Hierarchical Model Reduction: Three Different Approaches

S. Perotto and A. Zilio

Abstract We present three different approaches to model, in a computationally cheap way, problems characterized by strong horizontal dynamics, even though in the presence of transverse heterogeneities. The three approaches are based on the hierarchical model reduction setting introduced in Ern et al. (Hierarchical model reduction for advection-diffusion-reaction problems. In: Kunisch K, Of G, Steinbach O (eds) Numerical mathematics and advanced applications. Springer (2008), pp 703–710) and Perotto et al. (Multiscale Model Simul 8(4):1102–1127, 2010).

1 Motivations

We focus on the modeling of engineering applications which exhibit a dominant dynamic (e.g., flows in tubular domains as in haemodynamics or in a channel network as in hydrodynamics, flows through anisotropic porous media). For this modeling, downscaled models, where only the dominant space dependence is considered, are sometimes advisable. Nevertheless, in the presence of significant transverse dynamics, these downscaled models may become uneffective (see, e.g., [2]).

We move consequently to a different approach, known as Hierarchical Model (Hi-Mod) reduction to get a sort of trade-off between accuracy and efficiency

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[1, 3]. We suitably rewrite the *full problem* as a set of coupled 1D differential problems (i.e., the *reduced model*) associated with the dominant dynamic, while the information along the transverse directions are lumped in the coefficients of the reduced formulation. We focus on a generic second-order elliptic full problem, given by

$$\text{find } u \in V \quad : \quad a(u, v) = \mathscr{F}(v) \quad \forall v \in V, \tag{1}$$

with $V \subseteq H^1(\Omega)$ a Hilbert space, $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ a continuous and coercive bilinear form and $\mathscr{F}(\cdot) : V \to \mathbb{R}$ a continuous linear functional.

In this paper we propose three different techniques for a Hi-Mod reduction. The first two approaches have already been validated with good results (see [1,3]). The nodewise Hi-Mod reduction represents the novelty of this paper.

2 Hierarchical Model Reduction Techniques

We fix the basic ingredients to perform a Hi-Mod reduction [1,3]. We first introduce a constraint on the computational domain. We assume $\Omega = \bigcup_{x \in \Omega_{1D}} \{x\} \times \gamma_x$, i.e., Ω coincides with a *fiber bundle*, where $\Omega_{1D} = (x_0, x_1)$ is the supporting fiber (parallel to the dominant dynamics) while γ_x is the transverse fiber at x (parallel to the secondary transverse dynamics). In particular, we focus on 2D domains.

Then, for any $x \in \Omega_{1D}$, we introduce the map $\psi_x : \gamma_x \to \widehat{\gamma}$ between the generic fiber γ_x and the reference fiber $\widehat{\gamma}$, so that the physical domain Ω is mapped into the reference domain $\widehat{\Omega} = \Omega_{1D} \times \widehat{\gamma}$ via the map $\Psi : \Omega \to \widehat{\Omega}$, given by $\Psi(\mathbf{z}) = \widehat{\mathbf{z}}$, where $\mathbf{z} = (x, y), \widehat{\mathbf{z}} = (\widehat{x}, \widehat{y})$ with $\widehat{x} = x$ and $\widehat{y} = \psi_x(y)$. We assume that ψ_x is a C^1-diffeomorphism for all $x \in \Omega_{1D}$ and that Ψ is differentiable with respect to $\mathbf{z}$. A standard choice for ψ_x is an affine map.

The fiber structure on Ω is at the basis of all the Hi-Mod reduction techniques below. The common idea is to differently tackle the dependence of the full solution on the dominant and on the transverse directions. The former is spanned via a standard (1D) finite element basis. The latter are expanded into a modal basis $\{\varphi_k\}_{k \in \mathbb{N}^+}$ of functions in $H^1(\widehat{\gamma})$, orthonormal with respect to the $L^2(\widehat{\gamma})$-scalar product and compatible with the boundary conditions along the horizontal sides of Ω.

