Dynamics and Oscillations of GHNNs with Time-Varying Delay

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Abstract. In this paper, we investigate the dynamics and the global exponential stability of the Hopfield Neural network with time-varying delay and variable coefficients. For this purpose, the activation functions are assumed to be globally Lipschitz continuous. The properties of norms and the contraction principle are adjusted to ensure the existence as well as the uniqueness of the the pseudo almost automorphic solution. Then by employing suitable analytic techniques, global attractivity of the unique pseudo almost automorphic solution is established.

Keywords: Hopfield neural networks, Pseudo almost automorphic, Existence of solution, Stability.

1 Introduction

In the past two decades, neural networks has been received considerable attention, and there have been extensive research results presented about the stability analysis of neural network and its applications (See, e.g., [5], [2], [11], [12]). In particular, the stability research related to Hopfield neural networks have been extensively studied and developed in recent years since it has been widely used to model many of the phenomena arising in areas such as signal processing, pattern recognition, static image processing, associative memory, especially for solving some difficult optimization problems, we refer the reader to ([7], [8], [2], [10], [13]) and the references cited therein. As we all know, many phenomena in nature have oscillatory character and their mathematical models have led to the introduction of certain classes of functions to describe them. Such a class form pseudo almost periodic functions which a natural generalization of the concept of almost periodicity. Recently, the concept of almost automorphic functions has widely been used in the investigation of the existence of almost automorphic solutions of various kinds of evolution equations ([3], [4], [9], [10], [13]). Some fundamental properties of almost periodic functions are not verified by them almost automorphic functions, as example the property of uniform continuity. Consequently, the research for the solutions almost automorphic for dynamic systems are more complicated. It should be mentioned that the criteria obtained in this paper extend or improve the results given in [1] since the delays $\tau_j\,(.)$ and

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the neuron firing rate $d_i(.)$ are time-varying. Further, our goal in this paper is to study the pseudo almost automorphic solution of Hopfield model (1.1).

2 Preliminaries: The Functions Spaces

Let $BC(\mathbb{R}, \mathbb{R}^n)$ denote the set of bounded continued functions from $\mathbb{R}$ to $\mathbb{R}^n$. Note that $(BC(\mathbb{R}, \mathbb{R}^n), \|.\|)$ is a Banach space where $\|.\|$ denotes the sup norm $\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|$.

Definition 1. *A continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that for each $t \in \mathbb{R}$*

$$g(t) := \lim_{n\to\infty} f(t + s_n), \lim_{n\to\infty} g(t - s_n) = f(t).$$

Denote by $AA(\mathbb{R}, \mathbb{R}^n)$ the collection of all almost automorphic functions $\mathbb{R} \to \mathbb{R}^n$. The notation $PAA_0(\mathbb{R}, \mathbb{R}^n)$ stands for the spaces of functions

$$PAA_0(\mathbb{R}, \mathbb{R}^n) = \left\{ f \in BC(\mathbb{R}, \mathbb{R}^n) / \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t)\| \, dt = 0 \right\}.$$

Definition 2. *A function $f : \mathbb{R} \to \mathbb{R}^n$ is called pseudo-almost automorphic if it can be decomposed as $f = g + \varphi$, where $g \in AA(\mathbb{R}, \mathbb{R}^n)$ and $\varphi \in PAA_0(\mathbb{R}^n, \mathbb{R}^n)$. The class of all such functions will be denote by $PAA(\mathbb{R}, \mathbb{R}^n)$.*

Remark 1. The function $t \longmapsto \sin\left(\frac{1}{\pi - \sin t - \sin \pi t}\right) + \frac{1}{1+t^2}$ shows that the set of pseudo-almost automorphic functions contains stictly the almost automorphic and the pseudo almost periodic functions. It should be mentionned that $PAA(\mathbb{R}, \mathbb{R}^n)$ is a translation invariant closed subspace of $BC(\mathbb{R}, \mathbb{R}^n)$ containing the constant functions. Furthermore, $PAA(\mathbb{R}, \mathbb{R}^n) = AA(\mathbb{R}, \mathbb{R}^n) \oplus PAA_0(\mathbb{R}, \mathbb{R}^n)$.

