2}(r) := \frac{1}{v_d(W_n)} \sum_{i,j\ge 1}^{\neq} \mathbf{1}(X_i \in W_n)\,\mathbf{1}(X_j \in W_n)\,\mathbf{1}(\mathrm{N}_B(X_j - X_i) \in [0,r])\,,$$

$$\widehat{\big(\lambda^2 K_B\big)}_{n,3}(r) := \sum_{i,j\ge 1}^{\neq} \frac{\mathbf{1}(X_i \in W_n)\,\mathbf{1}(X_j \in W_n)\,\mathbf{1}(\mathrm{N}_B(X_j - X_i) \in [0,r])}{v_d((W_n - X_i)\cap(W_n - X_j))}\,.$$

Each of these empirical processes is non-decreasing, right-continuous, and piecewise constant with jumps of magnitude $1/\nu_d(W_n)$ (for $i = 1, 2$) at random positions $\mathrm{N}_B(X_j - X_i)$ arranged in order of size for $i \neq j$. Quite analogously, by substituting the indicators of the events $\{\mathrm{N}_B(X_j - X_i) \in [0, r]\}$ by the indicators of $\{X_j - X_i \in \times_{k=1}^{d}[-r_k, r_k]\}$ we obtain the multivariate empirical processes $\widehat{\big(\lambda^2 K\big)}_{n,i}(r_1, \ldots, r_d)$ for $i = 1, 2, 3$ as empirical counterparts of $\lambda^2\, K(r_1, \ldots, r_d)$. By (4.8) resp. (4.9), $\widehat{\big(\lambda^2 K_B\big)}_{n,1}(r)$ is easily seen to be an unbiased estimator for $\lambda^2\, K_B(r)$ but it ignores the edge effect problem, that is, we need information from the dilated sampling window $W_n \oplus r_0\, B$ to calculate this estimator for $0 \le r \le r_0$. If this information is not available then one has to reduce the original window to the eroded set $W_n \ominus r_0\, B$ which is known as *minus sampling*. The second estimator needs only the positions of points within W_n, however, its bias disappears only asymptotically, i.e.

$$\mathbf{E}\widehat{\big(\lambda^2 K_B\big)}_{n,2}(r) = \lambda \int\limits_{r\,B} \frac{\nu_d(W_n \cap (W_n - x))}{\nu_d(W_n)} \alpha^{(2)}_{red}(dx) \underset{n\to\infty}{\longrightarrow} \lambda^2\, K_B(r)\,. \tag{4.16}$$

Finally, $\widehat{\big(\lambda^2 K_B\big)}_{n,3}(r)$ is a so-called *edge-corrected* or *Horvitz–Thompson-type estimator* which also needs only the points located within W_n. The pairs $(X_i, X_j) \in W_n \times W_n$ are weighted according to the length and direction of the difference vector $X_j - X_i$ providing the unbiasedness of the estimator $\mathbf{E}\widehat{\big(\lambda^2 K_B\big)}_{n,3}(r)$.

Exercise 4.2. Show (4.16) by applying the inequality (4.13) and prove

$$\mathbf{E}\widehat{\big(\lambda^2 K_B\big)}_{n,3}(r) = \lambda^2\, K_B(r)$$

by means of (4.8) resp. (4.9).

For further details and more sophisticated edge corrections we refer to [265, 385, 489] and references therein.

Before regarding consistency properties of the empirical K-functions we have a short look at the estimation of the simplest summary characteristic—the intensity λ—and its powers λ^k given by

$$\widehat{\lambda}_n := \frac{\Psi(W_n)}{\nu_d(W_n)} \text{ and } (\widehat{\lambda^k})_n := \prod_{j=0}^{k-1} \frac{\Psi(W_n) - j}{\nu_d(W_n)} \tag{4.17}$$

for any fixed integer $k \ge 2$. A simple application of the Campbell formula (4.6) (or (4.8)) and the definition of the kth-order factorial moment measure yields

$$\mathbf{E}\widehat{\lambda}_n = \lambda \text{ and } \mathbf{E}(\widehat{\lambda^k})_n = \frac{\alpha^{(k)}(W_n \times \cdots \times W_n)}{\nu_d^k(W_n)}$$

which shows the unbiasedness of $\widehat{\lambda}_n$ for any stationary PP, whereas $(\widehat{\lambda^k})_n$ for $k \geq 2$ is unbiased only for the Poisson process $\Psi \sim \Pi_{\lambda\, \nu_d}$.

Exercise 4.3. For a stationary Poisson process $\Psi \sim \Pi_{\lambda\, \nu_d}$ show that

$$\alpha^{(k)}(B_1 \times \cdots \times B_k) = \lambda^k\, \nu_d(B_1) \cdots \nu_d(B_k)$$

for any (not necessarily disjoint) bounded sets $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{R}^d)$.

The decomposition $\alpha^{(2)}(W_n \times W_n) = \gamma^{(2)}(W_n \times W_n) + \lambda^2\, \nu_d^2(W_n)$ and reduction reveal the asymptotic unbiasedness of $(\widehat{\lambda^2})_n$

$$\mathbf{E}(\widehat{\lambda^2})_n = \lambda^2 + \frac{\lambda}{\nu_d^2(W_n)} \int\limits_{W_n} \gamma_{red}^{(2)}(W_n - x)\, dx \underset{n\to\infty}{\longrightarrow} \lambda^2$$

provided that the total variation $\| \gamma_{red}^{(2)} \|_{\mathrm{TV}}$ is finite. This motivates the assumption of bounded total variation of the reduced factorial cumulant measure $\gamma_{red}^{(k)}(\cdot)$ for some $k \geq 2$ to express short-range correlation of the point process. To be precise, we rewrite the locally finite (in general not finite) signed measure $\gamma_{red}^{(k)}(\cdot)$ on $\left(\mathbb{R}^{d(k-1)}, \mathcal{B}(\mathbb{R}^{d(k-1)})\right)$ as difference of the positive and negative part $\gamma_{red}^{(k)+}(\cdot)$ resp. $\gamma_{red}^{(k)-}(\cdot)$ (Jordan decomposition) and define the corresponding *total variation measure* $\left| \gamma_{red}^{(k)} \right|(\cdot)$ as a sum of the positive and negative part:

$$\gamma_{red}^{(k)}(\cdot) = \gamma_{red}^{(k)+}(\cdot) - \gamma_{red}^{(k)-}(\cdot) \text{ and } \left| \gamma_{red}^{(k)} \right|(\cdot) := \gamma_{red}^{(k)+}(\cdot) + \gamma_{red}^{(k)-}(\cdot)\,.$$

Note that the locally finite measures $\gamma_{red}^{(k)+}(\cdot)$ and $\gamma_{red}^{(k)-}(\cdot)$ are concentrated on two disjoint Borel sets H^+ resp. H^- with $H^+ \cup H^- = \mathbb{R}^{d(k-1)}$ (Hahn decomposition) which leads to the *total variation* of $\gamma_{red}^{(k)}(\cdot)$:

$$\| \gamma_{red}^{(k)} \|_{\mathrm{TV}} := \left| \gamma_{red}^{(k)} \right|(\mathbb{R}^{d(k-1)}) = \gamma_{red}^{(k)+}(H^+) + \gamma_{red}^{(k)-}(H^-) = \int\limits_{\mathbb{R}^{d(k-1)}} |\, c^{(k)}(x)\, |\, d\, x\,,$$

where $c^{(k)} : \mathbb{R}^{d(k-1)} \mapsto [-\infty, \infty]$ is the kth-order cumulant density, if it exists.