2.1 Uniform Hi-Mod Reduction

This approach resorts to a global one-dimensional space $V_{1D} \subseteq H^1(\Omega_{1D})$ to describe the solution along the fiber Ω_{1D} as well as to the same number of modal functions along the transverse directions [1, 3]. In particular, the functions in V_{1D} take into account the boundary conditions on the vertical sides of Ω.

The discrete uniform Hi-Mod reduced formulation for (1) reads: for a certain modal index $m \in \mathbb{N}^+$,

$$\text{find } u_m^h \in V_m^h \quad : \quad a(u_m^h, v_m^h) = \mathscr{F}(v_m^h) \quad \forall v_m^h \in V_m^h, \tag{2}$$

where the discrete reduced space

$$V_m^h = \Big\{ v_m^h(x,y) = \sum_{k=1}^{m} \widetilde{v}_k^{\,h}(x)\, \varphi_k(\psi_x(y)), \text{ with } \widetilde{v}_k^{\,h} \in V_{1D}^h,\ x \in \Omega_{1D},\ y \in \gamma_x \Big\} \tag{3}$$

establishes an actual *hierarchy* of reduced models marked by the modal index m, i.e., by the different level of detail in describing the transverse dynamics of the full problem. Space $V_{1D}^h \subset V_{1D}$ is a finite element space associated with a subdivision $\mathscr{T}_h$ of Ω_{1D}, with $\dim(V_{1D}^h) = N_h < +\infty$. A standard density assumption is made on V_{1D}^h. Then, suitable hypotheses of conformity and of spectral approximability guarantee the inclusion $V_m^h \subset V$ as well as the well-posedness of the reduced formulation (2).

If we replace in (2) the reduced solution with the corresponding discrete modal representation ($u_m^h(\mathbf{z}) = \sum_{k=1}^m \widetilde{u}_k^{\,h}(x)\, \varphi_k(\psi_x(y))$) and choose $v_m^h = \vartheta_i \varphi_j$, with ϑ_i the generic finite element basis function, we are led to solve

$$\sum_{k=1}^{m} a(\widetilde{u}_k^{\,h} \varphi_k, \vartheta_i \varphi_j) = \mathscr{F}(\vartheta_i \varphi_j) \quad j = 1, \ldots, m,\ i = 1, \ldots, N_h \tag{4}$$

i.e., a set of coupled 1D problems instead of the full 2D problem. From an algebraic viewpoint, (4) coincides with a linear system with an $m \times m$ block matrix, where each block is an $N_h \times N_h$ matrix exhibiting the sparsity of the finite element space.

An appropriate choice of the modal index m in (3) is certainly the most critical issue of the uniform Hi-Mod reduction. This choice can be driven, e.g., by an a priori knowledge of the phenomenon at hand. In [3] a "trial and error" approach is suggested: we move from the computationally cheapest choice for m ($m = 1$) and then we gradually increase such a value. We stop when the addition of the successive modal function does not significantly improve the accuracy of the reduced solution. This choice may become really uneffective when strongly localized transverse dynamics are present. In such a case a large number of modal functions is required on the whole Ω, even though it would be strictly necessary only on the portion of Ω where the strong dynamics occur.

2.2 *Piecewise Hi-Mod Reduction*

To overcome the intrinsic limit of a uniform Hi-Mod reduction, we move in [3] to a new formulation, where a different number of modes is employed in different parts of Ω: essentially, large values of m are used where the transverse dynamics are relevant, small values where the dynamics are less important. In particular, we resort to a domain decomposition approach to glue the models associated with a different number of modes: the reduced problem is thus split and iteratively solved on subdomains of Ω. The modal index m becomes therefore a piecewise constant vector: this justifies the name of this approach.