3 The Model

Let us consider the following GHNNs

$$\begin{cases} \dot{x}_i(t) = -d_i(t) x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) + b_{ij}(t) g_j(x_j(t - \tau_j(t))) + I_i(t) \\ x_i(t) = \qquad \psi_i(t), -\tau \le t \le 0, 1 \le i \le n. \end{cases} \tag{1.1}$$

where n denotes the total number of units in the GHNNs, $x_i(t)$ corresponds to the state of the $i-$th unit at time t; $d_i(\cdot) > 0$ represents the neuron firing rate, $f(x_j(t))$ and $g(x_j(t - \tau_j))$ denote the outpouts of the $j-$th unit at time t and $(t - \tau_j)$ respectively; $a_{ij}(\cdot)$ and $b_{ij}(\cdot)$ denote the connection weights between the $j-$th unit

and the $i-$th unit with which the $i-$th unit at time t and $(t-\tau_j(t))$ respectively. The function $I_i(t)$ is an external input on the $i-$th unit at time t. $\tau_j(t)$ denotes the transmission delay along the axon of the $j-$th unit and $0 \leq \tau_j(t)$.
(H_1) For all $1 \leq i \leq n$, the functions $d_i(\cdot) > 0$.
(H_2) For all $1 \leq i, j \leq n$, the functions $d_i(\cdot), a_{ij}(\cdot), b_{ij}(\cdot)$ and $I_i(\cdot) \in PAA(\mathbb{R}, \mathbb{R})$.
(H_3) The functions $f_j(\cdot)$ and $g_j(\cdot)$ are pseudo almost automorphic and satisfy the Lipschitz condition, i.e., there are constants $L_j^{f_j} > 0, L_j^{g_j} > 0$ such that for all $x, y \in \mathbb{R}$ and for all $1 \leq j \leq n$, one has

$$|f_j(x) - f_j(y)| \leq L_j^{f_j} |x-y|, |g_j(x) - g_j(y)| \leq L_j^{g_j} |x-y|.$$

(H_4) Denote $a_{ij}^+ = \max_{t\in\mathbb{R}} a_{ij}(t), b_{ij}^+ = \max_{t\in\mathbb{R}} b_{ij}(t), \widetilde{d_i} = \min_{t\in\mathbb{R}} \widetilde{d_i}(t), \widetilde{d} = \min_{1\leq i\leq n} \widetilde{d_i}$ and $r = \max_{1\leq i\leq n} \left[\frac{\sum_{j=1}^{n} L_j^{f_j} a_{ij}^+ + \sum_{j=1}^{n} b_{ij}^+ L_j^{g_j}}{\widetilde{d}} \right] < 1$.

4 Existence and Uniqueness of Pseudo Almost Automorphic Solution

In this section, we establish some results for the existence, uniqueness of pseudo almost automorphic solution of the model (1.1).

Lemma 1. *If $\varphi, \psi \in PAA(\mathbb{R}, \mathbb{R})$, then $\varphi \times \psi \in PAA(\mathbb{R}, \mathbb{R})$.*

Lemma 2. *([15]) $(PAA(\mathbb{R}, \mathbb{R}^n), \|\cdot\|)$ is a Banach space.*

Following along the same lines as in the proof of ([1]) it follows that:

Lemma 3. *If $f(\cdot) \in PAA(\mathbb{R}, \mathbb{R}^n)$ then $f(\cdot - h) \in PAA(\mathbb{R}, \mathbb{R}^n)$ where h is a fixed constant.*

Theorem 1. *Suppose that assumptions $(H_1), (H_3)$ and (H_3) hold. Define the nonlinear operator Γ by: for each $x \in PAA(\mathbb{R}, \mathbb{R}^n)$*

$$(\Gamma x)(t) = \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_s^t d_1(\xi)d\xi} \left[\sum_{j=1}^{n} a_{1j}(s) f_j(x_j(s)) + b_{1j}(s) g_j(x_j(s-\tau_j(s))) \right] ds \\ \vdots \\ \int_{-\infty}^{t} e^{-\int_s^t d_n(\xi)d\xi} \left[\sum_{j=1}^{n} a_{nj}(s) f_j(x_j(s)) + b_{nj}(s) g_j(x_j(s-\tau_j(s))) \right] ds \end{pmatrix}$$

Then Γ maps $PAA(\mathbb{R}, \mathbb{R}^n)$ into itself.