Definition 4.5. A stationary PP $\Psi \sim P$ on $\mathbb{R}^d$ satisfying $\mathbf{E}\Psi^k([0,1]^d) < \infty$ for some integer $k \geq 2$ is said to be $\mathbf{B}_k$*-mixing* if $\| \gamma_{red}^{(j)} \|_{\mathrm{TV}} < \infty$ for $j = 2, \ldots, k$. A $\mathbf{B}_\infty$-mixing stationary PP is called *Brillinger-mixing* or briefly $\mathbf{B}$-mixing.

Example 4.1. From (4.12) it is easily seen that a Neyman–Scott process is $\mathbf{B}_k$-mixing iff $\mathbf{E}N_0^k < \infty$ without restrictions on f. This remains true for any Poisson-cluster process. Moreover, a general cluster process is $\mathbf{B}_k$-mixing if the PP

Ψ_c of cluster centres is $\mathbf{B}_k$-mixing and the typical cluster size N_0 has a finite kth moment, see [227] for details and further examples like Cox processes.

Proposition 4.1. *For any $\mathbf{B}_k$-mixing stationary PP we have $\mathbf{E}(\widehat{\lambda^k})_n \underset{n\to\infty}{\longrightarrow} \lambda^k$ for $k \geq 2$.*

We next state the mean-square consistency of the above-defined empirical K-functions under mild conditions. Furthermore, it can be shown that a possible weak Gaussian limit (after centering with mean and scaling with $\sqrt{v_d(W_n)}$) is for each of the estimators of $\lambda^2 K_B(r)$ the same.

Theorem 4.6. *Let $\Psi \sim P$ be a $\mathbf{B}_4$-mixing stationary PP with intensity λ. Then*

$$\mathbf{E}\Big(\big(\widehat{\lambda^2 K_B}\big)_{n,i}(r) - \lambda^2 K_B(r)\Big)^2 \underset{n\to\infty}{\longrightarrow} 0 \text{ for } i = 1,2,3$$

$$v_d(W_n)\ \mathbf{var}\Big(\big(\widehat{\lambda^2 K_B}\big)_{n,1}(r) - \big(\widehat{\lambda^2 K_B}\big)_{n,i}(r)\Big) \underset{n\to\infty}{\longrightarrow} 0 \text{ for } i = 2,3\,.$$

In other words, the boundary effects are asymptotically neglectable which can be considered as a general rule of thumb in large domain statistics.

Finally, we mention that also higher-order reduced moment measures can be estimated in quite the same way, see for example [274, 289]. Further second-order summary characteristics and their empirical counterparts (called summary statistics) such as the second-order product density $\rho^{(2)}(x)$, the *pair correlation function* $g(r)$ and the asymptotic variance $\sigma^2 := \lim_{n\to\infty} v_d(W_n)\,\mathbf{E}(\widehat{\lambda}_n - \lambda)^2$, see (4.17), are briefly discussed in Sect. 4.2.3. Summary statistics are used in all branches of statistics to summarize data sets—in our case data from point patterns or from realizations of random sets—to describe the underlying models by a small number of parametric and non-parametric estimates. Further summary characteristics frequently used in point process statistics are the *empty space function* (or *contact distribution function*) F, the *nearest-neighbour distance function* G and the *J-function* defined for a stationary PP $\Psi = \sum_{i\geq 1} \delta_{X_i} \sim P$ by

$$F(r) = \mathbf{P}(\Psi(B_r(o)) > 0) = P(\{\psi \in \mathcal{N} : \psi(B_r(o)) > 0\})\,,$$

$$G(r) = P^{o\,!}(\{\psi \in \mathcal{N}^o : \psi(B_r(o)) > 0\}) \quad \text{and} \quad J(r) = (1 - G(r))/(1 - F(r))\,.$$

F is the distribution function of the distance $\mathrm{dist}(x, \Psi)$ from a fixed point $x \in \mathbb{R}^d$ to the nearest atom of Ψ, whereas G is the distribution function of the corresponding distance from a typical atom of Ψ to the nearest other atom of Ψ. Unbiased non-parametric estimators of $F(r)$ and $\lambda\, G(r)$ are

$$\widehat{F}_n(r) = \frac{v_d\big(\bigcup_{i\geq 1} B_r(X_i) \cap W_n\big)}{v_d(W_n)},\quad \widehat{(\lambda\, G)}_n(r) = \sum_{X_i \in W_n} \frac{\mathbf{1}(\mathrm{dist}(X_i, \Psi - \delta_{X_i}) \in [0, r])}{v_d(W_n)}.$$

The empirical J-function $\widehat{J}_n(r)$ is defined as ratio $\widehat{\lambda}_n\,\widehat{F}_n(r)/(\widehat{\lambda\,G}\,)_n(r)$. To avoid boundary effects we replace W_n by $W_n \ominus B_r(o)$ for $0 \le r \le \mathrm{diam}(W_n)/2$, if the point pattern is observable only inside W_n. In case of $\Psi \sim \Pi_{\lambda\nu_d}$ Slivnyak's theorem yields $F(r) = G(r) = 1 - \exp\{-\lambda\,\omega_d\,r^d\}$ so that $J(r) \equiv 1$. This fact can be used for testing CSR just by regarding the plot of the empirical version $\widehat{J}_n(r)$ in some interval $[0, r_0]$.

