On ε-Uniform Error Estimates For Singularly Perturbed Problems in the DG Method

V. Kučera

Abstract In this paper we present the analysis of the discontinuous Galerkin (DG) finite element method applied to a nonstationary nonlinear convection-diffusion problem. Using the technique of Zhang and Shu (SIAM J Numer Anal 42(2):641–666, 2004), originally for explicit schemes, we prove apriori error estimates uniform with respect to the diffusion coefficient and valid even in the purely convective case. We extend the cited analysis to the method of lines using continuous mathematical induction and a nonlinear Gronwall-type lemma. For an implicit scheme, we prove that there does not exist a Gronwall-type lemma capable of proving the desired estimates using standard arguments. Next, we use a suitable continuation of the implicit solution and use continuous mathematical induction to prove error estimates under a CFL-like condition.

1 Continuous Problem

Let $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$ be a bounded open (polyhedral) domain. We treat the following nonlinear convection-diffusion problem: find $u : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\text{(a)} \quad \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = \varepsilon \Delta u + g \quad \text{in } \Omega \times (0, T), \tag{1}$$

$$\text{(b)} \quad u|_{\Gamma_D \times (0,T)} = u_D, \quad \varepsilon \frac{\partial u}{\partial n}\Big|_{\Gamma_N \times (0,T)} = g_N, \tag{2}$$

V. Kučera (✉)

Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Prague, Czech Republic

e-mail: vaclav.kucera@email.cz

along with the initial condition $u(x, 0) = u^0(x)$ in Ω. The diffusion coefficient $\varepsilon \geq 0$ is a given constant, g, u_D, g_N, and u^0 are given functions.

We assume that the convective fluxes $\mathbf{f} = (f_1, \cdots, f_d) \in (C_b^2(\mathbb{R}))^d = (C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R}))^d$, hence $\mathbf{f}$ and $\mathbf{f}' = (f_1', \cdots, f_d')$ are *globally Lipschitz continuous*. For improved estimates via Remark 1, we shall assume $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$. In [4], the error analysis is extended, assuming only local properties, i.e. $\mathbf{f} \in (C^2(\mathbb{R}))^d$ and $\mathbf{f} \in (C^3(\mathbb{R}))^d$.

In our analysis, we need to assume that Γ_N is an outflow boundary for either u or u_h, i.e. e.g. for u, we assume $\Gamma_N^{(t)} \subseteq \{x \in \partial\Omega; \mathbf{f}'(u(x,t)).\mathbf{n} \geq 0\}$ and $\Gamma_D^{(t)} := \partial\Omega \setminus \Gamma_N$.

2 Discretization

Let $\mathcal{T}_h$ be (generally nonconforming) triangulation of $\overline{\Omega}$. For $K \in \mathcal{T}_h$ we set $h = \max_{K \in \mathcal{T}_h} \text{diam}(K)$. By $\mathcal{F}_h$ we denote the system of all faces of all elements $K \in \mathcal{T}_h$. By $\mathcal{F}_h^I, \mathcal{F}_h^D, \mathcal{F}_h^N, \mathcal{F}_h^B$ we denote the sets on interior, Dirichlet, Neumann and boundary edges, respectively. For each $\Gamma \in \mathcal{F}_h$ we define a fixed unit normal $\mathbf{n}_\Gamma$, which has the same orientation as the outer normal to $\partial\Omega$ if $\Gamma \in \mathcal{F}_h^B$.

Over a triangulation $\mathcal{T}_h$ we define the *broken Sobolev spaces* $H^k(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^k(K), \forall K \in \mathcal{T}_h\}$. For $\Gamma \in \mathcal{F}_h^I$ we have two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_h$, where $\mathbf{n}_\Gamma$ is the outer normal to $K_\Gamma^{(L)}$. For $v \in H^1(\Omega, \mathcal{T}_h)$ we define on $\Gamma \in \mathcal{F}_h^I$: $v|_\Gamma^{(L)} = $ the trace of $v|_{K_\Gamma^{(L)}}$ on Γ, $v|_\Gamma^{(R)} = $ the trace of $v|_{K_\Gamma^{(R)}}$ on Γ, $\langle v \rangle_\Gamma = \frac{1}{2}(v|_\Gamma^{(L)} + v|_\Gamma^{(R)})$ and $[v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}$. On $\Gamma \in \mathcal{F}_h^B$ we set $v_\Gamma = v|_\Gamma^{(L)} = $ the trace of $v|_{K_\Gamma^{(L)}}$ on Γ, while $v|_\Gamma^{(R)} = u_D$ on Γ_D, $v|_\Gamma^{(R)} = v|_\Gamma^{(L)}$ on Γ_N.