Following [4], the discrete piecewise Hi-Mod reduced formulation for (1) reads: for a certain modal multi-index $\mathbf{m} \in [\mathbb{N}^+]^s$,

$$\text{find } u_{\mathbf{m}}^{b,h} \in V_{\mathbf{m}}^{b,h} \quad : \quad a_\Omega(u_{\mathbf{m}}^{b,h}, v_{\mathbf{m}}^{b,h}) = \mathscr{F}_\Omega(v_{\mathbf{m}}^{b,h}) \quad \forall v_{\mathbf{m}}^{b,h} \in V_{\mathbf{m}}^{b,h}, \tag{5}$$

with $a_\Omega(u_{\mathbf{m}}^{b,h}, v_{\mathbf{m}}^{b,h}) = \sum_{i=1}^{s} a_i(u_{\mathbf{m}}^{b,h}\big|_{\Omega_i}, v_{\mathbf{m}}^{b,h}\big|_{\Omega_i})$, $\mathscr{F}_\Omega(v_{\mathbf{m}}^{b,h}) = \sum_{i=1}^{s} \mathscr{F}_i(v_{\mathbf{m}}^{b,h}\big|_{\Omega_i})$ where $a_i(\cdot,\cdot)$ and $\mathscr{F}_i(\cdot)$ are the restrictions of the bilinear and linear forms in (1) to the s subdomain Ω_i of Ω, such that $\overline{\Omega} = \cup_{i=1}^{s} \overline{\Omega}_i$. The modal multi-index $\mathbf{m} = \{m_i\}_{i=1}^{s}$ collects the number of modes employed on each Ω_i. The discrete reduced space $V_{\mathbf{m}}^{b,h}$ is defined by

$$\begin{aligned} V_{\mathbf{m}}^{b,h} = \Big\{ v_{\mathbf{m}}^{b,h} \in L^2(\Omega) \; : \; v_{\mathbf{m}}^{b,h}|_{\Omega_i} = \sum_{k=1}^{m_i} \widetilde{v}_k^{i,h}|_{\Omega_{1D,i}}(x)\, \varphi_k(\psi_x(y)) \in H^1(\Omega_i) \\ \forall i = 1,\ldots,s, \text{ with } \widetilde{v}_k^{i,h} \in V_{1D}^{b,h} \text{and s.t., } \forall k = 1,\ldots,m_\perp^j \text{ with } j = 1,\ldots,s-1, \\ \int_{\widehat{\gamma}} \big[v_{\mathbf{m}}^{b,h}|_{\Omega_{j+1}}(\sigma_j, \psi_{\sigma_j}^{-1}(\widehat{y})) - v_{\mathbf{m}}^{b,h}|_{\Omega_j}(\sigma_j, \psi_{\sigma_j}^{-1}(\widehat{y})) \big] \varphi_k(\widehat{y})\, d\widehat{y} = 0 \Big\}, \end{aligned} \tag{6}$$

with $m_\perp^j = \min(m_j, m_{j+1})$, $\Omega_{1D,i} = \Omega_{1D} \cap \Omega_i$, $\sigma_j = \overline{\Omega}_j \cap \overline{\Omega}_{j+1}$. Space $V_{1D}^{b,h}$ is a suitable discrete space associated with the finite element partition $\mathscr{T}_h$: it represents a subset of the one-dimensional broken Sobolev space $H^1(\Omega_{1D}, \mathscr{T}_{\Omega_{1D}})$ depending on the partition $\mathscr{T}_{\Omega_{1D}} = \{\Omega_{1D,i}\}_{i=1}^{s}$ of the supporting fiber Ω_{1D}. Likewise, the space $V_{\mathbf{m}}^{b,h}$ is a subset of the two-dimensional broken Sobolev space $H^1(\Omega, \mathscr{T}_\Omega)$ associated with the partition $\mathscr{T}_\Omega = \{\Omega_i\}_{i=1}^{s}$ of Ω.