4.2.2 The Role of Ergodicity and Mixing in Point Process Statistics

The assumption of (strict) stationarity of a point process or random closed set under consideration is frequently accompanied by the requirement of ergodicity. It is beyond the scope of this survey to capture the full depth of this notion. We only say that ergodicity is always connected with a group of measure preserving transformations acting on the probability space. In our situation we take quite naturally the group of translations $\{T_x : x \in \mathbb{R}^d\}$ as defined in Sect. 4.1.1 on the space of (marked) locally-finite counting measures or the corresponding shifts on the space of closed sets in $\mathbb{R}^d$. To be precise, we define besides ergodicity also the somewhat stronger condition of mixing for stationary (unmarked) PP's:

Definition 4.6. A (strictly) stationary PP $\Psi \sim P$ is said to be *ergodic* resp. *mixing* if

$$\frac{1}{\nu_d(W_n)} \int\limits_{W_n} P(T_x Y_1 \cap Y_2)\,dx \underset{n\to\infty}{\longrightarrow} P(Y_1)\,P(Y_2) \quad \text{resp.} \quad P(T_x\,Y_1 \cap Y_2) \underset{\|x\|\to\infty}{\longrightarrow} P(Y_1)\,P(Y_2)$$

for any $Y_1, Y_2 \in \mathfrak{N}$.

Loosely speaking, mixing means that two events becomes nearly independent when they occur over parts of $\mathbb{R}^d$ being separated by a great distance and ergodicity weakens this distributional property in the sense of Cesaro limits. In physics and engineering one says that an ergodic stochastic process allows to detect its distribution after very long time of observation which carries over to spatial ergodic processes when the observation window expands unboundedly in all directions. This interpretation is rigorously formulated by ergodic theorems which state the **P**-a.s. convergence of spatial means to expectations with respect to the underlying distribution. The following ergodic theorem by X.X. Nguyen and H. Zessin [382] is of particular importance in the theory as well as in statistics of stationary PP's.

Theorem 4.7. *Let $\Psi \sim P$ be a stationary ergodic PP on $\mathbb{R}^d$ with intensity λ, and let $g : \mathcal{N} \mapsto [0,\infty]$ be $(\mathfrak{N}, \mathcal{B}(\mathbb{R}^d))$-measurable such that $\int_{\mathcal{N}^o} g(\psi)\,P^{o\,!}(d\psi) < \infty$. Then*

$$\frac{1}{\nu_d(W_n)} \int\limits_{W_n} g(T_x\psi - \delta_o)\,\psi(dx) \underset{n\to\infty}{\longrightarrow} \lambda \int\limits_{\mathcal{N}^o} g(\psi) P^{o\,!}(d\psi)$$

for P-almost every $\psi \in \mathcal{N}$.

This result can be applied to prove strong consistency for many estimators, in particular, for various empirical Palm characteristics. In the special cases (a) $g(\psi) \equiv 1$ and (b) $g(\psi) = \psi(r\,B)$ we obtain strong consistency of the intensity estimator (4.17) and $\widehat{\left(\lambda^2 K_B\right)}_{n,1}(r)$ for any $r \geq 0$, which implies even uniformly strong consistency:

$$\widehat{\lambda}_n \underset{n\to\infty}{\overset{\mathbf{P}-\text{a.s.}}{\longrightarrow}} \lambda \text{ and } \sup_{0\leq r\leq R} \left| \widehat{\left(\lambda^2 K_B\right)}_{n,1}(r) - \lambda^2 K_B(r) \right| \underset{n\to\infty}{\overset{\mathbf{P}-\text{a.s.}}{\longrightarrow}} 0\,.$$

We mention just one asymptotic relationship which requires mixing instead of ergodicity, namely the generalized version of Blackwell's renewal theorem. If the stationary second-order PP $\Psi \sim P$ is mixing, then, for any bounded $B \in \mathcal{B}(\mathbb{R}^d)$ satisfying $\nu_d(\partial B) = 0$, it holds

$$\alpha^{(2)}_{red}(B + x) \underset{\|x\|\to\infty}{\longrightarrow} \lambda\, \nu_d(B),$$

see [140]. Note that a renewal process is just mixing if the length of the typical renewal interval has a non-arithmetic distribution and thus, the latter result (applied to an bounded interval $B = [a, b]$) contains the mentioned classical result from renewal theory. For related results concerning the weak convergence of the shifted Palm distribution $P^{o\,!}(T_x(\cdot))$ to the stationary distribution $P(\cdot)$ as $\|x\| \to \infty$ we refer the reader to [140, 346].

4.2.3 *Kernel-Type Estimators for Product Densities and the Asymptotic Variance of Stationary Point Processes*

The Lebesgue density $\rho^{(2)}(x)$ of $\alpha^{(2)}_{red}(\cdot)$—introduced in Sect. 4.1.2 as second-order product density—and, if $\Psi \sim P$ is motion-invariant, the *pair correlation function* $g(r)$ defined by

$$g(r) = \frac{\rho^{(2)}(x)}{\lambda} \text{ for } \|x\| = r > 0 \quad \text{or equivalently } g(r) = \frac{1}{d\,\omega_d\, r^{d-1}} \frac{dK(r)}{dr}$$

are very popular second-order characteristics besides the cumulative K-function. Note that $g(r)$ is understood as derivative (existing for ν_1-almost every $r \geq 0$) of an absolutely continuous K-function (4.14). Since the numerical differentiation of the empirical versions of $K(r)$ as well as of the multiparameter K-function

$K(r_1, \ldots, r_d)$ leads to density estimators of minor quality, the most statisticians prefer the established method of kernel estimation in analogy to probability density functions. The corresponding edge-corrected kernel estimators for $\lambda\, \rho^{(2)}(x)$ and $\lambda^2\, g(r)$ are

$$\widehat{(\lambda\, \rho^{(2)})}_n(x) = \frac{1}{b_n^d} \sum_{i,j \geq 1}^{\neq} \frac{\mathbf{1}(X_i \in W_n)\, \mathbf{1}(X_j \in W_n)}{\nu_d((W_n - X_i) \cap (W_n - X_j))}\, k_d\Big(\frac{X_j - X_i - x}{b_n}\Big)$$

resp.

