A Note on Large Deviations of Random Sets and Random Upper Semicontinuous Functions

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Abstract. In this paper, we show a sufficient condition under which the law of sums of i.i.d. compact random sets in a separable type p Banach space (resp. compact random upper semicontimuous functions) satisfies large deviations if the law of sums of its corresponding convex hull of compact random sets(resp. quasiconcave envelope of compact random upper semicontimuous functions) satisfies large deviations.

Keywords: Large deviations, random sets, random upper semicontinuous functions.

1 Introduction

The theory of large deviation principle (LDP) deals with the asymptotic estimation of probabilities of rare events and provides exponential bound on probability of such events. Some authors have discussed LDP on random sets and random upper semicontinuous functions. In 1999, Cerf [3] proved Cramér type LDP for sums of i.i.d. compact random sets in a separable type p Banach space with respect to the Hausdorff distance d_H. In 2006, Terán obtained Cramér type LDP of compact random upper semicontinuous functions [9], and Bolthausen type LDP of compact convex random upper semicontinuous functions [10] on a separable Banach space in the sense of the uniform Hausdorff distance d_H^∞. In 2009, Ogura and Setokuchi [7] proved a Cramér type LDP for compact random upper semicontiunous functions on the underling separable Banach space with respect to the metric d_Q (see [7] for the notation) in a different method, which is weaker than the uniform Hausdorff distance

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d_H^∞. In 2010, Ogura, Li and Wang [6] also discussed LDP for random upper semicontinuous functions whose underlying space is d-dimensional Euclidean space $\mathbb{R}^d$ under various topologies for compact covex random sets and random upper semicontinuous functions, Wang [12] considered functional LDP of compact random sets, Wang and Li [11] obtained LDP for bounded closed convex random sets and related random upper semicontiunous functions. In fact, about these work above, some work of papers extended compact convex random sets(resp. compact convex random upper semicontinuous functions) to the non-convex case in a separable type p Banach space (see [3, 7, 9, 12]). So we hope the LDP of the law of sums of i.i.d. compact random sets(resp. compact random upper semicontimuous functions) still holds if the law of sums of its corresponding convex hull of compact random sets(resp. quasiconcave envelope of compact random upper semicontimuous functions) satisfies large deviations. However, until now, all ideal of "deconvexification" comes from Cerf's basic work(Lemma 2 in [3]). In [3], Cerf gives a sufficient condition for the case of compact random sets : $E[\exp\{\lambda\|X\|_{\mathcal{K}}\}] < \infty$ for any $\lambda > 0$. In [9], Terán gives a sufficient condition(see Lemma 4.4 in [9]) for the case of compact random upper semicontinuous functions : $E[\exp\{\lambda\|X_0\|_{\mathcal{K}}\}] < \infty$ for some $\lambda > 0$. In [9], the author doesn't give the proof of Lemma 4.4, and he said the basic idea is the same as Cerf's paper. I think, if the author use Cerf's idea, the Lemma 4.4 can't be obtained under the condition: $E[\exp\{\lambda\|X_0\|_{\mathcal{K}}\}] < \infty$ for some $\lambda > 0$. So in our paper, we don't use Cerf's idea and use another method to give another condition for compact random sets: $E[\exp\{\lambda\|X\|_{\mathcal{K}}^p\}] < \infty$ for some $\lambda > 0$, and another condition for compact random upper semicontinuous functions: $E[\exp\{\lambda\|X\|_{\mathcal{F}}^p\}] < \infty$ for some $\lambda > 0$. Under these conditions, we prove the laws of sums of i.i.d. compact random sets and compact random upper semicontimuous functions satisfy large deviations if the laws of sums of its corresponding convex hull of compact random sets and quasiconcave envelope of compact random upper semicontimuous functions satisfy large deviations.

The paper is structured as follows. Section 2 will give some preliminaries about compact random sets and compact random upper semicontinuous functions. In section 3, we will give and prove our main results.

2 Preliminaries

Throughout this paper, we assume that $(\Omega, \mathcal{A}, P)$ is a complete probability space, $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ is a real separable Banach space with its dual space $\mathfrak{X}^*$. We suppose that $\mathfrak{X}$ is of type $p > 1$, i.e., there exists a constant c such that

$$E\Big[\|\sum_{i=1}^n f_i\|_{\mathfrak{X}}^p\Big] \le c\sum_{i=1}^n E\Big[\|f_i\|_{\mathfrak{X}}^p\Big],$$

for any independent random variables $f_1, f_2, \cdots, f_n$ with values in $\mathfrak{X}$ and mean zero. Every Hilbert space is type 2; the space L^p with $1 < p < \infty$ are of type $\min(p, 2)$. However, the space of continuous functions on $[0,1]$ equipped with supremum norm is of type 1 and not of type p for any $p > 1$.