Notice that the integral condition in (6) weakly enforces the continuity of the solution in correspondence with the minimum number of modes employed on the whole Ω. This does not guarantee a priori the conformity of the reduced solution $u_{\mathbf{m}}^{b,h}$ in (5). Different strategies can be adopted to impose this interface condition: in [4] we resort to an iterative substructuring Dirichlet/Neumann method (with relaxation).

From a computational viewpoint, at each iteration of the Dirichlet/Neumann scheme, we apply, separately, a uniform Hi-Mod reduction on the subdomains Ω_i. This leads to solve s systems of coupled 1D problems as in (4), with an $m_i N_h^i \times m_i N_h^i$ block matrix, whose factorization is stored once and for all at the first iteration and with $N_h^i < +\infty$ the dimension of the finite element space associated with $\Omega_{1D,i}$.

The choice of the modal multi-index $\mathbf{m}$ in (5) can be made a priori, as in [3], when we have some hints about u, or automatically, as in [4], if a suitable a posteriori modeling error estimator drives the selection of both Ω_i and $\mathbf{m}$.

2.3 Nodewise Hi-Mod Reduction

The piecewise Hi-Mod reduction represents a significative computational improvement with respect to the uniform Hi-Mod approach. Yet, it exhibits some limitations especially when dealing with extremely localized (almost pointwise) transverse dynamics or, on the contrary, with dynamics which involve the whole domain, even though with a different intensity (see Sect. 3 for an example). In the former case, a sufficiently large number of modes is assigned to a subdomain around the localized dynamic but, likely, the size of this domain will be excessively large compared with the entity of the dynamic; in the latter case, a piecewise Hi-Mod reduction may become uneffective so that the only feasible way is the uniform approach.

These considerations prompt us to set up a third Hi-Mod reduction procedure: the novelty is that now the modal functions are associated with the nodes of the finite element partition, in contrast to the piecewise approach where the modes are associated with subdomains of Ω. The association of the modes with the finite element nodes motivates the name chosen for this approach.

The trick which inspired us in setting up the nodewise approach consists of properly rewriting the modal expansion in the discrete space (3). By exploting the finite element basis $\{\vartheta_i\}$, we have indeed

$$v_m^h(x,y) = \sum_{k=1}^{m} \widetilde{v}_k^h(x)\,\varphi_k(\psi_x(y)) = \sum_{k=1}^{m} \Big[\sum_{i=1}^{N_h} \widetilde{v}_{k,i}^h\,\vartheta_i(x) \Big] \varphi_k(\psi_x(y)). \tag{7}$$

Notice that the leading role in such an expansion is taken by the summation on the modes. Simply by exchanging the two summations, we get

$$v_m^h(x,y) = \sum_{i=1}^{N_h} \Big[\sum_{k=1}^{m} \widetilde{v}_{k,i}^h\,\varphi_k(\psi_x(y)) \Big] \vartheta_i(x), \tag{8}$$

i.e., a representation for v_m^h, equivalent to (7), where the expansion runs over the finite element nodes. This leads us to define, in a straightforward way, a new discrete reduced space $V_{\mathbf{M}}^h$ where, ideally, the number of the modal basis functions may vary on each finite element node:

$$V_{\mathbf{M}}^h = \Big\{ v_{\mathbf{M}}^h(x,y) = \sum_{i=1}^{N_h} \Big[\sum_{k=1}^{m_i^N} \widetilde{v}_{k,i}^h\,\varphi_k(\psi_x(y)) \Big] \vartheta_i(x),\ \text{with } x \in \Omega_{1D},\ y \in \gamma_x \Big\}. \tag{9}$$

The global modal index m in (8) is here replaced by the nodewise modal index m_i^N, with $\mathbf{M} = \{m_i^N\}_{i=1}^{N_h}$ the vector of the modes for each finite element node.