$$\widehat{(\lambda^2\, g)}_n(r) = \frac{1}{d\, \omega_d\, r^{d-1}\, b_n} \sum_{i,j \geq 1}^{\neq} \frac{\mathbf{1}(X_i \in W_n)\, \mathbf{1}(X_j \in W_n)}{\nu_d((W_n - X_i) \cap (W_n - X_j))}\, k_1\Big(\frac{\|X_j - X_i\| - r}{b_n}\Big),$$

where the *kernel function* $k_d \mid \mathbb{R}^d \mapsto \mathbb{R}$ is integrable (and mostly symmetric, bounded with bounded support) such that $\int_{\mathbb{R}^d} k_d(x)\, dx = 1$ and the sequence of *bandwidths* is chosen such that $b_n \downarrow 0$ and $b_n^d\, \nu_d(W_n) \underset{n\to\infty}{\longrightarrow} \infty$. These conditions imply the pointwise asymptotic unbiasedness of the kernel estimators, namely

$$\mathbf{E}\widehat{(\lambda\, \rho^{(2)})}_n(x) \underset{n\to\infty}{\longrightarrow} \lambda\, \rho^{(2)}(x) \quad \text{and} \quad \mathbf{E}\widehat{(\lambda^2\, g)}_n(r) \underset{n\to\infty}{\longrightarrow} \lambda^2\, g(r)$$

at any continuity point $x \neq o$ of $\rho^{(2)}$ resp. at any continuity point $r > 0$ of g, see e.g. [232–234, 275]. Under some further additional conditions one can show that

$$b_n^d\, \nu_d(W_n)\, \mathbf{var}\, \widehat{(\lambda\, \rho^{(2)})}_n(x) \underset{n\to\infty}{\longrightarrow} \lambda \rho^{(2)}(x) \int_{\mathbb{R}^d} k_d^2(x)\, dx$$

and also central limit theorems (briefly CLT's) and optimal bandwidths can be derived, see for example [232] for an application to testing point process models. Furthermore, various asymptotic results for higher-order kernel-type product density estimators (among them rates of convergence, **P**-a.s. convergence) have been obtained under stronger mixing assumptions, see [234, 275].

Finally, we regard a kernel-type estimator of the limit

$$\sigma^2 = \lim_{n\to\infty} \nu_d(W_n)\, \mathbf{var}\big(\widehat{\lambda}_n\big)$$

which exists for all $\mathbf{B}_2$-mixing stationary PP's. The following estimator has been studied in [238]:

$$\widehat{(\sigma^2)}_n := \widehat{\lambda}_n + \sum_{i,j \geq 1}^{\neq} \frac{\mathbf{1}(X_i \in W_n)\, \mathbf{1}(X_j \in W_n)\, w((X_j - X_i)/c_n)}{\nu_d((W_n - X_i) \cap (W_n - X_j))} - c_n^d\, \widehat{(\lambda^2)}_n \int_{\mathbb{R}^d} w(x)\, dx,$$

where $c_n := b_n\,(v_d(W_n))^{1/d}$ and $w : \mathbb{R}^d \mapsto \mathbb{R}^1$ is a non-negative, symmetric, bounded function with bounded support satisfying $\lim_{x\to o} w(x) = w(o) = 1$.

Theorem 4.8. *For a* $\mathbf{B}_4$*-mixing stationary PP the estimator* $(\widehat{\sigma^2})_n$ *is asymptotically unbiased and mean-square consistent if* $b_n \underset{n\to\infty}{\longrightarrow} 0$, $c_n/r(W_n) \underset{n\to\infty}{\longrightarrow} 0$, $c_n \underset{n\to\infty}{\longrightarrow} \infty$, *and* $b_n\,c_n \underset{n\to\infty}{\longrightarrow} 0$. *If the PP is even* $\mathbf{B}$*-mixing, then* $\sqrt{v_d(W_n)}\,\big(\widehat{\lambda}_n - \lambda\big)/\sigma$ *is asymptotically* $N(0,1)$*-distributed, where* σ *can be replaced by the square root of* $(\widehat{\sigma^2})_n$.

In this way one can construct an asymptotic confidence interval which covers the intensity λ with given probability $1-\alpha$.

4.3 Mixing and *m*-Dependence in Random Point Processes

Large domain statistics requires weak dependence assumptions of the observed spatial process to derive properties of the estimators and to construct asymptotic tests for checking statistical hypotheses. We formulate and apply a spatial ergodic theorem. The notion of m-dependence plays an important role to prove limits theorems for Poisson-driven models demonstrated in particular for the Boolean model and statistics taken from Poisson procesess. We consider also some examples which exhibit appropriate spatial versions of the α- and β-mixing condition.

4.3.1 Poisson-Based Spatial Processes and m-Dependence

Definition 4.7. A family of random variables $\{\xi(t),\ t \in \mathbb{Z}^d\}$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$ is called *m-dependent (d-dimensional) random field* for some $m \in \mathbb{N}$ if for any finite $U, V \subset \mathbb{Z}^d$ the random vectors $(\xi(u))_{u\in U}$ and $(\xi(v))_{v\in V}$ are independent whenever $\max\limits_{1\le i\le d} |\,u_i - v_i\,| > m$ for all $u = (u_1,\dots,u_d)^\top \in U$ and $v = (v_1,\dots,v_d)^\top \in V$, see also Sect. 10.1.2.

For $d = 1$ we use the term "sequence" instead of "field" and in what follows we shall fix the dimension $d \ge 1$. In particular, in the theory of limit theorems for sums of random fields the particular case of m-dependent random variables indexed by a subset of $\mathbb{Z}^d$ plays an important role because most of the classical limit theorems known for sums of independent random variables remain valid with obvious modifications for m-dependent sequences and fields. This includes also a number of refined results such as Berry–Esseen bounds and asymptotic expansions of the remainder term in the CLT, see [226], or Donsker's invariance principle and functional CLT's for empirical m-dependent processes with càdlàg-trajectories, see for example [69].

In stochastic geometry and point process statistics, m-dependent random fields appear in connection with models which are defined by independently marked Poisson processes. We discuss here two examples which exhibit the main idea. This approach has been successfully applied to derive CLT's for functionals of Poisson-cluster processes and Poisson-grain models, see for example [227, 236]. For notational ease, let $W_n = \times_{i=1}^{d}[0, a_i^{(n)})$ be a rectangle with large enough edges $a_1^{(n)}, \ldots, a_d^{(n)}$.

Example 4.2. Let $\Xi = \bigcup_{i\geq 1}(\Xi_i + X_i)$ be a Boolean model generated by the stationary Poisson process $\Psi \sim \Pi_{\lambda\, \nu_d}$ and a bounded typical grain satisfying $\Xi_0 \subseteq [-r, r]^d$ **P**-a.s. for some fixed $r > 0$. We are interested in the asymptotic behaviour of the random d-volume $S_n = \nu_d(\Xi \cap W_n)$ which is closely connected with the empirical volume fraction $\widehat{p}_n = S_n/\nu_d(W_n)$.