$\mathcal{K}_k(\mathfrak{X})$(resp. $\mathcal{K}_c(\mathfrak{X}), \mathcal{K}_{kc}(\mathfrak{X})$) is the family of all non-empty compact (resp. convex, compact convex) subsets of $\mathfrak{X}$.

Let A and B be two non-empty subsets of $\mathfrak{X}$ and let $\lambda \in \mathbb{R}$, we can define addition and scalar multiplication by $A+B = cl\{a+b : a \in A,\ b \in B\}, \lambda A = \{\lambda a :\ a \in A\}$, where clA is the closure of set A taken in $\mathfrak{X}$. The Hausdorff distance on $\mathcal{K}_k(\mathfrak{X})$ is defined by

$$d_H(A,B) = \max\Big\{\sup_{a\in A}\inf_{b\in B}\|a-b\|_{\mathfrak{X}}, \sup_{b\in B}\inf_{a\in A}\|a-b\|_{\mathfrak{X}}\Big\}.$$

In particular, we denote $\|A\|_{\mathcal{K}} = d_H(\{0\}, A) = \sup_{a\in A}\{\|a\|_{\mathfrak{X}}\}$.

X is called compact random set (resp. compact convex random set), if it is a measurable mapping from the space $(\Omega, \mathcal{A}, P)$ to $(\mathcal{K}_k(\mathfrak{X}), \mathfrak{B}(\mathcal{K}_k(\mathfrak{X})))$, (resp. $(\mathcal{K}_k(\mathfrak{X}), \mathfrak{B}(\mathcal{K}_{kc}(\mathfrak{X}))))$ where $\mathfrak{B}(\mathcal{K}_k(\mathfrak{X}))$ (resp. $\mathfrak{B}(\mathcal{K}_{kc}(\mathfrak{X})))$ is the Borel σ-field of $\mathcal{K}_k(\mathfrak{X})$ (resp. $\mathcal{K}_{kc}(\mathfrak{X})$) generated by the Hausdorff distance d_H.

In the following, we introduce the definition of a random upper semicontinuous function. Let $\mathcal{F}_k(\mathfrak{X})$ denote the family of all functions $u :\ \mathfrak{X} \to [0,1]$ satisfying the conditions: (1) the 1-level set $[u]_1 = \{x \in \mathfrak{X} :\ u(x) = 1\} \neq \emptyset$, (2) each u is upper semicontinuous, i.e., for each $\alpha \in (0,1]$, the α level set $[u]_\alpha = \{x \in \mathfrak{X} :\ u(x) \geq \alpha\}$ is a compact subset of $\mathfrak{X}$, (3) the support set $[u]_0 = \mathrm{cl}\{x \in \mathfrak{X} :\ u(x) > 0\}$ is compact.

The subfamily of all u such that $[u]_\alpha$ is in $\mathcal{K}_c(\mathfrak{X})$ for all $\alpha \in [0,1]$ will be denoted of $\mathcal{F}_c(\mathfrak{X})$. Let $\mathcal{F}_{kc}(\mathfrak{X})$ denote the subfamily of all u such that u is in both $\mathcal{F}_k(\mathfrak{X})$ and $\mathcal{F}_c(\mathfrak{X})$. For every $u \in \mathcal{F}_k(\mathfrak{X})$, denote by $cou \in \mathcal{F}_{kc}(\mathfrak{X})$ the quasiconcave envelope of u, we have $[cou]_\alpha = co[u]_\alpha$ for all $\alpha \in (0,1]$.

For any two upper semicontinuous functions u_1, u_2, define

$$(u_1 + u_2)(x) = \sup_{x_1+x_2=x} \min\{u_1(x_1), u_2(x_2)\} \quad \text{for any} \quad x \in \mathfrak{X}.$$

Similarly, for any upper semicontinuous function u and for any $\lambda \geq 0$ and $x \in \mathfrak{X}$, define

$$(\lambda u)(x) = \begin{cases} u(\dfrac{x}{\lambda}), & \text{if } \lambda \neq 0, \\ I_0(x), & \text{if } \lambda = 0, \end{cases}$$

where I_0 is the indicator function of 0.

The following distance is the uniform Hausdorff distance which is extension of the Hausdorff distance d_H : for $u, v \in \mathcal{F}_b(\mathfrak{X})$, $d_H^\infty(u,v) = \sup_{\alpha\in[0,1]} d_H([u]_\alpha, [v]_\alpha)$, this distance is the strongest one considered in the literatures.