The discrete nodewise Hi-Mod reduced formulation for (1) thus reads: for a certain modal multi-index $\mathbf{M} \in [\mathbb{N}^+]^{N_h}$,

$$\text{find } u_{\mathbf{M}}^h \in V_{\mathbf{M}}^h \quad : \quad a(u_{\mathbf{M}}^h, v_{\mathbf{M}}^h) = \mathscr{F}(v_{\mathbf{M}}^h) \quad \forall v_{\mathbf{M}}^h \in V_{\mathbf{M}}^h \tag{10}$$

where $a(\cdot,\cdot)$ and $\mathscr{F}(\cdot)$ coincides with the bilinear and linear forms in (1).

The algebraic counterpart of (10) is represented by a linear system whose matrix has a structure similar to that of the uniform case (with $m = \max_i m_i^N$), except that some rows and columns are deleted where $m_i^N < \max_i m_i^N$.

The change of perspective introduced by the nodewise Hi-Mod reduction relieves us from using a domain decomposition scheme in the presence of a different number of modal functions in Ω. This represents a significative improvement with respect to the piecewise Hi-Mod approach. No iterative procedure is now required to get the reduced solution; on the contrary, a domain decomposition scheme could now be employed to deal with more complex geometries (e.g., a bifurcation) not taken into account by the setting in Sect. 2.

The nodewise Hi-Mod reduction yields a reduced solution which is continuous, i.e., H^1-conformal, in Ω unlike the piecewise approach, where model discontinuities may occur. Moreover, the nodewise formulation makes sense, by definition, only after introducing the finite element basis. Spaces V_m^h and $V_{\mathbf{m}}^{b,h}$ have, on the contrary, a continuous counterpart obtained by replacing the modal coefficients in (3) and (6) with functions in V_{1D} and $H^1(\Omega_{1D}, \mathscr{T}_{\Omega_{1D}})$, respectively (see [1, 3] for the details).

Concerning the choice of the modal multi-index $\mathbf{M}$ in (10), we can ideally proceed via an a priori or an automatic selection, exactly as for the piecewise approach.

3 Numerical Assessment

We numerically validate the proposed Hi-Mod reduction procedures, to focus on the corresponding advantages and limits. In particular, we use affine finite elements to discretize the problem along Ω_{1D}, while employing sinusoidal functions to model the transverse dynamics. We evaluate the integrals of the sine functions via Gaussian quadrature formulas, based on, at least, four quadrature nodes per wavelenght.

First test case. This test case is meant to compare the three approaches onto the same full configuration. For this purpose, we consider a problem characterized by an analytical solution. We solve the Poisson problem $-\Delta u = f$ on $\Omega = (0,2) \times (0,\pi)$, completed with full homogeneous Dirichlet boundary conditions, so that $V \equiv H_0^1(\Omega)$, $V_{1D} \equiv H_0^1((0,2))$. The source term f is chosen such that the full solution is

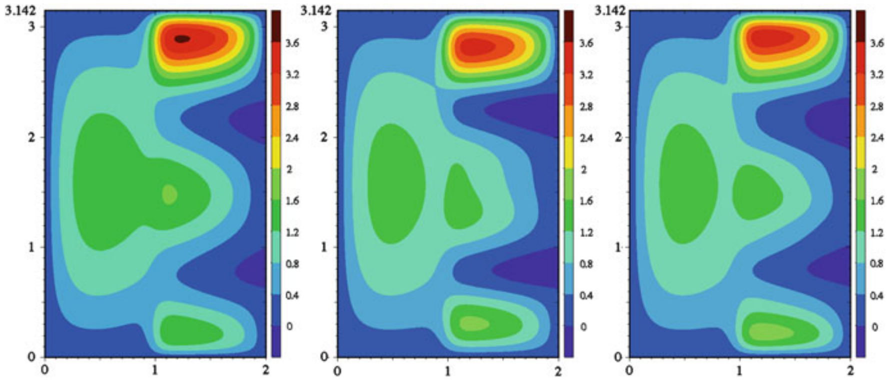

Fig. 1 First test case: full solution (*left*); uniformly reduced solutions, u_7^h (*center*), u_{16}^h (*right*)