Example 4.3. We consider the random sum

$$S_n(r) = \sum_{i\geq 1} \mathbf{1}(X_i \in W_n)\,\big(\Psi - \delta_{X_i}\big)(r\,B + X_i)$$

which coincides up to the scaling factor $1/\nu_d(W_n)$ with the empirical K-function $\big(\widehat{\lambda^2 K_B}\big)_{n,1}(r)$. We are able to derive the Gaussian limit distribution using the CLT for m-dependent field provided that $\Psi \sim \Pi_{\lambda\, \nu_d}$. For simplicity assume that $B \subseteq [-1, 1]^d$.

In both cases take the smallest number $r_i \geq r$ such that the ratio $v_i^{(n)} = a_i^{(n)}/2r_i$ is an integer for $i = 1, \ldots, d$ and decompose W_n into blocks E_t with $t = (t_1, \ldots, t_d)^\top$ as follows:

$$W_n = \bigcup_{t\in V_n} E_t \quad , \quad E_t = \overset{d}{\underset{i=1}{\times}}\big[\,2r_i\,t_i, 2r_i\,(t_i + 1)\big) \quad , \quad V_n = \overset{d}{\underset{i=1}{\times}}\{1, \ldots, v_i^{(n)}\}\,.$$

Then we may write $S_n = \sum\limits_{t\in V_n} \xi(t)$ and $S_n(r) = \sum\limits_{t\in V_n} \xi_r(t)$ with the random variables

$$\xi(t) = \nu_d(\Xi \cap E_t) \text{ and } \xi_r(t) = \sum_{i\geq 1} \mathbf{1}(X_i \in E_t)\,\big(\Psi - \delta_{X_i}\big)(r\,B + X_i), \quad t \in V_n\,,$$

forming a stationary 1-dependent random field due to the independence properties of the stationary Poisson process Ψ and the fact that grains $\{\Xi_i,\ i \geq 1\}$ are i.i.d. and independent of Ψ. By the same arguments we get an i.i.d. sequence of random marked counting measures

$$\Psi_t = \sum_{i\geq 1} \mathbf{1}(X_i \in E_t)\,\delta_{(X_i, \Xi_i)} \text{ for } t \in \mathbb{Z}_d$$

and, in addition, the $\xi(t)$'s admit a representation $\xi(t) = f(\Psi_y\,,\ |t-y| \le 1)$ in terms of a measurable function $f : (\mathcal{N}_{\mathsf{M}}^0)^{3^d} \mapsto \mathbb{R}^1$, where $\mathcal{N}_{\mathsf{M}}^0$ denotes the space of locally-finite marked counting measures on $\times_{i=1}^d [0, 2r_i) \times \mathcal{K}$. In this way $\{\xi(t),\ t \in V_n\}$ becomes a two-dependent random field with *block representation*, see [199, 226] for details. This representation of the field by functions of finite blocks of independent random elements allows to check simple conditions that imply explicit bounds of the remainder terms of asymptotic expansions in the CLT for S_n and $S_n(r)$ as well.

The CLT for (stationary) m-dependent random fields, see for example [69] or references in [226], combined with $|V_n| = v_d(W_n)/(2r)^d$ and $p = \mathbf{E}v_d(\Xi \cap [0,1)^d)$ yields

$$\sqrt{v_d(W_n)}\,\big(\,\widehat{p}_n - p\,\big) \underset{n\to\infty}{\Longrightarrow} N(0,\sigma_p^2) \ \text{ with } \ \sigma_p^2 = (1-p)^2 \int\limits_{\mathbb{R}^d} \big(\mathrm{e}^{\lambda\,\mathbf{E}v_d(\Xi_0\cap(\Xi_0-x))} - 1\big)\,dx.$$

If the compact typical grain Ξ_0 is not strictly bounded, then we first replace Ξ_0 by the truncated grain $\Xi_0 \cap [-r,r]^d$ and apply the above CLT to the corresponding truncated Boolean model $\Xi(r)$. In a second step we show that the ratio

$$\mathbf{var}\big(v_d((\Xi \setminus \Xi(r)) \cap W_n\big)/v_d(W_n)$$

becomes arbitrarily small uniformly in $n \in \mathbb{N}$ as r grows large provided that $\mathbf{E}v_d^2(\Xi_0) < \infty$. Finally, Slutsky's theorem completes the proof of the CLT in the general case.

In Example 4.3 we immediately obtain the normal convergence

$$\sqrt{v_d(W_n)}\,\big(\,\widehat{\big(\lambda^2 K_B\big)}_{n,1}(r) - \lambda^2\,K_B(r)\,\big) \underset{n\to\infty}{\Longrightarrow} N(0,\sigma_B^2(r)) \tag{4.18}$$

with $\sigma_B^2(r) = 2\,\lambda\,v_d(B)\,r^d\,\big(1 + 2\,\lambda\,v_d(B)\,r^d\,\big)$, see also [274] for related CLT's for **B**-mixing stationary PP's. Using the block representation of the random variables $\xi_r(t)$, $t \in V_n$, and the some results in [226], see also references therein, we obtain the optimal Berry–Esseen bound

$$\sup_{x\in\mathbb{R}^1} \Big|\,\mathbf{P}\big(\,\sqrt{v_d(W_n)}\,\big(\,\widehat{\big(\lambda^2 K_B\big)}_{n,1}(r) - \lambda^2\,K_B(r)\,\big) \le x\,\big) - \Phi\big(\frac{x}{\sigma_B(r)}\big)\,\Big| \le \frac{c(\lambda,B,r)}{\sqrt{v_d(W_n)}}\,,$$

where $\Phi(x) := \mathbf{P}(N(0,1) \le x)\,,\ x \in \mathbb{R}^1$, denotes the standard normal distribution function.

Moreover, for the random sum

$$\widehat{S}_n(r) = v_d(W_n)\,\widehat{\big(\lambda^2 K_B\big)}_{n,2}(r) = \sum_{i\ge 1} \mathbf{1}(X_i \in W_n)\,\big(\Psi - \delta_{X_i}\big)\big(\,(r\,B + X_i) \cap W_n\,\big)\,,$$

which equals twice the number of pairs of points having N_B-distance less than or equal to r, a local CLT with asymptotic expansion can be proved by methods developed in [199, 226]:

$$\bigl(1 + |x_n(k,r)|^3\bigr)\,\sqrt{v_d(W_n)}\,\Bigl|\,\frac{1}{2}\,\sqrt{\mathbf{var}\;\widehat{S}_n(r)}\,\mathbf{P}(\widehat{S}_n(r) = 2\,k) - \varphi_n\bigl(x_n(k,r)\bigr)\Bigr| \underset{n\to\infty}{\longrightarrow} 0$$

for any $k = 0, 1, 2, \ldots$, where $x_n(k,r) = \bigl(2\,k - \mathbf{E}\widehat{S}_n(r)\bigr)/\bigl(\mathbf{var}\;\widehat{S}_n(r)\bigr)^{1/2}$ and

$$\varphi_n(x) = \frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-x^2/2}\left(1 + \frac{(x^3 - 3\,x)\,\mathbf{E}\bigl(\widehat{S}_n(r) - \mathbf{E}\widehat{S}_n(r)\bigr)^3}{6\,\sqrt{2\,\pi}\,\bigl(\mathbf{var}\;\widehat{S}_n(r)\bigr)^{3/2}}\right).$$