Theorem 2. *Suppose that assumptions $(H_1) - (H_3)$ hold. Then the GHNNs (1.1) has a unique pseudo almost automorphic solution in the region*

$$\mathcal{B} = \left\{ \psi \in PAA(\mathbb{R}, \mathbb{R}^n), \|\psi - \varphi_0\| \leq \frac{r \|I\|_\infty}{\widetilde{d}(1-r)} \right\},$$

where

$$\varphi_0(t) = \begin{pmatrix} \int\limits_{-\infty}^{t} \exp\left(-\int\limits_{s}^{t} d_1(\xi)\,d\xi\right) I_1(s)\,ds \\ \vdots \\ \vdots \\ \int\limits_{-\infty}^{t} \exp\left(-\int\limits_{s}^{t} d_n(\xi)\,d\xi\right) I_n(s)\,ds \end{pmatrix}$$

Proof. Clearly, $\mathcal{B}$ is a closed convex subset of $PAA(\mathbb{R},\mathbb{R}^n)$ and one has $\|\varphi_0(t)\| = \max_{1\le i\le n}\sup_{t\in\mathbb{R}}\left\| \int\limits_{-\infty}^{t} e^{-\int\limits_{s}^{t} d_i(\xi)d\xi} I_i(s)\,ds \right\| \le \frac{\|I\|_\infty}{\widetilde{d}}$. Therefore, for any $\varphi\in\mathcal{B}$ by using the estimate just obtained, we see that

$$\|\varphi\| \le \|\varphi-\varphi_0\| + \|\varphi_0\| \le \frac{r\,\|I\|_\infty}{\widetilde{d}\,(1-r)} + \frac{\|I\|_\infty}{\widetilde{d}} = \frac{\|I\|_\infty}{\widetilde{d}\,(1-r)}.$$

Let us prove that the operator Γ is a self-mapping from $\mathcal{B}$ to $\mathcal{B}$. In fact, for any $\varphi\in\mathcal{B}$, we have $\|(\Gamma\varphi)(t)-\varphi_0(t)\| \le \frac{r\|I\|_\infty}{\widetilde{d}(1-r)}$, which implies that $(\Gamma\varphi)\in\mathcal{B}$. Next, we prove the mapping Γ is a contraction mapping of $\mathcal{B}$. In view of (H_3), for any $\varphi,\psi\in\mathcal{B}$, we have

$$\begin{aligned}
\|(\Gamma\varphi)(t)-(\Gamma\psi)(t)\| = \max_{1\le i\le n}\sup_{t\in\mathbb{R}} \Bigg\| \int_{-\infty}^{t} e^{-\int\limits_{s}^{t} d_i(\xi)d\xi} &\left\{ \sum_{j=1}^{n} a_{ij}(s)\, f_j(\varphi_j(s)) \right. \\
&+ \sum_{j=1}^{n} b_{ij}(s)\, g_j(\varphi_j(s-\tau_j(s))) \\
&+ \sum_{j=1}^{n} b_{ij}(s)\, g_j(\varphi_j(s-\tau_j(s))) \\
&- \sum_{j=1}^{n} a_{ij}(s)\, f_j(\psi_j(s)) + \sum_{j=1}^{n} b_{ij}(s)\, g_j(\psi_j(s-\tau_j(s))) \Bigg\| \\
\le \max_{1\le i\le n}\sup_{t\in\mathbb{R}} &\left[\frac{\sum\limits_{j=1}^{n} L_j^f\,|a_{ij}(t)| + \sum\limits_{j=1}^{n} L_j^g\,|b_{ij}(t)|}{\widetilde{d}} \right] \|\varphi-\psi\|
\end{aligned}$$

which proves that Γ is a contraction mapping. Consequently, Γ possess a unique fixed point $x^* \in \mathcal{B}$ that is $\Gamma(x^*) = x^*$. Hence, x^* is the unique pseudo almost automorphic solution of (1.1) in $\mathcal{B}$.