$$u(x,y) = \tfrac{(256-x^8)(256-(2-x)^8)}{64800}\left\{\tfrac{100}{247}y(\pi-y)(2-x)\right.$$
$$\left. + \, y\left(\tfrac{\pi}{5}-y\right)\left(\tfrac{\pi}{3}-y\right)\left(\tfrac{3}{5}\pi-y\right)\left(\tfrac{3}{4}\pi-y\right)(\pi-y)\big(1+\tanh(10x-10)\big)\right\}.$$

In Fig. 1 (left) we show the contour plot of u approximated via a finite element scheme on a uniform unstructured grid of about 25,300 elements. Solution u clearly exhibits a smooth behaviour on the left part of Ω and a more irregular trend on the right.

We first apply the uniform Hi-Mod approach, by selecting $m = 7$ and $m = 16$ modes and choosing a uniform partition $\mathscr{T}_h$ of Ω_{1D} into 20 subintervals. Figure 1 (center-right) gathers the contour plots of the corresponding reduced solution: as expected, 16 modes provide us with a more close approximation, even though the difference between u_7^h and u_{16}^h is not so striking.

We successively assess the piecewise approach, inspired by the intrinsic heterogeneity of u. We split Ω into the subdomains $\Omega_1 = (0, 0.9) \times (0, \pi)$ and $\Omega_2 = (0.9, 2) \times (0, \pi)$; then we employ $m_1 = 1$ and $m_2 = 7$ modes, respectively and the same partition $\mathscr{T}_h$ as above. The domain decomposition algorithm (with relaxation equal to 0.5) converges after three iterations to the reduced solution $u_{1,7}^{b,h}$ in Fig. 2 (left). The model discontinuity is evident: we are in the presence of a nonconformal reduced solution. Formulation (6) guarantees indeed the continuity on Ω of both the trace and the flux only of the first $m_\perp^G = \min_{j=1}^s m_j$ modal components of $u_{\mathbf{m}}^b$ (i.e., only of the first one in such a case). More in general, as proved in [3], for a partition $\mathscr{T}_\Omega = \{\Omega_i\}_{i=1}^s$ of Ω, an H^1-conforming approximation is yielded only if $m_i > m_{i+1}$, for any $i = 1, \ldots, s-1$. By comparing Fig. 2 (left), e.g., with Fig. 1 (center), we recognize that a single mode is enough to describe u on Ω_1 with sufficient accuracy.

Finally, we resort to the node-wise Hi-Mod approach. The adopted modal distribution is shown in Fig. 2 (right): it is based on a uniform partition $\mathscr{T}_h$ with 50 subintervals. The corresponding reduced solution (see Fig. 2 (center)) is fully

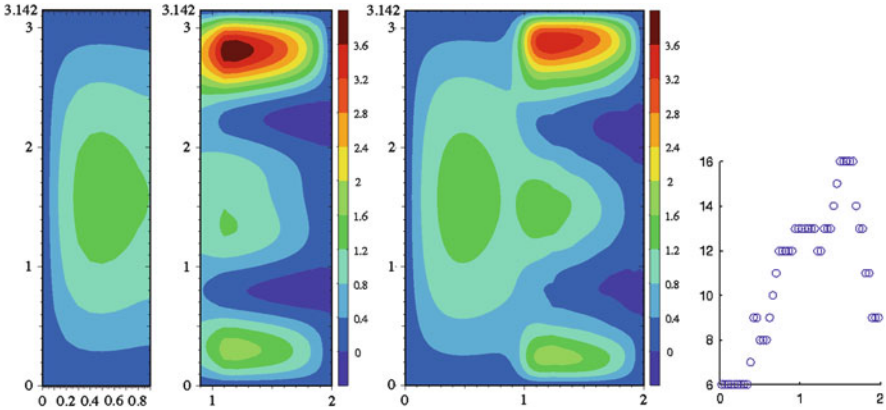

Fig. 2 First test case: piecewise reduced solution $u_{1,7}^{b,h}$ (*left*); nodewise reduced solution $u_{\mathbf{M}}^h$ (*center*) and corresponding modal ditribution (*right*)