4.3.2 *Strong Mixing and Absolute Regularity for Spatial Processes*

The quantitative assessment of (weak) dependence between parts of spatial processes (e.g. random fields, point processes, random closed sets) over disjoint subsets of $\mathbb{R}^d$ is based on *mixing coefficients*. These quantities provide uniform bounds of the dependence between σ-algebras generated by the spatial process over these disjoint set which include rates of decay when the distance between these subsets increases. These mixing coefficients permit to derive covariance estimates of the random variables measurable with respect to these σ-algebras. This in turn is essential in proving asymptotic normality for sums of these random fields defined over $(\Omega, \mathcal{F}, \mathbf{P})$. Here we shall briefly discuss two of the most relevant mixing coefficients, see also Sect. 10.1.2.

Definition 4.8. For any two sub-σ-algebras $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$ the α-*mixing* (or *strong*) coefficient $\alpha(\mathcal{A}, \mathcal{B})$ and the β-*mixing* (or *absolute regularity*) coefficient $\beta(\mathcal{A}, \mathcal{B})$ are defined by

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup_{A\in\mathcal{A}, B\in\mathcal{B}} |\,\mathbf{P}(A \cap B) - \mathbf{P}(A)\,\mathbf{P}(B)\,|\,,$$

$$\beta(\mathcal{A}, \mathcal{B}) := \mathbf{E}\sup_{B\in\mathcal{B}} |\,\mathbf{P}(B \mid \mathcal{A}) - \mathbf{P}(B)\,| = \sup_{C\in\mathcal{A}\otimes\mathcal{B}} |\,\mathbf{P}_{\mathcal{A}\otimes\mathcal{B}}(C) - (\mathbf{P}_{\mathcal{A}} \times \mathbf{P}_{\mathcal{B}})(C)\,|\,,$$

where $\mathcal{A} \otimes \mathcal{B}$ is the product σ-algebra generated by $\mathcal{A}$ and $\mathcal{B}$ and $\mathbf{P}_{\mathcal{A}} \times \mathbf{P}_{\mathcal{B}}$ denotes the product measure of the corresponding marginal distributions.

The inequality $2\,\alpha(\mathcal{A}, \mathcal{B}) \le \beta(\mathcal{A}, \mathcal{B})$ is immediately seen from the above definition, see [83] for an all-embracing discussion of mixing coefficients. As already mentioned the covariance $\mathbf{cov}(\xi, \eta)$ can be bounded by means of these mixing coefficient

with respect to the σ-algebras $\mathcal{A} = \sigma(\xi)$ and $\mathcal{B} = \sigma(\eta)$ generated by the random variables ξ and η, respectively. Such covariance bounds are known for long time and can be found in many papers and textbooks on limit theorems for sums of weakly dependent random variables. If ξ, η are real-valued and $p\,,\ q\ \in\ [1,\infty]$ such that $p^{-1} + q^{-1} \le 1$, then the inequality

$$|\ \mathbf{cov}\{\xi,\eta\}\ | \le C\ (\mathbf{E}|\xi|^p)^{1/p}\ (\mathbf{E}|\eta|^q)^{1/q}\ \big(2\,\alpha(\sigma(\xi),\sigma(\eta))\big)^{1-1/p-1/q}$$

holds which has been first proved by Yu.A. Davydov [146] with some positive constant $C\ (\ge\ 10)$. Recently, by improving the approximation technique used in [146], the author and M. Nolde could prove that $C\ =\ 2$ is possible, see also [423] for a different approach. A corresponding estimate with $\beta(\sigma(\xi),\sigma(\eta))$ rather than $\alpha(\sigma(\xi),\sigma(\eta))$ on the right-hand side goes back to K. Yoshihara [522], see also [236] for this and further references.

Let us consider a Voronoi-tessellation $V(\Psi) = \bigcup_{i\ge1} \partial C_i(\Psi)$ generated by a simple stationary PP $\Psi = \sum_{i\ge1} \delta_{X_i}$, where $\partial C_i(\Psi)$ denotes the boundary of the cell $C_i(\Psi)$ formed by all point in $\mathbb{R}^d$ which are closest to the atom X_i , i.e. $C_i(\Psi) = \{x \in \mathbb{R}^d : \|x - X_i\| < \|x - X_j\|, j \ne i\}$, and let denote by $\mathcal{A}_\Psi(F)$ resp. $\mathcal{A}_{V(\Psi)}(F)$ the σ-algebra generated by the PP Ψ restricted to $F\ \subset\ \mathbb{R}^d$ resp. the σ-algebra generated by the random closed set $V(\Psi) \cap F$. With the notation $F_a\ =\ [-a,a]^d$ and $\Delta = b/4$ the estimate

$$\beta\big(\mathcal{A}_{V(\Psi)}(F_a), \mathcal{A}_{V(\Psi)}(F^c_{a+b})\big) \le \beta\big(\mathcal{A}_\Psi(F_{a+\Delta}), \mathcal{A}_\Psi(F^c_{a+3\,\Delta})\big) + R(a,b) \tag{4.19}$$

has been obtained in [230], where $R(a,b)$ is a finite sum of certain void probabilities of the PP Ψ decaying to zero at some rate (depending on a) as $b \to \infty$. A similar estimate of $\beta\big(\mathcal{A}_\Xi(F_a), \mathcal{A}_\Xi(F^c_{a+b})\big)$ could be derived in [236] for stationary grain-germ models $\Xi = \bigcup_{i\ge1}(\Xi_i + X_i)$ in terms of a suitable β-mixing coefficient of the generating stationary PP $\Psi \sim P$ with intensity λ and the distribution function $D(x) = \mathbf{P}(\mathrm{diam}(\Xi_0) \le x)$ of the diameter of the typical grain Ξ_0 :

$$\begin{aligned}\beta\big(\mathcal{A}_\Xi(F_a), \mathcal{A}_\Xi(F^c_{a+b})\big) &\le \beta\big(\mathcal{A}_\Psi(F_{a+\Delta}), \mathcal{A}_\Psi(F^c_{a+3\,\Delta})\big) \\ &\quad + \lambda\, d\, 2^{d+1}\Big[\Big(1+\frac{a}{\Delta}\Big)^{d-1} + \Big(3+\frac{a}{\Delta}\Big)^{d-1}\Big]\int_\Delta^\infty x^d\, dD(x)\end{aligned} \tag{4.20}$$

Note that $\beta\big(\mathcal{A}_\Psi(F_{a+\Delta}), \mathcal{A}_\Psi(F^c_{a+3\,\Delta})\big) = 0$ in (4.19) and (4.20) if $\Psi \sim \Pi_{\lambda\, v_d}$, i.e. for the Poisson–Voronoi tessellation and for Boolean models. Furthermore, there exist precise estimates of this β-mixing coefficient for Poisson-cluster and Cox processes and some classes of dependently thinned Poisson processes. We only mention that both of the previous estimates can be reformulated with slight modifications in terms of α-mixing coefficients.