Let $p \geq 1$ be an integer. The approximate solution will be sought in the space of discontinuous piecewise polynomial functions $S_h = \{v; v|_K \in P^p(K), \forall K \in \mathcal{T}_h\}$, where $P^p(K)$ are polynomials on K of degree $\leq p$. By $(\cdot, \cdot)$ we denote the $L^2(\Omega)$-scalar product and by $\|\cdot\|$ the $L^2(\Omega)$-norm. By $\|\cdot\|_\infty$, we denote the $L^\infty(\Omega)$-norm.

We introduce the following forms defined for $v, \varphi \in H^2(\Omega, \mathcal{T}_h)$. *Diffusion form*:

$$a_h(v, \varphi) = \sum_{K \in \mathcal{T}_h} \int_K \nabla v \cdot \nabla \varphi \, \mathrm{d}x - \int_{\mathcal{F}_h^I} \langle \nabla v \rangle \cdot \mathbf{n}[\varphi] \, \mathrm{d}S - \Theta \int_{\mathcal{F}_h^I} \langle \nabla \varphi \rangle \cdot \mathbf{n}[v] \, \mathrm{d}S$$

$$- \int_{\mathcal{F}_h^D} \nabla v \cdot \mathbf{n} \varphi \, \mathrm{d}S - \Theta \int_{\mathcal{F}_h^D} \nabla \varphi \cdot \mathbf{n} v \, \mathrm{d}S.$$

Interior and boundary penalty jump terms:

$$J_h(v, \varphi) = \int_{\mathcal{F}_h^I} \sigma [v][\varphi] \, \mathrm{d}S + \int_{\mathcal{F}_h^D} \sigma v \varphi \, \mathrm{d}S.$$

Right-hand side form:

$$l_h(\varphi)(t) = \int_\Omega g(t)\varphi \, \mathrm{d}x + \int_{\mathscr{F}_h^N} g_N(t)\varphi \, \mathrm{d}S - \varepsilon\Theta \int_{\mathscr{F}_h^D} \nabla\varphi \cdot \mathbf{n} u_D(t) \, \mathrm{d}S + \varepsilon \int_{\mathscr{F}_h^D} \sigma u_D(t)\varphi \, \mathrm{d}S.$$

The parameter σ in the diffusion and right-hand side forms is defined by $\sigma|_\Gamma = C_W|\Gamma|^{-1}$, where $C_W > 0$ is a constant, which is chosen large enough to ensure coercivity of the diffusion form – cf. Lemma 2. Depending on the value of Θ in the diffusion form, we get the *symmetric* ($\Theta = 1$), *incomplete* ($\Theta = 0$) and *nonsymmetric interior penalty* ($\Theta = -1$) variants of the diffusion a right-hand side forms.

Finally we define the *convective form*

$$b_h(v, \varphi) = -\sum_{K \in \mathscr{T}_h} \int_K \mathbf{f}(v) \cdot \nabla v \, \mathrm{d}x + \int_{\mathscr{F}_h^I} H(v^{(L)}, v^{(R)}, \mathbf{n})[\varphi] \, \mathrm{d}S + \int_{\mathscr{F}_h^B} H(v^{(L)}, v^{(R)}, \mathbf{n})\varphi^{(L)} \, \mathrm{d}S.$$

The form b_h approximates convective terms with the aid of a numerical flux $H(v, w, \mathbf{n})$. We assume that H has the following standard properties: H is *Lipschitz-continuous, consistent, conservative* and H is an *E-flux*, i.e.

$$\big(H(v, w, \mathbf{n}) - \mathbf{f}(q) \cdot \mathbf{n}\big)(v - w) \geq 0, \quad \forall v, w \in \mathbb{R}, \ \mathbf{n} \in B_1 \text{ and all } q \text{ between } v, w.$$

The E-flux condition was introduced as a generalization of monotone fluxes by Osher in [5]. Many numerical fluxes used in practice are E-fluxes, e.g. Lax-Friedrichs, Godunov, Engquist-Osher and the Roe flux with entropy fix, cf. [5].