X is called a compact random upper semicontinuous function (or random fuzzy set or fuzzy set-valued random variable), if it is a measurable mapping $X : (\Omega, \mathcal{A}, P) \to (\mathcal{F}_k(\mathfrak{X}), \mathfrak{B}(\mathcal{F}_k(\mathfrak{X})))$ (where $\mathfrak{B}(\mathcal{F}_k(\mathfrak{X}))$ is the Borel σ-field of $\mathcal{F}_k(\mathfrak{X})$ generated by the uniform Hausdorff distance d_H^∞).

3 Main Results

Before giving our main results for random sets and random upper semicontinuous functions, we define rate functions and LDP. We refer to the books of Dembo and Zeitouni [4] and Deuschel and Stroock [5] for the general theory on large deviations (also see Yan, Peng, Fang and Wu [13]).

Let E be a regular Hausdorff topological and $\{\mu_n : n \geq 1\}$ be a family of probability measures on $(E, \mathcal{E})$, where $\mathcal{E}$ is the Borel σ-algebra. A *rate function* is a lower semicontinuous mapping $I : E \to [0, \infty]$. A *good rate function* is a rate function such that the level sets $\{x : I(x) \leq \alpha\}$ are compact subset of E. A family of probability measures $\{\mu_n : n \geq 1\}$ on the measurable space $(E, \mathcal{E})$ is said to satisfy the *LDP* with speed $\frac{1}{n}$ and with the rate function I if, for all open set $V \subset \mathcal{E}$, $\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n(V) \geq -\inf_{x\in V} I(x)$, for all closed set $U \subset \mathcal{E}$, $\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(U) \leq -\inf_{x\in U} I(x)$.

In the following, we give our main two results. We first present LDP for $(\mathcal{K}_k(\mathfrak{X}), d_H)$-valued *i.i.d.* random variables.

Theorem 1. Let $\mathfrak{X}$ be a Banach space of type $p > 1$. And $X_1, X_2, \ldots, X_n$ be $(\mathcal{K}_k(\mathfrak{X}), d_H)$-valued *i.i.d.* random variables satisfying $Ee^{\lambda\|X_1\|_{\mathcal{K}}^p} < \infty$ for some $\lambda > 0$. Let $S_n = \frac{X_1+X_2+\cdots+X_n}{n}, coS_n = \frac{coX_1+coX_2+\cdots+coX_n}{n}$. If the law of the random set coS_n satisfies a LDP with the good rate function I_1', then the law of the random set S_n also satisfies a LDP with the good rate function I_1(for $x \in \mathcal{K}_{kc}(\mathfrak{X}), I_1(x) = I_1'(x)$, for $x \in \mathcal{K}_k(\mathfrak{X}) \backslash \mathcal{K}_{kc}(\mathfrak{X}), I_1(x) = +\infty$,) i.e., Then for any open set $U \subset (\mathcal{K}_k(\mathfrak{X}), d_H)$,

$$\liminf_{n\to\infty} \frac{1}{n} \log P\left\{\frac{X_1 + X_2 + \cdots + X_n}{n} \in U\right\} \geq -\inf_{x\in U} I_1(x),$$

any for any closed set $V \subset (\mathcal{K}_k(\mathfrak{X}), d_H)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log P\left\{\frac{X_1 + X_2 + \cdots + X_n}{n} \in V\right\} \leq -\inf_{x\in V} I_1(x).$$

In the following, we give LDP for $(\mathcal{F}_k(\mathfrak{X}), d_H^\infty)$-valued *i.i.d.* random variables.

Theorem 2. Let $\mathfrak{X}$ be a Banach space of type $p > 1$. And $X_1, X_2, \ldots, X_n$ be $(\mathcal{F}_k(\mathfrak{X}), d_H^\infty)$-valued *i.i.d.* random variables satisfying $Ee^{\lambda\|X_1\|_{\mathcal{F}}^p} < \infty$ for some $\lambda > 0$. $S_n = \frac{X_1+X_2+\cdots+X_n}{n}, coS_n = \frac{coX_1+coX_2+\cdots+coX_n}{n}$. If the law of the random set coS_n satisfies a LDP with the good rate function I', then the law of the random set S_n also satisfies a LDP with the good rate function I(for $x \in \mathcal{F}_{kc}(\mathfrak{X}), I(x) = I'(x)$, for $x \in \mathcal{F}_k(\mathfrak{X})\backslash\mathcal{F}_{kc}(\mathfrak{X}), I(x) = +\infty$,) i.e., Then for any open set $U \subset (\mathcal{F}_k(\mathfrak{X}), d_H^\infty)$,

$$\liminf_{n\to\infty} \frac{1}{n} \log P\left\{\frac{X_1 + X_2 + \cdots + X_n}{n} \in U\right\} \geq -\inf_{x\in U} I(x), \quad (1)$$

any for any closed set $V \subset (\mathcal{F}_k(\mathfrak{X}), d_H^\infty)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log P\left\{\frac{X_1 + X_2 + \cdots + X_n}{n} \in V\right\} \leq -\inf_{x\in V} I(x). \quad (2)$$

In order to prove our two main theorems above, we need the following two lemmas.