In [236] a CLT for geometric functionals of β-mixing random closed sets has been proved. The conditions of this CLT can be expressed more explicitly for germ-grain models due to (4.20). CLT's for stationary random fields put assumptions on *mixing rates* derived from mixing coefficients between specific σ-algebras, see [267] in the case of PP's. An application of α-mixing to study empirical functionals of geostatistically marked point processes can be found in [392]. Besides the frequently used CLT of E. Bolthausen [74] the following CLT (first proved and applied in [230]) presents a meaningful alternative to verify asymptotic normality of estimators in stochastic-geometric models.

Let $\xi = \{\xi(t),\, t \in V_n\}$ be a stationary random field with index set $V_n = \{t \in \mathbb{Z}^d : ([0,1)^d + t) \subset W_n\}$, where $\{W_n\,,\, n \in \mathbb{N}\}$ is a convex averaging sequence in $\mathbb{R}^d$ implying $|V_n|/v_d(W_n) \underset{n\to\infty}{\longrightarrow} 1$. Further, $\mathcal{A}_\xi(F)$ denotes the σ-algebra generated by the random variables $\{\xi(t),\, t \in F \cap \mathbb{Z}^d\}$ and $S_n = \sum_{t\in V_n} \xi(t)$.

Theorem 4.9. *Assume that there are two functions β^*_ξ and β^{**}_ξ on $\mathbb{N}$ such that*

$$\beta\big(\mathcal{A}_\xi(F_p), \mathcal{A}_\xi(F^c_{p+q})\big) \;\le\; \begin{cases} \beta^*_\xi(q) & \text{for } p = 1,\ q \in \mathbb{N} \\ p^{d-1}\,\beta^{**}_\xi(q) & \text{for } p \in \mathbb{N},\ q = 1,\dots,p\,. \end{cases}$$

If, for some $\delta > 0$,

$$\mathbf{E}|\,\xi(o)\,|^{2+\delta} < \infty,\ \ \sum_{r=1}^{\infty} r^{d-1}\left(\beta^*_\xi(r)\right)^{\delta/(2+\delta)} < \infty \ \text{ and } \ r^{2d-1}\,\beta^{**}_\xi(r) \underset{r\to\infty}{\longrightarrow} 0\,,$$

then the asymptotic variance $\tau^2 = \lim\limits_{n\to\infty} \mathbf{var}\; S_n/v_d(W_n) = \sum_{t\in\mathbb{Z}^d} \mathbf{cov}(\xi(o),\xi(t))$ exists and the normal convergence $(v_d(W_n))^{-1/2}\left(S_n - |V_n|\,\mathbf{E}\xi(o)\right) \underset{n\to\infty}{\Longrightarrow} N(0,\tau^2)$ holds.

Note that the assertion of Theorem 4.9 remains valid if the slightly weaker α-mixing coefficient is used, see [235] and references therein.

On the other hand, there are situations which require the stronger β-mixing coefficient. For example, $\Psi \sim P$ can be shown to be $\mathbf{B}_k$-mixing for any fixed $k \ge 2$ if $\mathbf{E}\Psi([0,1]^d)^{k+\delta} < \infty$ and

$$\int_1^\infty r^{(k-1)d-1}\left(\beta_\Psi(r)\right)^{\delta/(k+\delta)} dr < \infty$$

for some $\delta > 0$, where the β-mixing coefficient $\beta_\Psi : [1,\infty) \to [0,1]$ is defined as a non-increasing function such that $\beta_\Psi(r) \ge \left(\min\{1, \frac{r}{a}\}\right)^{d-1} \beta\big(\mathcal{A}_\Psi(F_a), \mathcal{A}_\Psi(F^c_{a+r})\big)$ for all $a, r \ge 1$. This implies that $\Psi \sim P$ is Brillinger-mixing if $\beta_\Psi(r) \le e^{-g(r)}$ with $g : [1,\infty) \to [0,\infty]$ satisfying $g(r)/\log(r) \underset{r\to\infty}{\longrightarrow} \infty$.

4.3.3 Testing CSR Based on Empirical K-Functions

Let $\widehat{(\lambda^2 K_B)}_n(r)$ be any of the empirical K-functions $\widehat{(\lambda^2 K_B)}_{n,i}(r)$, $i = 1,2,3$, introduced and discussed in the above Sect. 4.2.1. Below we formulate two functional CLT's for the corresponding centered and scaled empirical process on the interval $[0,R]$ when $\Psi = \{X_i, i \geq 1\}$ is a stationary Poisson process. We distinguish between the cases of known intensity λ and estimated intensity $\widehat{\lambda}_n$ which leads to two distinct zero mean Gauss–Markov limit processes (in the sense of weak convergence in the Skorokhod-space $D[0,R]$, see [69]). For both limits the distribution function of the maximal deviation over $[0,R]$ can be calculated. This fact can be used to establish a Kolmogorov–Smirnov-type test for checking the null hypothesis of CSR via testing the suitably scaled maximal deviation of the empirical K-functions from $\lambda^2 \nu_d(B) r^d$ resp. $(\widehat{\lambda^2})_n \nu_d(B) r^d$, see (4.17). For the details of the proofs (in the particular case $B = B_1(o)$) and some extensions (among them, a Cramér-von Mises-type test for K-functions) the reader is referred to [228].