Definition 1. We say that $u_h \in C^1([0, T]; S_h)$ is a DG solution of (1) and (2), if $u_h(0) = u_h^0 \approx u^0$ and for all $\varphi_h \in S_h$, and $t \in (0, T)$

$$\frac{d}{dt}\big(u_h(t), \varphi_h\big) + b_h\big(u_h(t), \varphi_h\big) + \varepsilon J_h\big(u_h(t), \varphi_h\big) + \varepsilon a_h\big(u_h(t), \varphi_h\big) = l_h\big(\varphi_h\big)(t). \quad (3)$$

3 Some Necessary Results

We assume that the weak solution u is sufficiently regular, namely $u_t := \frac{\partial u}{\partial t} \in L^2\big(0, T; H^{p+1}(\Omega)\big)$, $u \in L^\infty(0, T; W^{1,\infty}(\Omega))$, where $p \geq 1$ is the degree of approximation. These conditions imply $u \in C\big([0, T]; H^{p+1}(\Omega)\big)$.

As for the mesh assumptions, we consider a system $\{\mathscr{T}_h\}_{h \in (0, h_0)}$, $h_0 > 0$, of triangulations, which are shape regular and satisfy the inverse assumption, cf. [2].

Now, for $v \in L^2(\Omega)$ we denote by $\Pi_h v$ the $L^2(\Omega)$-projection of v on S_h:

$$\Pi_h v \in S_h, \quad \big(\Pi_h v - v, \varphi_h\big) = 0, \qquad \forall \varphi_h \in S_h.$$

Let $\eta_h(t) = u(t) - \Pi_h u(t) \in H^{p+1}(\Omega, \mathscr{T}_h)$ and $\xi_h(t) = \Pi_h u(t) - u_h(t) \in S_h$ for $t \in (0, T)$. Then we can write the error e_h as $e_h(t) := u(t) - u_h(t) = \eta_h(t) + \xi_h(t)$. Standard approximation results give us estimates for $\eta_h(t)$ in terms of power of h, e.g. $||\eta||_{L^2(\Omega)} \le C h^{p+1} |u|_{H^{p+1}}$, cf. [2].

Lemma 1. *There exists a constant $C \ge 0$ independent of h, t, such that*

$$b_h\big(u_h(t), \xi_h(t)\big) - b_h\big(u(t), \xi_h(t)\big) \le C \Big(1 + \frac{\|e_h(t)\|_\infty^2}{h^2}\Big)\big(h^{2p+1}|u(t)|_{H^{p+1}}^2 + \|\xi_h(t)\|^2\big).$$

Proof. The proof follows the arguments of [7], where similar estimates are derived for periodic boundary conditions or compactly supported solutions. The proof for mixed Dirichlet-Neumann boundary conditions is contained in [4]. Here, we only note that the estimate is based on performing second order Taylor expansions of and using the *E-flux* properties for H. □

Remark 1. We can improve Lemma (1), if we suppose $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$ and $\Gamma_N = \emptyset$. Then we obtain a factor of $h^{-1}\|e_h\|_\infty^2$ instead of $h^{-2}\|e_h\|_\infty^2$ in the estimate of Lemma (1). This improved estimate will be useful in proving the resulting estimates for lower order polynomials and with a less restrictive CFL condition, cf. Remark 3.

Lemma 2 (Ellipticity and boundedness of A_h, cf. [3]). *Let the constant C_W be large enough. Then the form A_h is elliptic and bounded, i.e.*

$$\|v\|_{DG}^2 \le A_h(v, v), \quad \forall v \in H^2(\Omega, \mathscr{T}_h),$$

$$A_h(v, w) \le \|v\|_{DG}\|w\|_{DG}, \quad \forall v, w \in H^2(\Omega, \mathscr{T}_h),$$

where $\|w\|_{DG}^2 = \frac{1}{2}\big(\sum_{K \in \mathscr{T}_h} |w|_{H^k(K)}^2 + J_h(w, w)\big)$ and $A_h(\cdot, \cdot) = a_h(\cdot, \cdot) + J_h(\cdot, \cdot)$.