Lemma 3: Let $\mathfrak{X}$ is of type $p > 1$ and $X_1, X_2, \cdots, X_n$ be $(\mathcal{K}_k(\mathfrak{X}), d_H)$-valued i.i.d. random variables such that $Ee^{\lambda\|X_1\|_{\mathcal{K}}^p} < \infty$ for some $\lambda > 0$, then for any $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \ln P(d_H(\frac{X_1 + X_2 + \cdots + X_n}{n}, \frac{coX_1 + coX_2 + \cdots + coX_n}{n}) \geq \delta) = -\infty.$$

This proof is same as those of the following Lemma 4, so we omit it. But here we state the inequality of Puri and Ralescu we use in our proofs of Lemma 3 and Lemma 4.

Let A belong to $\mathcal{K}_k(\mathfrak{X})$, and its inner radius is $r(A)$, and we know $r(A) \leq 2\|A\|_{\mathcal{K}}$. In [8], Puri and Ralescu extended a result of Cassels [2] and proved the following inequality(we call it inequality of Puri and Ralescu): for any $A_1, A_2, \cdots, A_n$ in $\mathcal{K}_k(\mathfrak{X})$,

$$d_H(A_1 + A_2 + \cdots + A_n, coA_1 + coA_2 + \cdots + coA_n) \leq c^{\frac{1}{p}}(r(A_1)^p + r(A_2)^p + \cdots + r(A_n)^p)^{\frac{1}{p}}.$$

Lemma 4: Let $\mathfrak{X}$ is of type $p > 1$ and $X_1, X_2, \cdots, X_n$ be $(\mathcal{F}_k(\mathfrak{X}), d_H^\infty)$-valued i.i.d. random variables such that $Ee^{\lambda\|X_1\|_{\mathcal{F}}^p} < \infty$ for some $\lambda > 0$, then for some $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \ln P(d_H^\infty(\frac{X_1 + X_2 + \cdots + X_n}{n}, \frac{coX_1 + coX_2 + \cdots + coX_n}{n}) \geq \delta) = -\infty.$$

Proof: We apply the definition of d_H^∞ and the inequality of Puri and Ralescu and for any $A \in \mathcal{K}_k(\mathfrak{X})$, the inner radius has the property: $r(A) \leq 2\|A\|_{\mathcal{K}}$, then

$$d_H^\infty(\frac{X_1+X_2+\cdots+X_n}{n},\frac{coX_1+coX_2+\cdots+coX_n}{n})$$
$$=\frac{1}{n}\sup_{\alpha\in[0,1]}d_H(\sum_{i=1}^n[X_i]_\alpha,\sum_{i=1}^n[coX_i]_\alpha)$$

$$\leq\frac{1}{n}\cdot c^{\frac{1}{p}}\sup_{\alpha\in[0,1]}(r([X_1]_\alpha)^p+r([X_2]_\alpha)^p+\cdots+r([X_n]_\alpha)^p)^{\frac{1}{p}}$$
$$\leq\frac{1}{n}\cdot 2c^{\frac{1}{p}}(\|X_1\|_{\mathcal{F}}^p+\|X_2\|_{\mathcal{F}}^p+\cdots+\|X_n\|_{\mathcal{F}}^p)^{\frac{1}{p}}.$$