5 The Global Attractivity of the paa Solution

Let $x^*(\cdot) = (x_1^*(\cdot), \cdots, x_n^*(\cdot))^T$ the pseudo almost automorphic solution of theorem 1 and $x(\cdot) = (x_1(\cdot), \cdots, x_n(\cdot))^T$ be an arbitrary solution of (1.1). So, one has

$$\dot{x_i^*}(t) = -d_i(t) x_i^*(t) + \sum_{j=1}^{n} \left[a_{ij}(t) f\left(x_j^*(t)\right) + b_{ij}(t) g\left(x_j^*(t-\tau_j(t))\right)\right] + I_i(t)$$

and

$$\dot{x}_i(t) = -d_i(t) x_i(t) + \sum_{j=1}^{n} \left[a_{ij}(t) f(x_j(t)) + b_{ij}(t) g_j(x_j(t-\tau_j(t)))\right] + I_i(t).$$

Let us pose for all $1 \le i \le n$, $z_i(\cdot) = x_i(\cdot) - x_i^*(\cdot)$. Consequently, we obtain

$$\begin{cases} \dot{z}_i(t) = -d_i(t) z_i(t) + \sum_{j=1}^{n} a_{ij}(t) F_j(z_j(t)) + \sum_{j=1}^{n} b_{ij}(t) G_j(x_j(t-\tau_j(t))) \\ z_i(t) = \theta_i(t), -r \le t \le 0, 1 \le i \le n. \end{cases} \tag{1.2}$$

where for all $1 \le i, j \le n$,

$$F_j(z_j(\cdot)) = f_j(x_j(\cdot)) - f_j\left(x_j^*(\cdot)\right), G_j(z_j(\cdot)) = g_j(x_j(\cdot)) - g_j\left(x_j^*(\cdot)\right)$$

and $\theta_i(\cdot) = \psi_i(\cdot) - x_i^*(\cdot)$. Clearly, the pseudo almost automorphic solution $x^*(\cdot)$ of system (1.1) is global attractivity if and only if the equilibrium point O of system (1.2) is global attractivity. So let us study the global attractivity of the equilibrium point O for system (1.2).

Theorem 3. *Suppose that assumptions $(H_1) - (H_4)$ hold, then the equilibrium point O of the nonlinear system (1.2) is global attractive.*

Proof. First, let us prove that the solution of system (1.2) are uniformly bounded. In other words, there exists $M > 0$ such that for all $t \ge 0$ one has $\|z(t)\| \le M$. By the assumption (H_4), $1 - r > 0$. So for any given continuous function $\theta(\cdot)$, there exists a large number $M > 0$, such that $\|\theta\| < M$ and $(1-r) M > 0$. Let κ a real number, $\kappa < 1$. We shall prove that for all $t \ge 0$, $\|z(t)\| \le \kappa M$. Suppose the contrary, then there must be some $t' > 0$, such that $\begin{cases} \|z(t')\| = \kappa M \\ \|z(t)\| < \kappa M, 0 \le t \le t' \end{cases}$
In view of (H_3), (H_4) and the equation (1.2), we have

$$\|z(t')\| \le \max_{1 \le i \le n} \left\{ |\theta_i(0)| e^{-\int_0^{t'} d_i(u) du} + \int_0^{t'} e^{-\int_s^{t'} d_i(u) du} \times \right.$$
$$\left. \left(\sum_{j=1}^{n} a_{ij}(s) L_j^f |z_j(s)| + b_{ij}(s) L_j^g |z_j(s-\tau_j(s))| \right) ds \right\}$$