4 Error Analysis for the Method of Lines

We proceed in a standard way. Due to Galerkin orthogonality, we subtract the equations for u and u_h and set $\varphi_h := \xi_h(t) \in S_h$. Since $\big(\frac{\partial \xi_h}{\partial t}, \xi_h\big) = \frac{1}{2} \frac{d}{dt} \|\xi_h\|^2$, we get

$$\frac{1}{2} \frac{d}{dt} \|\xi_h(t)\|^2 + \varepsilon A_h\big(\xi_h(t), \xi_h(t)\big)$$

$$= -\varepsilon A_h\big(\eta_h(t), \xi_h(t)\big) + b_h\big(u_h(t), \xi_h(t)\big) - b_h\big(u(t), \xi_h(t)\big) - \Big(\frac{\partial \eta_h(t)}{\partial t}, \xi_h(t)\Big).$$

For the last right-hand side term, we use the Cauchy and Young's inequalities and standard estimates for η. For the convective and diffusion terms we use Lemmas 1 and 2. Integration from 0 to $t \in [0, T]$ yields

$$\|\xi_h(t)\|^2 + \int_0^t \varepsilon \|\xi_h(\vartheta)\|_{DG}^2 \, \mathrm{d}\vartheta \tag{4}$$

$$\leq C \int_0^t \Big(1 + \frac{\|e_h(\vartheta)\|_\infty^2}{h^2}\Big)\Big((h^{2p+1} + \varepsilon h^{2p})|u(\vartheta)|_{H^{p+1}}^2 + h^{2p+2}|u_t(\vartheta)|_{H^{p+1}}^2 + \|\xi_h(\vartheta)\|^2\Big)\mathrm{d}\vartheta.$$

For simplicity we have assumed $\xi_h(0) = 0$, i.e. $u_h^0 = \Pi_h u^0$. Otherwise, we must assume e.g. $\|\xi_h(0)\| = O(h^{p+1/2})$ and include this term in (4). We notice that if we knew apriori that $\|e_h\|_\infty = O(h)$ then the unpleasant term $h^{-2}\|e_h\|_\infty^2$ in (4) would be $O(1)$. Thus we could simply apply the standard Gronwall inequality to obtain the desired error estimates.

Lemma 3. *Let $t \in [0, T]$ and $p \geq d/2$. If $\|e_h(\vartheta)\| \leq h^{1+d/2}$ for all $\vartheta \in [0, t]$, then there exists a constant C_T independent of h, t and ε such that*

$$\max_{\vartheta \in [0,t]} \|e_h(\vartheta)\|^2 + \int_0^t \varepsilon \|e_h(\vartheta)\|_{DG}^2 \, \mathrm{d}\vartheta \leq C_T^2 \big(h^{2p+1} + \varepsilon h^{2p}\big). \tag{5}$$

Proof. The assumptions imply, using the inverse inequality and estimates of η, that

$$\|e_h(\vartheta)\|_\infty \leq \|\eta_h(\vartheta)\|_\infty + \|\xi_h(\vartheta)\|_\infty \leq Ch|u(t)|_{W^{1,\infty}(\Omega)} + C_I h^{-d/2}\|\xi_h(\vartheta)\|$$

$$\leq Ch + C_I h^{-d/2}\|e_h(\vartheta)\| + C_I h^{-d/2}\|\eta_h(\vartheta)\| \leq Ch + Ch^{p+1-d/2}|u(\vartheta)|_{H^{p+1}} \leq Ch,$$

where the constant C is independent of h, ϑ, t. Using this estimate in (4) gives us

$$\|\xi_h(t)\|^2 + \int_0^t \varepsilon \|\xi_h(\vartheta)\|_{DG}^2 \, \mathrm{d}\vartheta \leq \tilde{C}\big(h^{2p+1} + \varepsilon h^{2p}\big) + C \int_0^t \|\xi_h(\vartheta)\|^2 \, \mathrm{d}\vartheta,$$

Applying Gronwall's inequality gives us the desired estimate for ξ_h, which along with similar estimates for η gives us (5). $\qquad\square$