In view of the condition of Lemma 4: $Ee^{\lambda\|X_1\|_{\mathcal{K}}^p}<\infty$ for some $\lambda>0$, then for this positive $\lambda>0$, we have $Ee^{\lambda\|X_1\|_{\mathcal{K}}^p}<\infty$, so we can apply the chebyshev exponential inequality , then we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\ln P(d_H^\infty(\frac{X_1+X_2+\cdots+X_n}{n},\frac{coX_1+coX_2+\cdots+coX_n}{n})\geq\delta)$$
$$\leq\limsup_{n\to\infty}\frac{1}{n}\ln P(\|X_1\|_{\mathcal{F}}^p+|X_2\|_{\mathcal{F}}^p+\cdots+\|X_n\|_{\mathcal{F}}^p\geq\frac{n^p\delta^p}{2^pc})$$
$$\leq\limsup_{n\to\infty}\frac{1}{n}\ln[e^{-\frac{\lambda n^p\delta^p}{2^pc}}(Ee^{\lambda\|X_1\|_{\mathcal{F}}^p})^n]$$
$$=\limsup_{n\to\infty}(-\frac{\lambda n^{p-1}\delta^p}{2^pc}+Ee^{\lambda\|X_1\|_{\mathcal{F}}^p})$$
$$=-\infty.$$

So we complete the proof of this lemma.

Since random sets are particular cases of those for fuzzy random variables, then we omit the proof of Theorem 1, and only give the proof of Theorem 2.

Proof of theorem 2: Step 1: First we prove the upper bound of (1). Let $\mathcal{U}$ be a closed subset of $(\mathcal{F}_k(\mathfrak{X}),d_H^\infty)$. For any $\forall\ \delta>0$, let

$$\mathcal{U}^\delta=\{x\in\mathcal{F}_k(\mathfrak{X}):\ d_H^\infty(x,\mathcal{U})=\inf_{y\in\mathcal{U}}d_H^\infty(x,y)<\delta\}.$$

Then $P(S_n\in\mathcal{U})\leq P(coS_n\in\overline{\mathcal{U}^\delta})+P(d_H^\infty(S_n,coS_n)\geq\delta)$. So

$$\limsup_{n\to\infty}P(S_n\in\mathcal{U})$$
$$\leq\max\{\limsup_{n\to\infty}P(coS_n\in\overline{\mathcal{U}^\delta}),\limsup_{n\to\infty}P(d_H^\infty(S_n,coS_n)\geq\delta)\}$$
$$=-\inf_{x\in\overline{\mathcal{U}^\delta}}I(x).$$

Since $I(x)$ is a good rate function, by [1], we have

$$\lim_{\delta\downarrow 0}\inf_{x\in\overline{\mathcal{U}^\delta}} I(x) = \inf_{x\in\mathcal{U}} I(x).$$

So (1) holds.

Step 2: we prove the lower bound of (2). Let $\mathcal{U}$ be an open subset of $(\mathcal{F}_k(\mathfrak{X}), d_H^\infty)$. $\forall\ x \in \mathcal{U}$, then there exists a $\delta > 0$ and an open subset V of $(\mathcal{F}_k(\mathfrak{X}), d_H^\infty)$ such that $x \in V \subset V^\delta \subset \mathcal{U}$. So

$$P(S_n \in \mathcal{U}) \geq P(S_n \in V^\delta) \geq P(coS_n \in V) - P(d_H^\infty(S_n, coS_n) \geq \delta).$$

Hence $P(S_n \in V) \leq P(S_n \in \mathcal{U}) + P(d_H^\infty(S_n, coS_n) \geq \delta)$. By Lemma 4, we have

$$\liminf_{n\to\infty} P(S_n \in \mathcal{U}) \geq -\inf_{x'\in V} I(x') \geq -I(x).$$

Taking the supermum over all elements x in $\mathcal{U}$, we have

$$\liminf_{n\to\infty} P(S_n \in \mathcal{U}) \geq -\inf_{x\in\mathcal{U}} I(x).$$

This completes the proof of Theorem 2.

Remark: In 2010, Ogura, Li and Wang [6] have proved a Cramér type LDP for compact convex random upper semicontinuous functions whose underlying space is d-dimensional Euclidean space $\mathbb{R}^d$ under the condition $E[\exp\{\lambda\|X\|_\mathcal{F}\}] < \infty$, for some $\lambda > 0$ with respect to the metric d_Q(see the detailed notation in [6]). Since the d-dimensional Euclidean space $\mathbb{R}^d$ is type 2, then if $X_1, X_2, \cdots, X_n$ are $(\mathcal{F}_k(\mathbb{R}^d), d_H^\infty)$-valued i.i.d. random variables such that $Ee^{\lambda\|X_1\|_\mathcal{F}^2} < \infty$ for some $\lambda > 0$, then Lemma 4 holds. And the condition $E[\exp\{\lambda\|X\|_\mathcal{F}\}] < \infty$ also holds for this positive λ. By Theorem 3.4 in [6], we know the law of sums of quasiconcave envelope of compact random upper semicontimuous functions satisfies large deviations, then in view of Theorem 2 in our paper, the law of sums of compact random upper semicontimuous functions satisfies large deviations with the same rate function.

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