$$\leq \max_{1\leq i\leq n}\left\{e^{-\widetilde{d}_i t'}+\left(\frac{\sum\limits_{j=1}^{n} a_{ij}^{+}L_j^{f}+b_{ij}^{+}L_j^{g}}{\widetilde{d}_i}\right)\left(1-e^{-\underline{d_i}t'}\right)\right\}hM$$
$$< hM$$

which gives a contradiction. Consequently, for all $t \geq 0$, $\|z(t)\| \leq \kappa M$. Let us take $\kappa \longrightarrow 1$, then for all $t \geq 0$, $\|z(t)\| \leq M$. Thus, there is a constant $\sigma \geq 0$, such that $\limsup\limits_{t\longrightarrow+\infty}\|z(t)\| = \beta$. It follows that $\forall\varepsilon > 0, \exists t_2 < 0$, such that

$$(\forall t, t \geq t_2 \Longrightarrow \|z(t)\| \leq (1+\varepsilon)\,\beta)\,.$$

So,

$$\begin{aligned}\dot{z}_i(t)+d_i(t)\,z_i(t) &= \sum_{j=1}^{n} a_{ij}(t)\,F_j(z_j(t))+\sum_{j=1}^{n} b_{ij}(t)\,G_j(x_j(t-\tau_j(t)))\\ &\leq \sum_{j=1}^{n}|a_{ij}(t)|\,|F_j(z_j(t))|+\sum_{j=1}^{n}|b_{ij}(t)|\,|G_j(x_j(t-\tau_j(t)))|\\ &\leq \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} a_{ij}^{+}L_j^{f}+b_{ij}^{+}L_j^{g}\right)(1+\varepsilon)\,\beta.\end{aligned}$$

So, throug the integration, we obtain the inequality

$$|z_i(t)| \leq |\theta_i(0)|\,e^{-\int_0^t d_i(u)du}+\left\{\max_{1\leq i\leq n}\left(\sum_{j=1}^{n} a_{ij}^{+}L_j^{f}+b_{ij}^{+}L_j^{g}\right)(1+\varepsilon)\,\beta\right\}\int_0^t e^{-\int_s^t d_i(u)du}ds$$
$$\leq \|\theta\|\,e^{-\underline{d_i}t}+\left\{\max_{1\leq i\leq n}\left(\sum_{j=1}^{n} a_{ij}^{+}L_j^{f}+b_{ij}^{+}L_j^{g}\right)(1+\varepsilon)\,\beta\right\}\int_0^t e^{-\underline{d_i}(t-s)}ds$$
$$\leq \|\theta\|\,e^{-\underline{d_i}t}+\left\{\max_{1\leq i\leq n}\left(\frac{\sum\limits_{j=1}^{n} a_{ij}^{+}L_j^{f}+b_{ij}^{+}L_j^{g}}{\underline{d_i}}\right)\right\}(1+\varepsilon)\,\sigma\left(1-e^{-\underline{d_i}t}\right)$$

In particular,

$$\begin{aligned}\limsup_{t\longrightarrow+\infty}\|z(t)\| &\leq \limsup_{t\longrightarrow+\infty}\max_{1\leq i\leq n}\Bigg[\|\theta\|\,e^{-\widetilde{d}_i t}\\ &\quad+\left\{\max_{1\leq i\leq n}\left(\frac{\sum\limits_{j=1}^{n} a_{ij}^{+}L_j^{f_j}+b_{ij}^{+}L_j^{g_j}}{\widetilde{d}_i}\right)\right\}(1+\varepsilon)\,\beta\left(1-e^{-\underline{d_i}t}\right)\Bigg]\\ &= [r\,(1+\varepsilon)\,\sigma]\,.\end{aligned}$$

In other words, $\beta \leq r(1+\varepsilon)\beta$. Passing to limit when $\varepsilon \longrightarrow 0$, we obtain $\beta(1-r) \leq 0$. By condition (H_4), we obtain $\sigma = 0$ which imply that

$$\lim_{t \longrightarrow +\infty} \|z(t)\| = \lim_{t \longrightarrow +\infty} \|x_i(t) - x_i^*(t)\| = 0$$

and consequently the proof of this theorem is completed.