Now it remains to get rid of the *apriori* assumption $\|e_h\|_\infty = O(h)$. In [7] this is done for an explicit scheme using mathematical induction. Starting from $\|e_h^0\| = O(h^{p+1/2})$, the following induction step is proved:

$$\|e_h^n\| = O(h^{p+1/2}) \quad \Longrightarrow \quad \|e_h^{n+1}\|_\infty = O(h) \quad \Longrightarrow \quad \|e_h^{n+1}\| = O(h^{p+1/2}). \tag{6}$$

For the method of lines we have no discrete structure with respect to time and hence cannot use mathematical induction straightforwardly. However, we can divide $[0, T]$ into a finite number of sufficiently small intervals $[t_n, t_{n+1}]$ on which "e_h *does not change too much*" and use induction with respect to n. This is essentially a *continuous mathematical induction* argument, a concept introduced in [1].

Remark 2. Due to the regularity assumptions, $u, u_h \in C([0, T]; L^2(\Omega))$. Since $[0, T]$ is a compact set, $e_h(\cdot)$ is a *uniformly continuous* function from $[0, T]$ to $L^2(\Omega)$, i.e.

$$\forall \bar{\epsilon} > 0 \; \exists \delta > 0 : \; s, \bar{s} \in [0, T], |s - \bar{s}| \le \delta \Rightarrow \|e_h(s) - e_h(\bar{s})\| \le \bar{\epsilon}.$$

Theorem 1 (Main theorem). *Let $p > 1 + d/2$. Then there exists $h_1 > 0$ such that for all $h \in (0, h_1]$ we have the estimate*

$$\max_{\vartheta \in [0,T]} \|e_h(\vartheta)\|^2 + \int_0^T \varepsilon \|e_h(\vartheta)\|_{DG}^2 \, d\vartheta \le C_T^2 \big(h^{2p+1} + \varepsilon h^{2p} \big).$$

Proof. We have $p > 1 + d/2$, thus for all sufficiently small h, we have $C_T(h^{p+1/2} + \sqrt{\varepsilon}h^p) \le \frac{1}{2}h^{1+d/2}$. We fix an arbitrary h. By Remark 2, there exists $\delta > 0$, such that if $s, \bar{s} \in [0, T], |s - \bar{s}| \le \delta$, then $\|e_h(s) - e_h(\bar{s})\| \le \frac{1}{2}h^{1+d/2}$. We define $t_i = i\delta$, $i = 0, 1, \ldots$ and set $N := \max\{i = 0, 1, \ldots; t_i < T\}, t_{N+1} := T$. This defines a partition $0 = t_0 < t_1 < \cdots < t_{N+1} = T$ of $[0, T]$ into $N + 1$ intervals of length (at most) δ.

We shall now prove by induction that for all $n = 1, \ldots, N + 1$

$$\max_{\vartheta \in [0,t_n]} \|e_h(\vartheta)\|^2 + \int_0^{t_n} \varepsilon \|e_h(\vartheta)\|_{DG}^2 \, d\vartheta \le C_T^2 \big(h^{2p+1} + \varepsilon h^{2p} \big). \qquad (7)$$

The desired error estimate is thus obtained by taking $n := N + 1$ in (7).

(i) $n = 1$: Since $\|e_h(0)\| \le \frac{1}{2}h^{1+d/2}$. By uniform continuity, we have for all $s \in [0, t_1]$

$$\|e_h(s)\| \le \|e_h(0)\| + \|e_h(s) - e_h(0)\| \le \frac{1}{2}h^{1+d/2} + \frac{1}{2}h^{1+d/2} = h^{1+d/2}.$$

Therefore, by Lemma 3 we obtain estimate (7) on $[0, t_1]$, i.e. for $n = 1$.

(ii) Induction step: We assume that (7) holds for general $n < N + 1$. Therefore $\|e_h(t_n)\| \le C_T(h^{p+1/2} + \sqrt{\varepsilon}h^p) \le \frac{1}{2}h^{1+d/2}$. By uniform continuity, for all $s \in [t_n, t_{n+1}]$

$$\|e_h(s)\| \le \|e_h(t_n)\| + \|e_h(s) - e_h(t_n)\| \le \frac{1}{2}h^{1+d/2} + \frac{1}{2}h^{1+d/2} = h^{1+d/2}.$$

This and the induction assumption imply that $\|e_h(s)\| \le h^{1+d/2}$ for $s \in [0, t_n] \cup [t_n, t_{n+1}] = [0, t_{n+1}]$. By Lemma 3, we obtain estimate (7) on $[0, t_{n+1}]$. $\qquad \square$

Remark 3. If we assume $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$ then by Remark 1 we get the improved assumption $p > (1+d)/2$ in Theorem 1. If $\varepsilon = 0$ we need to assume only $p > d/2$.