Theorem 4.10. *Let the stationary Poisson process $\Psi \sim \Pi_{\lambda \nu_d}$ with intensity $\lambda > 0$ be observed in window $W_n = \times_{i=1}^d [0, a_i^{(n)}]$ with unboundedly increasing edges. Then*

$$\zeta_n(r) := \sqrt{\nu_d(W_n)/\lambda}\left(\widehat{(\lambda^2 K_B)}_n(r) - \lambda^2 \nu_d(B) r^d\right) \underset{n\to\infty}{\Longrightarrow} \zeta(r) \stackrel{d}{=} \frac{W(L(r))}{1-L(r)} \tag{4.21}$$

$$\eta_n(r) := \sqrt{\nu_d(W_n)/\widehat{\lambda}_n}\left(\widehat{(\lambda^2 K_B)}_n(r) - (\widehat{\lambda^2})_n \nu_d(B) r^d\right) \underset{n\to\infty}{\Longrightarrow} \eta(r) \stackrel{d}{=} W(2\nu_d(B) r^d)$$

for $0 \leq r \leq R$, where $\underset{n\to\infty}{\Longrightarrow}$ stands for weak convergence in the Skorokhod-space $D[0,R]$. Both weak limits $\zeta(r)$ and $\eta(r)$ for $r \in [0,R]$ are Gaussian diffusion processes with zero means and covariance functions

$$\mathbf{E}\zeta(s)\zeta(t) = 2\lambda \nu_d(B) s^d \left(1 + 2\lambda \nu_d(B) t^d\right) \quad \text{and} \quad \mathbf{E}\eta(s)\eta(t) = 2\nu_d(B) s^d$$

for $0 \leq s \leq t \leq R$. In (4.21), $\stackrel{d}{=}$ means stochastic equivalence, $W = \{W(t), t \geq 0\}$ denotes the one-dimensional standard Wiener process and $L(r) = 2\lambda \nu_d(B) r^d/(1 + 2\lambda \nu_d(B) r^d)$.

Corollary 4.1. *The* continuous mapping theorem*, see [69], applied to (4.21) implies that*

$$\max_{0\leq r\leq R} |\zeta_n(r)| \underset{n\to\infty}{\Longrightarrow} \max_{0\leq t\leq L} \frac{|W(t)|}{1-t} \sim F_L \quad \text{and} \quad \frac{\max\limits_{0\leq r\leq R} |\eta_n(r)|}{\sqrt{2\nu_d(B) R^d}} \underset{n\to\infty}{\Longrightarrow} \max_{0\leq t\leq 1} |W(t)| \sim G\,,$$

where $L = L(R)\,(<1)$ *and*

$$1 - F_L(x) = 2\Big(1 - \Phi\big(x(1-L)/\sqrt{L}\big)\Big) + 2\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{2nx^2}\Big(\Phi\big(x(2n+1-L)/\sqrt{L}\big) - \Phi\big(x(2n-1-L)/\sqrt{L}\big)\Big)$$

and $\quad 1 - G(x) \;=\; 4\big(1-\Phi(x)\big) + 4\sum_{n=1}^{\infty}(-1)^n\big(1-\Phi(\,(2n+1)x\,)\big)\,.$

Remark 4.5. The relevant quantiles of F_L and G are known. Obviously, testing the CSR-property via checking the goodness-of-fit of the K-function seems to be easier when λ is unknown. The convergence of the finite-dimensional distributions of $\{\zeta_n(\cdot),\, n \in \mathbb{N}\}$ follows from the CLT for m-dependent fields and the tightness in $\mathrm{D}[0,R]$ is seen by an exact bound of the mixed fourth-order moment of two consecutive increments. The convergence of the finite-dimensional distributions of $\{\eta_n(\cdot),\, n \in \mathbb{N}\}$ follows by applying a variant of Stein's method to an asymptotically degenerate U-statistic, see [228, 231]

In [231] an analogous test of CSR based on the multivariate K-function $K(r_1,..,r_d) = 2^d\, r_1 \cdot \ldots \cdot r_d$ and its empirical counterpart in case of a Poisson process has been developed. We only sketch the main result in the case of unknown intensity λ. In [231] the case of known λ is also treated in detail.

Let the assumptions of Theorem 4.10 be satisfied. Setting

$$\big(\widehat{\lambda^2 K}\big)_n(\mathbf{r}) := \frac{1}{v_d(W_n)}\sum_{j\geq 1}\mathbf{1}(X_j \in W_n)\,\big(\Psi - \delta_{X_j}\big)\big((\overset{d}{\underset{i=1}{\times}}[-r_i, r_i] + X_j) \cap W_n\big)$$

for $\mathbf{r} = (r_1, \ldots, r_d)^T \in [0,\infty)^d$, and

$$\eta_n(\mathbf{r}) := \sqrt{\frac{v_d(W_n)}{\widehat{\lambda}_n}}\,\Big(\big(\widehat{\lambda^2 K}\big)_n(\mathbf{r}) - (\widehat{\lambda^2})_n\; 2^d \prod_{i=1}^{d} r_i\Big)$$

we obtain a sequence $\{\eta_n(\mathbf{r}),\, \mathbf{r} \in [0,R]^d\}$ of empirical processes belonging to the Skorokhod-space $\mathrm{D}([0,R]^d)$ of d-parameter càdlàg-processes that converges weakly to a Gaussian random field $\{\eta(\mathbf{r}),\, \mathbf{r} \in [0,R]^d\} \overset{d}{=} \{\sqrt{2^{d+1}}\, W_d(\mathbf{r}),\, \mathbf{r} \in [0,R]^d\}$, where $\{W_d(\mathbf{r}),\, \mathbf{r} \in [0,\infty)^d\}$ denotes the d-dimensional *standard Wiener sheet* with mean value function $\mathbf{E}W_d(\mathbf{r}) = 0$ and covariance function $\mathbf{E}W_d(\mathbf{s})W_d(\mathbf{t}) = \prod_{i=1}^{d}(s_i \wedge t_i)$ for $\mathbf{s} = (s_1,\ldots,s_d)^\top$, $\mathbf{t} = (t_1,\ldots,t_d)^\top$. Hence, by the continuous mapping theorem it follows that

$$\max_{\mathbf{r}\in[0,R]^d}|\,\eta_n(\mathbf{r})\,| \Longrightarrow_{n\to\infty} \max_{\mathbf{r}\in[0,R]^d}|\,\eta(\mathbf{r})\,| \overset{d}{=} \sqrt{2^{d+1}R^d}\,\max_{\mathbf{r}\in[0,1]^d}|\,W_d(\mathbf{r})\,|\,.$$

The α-quantiles of the distribution function

$$G_2(x) = \mathbf{P}(|W_2(r_1, r_2)| \leq x \,,\ \forall\ (r_1, r_2) \in [0,1]^2)$$

can be determined only approximately via large-scale simulations of the planar Wiener sheet. In this way we found $m_{0.95} = 2.1165$, $m_{0.99} = 2.7105$, and $m_{0.995} = 2.9313$, where $G_2(m_\alpha) = \alpha$.