Example 1. Let us consider the following Hopfield neural network

$$\dot{x}_i(t) = -d_i(t)\, x_i(t) + \sum_{j=1}^{3} a_{ij}(t)\, f_j(x_j(t)) + \sum_{j=1}^{3} b_{ij}(t)\, g_j(x_j(t-\tau_j(t)))$$

where

$$\begin{pmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \end{pmatrix} = \begin{pmatrix} 3+\cos^2 \pi t \\ 7 + 2\cos\sqrt{2}t \\ 5 + 2\sin\sqrt{3}t \end{pmatrix} \Longrightarrow \widetilde{d} = 2$$

for all $x \in \mathbb{R}$, for all $t \in \mathbb{R}, \forall 1 \leq j \leq 3 f_j(t) = g_j(t) = \sin t$, and

$$(a_{ij}) = \begin{pmatrix} 0,2\cos t & 0,5\cos\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right) + \frac{0,5}{1+t^2} & 0,2\sin\left(\frac{1}{1+\sin t+\sin\sqrt{5}t}\right) \\ 0,1\cos\sqrt{3}t & 0,3\cos\left(\frac{1}{2+\sin t+\sin\sqrt{3}t}\right) + \frac{0,2}{1+t^2} & 0,2\cos\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right) \\ 0,2\sin\sqrt{2}t & 0,2\sin\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right) + \frac{0,1}{1+t^2} & 0,2\sin\left(\frac{1}{2+\sin t+\sin\sqrt{3}t}\right) \end{pmatrix}$$

$$(b_{ij}) = \begin{pmatrix} 0,1\sin\sqrt{2}t & 0,2\cos\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right) + \frac{0,1}{1+t^2} & 0,1\sin\left(\frac{1}{1+\sin t+\sin\sqrt{5}t}\right) \\ 0,2\cos\sqrt{5}t & 0,1\cos\left(\frac{1}{2+\sin t+\sin\sqrt{3}t}\right) + \frac{0,2}{1+t^2} & 0,2\cos\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right) \\ 0,2\sin\sqrt{3}t & 0,1\sin\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right) + \frac{0,1}{1+t^2} & 0,1\sin\left(\frac{1}{2+\sin t+\sin\sqrt{3}t}\right) \end{pmatrix}$$

$$I_i(t) = \begin{pmatrix} \cos\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right) + \frac{1}{1+t^2} \\ \sin\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right) + \frac{2}{1+t^2} \\ \cos\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right) + \frac{1}{1+t^2} \end{pmatrix} \Longrightarrow \left(\overline{I_i}\right)_{1\leq i\leq 3} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \Longrightarrow \beta = 3. \text{ Then}$$

$$r = \max_{1\leq i\leq n} \sup_{t\in\mathbb{R}} \left[\frac{\sum_{j=1}^{3} L_j^{f_j} |a_{ij}(t)| + \sum_{j=1}^{n} L_j^{g_j} |b_{ij}(t)|}{\widetilde{d}} \right]$$

$$= \max \frac{\sum_{j=1}^{3} a_{ij}^{+} + b_{ij}^{+}}{\widetilde{d}} = \max \sum_{j=1}^{3} \left(\frac{1.8}{2}, \frac{1.5}{2}, \frac{1.2}{2} \right) < 1.$$

Therefore, all conditions of Theorem 2 are satisfied, then the delayed Hopfield neural networks (1.1) has a unique pseudo almost automorphic solution in the region

$$\mathcal{B} = B(\varphi_0, r) = \left\{ x \in PAA(\mathbb{R}, \mathbb{R}^3), \|\varphi - \varphi_0\| \leq \frac{0,9 \times 3 + 0}{2\,(1 - 0,9)} = 0,135 \right\}.$$

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