Remark 4. For the method of lines we can use a *nonlinear Gronwall-type lemma* to prove Theorem 1 directly, cf. [4]. As stated in Remark 6, this is not possible for an implicit scheme, since an analogous discrete Gronwall lemma cannot exist.

5 Error Estimates for a Fully Implicit Scheme

In this section, we shall introduce and analyze the DG scheme with a standard implicit Euler time discretization. Here we cannot use the approach of [7] for the explicit scheme, since we were unable to prove the first implication in the induction step (6). On the other hand, in Lemma 6 we prove that for the implicit Euler scheme we cannot use a discrete Gronwall-type lemma as mentioned in Remark 4.

We consider a partition $0 = t_0 < t_1 < \cdots < t_{N+1} = T$ of $[0, T]$ and set $\tau_n = t_{n+1} - t_n$ for $n = 0, \cdots, N$. The exact solution $u(t_n)$ will be approximated by $u_h^n \in S_h$.

Definition 2. We say that $\{u_h^n\}_{n=0}^N \subset S_h$ is an implicit Euler DGFE solution of the convection-diffusion problem (1) and (2), if $u_h^0 = \Pi_h u^0$ and for all $\varphi_h \in S_h, n = 0, \cdots, N$

$$\left(\frac{u_h^{n+1} - u_h^n}{\tau_n}, \varphi_h\right) + b_h\left(u_h^{n+1}, \varphi_h\right) + \varepsilon A_h\left(u_h^{n+1}, \varphi_h\right) = l_h\left(\varphi_h\right)(t_{n+1}). \tag{8}$$

Similarly as in Sect. 3, we define $\eta_h^n = u(t_n) - \Pi_h u(t_n) \in H^{p+1}(\Omega, \mathcal{T}_h)$ and $\xi_h^n = \Pi_h u(t_n) - u_h^n \in S_h$. Then we can write the error e_h^n as $e_h^n := u(t_n) - u_h^n = \eta_h^n + \xi_h^n$.

First, we analyze problem (8), proving that u_h^{n+1} exists uniquely and depends continuously on τ_n. To this end we define an abstract formulation of problem (8):

Definition 3. (Auxiliary problem) Let $t \in [0, T], \tau \in [0, T]$ and $U_h \in S_h$. We seek $u_\tau \in S_h$ such that

$$\left(u_\tau - U_h, \varphi_h\right) + \tau b_h\left(u_\tau, \varphi_h\right) + \tau \varepsilon A_h\left(u_\tau, \varphi_h\right) = \tau l_h\left(\varphi_h\right)(t), \quad \forall \varphi_h \in S_h. \tag{9}$$

Remark 5. If we take $\tau := \tau_n, U_h := u_h^n, t := t_{n+1}$ and define $u_h^{n+1} := u_\tau$, the auxiliary problem (9) reduces to equation (8), which defines u_h^{n+1}. If we take $\tau := 0$ the solution of (9) is $u_\tau = u_h^n$. Between these two cases u_τ depends continuously on τ:

Lemma 4. *There exist constants $C_1, C_2 > 0$ independent of h, τ, t, ε, such that the following holds. Let $t \in [0, T], h \in (0, h_0), U_h \in S_h$ and $\tau \in [0, \tau_0)$, where $\tau_0 = \max\{C_1\varepsilon, C_2h\}$. Then the solution u_τ of (9) exists, is uniquely determined and $\|u_\tau\|$ depends continuously on $\tau \in [0, \tau_0)$.*

Proof. Problem (9) is a nonlinear equation for u_τ on the finite-dimensional space S_h. The statements follow from the nonlinear Lax-Milgram theorem, cf. [6]. For details of the proof, see [4]. □

Definition 4 (Continuated discrete solution). Let $\tilde{u}_h : [0, T] \to S_h$ such that for $s \in [t_n, t_{n+1}]$ we set $\tilde{u}_h(s) := u_\tau$, the solution of the auxiliary problem (9) with $\tau := s - t_n, \mathrm{t} := t_{n+1}$ and $U_h := u_h^n$. Furthermore, we define $\tilde{e}_h := u - \tilde{u}_h$ and $\tilde{\xi}_h := \Pi_h u - \tilde{u}_h$.