comparable with the uniform one, u_{16}^h, in Fig. 1: nevertheless, 16 modes are now employed only on few nodes with a reduction of the size of the corresponding linear system, i.e., of the whole computational cost. As expected, the current reduced solution is continuous.

To make the comparison among the three approaches more quantitative, we consider the L^2-norm of the corresponding errors: $\|u - u_7^h\| = 0.2028$; $\|u - u_{16}^h\| = 0.0388$; $\|u - u_{1,7}^{b,h}\| = 0.2506$; $\|u - u_{\mathbf{M}}^h\| = 0.0566$. As expected, $u_{1,7}^{b,h}$ is thoroughly comparable with u_7^h, while the nodewise reduced solution is not so far from u_{16}^h.

Second test case. This test case provides an example of nodewise Hi-Mod reduction applied to a strong dynamic involving the whole Ω. We solve on $\Omega = (0,4) \times (0,1)$ the advection-diffusion problem $-\nabla \cdot \big(a(\mathbf{z})\nabla u\big) + \mathbf{b} \cdot \nabla u = 1$, with $a(\mathbf{z}) = 5 + 4.8\sin(\pi x)\cos(\pi y)^{1/5}$ the diffusive coefficient, $\mathbf{b} = (100, 0)^T$ the advective field. We assign homogeneous Dirichlet boundary conditions along the horizontal sides, a nonhomogeneous Dirichlet datum, $u = 4\sin(\pi x)$, at the inflow, homogeneous Neumann conditions at the outflow. This problem may model the density u of a fluid flowing horizontally (from left to right) in a media with a nonhomogenous permeability a. A distributed source, $f = 1$, is also present.

Due to the complex dynamics involved, it turns out to be a hard task to identify, a priori, suitable subdomains with a view to a piecewise approach. We consequently resort to both a uniform and a nodewise Hi-Mod reduction, by comparing the corresponding performances. Figure 3 (top) shows the uniform solution obtained by employing ten modes on the whole Ω. In Fig. 3 (bottom-left) we show the nodewise solution based on the modal distribution on the right. The two reduced solutions are really similar, but in the latter case at most eight modes are associated with a node. The order of the system reduces from 501 to 251.

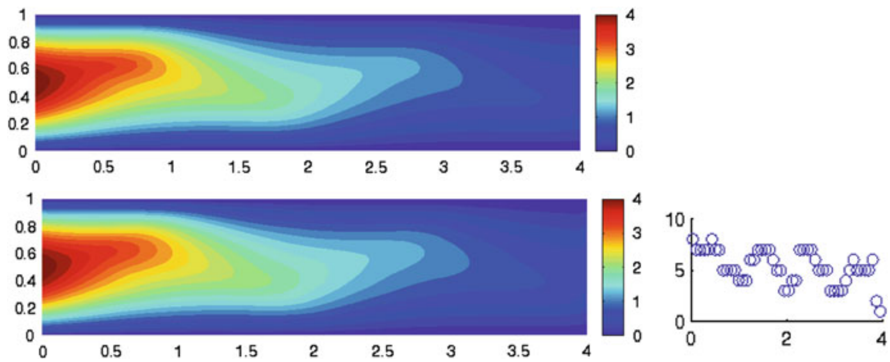

Fig. 3 Second test case: uniformly reduced solution u^h_{10} (*top*); nodewise reduced solution and associated modal distribution (*bottom*)

To summarize, the numerical assessment suggests that the nodewise Hi-Mod reduction is effective to deal with both localized and spread transverse dynamics.

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