Under the assumptions of Lemma 4, $\tilde{u}_h, \tilde{e}_h \in C([0, T]; L^2(\Omega))$ and $\tilde{u}_h$ is uniquely determined. Also, $\tilde{u}_h(t_n) = u_h^n$ and $\tilde{e}_h(t_n) = e_h^n$ for $n = 0, \cdots, N$. Therefore, estimates of $\tilde{e}_h(\cdot)$ imply estimates of e_h^n. Since $\tilde{u}_h$ is constructed using problem (9), which is essentially the implicit scheme (8) with special data, we can derive error estimates for $\tilde{u}_h$ in a standard manner. For simplicity we assume a uniform partition of $[0, T]$.

Lemma 5. *Let $p > d/2$ and $s \in (t_n, t_{n+1}]$ for some $n \in \{0, \cdots, N-1\}$. If $\|\tilde{e}_h(s)\| \le h^{1+d/2}$ and $\|\tilde{e}_h(t_k)\| \le h^{1+d/2}$ for all $k = 0, \cdots, n$, then there exists $C_T > 0$ independent of s, n, h, τ such that*

$$\max_{t \in \{t_0, \cdots, t_n, s\}} \|\tilde{e}_h(t)\|^2 + \sum_{k=1}^n \tau \varepsilon \|\tilde{e}_h(t_k)\|_{DG}^2 + (s - t_n)\varepsilon \|\tilde{e}_h(s)\|_{DG}^2 \le C_T^2 \big(h^{2p+1} + \varepsilon h^{2p} + \tau^2 \big).$$

Proof. We subtract (9) from the equation for the exact solution. Thus $\tilde{e}_h(s)$ satisfies

$$\big(\tilde{e}_h(s) - \tilde{e}_h(t_n), \varphi_h\big) + (s - t_n)\big(b_h(u(s), \varphi_h) - b_h(\tilde{u}_h(s), \varphi_h)\big) + (s - t_n)\varepsilon A_h(\tilde{e}_h(s), \varphi_h)$$
$$= \big(u(s) - u(t_n) - (s - t_n)u_t(s), \varphi_h\big). \tag{10}$$

We set $\varphi_h := \tilde{\xi}_h(s)$ and use the fact that $2(a - b, a) = \|a\|^2 - \|b\|^2 + \|a - b\|^2$. We estimate the convective terms using Lemma 1 and the diffusion terms using Lemma 2. The right-hand side represents the temporal error and is estimated as usual. Thus

$$\|\tilde{\xi}_h(s)\|^2 - \|\tilde{\xi}_h(t_n)\|^2 + \|\tilde{\xi}_h(s) - \tilde{\xi}_h(t_n)\|^2 + (s - t_n)\varepsilon \|\tilde{\xi}_h(s)\|_{DG}^2$$
$$\le C\tau \Big(1 + \frac{\|\tilde{e}_h(s)\|_\infty^2}{h^2} \Big) \Big(\big(h^{2p+1} + \varepsilon h^{2p}\big)|u|_{L^\infty(H^{p+1})}^2 + \tau^2 \|u_{tt}\|_{L^\infty(L^2(\Omega))}^2 + \|\tilde{\xi}_h(s)\|^2 \Big).$$

The assumptions imply $\|\tilde{e}_h(s)\|_\infty \le Ch$, eliminating the factor h^{-2}. Thus

$$\|\tilde{\xi}_h(s)\|^2 + (s - t_n)\varepsilon \|\tilde{\xi}_h(s)\|_{DG}^2 \le \|\tilde{\xi}_h(t_n)\|^2 + C\tau \big(h^{2p+1} + \varepsilon h^{2p} + \tau^2 + \|\tilde{\xi}_h(s)\|^2 \big).$$

Similarly, we may derive estimates at t_{k+1}:

$$\|\tilde{\xi}_h(t_{k+1})\|^2 + \tau \varepsilon \|\tilde{\xi}_h(t_{k+1})\|_{DG}^2 \le \|\tilde{\xi}_h(t_k)\|^2 + C\tau \big(h^{2p+1} + \varepsilon h^{2p} + \tau^2 + \|\tilde{\xi}_h(t_{k+1})\|^2 \big).$$

Combining these estimates and using the discrete Gronwall lemma gives us the desired estimate for $\tilde{\xi}_h$. Standard estimates for η give us the estimate for $\tilde{e}_h$. $\qquad\square$

Theorem 2 (Main theorem – implicit version). *Let $p > 1 + d/2$. Let $h_1, \tau_1 > 0$ be such that $C_T(h_1^{p+1/2} + \sqrt{\varepsilon}h_1^p + \tau_1) = \frac{1}{2}h_1^{1+d/2}$ and $\tau_1 < \tau_0$, where τ_0 is defined in Lemma 4. Then for all $h \in (0, h_1), \tau \in (0, \tau_1)$ we have the estimate*

$$\max_{n\in\{0,\cdots,N\}}\|e_h^n\|^2 + \sum_{n=1}^{N}\tau\Big(\varepsilon\|e_h^n\|_{DG}^2 + \widetilde{J}_h\big(e_h^n,e_h^n\big)\Big) \le C_T^2\big(h^{2p+1} + \varepsilon h^{2p} + \tau^2\big). \quad (11)$$

Proof. Again, $\tilde{e}_h(\cdot)$ is a *uniformly continuous* function from $[0,T]$ to $L^2(\Omega)$. This allows to use continuous mathematical induction to eliminate the apriori assumption $\|\tilde{e}_h(t)\| = O(h^{1+d/2})$ from Lemma 5. The proof thus follows that of Theorem 1. □

Remark 6. The reason we introduced the continuation of u_h^n is that a more standard, straightforward approach is insufficient. Specifically, we prove in [4] that there does not exist a Gronwall-type lemma which could prove the desired error estimate (11) *only* from the error equation of the implicit scheme tested by ξ_h^{n+1} and the derived estimates of individual terms contained therein.

6 Conclusion

We have presented an analysis of the DG method for a nonlinear convection-diffusion problem. Building on results from [7], which dealt with an explicit time discretization, we proved apriori $L^\infty(L^2)$ error estimates independent of the diffusion coefficient for the method of lines and a fully implicit scheme. We have derived the key estimates for the case of mixed Dirichlet-Neumann boundary conditions, improving the results of [7]. For the method of lines, the error estimates are derived using a continuous mathematical induction argument or a nonlinear Gronwall lemma. For the implicit time discretization, we show that a similar discrete Gronwall lemma does not exist and prove the error estimates using continuous mathematical induction applied to a suitable continuation of the discrete solution. However, using this technique, we obtain an unnatural CFL-like condition for the implicit scheme. In [4], the presented results are extended to of a *locally* Lipschitz continuous **f**.

References

1. Chao Y. R., *A note on "Continuous mathematical induction"*, Bull. Amer. Math. Soc., **26** (1), 17–18 (1919).
2. Ciarlet P.G., *The Finite Element Method for Elliptic Problems,* North-Holland, Amsterdam (1979).
3. Dolejší V., Feistauer M., Kučera V. and Sobotíková V., *An optimal $L^\infty(L^2)$-error estimate for the discontinuous Galerkin approximation of a nonlinear non-stationary convection-diffusion problem*, IMA J. Numer. Anal., **28**, 496–521 (2008).

4. Kučera V., *On diffusion-uniform error estimates for the DG method applied to singularly perturbed problems*, The Preprint Series of the School of Mathematics, preprint No. MATH-knm-2011/3 (2011), http://www.karlin.mff.cuni.cz/ms-preprints/prep.php.

5. Osher S., *Riemann solvers, the entropy condition, and difference approximations*, SIAM. J. Numer. Anal., **21**, 217–235 (1984).

6. Zeidler E., *Nonlinear Functional Analysis and Its Applications II/B: Nonlinear Monotone Operators*, Springer (1986).

7. Zhang Q. and Shu C.-W., *Error estimates to smooth solutions of Runge–Kutta discontinuous Galerkin methods for scalar conservation laws*, SIAM J. Numer. Anal., **42**(2), 641–666 (2004).